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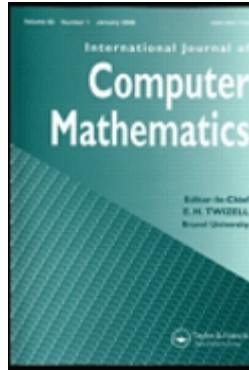
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# A family of integrals analytically solvable

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## Abstract

We consider the following family of definite integrals:

$$\int_0^{\infty} \frac{t^n}{(t-\lambda)^k} e^{-xt^m} dt, \quad \Re x > 0, \quad m, k \in \mathbb{N}, \quad \lambda \in \mathbb{C} \setminus \{\mathbb{R}^+ \cup 0\}, \quad n = 0, 1, 2 \dots$$

Its explicit computation cannot be found in the traditional table books for general values of the parameters. It cannot be computed by algebraic manipulators either. *Mathematica 7.0* computes the integral for some particular (low) values of  $m$ . In this paper we compute this family of integrals for general values of the parameters in terms of well known special functions: Gamma and Exponential integrals. From this computation, it is then straightforward to design a code for its evaluation with *Mathematica* and incorporate those integrals to the *Mathematica* library of integrals. We have compared the speed of computation of that integral for  $m = 1, 3$  by using our code on the one hand and the command *Integrate* of *Mathematica 7.0* on the other hand, concluding that our code is much faster.

Keywords & Phrases: Evaluation of definite integrals; Gamma function and Exponential integral; *Mathematica* codes.

## 1 Introduction

Because of its frequent use in many branches of pure and applied sciences, it is interesting to have at our disposal a data base of integrals as large as possible. Important table books of integrals like for example [3] or [7] contain many integrals that frequently appear in practice, although obviously those books are far from being exhaustive. The symbolic manipulator *Mathematica* incorporates most of the integrals contained in the table books: a large selection of definite and indefinite integrals of elementary as well as special functions. The purpose of this paper is to enlarge that selection of known integrals, adding the following family of integrals to the list:

$$\Omega_{n,k}^m(x, \lambda) := \int_0^{\infty} \frac{t^n}{(t-\lambda)^k} e^{-xt^m} dt, \quad \Re x > 0, \quad m, k \in \mathbb{N}, \quad \lambda \in \mathbb{C} \setminus \{\mathbb{R}^+ \cup 0\}, \quad n = 0, 1, 2 \dots \quad (1)$$

This family of integrals contains two complex parameters and three integer parameters. The computation of these integrals has an interest on its own because of the simplicity of the integrand. This family of integrals is the generalized Stieltjes transform [9] of the generalized gamma distribution  $t^n e^{-xt^m}$ . These integrals also have an interest in the design of asymptotic methods for integrals of the form

$$\int_{\mathcal{C}} e^{-xf(t)} g(t) dt, \quad \Re x > 0, \quad |x| \rightarrow \infty, \quad (2)$$

with  $\mathcal{C}$  a real interval or complex path. Traditional text books in asymptotic like for example [2] or [8] contain several chapters devoted to this kind of integrals (Laplace and saddle point methods). Many special functions have integral representations of the form (2). These facts justify the interest on the approximation of this kind of integrals. More precisely, in [6] we have designed a new method to calculate asymptotic expansions of this kind of integrals whose phase function  $f(t)$  has a minimum near of a pole of the function  $g(t)$  (they are also allowed to coalesce). And this asymptotic method requires the computation of the family of integrals (1) because the coefficients of the asymptotic expansion are given in terms of (1).

The family of integrals (1) satisfy the following properties:

1. Consider  $m, k \in \mathbb{N}$  and  $n = 0, 1, 2, \dots$

1.1. For  $\lambda = 0$ ,  $\Re x > 0$  and  $n + 1 > k$ ,

$$\Omega_{n,k}^m(x, 0) = \frac{1}{m} \Gamma\left(\frac{n-k+1}{m}\right) x^{\frac{k-n-1}{m}}. \quad (3)$$

1.2. For  $x = 0$ ,  $\lambda \in \mathbb{C} \setminus \{\mathbb{R}^+ \cup 0\}$  and  $n + 1 < k$ ,

$$\Omega_{n,k}^m(0, \lambda) = (-\lambda)^{n-k+1} \beta(n+1, k-n-1). \quad (4)$$

1.3. For  $\Re x > 0$ ,  $\lambda \in \mathbb{C} \setminus \{\mathbb{R}^+ \cup 0\}$  and  $m = 1$ ,

$$\Omega_{n,k}^1(x, \lambda) = n! (-\lambda)^{n-k+1} U(n+1, n+2-k, -x\lambda). \quad (5)$$

