A comment to the talk by E. Seiler

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The formula given by Wegner may be rigorously proven as follows (for instance).

Let Σ_L be a square surface with side L laying on a coordinate plane of the lattice $\mathbf{Z}^3$ on which an Ising model is resting.

Put for every pair of nearest neighbours $f = (i, j)$:

$$\sigma_f = \sigma_i \sigma_j$$

and let

$$\begin{aligned} \beta_f &= \beta^* \quad \text{if} \quad f \cap \Sigma_L = \emptyset \\ \beta_f &= -\beta^* \quad \text{if} \quad f \cap \Sigma_L \neq \emptyset \end{aligned}$$

We wish to show that:

$$\lim_{\Lambda \uparrow \mathbf{R}^3} \frac{Z_\Lambda}{Z_\Lambda^0} = \frac{\sum_\sigma \exp \sum_{f \subset \Lambda} \beta_f^* \sigma_f}{\sum_\sigma \exp \sum_{f \subset \Lambda} \beta^* \sigma_f} = \exp(O(\beta^4))L$$

This is an immediate consequence of the Gruber-Kunz cluster expansion which in our case may be performed as follows (if $\mathcal{B}_\Lambda$ = set of subset of "bonds", i.e., sets of n.n. pairs) :

$$\begin{aligned} Z_\Lambda &= \sum_\sigma \prod_f \Big(1 + (e^{\beta^* \sigma_f} - 1)\Big) = \sum_{F \subset \mathcal{B}_\Lambda} \sum_\sigma \prod_{f \in F} (e^{\beta^* \sigma_f} - 1) \\ &= 2^{|\Lambda|} \sum_{\substack{x_1, \ldots, x_k \\ x_i \cap x_j = \emptyset, i \neq j}} \prod_{i=1}^{k} \zeta(x_i) \end{aligned}$$

where $x_1, \ldots, x_k$ are the "connected components" of the family F of bonds and

$$\zeta(x) = \sum_{\sigma_x} 2^{-|x|} \prod_{f \in x} (e^{\beta^* \sigma_f} - 1)$$

The logarithm of the sum expressing Z_Λ can be expanded into a series of products of the ζ's and it is well known that only "connected diagrams" appear in this expansion. More precisely let N_Λ = set of the families $\Gamma = (x_1, \ldots, x_p)$ of connected components such that if $x_i, x_j \in \Gamma$ then there is a chain $i_1 = i, i_2, \ldots, i_p = j$ for which $x_{i_k} \cap x_{i_{k+1}} \neq \emptyset$, $k = 1, \ldots, p-1$.

The elements of Γ may contain several times the same x: so we may represent Γ as $(x_1^{n_1}, \ldots, x_p^{n_p})$ where $n_1, \ldots, n_p > 0$ are the (integer) multiplicities and $x_i \neq x_j$. Let $|\Gamma| = n_1 + \ldots + n_p$. Gruber and Kunz results read in this case

$$Z_\Lambda = 2^{|\Lambda|} \exp \sum_{\Gamma \subset N_\Lambda} \phi^{\mathrm{T}}(\Gamma)\zeta(\Gamma) \tag{1}$$

provided the series converges absolutely, where if $\Gamma = (x_1^{n_1}, \ldots, x_p^{n_p})$:

$$\zeta(\Gamma) = \prod_{i=1}^{p} \zeta(x_i)^{n_i} \tag{2}$$

and the numbers $\phi^{\mathrm{T}}(\Gamma)$ are certain combinatorial constants (independent of Λ if $\Gamma \in N_\Lambda$) verifying

$$|\phi^{\mathrm{T}}(\Gamma)| \leq |\Gamma|^{-1} \tag{3}$$

An expansion similar to the (1) above can be made for the Z_Λ^0 and calling $\zeta^0(x)$ the corresponding ζ's

$$Z_\Lambda / Z_\Lambda^0 = \langle \exp -2\beta^* \sum_{f \subset \Sigma_\Lambda} \sigma_f \rangle_0 = \exp \sum_{\Gamma \cap \Sigma_L \neq \emptyset} \phi^{\mathrm{T}}(\Gamma)(\zeta(\Gamma) - \zeta^0(\Gamma))$$

since $\zeta(\Gamma) = \zeta^0(\Gamma)$ if $\Gamma \cap \Sigma_L = \emptyset$ (provided the series in (1) for Z_Λ and the analogous for Z_Λ^0 converge).

It is obvious, however, that

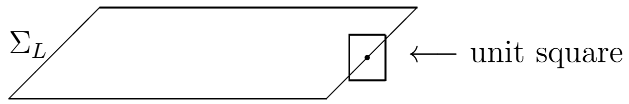
$$|\zeta(\Gamma)| \leq |e^{\beta^*} - 1| = \xi(\Gamma), \quad |\zeta^0(\Gamma)| \leq \xi(\Gamma)$$

which implies the convergence of the series (because of (2)) if β^* is small and also (since Γ is connected):

$$\sum_{\Gamma \ni 0} |\phi^{\mathrm{T}}(\Gamma)\xi(\Gamma) \leq O(\beta^*)^\ell$$

if β^* is small.

Finally remark that the Γ's for which $\zeta(\Gamma) \neq \zeta^0(\Gamma)$ are those which contain at least one x containing a closed circuit intersecting Σ_L an odd number of times (recall that x is a set of bonds and therefore it may contain "closed circuits").



The convergence of the series implies that the lowest order dominates which corresponds to the sum

$$\exp \sum_{\gamma \in \partial \Sigma_L} \zeta(\gamma) \phi^{\mathrm{T}}(\gamma)$$

where γ is the set of four pairs of n.n. forming a square as 1234 in the picture.

The $\phi^{\mathrm{T}}(\gamma)$ for such a configuration is $+1$ and

$$\begin{aligned} \zeta(f) &= \sum_{\sigma_1,\sigma_2,\sigma_3,\sigma_4} 2^{-4} (e^{-\beta^* \sigma_1 \sigma_4} - 1)(e^{-\beta^* \sigma_4 \sigma_3} - 1)(e^{-\beta^* \sigma_3 \sigma_2} - 1)(e^{-\beta^* \sigma_2 \sigma_1} - 1) \\ &= -\beta^{*\,4} + O(\beta^{*\,6}) \end{aligned}$$

while

$$\zeta^0(f) = +\beta^{*\,4} + O(\beta^{*\,6})$$

hence

$$Z_\Lambda / Z_\Lambda^0 = \exp\big(-\beta^{*\,4}(4L) + O(\beta^{*\,6})L \big)$$

since also the contribution for the more complicated Γ's must be at least of order $\beta^{*\,6}$.

It seems clear to us that the formula that Wegner gives without proof in his work has been obtained by an expansion of the above type which is a very familiar expansion technique for the physicists.

References:

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The talk by E. Seiler, entitled *Lattice Gauge Theories*, appears in the same volume, pp. 26–36.