On the Stokes phenomenon of a family of multi-perturbed level-one meromorphic linear differential systems

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Abstract

Given a level-one meromorphic linear differential system, we are interested in the behavior of its Stokes-Ramis matrices under the action of a holomorphic perturbation acting on the non-zero Stokes values. In particular, we show that the Stokes-Ramis matrices of the given system can be expressed as limit of convenient connection matrices of the perturbed systems. We believe that this result could provide an efficient tool for the numerical calculation of the Stokes-Ramis matrices. No assumption of genericity is made.

Keywords. Linear differential system, regular perturbation, holomorphic perturbation, Stokes phenomenon, summability

AMS subject classification. 34M03, 34M30, 34M35, 34M40

Introduction

All along the article, we are given a linear differential system (in short, a differential system or a system)

$$x^2\frac{dY}{dx} = A(x)Y \quad , \; A(x) \in M_n(\mathbb{C}\{x\}), \; A(0) \neq 0 \tag{0.1}$$

of dimension $n \geq 2$ with meromorphic coefficients of order 2 at the origin $0 \in \mathbb{C}$. Under the assumption of "*single level equal to* 1", system (0.1) admits a formal fundamental solution $\widetilde{Y}(x) = \widetilde{F}(x)x^L e^{Q(1/x)}$ where

- $\widetilde{F}(x) \in M_n(\mathbb{C}[[x]][x^{-1}])$ is an invertible formal meromorphic matrix,
- $L = \bigoplus_{j=1}^{J}(\lambda_j I_{n_j} + J_{n_j})$ where J is an integer ≥ 2, I_{n_j} is the identity matrix of size n_j and where

$$J_{n_j} = \begin{cases} 0 & \text{if } n_j = 1 \\ \begin{bmatrix} 0 & 1 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & 1 \\ 0 & \cdots & \cdots & 0 \end{bmatrix} & \text{if } n_j \geq 2 \end{cases}$$

  is an irreductible Jordan block of size n_j,
- $Q\left(\frac{1}{x}\right) = \bigoplus_{j=1}^{J} \left(-\frac{a_j}{x}\right) I_{n_j}$ where the $a_j \in \mathbb{C}$ are not equal to a same a.

To simplify calculations below, we assume besides that the following normalizations of $\widetilde{Y}(x)$ are satisfied:

(0.2) $\widetilde{F}(x) \in M_n(\mathbb{C}[[x]])$ is a formal power series in x verifying $\widetilde{F}(0) = I_n$,

(0.3) $0 \leq \mathrm{Re}(\lambda_j) < 1$ for all $j = 1, ..., J$,

(0.4) $a_1 = \lambda_1 = 0$.

Recall that such conditions can always be fulfilled by means of a jauge transformation of the form $Y \longmapsto T(x)x^{-\lambda_1}e^{a_1/x}Y$ where $T(x)$ has explicit computable polynomial entries in x and $1/x$. Recall also that such a gauge transformation does not affect the Stokes phenomenon of system (0.1).

Conditions (0.2) and (0.3) guarantee the unicity of $\widetilde{F}(x)$ as formal series solution of the homological system of system (0.1) ([1]). Condition (0.4) is for notational convenience. Notice however that the a_j's are not supposed distinct. Notice also that there exists j such that $a_j \neq 0$.

The Stokes phenomenon of system (0.1) stems from the fact that the sums $\widetilde{F}(x)$ on each side of a same singular direction (or anti-Stokes direction) of system (0.1) are not analytic continuations from each other in general; this defect of analyticity is quantified by the Stokes-Ramis matrices (see def. 1.1 below for a precise definition).

The aim of this paper is to study the behavior of these matrices under the action of a *holomorphic* perturbation acting on the Stokes values $a_j \neq 0$ of system (0.1). In particular, we show that they can be expressed as limit of convenient connection matrices of the perturbed systems. Recall that a result of the same type has already been proved in [5] for single-leveled systems in which some Stokes values were *continuously* perturbed.

In section 1, we recall for the convenience of the reader some definitions about the summation theory.

In section 2, we define the perturbed systems studied in this article and we give some basic properties of their Stokes values and their anti-Stokes directions. The main result of the paper is stated in theorem 2.5.

Section 3 is devoted to the proof of theorem 2.5. This proof is essentially based on an adequate variant of the proof of summable-resurgence theorem following Écalle's method by regular perturbation and majorant series which was given by M. Loday-Richaud and the author in [3].

1 Some definitions and notations

1.1 Stokes values and anti-Stokes directions

Split the matrix $\widetilde{F}(x) = \left[\widetilde{F}^{\bullet;1}(x) \quad \cdots \quad \widetilde{F}^{\bullet;J}(x)\right]$ into J column-blocks fitting to the Jordan structure of L (the size of $\widetilde{F}^{\bullet;k}(x)$ is $n \times n_k$ for all k).

Let $\Omega := \{a_j\ ,\ j = 1, ..., J\}$ denote the set of *Stokes values of system (0.1)*. The directions determined by the elements of $\Omega^* := \Omega \backslash \{0\}$ from 0 are called *anti-Stokes directions associated with* $\widetilde{F}^{\bullet;1}(x)$.

The anti-Stokes directions associated with the k-th column-block $\widetilde{F}^{\bullet;k}(x)$ *of* $\widetilde{F}(x)$ are given by the non-zero elements of $\Omega - a_k$ (to normalize the k-th column-block, one has to multiply by $e^{a_k/x}$); *the anti-Stokes directions of system (0.1)*, *i.e.*, associated with the full matrix $\widetilde{F}(x)$, are given by the non-zero elements of $\boldsymbol{\Omega} := \{a_j - a_k\ ,\ j, k = 1, ..., J\}$. Recall that the elements of $\boldsymbol{\Omega}$ are the Stokes values of the homological system of system (0.1).

1.2 Summation, Stokes phenomenon and Stokes-Ramis matrices

• Given a non anti-Stokes direction $\theta \in \mathbb{R}/2\pi\mathbb{Z}$ of system (0.1) and a choice of an argument of θ, say its principal determination $\theta^\star \in]-2\pi, 0]$ [1], we consider the sum of $\widetilde{Y}$ in the direction θ given by

$$Y_\theta(x) = s_{1;\theta}(\widetilde{F})(x) Y_{0,\theta^\star}(x)$$

where $s_{1;\theta}(\widetilde{F})(x)$ is the uniquely determined 1-sum (or Borel-Laplace sum) of $\widetilde{F}$ at θ and where $Y_{0,\theta^\star}(x)$ is the actual analytic function $Y_{0,\theta^\star}(x) := x^L e^{Q(1/x)}$ defined by the choice $\arg(x)$ close to $\theta^\star$ (denoted below by $\arg(x) \simeq \theta^\star$).

Recall that $s_{1;\theta}(\widetilde{F})$ is an analytic function defined and 1-Gevrey asymptotic to $\widetilde{F}$ on a germ of sector bisected by θ and opening larger than π.

Recall also that $s_{1;\theta}(\widetilde{F})(x)$ is given by the Borel-Laplace integral

$$\int_0^{\infty e^{i\theta}} \widehat{F}(\xi) e^{-\xi/x} d\xi$$

where $\widehat{F}(\xi)$ denotes the Borel transform of $\widetilde{F}(x)$.

• When $\theta \in \mathbb{R}/2\pi\mathbb{Z}$ is an anti-Stokes direction of system (0.1), we consider the two lateral sums $s_{1;\theta^-}(\widetilde{F})$ and $s_{1;\theta^+}(\widetilde{F})$ respectively obtained as analytic continuations of $s_{1;\theta-\eta}(\widetilde{F})$ and $s_{1;\theta+\eta}(\widetilde{F})$ to a germ of half-plane bisected by θ. Notice that such analytic continuations exist without ambiguity when $\eta > 0$ is small enough.

