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An inf-sup stable and robust discretization of the Stokes equations with large irrotational sources on general meshes

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Abstract

In this note we propose a discretization of the Stokes equations which is inf-sup stable on general polygonal or polyhedral meshes and robust with respect to the presence of large irrotational source terms. The key idea is to construct a discrete space for the velocity which extends two important properties of the Crouzeix–Raviart element to general meshes, namely the continuity of mean values at interfaces and the approximation of nontrivial solenoidal fields. As a result, it is proved that the approximation of the velocity is not affected by the presence of the irrotational part of the source term. A numerical validation of the theoretical results is provided.

*To cite this article: *****.*

Résumé en Français

Dans cette note on propose une discrétisation des équations de Stokes inf-sup stable sur des maillages polygonaux ou polyédriques généraux et robuste par rapport à la présence de termes sources irrotationnels. L'idée consiste à construire un espace discret pour la vitesse qui étend deux propriétés importantes de l'élément de Crouzeix–Raviart à des maillages généraux, à savoir la continuité des valeurs moyennes aux interfaces et l'approximation de champs solénoïdaux non triviaux. On prouve, en particulier, que l'approximation de la vitesse n'est pas affectée par la partie irrotationnelle du terme source. Les résultats théoriques sont validés par des exemples numériques.

*Pour citer cet article : *****.*

1. Introduction

Dans cette note, inspirée par le travail récent de Galvin, Linke *et al.* [7], on considère le problème de la conservation de la masse discrète dans le cadre des équations de Stokes avec un terme source irrotationnel,

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$$-\Delta \mathbf{u} + \nabla p = \mathbf{f} + \nabla \psi \quad \text{in } \Omega, \quad (1a)$$

$$\nabla \cdot \mathbf{u} = 0 \quad \text{in } \Omega, \quad (1b)$$

$$\mathbf{u} = \mathbf{0} \quad \text{on } \partial\Omega, \quad (1c)$$

$$\int_{\Omega} p = 0, \quad (1d)$$

où $\Omega \subset \mathbb{R}^d$, $d \geq 2$, est un domaine polygonal ou polyédrique borné, $\mathbf{f} \in L^2(\Omega)^d$ et $\psi \in L_0^2(\Omega) \cap H^1(\Omega)$ est un potentiel scalaire à moyenne nulle. Soit (\mathbf{u}_f, p_f) la solution du problème (1) où on a remplacé $\mathbf{f} + \nabla \psi \rightarrow \mathbf{f}$. Une propriété importante du problème continu est que la vitesse \mathbf{u} ne dépend pas de la partie irrotationnelle du terme source, qui intervient uniquement sur la pression :

$$\mathbf{u} = \mathbf{u}_f, \quad p = p_f + \psi. \quad (2)$$

L'objet de cette note est une discrétisation inf-sup stable du problème (1) sur maillages polyédriques généraux qui permet de mimer cette propriété au niveau discret. Une conséquence importante de ce fait est que l'estimation d'erreur pour la vitesse ne dépend pas de la quantité $\|\psi\|_{H^1(\Omega)}$, à savoir, la précision de la vitesse discrète n'est pas affectée par la partie irrotationnelle du terme source.

Ce travail s'inspire des idées récentes de [5], où l'on peut trouver l'ensemble des preuves concernant la construction de la méthode. L'idée clé pour la construction de la méthode est de définir un espace discret dans l'esprit des travaux de Eymard, Gallouët, Herbin [6] et Di Pietro [3]. Cet espace possède les propriétés optimales d'approximation ainsi que la propriété de continuité des valeurs moyennes aux interfaces caractérisant l'élément de Crouzeix–Raviart dans le cas de maillages simpliciaux conformes. Cette propriété intervient à la fois dans la version discrète de (2) et dans l'estimation optimale de l'erreur de consistance.

