HAL
open science

A central limit theorem for the sample autocorrelations of a Lévy driven continuous time moving average process

Serge Cohen, Alexander Lindner

▶ To cite this version:

Serge Cohen, Alexander Lindner. A central limit theorem for the sample autocorrelations of a Lévy driven continuous time moving average process. Journal of Statistical Planning and Inference, 2013, 143 (8), pp.1295-1306. 10.1016/j.jspi.2013.03.022 . hal-00708264

HAL Id: hal-00708264
https://hal.science/hal-00708264

Submitted on 14 Jun 2012

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

A central limit theorem for the sample autocorrelations of a Lévy driven continuous time moving average process

Serge Cohen* Alexander Lindner†

Abstract

In this article we consider Lévy driven continuous time moving average processes observed on a lattice, which are stationary time series. We show asymptotic normality of the sample mean, the sample autocovariances and the sample autocorrelations. A comparison with the classical setting of discrete moving average time series shows that in the last case a correction term should be added to the classical Bartlett formula that yields the asymptotic variance. An application to the asymptotic normality of the estimator of the Hurst exponent of fractional Lévy processes is also deduced from these results.

Keywords: Bartlett's formula, continuous time moving average process, estimation of the Hurst index, fractional Lévy process, Lévy process, limit theorem, sample autocorrelation, sample autocovariance, sample mean.

1 Introduction

Statistical models are often written in a continuous time setting for theoretical reasons (e.g. diffusions). But if one wants to estimate the parameters of these models, one usually assumes only the observation of a discrete sample. At this point a very general question, the answer of which depends on the model chosen, is to know if the estimation should not have been performed with an underlying discrete model in the beginning. In this article we will consider this for moving average processes and

*Institut de Mathématiques de Toulouse, Université Paul Sabatier, Université de Toulouse, 118 route de Narbonne F-31062 Toulouse Cedex 9. E-mail: `Serge.Cohen@math.univ-toulouse.fr`

†Institut für Mathematische Stochastik, Technische Universität Braunschweig, Pockelsstraße 14, D-38106 Braunschweig, Germany. E-mail: `a.lindner@tu-bs.de` (Corresponding author)

we refer to the classical moving average time series models as a discrete counterpart of this continuous model.

To be more specific, let $L = (L_t)_{t\in\mathbb{R}}$ be a two sided one-dimensional Lévy process, i.e. a stochastic process with independent and stationary increments, càdlàg sample paths and which satisfies $L_0 = 0$. Assume further that L has finite variance and expectation zero, and let $f : \mathbb{R} \to \mathbb{R}$ be in $L^2(\mathbb{R})$. Let $\mu \in \mathbb{R}$. Then the process $(X_t)_{t\in\mathbb{R}}$, given by

$$X_t = \mu + \int_{\mathbb{R}} f(t-s)\, dL_s, \quad t \in \mathbb{R}, \tag{1.1}$$

can be defined in the L^2 sense and is called a *continuous time moving average process with mean μ and kernel function f, driven by L.* See also [6] for more information on such processes, in particular fractional Lévy processes. The process $(X_t)_{t\in\mathbb{R}}$ is then strictly stationary. Equation (1.1) is the natural continuous time analogue of discrete time moving average processes

$$\widetilde{X}_t = \mu + \sum_{i\in\mathbb{Z}} \psi_{t-i} Z_i, \quad t \in \mathbb{Z}, \tag{1.2}$$

where $(Z_t)_{t\in\mathbb{Z}}$ is an independent and identically distributed (i.i.d.) noise sequence with finite variance and expectation zero, and $(\psi_i)_{i\in\mathbb{Z}}$ is a square summable sequence of real coefficients. The asymptotic behaviour of the sample mean and sample autocorrelation function of $\widetilde{X}_t$ in (1.2) has been studied for various cases of noise sequences $(Z_i)_{i\in\mathbb{Z}}$, such as regularly varying noise (cf. Davis and Mikosch [7]), martingale difference sequences (cf. Hannan [9]), or i.i.d. sequences with finite fourth moment or finite variance but more restrictive conditions on the decay of the sequence $(\psi_i)_{i\in\mathbb{Z}}$ (cf. Section 7 of Brockwell and Davis [3]).

Another approach to obtain limit theorems for sample autocovariances is to prove strong mixing properties of the time series under consideration, and provided it has finite $(4+\delta)$-moment, use the corresponding central limit theorems (such as in Ibragimov and Linnik [10], Theorem 18.5.3). If even stronger strong mixing conditions hold, then existence of a fourth moment may be enough. Observe however that processes with long memory are often not strongly mixing, and in this paper we are aiming also at applications with respect to the fractional Lévy noise, which is not strongly mixing.

In this paper we shall study the asymptotic behaviour as $n \to \infty$ of the sample mean

$$\overline{X}_{n;\Delta} := n^{-1} \sum_{i=1}^{n} X_{i\Delta}, \tag{1.3}$$

of the process $(X_t)_{t\in\mathbb{R}}$ defined in (1.1) when sampled at $(\Delta n)_{n\in\mathbb{N}}$, where $\Delta > 0$ is

fixed, and of its sample autocovariance and sample autocorrelation function

$$\widehat{\gamma}_{n;\Delta}(\Delta h) := n^{-1}\sum_{i=1}^{n-h}(X_{i\Delta}-\overline{X}_{n;\Delta})(X_{(i+h)\Delta}-\overline{X}_{n;\Delta}), \quad h\in\{0,\dots,n-1\}, \tag{1.4}$$

$$\widehat{\rho}_{n;\Delta}(\Delta h) := \widehat{\gamma}_{n;\Delta}(\Delta h)/\widehat{\gamma}_{n;\Delta}(0), \quad h\in\{0,\dots,n-1\}. \tag{1.5}$$

We write $\mathbb{N}=\{0,1,2,\dots\}$. Under appropriate conditions on f and L, in particular assuming L to have finite fourth moment for the sample autocorrelation functions, it will be shown that $\overline{X}_{n;\Delta}$ and $(\widehat{\rho}_{n;\Delta}(\Delta),\dots,\widehat{\rho}_{n;\Delta}(h\Delta))$ are asymptotically normal for each $h\in\mathbb{N}$ as $n\to\infty$. This is similar to the case of discrete time moving average processes of the form (1.2) with i.i.d. noise, but unlike for those, the asymptotic variance of the sample autocorrelations of model (1.1) will turn out to be given by Bartlett's formula *plus* an extra term which depends explicitly on the fourth moment of L, and in general this extra term does not vanish. This also shows that the "naive" approach of trying to write the sampled process $(X_{n\Delta})_{n\in\mathbb{Z}}$ as a discrete time moving average process as in (1.2) with i.i.d. noise does not work in general, since for such processes the asymptotic variance would be given by Bartlett's formula only. If $\mu=0$, then further natural estimators of the autocovariance and autocorrelation are given by

$$\gamma^*_{n;\Delta}(\Delta h) := n^{-1}\sum_{i=1}^{n}X_{i\Delta}X_{(i+h)\Delta}, \quad h\in\{0,\dots,n-1\}, \tag{1.6}$$

$$\rho^*_{n;\Delta}(\Delta h) := \gamma^*_{n;\Delta}(\Delta h)/\gamma^*_{n;\Delta}(0), \quad h\in\{0,\dots,n-1\}, \tag{1.7}$$

and the conditions we have to impose to get asymptotic normality of $\gamma^*_{n;\Delta}$ and $\rho^*_{n;\Delta}$ are less restrictive than those for $\widehat{\gamma}_{n;\Delta}$ and $\widehat{\rho}_{n;\Delta}$.