We denote by Y_{θ^-} and Y_{θ^+} the sums of $\widetilde{Y}$ respectively defined for $\arg(x) \simeq \theta^\star$ by $Y_{\theta^-}(x) := s_{1;\theta^-}(\widetilde{F})(x) Y_{0,\theta^\star}(x)$ and $Y_{\theta^+}(x) := s_{1;\theta^+}(\widetilde{F})(x) Y_{0,\theta^\star}(x)$.

The *Stokes phenomenon* stems from the fact that the sums $s_{1;\theta^-}(\widetilde{F})$ and $s_{1;\theta^+}(\widetilde{F})$ of $\widetilde{F}$ are not analytic continuations from each other in general. This defect of analyticity is quantified by the collection of *Stokes-Ramis automorphisms*

$$St_{\theta^\star} : Y_{\theta^+} \longmapsto Y_{\theta^-}$$

for all the anti-Stokes directions $\theta \in \mathbb{R}/2\pi\mathbb{Z}$ of system (0.1).

The *Stokes-Ramis matrices* are defined as matrix representations of the $St_{\theta^\star}$'s in $GL_n(\mathbb{C})$.

[1] Any choice is convenient. However, to be compatible, on the Riemann sphere, with the usual choice $0 \leq \arg(z = 1/x) < 2\pi$ of the principal determination at infinity, we suggest to choose $-2\pi < \arg(x) \leq 0$ as principal determination about 0.

Definition 1.1 (Stokes-Ramis matrices)
One calls Stokes-Ramis matrix associated with $\widetilde{Y}$ in the direction θ *the matrix of* $St_{\theta^\star}$ *in the basis* Y_{θ^+} [2]. *We still denote it* $St_{\theta^\star}$.

Notice that the matrix $St_{\theta^\star}$ is uniquely determined by the relation

$$\boxed{Y_{\theta^-}(x) = Y_{\theta^+}(x) St_{\theta^\star} \qquad \text{for } \arg(x) \simeq \theta^\star}$$

2 Setting the problem

We denote below by

- $D(\alpha, \rho) := \{x \in \mathbb{C} \; ; \; |x - \alpha| < \rho\}$ the open disc centered at $\alpha \in \mathbb{C}$ with radius $\rho > 0$,
- $\overline{D}(\alpha, \rho) := \{x \in \mathbb{C} \; ; \; |x - \alpha| \leq \rho\}$ the closed disc centered at $\alpha \in \mathbb{C}$ with radius $\rho > 0$,
- $\Sigma_{\theta,\eta} := \{x \in \mathbb{C}^* \; ; \; |\arg(x) - \theta| < \eta/2\}$ the open sector with vertex 0, bisected by $\theta \in \mathbb{R}/2\pi\mathbb{Z}$ and opening $\eta > 0$,
- $\overline{\Sigma}_{\theta,\eta} := \{x \in \mathbb{C}^* \; ; \; |\arg(x) - \theta| \leq \eta/2\}$ the closed sector with vertex 0, bisected by $\theta \in \mathbb{R}/2\pi\mathbb{Z}$ and opening $\eta > 0$.

2.1 A multi-perturbed system

In addition to notations above, we denote in this section by $\omega_1, ..., \omega_p$ with $p \geq 1$ the non-zero Stokes values of system (0.1). Hence, $\Omega = \{\omega_0 = 0\} \cup \{\omega_k \; , \; k = 1, ..., p\}$ and $\boldsymbol{\Omega} = \{\omega_0 = 0\} \cup \{\omega_k - \omega_\ell \; , \; k, \ell = 0, ..., p \text{ and } k \neq \ell\}$. Notice that $\omega_k - \omega_\ell \neq 0$ for all $k \neq \ell$.

According to the normalizations of $\widetilde{Y}(x)$ (*cf.* page 2), the matrix $A(x)$ of system (0.1) reads

$$A(x) = \bigoplus_{j=1}^{J} \left(a_j I_{n_j} + x L_j\right) + B(x)$$

[2]In the literature, a Stokes matrix has a more general meaning where one allows to compare any two asymptotic solutions whose domains of definition overlap. According to the custom initiated by J.-P. Ramis ([4]) in the spirit of Stokes' work, we exclude this case here. We consider only matrices providing the transition between the sums on each side of a same anti-Stokes direction.

where $a_j = \omega_k$ for a certain $k = 0, ..., p$, $L_j := \lambda_j I_{n_j} + J_{n_j}$ denotes the j-th Jordan block of the matrix L of exponents of formal monodromy and where $B(x)$ is analytic at the origin $0 \in \mathbb{C}$.

From now on, we are given

(1) a parameter $\boldsymbol{\varepsilon} := (\varepsilon_1, ..., \varepsilon_p)$ in a polydisc $\mathcal{D}_p := D(1, \rho_1) \times ... \times D(1, \rho_p)$ of $\mathbb{C}^p$; conditions on the ρ_k's are precised below,

(2) the regularly multi-perturbed system

$$x^2 \frac{dY}{dx} = A^{\boldsymbol{\varepsilon}}(x) Y \tag{2.1}$$

where

$$A^{\boldsymbol{\varepsilon}}(x) = \bigoplus_{j=1}^{J} \left(a_j^{\boldsymbol{\varepsilon}} I_{n_j} + x L_j\right) + B(x)$$

with

$$a_j^{\boldsymbol{\varepsilon}} = \begin{cases} 0 & \text{if } a_j = \omega_0 = 0 \\ \omega_k \varepsilon_k & \text{if } a_j = \omega_k \text{ for } k = 1, ..., p \end{cases}$$

Notice that, for $\boldsymbol{\varepsilon} = \mathbf{1} := (1, ..., 1)$, $A^{\mathbf{1}} \equiv A$ and systems (0.1) and (2.1) coincide. Notice also that

$$\omega_k \varepsilon_k \in D(\omega_k, |\omega_k| \rho_k) \quad \text{for all } k = 1, ..., p$$

Under the two conditions

$(C1)$ $0 \notin \overline{D}(\omega_k, |\omega_k| \rho_k)$ for all $k = 1, ..., p$,

$(C2)$ $\overline{D}(\omega_k, |\omega_k| \rho_k) \cap \overline{D}(\omega_\ell, |\omega_\ell| \rho_\ell) = \emptyset$ for all $k, \ell = 1, ..., p$, $k \neq \ell$,

which are always satisfied as soon as the ρ_k's are small enough, system (2.1) has, for all $\boldsymbol{\varepsilon} \in \mathcal{D}_p$, the unique level 1 and has for formal fundamental solution the matrix $\widetilde{Y}^{\boldsymbol{\varepsilon}}(x) = \widetilde{F}^{\boldsymbol{\varepsilon}}(x) x^L e^{Q^{\boldsymbol{\varepsilon}}(1/x)}$ where

- $\widetilde{F}^{\boldsymbol{\varepsilon}}(x) \in M_n(\mathbb{C}[[x]])$ is a power series in x verifying $\widetilde{F}^{\boldsymbol{\varepsilon}}(0) = I_n$,
- L is the matrix of exponents of formal monodromy of system (0.1),
- $Q^{\boldsymbol{\varepsilon}}(1/x) = \bigoplus_{j=1}^{J} (-a_j^{\boldsymbol{\varepsilon}}/x) I_{n_j}$.

Notice that, for $\boldsymbol{\varepsilon} = \mathbf{1}$, the two formal fundamental solutions $\widetilde{Y}^{\mathbf{1}}(x)$ and $\widetilde{Y}(x)$ coincide.

We shall now give some basic properties of the Stokes values and the anti-Stokes directions of systems (2.1).

For fixed $\boldsymbol{\varepsilon} \in \mathcal{D}_p$, we denote as before by

- $\Omega^{\boldsymbol{\varepsilon}}$ the set of Stokes values of system (2.1),
- $\boldsymbol{\Omega}^{\boldsymbol{\varepsilon}}$ the set of Stokes values of the homological system of system (0.1).