2. Discretization

2.1. Admissible mesh sequences

Let $\mathcal{H} \subset \mathbb{R}_+^*$ denote a countable set having 0 as its unique accumulation point. For all $h \in \mathcal{H}$ we denote by $\mathcal{K}_h = \{K\}$ a finite collection of nonempty, disjoint polyhedra forming a partition of Ω s.t. $h = \max_{K \in \mathcal{K}_h} h_K$, with h_K denoting the diameter of the element K . We say that a hyperplanar closed subset F of $\bar{\Omega}$ is a mesh face if it has positive $(d-1)$ -dimensional Lebesgue measure and if either there exist $K_1, K_2 \in \mathcal{K}_h$ s.t. $F \subset \partial K_1 \cap \partial K_2$ (and F is called an *interface*) or there exists $K \in \mathcal{K}_h$ s.t. $F \subset \partial K \cap \partial\Omega$ (and F is called a *boundary face*). Interfaces are collected in the set \mathcal{F}_h^i , boundary faces in \mathcal{F}_h^b and we let $\mathcal{F}_h := \mathcal{F}_h^i \cup \mathcal{F}_h^b$. For every $K \in \mathcal{K}_h$, we let $\mathcal{F}_K := \{F \in \mathcal{F}_h \mid F \subset \partial K\}$. For all $F \in \mathcal{F}_h$ we let $\bar{\mathbf{x}}_F := \int_F \mathbf{x} / |F|_{d-1}$ where, for $0 \leq l \leq d$, $|\cdot|_l$ denotes the l -dimensional Lebesgue measure. We assume in what follows that the mesh sequence $(\mathcal{K}_h)_{h \in \mathcal{H}}$ is regular in the sense of [3, Definition 2.8], *i.e.*, (i) there exists a sequence of matching simplicial submeshes shape-regular in the sense of Ciarlet [2] and the mesh sequence $(\mathcal{K}_h)_{h \in \mathcal{H}}$ is contact-regular; (ii) the L^2 -orthogonal projector Π_h^l , $l \geq 0$, on $\mathbb{P}_d^l(\mathcal{K}_h)$ has optimal polynomial approximation properties; (iii) for all $h \in \mathcal{H}$ there exists a set of cell centers $\{\mathbf{x}_K\}_{K \in \mathcal{K}_h}$ s.t., for all $K \in \mathcal{K}_h$, K is star-shaped with respect to \mathbf{x}_K , and, for all $F \in \mathcal{F}_K$, the orthogonal distance between \mathbf{x}_K and F , henceforth denoted by $d_{K,F}$, is positive and uniformly comparable to h_K . This definition encompasses fairly general meshes featuring polyhedral elements and nonconforming interfaces. Two examples are provided in Figure 1. Elements with nonplanar faces can be treated by introducing

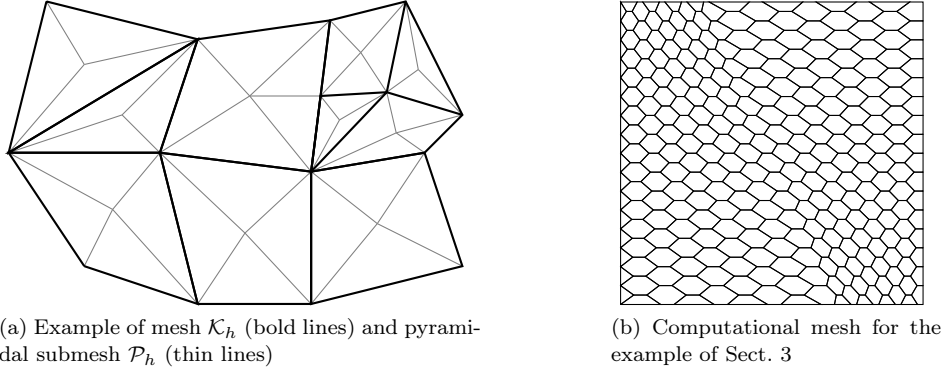


Figure 1. Two examples of admissible meshes in $d = 2$ featuring polygonal elements and nonconforming interfaces

a piecewise planar decomposition of such faces. Further details on broken polynomial spaces on general meshes can be found in [4, Chapter 1].

