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# PINCZON ALGEBRAS

DIDIER ARNAL

*To my friend Georges:  
Forty years ago, we discover together Mathematics...  
And the existence of the trilinear form  $I$  for semisimple Lie algebras.*

ABSTRACT. After recalling the construction of a graded Lie bracket on the space of cyclic multilinear forms on a vector space  $V$ , due to Georges Pinczon, we prove the equivalence of this construction to a structure of quadratic associative algebra, up to homotopy, on  $V$ . In the associative case, it is easy to refine the associated usual Hochschild cohomology. By considering restriction to a subspace or a quotient space of forms, we can present in a completely similar way the cases of quadratic commutative and quadratic Lie algebras, up to homotopy, and the corresponding Harrison and Chevalley cohomologies.

## 1. INTRODUCTION

In his last papers [PU, DPU], see also [D, MPU], Georges Pinczon introduced what he called a Poisson bracket on the space of forms on a finite dimensional vector space  $V$ , equipped with a symmetric, non degenerated, bilinear form  $b$ . If  $(e_i)$  is a basis in  $V$  and  $(e'_j)$  the basis defined by the relations  $b(e'_j, e_i) = \delta_{ij}$ , the bracket is:

$$\{\alpha, \beta\} = \sum_i \iota_{e_i} \alpha \wedge \iota_{e'_i} \beta.$$

Especially, if  $\alpha$  is a  $(k + 1)$ -form, and  $\beta$  a  $(k' + 1)$ , then  $\{\alpha, \beta\}$  is a  $(k + k')$ -form.

In fact, Pinczon proved that a structure of quadratic Lie algebra  $(V, [ , ], b)$  on  $V$  is completely characterized by a 3-form  $I$ , such that  $\{I, I\} = 0$ . The relation between the Lie bracket and  $I$  is simply:

$$I(x, y, z) = b([x, y], z),$$

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and the equation  $\{I, I\} = 0$  is the structure equation. A direct consequence of this construction is the existence of a cohomology on the space of form, given by:

$$d\alpha = \{\alpha, I\}.$$

It is easy to characterize the problem of deformation of the structure of quadratic Lie algebra on  $(V, b)$  with the use of this cohomology on forms.

In this paper, we shall just give a generalization of this construction, defining what we call the Pinczon bracket on the space of cyclic multilinear forms on  $V$ . More precisely, as G. Pinczon did, and as it is always the case in these sort of definitions (see [NR]), we prefer to make a shift on degree to present naturally the structure equation. Thus we consider the case of a graded vector space  $V$ , we study the space  $\mathcal{C}$  of cyclic multilinear forms on the shifted space  $V[1]$ ,  $\mathcal{C}$  is spanned by the  $k$ -linear forms:

$$(\beta_1 \otimes \dots \otimes \beta_k)^{Cycl} = \sum_j \beta_j \otimes \dots \otimes \beta_k \otimes \beta_1 \otimes \dots \otimes \beta_{j-1},$$

where the  $\beta_i$  are linear forms.

Let us define a Pinczon bracket on  $\mathcal{C}$  as a (graded) antisymmetric bilinear operator  $(\Omega, \Omega') \mapsto \{\Omega, \Omega'\}$  such that,  $\{\Omega, \Omega'\}$  is  $(k + k')$ -linear, if  $\Omega$  is  $(k + 1)$ -linear, and  $\Omega'$   $(k' + 1)$ . We moreover suppose that, if  $\alpha$  is a linear form,  $\{\alpha, \cdot\}$  is like a derivation:

$$\{\alpha, (\beta_1 \otimes \dots \otimes \beta_k)^{Cycl}\} = \sum_j \{\alpha, \beta_j\} (\beta_1 \otimes \dots \otimes \hat{\beta}_j \dots \otimes \beta_k)^{Cycl},$$

(with an usual rule for signs), and defines a (graded) Lie algebra structure on  $\mathcal{C}$ , whose center is the space of 0-forms.

In a slight generalization of [PU], we shall prove there is a one-to-one correspondence between the set of Pinczon bracket on  $\bigotimes(V[1])^*$  and the space of symplectic bilinear forms  $B^*$  on  $(V[1])^*$ , or between the space of such bracket and the space of non degenerated, bilinear symmetric forms  $b$  on  $V$ , and the correspondence is given through the Pinczon formula:

$$\{\Omega, \Omega'\} = \sum_i \iota_{e_i} \Omega \odot \iota_{e'_i} \Omega'.$$

On the other hand, it is well known (see for instance [AAC2, BGHHW]) that the space  $\bigotimes V[1]$  is a cogebra for the comultiplication given by the deconcatenation map:

$$\Delta(x_1 \otimes \dots \otimes x_k) = \sum_{i=1}^{k-1} (x_1 \otimes \dots \otimes x_i) \bigotimes (x_{i+1} \otimes \dots \otimes x_k).$$

Moreover, the coderivations of  $\Delta$  are characterized by their Taylor series  $(Q_n)$  where  $Q_n : \bigotimes^n V[1] \longrightarrow V[1]$ , and the bracket of such coderivation is still a coderivation.

If  $B$  is the bilinear form defined on  $V[1]$ , which corresponds to  $b$ , there is a bijective map between the cyclic forms  $\Omega$  and what we call cyclic coderivation  $Q$ , given by the formula:

$$\Omega_Q(x_1, \dots, x_{k+1}) = B(Q(x_1, \dots, x_k), x_{k+1}).$$

This relation is an isomorphism of Lie algebra:  $\{\Omega_Q, \Omega_{Q'}\} = \Omega_{[Q, Q']}$ .

With this construction, the notion of quadratic associative algebra, respectively quadratic algebra up to homotopy does coincide with the notion of Pinzon algebra structure on  $\mathcal{C}$ . We give also an explicit way to refine the Hochschild cohomology defined by the algebra structure.

For any Pinczon bracket, the subspace  $\mathcal{C}_{vsp}$  of cyclic forms, vanishing on shuffle products is a Lie subalgebra. The restriction to this subalgebra of the above construction gives us the notion of quadratic commutative algebra (up to homotopy): it is a Pinczon algebra structure on  $\mathcal{C}_{vsp}$ . Similarly, we refine the Harrison cohomology associated to this notion of commutative algebra.