By construction, $\Omega^{\boldsymbol{\varepsilon}}$ (resp. $\boldsymbol{\Omega}^{\boldsymbol{\varepsilon}}$) is deduced from Ω (resp. $\boldsymbol{\Omega}$) by replacing the Stokes values ω_k, $k = 1, ..., p$ (resp. $\omega_k - \omega_\ell$, $k, \ell = 0, ..., p$ and $k \neq \ell$) by the perturbed Stokes values $\omega_k \varepsilon_k$ (resp. $\omega_k \varepsilon_k - \omega_\ell \varepsilon_\ell$). Hence,

- $\Omega^{\boldsymbol{\varepsilon}} = \{0\} \cup \{\omega_k \varepsilon_k \ , \ k = 1, ..., p\}$,
- $\boldsymbol{\Omega}^{\boldsymbol{\varepsilon}} = \{0\} \cup \{\omega_k \varepsilon_k - \omega_\ell \varepsilon_\ell \ , \ k, \ell = 0, ..., p$ and $k \neq \ell\}$; we set $\varepsilon_0 := 1$.

Notice that, due to conditions $(C1)$ and $(C2)$, $\omega_k \varepsilon_k - \omega_\ell \varepsilon_\ell \neq 0$ for all $k \neq \ell$.

We denote also by

- $\Omega(\mathcal{D}_p) := \bigcup_{\boldsymbol{\varepsilon} \in \mathcal{D}_p} \Omega^{\boldsymbol{\varepsilon}}$ the set of all the Stokes values of all the systems (2.1) when $\boldsymbol{\varepsilon}$ runs $\mathcal{D}_p$,
- $\boldsymbol{\Omega}(\mathcal{D}_p) := \bigcup_{\boldsymbol{\varepsilon} \in \mathcal{D}_p} \boldsymbol{\Omega}^{\boldsymbol{\varepsilon}}$ the set of all the Stokes values of all the homological systems of all the systems (2.1) when $\boldsymbol{\varepsilon}$ runs $\mathcal{D}_p$.

The sets $\Omega(\mathcal{D}_p)$ and $\boldsymbol{\Omega}(\mathcal{D}_p)$ are the respective "images" of Ω and $\boldsymbol{\Omega}$ under the action of the perturbation in $\boldsymbol{\varepsilon}$. More precisely,

- $\Omega(\mathcal{D}_p) = \{0\} \cup \left(\bigcup_{k=1}^{p} D(\omega_k, |\omega_k| \, \rho_k) \right)$,
- $\boldsymbol{\Omega}(\mathcal{D}_p) = \{0\} \cup \left(\bigcup_{\substack{k,\ell=0 \\ k \neq \ell}}^{p} D(\omega_k - \omega_\ell, |\omega_k| \, \rho_k + |\omega_\ell| \, \rho_\ell) \right)$; we set $\rho_0 := 1$.

Notice that $\Omega \subset \boldsymbol{\Omega}$ implies $\Omega(\mathcal{D}_p) \subset \boldsymbol{\Omega}(\mathcal{D}_p)$. Notice also that, contrarily to $\Omega(\mathcal{D}_p)$, some discs of $\boldsymbol{\Omega}(\mathcal{D}_p)$ may overlap.

By construction, the disc $D_{\omega_k-\omega_\ell} := D(\omega_k-\omega_\ell, |\omega_k|\,\rho_k+|\omega_\ell|\,\rho_\ell)$ is formed, for all $k \neq \ell$, by all the points $\omega_k\varepsilon_k - \omega_\ell\varepsilon_\ell \in \boldsymbol{\Omega}(\mathcal{D}_p)$ issuing from the non-zero Stokes value $\omega_k - \omega_\ell \in \boldsymbol{\Omega}$ under the action of the perturbation. Hence, the following definition:

Definition 2.1 (Singular disc of $\boldsymbol{\Omega}(\mathcal{D}_p)$)
Let $\omega := \omega_k - \omega_\ell$ be a non-zero Stokes value of $\boldsymbol{\Omega}$. The disc $D_\omega := D_{\omega_k-\omega_\ell}$ is called the singular disc of $\boldsymbol{\Omega}(\mathcal{D}_p)$ associated with $\omega_k - \omega_\ell$.

Observe that, due to conditions $(C1)$ and $(C2)$, none of the closed singular disc $\overline{D}_\omega$ contains 0.

Remark 2.2 According to calculations above, any anti-Stokes direction $\theta \in \mathbb{R}/2\pi\mathbb{Z}$ of system (0.1) is transformed, under the action of the perturbation, into the set of anti-Stokes directions of systems (2.1) given by all the points of all the singular discs of $\boldsymbol{\Omega}(\mathcal{D}_p)$ centered on θ.

Remark 2.2 is precised in lemma 2.3 below.

2.2 Action of the perturbation on the anti-Stokes directions

We denote by

- Θ the set of anti-Stokes directions of system (0.1),
- $\boldsymbol{\Omega}_\theta$ the set of non-zero Stokes values of $\boldsymbol{\Omega}$ with argument θ.

Obviously, $\theta \in \Theta$ if and only if $\boldsymbol{\Omega}_\theta \neq \emptyset$. For $\theta \in \Theta$, we consider

- $\boldsymbol{\Omega}_\theta(\mathcal{D}_p) := \bigcup_{\omega \in \boldsymbol{\Omega}_\theta} D_\omega$ the set of singular discs of $\boldsymbol{\Omega}(\mathcal{D}_p)$ associated with $\omega \in \boldsymbol{\Omega}_\theta$, *i.e.*, the set of all the singular discs of $\boldsymbol{\Omega}(\mathcal{D}_p)$ centered on θ.

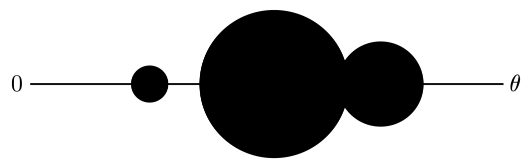

Figure 2.1 - A set $\boldsymbol{\Omega}_\theta(\mathcal{D}_p)$

Since all these discs D_ω are symmetrical about θ, we also consider

- $\eta(\theta)$ the minimal opening of sectors $\Sigma_{\theta,\eta}$ containing $\mathbf{\Omega}_\theta(\mathcal{D}_p)$.

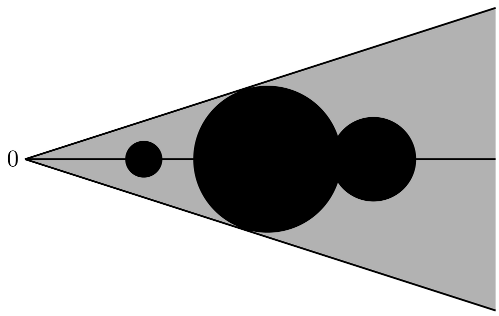

Figure 2.2 - A sector $\Sigma_{\theta,\eta(\theta)}$

By construction, the sector $\Sigma_{\theta,\eta(\theta)}$ is formed by all the anti-Stokes directions of systems (2.1) given by the points of $\mathbf{\Omega}_\theta(\mathcal{D}_p)$. Consequently, remark 2.2 implies the following result:

Lemma 2.3 (Action of the perturbation on $\theta \in \Theta$)
Let $\theta \in \Theta$ be an anti-Stokes direction of system (0.1).
Then, the "image" of θ by the perturbation is the sector $\Sigma_{\theta,\eta(\theta)}$.