A consequence of the assumption on cell centers is that, for all $K \in \mathcal{K}_h$ and all $F \in \mathcal{F}_K$, the open pyramid K_F of base F and apex \mathbf{x}_K is nondegenerated, and we can define the *pyramidal submesh* \mathcal{P}_h (see Figure 1a), which can be shown to inherit the shape- and contact-regularity properties of \mathcal{K}_h , as

$$\mathcal{P}_h := \{K_F\}_{K \in \mathcal{K}_h, F \in \mathcal{F}_K}.$$

Each element of \mathcal{P}_h is associated to a unique element $K \in \mathcal{K}_h$ and a unique face $F \in \mathcal{F}_K$. When this link is irrelevant, the notation P replaces K_F . The set of faces of \mathcal{P}_h (including the mesh faces in \mathcal{F}_h as well as the lateral faces of the pyramids) is denoted by Σ_h and we let $\Sigma_h^i := \Sigma_h \setminus \mathcal{F}_h^b$.

For an integer $k \geq 0$ and $\mathcal{S}_h \in \{\mathcal{K}_h, \mathcal{P}_h\}$ we introduce the broken polynomial space

$$\mathbb{P}_d^k(\mathcal{S}_h) := \{v \in L^2(\Omega) \mid v|_S \in \mathbb{P}_d^k(S), \forall S \in \mathcal{S}_h\},$$

where \mathbb{P}_d^k denotes the space of polynomial functions of total degree at most k . For all $\sigma \in \Sigma_h^i$ we fix an orientation for the normal \mathbf{n}_σ and denote by $P^-, P^+ \in \mathcal{P}_h$ the elements s.t. $\sigma = \partial P^- \cap \partial P^+$ and \mathbf{n}_σ points out of P^- . For boundary faces $F \in \mathcal{F}_h^b$ the normal \mathbf{n}_F points out of Ω . Finally, for any function φ regular enough to admit a (possibly two-valued) trace on $\sigma \in \Sigma_h^i$ we let

$$\llbracket \varphi \rrbracket := \varphi|_{P^-} - \varphi|_{P^+}, \quad \{\varphi\} := \frac{1}{2} (\varphi|_{P^-} + \varphi|_{P^+}).$$

On boundary faces $F \in \mathcal{F}_h^b$ we conventionally set $\llbracket \varphi \rrbracket = \{\varphi\} := \varphi$.

2.2. An extension of the Crouzeix–Raviart element to general polyhedral meshes

On matching simplicial (resp. quadrangular) meshes, good mass conservation properties are classically obtained using the Crouzeix–Raviart (resp. Rannacher–Turek) elements for the velocity together with a piecewise constant approximation for the pressure. In this section we present an extension of these finite elements to general polyhedral meshes inspired by the recent works on Hybrid Finite Volume (HFV) [6] and Cell Centered Galerkin methods (ccG) [3]. Further details can be found in [5].

As for HFV methods, the space of degrees of freedom (DOFs) contains both cell- and face-unknowns,

$$\mathbb{V}_h := \mathbb{R}^{\mathcal{K}_h} \times \mathbb{R}^{\mathcal{F}_h}.$$

The generic element $\mathbb{v}_h \in \mathbb{V}_h$ is indexed as $\mathbb{v}_h = ((v_K)_{K \in \mathcal{K}_h}, (v_F)_{F \in \mathcal{F}_h})$. When dealing with homogeneous Dirichlet problems, boundary conditions can be strongly enforced by introducing the space

$$\mathbb{V}_{h,0} := \{\mathbb{v}_h \in \mathbb{V}_h \mid v_F = 0, \forall F \in \mathcal{F}_h^b\}.$$

The first step to define a discrete functional space in the spirit of ccG methods is to introduce the discrete gradient operator $\mathfrak{G}_h : \mathbb{V}_h \rightarrow \mathbb{P}_d^0(\mathcal{P}_h)^d$ such that, for all $\mathbf{v}_h \in \mathbb{V}_h$, all $K \in \mathcal{K}_h$ and all $F \in \mathcal{F}_K$, $\mathfrak{G}_h(\mathbf{v}_h)|_{K_F} = \mathbf{G}_K(\mathbf{v}_h) + \mathbf{R}_{K,F}(\mathbf{v}_h)$ with

$$\mathbf{G}_K(\mathbf{v}_h) = \sum_{F \in \mathcal{F}_K} \frac{|F|^{d-1}}{|K|^d} v_F \mathbf{n}_{K,F}, \quad \mathbf{R}_{K,F}(\mathbf{v}_h) = \frac{\eta}{d_{K,F}} (v_F - v_K - \mathbf{G}_K(\mathbf{v}_h) \cdot (\bar{\mathbf{x}}_F - \mathbf{x}_K)) \mathbf{n}_{K,F}, \quad (3)$$