We will be particularly interested in the case when f decays like a polynomial, which is e.g. the case for fractional Lévy noises. For a given Lévy process with expectation zero and finite variance, and a parameter $d\in(0,1/2)$, the *(moving average) fractional Lévy process* $(M_{t;d})_{t\in\mathbb{R}}$ *with Hurst parameter* $H:=d+1/2$ is given by

$$M^1_{t;d} := \frac{1}{\Gamma(d+1)}\int_{-\infty}^{\infty}\left[(t-s)_+^d-(-s)_+^d\right]dL_s, \quad t\in\mathbb{R} \tag{1.8}$$

(cf. Marquardt [11]). A process also called fractional Lévy process was introduced before by Benassi et al. [2], where $(x)_+=\max(x,0)$ is replaced by an absolute value in (1.8),

$$M^2_{t;d} := \int_{-\infty}^{\infty}\left[|t-s|^d-|s|^d\right]dL_s, \quad t\in\mathbb{R}. \tag{1.9}$$

Although both processes have different distributions, they enjoy similar properties. For instance the sample paths of both versions are Hölder continuous, have the same pointwise Hölder exponent, and they are both locally self-similar (see [2] for the

definition of this local property of their distributions). The corresponding *fractional Lévy noises* based on increments of length $\Delta > 0$ are given by

$$X_t^i = M_{t;d}^i - M_{t-\Delta;d}^i, \quad t \in \mathbb{R} \quad i = 1,\ 2.$$

Hence the fractional Lévy noise is a Lévy driven moving average process with kernel function

$$f_{d,\Delta}^1(s) = \frac{1}{\Gamma(d+1)} \left(s_+^d - (s-\Delta)_+^d \right), \quad s \in \mathbb{R}, \tag{1.10}$$

or

$$f_{d,\Delta}^2(s) = |s|^d - |s-\Delta|^d, \quad s \in \mathbb{R}. \tag{1.11}$$

While the kernel functions $f_{d,\Delta}^i$, $i \in \{1,2\}$, do not satisfy the assumptions we will impose for the theorems regarding the sample mean $\overline{X}_{n;\Delta}$ and the sample autocorrelation function $\widehat{\rho}_{n;\Delta}$, for $d \in (0, 1/4)$ they do satisfy the assumptions we impose for the asymptotic behaviour of $\rho_{n;\Delta}^*$, so that an asymptotically normal estimator of the autocorrelation and hence of the Hurst index can be obtained if $d \in (0, 1/4)$. For general $d \in (0, 1/2)$, one may take the differenced fractional Lévy noises $M_{t;d}^i - 2M_{t-\Delta;d}^i + M_{t-2\Delta;d}^i$, $t \in \mathbb{R}$, and our theorems give asymptotically normal estimators for the autocorrelation function of these processes. Please note that asymptotically normal estimators of the Hurst exponent for $M_{t;d}^2$ are already described in [2] but they use fill-in observations of the sample paths $X^2(k/2^n)$ for $k = 1, \ldots, 2^n - 1$. If L is a Brownian motion, then $X^1 \stackrel{d}{=} CX^2$, where $\stackrel{d}{=}$ means equality in distribution for processes and C is a constant, is the fractional Brownian motion and it is self-similar. Except in this case, fractional Lévy processes are not self-similar and therefore observations on a grid $k/2^n$ do not yield the same information as the time series $X^i(t)$, $t \in \mathbb{Z}$.

The paper is organised as follows: in the next section we will derive asymptotic normality of the sample mean. Then, in Section 3 we will derive central limit theorems for the sample autocovariance $\widehat{\gamma}_{n;\Delta}$ and the sample autocorrelation $\widehat{\rho}_{n;\Delta}$, as well as for the related estimators $\gamma_{n;\Delta}^*$ and $\rho_{n;\Delta}^*$ of (1.6) and (1.7). As a byproduct of the asymptotic normality, these quantities are consistent estimators of the autocovariance and autocorrelation. Section 4 presents an application of our results to the estimation of the parameters of fractional Lévy noises, where the underlying Hurst parameter is estimated. We also recall there that fractional Lévy noises are mixing in the ergodic-theoretic sense, and we prove that they fail to be strongly mixing.

Throughout the paper, unless indicated otherwise, L will be a Lévy process with mean zero and finite variance $\sigma^2 = EL_1^2$, and $X = (X_t)_{t\in\mathbb{R}}$ denotes the process defined in (1.1) with kernel $f \in L^2(\mathbb{R})$, $f : \mathbb{R} \to \mathbb{R}$. Its autocovariance at lag $h \in \mathbb{R}$ will be denoted by

$$\gamma(h) = \gamma_f(h) = \mathrm{Cov}\,(X_0, X_h) = \sigma^2 \int_{\mathbb{R}} f(-s) f(h-s)\, ds, \tag{1.12}$$

where the last equation follows from the Itô isometry.

Let us set some notations used in the sequel.

If v is vector or A a matrix the transposed is denoted by v', respectively by A'.

Convergence in distribution is denoted by $\xrightarrow{d}$.

The function $\mathbf{1}_A$ for a set A is one for $x \in A$, and vanishing elsewhere.

The autocorrelation of X at lag h will be denoted by $\rho(h) = \rho_f(h) = \gamma(h)/\gamma(0)$.

2 Asymptotic normality of the sample mean

The sample mean $\overline{X}_{n;\Delta}$ of the moving average process X of (1.1) behaves like the sample mean of a discrete time moving average process with i.i.d. noise, in the sense that it is asymptotically normal with variance $\sigma^2 \sum_{k=-\infty}^{\infty} \gamma(k\Delta)$, provided the latter is absolutely summable.

Theorem 2.1. *Let L have zero mean and variance σ^2, let $\mu \in \mathbb{R}$ and $\Delta > 0$. Suppose that*

$$\left(F_\Delta : [0,\Delta] \to [0,\infty], \quad u \mapsto F_\Delta(u) = \sum_{j=-\infty}^{\infty} |f(u+j\Delta)| \right) \in L^2([0,\Delta]). \tag{2.1}$$

Then $\sum_{j=-\infty}^{\infty} |\gamma(\Delta j)| < \infty$,

$$\sum_{j=-\infty}^{\infty} \gamma(\Delta j) = \sigma^2 \int_0^\Delta \left(\sum_{j=-\infty}^{\infty} f(u+j\Delta) \right)^2 du, \tag{2.2}$$

and the sample mean of $X_\Delta, \ldots, X_{n\Delta}$ is asymptotically normal as $n \to \infty$, more precisely

$$\sqrt{n}\, \overline{X}_{n;\Delta} \xrightarrow{d} N\left(\mu, \sigma^2 \int_0^\Delta \left(\sum_{j=-\infty}^{\infty} f(u+\Delta j) \right)^2 du \right) \quad \text{as } n \to \infty.$$

Remark 2.2. *Throughout the paper the assumption that L has zero mean can be dropped very often. For instance, if $f \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$, then the assumption of zero mean of L presents no restriction, for in that case $L'_t := L_t - tE(L_1)$, $t \in \mathbb{R}$, defines another Lévy process with mean zero and the same variance, and it holds*

$$X_t = \mu + E(L_1) \int_{\mathbb{R}} f(s)\, ds + \int_{\mathbb{R}} f(t-s)\, dL'_s, \quad t \in \mathbb{Z}, \tag{2.3}$$

which has mean $\mu + E(L_1) \int_{\mathbb{R}} f(s)\, ds$.

Proof. For simplicity in notation, assume that $\Delta = 1$, and write $F = F_1$. Continue F periodically on $\mathbb{R}$ by setting

$$F(u) = \sum_{j=-\infty}^{\infty} |f(u+j)|, \quad u \in \mathbb{R}.$$

Since

$$|\gamma_f(h)| \le \sigma^2 \int_{-\infty}^{\infty} |f(-s)|\,|f(h-s)|\,ds$$

by (1.12), we have

$$\begin{aligned}
\frac{1}{\sigma^2}\sum_{h=-\infty}^{\infty} |\gamma_f(h)| &\le \int_{-\infty}^{\infty} |f(-s)| \sum_{h=-\infty}^{\infty} |f(h-s)|\,ds \\
&= \int_{-\infty}^{\infty} |f(s)|F(s)\,ds \\
&= \sum_{j=-\infty}^{\infty} \int_0^1 |f(s+j)|F(s)\,ds \\
&= \int_0^1 F(s)\,F(s)\,ds < \infty. \qquad (2.4)
\end{aligned}$$

The same calculation without the modulus gives (2.2).