Before to state the main result of the article (see theorem 2.5 below), let us make some remarks about sectors $\Sigma_{\theta,\eta(\theta)}$. First, their openings $\eta(\theta)$ only depend on the radius of the singular discs D_ω associated with $\omega \in \mathbf{\Omega}_\theta$. In particular, the $\eta(\theta)$'s tend to 0 when the ρ_k's go to 0. Second, the size of the $\eta(\theta)$'s will play a fundamental role in theorem 2.5 (see section 2.3 below). Henceforth, we suppose that the radius ρ_k's are chosen small enough so that, in addition to conditions $(C1)$ and $(C2)$ above, the following conditions be satisfied:

$(C3)$ $\overline{\Sigma}_{\theta,\eta(\theta)} \cap \overline{\Sigma}_{\theta',\eta(\theta')} = \emptyset$ for all $\theta, \theta' \in \Theta$, $\theta \neq \theta'$,

$(C4)$ for all $\theta \in \Theta$, $\eta(\theta) < \dfrac{\pi}{2}$,

$(C5)$ for all $\theta \in \Theta$, the principal determination $\theta^\star$ of θ and the principal determination $(\theta - \eta(\theta)/2)^\star$ of $\theta - \eta(\theta)/2$ satisfy

$$-2\pi < (\theta - \eta(\theta)/2)^\star < \theta^\star \leq 0$$

Remark 2.4 Condition $(C3)$ implies that the closed sector $\overline{\Sigma}_{\theta,\eta(\theta)}$ contains no other anti-Stokes directions of systems (2.1) than those issuing from θ under the action of the perturbation. In particular, since systems (0.1) and (2.1) coincide for $\boldsymbol{\varepsilon} = \mathbf{1}$, the sector $\overline{\Sigma}_{\theta,\eta(\theta)}$ just contains θ as anti-Stokes directions of system (0.1).

2.3 The main result

As before, we denote by Θ the set of anti-Stokes directions of system (0.1).

Let us consider a direction $\theta \in \Theta$ and its "image" $\Sigma_{\theta,\eta(\theta)}$ by the perturbation (*cf.* lemma 2.3). Under conditions $(C3)$, $(C4)$ and $(C5)$ above, there exists $\eta \in]\eta(\theta), \pi/2[$ such that

1. $\Sigma_{\theta,\eta(\theta)} \subsetneqq \Sigma_{\theta,\eta} \subsetneqq \Sigma_{\theta,\pi-\eta}$,
2. $\overline{\Sigma}_{\theta,\eta} \cap \overline{\Sigma}_{\theta',\eta(\theta')} = \emptyset$ for all $\theta' \in \Theta$, $\theta' \neq \theta$,
3. the principal determination $(\theta - \eta/2)^\star$ of $\theta - \eta/2$ satisfies
$$-2\pi < (\theta - \eta/2)^\star < (\theta - \eta(\theta)/2)^\star < \theta^\star \leq 0$$

Notice that point 1. results from the choice η in $]\eta(\theta), \pi/2[$ and that points 2. and 3. hold as soon as η is close enough to $\eta(\theta)$. Notice also that point 2. guarantees that the closed sector $\overline{\Sigma}_{\theta,\eta}$ contains no other anti-Stokes directions of systems (2.1) when $\boldsymbol{\varepsilon}$ runs $\mathcal{D}_p$ than those given by $\Sigma_{\theta,\eta(\theta)}$.

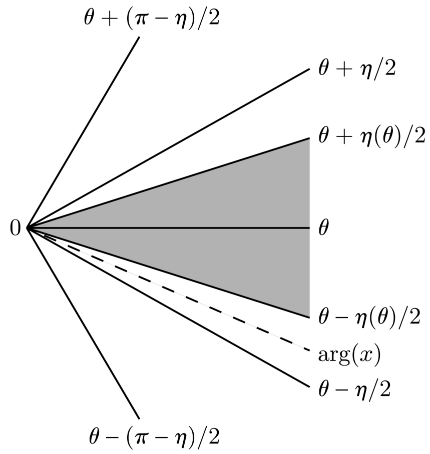

Figure 2.3 - Sector $\Sigma_{\theta,\eta(\theta)}$ and associated directions

Let us now fix $\boldsymbol{\varepsilon} \in \mathcal{D}_p$ and η as above. Then, according to points 1. to 3., directions $\theta \pm \eta/2$ are not anti-Stokes directions of system (2.1) and the 1-sums $s_{1;\theta\pm\eta/2}(\widetilde{F}^{\boldsymbol{\varepsilon}})$ are both defined and analytic on a same germ of sector $\Sigma_{\theta,\pi-\eta}$. Consequently, the sums

$$Y^{\boldsymbol{\varepsilon}}_{\theta\pm\eta/2}(x) := s_{1;\theta\pm\eta/2}(\widetilde{F}^{\boldsymbol{\varepsilon}})(x)x^L e^{Q^{\boldsymbol{\varepsilon}}(1/x)}$$

are related for $\arg(x) \in](\theta-\eta/2)^\star, (\theta-\eta(\theta)/2)^\star[$ (see figure 2.3 above) by the relation

$$\boxed{Y^{\boldsymbol{\varepsilon}}_{\theta-\eta/2}(x) = Y^{\boldsymbol{\varepsilon}}_{\theta+\eta/2}(x)\mathfrak{S}^{\boldsymbol{\varepsilon}}_{\theta^\star}} \tag{2.2}$$

The matrix $\mathfrak{S}^{\boldsymbol{\varepsilon}}_{\theta^\star} \in GL_n(\mathbb{C})$ denotes the *(perturbed) connection matrix* between $Y^{\boldsymbol{\varepsilon}}_{\theta+\eta/2}$ and $Y^{\boldsymbol{\varepsilon}}_{\theta-\eta/2}$. It is uniquely determined by identity (2.2) above. Notice that remark 2.4 and point 3. above imply that $\mathfrak{S}^{\boldsymbol{\varepsilon}}_{\theta^\star}$ is defined as a (finite) product of Stokes-Ramis matrices associated with $\widetilde{Y}^{\boldsymbol{\varepsilon}}$ in the anti-Stokes directions of system (2.1) contained in $\Sigma_{\theta,\eta(\theta)}$. Notice also that, for $\boldsymbol{\varepsilon} = \mathbf{1}$,

$$Y^{\mathbf{1}}_{\theta\pm\eta/2}(x) = Y_{\theta\pm\eta/2}(x) = Y_{\theta^\pm}(x) \quad \text{and} \quad \mathfrak{S}^{\mathbf{1}}_{\theta^\star} = St_{\theta^\star}$$

The aim of this article is to prove the following theorem:

Theorem 2.5 *Let $\theta \in \Theta$ be an anti-Stokes direction of system (0.1). Then, the function $\boldsymbol{\varepsilon} \longmapsto \mathfrak{S}^{\boldsymbol{\varepsilon}}_{\theta^\star}$ is holomorphic on $\mathcal{D}_p$.*

Note that theorem 2.5 implies the following result which could provide an efficient tool for the numerical calculation of the Stokes-Ramis matrices $St_{\theta^\star}$ of the initial system (0.1):

Corollary 2.6 *Let $\theta \in \Theta$ be an anti-Stokes direction of system (0.1). Then,*

$$\boxed{\lim_{\boldsymbol{\varepsilon}\to\mathbf{1}} \mathfrak{S}^{\boldsymbol{\varepsilon}}_{\theta^\star} = St_{\theta^\star}}$$

Theorem 2.5 is proved in section 3 below. More precisely, it results from section 3.2 which asserts, on the one hand, that the 1-sums $s_{1;\theta\pm\eta/2}(\widetilde{F}^{\boldsymbol{\varepsilon}})(x)$ are defined for all $\boldsymbol{\varepsilon} \in \mathcal{D}_p$ on a same germ Σ of sector

$$\left\{ x \in \mathbb{C}^* \ ; \ \left(\theta - \frac{\eta}{2}\right)^\star < \arg(x) < \left(\theta - \frac{\eta(\theta)}{2}\right)^\star \right\}$$

and, on the other hand, that the functions $\boldsymbol{\varepsilon} \longmapsto s_{1;\theta\pm\eta/2}(\widetilde{F}^{\boldsymbol{\varepsilon}})(x)$ with $x \in \Sigma$ are holomorphic on $\mathcal{D}_p$ (see proposition 3.10). To prove these two points, we shall first study the dependence in ε of the Borel transforms $\widehat{F}^{\varepsilon}(\xi)$ of $\widetilde{F}^{\varepsilon}(x)$ with respect to x (see section 3.1). Recall indeed that $s_{1;\theta\pm\eta/2}(\widetilde{F}^{\varepsilon})(x)$ is given by the Borel-Laplace integral

$$\int_0^{\infty e^{i(\theta\pm\eta/2)}} \widehat{F}^{\varepsilon}(\xi)e^{-\xi/x}d\xi$$