where $\eta > 0$ is a stabilization parameter which will play a key role in Lemma 1. An important remark is that, by definition,

$$\forall \mathbf{v}_h \in \mathbb{V}_h, \forall K \in \mathcal{K}_h, \quad \sum_{F \in \mathcal{F}_K} |K_F|_d \mathbf{R}_{K,F}(\mathbf{v}_h) = \mathbf{0}, \quad (4)$$

i.e., the residual contribution in \mathfrak{G}_h is orthogonal to piecewise constant functions on \mathcal{K}_h . The discrete gradient \mathfrak{G}_h is then used to construct a linear perturbation of face values resulting in the piecewise affine reconstruction operator $\mathfrak{R}_h : \mathbb{V}_h \rightarrow \mathbb{P}_d^1(\mathcal{P}_h)$ s.t., for all $\mathbf{v}_h \in \mathbb{V}_h$, all $K \in \mathcal{K}_h$, all $F \in \mathcal{F}_K$, and all $\mathbf{x} \in K_F$,

$$\mathfrak{R}_h(\mathbf{v}_h)|_{K_F}(\mathbf{x}) = v_F + \mathfrak{G}_h(\mathbf{v}_h)|_{K_F} \cdot (\mathbf{x} - \bar{\mathbf{x}}_F). \quad (5)$$

The discrete space is defined as follows:

$$\mathfrak{CR}(\mathcal{K}_h) := \mathfrak{R}_h(\mathbb{V}_h).$$

When considering homogeneous Dirichlet boundary conditions, the relevant space is $\mathfrak{CR}_0(\mathcal{K}_h) := \mathfrak{R}_h(\mathbb{V}_{h,0})$.

Lemma 1 (Semicontinuity at interfaces) *Let $\eta = d$ in (3). Then, for all $\mathbf{v}_h \in \mathfrak{CR}(\mathcal{K}_h)$ (resp. $v_h \in \mathfrak{CR}_0(\mathcal{K}_h)$) there holds for all $\sigma \in \Sigma_h^i$ (resp. $\sigma \in \Sigma_h$) with $\langle \varphi \rangle_\sigma := \int_\sigma \varphi / |\sigma|_{d-1}$,*

$$\langle \llbracket v_h \rrbracket \rangle_\sigma = 0.$$

Observe that (i) the continuity of average values across mesh faces in \mathcal{F}_h^i is a trivial consequence of choosing v_F as a starting point in (5) for the affine reconstruction in $K_F \in \mathcal{P}_h$ (unlike [3, eq. (2.14)], where v_K is used instead), while (ii) the continuity of average values across lateral pyramidal faces $\sigma \in \Sigma_h^i \setminus \mathcal{F}_h^i$ is obtained by choosing $\eta = d$ in the residual $\mathbf{R}_{K,F}(\mathbf{v}_h)$ defined by (3) (unlike [6, eq. (25)], where the value $\eta = d^{1/2}$ is proposed to recover the two-point finite volume scheme on superadmissible meshes).

The interpolator $\mathcal{I}_h : H^1(\Omega) \rightarrow \mathfrak{CR}(\mathcal{K}_h)$ realizes the mapping $H^1(\Omega) \ni v \mapsto \mathcal{I}_h(v) \in \mathfrak{CR}(\mathcal{K}_h)$ s.t. $\mathcal{I}_h(v) = \mathfrak{R}_h(\mathbf{v}_h)$ for $\mathbf{v}_h \in \mathbb{V}_h$ with (i) $v_K = \Pi_h^1 v(\mathbf{x}_K) \forall K \in \mathcal{K}_h$, (ii) $v_F = \langle v \rangle_F \forall F \in \mathcal{F}_h$. When applied to vector-valued functions, \mathcal{I}_h acts component-wise.