A natural quotient of  $\mathcal{C}$  is the space  $\mathcal{S}$  of totally symmetric multilinear forms on  $V[1]$ . Any Poisson bracket on  $\mathcal{C}$  induces by symmetrization a bracket on  $\mathcal{S}$ , denoted  $\{, \}$ . This allows us to refine the notion of quadratic Lie algebra (up to homotopy), and the corresponding Chevalley cohomology.

## 2. CYCLIC FORMS

### 2.1. Koszul's rule.

In this paper, we consider a finite dimensional graded vector space  $V$ , on a characteristic 0 field. Denote  $|x|$  the degree of a vector  $x$  in  $V$ . As usual,  $V[1]$  is the space  $V$  with a shifted degree. If  $x$  is homogeneous, we put, in  $V[1]$ ,

$$\deg(x) = |x| - 1,$$

and note simply  $x$  for  $\deg(x)$  when no confusion is possible. Very generally, we use small letters for mapping defined on  $V$ , and capital letter for 'corresponding' mapping, defined on  $V[1]$ . Let us define now these corresponding mappings.

For any real number  $a$ , define a ' $k$ -cocycle'  $\eta_a$  by putting:

$$\eta_a(x_1, \dots, x_k) = (-1)^{\sum_{j \leq k} (a-j)x_j}.$$

For any permutation  $\sigma$  in  $\mathfrak{S}_k$ , we define  $\varepsilon_{|x|}(\sigma)$  as the sign of  $\sigma$ , taking in account only the positions of the  $x_j$  with odd degree  $|$ , and  $\varepsilon_{\deg(x)}(\sigma) = \varepsilon_x(\sigma)$  is the sign of the permutation  $\sigma$ , taking in account only the positions of the  $x_j$  with odd degree

deg. Let  $\varepsilon(\sigma)$  be the sign of  $\sigma$ . Then a direct computation shows (see for instance [AAC1]):

$$\eta_a(x_{\sigma(1)}, \dots, x_{\sigma(k)})\eta_a(x_1, \dots, x_k) = \varepsilon(\sigma)\varepsilon_{|x|}(\sigma)\varepsilon_x(\sigma).$$

If  $q$  is a  $k$ -linear mapping from  $V$  into a graded vector space  $W$ , we define the associated  $k$ -linear mapping from  $V[1]$  into  $W[1]$ , with  $\deg(Q) = |q| + k - 1$ , by

$$Q(x_1, \dots, x_k) = \eta_k(x_1, \dots, x_k)q(x_1, \dots, x_k).$$

If  $W$  is the base field  $\mathbb{K}$ , we generally prefer to keep the 0 degree for scalars. If  $\alpha$  is a  $k$ -linear form on  $V$ , we associate to  $\alpha$  a form  $A$ , on  $V[1]$ , with degree  $\deg(A) = |A| + k$ .

Let us now recall the sign rule due to Koszul ([K]). For any relation using letters representing graded objects, in which the ordering of the letters is modified in some term, we always undermean that, in front of the first term, there is a  $+$  sign and in front of every other term, there is an (implicit) sign  $\varepsilon_{letters}(\sigma)$  where  $\sigma$  is the permutation of the letters between the first term and the other.

For instance, the relation:

$$F(A(x, y), z) = A(F(x, z), y) + F(x, A(y, z))$$

means in an explicit form:

$$F(A(x, y), z) = (-1)^{Ax}A(F(x, z), y) + (-1)^{Ay}F(x, A(y, z)).$$

## 2.2. Cyclic product.

### Definition 2.1.

If  $\Omega$  is a  $(k+1)$ -linear form, we say that  $\Omega$  is a cyclic form on  $V[1]$ , if it satisfies (with the above rule):

$$\Omega(x_{k+1}, x_1, \dots, x_k) = \Omega(x_1, \dots, x_{k+1}).$$

Denote  $\mathcal{C}$  the space of cyclic forms on  $V[1]$ .

For any  $k+1$ -linear form  $\Omega$ , we define the action of  $\sigma$  in  $\mathfrak{S}_{k+1}$  on  $\Omega$  as:

$$(\Omega^\sigma)(x_1, \dots, x_{k+1}) = \Omega(x_{\sigma^{-1}(1)}, \dots, x_{\sigma^{-1}(k+1)}).$$

Then  $(\Omega^\sigma)^\tau = \Omega^{\tau\sigma}$ , and we can build a cyclic form  $Cycl(\Omega)$  as follows: denote  $Cycl$  the subgroup of  $\mathfrak{S}_{k+1}$  generated by the cycle  $\tau_k = (1, 2, \dots, k+1)$  and put

$$\Omega^{Cycl} = \sum_{\tau \in Cycl} \Omega^\tau.$$

Thanks to this operator, it is easy to ‘cyclicize’ the tensor product of multilinear forms:

**Definition 2.2.**

We define the cyclic product of multilinear forms  $A, B$  on  $V[1]$  as:

$$A \odot B = (A \otimes B)^{Cycl}.$$

**Lemma 2.3.**

The cyclic product is (graded) commutative.

The cyclic product is non associative.

*Proof.* Suppose  $A$  is a  $p$ -linear cyclic form and  $B$  a  $q$ -linear one. Then,  $A \odot B$  is a cyclic  $p + q$ -form. Consider the element in  $Cycl$ :

$$\sigma = \begin{pmatrix} 1 & \dots & \dots & p+q \\ q+1 & \dots & p+q & 1 & \dots & q \end{pmatrix},$$

then

$$(B \otimes A)(x_1, \dots, x_{p+q}) = A(x_{\sigma^{-1}(1)}, \dots, x_{\sigma^{-1}(p)})B(x_{\sigma^{-1}(q+1)} \dots x_{\sigma^{-1}(p+q)}),$$

or  $B \otimes A = (A \otimes B)^\sigma$ . Therefore

$$B \odot A = \sum_{\tau \in Cycl} (B \otimes A)^\tau = \sum_{\tau \in Cycl} (A \otimes B)^\tau = A \odot B.$$