The proof for asymptotic normality is now much in the same spirit as for discrete time moving average processes, by reducing the problem to m-dependent sequences first and then applying an appropriate variant of Slutsky's theorem. By subtracting the mean we may assume without loss of generality that $\mu = 0$. For $m \in \mathbb{N}$, let $f_m := f\,\mathbf{1}_{(-m,m)}$, and denote

$$X_t^{(m)} := \int_{\mathbb{R}} f_m(s)\,dL_s = \int_{t-m}^{t+m} f(t-s)\,dL_s, \quad t \in \mathbb{Z}.$$

Observe that $(X_t^{(m)})_{t\in\mathbb{Z}}$ is a $(2m-1)$-dependent sequence, i.e. $(X_j^{(m)})_{j\le t}$ and $(X_j^{(m)})_{j\ge t+2m}$ are independent for each $t \in \mathbb{Z}$. From the central limit theorem for strictly stationary $(2m-1)$-dependent sequences (cf. Theorem 6.4.2 in Brockwell and Davis [3]) we then obtain that

$$\sqrt{n}\,\overline{X}_{n;1}^{(m)} = n^{-1/2}\sum_{t=1}^{n} X_t^{(m)} \xrightarrow{d} Y^{(m)}, \quad n \to \infty, \qquad (2.5)$$

where $Y^{(m)}$ is a random variable such that

$$Y^{(m)} \stackrel{d}{=} N(0, v_m)$$

with $v_m = \sum_{j=-2m}^{2m} \gamma_{f_m}(j)$. Since $\lim_{m\to\infty} \gamma_{f_m}(j) = \gamma_f(j)$ for each $j \in \mathbb{Z}$ by (1.12), since

$$|\gamma_{f\mathbf{1}_{(-m,m)}}(j)| \le \sigma^2 \int_{-\infty}^{\infty} |f(-s)|\,|f(j-s)|\,ds,$$

and $\sum_{j=-\infty}^{\infty}\int_{-\infty}^{\infty} |f(-s)|\,|f(j-s)| < \infty$ by (2.4), it follows from Lebesgue's dominated convergence theorem that $\lim_{m\to\infty} v_m = \sum_{j=-\infty}^{\infty} \gamma_f(j)$. Hence by (2.2),

$$Y^{(m)} \xrightarrow{d} Y, \quad m \to \infty, \quad \text{where} \quad Y \stackrel{d}{=} N\left(0, \sigma^2 \int_0^1 \left(\sum_{j=-\infty}^{\infty} f(u+j)\right)^2 du\right). \qquad (2.6)$$

A similar argument gives $\lim_{m\to\infty}\sum_{j=-\infty}^{\infty}\gamma_{f-f_m}(j)=0$, so that

$$\begin{aligned}
&\lim_{m\to\infty}\lim_{n\to\infty}\operatorname{Var}\left(n^{1/2}(\overline{X}_{n;1}-\overline{X}_{n;1}^{(m)})\right)\\
&\quad=\lim_{m\to\infty}\lim_{n\to\infty}n\operatorname{Var}\left(n^{-1}\sum_{t=1}^{n}\int_{-\infty}^{\infty}(f(t-s)-f_m(t-s))\,dL_s\right),\\
&\quad=\lim_{m\to\infty}\sum_{j=-\infty}^{\infty}\gamma_{f-f_m}(j)=0,
\end{aligned}$$

where we used Theorem 7.1.1 in Brockwell and Davis [3] for the second equality. An application of Chebychef's inequality then shows that

$$\lim_{m\to\infty}\limsup_{n\to\infty}P(n^{1/2}|\overline{X}_{n;1}-\overline{X}_{n;1}^{(m)}|>\varepsilon)=0$$

for every $\varepsilon>0$. Together with (2.5) and (2.6) this implies the claim by a variant of Slutsky's theorem (cf. [3], Proposition 6.3.9). □

Remark 2.3. *Let us start with an easy remark on a necessary condition on the kernel f to apply the previous theorem. Obviously $F_\Delta\in L^1([0,\Delta])$ is equivalent to $f\in L^1(\mathbb{R})$. Hence $F_\Delta\in L^2([0,\Delta])\Rightarrow f\in L^1(\mathbb{R})$.*

Remark 2.4. *Unlike for the discrete time moving average process of* (1.2)*, where absolute summability of the autocovariance function is guaranteed by absolute summability of the coefficient sequence, for the continuous time series model* (1.1) *it is not enough to assume that the kernel satisfies $f\in L^1(\mathbb{R})\cap L^2(\mathbb{R})$. An example is given by taking $\Delta=1$ and*

$$f(u):=\begin{cases}0, & u\le 0,\\ 1, & u\in[0,1),\\ \frac{1\cdot 3\cdots(2j-1)}{2^j j!}(u-j)^j, & u\in[j,j+1),\quad j\in\mathbb{N}.\end{cases}$$

For then the function F_1 is given by

$$F_1(u)=\sum_{j\in\mathbb{Z}}f(u+j)=(1-u)^{-1/2},\quad u\in[0,1),$$

so that $F_1\in L^1([0,1])\setminus L^2([0,1])$. But $F_1\in L^1([0,1])$ is equivalent to $f\in L^1(\mathbb{R})$, and since $|f(u)|\le 1$ for all $u\in\mathbb{R}$, this implies also $f\in L^2(\mathbb{R})$. Observe further that for non-negative f, condition (2.1) *is indeed necessary and sufficient for absolute summability of the autocovariance function.*

3 Asymptotic normality of the sample autocovariance

As usual, we consider the stationary process

$$X_t = \int_{-\infty}^{\infty} f(t-s)\, dL_s, \quad t \in \mathbb{R}. \tag{3.1}$$

We recall that

$$\gamma^*_{n;\Delta}(h\Delta) = n^{-1} \sum_{t=1}^{n} X_{t\Delta} X_{(t+h)\Delta}, \quad h \in \mathbb{N}$$

and first we establish an asymptotic result for $\mathrm{Cov}\,(\gamma^*_{n;\Delta}(p\Delta), \gamma^*_{n;\Delta}(q\Delta))$.

Proposition 3.1. *Let L be a (non-zero) Lévy process, with expectation zero, and finite fourth moment, and denote $\sigma^2 := EL_1^2$ and $\eta := \sigma^{-4} EL_1^4$. Let $\Delta > 0$, and suppose further that $f \in L^2(\mathbb{R}) \cap L^4(\mathbb{R})$ and that*

$$\left([0,\Delta] \to \mathbb{R}, \quad u \mapsto \sum_{k=-\infty}^{\infty} f(u+k\Delta)^2 \right) \in L^2([0,\Delta]). \tag{3.2}$$

For $q \in \mathbb{Z}$ denote

$$g_{q;\Delta} : [0,\Delta] \to \mathbb{R}, \quad u \mapsto \sum_{k=-\infty}^{\infty} f(u+k\Delta) f(u+(k+q)\Delta),$$

which belongs to $L^2([0,\Delta])$, by the previous assumption. If further

$$\sum_{h=-\infty}^{\infty} |\gamma(h\Delta)|^2 < \infty, \tag{3.3}$$

then we have for each $p, q \in \mathbb{N}$

$$\lim_{n\to\infty} n\mathrm{Cov}\,(\gamma^*_{n;\Delta}(p\Delta), \gamma^*_{n;\Delta}(q\Delta)) = (\eta-3)\sigma^4 \int_0^{\Delta} g_{p;\Delta}(u) g_{q;\Delta}(u)\, du + \sum_{k=-\infty}^{\infty} \left[\gamma(k\Delta)\gamma((k-p+q)\Delta) + \gamma((k+q)\Delta)\gamma((k-p)\Delta) \right]. \tag{3.4}$$

Proof. For simplicity in notation we assume that $\Delta = 1$. The general case can be proved analogously or reduced to the case $\Delta = 1$ by a simple time change. We shall first show that for $t, p, h, q \in \mathbb{Z}$

$$\begin{aligned} & E(X_t X_{t+p} X_{t+h+q} X_{t+h+p+q}) \\ = \; & (\eta-3)\sigma^4 \int_{-\infty}^{\infty} f(u) f(u+p) f(u+h+p) f(u+h+p+q)\, du \\ & + \gamma(p)\gamma(q) + \gamma(h+p)\gamma(h+q) + \gamma(h+p+q)\gamma(h). \end{aligned} \tag{3.5}$$

To show this, assume first that f is of the form

$$f(s) = f_{m,\epsilon}(s) = \sum_{i=-m/\epsilon}^{m/\epsilon} \psi_i \mathbf{1}_{(i\epsilon,(i+1)\epsilon]}(s), \tag{3.6}$$

where $m \in \mathbb{N}$, $\epsilon > 0$ such that $1/\epsilon \in \mathbb{N}$, and $\psi_i \in \mathbb{R}$, $i = -m/\epsilon, \ldots, m/\epsilon$. Denote

$$X_{t;m,\epsilon} := \int_{-\infty}^{\infty} f_{m,\epsilon}(t-s)\, dL_s = \sum_{i=-m/\epsilon}^{m/\epsilon} \psi_i (L_{t-i\epsilon} - L_{t-(i+1)\epsilon}), \quad t \in \mathbb{R}.$$

Denote further

$$Z_i := L_{i\epsilon} - L_{(i-1)\epsilon}, \quad i \in \mathbb{Z}.$$

Then $(Z_i)_{i\in\mathbb{Z}}$ is i.i.d. and we have

$$X_{t\epsilon;m,\epsilon} = \sum_{i=-m/\epsilon}^{m/\epsilon} \psi_i Z_{t-i}, \quad t \in \mathbb{Z}.$$

At this point we will need to compute the fourth moment of integrals of the Lévy process. Let us state an elementary result that yields a formula for this moment.