3 Proof of theorem 2.5

Recall that the formal Borel transformation is an isomorphism from the $\mathbb{C}$-differential algebra $\left(\mathbb{C}[[x]],+,\cdot,x^2\frac{d}{dx}\right)$ to the $\mathbb{C}$-differential algebra $(\delta\mathbb{C}\oplus\mathbb{C}[[\xi]],+,*,\xi\cdot)$ that changes ordinary product $\cdot$ into convolution product $*$ and changes derivation $x^2\frac{d}{dx}$ into multiplication by ξ. It also changes multiplication by $\frac{1}{x}$ into derivation $\frac{d}{d\xi}$.

Recall also that the formal Borel transform $\widehat{g}(\xi)$ of an analytic function $g(x)\in\mathbb{C}\{x\}$ at 0 defines an entire function on all $\mathbb{C}$ with exponential growth at infinity.

3.1 Dependence in ε and Borel transform

Recall that $\mathcal{D}_p$ denotes the polydisc $D(1,\rho_1)\times ...\times D(1,\rho_p)$ in $\mathbb{C}^p$ where the radius ρ_k are chosen so that conditions $(C1)$ to $(C5)$ hold. Recall also that, for any non-zero Stokes value $\omega\in\mathbf{\Omega}$, D_ω denotes the singular disc of $\mathbf{\Omega}(\mathcal{D}_p)$ associated with ω, *i.e.*, the open disc formed by all the Stokes values of $\mathbf{\Omega}(\mathcal{D}_p)$ issuing from ω under the action of the perturbation.

In this section, we consider a domain $V\subset\mathbb{C}$ defined by the data of an open disc centered at $0\in\mathbb{C}$ and an open sector in $\mathbb{C}$ with vertex 0 such that

$$\overline{V}\cap\overline{D}_\omega=\emptyset \quad \text{for all } \omega\in\mathbf{\Omega}\backslash\{0\} \tag{3.1}$$

($\overline{V}$ and $\overline{D}_\omega$ denote respectively the closure of V and D_ω in $\mathbb{C}$). Observe that the existence of such a domain V is guaranteed by conditions $(C1)$, $(C2)$ and

$(C3)$.

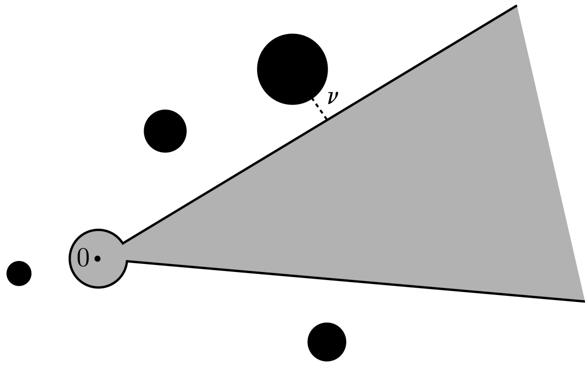

Figure 3.1 - A domain V and the singular discs of $\mathbf{\Omega}(\mathcal{D}_p)$

Our aim is to prove the following result:

Proposition 3.1 *Let V be a domain as above.*
Then, the function $(\xi, \boldsymbol{\varepsilon}) \longmapsto \widehat{F}^{\boldsymbol{\varepsilon}}(\xi)$ is holomorphic on $V \times \mathcal{D}_p$.

Proposition 3.1 is proved below by using an adequate variant of the proof of summable-resurgence theorem following Écalle's method by regular perturbation and majorant series which was given by M. Loday-Richaud and the author in [3].

Remark 3.2 For all $\boldsymbol{\varepsilon} \in \mathcal{D}_p$, any of the J column-blocks of $\widetilde{F}^{\boldsymbol{\varepsilon}}(x)$ associated with the Jordan structure of L (matrix of exponents of formal monodromy) can be positionned at the first place by means of a same permutation (hence, independent of $\boldsymbol{\varepsilon}$) acting on the columns of $\widetilde{Y}^{\boldsymbol{\varepsilon}}(x)$. Consequently, it is sufficient to prove proposition 3.1 in restriction to the column-block $\widetilde{f}^{\boldsymbol{\varepsilon}}(x)$ formed by the first n_1 (= dimension of the first Jordan block of L) columns of $\widetilde{F}^{\boldsymbol{\varepsilon}}(x)$.

For all $\boldsymbol{\varepsilon} \in \mathcal{D}_p$, the system

$$x^2 \frac{dY}{dx} = A_0^{\boldsymbol{\varepsilon}}(x) Y \quad \text{with} \quad A_0^{\boldsymbol{\varepsilon}}(x) = \bigoplus_{j=1}^{J} \left(a_j^{\boldsymbol{\varepsilon}} I_{n_j} + x L_j \right)$$

has for formal fundamental solution the matrix $x^L e^{Q^{\boldsymbol{\varepsilon}}(1/x)}$ (recall that $L_j := \lambda_j I_{n_j} + J_{n_j}$ denotes the j-th Jordan block of L). According to the normalizations of $\widetilde{Y}^{\boldsymbol{\varepsilon}}(x)$ (*cf.* page 6), $\widetilde{f}^{\boldsymbol{\varepsilon}}(x)$ is uniquely determined by the first n_1

columns of the homological system

$$x^2\frac{dF}{dx} = A^{\varepsilon}(x)F - FA_0^{\varepsilon}(x)$$

of system (2.1) jointly with the initial condition $\widetilde{f}^{\varepsilon}(0) = I_{n,n_1}$ (= the first n_1 columns of the identity matrix I_n of size n) (see [1]). Hence, the system

$$\boxed{x^2\frac{df}{dx} = A^{\varepsilon}(x)f - xfJ_{n_1}} \tag{3.2}$$

Recall that $a_1^{\varepsilon} = \lambda_1 = 0$. Recall also that

$$A^{\varepsilon}(x) = \bigoplus_{j=1}^{J}\left(a_j^{\varepsilon}I_{n_j} + xL_j\right) + B(x)$$

where $B(x)$ is analytic at 0. More precisely, split $B(x) = \left[B^{j;\ell}(x)\right]$ into blocks fitting to the Jordan structure of L. Then,

$$B^{j;\ell}(x) = \left\{\begin{array}{ll} O(x) & \text{if } a_j^{\varepsilon} \neq a_\ell^{\varepsilon} \\ O(x^2) & \text{if } a_j^{\varepsilon} = a_\ell^{\varepsilon} \end{array}\right. \tag{3.3}$$

Notation 3.3 From now on, given a matrix M split into blocks fitting to the Jordan structure of L, we denote by $M^{j;\bullet}$ the j-th row-block of M. So, $M^{j;\bullet}$ is a $n_j \times p$-matrix when M is a $n \times p$-matrix.

3.1.1 Regular perturbation

Following J. Écalle ([2]), we consider, instead of system (3.2), the regularly perturbed system

$$x^2\frac{df}{dx} = A^{\varepsilon}(x,\alpha)f - xfJ_{n_1} \tag{3.4}$$

obtained by substituting αB for B in the matrix $A^{\varepsilon}(x)$.