To prove the product is non associative (even on the space  $\mathcal{C}$ ), we suppose  $V$  is 6 dimensional, with a basis  $(e_i)$ . Denote  $\epsilon_i$  the dual basis, and consider the symmetric bilinear forms:

$$A = \epsilon_1 \odot \epsilon_2, \quad B = \epsilon_3 \odot \epsilon_5, \quad \Gamma = \epsilon_6 \odot \epsilon_4.$$

Then a direct computation gives

$$((A \odot B) \odot C)(e_1, \dots, e_6) = 1, \quad \text{and} \quad (A \odot (B \odot C))(e_1, \dots, e_6) = 0.$$

□

**2.3. Pinczon bracket.**
**Definition 2.4.**

A Pinczon bracket on the space  $\mathcal{C}$  of cyclic multilinear forms on a graded space  $V[1]$  is a bilinear map:

$$(\Omega, \Omega') \mapsto \{\Omega, \Omega'\}$$

such that

1. If  $\mathcal{C}^k$  is the space of  $k$ -linear cyclic forms,  $\{\mathcal{C}^{k+1}, \mathcal{C}^{k'+1}\} \subset \mathcal{C}^{k+k'}$ ,
2.  $\mathcal{C}$ , equipped with  $\{, \}$  is a graded Lie algebra:  $\{\Omega, \Omega'\} = -\{\Omega', \Omega\}$ , and
 
$$\{\Omega, \{\Omega', \Omega''\}\} + \{\Omega', \{\Omega'', \Omega\}\} + \{\Omega'', \{\Omega, \Omega'\}\} = 0.$$

3. for any linear form  $\alpha$ ,  $\{ \alpha, \cdot \}$  is like a derivation: for any  $\beta_1, \dots, \beta_k$  linear,

$$\{\alpha, (\beta_1 \otimes \dots \otimes \beta_k)^{Cycl}\} = \sum_j \{\alpha, \beta_j\} (\beta_1 \otimes \dots \hat{\beta}_j \dots \otimes \beta_k)^{Cycl},$$

4.  $\{\Omega, \mathcal{C}\} = 0$  if and only if  $\Omega$  is in  $\mathcal{C}^0$ .

The space of such bracket is in fact in one-to-one correspondence with the space of symmetric, non degenerated bilinear forms  $b$  on  $V$  (see [PU]).

**Theorem 2.5.**

1. There is a bijective map between the space  $\mathcal{P}$  of Pinczon bracket on  $\mathcal{C}$  and the space  $\mathcal{B}$  of degree 0, symmetric, non degenerated bilinear forms  $b$  on  $V$ ,
2. Let  $b$  be an element in  $\mathcal{B}$ ,  $(e_i)$  be any basis for  $V$ , if  $(e'_i)$  is the basis defined by  $b(e_i, e'_j) = \delta_{ij}$ , then the Pinczon bracket associated to  $b$  is:

$$\{\Omega, \Omega'\} = \sum_i \iota_{e_i} \Omega \odot \iota_{e'_i} \Omega'.$$

*Proof.* Let  $\{ \cdot, \cdot \}$  be a Pinczon bracket on  $\mathcal{C}$ . We have:

$$\{\mathcal{C}^0, \mathcal{C}\} = 0, \quad \text{and} \quad \{\mathcal{C}^1, \mathcal{C}^1\} \subset \mathcal{C}^0 = \mathbb{K}.$$

Thus the bracket defines a degree 0 bilinear, antisymmetric form  $B^*$  on  $\mathcal{C}^1 = (V[1])^*$ . As above this defines a symmetric bilinear form  $b^*$  on the space  $(V[1])^*[1]$ , which is canonically isomorphic to  $V^*$ .

Let us suppose that  $b^*$  (or  $B^*$ ) is degenerated, then there is  $\alpha$  in  $\mathcal{C}^1$  such that

$$B^*(\alpha, \mathcal{C}^1) = \{\alpha, \mathcal{C}^1\} = 0.$$

Then, for any linear forms  $\beta_1, \dots, \beta_k$ ,  $\{\alpha, (\beta_1 \otimes \dots \otimes \beta_k)^{Cycl}\} = 0$ , or  $\{\alpha, \mathcal{C}^k\} = 0$ , and  $\alpha$  is a central element. This is impossible,  $b^*$  is non degenerated and allows to identify  $V^*$  and  $V$  and to define a non degenerated bilinear symmetric form  $b$  on  $V$ .

Let  $B^*$  be a bilinear form on  $(V[1])^*$ . For any basis  $(e_i)$  of  $V[1]$  there are vectors  $e'_i$  such that we can write:

$$B^*(\alpha, \beta) = \sum_i \iota_{e_i} \alpha \otimes \iota_{e'_i} \beta = \sum_i \alpha(e_i) \beta(e'_i) = \sum_i \iota_{e_i} \alpha \odot \iota_{e'_i} \beta,$$

(with the rule on signs).

Coming back to  $V^*$  this means, if all the objects are homogeneous,

$$\begin{aligned} (-1)^{\alpha+1}b^*(\alpha, \beta) &= \sum_i (-1)^{\alpha e_i} (-1)^{\beta e'_i} \alpha(e_i) \beta(e'_i) \\ b^*(\alpha, \beta) &= \sum_i (-1)^{|\beta|} \alpha(e_i) \beta(e'_i). \end{aligned}$$

We identify  $V^*$  to  $V$  by defining, for any  $\gamma$  in  $V^*$ , the vector  $x_\gamma$  in  $V$  such that  $\alpha(x_\gamma) = b^*(\alpha, \gamma)$ , for any  $\alpha$  in  $V^*$ . Then, if  $(\epsilon_i)$  is the dual basis of  $(e_i)$  ( $\epsilon_i(e_j) = \delta_{ij}$ ), we get

$$\begin{aligned} b^*(\epsilon_i, \beta) &= (-1)^{|\beta||\epsilon_i|} b^*(\beta, \epsilon_i) = (-1)^{|\beta||\epsilon_i|} \beta(x_{\epsilon_i}) \\ &= (-1)^{|\beta|} \beta(e'_i). \end{aligned}$$