Lemma 3.2. *Let $\phi \in L^2(\mathbb{R}) \cap L^4(\mathbb{R})$, then, with the assumptions and notations on L used in Proposition 3.1,*

$$E(\int_{\mathbb{R}} \phi(s) dL_s)^4 = (\eta - 3)\sigma^4 \int_{\mathbb{R}} \phi^4(s) ds + 3\sigma^4 (\int_{\mathbb{R}} \phi^2(s) ds)^2. \tag{3.7}$$

Proof. If ν is the Lévy measure of L and A its Gaussian variance, then by the Lévy Khintchine formula we get

$$\begin{aligned} \xi(u) &= E \exp\left(iu \int_{\mathbb{R}} \phi(s) dL_s \right) \\ &= \exp\left(-\frac{1}{2} A u^2 \int_{\mathbb{R}} \phi^2(s) ds + \int_{\mathbb{R}\times\mathbb{R}} [e^{iu\phi(s)x} - 1 - iu\phi(s)x] \nu(dx) ds \right). \end{aligned}$$

Then $E(\int_{\mathbb{R}} \phi(s) dL_s)^4$ is obtained as the fourth derivative of ξ at $u = 0$. If we recall that $(\eta - 3)\sigma^4 = \int_{\mathbb{R}} x^4 \nu(dx)$, and $\sigma^2 = A + \int_{\mathbb{R}} x^2 \nu(dx)$, we get (3.7), after elementary but tedious computations. □

To continue with the proof of Proposition 3.1, we now apply (3.7) to the special case where $f(s) = \mathbf{1}_{(0,\epsilon]}(s)$ and we get

$$EZ_i^2 = EL_\epsilon^2 = \sigma^2 \epsilon, \quad EZ_i^4 = EL_\epsilon^4 = \eta\sigma^4\epsilon - 3\sigma^4\epsilon + 3\sigma^4\epsilon^2. \tag{3.8}$$

As shown in the proof of Proposition 7.3.1 in [3], we then have

$$\begin{aligned} &E(X_{t;m,\epsilon} X_{t+p;m,\epsilon} X_{t+h+p;m,\epsilon} X_{t+h+p+q;m,\epsilon}) \\ &= \left(EZ_i^4 - 3(EZ_i^2)^2\right) \sum_{i=-m/\epsilon}^{m/\epsilon} \psi_i \psi_{i+p/\epsilon} \psi_{i+h/\epsilon+p/\epsilon} \psi_{i+h/\epsilon+p/\epsilon+q/\epsilon} \\ &\quad + \gamma_{m,\epsilon}(p)\gamma_{m,\epsilon}(q) + \gamma_{m,\epsilon}(h+p)\gamma_{m,\epsilon}(h+q) + \gamma_{m,\epsilon}(h+p+q)\gamma_{m,\epsilon}(h), \end{aligned}$$

where $\gamma_{m,\epsilon}(u) = E(X_{0;m,\epsilon}X_{u;m,\epsilon})$, $u \in \mathbb{R}$. By (3.8),

$$EZ_i^4 - 3(EZ_i^2)^2 = (\eta - 3)\sigma^4\epsilon,$$

and

$$\begin{aligned}
&\epsilon \sum_{i=-m/\epsilon}^{m/\epsilon} \psi_i\psi_{i+p/\epsilon}\psi_{i+h/\epsilon+p/\epsilon}\psi_{i+h/\epsilon+p/\epsilon+q/\epsilon} \\
&= \int_{-\infty}^{\infty} f(u)f(u+p)f(u+h+p)f(u+h+p+q)\,du,
\end{aligned}$$

so that (3.5) follows for f of the form $f = f_{m,\epsilon}$. Now let $f \in L^2(\mathbb{R}) \cap L^4(\mathbb{R})$ and X_t, $t \in \mathbb{R}$, defined by (3.1). Then there is a sequence of functions $(f_{m_k,\epsilon_k})_{k\in\mathbb{N}}$ of the form (3.6) such that f_{m_k,ϵ_k} converges to f both in $L^2(\mathbb{R})$ and in $L^4(\mathbb{R})$ as $k \to \infty$. Then for each fixed $t \in \mathbb{R}$, we have that $X_{t;m_k,\epsilon_k} \to X_t$ in $L^2(P)$ (P the underlying probability measure) as $k \to \infty$, where we used the Itô isometry. Further, by Lemma 3.2, and convergence of f_{m_k,ϵ_k} both in $L^2(\mathbb{R})$ and in $L^4(\mathbb{R})$, we get convergence of $X_{t;m_k,\epsilon_k}$ to X_t in $L^4(P)$. This then shows (3.5), by letting f_{m_k,ϵ_k} converge to f both in $L^2(\mathbb{R})$ and $L^4(\mathbb{R})$ and observing that $\gamma_{m_k,\epsilon_k}(u) \to \gamma(u)$ for each $u \in \mathbb{R}$. From (3.5) we conclude that, with $p, q \in \mathbb{N}$,

$$\mathrm{Cov}\,(\gamma^*_{n;1}(p), \gamma^*_{n;1}(q)) = n^{-1} \sum_{|k|<n} (1 - n^{-1}|k|)T_k, \tag{3.9}$$

where

$$\begin{aligned}
T_k &= \gamma(k)\gamma(k-p+q) + \gamma(k+q)\gamma(k-p) \\
&\quad +(\eta-3)\sigma^4 \int_{-\infty}^{\infty} f(u)f(u+p)f(u+k)f(u+q+k)\,du.
\end{aligned}$$

Now by (3.3), $\sum_{k=-\infty}^{\infty} |T_k| < \infty$ if

$$\sum_{k=-\infty}^{\infty} \left| \int_{-\infty}^{\infty} f(u)f(u+p)f(u+k)f(u+q+k)\,du \right| \tag{3.10}$$

is finite. Denote

$$G_r(u) := \sum_{k=-\infty}^{\infty} |f(u+k)f(u+k+r)|, \quad u \in \mathbb{R}, \quad r \in \mathbb{N}.$$

Then G_r is periodic, and by assumption, G_r restricted to $[0,1]$ is square integrable.