1.4. For  $\Re x > 0$ ,  $\lambda \in \mathbb{C} \setminus \{\mathbb{R}^+ \cup 0\}$  and  $k = 1$ ,

$$\Omega_{n,1}^m(x, \lambda) = \frac{\lambda^n}{m} \left[ \sum_{j=0}^{m-1} \frac{\lambda^j x^{j/m} \Gamma(1-j/m)}{e^{\lambda^m x}} E_{1-j/m}[-\lambda^m x] + \sum_{j=0}^n \frac{\Gamma(j/m)}{\lambda^j x^{j/m}} \right]. \quad (6)$$

In these equations  $U(a, b, x)$  is the confluent hypergeometric function of the second kind [[1], Chap. 13] and  $E_a[x]$  is the Generalized exponential integral [[1], eq. (5.1.4), p. 228]:

$$E_r[z] := \int_1^\infty \frac{e^{-zt}}{t^r} dt, \quad \Re z > 0, \quad r \in \mathbb{C}.$$

2. For  $\Re x > 0$ ,  $\lambda \in \mathbb{C} \setminus \{\mathbb{R}^+ \cup 0\}$ ,  $m, k \in \mathbb{N}$  and  $n = 0, 1, 2, \dots$ ,

$$\frac{\partial^r}{\partial x^r} \Omega_{n,k}^m(x, \lambda) = (-1)^r \Omega_{n+r,m,k}^m(x, \lambda), \quad \forall r \in \mathbb{N} \quad (7)$$

and

$$\frac{\partial^r}{\partial \lambda^r} \Omega_{n,k}^m(x, \lambda) = (k)_r \Omega_{n,k+r}^m(x, \lambda), \quad \forall r \in \mathbb{N}. \quad (8)$$

3. For  $m, k \in \mathbb{N}$  and  $n = 0, 1, 2, \dots$ ,  $\Omega_{n,k}^m(x, \lambda)$  verifies the recurrence relations

$$\Omega_{n,k}^m(x, \lambda) = \frac{m x}{n+1} \Omega_{n+m,k}^m(x, \lambda) + \frac{k}{n+1} \Omega_{n+1,k+1}^m(x, \lambda) \quad (9)$$

and

$$\Omega_{n,k}^m(x, \lambda) = \Omega_{n+1,k+1}^m(x, \lambda) - \lambda \Omega_{n,k+1}^m(x, \lambda). \quad (10)$$

4. For  $\Re x > 0$ ,  $\lambda \in \mathbb{C} \setminus \{\mathbb{R}^+ \cup 0\}$ ,  $m, k \in \mathbb{N}$  and  $n = 0, 1, 2, \dots$ ,  $\Omega_{n,k}^m(x, \lambda)$  verifies the following linear partial differential equation

$$\lambda \frac{\partial}{\partial \lambda} \Omega_{n,k}^m(x, \lambda) - m x \frac{\partial}{\partial x} \Omega_{n,k}^m(x, \lambda) = (n - k + 1) \Omega_{n,k}^m(x, \lambda). \quad (11)$$

In the following section we prove the properties listed above. Section 3 is the core of the paper, where we give an elementary formula to compute  $\Omega_{n,k}^m(x, \lambda)$  in terms of exponential integrals or incomplete gamma functions. Finally, in Section 4 of we propose an elementary *Mathematica* code based on that formula to compute  $\Omega_{n,k}^m(x, \lambda)$ . Also in Section 4, for  $m = 1, 3$  and using *Mathematica 7.0*, we compare the speed of computation of this code and the speed of computation of the command *Integrate*.

## 2 Proof of properties 1-4.

1.1. Formula (3) is obtained from  $\Omega_{n,k}^m(x, 0) = \int_0^\infty e^{-xt^m} t^{n-k} dt$  after the change of variable  $xt^m \rightarrow z$ .

1.2. Formula (4) is obtained after the change of variable  $\frac{t}{t-\lambda} \rightarrow z$  in

$$\Omega_{n,k}^m(0, \lambda) = \int_0^\infty \frac{t^n}{(t-\lambda)^k} dt.$$

1.3. Formula (5) is trivial if we consider the integral representation of the confluent hypergeometric function of the second kind [[1], eq.(13.2.5)].