Therefore  $e'_i = x_{\epsilon_i}$ , the  $(e'_i)$  are a basis for  $V$ , and with the same computation, if  $(\epsilon'_i)$  is the dual basis,  $x_{\epsilon'_i} = (-1)^{|\epsilon_i|} e_i$ , we get:

$$b(e'_j, e_i) = (-1)^{|\epsilon_i|} b^*(\epsilon_j, \epsilon'_i) = \delta_{ij}$$

Consider now the bracket:

$$\{\Omega, \Omega'\}_P = \sum_i \iota_{e_i} \Omega \odot \iota_{e'_i} \Omega',$$

where  $\Omega$  is a  $k+1$ -linear cyclic form and  $\Omega'$  a  $k'+1$ -linear one. In a following section, we shall prove this bracket defines a graded Lie algebra structure on  $\mathcal{C}$ . It is clear that, for any  $\alpha$  and  $\beta_j$  linear,

$$\{\alpha, (\beta_1 \otimes \dots \otimes \beta_k)^{Cycl}\}_P = \sum_j \{\alpha, \beta_j\}_P (\beta_1 \otimes \dots \otimes \hat{\beta}_j \otimes \dots \otimes \beta_k)^{Cycl}.$$

Moreover the center of the Lie algebra  $(\mathcal{C}, \{ , \}_P)$  is  $\mathcal{C}^0$ . In other word,  $\{ , \}_P$  is a Pinczon bracket.

If  $k+k' \leq 0$ ,  $\{\Omega, \Omega'\} = \{\Omega, \Omega'\}_P$ . Suppose by induction this relation holds for  $k+k' < N$  and consider  $\Omega$  and  $\Omega'$  such that  $k+k' = N$ . For any  $i$ ,

$$\begin{aligned} \{\epsilon_i, \{\Omega, \Omega'\}\} &= -\{\Omega, \{\Omega', \epsilon_i\}\} - \{\Omega', \{\epsilon_i, \Omega\}\} \\ &= -\{\Omega, \{\Omega', \epsilon_i\}_P\}_P - \{\Omega', \{\epsilon_i, \Omega\}_P\}_P \\ &= \{\epsilon_i, \{\Omega, \Omega'\}_P\}_P \\ &= \iota_{e'_i} \{\Omega, \Omega'\}_P \end{aligned}$$



On the other hand, if  $\{\Omega, \Omega'\} = \sum_{\beta} (\beta_1 \otimes \dots \otimes \beta_{k+k'})^{Cycl}$ ,

$$\begin{aligned} \{\epsilon_i, \{\Omega, \Omega'\}\} &= \sum_{\beta, j} \beta_j(e'_i)(\beta_1 \otimes \dots \otimes \hat{\beta}_j \dots \otimes \beta_{k+k'})^{Cycl} \\ &= \sum_{\beta} \iota_{e'_i}(\beta_1 \otimes \dots \otimes \beta_{k+k'})^{Cycl} \\ &= \iota_{e'_i}\{\Omega, \Omega'\}. \end{aligned}$$

Since  $(e'_i)$  is a basis for  $V[1]$ , this implies  $\{\Omega, \Omega'\}_P = \{\Omega, \Omega'\}$ , this proves the existence and unicity of the Pinczon bracket associated to the symmetric, non degenerated bilinear form  $b$  on  $V$  and the expression of this bracket.

□

### 3. CODIFFERENTIAL

#### 3.1. General construction.

The tensor algebra  $\bigotimes^+ V[1]$  has a natural comultiplication: the deconcatenation map  $\Delta$ :

$$\Delta(x_1 \otimes \dots \otimes x_k) = \sum_{r=1}^{k-1} (x_1 \otimes \dots \otimes x_r) \bigotimes (x_{r+1} \otimes \dots \otimes x_k).$$

It is well known (see for instance [AAC2, BGHHW]) that any multilinear mapping  $Q$  can be extended in an unique way into a coderivation  $D_Q$  of the  $\Delta$ -operation, moreover to any coderivation  $D$  corresponds a sequence  $(Q_k)_{k \geq 1}$  of  $k$ -linear mappings such that  $D = \sum_k D_{Q_k}$  (this sequence can be viewed as the Taylor series of  $D$ ). Moreover, the space of coderivation is a natural graded Lie algebra, thus the space of multilinear mapping is a graded Lie algebra for the bracket (with sign rule):

$$[Q, Q'] = Q \circ Q' - Q' \circ Q,$$

with, if we note  $x_{[a,b]}$  the tensor product  $x_a \otimes x_{a+1} \otimes \dots \otimes x_b$ ,

$$Q \circ Q'(x_{[1, k+k'-1]}) = \sum_{r=0}^{k-1} Q(x_{[1, r]}, Q'(x_{[r+1, r+k']}, x_{[r+k'+1, k+k'-1]}).$$

**Lemma 3.1.** *The space  $\mathcal{D}$  of multilinear mappings is a Lie algebra for the bracket.*

*Proof.* This is well known ([NR]), we give a sketch of the proof. We thus have:

$$\begin{aligned}
 (Q \circ Q') \circ Q''(x_{[1, k+k'+k''-2]}) &= \\
 &= \sum Q(x_{[1, s]}, Q''(x_{[s, s+k'']}), x_{[s+k'', r]}, Q'(x_{[r, r+k']}), x_{[r+k', k+k'+k''-2]}) \\
 &+ \sum Q(x_{[1, r]}, Q'(x_{[r, s]}), Q''(x_{[s, s+k'']}), x_{[s+k'', r+k'-1]}, x_{[r+k'-1, k+k'-2]}) \\
 &+ \sum Q(x_{[1, r]}, Q'(x_{[r, r+k']}), x_{[r+k', s]}, Q''(x_{[s, s+k'']}), x_{[s+k'', k+k'+k''-2]}) .
 \end{aligned}$$

From this relation, we get:

$$(Q \circ Q') \circ Q'' - (Q \circ Q'') \circ Q = Q \circ (Q' \circ Q'' - Q'' \circ Q'),$$

and the Jacobi identity:

$$[[Q, Q'], Q''] + [[Q', Q''], Q] + [[Q'', Q], Q'] = 0.$$

□

### 3.2. Relation with the Pinczon bracket.

Let us consider a vector space  $V$  equipped with a symmetric, non degenerated bilinear form  $b$ , with degree 0. We define, as usual a symplectic form  $B$  on  $V[1]$ :

$$B(x, y) = (-1)^x b(x, y) = (-1)^y b(x, y).$$

For any  $k$  linear map  $Q$  from  $V[1]^k$  into  $V[1]$ , we define the  $(k+1)$ -linear form:

$$\Omega_Q(x_1, \dots, x_{k+1}) = B(Q(x_1, \dots, x_k), x_{k+1}),$$

and say that  $Q$  is cyclic if and only if  $\Omega_Q$  is cyclic. Denote  $\mathcal{D}_B$  the space of cyclic multilinear maps, or cyclic coderivations.