Hence we can estimate (3.10) by

$$\begin{aligned}
&\sum_{k=-\infty}^{\infty}\int_{-\infty}^{\infty}|f(u)f(u+p)|\,|f(u+k)f(u+q+k)|\,du\\
=&\sum_{h=-\infty}^{\infty}\int_{h}^{h+1}|f(u)f(u+p)|\,G_q(u)\,du\\
=&\sum_{h=-\infty}^{\infty}\int_{0}^{1}|f(u+h)f(u+p+h)|\,G_q(u)\,du\\
=&\int_0^1 G_p(u)G_q(u)\,du<\infty.
\end{aligned}$$

The same calculation without the modulus and an application of the dominated convergence theorem to (3.9) then shows (3.4). □

Remark 3.3. *A sufficient condition for* (3.3) *is that* $\sum_{h=-\infty}^{\infty}|\gamma(h\Delta)|<\infty$*, which is implied by the function* F_Δ *in Theorem 2.1 belonging to* $L^2([0,\Delta])$*. Another sufficient condition is that* $\Phi: u\mapsto\sum_{k\in\mathbb{Z}}|\mathcal{F}(f)(u+2\pi k/\Delta)|^2$*,* $u\in[0,2\pi/\Delta]$ *is in* $L^\infty([0,2\pi/\Delta])$*, where* $\mathcal{F}(f)$ *is the Fourier transform of* $f\in L^2(\mathbb{R})$ *in the form* $z\mapsto\int_{-\infty}^{\infty}e^{izt}f(t)\,dt$ *(for* L^1*-functions). For if* $\|\Phi\|_\infty\le B$*, then* $(f(\cdot+h\Delta))_{h\in\mathbb{Z}}$ *is a Bessel sequence in* $L^2(\mathbb{R})$ *with bound* B/Δ*, i.e.*

$$\sum_{h=-\infty}^{\infty}\left|\int_{-\infty}^{\infty}\varphi(u)f(u+h\Delta)\,du\right|^2\le B\Delta^{-1}\int_{-\infty}^{\infty}\varphi(u)^2\,du\quad\forall\,\varphi\in L^2(\mathbb{R}),$$

see e.g. Theorem 7.2.3 in Christensen [5]. Taking $\varphi=f$ *then gives the square summability of the autocovariance functions by* (1.12)*.*

Please remark that $\sum_{h=-\infty}^{\infty}\gamma(h\Delta)^2<\infty$ *cannot be deduced from the condition that* $u\mapsto\sum_{k=-\infty}^{\infty}f(u+k\Delta)^2$ *is in* $L^2([0,\Delta])$*. One can take* $\Delta=1$ *and* $f(s)=\sum_{i\ge1}\frac{\mathbf{1}_{(i,i+1]}(s)}{i^H}$ *for* $\frac12<H\le\frac34$*, to get the latter condition but not* $\sum_{h=-\infty}^{\infty}\gamma(h)^2<\infty$*.*

Remark 3.4. *By* (1.12)*, the condition* (3.3) *can be written as* $\sum_{k=-\infty}^{\infty}\left(\int_{-\infty}^{\infty}f(s)f(s+k\Delta)\,ds\right)^2<\infty$*. The assumption* (3.11) *used in Theorem 3.5 below is slightly stronger than* (3.3)*, but equivalent to* (3.3) *if* $f\ge0$*.*

The following theorem gives asymptotic normality of the sample autocovariance and sample autocorrelation and the related estimators $\gamma^*_{n;\Delta}$ and $\rho^*_{n;\Delta}$.

Theorem 3.5. *(a) Suppose the assumptions of Proposition 3.1 are satisfied and suppose further that*

$$\sum_{k=-\infty}^{\infty}\left(\int_{-\infty}^{\infty}|f(s)f(s+k\Delta)|\,ds\right)^2<\infty. \tag{3.11}$$

Then we have for each $h\in\mathbb{N}$

$$\sqrt{n}(\gamma^*_{n;\Delta}(0)-\gamma(0),\dots,\gamma^*_{n;\Delta}(h\Delta)-\gamma(h))'\xrightarrow{d}N(0,V),\quad n\to\infty, \tag{3.12}$$

where $V = (v_{pq})_{p,q=0,\dots,h} \in \mathbb{R}^{h+1,h+1}$ *is the covariance matrix defined by*

$$v_{pq} = (\eta - 3)\sigma^4 \int_0^\Delta g_{p;\Delta}(u) g_{q;\Delta}(u)\, du + \sum_{k=-\infty}^{\infty} \big[\gamma(k\Delta)\gamma((k-p+q)\Delta) + \gamma((k+q)\Delta)\gamma((k-p)\Delta)\big]. \quad (3.13)$$

(b) In addition to the assumptions of (a), assume that the function

$$u \mapsto \sum_{j=-\infty}^{\infty} |f(u + j\Delta)|$$

is in $L^2([0,\Delta])$*. Denote by*

$$\widehat{\gamma}_{n;\Delta}(j\Delta) = n^{-1} \sum_{t=1}^{n-j} (X_{t\Delta} - \overline{X}_{n;\Delta})(X_{(t+j)\Delta} - \overline{X}_{n;\Delta}), \quad j = 0, 1, \dots, n-1,$$

the sample autocovariance, as defined in (1.4)*. Then we have for each* $h \in \mathbb{N}$

$$\sqrt{n}(\widehat{\gamma}_{n;\Delta}(0) - \gamma(0), \dots, \widehat{\gamma}_{n;\Delta}(h\Delta) - \gamma(h))' \xrightarrow{d} N(0, V), \quad n \to \infty,$$

where $V = (V_{pq})_{p,q=0,\dots,h}$ *is defined by* (3.13)*.*
(c) For $j \in \mathbb{N}$ *let* $\rho^*_{n;\Delta}(j\Delta) = \gamma^*_{n;\Delta}(j\Delta)/\gamma^*_{n;\Delta}(0)$ *and* $\widehat{\rho}_n(j\Delta) = \widehat{\gamma}_{n;\Delta}(j\Delta)/\widehat{\gamma}_{n;\Delta}(0)$*, the latter being the sample autocorrelation at lag* $j\Delta$*. Suppose that* f *is not almost everywhere equal to zero. Then, under the assumptions of (a), we have for each* $h \in \mathbb{N}$*, that*

$$\sqrt{n}(\rho^*_{n;\Delta}(\Delta) - \rho(\Delta), \dots, \rho^*_{n;\Delta}(h\Delta) - \rho(h\Delta))' \xrightarrow{d} N(0, W), \quad n \to \infty, \quad (3.14)$$

where $W = W_\Delta = (w_{ij;\Delta})_{i,j=1,\dots,h}$ *is given by*

$$w_{ij;\Delta} = \widetilde{w}_{ij;\Delta} + \frac{(\eta-3)\sigma^4}{\gamma(0)^2} \int_0^\Delta \big(g_{i;\Delta}(u) - \rho(i\Delta) g_{0;\Delta}(u)\big)\big(g_{j;\Delta}(u) - \rho(j\Delta) g_{0;\Delta}(u)\big)\, du,$$

and

$$\begin{aligned}
\widetilde{w}_{ij;\Delta} &= \sum_{k=-\infty}^{\infty} \big(\rho((k+i)\Delta)\rho((k+j)\Delta) + \rho((k-i)\Delta)\rho((k+j)\Delta) + 2\rho(i\Delta)\rho(j\Delta)\rho(k\Delta)^2 \\
&\quad -2\rho(i\Delta)\rho(k\Delta)\rho((k+j)\Delta) - 2\rho(j\Delta)\rho(k\Delta)\rho((k+i)\Delta)\big) \\
&= \sum_{k=1}^{\infty} \big(\rho((k+i)\Delta) + \rho((k-i)\Delta) - 2\rho(i\Delta)\rho(k\Delta)\big) \times \\
&\qquad \big(\rho((k+j)\Delta) + \rho((k-j)\Delta) - 2\rho(j\Delta)\rho(k\Delta)\big)
\end{aligned}$$

is given by Bartlett's formula. If additionally the function $u \mapsto \sum_{j=-\infty}^{\infty} |f(u + j\Delta)|$ *is in* $L^2([0,\Delta])$*, then it also holds that*

$$\sqrt{n}(\widehat{\rho}_{n;\Delta}(\Delta) - \rho(\Delta), \dots, \widehat{\rho}_{n;\Delta}(h\Delta) - \rho(h\Delta))' \xrightarrow{d} N(0, W), \quad n \to \infty. \quad (3.15)$$

Proof. For simplicity in notation we assume again $\Delta = 1$ in this proof.