Like in [3], an identification of equal power in α shows that system (3.4) admits, for all $\boldsymbol{\varepsilon} \in \mathcal{D}_p$, a unique formal solution of the form

$$\widetilde{f}^{\varepsilon}(x,\alpha) = \sum_{m\geq 0}\widetilde{f}_m^{\varepsilon}(x)\alpha^m$$

satisfying $\widetilde{f}_0^{\boldsymbol{\varepsilon}}(x) = I_{n,n_1}$ and $\widetilde{f}_m^{\boldsymbol{\varepsilon}}(x) \in M_{n,n_1}(\mathbb{C}[[x]])$ for all $m \geq 1$. More precisely, the components $\widetilde{f}_m^{\boldsymbol{\varepsilon}\ j;\bullet}(x) \in M_{n_j,n_1}(\mathbb{C}[[x]])$ of $\widetilde{f}_m^{\boldsymbol{\varepsilon}}(x)$ are uniquely determined for all $m \geq 1$ and $j = 1, ..., J$ as formal solutions of systems

$$x^2 \frac{d\widetilde{f}_m^{\boldsymbol{\varepsilon}\ j;\bullet}}{dx} - a_j^{\boldsymbol{\varepsilon}} \widetilde{f}_m^{\boldsymbol{\varepsilon}\ j;\bullet} + xL_j \widetilde{f}_m^{\boldsymbol{\varepsilon}\ j;\bullet} = B^{j;\bullet} \widetilde{f}_{m-1}^{\boldsymbol{\varepsilon}\ j;\bullet} - x\widetilde{f}_m^{\boldsymbol{\varepsilon}\ j;\bullet} J_{n_1} \tag{3.5}$$

Relations (3.5) and normalizations (3.3) of $B(x)$ show in particular that

$$\widetilde{f}_{2m-1}^{\boldsymbol{\varepsilon}\ j;\bullet}(x) = O(x^m) \quad \text{and} \quad \widetilde{f}_{2m}^{\boldsymbol{\varepsilon}\ j;\bullet} = \begin{cases} O(x^m) & \text{if } a_j^{\boldsymbol{\varepsilon}} = 0 \\ O(x^{m+1}) & \text{if } a_j^{\boldsymbol{\varepsilon}} \neq 0 \end{cases}$$

for all $m \geq 1$ and $j = 1, ..., J$.

As a result, the series $\widetilde{f}^{\boldsymbol{\varepsilon}}(x, \alpha)$ can be rewritten as a series in x with polynomial coefficients in α. Consequently, for all $\boldsymbol{\varepsilon} \in \mathcal{D}_p$, $\widetilde{f}^{\boldsymbol{\varepsilon}}(x) = \widetilde{f}^{\boldsymbol{\varepsilon}}(x, 1)$ (by unicity of $\widetilde{f}^{\boldsymbol{\varepsilon}}(x)$ and $\widetilde{f}^{\boldsymbol{\varepsilon}}(x, 1)$) and, for all α, the series $\widetilde{f}^{\boldsymbol{\varepsilon}}(x, \alpha)$ admits a formal Borel transform $\varphi^{\boldsymbol{\varepsilon}}(\xi, \alpha)$ with respect to x of the form

$$\varphi^{\boldsymbol{\varepsilon}}(\xi, \alpha) = \delta I_{n,n_1} + \sum_{m \geq 1} \varphi_m^{\boldsymbol{\varepsilon}}(\xi) \alpha^m$$

where $\varphi_m^{\boldsymbol{\varepsilon}}(\xi) \in M_{n,n_1}(\mathbb{C}[[\xi]])$ denotes, for all $m \geq 1$, the Borel transform of $\widetilde{f}_m^{\boldsymbol{\varepsilon}}(x)$. In particular, the components $\varphi_m^{\boldsymbol{\varepsilon}\ j;\bullet}(\xi) \in M_{n_j,n_1}(\mathbb{C}[[\xi]])$ of $\varphi_m^{\boldsymbol{\varepsilon}}(\xi)$ are iteratively determined for all $m \geq 1$ and $j = 1, ..., J$ as solutions of systems

$$(\xi - a_j^{\boldsymbol{\varepsilon}}) \frac{d\varphi_m^{\boldsymbol{\varepsilon}\ j;\bullet}}{d\xi} + (L_j - I_{n_j}) \varphi_m^{\boldsymbol{\varepsilon}\ j;\bullet} = \frac{d}{d\xi}(\widehat{B}^{j;\bullet} * \varphi_{m-1}^{\boldsymbol{\varepsilon}}) - \varphi_m^{\boldsymbol{\varepsilon}\ j;\bullet} J_{n_1} \tag{3.6}$$

We set $\varphi_0^{\boldsymbol{\varepsilon}} := \delta I_{n,n_1}$. Note that the Borel transforms $\widehat{B}^{j;\bullet}$ of $B^{j;\bullet}$ are entire functions on all $\mathbb{C}$ since B is analytic at 0. Note also that normalizations (3.3) of $B(x)$ imply that the only singularities in $\mathbb{C}$ of systems (3.6) when $\boldsymbol{\varepsilon}$ runs $\mathcal{D}_p$ are the Stokes values $a_j^{\boldsymbol{\varepsilon}} \neq 0$ of $\Omega(\mathcal{D}_p) \subset \boldsymbol{\Omega}(\mathcal{D}_p)$. Hence, since the domain V does not meet $\boldsymbol{\Omega}(\mathcal{D}_p) \backslash \{0\}$ and since system (3.6) depends holomorphically on the parameter $\boldsymbol{\varepsilon} \in \mathcal{D}_p$, the following lemma:

Lemma 3.4 *The function* $(\xi, \boldsymbol{\varepsilon}) \longmapsto \varphi_m^{\boldsymbol{\varepsilon}}(\xi)$ *is holomorphic on* $V \times \mathcal{D}_p$ *for all* $m \geq 1$*.*

We are left to prove that the function

$$(\xi, \boldsymbol{\varepsilon}) \longmapsto \widehat{f}^{\boldsymbol{\varepsilon}}(\xi) = \varphi^{\boldsymbol{\varepsilon}}(\xi, 1) = \sum_{m \geq 1} \varphi_m^{\boldsymbol{\varepsilon}}(\xi)$$

is well-defined and holomorphic on $V \times \mathcal{D}_p$. This point is proved below by using a technique of majorant series satisfying a convenient system. There exists, of course, many possible majorant systems. Here, we make explicit a possible one.

3.1.2 A convenient majorant system

Let ν denote the minimal distance between the elements of V and the elements of $\mathbf{\Omega}(\mathcal{D}_p)\backslash\{0\}$. Observe that condition (3.1) implies $\nu > 0$ (see figure 3.1).

We consider, for $j = 1, ..., J$, the regularly perturbed linear system

$$(3.7) \quad \begin{cases} C_j(g^{j;\bullet} - I^{j;\bullet}_{n,n_1}) = J_{n_j}g^{j;\bullet} + g^{j;\bullet}J_{n_1} - 2I^{j;\bullet}_{n,n_1}J_{n_1} + \alpha\dfrac{|B^{j;\bullet}|(x)}{x}g & \text{if } a_j = 0 \\ (\nu - x\,|\lambda_j - 1|\,I_{n_j})g^{j;\bullet} = xJ_{n_j}g^{j;\bullet} + xg^{j;\bullet}J_{n_1} + \alpha\,|B^{j;\bullet}|\,(x)g & \text{if } a_j \neq 0 \end{cases}$$

where

- the unknown g is, as previously, a $n \times n_1$-matrix split into row-blocks $g^{j;\bullet}$ fitting to the Jordan structure of L,
- $|B|\,(x)$ denotes the series $B(x)$ in which the coefficients of the powers of x are replaced by their module,
- the C_j's are positive constants which are to be adequatly chosen below (see lemma 3.5).

Recall that λ_j denotes the eigenvalue of the j-th Jordan block L_j of L.

Observe that the so-defined system depends on the domain V but not on the parameter $\boldsymbol{\varepsilon}$.