The fundamental examples of cyclic maps are mapping associated to a Lie bracket or an associative multiplication on  $V$ . More precisely, if  $(x, y) \mapsto q(x, y)$  is any internal law, with degree 0, we have  $Q(x, y) = (-1)^x q(x, y)$ , then  $\Omega_Q$  is cyclic if and only if:

$$\begin{aligned}
 b(q(x, y), z) &= (-1)^{x+z} \Omega_Q(x, y, z) \\
 &= (-1)^{x+z+x(y+z)} \omega_Q(y, z, x) \\
 &= (-1)^{x+z+x(y+z)+y+3x} b(q(y, z), x) \\
 &= (-1)^{(x+1)(y+z)} b(q(y, z), x) \\
 &= b(x, q(y, z)),
 \end{aligned}$$

if and only if  $(V, q, b)$  is an algebra with an invariant bilinear form, *i.e.* a 'quadratic' algebra.

Remark that if  $q$  is a Lie bracket (antisymmetric), then  $\omega_q$  is completely antisymmetric,  $\Omega_Q$  is completely symmetric.

Now, if  $q$  is a commutative (and associative) product, and  $b$  is invariant,  $\omega_q$  is vanishing on the image of the twisted shuffle product on the 2 first variables. In fact there is an unique such product, defined by

$$(x_1 \otimes x_2) \mapsto (x_1 \otimes x_2) - (-1)^{|x_1||x_2|}(x_2 \otimes x_1).$$

$\Omega_Q$  is vanishing on the image of the shuffle product on the 2 first variables:

$$sh_{(1,1)}(x_1 \otimes x_2) = (x_1 \otimes x_2) + (-1)^{x_1 x_2}(x_2 \otimes x_1).$$

**Proposition 3.2.** *The space  $\mathcal{D}_B$  of cyclic maps  $Q$  is a Lie subalgebra of  $\mathcal{D}$ .*

*The space  $\mathcal{C}$  of cyclic forms, equipped with the Pinczon bracket is a graded Lie algebra, isomorphic to  $\mathcal{D}_B$ .*

*Proof.* First if  $I = \{i_1, \dots, i_k\}$  is a sequence of indices, we shall denote  $x_I$  the tensor  $x_{i_1} \otimes \dots \otimes x_{i_k}$ .

Remark that (with our notations), if  $Q$  is  $k$ -linear and  $Q'$   $k'$ -linear,

$$\begin{aligned} A &= (-1)^{Q' + \sum_{j>k} x_j} b(Q(x_{[1,k]}), Q'(x_{[k,k+k']})) \\ &= (-1)^{Q' + \sum_{j>k} x_j + (Q' + \sum_{j>k} x_j + 1)(Q + \sum_{i \leq k} x_i + 1)} b(Q'(x_{[k,k+k']}), Q(x_{[1,k]})) \\ &= -(-1)^{Q + \sum_{i \leq k} x_i} (-1)^{(Q' + \sum_{j>k} x_j)(Q + \sum_{i \leq k} x_i)} b(Q'(x_{[k,k+k']}), Q(x_{[1,k]})), \end{aligned}$$

Suppose  $Q$  and  $Q'$  cyclic, thus, with the Koszul rule, this is just:

$$\begin{aligned} B(Q(x_{[1,k]}), Q'(x_{[k,k+k']})) &= -B(Q'(Q(x_{[1,k]}), x_{[k,k+k'-1]}), x_{k+k'}) \\ &= B(Q(Q'(x_{[k,k+k']}), x_{[1,k-1]}), x_k). \end{aligned}$$

Now we get:

$$\begin{aligned} B(Q \circ Q'(x_{[2,k+k']}), x_1) &= \sum_{1 \leq r} B(Q(x_{[2,r]}, Q'(x_{[r,r+k']}), x_{[r+k',k+k']}), x_1) \\ &= \sum_{1 \leq r < k} B(Q(x_{[1,r]}, Q'(x_{[r,r+k']}), x_{[r+k',k+k'-1]}), x_{k+k'}) \\ &\quad + B(Q(x_{[1,k]}), Q'(x_{[k,k+k']})) \\ &= \sum_{1 \leq r < k} B(Q(x_{[1,r]}, Q'(x_{[r,r+k']}), x_{[r+k',k+k'-1]}), x_{k+k'}) \\ &\quad - B(Q'(Q(x_{[1,k]}), x_{[k,k+k'-1]}), x_{k+k'}) \end{aligned}$$

Or

$$\begin{aligned}
 B([Q, Q'](x_{[2, k+k']}), x_1) &= \sum_{1 \leq r < k} B(Q(x_{[1, r]}, Q'(x_{[r, r+k']}), x_{[r+k', k+k'-1]}), x_{k+k'}) \\
 &\quad + B(Q(Q'(x_{[1, k']}), x_{[k', k+k'-1]}), x_{k+k'}) \\
 &\quad - \sum_{1 \leq r < k'} B(Q'(x_{[1, r]}, Q(x_{[r, r+k]}), x_{[r+k, k+k'-1]}), x_{k+k'}) \\
 &\quad - B(Q'(Q(x_{[1, k]}), x_{[k, k+k'-1]}), x_{k+k'}) \\
 &= B([Q, Q'](x_{[1, k+k'-1]}), x_{k+k'})
 \end{aligned}$$

Thus  $[Q, Q']$  is cyclic,  $\mathcal{D}_B$  is a Lie subalgebra of  $\mathcal{D}$ .