(a) Using Proposition 3.1 it follows as in the proof of Proposition 7.3.2 in [3], that the claim is true if f has additionally compact support. For general f and $m \in \mathbb{N}$ let $f_m := f\mathbf{1}_{(-m,m)}$. Hence we have that

$$n^{1/2}(\gamma^*_{n;(m)}(0) - \gamma_m(0), \ldots, \gamma^*_{n;(m)}(h) - \gamma_m(h))' \xrightarrow{d} \mathbf{Y}_m, \quad m \to \infty,$$

where γ_m is the autocovariance function of the process $X_{t;m} = \int_{-\infty}^{\infty} f_m(t-s)\, dL_s$, $\gamma^*_{n;(m)}(p) = n^{-1}\sum_{t=1}^n X_{t;m}X_{t+p;m}$ the corresponding autocovariance estimate, and $\mathbf{Y}_m \stackrel{d}{=} N(0, V_m)$ with $V_m = (v_{pq;m})_{p,q=0,\ldots,h}$ and

$$v_{pq;m} = (\eta-3)\sigma^4 \int_0^1 g_{p;(m)}(u) g_{q;(m)}(u)\, du + \sum_{k=-\infty}^{\infty} \big[\gamma_m(k)\gamma_m(k-p+q) + \gamma_m(k+q)\gamma_m(k-p)\big].$$

Here, $g_{p;(m)}(u) = \sum_{k=-\infty}^{\infty} f_m(u+k) f_m(u+k+p)$, $u \in [0,1]$.
Next, we want to show that $\lim_{m\to\infty} V_m = V$. Observe first that

$$g_{p;(m)}(u) = \sum_{k=-\infty}^{\infty} f_m(u+k) f_m(u+k+p) \to \sum_{k=-\infty}^{\infty} f(u+k) f(u+k+p) = g_{p,1}(u) =: g_p(u)$$

almost surely in the variable u as $m \to \infty$ by Lebesgue's dominated convergence theorem, since $u \mapsto \sum_{k=-\infty}^{\infty} |f(u+k)f(u+k+p)|$ is in $L^2([0,1])$ by (3.2) and hence is almost surely finite. Further we have

$$|g_{p;(m)}(u)| \le \sum_{k=-\infty}^{\infty} |f(u+k)f(u+k+p)|$$

uniformly in u and m, so that again by the dominated convergence theorem we have that $g_{p;(m)} \to g_p$ in $L^2([0,1])$ as $m \to \infty$. Next, observe that

$$|\gamma_m(k)| \le \int_{-\infty}^{\infty} |f(s)f(s+k)|\, ds \quad \forall\, m \in \mathbb{N} \quad \forall\, k \in \mathbb{Z}.$$

Since $\lim_{m\to\infty} \gamma_m(k) = \gamma(k)$ for every $k \in \mathbb{Z}$, it follows from the dominated convergence theorem and (3.11) that $(\gamma_m(k))_{k\in\mathbb{Z}}$ converges in $l^2(\mathbb{Z})$ to $(\gamma(k))_{k\in\mathbb{Z}}$. This together with the convergence of $g_{p;(m)}$ gives the desired $\lim_{m\to\infty} V_m = V$, so that

$$\mathbf{Y}_m \xrightarrow{d} \mathbf{Y}, \quad m \to \infty,$$

where $\mathbf{Y} \stackrel{d}{=} N(0, V)$. Finally, that

$$\lim_{m\to\infty} \limsup_{n\to\infty} P(n^{1/2}|\gamma^*_{n;(m)}(p) - \gamma_m(p) - \gamma^*(p) + \gamma(p)| > \varepsilon) = 0 \quad \forall\, \varepsilon > 0,\ p \in \{0, \ldots, h\}$$

follows as in Equation (7.3.9) in [3]. An application of a variant of Slutsky's theorem (cf. [3], Proposition 6.3.9) then gives the claim.
(b) This follows as in the proof of Proposition 7.3.4 in [3]. One only has to observe

that by Theorem 2.1, $\sqrt{n}\,\overline{X}_{n;1}$ converges in distribution to a normal random variable as $n \to \infty$. In particular, $\overline{X}_{n;1}$ must converge to 0 in probability as $n \to \infty$.
(c) The limit theorem follows as in the proof of Theorem 7.2.1 in [3], and for w_{ij} we have the representation

$$\begin{aligned} w_{ij;\Delta} &= (v_{ij} - \rho(i)v_{0j} - \rho(j)v_{i0} + \rho(i)\rho(j)v_{00})/\gamma(0)^2 \\ &= \widetilde{w}_{ij;\Delta} + \frac{(\eta-3)\sigma^4}{\gamma(0)^2} \times \\ &\qquad \int_0^1 \left(g_i(u)g_j(u) - \rho(i)g_0(u)g_j(u) - \rho(j)g_i(u)g_0(u) + \rho(i)\rho(j)g_0(u)^2\right) du, \end{aligned}$$

giving the claim. □

Remark 3.6. *It is easy to check that $w_{ij;\Delta} = \widetilde{w}_{ij;\Delta}$ if f is of the form $f = \sum_{i=-\infty}^{\infty} \psi_i \mathbf{1}_{(i\Delta,(i+1)\Delta]}$, in accordance with Bartlett's formula, since then $(X_{t\Delta})_{t\in\mathbb{Z}}$ has a discrete time moving average representation with i.i.d. coefficients.*

Remark 3.7. *Another case when $w_{ij;\Delta} = \widetilde{w}_{ij;\Delta}$ is when $\eta = 3$, which happens if and only if L is Brownian motion. However, in general we do not have $w_{ij;\Delta} = \widetilde{w}_{ij;\Delta}$. An example is given by $f = \mathbf{1}_{(0,1/2]} + \mathbf{1}_{(1,2]}$ and $\Delta = 1$, in which case $g_{1;1} = \mathbf{1}_{(0,1/2]}$ and $g_{0;1} = 2 \cdot \mathbf{1}_{(0,1/2]} + \mathbf{1}_{(1/2,1]}$, and it is easy to see that $g_{1;1} - \rho(1)g_{0;1}$ is not almost everywhere zero, so that $w_{11;1} \neq \widetilde{w}_{11;1}$ if $\eta \neq 3$. The latter example corresponds to a moving average process, which is varying at the scale $\frac{1}{2}$, but sampled at integer times. Observe however that $w_{11;1/2} = \widetilde{w}_{11;1/2}$ by Remark 3.6. A more detailed study of such phenomena in discrete time can be found in Niebuhr and Kreiss [13].*

Remark 3.8. *Recently, sophisticated and powerful results on the normal approximation of Poisson functionals using Malliavan calculus have been obtained. E.g., Peccati and Taqqu [15, Theorems 2, 3 and 5] prove a central limit theorem for double Poisson integrals and apply this to a specific quadratic functional of a Lévy driven Ornstein–Uhlenbeck process, and Peccati et al. [14, Section 4] obtain bounds for such limit theorems, to name just of few of some recent publications on this subject. It may be possible to apply the results of [14, 15] to obtain another proof of Theorem 3.5 under certain conditions such as finite 6th moment, but we have not investigated this issue further. Note that our proof uses only basic knowledge of stochastic integrals and methods from time series analysis.*

4 An application to fractional Lévy noise

We will now apply the previous results to fractional Lévy processes. Recall from (1.8) and (1.9) that these were denoted by

$$M^1_{t;d} := \frac{1}{\Gamma(d+1)} \int_{-\infty}^{\infty} \left[(t-s)_+^d - (-s)_+^d\right] dL_s, \quad t \in \mathbb{R}, \quad \text{and}$$

$$M^2_{t;d} = \int_{-\infty}^{\infty} \left[|t-s|^d - |s|^d \right] dL_s, \quad t \in \mathbb{R},$$

respectively, and the corresponding *fractional Lévy noises* based on increments of length $\Delta > 0$ by

$$X^i_t = M^i_{t;d} - M^i_{t-\Delta;d}, \quad t \in \mathbb{R}, \quad i = 1,\ 2.$$

Hence the fractional Lévy noises are Lévy driven moving average processes with kernel functions

$$f^1_{d,\Delta}(s) = \frac{1}{\Gamma(d+1)} \left(s^d_+ - (s-\Delta)^d_+ \right), \quad s \in \mathbb{R},$$

and

$$f^2_{d,\Delta}(s) = |s|^d - |s-\Delta|^d, \quad s \in \mathbb{R},$$

respectively. Neither $f^1_{d,\Delta}$ nor $f^2_{d,\Delta}$ are in $L^1(\mathbb{R})$, so Theorem 2.1 cannot be applied because of Remark 2.3. Please note that for the same reason the assumptions for *(b)* of Theorem 3.5 and for (3.15) are not fulfilled.