1.4. Formula (6) can be proved by induction over  $n$ . It's trivial to see that it holds for  $n = 0$ . Now, suppose that (6) is true for a given  $n > 0$  and write

$$\Omega_{n+1,1}^m(x, \lambda) = \int_0^\infty \frac{e^{-xt^m}}{t-\lambda} t^{n+1} dt = \int_0^\infty \frac{e^{-xt^m}}{t-\lambda} (t-\lambda + \lambda) t^n dt.$$

Split the last integral in two integrals:

$$\Omega_{n+1,1}^m(x, \lambda) = \lambda \int_0^\infty \frac{e^{-xt^m}}{t-\lambda} t^n dt + \int_0^\infty e^{-xt^m} t^n dt.$$

The first integral above is  $\Omega_{n,1}^m(x, \lambda)$  and the last integral is a gamma function. From here, we immediately find that (6) is also true with  $n$  replaced by  $n + 1$ .

2. Both formulas can be proved straightforwardly by induction over  $r$  deriving under the integral sign with respect to  $x$  and  $\lambda$  respectively.
3. Formula (9) is obtained from (1) integrating by parts:

$$\int_0^{\infty} \frac{t^n e^{-xt^m}}{(t-\lambda)^k} dt = \frac{e^{-xt^m} t^{n+1}}{(n+1)(t-\lambda)^k} \Big|_0^{\infty} + \frac{m x}{n+1} \int_0^{\infty} \frac{e^{-xt^m} t^{n+m}}{(t-\lambda)^k} dt + \frac{k}{n+1} \int_0^{\infty} \frac{e^{-xt^m} t^{n+1}}{(t-\lambda)^{k+1}} dt.$$

Formula (10) is obtained from (1) multiplying numerator and denominator in the integrand by  $(t-\lambda)$  and separating into two integrals.

4. From formula (7):  $\Omega_{n+m,k}^m(x, \lambda) = -\frac{\partial}{\partial x} \Omega_{n,k}^m(x, \lambda)$ . From the recurrence relation (9):  $\Omega_{n+1,k+1}^m(x, \lambda) = \Omega_{n,k}^m(x, \lambda) + \lambda \Omega_{n,k+1}^m(x, \lambda)$ . Applying formula (8):  $\Omega_{n,k+1}^m(x, \lambda) = \frac{1}{k} \frac{\partial}{\partial \lambda} \Omega_{n,k}^m(x, \lambda)$ . If we insert these last three results into (9) we obtain (11).

### 3 Evaluation of the integral (1)

The computation of the family of integrals (1) is given in Theorem 1 below. That computation requires the following two preliminary lemmas.

**Lemma 1** For  $\Re x > 0$ ,  $m \in \mathbb{N}$  and  $\lambda \in \mathbb{C} \setminus \{\mathbb{R}^+ \cup 0\}$  the following identity holds:

$$\int_0^{\infty} \frac{e^{-xt^m}}{t-\lambda} dt = \frac{1}{m} \sum_{j=0}^{m-1} \frac{\lambda^j x^{j/m} \Gamma(1-j/m)}{e^{\lambda^m x}} E_{1-j/m}[-\lambda^m x]. \quad (12)$$

**Proof.** Denote

$$I_m(x) := \int_0^{\infty} \frac{e^{-xt^m}}{t-\lambda} dt$$

and derive with respect to  $x$  under the integral sign. It is straightforward to see that  $I_m(x)$  is a solution of the following linear differential equation:

$$y'(x) + \lambda^m y(x) = F_m(x), \quad (13)$$

with

$$F_m(x) := -\sum_{j=0}^{m-1} \lambda^j \int_0^{\infty} e^{-xt^m} t^{m-j-1} dt = -\sum_{j=0}^{m-1} \frac{\lambda^j \Gamma(1-j/m)}{m x^{1-j/m}}.$$

The integral  $I_m(x)$  also satisfies the boundary condition:

$$\lim_{\Re x \rightarrow \infty} I_m(x) = 0, \quad \forall m \in \mathbb{N}, \quad \lambda \in \mathbb{C} \setminus \{\mathbb{R}^+ \cup 0\}. \quad (14)$$

The general solution of the differential equation (13) is:

$$y(x) = \sum_{j=0}^{m-1} \frac{\lambda^j \Gamma(1-j/m)}{m e^{\lambda^m x}} \left[ x^{j/m} E_{1-j/m}[-\lambda^m x] + K \right], \quad K \in \mathbb{R}.$$

The integral  $I_m(x)$  is one of these solutions, the solution that we select by imposing the boundary condition (14). This condition imposes  $K = 0$  and then we obtain (12).