On the other hand,  $\Omega_Q$  (resp.  $\Omega_{Q'}$ ) is a form  $k + 1$ -linear (resp.  $k' + 1$ -linear), and

$$\begin{aligned}
 \{\Omega_Q, \Omega_{Q'}\}(x_1, \dots, x_{k+k'}) &= \left( \sum_i \iota_{e_i} \Omega_Q \otimes \iota_{e'_i} \Omega_{Q'} \right)^{Cycl} (x_1, \dots, x_{k+k'}) \\
 &= \sum_{\sigma \in Cycl} \sum_i B(Q(x_{\sigma^{-1}(1)}, \dots, x_{\sigma^{-1}(k)}), e_i) \\
 &\quad B(Q'(x_{\sigma^{-1}(k+1)}, \dots, x_{\sigma^{-1}(k+k')}), e'_i) \\
 &= \sum_{\sigma \in Cycl} B(Q(x_{\sigma^{-1}([1, k])}, Q'(x_{\sigma^{-1}([k+1, k+k']]}))),
 \end{aligned}$$

with a little abuse of notations.

Consider a term in this sum, such that  $k + k'$  belongs to  $\sigma^{-1}([1, k])$ . This term is:

$$B(Q(x_{[r+k'+1, k+k']}, x_{[1, r]}), Q'(x_{[r+1, r+k']})) = B(Q(x_{[1, r]}, Q'(x_{[r+1, r+k']}), x_{[r+k'+1, k+k'-1]}), x_{k+k'}),$$

and the sum of all these terms is just  $B((Q \circ Q')(x_{[1, k+k'-1]}), x_{k+k'})$ .

Similarly, a term such that  $k + k'$  is in  $\sigma^{-1}([k + 1, k + k'])$  is:

$$B(Q(x_{[r+1, r+k']}), Q'(x_{[r+k+1, k+k']}, x_{[1, r]})) = -B(Q'(x_{[1, r]}, Q(x_{[r+1, r+k]}), x_{[r+k+1, k+k'-1]}), x_{k+k'}),$$

and the corresponding sum is  $-B((Q' \circ Q)(x_{[1, k+k'-1]}), x_{k+k'})$ .

This proves:

$$\{\Omega_Q, \Omega_{Q'}\} = \Omega_{[Q, Q']}.$$

Since the map  $Q \mapsto \Omega_Q$  is bijective, this proves the proposition.  $\square$

We know study in a more detailed way the three main cases, when  $Q$  corresponds to an associative, or commutative or Lie structure, or to such a structure up to homotopy.

## 4. ASSOCIATIVE PINCZON ALGEBRAS

## 4.1. Associative quadratic algebras.

Suppose now  $q$  is an associative law (with degree 0) on  $V$ , and  $b$  is invariant, then we get a coderivation  $Q$  of  $\Delta$  on  $\otimes^+ V[1]$ , with degree 1, which is the Bar resolution of the associative algebra  $(V, q)$ . The associativity of  $q$  is equivalent to the relation  $[Q, Q] = 0$  ( $Q$  is the map associated to  $q$ ).

More generally, a structure of  $A_\infty$  algebra (or associative algebra up to homotopy) on the space  $V$  is a degree 1 coderivation  $Q$  of  $\Delta$  on  $\otimes^+ V[1]$ , such that  $[Q, Q] = 0$ . With this last relation, the map

$$\Lambda \mapsto \{\Omega_Q, \Lambda\}$$

is a degree 1 differential on the (graded) Lie algebra  $\mathcal{C}$ . The corresponding cohomology is the Pinczon cohomology of multilinear forms.

**Definition 4.1.**

We call associative Pinczon algebra any vector space  $V$ , such that  $\otimes(V[1])^*$  is equipped with a Pinczon bracket  $\{ , \}$ , and an element  $\Omega$  in  $\mathcal{C}$ , with degree 3, such that  $\{\Omega, \Omega\} = 0$ .

If  $\Omega$  is trilinear, then an associative Pinczon algebra is simply a quadratic associative algebra  $(V, b, q)$ , where  $b$  is the symmetric non degenerated form coming from the restriction of the Pinczon bracket to  $((V[1])^*)^2$ , and  $q$  is the bilinear mapping associated to  $Q$  such that  $\Omega = \Omega_Q$ .

**Proposition 4.2.**

Let  $V$  be a finite dimensional graded vector space, let  $\{ , \}$  a Pinczon bracket on the space  $\mathcal{C}$  of cyclic multilinear forms on  $V[1]$ , and  $\Omega$  a degree 3 element of  $\mathcal{C}$  such that  $\{\Omega, \Omega\} = 0$ .

Write  $\Omega = \sum \Omega_{Q_k}$ , where each  $Q_k$  is a  $k$  linear map from  $(V[1])^k$  into  $V[1]$ , then  $(\otimes^+ V[1], \sum_k Q_k)$  is a structure of  $A_\infty$  algebra on  $V$ , and, for the bilinear form  $B$  on  $V[1]$ , coming from the bracket, each  $Q_k$  is cyclic (correspondingly, each  $q_k$  is anticyclic for  $b$ ).

Conversely, if  $(V, b)$  is a vector space with a non degenerated symmetric bilinear form, any cyclic structure  $Q$  of  $A_\infty$  algebra on  $V[1]$  defines an unique structure of associative Pinczon algebra on  $V$ .

In other words, the associative Pinczon algebra  $(V, \{ , \}, \Omega)$  defines a structure of quadratic associative algebra up to homotopy on  $V$ .

#### 4.2. Bimodules and Hochschild cohomology.

In this situation, we can recover the usual Hochschild cohomology of any (graded) bimodule  $M$  for a quadratic associative algebra  $(V, b, q)$ .