Lemma 2 (Approximation in $\mathfrak{CR}(\mathcal{K}_h)$) *Let $l \in \{0, 1\}$. Then, there holds with ∇_h broken gradient on \mathcal{P}_h and $C > 0$ independent of h ,*

$$\forall v \in H^{l+1}(\Omega), \quad \|v - \mathcal{I}_h(v)\|_{L^2(\Omega)} + h \|\nabla_h(v - \mathcal{I}_h(v))\|_{L^2(\Omega)^d} \leq Ch^{l+1} \|v\|_{H^{l+1}(\Omega)}. \quad (6)$$

Moreover, for all $\mathbf{v} \in H^1(\Omega)^d$ with $\mathbf{v}_h := \mathcal{I}_h(\mathbf{v})$, there holds

$$\Pi_h^0(\nabla_h \cdot \mathbf{v}_h) = \Pi_h^0(\nabla \cdot \mathbf{v}). \quad (7)$$

Inequality (6) can be proved in the spirit of [3, Theorem 3.11], whereas (7) is a consequence of property (4), which allows to infer that there holds, for all $\mathbf{v}_h \in \mathfrak{CR}(\mathcal{K}_h)^d$, $\Pi_h^0(\nabla_h \cdot \mathbf{v}_h)|_K = \sum_{F \in \mathcal{F}_K} \frac{|F|^{d-1}}{|K|^d} \mathbf{v}_F \cdot \mathbf{n}_{K,F}$ (for all $F \in \mathcal{F}_h$, \mathbf{v}_F is the vector containing the degrees of freedom for the components of \mathbf{v}_h located at F).

Remark 3 (Finite element interpretation) *The proposed method can be shown to fit in the classical definition of finite element; see, e.g., [1, p. 93]. Indeed, let $K \in \mathcal{K}_h$, $\mathbb{V}_K := \mathbb{R} \times \mathbb{R}^{\mathcal{F}_K}$, $\mathcal{P}_K := \{K_F\}_{F \in \mathcal{F}_K}$, and introduce the operator $\tilde{\mathfrak{R}}_K : \mathbb{V}_K \rightarrow \mathbb{P}_d^1(\mathcal{P}_K)$ which realizes the mapping $\mathbb{V}_K \ni \mathbf{v}_K \mapsto \tilde{\mathfrak{R}}_K(\mathbf{v}_K) \in \mathbb{P}_d^1(\mathcal{P}_K)$ according to the principle of (5). Let $P_K := \tilde{\mathfrak{R}}_K(\mathbb{V}_K)$. It follows from [6, Lemma 4.1] that $\tilde{\mathfrak{R}}_K$ is injective, hence the operator \mathfrak{R}_K obtained by restricting the codomain of $\tilde{\mathfrak{R}}_K$ to P_K is bijective. The triplet defining the finite element is then $(K, P_K, \mathfrak{R}_K^{-1})$. In practical implementations, this remark can be exploited when reference elements are available, as it is usually the case in finite element codes.*

2.3. Discrete problem

Let $\mathbf{U}_h := \mathfrak{C}\mathfrak{R}_0(\mathcal{K}_h)^d$, $P_h := \mathbb{P}_d^0(\mathcal{K}_h) \cap L_0^2(\Omega)$. We equip the space \mathbf{U}_h with the norm $\|\mathbf{v}\|_{\mathbf{U}} := \|\nabla_h \mathbf{v}\|_{L^2(\Omega)^{d,d}}$ and the space P_h with the norm $\|q\|_P := \|q\|_{L^2(\Omega)}$. The fact that $\|\cdot\|_{\mathbf{U}}$ is a norm is a consequence of the semicontinuity property stated in Lemma 1. The discrete problem reads: Find $(\mathbf{u}_h, p_h) \in \mathbf{U}_h \times P_h$ s.t.

$$a_h(\mathbf{u}_h, \mathbf{v}_h) + b_h(\mathbf{v}_h, p_h) - b_h(\mathbf{u}_h, q_h) = l_h(\mathbf{v}_h), \quad \forall (\mathbf{v}_h, q_h) \in \mathbf{U}_h \times P_h, \quad (8)$$

with $a_h(\mathbf{w}, \mathbf{v}) := \int_{\Omega} \nabla_h \mathbf{w} : \nabla_h \mathbf{v}$, $b_h(\mathbf{v}, q) := - \int_{\Omega} (\nabla_h \cdot \mathbf{v}) q$, and $l_h(\mathbf{v}) := (\mathbf{f}, \mathbf{v})_{L^2(\Omega)^d} + b_h(\mathbf{v}, \Pi_h^0 \psi)$.