System (3.7) above has already been studied in [3] since it is actually the majorant system which has been used to prove the summable-resurgence theorem for level-one linear differential systems. In particular, one has been shown that its Borel transformed system admits, for $\alpha = 1$, a solution of the form

$$\widehat{g}(\xi) = \delta I_{n,n_1} + \sum_{m\geq 1}\Phi_m(\xi)$$

which is entire on all $\mathbb{C}$ with exponential growth at infinity. Moreover, for any $m \geq 1$, $\Phi_m(\xi)$ belongs to $M_{n,n_1}(\mathbb{R}^+[[\xi]])$ and is also an entire function on all $\mathbb{C}$ with exponential growth at infinity. More precisely, the components $\Phi^{j;\bullet}_m(\xi) \in M_{n_j,n_1}(\mathbb{R}^+[[\xi]])$ of $\Phi_m(\xi)$ are iteratively determined for all $m \geq 1$ and $j = 1, ..., J$ as solutions of systems

- Case $a_j = 0$:

$$C_j \Phi_m^{j;\bullet} = J_{n_j} \Phi_m^{j;\bullet} + \Phi_m^{j;\bullet} J_{n_1} + \frac{d}{d\xi}\left(\widehat{|B^{j;\bullet}|} * \Phi_{m-1}\right)$$

- Case $a_j \neq 0$:

$$\nu \frac{d\Phi_m^{j;\bullet}}{d\xi} = |\lambda_j - 1|\, \Phi_m^{j;\bullet} + J_{n_j} \Phi_m^{j;\bullet} + \Phi_m^{j;\bullet} J_{n_1} + \frac{d}{d\xi}\left(\widehat{|B^{j;\bullet}|} * \Phi_{m-1}\right)$$

We set $\Phi_0 := \delta I_{n,n_1}$.

One can besides verify that *all* the calculations made in [3, section 2.5.5] to prove that system (3.7) was a convenient majorant system can still be applied in the present case. Indeed, the parameter $\boldsymbol{\varepsilon}$ just acts on the Stokes values $a_j^{\boldsymbol{\varepsilon}}$ in systems (3.6) and condition $a_j^{\boldsymbol{\varepsilon}} = 0$ (resp. $a_j^{\boldsymbol{\varepsilon}} \neq 0$) is equivalent to the condition $a_j = 0$ (resp. $a_j \neq 0$). Hence, the following lemma:

Lemma 3.5 (Majorant series, [3, lemma 2.9])
Let a be a positive constant such that $|\arg(\xi)| \leq a$ for all $\xi \in V$.
Let

$$C_j = \frac{1 - \mathrm{Re}(\lambda_j)}{\max\limits_{1\leq j\leq J} \exp(2a\,|\mathrm{Im}(\lambda_j)|)}$$

Then, for all $m \geq 1$, $\xi \in V$, $\boldsymbol{\varepsilon} \in \mathcal{D}_p$ and $j = 1, ..., J$, the following inequalities hold:

$$\left|\varphi_m^{\boldsymbol{\varepsilon}\ j;\bullet}(\xi)\right| \leq \Phi_m^{j;\bullet}(|\xi|)$$

In particular, for all $\boldsymbol{\varepsilon} \in \mathcal{D}_p$, the series

$$\widehat{g}(|\xi|) = \sum_{m\geq 1} \Phi_m(|\xi|)$$

is a majorant series of $\widehat{f}^{\boldsymbol{\varepsilon}}(\xi)$.

Recall that lemma 3.5 is proved by applying Grönwall lemma to systems (3.6) defining the $\varphi_m^{\boldsymbol{\varepsilon}\ j;\bullet}$'s and systems above defining the $\Phi_m^{j;\bullet}$'s.

Notice besides that the constant K given in [3, lemma 2.9] is equal to 1 in our case. Indeed, according to the definition of domain V, the "optimal" path γ_ξ from 0 to any $\xi \in V$ used in the proof of [3, lemma 2.9] is the straigth line $[0, \xi]$.

Before to prove proposition 3.1, let us make some remarks about lemma 3.5 and calculations above.

Remark 3.6 Like system (3.7), the function $\widehat{g}(\xi)$ depends on domain V but not on the parameter $\boldsymbol{\varepsilon}$.

Remark 3.7 Lemma 3.5 and calculations above imply the following property: there exist $c, k > 0$ such that

$$\left|\widehat{f}^{\boldsymbol{\varepsilon}}(\xi)\right| \leq ce^{k|\xi|} \tag{3.8}$$

for all $\xi \in V$ and $\boldsymbol{\varepsilon} \in \mathcal{D}_p$.

Remark 3.8 According to remark 3.2, property (3.8) can be extended to other columns of $\widehat{F}^{\boldsymbol{\varepsilon}}$: there exist $C, K > 0$ such that

$$\left|\widehat{F}^{\boldsymbol{\varepsilon}}(\xi)\right| \leq Ce^{K|\xi|}$$

for all $\xi \in V$ and all $\boldsymbol{\varepsilon} \in \mathcal{D}_p$.

3.1.3 Proof of proposition 3.1

We shall now prove proposition 3.1: lemmas 3.4 and 3.5 tell us that the series

$$(\xi, \boldsymbol{\varepsilon}) \longmapsto \widehat{f}^{\boldsymbol{\varepsilon}}(\xi) = \sum_{m \geq 1} \varphi_m^{\boldsymbol{\varepsilon}}(\xi)$$

is a series of holomorphic functions on $V \times \mathcal{D}_p$ which normally converges on all compact sets of $V \times \mathcal{D}_p$. Hence, the function $(\xi, \boldsymbol{\varepsilon}) \longmapsto \widehat{f}^{\boldsymbol{\varepsilon}}(\xi)$ is well-defined and also holomorphic on $V \times \mathcal{D}_p$ which achieves the proof of proposition 3.1 (*cf.* remark 3.2).

3.2 Dependence in $\boldsymbol{\varepsilon}$ and summation

Let us now consider an anti-Stokes direction $\theta \in \Theta$ of system (0.1) and its associated sector $\Sigma_{\theta,\eta(\theta)}$ (*cf.* section 2.2). Recall that the directions determined by $\Sigma_{\theta,\eta(\theta)}$ are all the anti-Stokes directions of all systems (2.1) associated with θ under the action of the perturbation.

We also consider two directions $\theta + \eta/2$ and $\theta - \eta/2$ as in section 2.3 (*cf.* figure 2.3). Let V^+ (resp. V^-) be a domain in $\mathbb{C}$ verifying condition (3.1) above and defined by the data of an open disc centered at $0 \in \mathbb{C}$ and an open sector in $\mathbb{C}$ with vertex 0 and bisected by $\theta + \eta/2$ (resp. $\theta - \eta/2$). Such domains exist since $\theta \pm \eta/2$ are singular directions for none of systems (2.1).

Since V^+ and V^- are domains as in section 3.1, proposition 3.1 and remark 3.8 imply the following lemma:

Lemma 3.9

1. *Domain V^+*

   (a) *For all $\xi \in V^+$, the function $\boldsymbol{\varepsilon} \longmapsto \widehat{F}^{\boldsymbol{\varepsilon}}(\xi)$ is holomorphic on $\mathcal{D}_p$.*

   (b) *There exist $C^+, K^+ > 0$ such that*

   $$\left|\widehat{F}^{\boldsymbol{\varepsilon}}(\xi)\right| \leq C^+ e^{K^+|\xi|}$$

   for all $\xi \in V^+$ and all $\boldsymbol{\varepsilon} \in \mathcal{D}_p$.