First, we consider the semidirect product  $W = V \rtimes M$ , that is the vector space  $V \oplus M$ , equipped with the associative multiplication:

$$q_W(x + a, y + b) = q(x, y) + x \cdot b + a \cdot x,$$

then  $V$  has a natural  $(W, q_W)$  bimodule structure, namely:

$$(x + a) \cdot y = xy, \quad y \cdot (x + a) = yx.$$

Now, since  $W$  is a  $(W, q_W)$  bimodule, its dual  $W^*$  is also a  $(W, q_W)$ -bimodule, with:

$$((x + a) \cdot f)(z + c) = f((z + c)(x + a)), \quad (f \cdot (x + a))(z + c) = f((x + a)(z + c)).$$

With these objects, we define a structure of associative algebra on the space  $\tilde{V} = W \times V \times W^*$ , by putting:

$$\tilde{q}((x+a)+y+f, (x'+a')+y'+f') = (xx'+xa'+ax')+(yy'+xy'+yx')+((x+a) \cdot f' + f \cdot (x'+a')),$$

and a non degenerated symmetric bilinear form  $\tilde{b}$ , by

$$\tilde{b}((x + a) + y + f, (x' + a') + y' + f') = b(y, y') + f(x' + a') + f'(x + a).$$

A direct computation shows that  $(\tilde{V}, \tilde{b}, \tilde{q})$  is a quadratic associative algebra, we call it the double extension of  $(V, b, q)$  by the bimodule  $M$ .

#### Definition 4.3.

Let  $(V, b, q)$  be a quadratic associative algebra and  $M$  a bimodule, the quadratic associative algebra:

$$(\tilde{V}, \tilde{b}, \tilde{q})$$

is the double semi-direct product of  $V$  by  $M$ .

Similarly, the associative Pinczon algebra  $(\tilde{V}[1], \{ , \}^{\sim}, \Omega_{\tilde{Q}})$  associated to  $(\tilde{V}, \tilde{b}, \tilde{q})$  is the associative Pinczon algebra double semi-direct product of  $(V[1], \{ , \}, \Omega_Q)$  by  $M$ .

We look now for a  $k$  linear mapping  $C$  from  $V^k$  into  $M$ , and consider it as a mapping from  $\tilde{V}^k$  into  $\tilde{V}$ , as follows:

$$\tilde{C}((x_1 + a_1) + y_1 + f_1, \dots, (x_k + a_k) + y_k + f_k) = (0 + C(x_1, \dots, x_k)) + 0 + 0.$$

Then, if  $\tilde{Q}$  is associated to  $\tilde{q}$  as above, we get:

$$[\tilde{Q}, \tilde{C}] = \widetilde{d_H C},$$

where  $d_H$  is the Hochschild coboundary operator on the bimodule  $M$ , and similiary:

$$\{\Omega_{\tilde{Q}}, \Omega_{\tilde{C}}\}^{\sim} = \Omega_{\widetilde{d_H C}}.$$

**Proposition 4.4.**

Let  $(V, \{ , \}, \Omega_Q)$  be an associative Pinczon algebra, with  $\Omega_Q$  trilinear. Then the restriction of the Pinczon cohomology

$$\Omega \mapsto \{\Omega_{\tilde{Q}}, \Omega_{\tilde{C}}\}^{\sim}$$

to the cyclic forms  $\Omega_{\tilde{C}}$  coming from multilinear mappings  $C$  from  $V^k$  into  $M$  is the Hochschild cohomology of the  $(V, q)$  bimodule  $M$ .

## 5. COMMUTATIVE PINCZON ALGEBRAS

### 5.1. Commutative quadratic algebras.

Let us consider a quadratic associative algebra  $(V, b, q)$ , we suppose now  $q$  is commutative, that means, without any sign rule:

$$q(x_1, x_2) = (-1)^{|x_1||x_2|} q(x_2, x_1),$$

and consider, as above the corresponding coderivation  $Q$ . It is now anticommutative, with degree 1, and we see it as a map from the quotient  $\underline{\otimes}^2 V[1]$  of  $\otimes^2 V[1]$  by the image, into  $V[1]$ , of the 1, 1 shuffle product (with our sign rule):

$$sh_{1,1}(x_1, x_2) = x_1 \otimes x_2 + x_2 \otimes x_1.$$

Recall that a  $p, q$  shuffle  $\sigma$  is a permutation  $\sigma \in \mathfrak{S}_{p+q}$  such that  $\sigma(1) < \dots < \sigma(p)$  and  $\sigma(p+1) < \dots < \sigma(p+q)$ . Denote  $Sh(p, q)$  the set of all such shuffles. Then the  $p, q$  shuffle product on  $\otimes^+ V[1]$  is (with the Koszul rule)

$$sh_{p,q}(x_{[1,p]}, x_{[p+1,p+q]}) = \sum_{\sigma \in Sh(p,q)} x_{\sigma^{-1}(1)} \otimes \dots \otimes x_{\sigma^{-1}(p+q)}.$$

Denote  $\underline{\otimes}^n(V[1])$  the quotient of  $\otimes^n V[1]$  by the sum of all the image of the maps  $sh_{p,n-p}$  ( $0 < p < n$ ). Similarly, we abusively note  $\underline{x}_{[1,n]} = x_1 \underline{\otimes} \dots \underline{\otimes} x_n$  the class of  $x_1 \otimes \dots \otimes x_n$ , where the  $x_i$  belong to  $V[1]$ . Finally,  $\underline{\otimes}^+(V[1])$  is the sum of all  $\underline{\otimes}^n(V[1])$  ( $n > 0$ ).

After replacing  $\otimes^+ V[1]$  by  $\underline{\otimes}^+ V[1]$ , we replace the comultiplication  $\Delta$  by a Lie cocrochet  $\delta$ , defined by

$$\delta(\underline{x}_{[1,n]}) = \sum_{j=1}^{n-1} \underline{x}_{[1,j]} \otimes \underline{x}_{[j+1,n]} - \underline{x}_{[j+1,n]} \otimes \underline{x}_{[1,j]}.$$

In fact  $\delta$  is well defined on the quotient and any coderivation  $Q$  of  $\delta$  is characterized by its Taylor expansion  $Q = \sum_k Q_k$  where each  $Q_k$  is a linear map from  $\underline{\otimes}^k V[1]$  into  $V[1]$  ([AAC1, BGHHW]).

**Definition 5.1.**

A structure of  $C_\infty$  algebra, or up to homotopy commutative algebra, on  $V$  is a degree 1 coderivation  $Q$  of  $\delta$ , on  $\underline{\otimes}^+ V[1]$ , such that  $[Q, Q] = 0$ .