Lemma 4 (Stability) *There exist two reals $\alpha, \beta > 0$ independent of h s.t., for all $(\mathbf{v}_h, q_h) \in \mathbf{U}_h \times P_h$*

$$a_h(\mathbf{v}_h, \mathbf{v}_h) \geq \alpha \|\mathbf{v}_h\|_{\mathbf{U}}^2, \quad \beta \|q_h\|_P \leq \sup_{\mathbf{w}_h \in \mathbf{U}_h \setminus \{\mathbf{0}\}} \frac{b_h(\mathbf{w}_h, q_h)}{\|\mathbf{w}_h\|_{\mathbf{U}}}.$$

Observe that the inf-sup stability of the velocity-pressure coupling is an immediate consequence of (7) since we have chosen to work with piecewise constant pressures. The proof of this lemma can be found in [5].

Lemma 5 (Weak consistency) *Denote by $(\mathbf{u}_f, p_f) = (\mathbf{u}, p_f)$ the weak solution of (1) where we have substituted $\mathbf{f} + \nabla \psi \rightarrow \mathbf{f}$, and assume $\eta = d$, $\mathbf{u} \in [H_0^1(\Omega) \cap H^2(\Omega)]^d$, and $p_f \in L_0^2(\Omega) \cap H^1(\Omega)$. Then,*

$$\forall (\mathbf{v}_h, q_h) \in \mathbf{U}_h \times P_h, \quad a_h(\mathbf{u}, \mathbf{v}_h) + b_h(\mathbf{v}_h, p_f) - b_h(\mathbf{u}, q_h) = (\mathbf{f}, \mathbf{v}_h) + \mathcal{E}_{h,f}(\mathbf{v}_h),$$

with consistency error $\mathcal{E}_{h,f}(\mathbf{v}_h) := \sum_{\sigma \in \Sigma_h} \int_{\sigma} (\nabla \mathbf{u} \mathbf{n}_{\sigma} - p_f \mathbf{n}_{\sigma}) \cdot \llbracket \mathbf{v}_h \rrbracket$. Moreover, there holds with C independent of h and $\mathcal{N} := \|\mathbf{u}\|_{H^2(\Omega)^d} + \|p_f\|_{H^1(\Omega)}$,

$$\sup_{\mathbf{w}_h \in \mathbf{U}_h \setminus \{\mathbf{0}\}} \frac{\mathcal{E}_{h,f}(\mathbf{w}_h)}{\|\mathbf{w}_h\|_{\mathbf{U}}} \leq Ch\mathcal{N}. \quad (9)$$

The property stated in Lemma 1 is instrumental in obtaining the optimal bound (9) since it allows to infer, for all $\mathbf{v}_h \in \mathbf{U}_h$, $\mathcal{E}_{h,f}(\mathbf{v}_h) = \sum_{\sigma \in \Sigma_h} \int_{\sigma} (\{\nabla \mathbf{u} - \Pi_h^0 \nabla \mathbf{u}\} \mathbf{n}_{\sigma} - \{p_f - \Pi_h^0 p_f\} \mathbf{n}_{\sigma}) \cdot \llbracket \mathbf{v}_h \rrbracket$. The estimate (9) then results from the Cauchy-Schwarz and trace inequalities together with the approximation properties of the L^2 -orthogonal projector.

A classical consequence of Lemmata 4–5 together with the boundedness of the bilinear forms a_h and b_h is the error estimate

$$\|\mathbf{u}_f - \mathbf{u}_{h,f}\|_{\mathbf{U}} + \|p_f - p_{h,f}\|_P \leq Ch\mathcal{N}. \quad (10)$$

The following property reflects the fact that the velocity approximation is invariant with respect to the irrotational source term $\nabla \psi$ in (1a) as it is the case for the continuous velocity; cf. (2).