2. *Domain V^-*

   (a) *For all $\xi \in V^-$, the function $\boldsymbol{\varepsilon} \longmapsto \widehat{F}^{\boldsymbol{\varepsilon}}(\xi)$ is holomorphic on $\mathcal{D}_p$.*

   (b) *There exist $C^-, K^- > 0$ such that*

   $$\left|\widehat{F}^{\boldsymbol{\varepsilon}}(\xi)\right| \leq C^- e^{K^-|\xi|}$$

   for all $\xi \in V^-$ and all $\boldsymbol{\varepsilon} \in \mathcal{D}_p$.

As a result of point 1.(b) (resp. 2.(b)), the 1-sums $s_{1;\theta+\eta/2}(\widetilde{F}^{\boldsymbol{\varepsilon}})$ (resp. $s_{1;\theta-\eta/2}(\widetilde{F}^{\boldsymbol{\varepsilon}})$) are holomorphic for all $\boldsymbol{\varepsilon} \in \mathcal{D}_p$ on the sector

$$\Sigma_{\theta+\eta/2}\left(\frac{1}{K^+}\right) := \left\{x \in \mathbb{C}^* \ ; \ |x| < \frac{1}{K^+} \text{ and } \left|\arg(x) - \theta - \frac{\eta}{2}\right| < \frac{\pi}{2}\right\}$$

(resp. $\Sigma_{\theta-\eta/2}\left(\frac{1}{K^-}\right) := \left\{x \in \mathbb{C}^* \ ; \ |x| < \frac{1}{K^-} \text{ and } \left|\arg(x) - \theta + \frac{\eta}{2}\right| < \frac{\pi}{2}\right\}$)

Hence, according to the choice of η (*cf.* section 2.3), the 1-sums $s_{1;\theta+\eta/2}(\widetilde{F}^{\boldsymbol{\varepsilon}})$ and $s_{1;\theta-\eta/2}(\widetilde{F}^{\boldsymbol{\varepsilon}})$ are holomorphic for all $\boldsymbol{\varepsilon} \in \mathcal{D}_p$ on the sector

$$\Sigma := \left\{x \in \mathbb{C}^* \ ; \ |x| < \min\left(\frac{1}{K^-}, \frac{1}{K^+}\right) \text{ and } \left(\theta - \frac{\eta}{2}\right)^\star < \arg(x) < \left(\theta - \frac{\eta(\theta)}{2}\right)^\star\right\}$$

Observe that Σ does not depend on the parameter $\boldsymbol{\varepsilon}$.

We are now able to state the result in view in this section:

Proposition 3.10 *For all $x \in \Sigma$, the functions $\boldsymbol{\varepsilon} \longmapsto s_{1;\theta+\eta/2}(\widetilde{F}^{\boldsymbol{\varepsilon}})(x)$ and $\boldsymbol{\varepsilon} \longmapsto s_{1;\theta-\eta/2}(\widetilde{F}^{\boldsymbol{\varepsilon}})(x)$ are holomorphic on $\mathcal{D}_p$.*

Proof. $\star$ Fix $x \in \Sigma$. For all $\boldsymbol{\varepsilon} \in \mathcal{D}_p$, the 1-sum $s_{1;\theta+\eta/2}(\widetilde{F}^{\boldsymbol{\varepsilon}})(x)$ is given by the Borel-Laplace integral

$$s_{1;\theta+\eta/2}(\widetilde{F}^{\boldsymbol{\varepsilon}})(x) = \int_0^{\infty e^{i(\theta+\eta/2)}} \widehat{F}^{\boldsymbol{\varepsilon}}(\xi) e^{-\xi/x} d\xi = \int_0^{+\infty} \widehat{G}_+^{\boldsymbol{\varepsilon}}(\xi) d\xi$$

where

$$\widehat{G}_+^{\boldsymbol{\varepsilon}}(\xi) := \widehat{F}^{\boldsymbol{\varepsilon}}(\xi e^{i(\theta+\eta/2)}) e^{-\xi \exp(i(\theta+\eta/2))/x}$$

Since $\xi e^{i(\theta+\eta/2)} \in V^+$ for all $\xi \geq 0$, we can apply lemma 3.9 to $\widehat{G}_+^{\boldsymbol{\varepsilon}}(\xi)$:

– for all $\xi \geq 0$, the function $\boldsymbol{\varepsilon} \longmapsto \widehat{G}_+^{\boldsymbol{\varepsilon}}(\xi)$ is holomorphic on $\mathcal{D}_p$,

– for all $\xi \geq 0$ and all $\boldsymbol{\varepsilon} \in \mathcal{D}_p$,

$$\begin{aligned}\left|\widehat{G}_+^{\boldsymbol{\varepsilon}}(\xi)\right| &\leq \left|\widehat{F}^{\boldsymbol{\varepsilon}}(\xi e^{i(\theta+\eta/2)})\right| e^{-\xi \operatorname{Re}(\exp(i(\theta+\eta/2))/x)} \\ &\leq C^+ e^{-\xi(\operatorname{Re}(\exp(i(\theta+\eta/2))/x)-K^+)} := M_+(\xi)\end{aligned}$$

Notice that M_+ does not depend on $\boldsymbol{\varepsilon}$. Notice also that the choice "$x \in \Sigma$" implies that $\xi \longmapsto M_+(\xi)$ is integrable on $[0, +\infty[$. Then, Lebesgues dominated convergence theorem applies; hence, the function $\boldsymbol{\varepsilon} \longmapsto s_{1;\theta+\eta/2}(\widetilde{F}^{\boldsymbol{\varepsilon}})(x)$ is holomorphic on $\mathcal{D}_p$.

$\star$ The holomorphy of $\boldsymbol{\varepsilon} \longmapsto s_{1;\theta-\eta/2}(\widetilde{F}^{\boldsymbol{\varepsilon}})(x)$ is proved in a similar way. ■

We are now able to prove the main result of this paper:

3.3 Proof of theorem 2.5

Let us fix $x \in \Sigma$. Recall (*cf.* page 11) that the perturbed connection matrices $\mathfrak{S}_{\theta^\star}^{\boldsymbol{\varepsilon}}$ are uniquely determined, for all $\boldsymbol{\varepsilon} \in \mathcal{D}_p$, by the relation

$$Y_{\theta-\eta/2}^{\boldsymbol{\varepsilon}}(x) = Y_{\theta+\eta/2}^{\boldsymbol{\varepsilon}}(x) \mathfrak{S}_{\theta^\star}^{\boldsymbol{\varepsilon}} \tag{2.2}$$

where

$$Y_{\theta\pm\eta/2}^{\boldsymbol{\varepsilon}}(x) = s_{1;\theta\pm\eta/2}(\widetilde{F}^{\boldsymbol{\varepsilon}})(x) x^L e^{Q^{\boldsymbol{\varepsilon}}(1/x)}$$

Obviously, the function $\boldsymbol{\varepsilon} \longmapsto Q^{\boldsymbol{\varepsilon}}(1/x)$ is holomorphic on $\mathcal{D}_p$. Consequently, proposition 3.10 above implies that the functions $\boldsymbol{\varepsilon} \longmapsto Y_{\theta\pm\eta/2}^{\boldsymbol{\varepsilon}}(x)$ are also holomorphic on $\mathcal{D}_p$.

On the other hand, for any $\varepsilon \in \mathcal{D}_p$, the matrices $Y^{\varepsilon}_{\theta\pm\eta/2}$ are formal fundamental solutions of system (2.1). Hence, $Y^{\varepsilon}_{\theta\pm\eta/2}(x) \neq 0$ for all $\varepsilon \in \mathcal{D}_p$ and $\varepsilon \longmapsto Y^{\varepsilon}_{\theta\pm\eta/2}(x)^{-1}$ keep being holomorphic on $\mathcal{D}_p$.

Theorem 2.5 follows since identity (2.2) implies

$$\mathfrak{S}^{\varepsilon}_{\theta^\star} = Y^{\varepsilon}_{\theta+\eta/2}(x)^{-1} Y^{\varepsilon}_{\theta-\eta/2}(x)$$

for all $\varepsilon \in \mathcal{D}_p$.

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