For simplicity in notation we assume $\Delta = 1$, and drop the subindex Δ. Although the fractional noises X^i have different distributions for $i = 1$ and $i = 2$, they are both stationary with the autocovariance

$$\mathbb{E}(X^i_{t+h} X^i_t) = \gamma_{X^i}(h) = \frac{C^i(d)\sigma^2}{2} \left(|h+1|^{2d+1} - 2|h|^{2d+1} + |h-1|^{2d+1} \right), \tag{4.1}$$

where $C^i(d)$ is a normalising multiplicative constant depending on d. Both processes X^i are infinitely divisible and of moving average type, hence we know from [4, 8] that $(X^i_t)_{t\in\mathbb{Z}}$ is mixing in the ergodic-theoretic sense. For fixed $h \in \mathbb{Z}$, define the function

$$F : \mathbb{R}^{\mathbb{Z}} \to \mathbb{R}, \quad (x_n)_{n\in\mathbb{Z}} \mapsto x_0 x_h.$$

If T denotes the forward shift operator, then

$$F(T^k (X^i_t)_{t\in\mathbb{Z}}) = X^i_k X^i_{k+h},$$

and from Birkhoff's ergodic theorem (e.g. Ash and Gardner [1], Theorems 3.3.6 and 3.3.10) we know that

$$\frac{1}{n} \sum_{k=1}^{n} X^i_k X^i_{k+h} \to E\left(F((X^i_t)_{t\in\mathbb{Z}}) \right) = E X^i_0 X^i_h, \quad n \to \infty,$$

for $i = 1,\ 2$, and the convergence is almost sure and in L^1 ([1], Theorems 3.3.6 and 3.3.7). Hence, with $\gamma^*_n = \gamma^*_{n;1}$ as defined in (1.6),

$$\lim_{n\to\infty} \gamma^*_n(h) = \frac{C^i(d)\sigma^2}{2} \left(|h+1|^{2d+1} - 2|h|^{2d+1} + |h-1|^{2d+1} \right) \quad \text{a.s.}$$

Since $\gamma(0) = C^i(d)\sigma^2$ and $\gamma(1) = C^i(d)\sigma^2(2^{2d}-1)$, $\rho_n^*(1) = \frac{\gamma_n^*(1)}{\gamma_n^*(0)}$ is a strongly consistent estimator for $2^{2d}-1$. Hence,

$$\widehat{d} := \frac{1}{2}\left(\frac{\log(\rho_n^*(1)+1)}{\log 2}\right) \tag{4.2}$$

is a strongly consistent estimator for d.

The question of the asymptotic normality of these estimators arises naturally. There are many classical techniques to show the asymptotic normality of an ergodic stationary sequence by assuming some stronger mixing assumption. As far as we know, they do not work in our setting. To illustrate this point, we would like to show that fractional Lévy noises are not strongly mixing. Let us first recall the definition.

Definition 4.1. Let $(X_n)_{n\in\mathbb{Z}}$ be a stationary sequence, and let

$$\alpha_X(n) = \sup\{|P(A\cap B) - P(A)P(B)|,\ A \in \sigma(X_k,\ k \le 0),\ B \in \sigma(X_k,\ k \ge n)\}.$$

The sequence $(X_n)_{n\in\mathbb{Z}}$ is *strongly mixing* if $\lim_{n\to\infty} \alpha_X(n) = 0$.

In our case we know the weak mixing property $\lim_{n\to\infty} |P(A\cap B) - P(A)P(B)| = 0$ for $A \in \sigma(X_0)$, $B \in \sigma(X_n)$, because of [4, 8]. There are classical central limit theorems for strongly mixing sequences, see e.g. [12] for an overview. The following result, which is stated as Proposition 34 in [12], will be particularly useful for us.

Theorem 4.2. *Suppose that $(X_t)_{t\in\mathbb{Z}}$ is a mean zero, strongly mixing sequence and that there exists some $\delta > 0$ and a constant $K > 0$ such that*

$$E|X_0|^{2+\delta} < \infty, \tag{4.3}$$

$$\lim_{m\to\infty} \mathrm{Var}\,(\sum_{i=1}^m X_i) = \infty, \tag{4.4}$$

$$E|\sum_{i=1}^m X_i|^{2+\delta} \le K\left(\mathrm{Var}\,(\sum_{i=1}^m X_i)\right)^{1+\delta/2} \quad \forall\, m \in \mathbb{N}. \tag{4.5}$$

Write

$$S^{(n)}(t) := \sum_{i=1}^{\lfloor nt\rfloor} X_i, \quad n \in \mathbb{N}, \quad t \in [0,1],$$

where $\lfloor x\rfloor$ is the integer part of the real number x. Then

$$\left(\mathrm{Var}\,(\sum_{i=1}^n X_i)\right)^{-1/2} S^{(n)} \xrightarrow{d} B \quad \textit{weakly in } D[0,1], \tag{4.6}$$

where B is a standard Brownian motion.

Lemma 4.3. *Let L be a two sided non-zero Lévy process with expectation zero and finite fourth moment, and let $d \in (0, 1/2)$. Let*

$$X_t^i := M_{t;d}^i - M_{t-1;d}^i, \quad t \in \mathbb{Z},$$

be the corresponding fractional Lévy noises, for $i = 1,\ 2$. Then $(X_t^i)_{t\in\mathbb{Z}}$ satisfies (4.3) – (4.5) with $\delta = 2$, for $i = 1,\ 2$.

Proof. The proof is only written for the fractional noise X^1 denoted by X but it is similar for X^2. Equation (4.3) holds since L has finite fourth moment and since the kernel function of fractional noise is in $L^2(\mathbb{R}) \cap L^4(\mathbb{R})$, and (4.4) follows from the fact that

$$\sum_{i=1}^{m} X_i = M_{m;d}, \tag{4.7}$$

and

$$\mathrm{Var}\,(M_{m;d}) = Cm^{2d+1} \quad \forall\, m \in \mathbb{N} \tag{4.8}$$

for some constant C. To see (4.5) for $\delta = 2$, we use (4.7) and

$$f_m(s) := \frac{1}{\Gamma(d+1)}[(m-s)_+^d - (-s)_+^d], \quad s \in \mathbb{R}.$$

Then by Lemma 3.2,

$$E|M_d(m)|^4 \;=\; (\eta - 3)\sigma^4 \int_{\mathbb{R}} f_m^4(s)ds + 3\sigma^4 \left(\int_{\mathbb{R}} f_m^2(s)ds\right)^2.$$

Observe that

$$\left(\int_{\mathbb{R}} f_m^2(s)\,ds\right)^2 = Cm^{4d+2}$$

and that

$$\int_{\mathbb{R}} f_m^4(s)\,ds \le C'm^{4d+1}$$

for positive constants $C,\ C'$, which gives the claim. □

A consequence of the previous theorem and lemma is the following negative result.

Corollary 4.4. *Assume the assumptions of Lemma 4.3. Then the fractional Lévy noises X^1 and X^2 are not strongly mixing.*

Proof. If fractional Lévy noises were strongly mixing, then (4.6) would follow. Please remark that, since fractional noises are increments, $S^{(n)}(t) = M_d^i(\lfloor nt \rfloor) - M_d^i(0)$, and $(\mathrm{Var}\,(\sum_{i=1}^n X_i)) = Cn^{2d+1}$ by (4.8). Owing to the asymptotic self-similarity of fractional Lévy processes (Proposition 3.1 in [2]), we know that

$$\frac{M_d^2(\lfloor nt \rfloor) - M_d^2(0)}{n^{d+1/2}} \xrightarrow{d} B_{d+1/2}(t),$$

where the limit is a fractional Brownian motion with Hurst exponent $d + 1/2$. A similar asymptotic self-similarity holds for M_d^1. Hence (4.6) is violated and the corollary is proved by contradiction. $\square$

Nevertheless one can apply Theorem 3.5 to get the asymptotic normality of the estimator (4.2).

Proposition 4.5. *Assume the assumptions of Lemma 4.3, and let $\widehat{d}$ be defined by (4.2). If $d \in (0, 1/4)$ then $\sqrt{n}(\widehat{d} - d)$ converges in distribution to a Gaussian random variable as $n \to \infty$.*

Proof. The proof is only written for the fractional noise X^1 denoted by X but it is similar for X^2.