**Remark 1.** By using the relation  $E_n[z] = z^{n-1}\Gamma(1-n, z)$  (see for example [1], equation (5.1.45), p. 230), we have that  $I_m(x)$  can be cast in terms of the Incomplete Gamma functions instead of exponential integrals:

$$\int_0^\infty \frac{e^{-xt^m}}{t-\lambda} dt = \sum_{j=0}^{m-1} \frac{\lambda^j \Gamma(1-j/m)}{m(-\lambda^m)^{j/m} e^{\lambda^m x}} \Gamma(j/m, -\lambda^m x).$$

**Lemma 2** Consider the generalized Stieltjes transform on  $[0, \infty)$  of a function  $f(t) \in L_1[0, \infty)$ :

$$F(\lambda, k) = \int_0^\infty \frac{f(t)}{(t-\lambda)^k} dt, \quad k = 1, 2, 3 \dots$$

Then

$$F(\lambda, k) = \frac{1}{(k-1)!} \frac{\partial^{k-1} F(\lambda, 1)}{\partial \lambda^{k-1}}, \quad \forall k = 2, 3 \dots \quad (15)$$

**Proof.** It proceeds by induction over  $k$ . By deriving  $F(\lambda, 1)$  (under the integral sign) with respect to  $\lambda$  we can see that (15) holds for  $k = 2$ . Now, suppose that (15) is true for a given  $k > 2$ :

$$\int_0^\infty \frac{f(t)}{(t-\lambda)^k} dt = \frac{1}{(k-1)!} \frac{\partial^{k-1} F(\lambda, 1)}{\partial \lambda^{k-1}}.$$

If we derive both sides of the above equation with respect to  $\lambda$ ,

$$k \int_0^\infty \frac{f(t)}{(t-\lambda)^{k+1}} dt = \frac{1}{(k-1)!} \frac{\partial^k F(\lambda, 1)}{\partial \lambda^k},$$

we find that (15) is also true with  $k$  replaced by  $k + 1$ .

**Theorem 1** For  $\lambda \in \mathbb{C} \setminus \{\mathbb{R}^+ \cup 0\}$  and  $\Re x > 0$ , the family of integrals  $\Omega_{n,k}^m(x, \lambda)$  defined in (1) may be computed in terms of exponential integrals:

1. For  $k = 1$  and  $\forall m = 1, 2, \dots$ ,

$$1.1. \quad \Omega_{0,1}^m(x, \lambda) = \frac{1}{m} \sum_{j=0}^{m-1} \frac{\lambda^j x^{j/m} \Gamma(1-j/m)}{e^{\lambda^m x}} E_{1-j/m}[-\lambda^m x].$$

$$1.2. \quad \Omega_{n,1}^m(x, \lambda) = \frac{\Gamma(n/m)}{m x^{n/m}} + \lambda \Omega_{n-1,1}^m(x, \lambda), \quad \forall n = 1, 2, \dots$$

2. For  $k = 2, 3, \dots$  and  $\forall m \in \mathbb{N}$ ,

$$\bullet \Omega_{n,k}^m(x, \lambda) = \frac{1}{(k-1)!} \frac{\partial^{k-1} \Omega_{n,1}^m(x, \lambda)}{\partial \lambda^{k-1}}, \quad n = 0, 1, 2, \dots$$

**Proof.** Case 1.1 is proved in Lemma 1. To prove case 1.2 we write

$$\Omega_{n,1}^m(x, \lambda) = \int_0^\infty \frac{e^{-xt^m}}{t-\lambda} t^n dt = \int_0^\infty \frac{e^{-xt^m}}{t-\lambda} (t-\lambda + \lambda) t^{n-1} dt$$

and split the last integral in two integrals:

$$\Omega_{n,1}^m(x, \lambda) = \lambda \int_0^\infty \frac{e^{-xt^m}}{t-\lambda} t^{n-1} dt + \int_0^\infty e^{-xt^m} t^{n-1} dt.$$

The first integral above is  $\Omega_{n-1,1}^m(x, \lambda)$  and the last integral is a gamma function. From here, we immediately obtain the recurrence relation 1.2.

Case 2 is proved in Lemma 2 with  $f(t) = t^n e^{-xt^m}$ .