Proposition 1 (Robustness with respect to irrotational source terms) *Let $(\mathbf{u}_{h,f}, p_{h,f})$ denote the solution to (8) where we have substituted $l_h(\mathbf{v}_h) \rightarrow (\mathbf{f}, \mathbf{v}_h)_{L^2(\Omega)^d}$ in the right-hand side. Then, there holds*

$$\mathbf{u}_h = \mathbf{u}_{h,f}, \quad p_h = p_{h,f} + \Pi_h^0 \psi.$$

Theorem 6 (Error estimate) *Under the assumptions of Lemma 5 and using the notations of Lemma 5 and Proposition 1, there holds with C independent of h and \mathcal{N} defined in Lemma 5,*

$$\|\mathbf{u} - \mathbf{u}_h\|_{\mathbf{U}} \leq Ch\mathcal{N}, \quad \|p - p_h\|_P \leq Ch(\mathcal{N} + \|\psi\|_{H^1(\Omega)}). \quad (11)$$

The first estimate in (11) is a direct consequence of (10) together with (2) and Proposition 1. To prove the second estimate, we use again (2) and Proposition 1 together with the triangular inequality to infer

$$\|p - p_h\|_P = \|(p - \psi) - (p_h - \Pi_h^0 \psi) + (\psi - \Pi_h^0 \psi)\|_P \leq \|p_f - p_{h,f}\|_P + \|\psi - \Pi_h^0 \psi\|_P,$$

and conclude using the error estimate (10) together with the approximation properties of the L^2 -orthogonal projector. The key point is that the estimate for the velocity error in the $\|\cdot\|_{\mathbf{U}}$ norm does not depend on the irrotational part of the source term, thereby proving the robustness of the discretization (8).

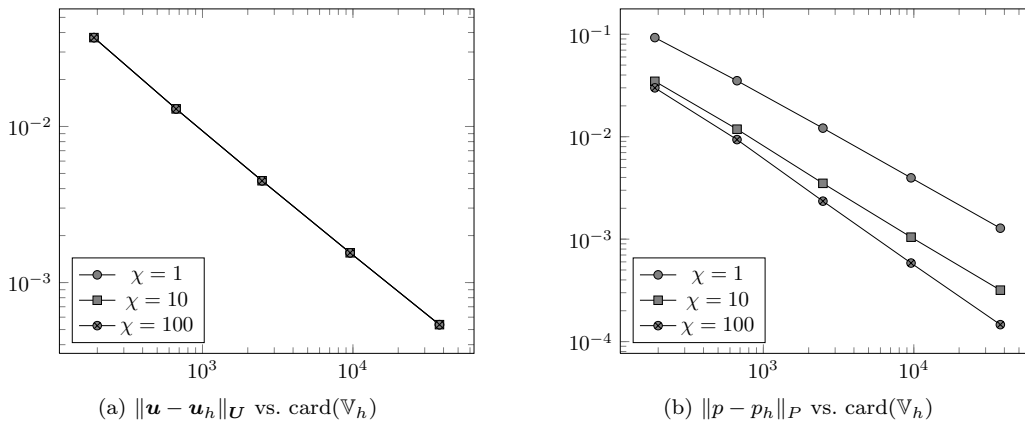


Figure 2. The convergence results confirm that the velocity approximation is independent from the irrotational source term.

3. Numerical examples

To assess the performance of the method we consider the following exact solution on the unit square domain $\Omega = (0, 1)^2$:

$$u_1 = -e^x(y \cos(y) + \sin(y)), \quad u_2 = e^x y \sin(y), \quad p_{\mathbf{f}} = p_{\mathbf{f}}^* - \langle p_{\mathbf{f}}^* \rangle_{\Omega} \text{ with } p_{\mathbf{f}}^* = 2 \exp(x) \sin(y),$$

with $\mathbf{f} \equiv \mathbf{0}$ and potential $\psi = \chi \sin(2\pi x) \sin(2\pi y)$, $\chi > 0$. The parameter χ allows to vary the value of the norm of the potential, which appears in the error estimate for the pressure (but not for the velocity); cf. Theorem 6. The discrete problem (8) is solved on a sequence of meshes containing all polygonal elements from the triangle to the hexagon; cf. Figure 1b. The numerical results of Figure 2a confirm the theoretical prediction that the approximation of the velocity is not affected by the presence of an irrotational source term, whereas this is not the case for the pressure (cf. Figure 2b). In both cases the errors are normalized with respect to the corresponding norm of the exact solution.

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