We shall apply (3.14) to get convergence of $\sqrt{n}(\rho_n^*(1) - \rho(0))$ to a Gaussian random variable. First, observe that

$$f_{d,1}^1 \in L^2(\mathbb{R}) \cap L^4(\mathbb{R}), \tag{4.9}$$

and $\sum_{k=-\infty}^{\infty} \gamma(k)^2 < \infty$. The latter inequality is classical for the fractional Gaussian noise, when $d < 1/4$, and holds for fractional Lévy noises since they have the same autocorrelation as fractional Gaussian noise. This also implies (3.11) by Remark 3.4 since $f_{d,1}^1 \geq 0$. Let us check that $g_0 := g_{0;1} \in L^2(0, 1)$. Since

$$\Gamma^2(d+1) g_0(u) \quad = \quad \sum_{k=-\infty}^{\infty} ((u+k)_+^d - (u+k-1)_+^d)^2,$$

it follows for all $u \in (0, 1)$ that

$$\begin{aligned} \Gamma^2(d+1) g_0(u) \quad &\leq \quad \sum_{k=0}^{\infty} ((1+k)_+^d - (k-1)_+^d)^2 \\ &= \quad 1 + \sum_{k=1}^{\infty} |k|^{2d} \left(\left(1 + \frac{1}{|k|}\right)^d - \left(1 - \frac{1}{|k|}\right)^d \right)^2 \\ &< \quad \infty, \end{aligned}$$

so that even $g_0 \in L^\infty([0, 1])$. Hence the assumptions of Theorem 3.5 (a) are fulfilled, and the result follows from (3.14). $\square$

If $d \geq 1/4$ then $\sum_{k=-\infty}^{\infty} \gamma(k)^2 = \infty$, since fractional Lévy noises have the same autocorrelation as the fractional Gaussian noise. Hence we consider

$$Z_t^i = X_t^i - X_{t-1}^i, \quad t \in \mathbb{Z},$$

for which it holds $\sum_{k=-\infty}^{\infty} \gamma_{Z^i}(k)^2 < \infty$. We conclude from Birkhoff's ergodic theorem that

$$\gamma_{n,Z}^*(h) := \frac{1}{n} \sum_{k=2}^{n} Z_k Z_{k+h} \to E(Z_0 Z_h), \quad n \to \infty,$$

is a strongly consistent estimator of

$$E(Z_0 Z_h) = \frac{C^i(d)\sigma^2}{2}\left(-|h+2|^{2d+1} + 4|h+1|^{2d+1} - 6|h|^{2d+1} + 4|h-1|^{2d+1} - |h-2|^{2d+1}\right).$$

Therefore $\rho^*_{n,Z}(1) = \frac{\gamma^*_{n,Z}(1)}{\gamma^*_{n,Z}(0)}$ is a strongly consistent estimator for $\phi(d) = \frac{-3^{2d+1}+4\cdot 2^{2d+1}-7}{8-2^{2d+2}}$. It turns out that ϕ is increasing on $(0, 1/2)$. Therefore we can define the estimator

$$\tilde{d} := \phi^{-1}(\rho^*_{n,Z}(1)). \tag{4.10}$$

Proposition 4.6. *Assume the assumptions of Lemma 4.3, and let $\tilde{d}$ be defined by* (4.10)*. If $d \in (0, 1/2)$ then $\sqrt{n}(\tilde{d}-d)$ converges in distribution to a Gaussian random variable as $n \to \infty$.*

Proof. The proof is only written for the fractional noise X^1 denoted by X but it is similar for X^2. Let us remark that $Z_t = \int_{-\infty}^{\infty} \tilde{f}^1_{d,1}(t-s)\, dL_s$, where

$$\tilde{f}^1_{d,1}(s) = f^1_{d,1}(s) - f^1_{d,1}(s-1),$$

To apply Theorem 3.5, we have to check that

$$\tilde{f}^1_{d,1}(s) \in L^2(\mathbb{R}) \cap L^4(\mathbb{R}),$$

which is obvious from (4.9). Moreover we already know that $\sum_{k=-\infty}^{\infty} \gamma_{Z^i}(k)^2 < \infty$. This time, however, the kernel function $\tilde{f}^1_{d,1}$ is not nonnegative, but it is easy to see that $|\tilde{f}^1_{d,1}(t)| \le C \min(1, |t|^{d-2})$ and hence that

$$\int_{-\infty}^{\infty} |\tilde{f}^1_{d,1}(t)\, \tilde{f}^1_{d,1}(t+k)|\, dt \le C' \min(1, |k|^{d-1}), \quad \forall\, k \in \mathbb{Z},$$

for some constants C, C', giving (3.11). Finally,

$$\Gamma^2(d+1) g_0(u) \;=\; \sum_{k=-\infty}^{\infty} ((u+k)^d_+ + (u+k-2)^d_+ - 2(u+k-1)^d_+)^2,$$

and estimating the summands separately for $k < 0$, $k = 0, 1$ and $k \ge 2$ we obtain for $u \in [0, 1]$

$$\begin{aligned} \Gamma^2(d+1) g_0(u) &\le 1 + (2^d+2)^2 + \sum_{k=2}^{\infty} (k^d - (k-1)^d)^2 \\ &< \infty, \end{aligned}$$

so that $g_0 \in L^\infty([0,1]) \subset L^2([0,1])$. The claim now follows from Theorem 3.5, using (3.14). □

Acknowledgement

Major parts of this research where carried out while AL was visiting the Institut de Mathématiques de Toulouse as guest professor in 2008 and 2009. He takes pleasure in thanking them for the kind hospitality and their generous financial support.

References

[1] R.B. Ash and M.F. Gardner. *Topics in Stochastic Processes.* Academic Press [Harcourt Brace Jovanovich Publishers], New York, 1975. Probability and Mathematical Statistics, Vol. 27.

[2] A. Benassi, S. Cohen, and J. Istas. On roughness indices for fractional fields. *Bernoulli*, 10(2):357–373, 2004.

[3] P.J. Brockwell and R.A. Davis. *Time Series: Theory and Methods.* Springer Series in Statistics. Springer-Verlag, New York, 1987.

[4] S. Cambanis, K. Podgórski, and A. Weron. Chaotic behavior of infinitely divisible processes. *Studia Math.*, 115(2):109–127, 1995.

[5] O. Christensen. *An Introduction to Frames and Riesz Bases.* Applied and Numerical Harmonic Analysis. Birkhäuser Boston Inc., Boston, MA, 2003.

[6] S. Cohen. Fractional Lévy fields. To appear in Springer Verlag collection *Lévy Matters.* Available at http://perso.math.univ-toulouse.fr/cohen/, 2012.

[7] R.A. Davis and Th. Mikosch. The sample autocorrelations of heavy-tailed processes with applications to ARCH. *Ann. Statist.*, 26(5):2049–2080, 1998.

[8] F. Fuchs and R. Stelzer. Mixing conditions for multivariate infinitely divisible processes with an application to mixed moving averages and the sup ou stochastic volatility model. *ESAIM Probab. Stat.* To appear. ESAIM:PS.doi 10.1051/ps2011158.

[9] E. J. Hannan. The asymptotic distribution of serial covariances. *Ann. Statist.*, 4(2):396–399, 1976.

[10] I. A. Ibragimov and Yu. V. Linnik. *Independent and Stationary Sequences of Random Variables.* Wolters-Noordhoff Publishing, Groningen, 1971. With a supplementary chapter by I. A. Ibragimov and V. V. Petrov, Translation from the Russian, edited by J. F. C. Kingman.

[11] T. Marquardt. Fractional Lévy processes with an application to long memory moving average processes. *Bernoulli*, 12(6):1099–1126, 2006.

[12] F. Merlevéde, M. Peligrad, and S. Utev. Recent advances in invariance principles for stationary sequences. *Probab. Surveys*, 3:1–36, 2006.

[13] T. Niebuhr and J.-P. Kreiss. Asymptotics for autocovariances and integrated periodograms for linear processes observed at lower frequencies. *Preprint*, 2012.

[14] G. Peccati, J.-L. Solé, M.S. Taqqu, and F. Utzet. Stein's method and normal approximation of Poisson functionals. *Ann. Probab.*, 37:2231–2261, 2010.

[15] G. Peccati and M.S. Taqqu. Central limit theorems for double Poisson integrals. *Bernoulli*, 14:791–821, 2008.