## 4 Implementation in *Mathematica*

As discussed above, the family of integrals (1) does not appear in the classical table books of integrals for general values of the parameters. It appears only for special cases, like for example in [3], equations (3.353.2), or (3.353.5), for  $m = 1$ . Likewise, *Mathematica 7.0* only computes  $\Omega_{n,k}^m(x, \lambda)$  for some particular (small) values of  $m$ . Theorem 1 invites to implement the following code in *Mathematica*:

$$\begin{aligned} \text{In[1]:= } \Omega[\mathbf{m}_-, 0, 1, \lambda_-, \mathbf{x}_-] &:= \frac{e^{-x \lambda^m}}{\mathbf{m}} \sum_{j=0}^{\mathbf{m}-1} \lambda^j \mathbf{x}^{j/m} \text{ExpIntegralE}\left[\frac{\mathbf{m}-j}{\mathbf{m}}, -x \lambda^m\right] \text{Gamma}\left[\frac{\mathbf{m}-j}{\mathbf{m}}\right]; \\ \Omega[\mathbf{m}_-, \mathbf{n}_-, 1, \lambda_-, \mathbf{x}_-] &:= \frac{\text{Gamma}[\mathbf{n}/\mathbf{m}]}{\mathbf{m} \mathbf{x}^{\mathbf{n}/\mathbf{m}}} + \lambda \Omega[\mathbf{m}, \mathbf{n}-1, 1, \lambda, \mathbf{x}]; \\ \Omega[\mathbf{m}_-, \mathbf{n}_-, \mathbf{k}_-, \lambda_-, \mathbf{x}_-] &:= \text{If}[\mathbf{k} > 1, \frac{1}{(\mathbf{k}-1)!} \text{D}[\Omega[\mathbf{m}, \mathbf{n}, 1, \lambda, \mathbf{x}], \{\lambda, \mathbf{k}-1\}], \Omega[\mathbf{m}, \mathbf{n}, 1, \lambda, \mathbf{x}]]; \end{aligned}$$

With this code we can enlarge the data base of integrals that *Mathematica* can compute analytically. But moreover, we improve the speed of computation of the *Mathematica* command *Integrate* for this family of integrals. To illustrate this fact, in the following tables we compare the time of computation of the integral (1) using the command *Integrate* of *Mathematica 7.0* and the time of computation of (1) using our code. Each entry contains two values: the first value is the computing time of *Integrate* and the second value is the computing time of our code. In general, our algorithm is about  $10^{14}$  faster than the command *Integrate*. These times have been calculated by *Mathematica 7.0* using the command *Timing* in a processor Intel Core 2 Duo E6420 / 2.13 GHz. The word *No Result* appearing in some entrances means that *Mathematica* has not found an analytical expression for  $\Omega_{n,k}^m(x, \lambda)$  for the particular values of  $k$  and  $n$  there detailed.



Out[10]/TableForm=

	k=1	k=2	k=3
n=0	{0.766, <math>10^{-16}</math>}	{1.125, $1.13798 \times 10^{-15}$ }	{1., <math>10^{-16}</math>}
n=1	{0.86, <math>10^{-16}</math>}	{4.985, $2.30371 \times 10^{-15}$ }	{5.343, <math>10^{-16}</math>}
n=2	{1.032, <math>10^{-16}</math>}	{6.234, <math>10^{-16}</math>}	{10.265, <math>10^{-16}</math>}
n=3	{1.125, <math>10^{-16}</math>}	{18.625, <math>10^{-16}</math>}	{No Result, $7.64666 \times 10^{-15}$ }

**Table 1:** Computation time required by our CPU to calculate the analytical expression of the integral (1) for  $m = 1$ , several values of  $n$  and  $k$  and general values of  $x$  and  $\lambda$ .

Out[7]/TableForm=

	k=1	k=2	k=3
n=0	{1.016, <math>10^{-16}</math>}	{5.078, $1.1241 \times 10^{-15}$ }	{9.641, <math>10^{-16}</math>}
n=1	{1.109, <math>10^{-16}</math>}	{9.234, <math>10^{-16}</math>}	{9.281, <math>10^{-16}</math>}
n=2	{0.953, $1.1241 \times 10^{-15}$ }	{9.641, <math>10^{-16}</math>}	{12.031, $4.08007 \times 10^{-15}$ }
n=3	{5.047, $9.99201 \times 10^{-15}$ }	{5.719, $4.08007 \times 10^{-15}$ }	{5.812, <math>10^{-16}</math>}

**Table 2:** This table contains the time that our CPU takes to calculate the analytical expression of the integral (1) for  $m = 3$ , several values of  $n$  and  $k$  and general values of  $x$  and  $\lambda$ .

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