Such mappings  $Q$  are vanishing on the image of shuffle products. Consider now cyclic forms  $\Omega_Q$  such that  $Q$  is vanishing on shuffle products, or:

**Definition 5.2.**

A  $(k + 1)$ -linear cyclic form  $\Omega$  on the vector space  $V[1]$  is vanishing on shuffle product if and only if, for any  $y, (x_1, \dots, x_k) \mapsto \Omega(x_1, \dots, x_k, y)$  is vanishing on shuffle product.

Denote  $\mathcal{C}_{vsp}$  the space of cyclic, vanishing on shuffle product multilinear forms on  $V[1]$ .

**Proposition 5.3.**

Suppose  $\{ , \}$  is a Pinczon bracket on the space  $\mathcal{C}$  of cyclic multilinear forms on  $V[1]$ . Then  $\mathcal{C}_{vsp}$  is a Lie subalgebra of  $(\mathcal{C}, \{ , \})$ .

*Proof.* In fact, having a Pinczon bracket, we have a non degenerate form  $b$  on  $V$ , thus  $B$  on  $V[1]$ , we can write any form  $\Omega$  as  $\Omega = \Omega_Q$ , with

$$\Omega_Q(x_1, \dots, x_{k+1}) = B(Q(x_1, \dots, x_k), x_{k+1}).$$

Therefore  $\Omega$  is in  $\mathcal{C}_{vsp}$  if and only if  $Q$  is vanishing on shuffle products. Now, in [AAC2] it is shown that if  $Q, Q'$  are vanishing on shuffle products, then  $[Q, Q']$  is also vanishing on shuffle products. This proves the proposition. □

**Definition 5.4.**

We call commutative Pinczon algebra any vector space  $V$ , such that  $\otimes(V[1])^*$  is equipped with a Pinczon bracket  $\{ , \}$ , and an element  $\Omega$  in  $\mathcal{C}_{vsp}$ , with degree 3, such that  $\{\Omega, \Omega\} = 0$ .

If  $\Omega$  is trilinear, then a commutative Pinczon algebra is simply a quadratic commutative algebra  $(V, q, b)$ , where  $b$  is the symmetric non degenerated form coming from the restriction of the Pinczon bracket to  $((V[1])^*)^2$ , and  $q$  is the bilinear mapping associated to  $Q$  such that  $\Omega = \Omega_Q$ .



**Proposition 5.5.**

Let  $V$  be a finite dimensional graded vector space, let  $\{ , \}$  be a Pinczon bracket on the space  $\mathcal{C}$  of cyclic multilinear forms on  $V[1]$ , and  $\Omega$  a degree 3 element of  $\mathcal{C}_{vsp}$  such that  $\{\Omega, \Omega\} = 0$ .

Write  $\Omega = \sum \Omega_{Q_k}$ , where each  $Q_k$  is a  $k$  linear map from  $(V[1])^k$  into  $V[1]$ , then  $(\otimes^+ V[1], \sum_k Q_k)$  is a structure of  $C_\infty$  algebra on  $V$ , and, for the bilinear form  $B$  on  $V[1]$ , coming from the bracket, each  $Q_k$  is cyclic (correspondingly, each  $q_k$  is anticyclic for  $b$ ).

Conversely, if  $(V, b)$  is a vector space with a non degenerated symmetric bilinear form, any cyclic structure  $Q$  of  $C_\infty$  algebra on  $V[1]$  defines an unique structure of commutative Pinczon algebra on  $V$ .

In other words, the commutative Pinczon algebra  $(V, \{ , \}, \Omega)$  defines a structure of quadratic commutative algebra up to homotopy on  $V$ .

**5.2. (Bi)modules and Harrison cohomology.**

Any module  $M$  on a commutative algebra  $(V, q)$  can be viewed as a natural bi-module, the right action coinciding with the left one.

Let now  $(V, b, q)$  be a quadratic commutative algebra, and  $M$  a  $(V, q)$ -bimodule. We can repeat exactly the preceding construction of the semidirect product  $W = V \rtimes M$ , with the associative and commutative multiplication:

$$q_W(x + a, y + b) = q(x, y) + x \cdot b + a \cdot x.$$

Consider then the quadratic associative algebra  $(\tilde{V}, \tilde{b}, \tilde{q})$ , where  $\tilde{V} = W \times V \times W^*$ ,  $\tilde{q}$  is

$$\tilde{q}((x+a)+y+f, (x'+a')+y'+f') = (xx'+xa'+ax')+(yy'+xy'+yx')+((x+a) \cdot f' + f \cdot (x'+a')),$$

and  $\tilde{b}$ ,

$$\tilde{b}((x + a) + y + f, (x' + a') + y' + f') = b(y, y') + f(x' + a') + f'(x + a).$$

By construction,  $(\tilde{V}, \tilde{b}, \tilde{q})$  is a quadratic commutative algebra, we call it the double extension of  $(V, b, q)$  by the bimodule  $M$ .

**Definition 5.6.**

Let  $(V, b, q)$  be a quadratic commutative algebra and  $M$  a bimodule, the quadratic commutative algebra:

$$(\tilde{V}, \tilde{b}, \tilde{q})$$

is the double semi-direct product of  $V$  by  $M$ .

Similarly, the commutative Pinczon algebra  $(\tilde{V}[1], \{ , \}^{\sim}, \Omega_{\tilde{Q}})$  associated to  $(\tilde{V}, \tilde{b}, \tilde{q})$  is the commutative Pinczon algebra double semi-direct product of  $(V[1], \{ , \}, \Omega_Q)$  by  $M$ .

We look now for a  $k$  linear mapping  $C$  from  $V^k$  into  $M$ , vanishing on shuffle products, and consider it as a mapping from  $\tilde{V}^k$  into  $\tilde{V}$ , as follows:

$$\tilde{C}((x_1 + a_1) + y_1 + f_1, \dots, (x_k + a_k) + y_k + f_k) = (0 + C(x_1, \dots, x_k)) + 0 + 0.$$

Then, if  $\tilde{Q}$  is associated to  $\tilde{q}$  as above, we get:

$$[\tilde{Q}, \tilde{C}] = \widetilde{d_{Ha}C},$$

where  $d_{Ha}$  is the Harrison coboundary operator on the bimodule  $M$ , and similiary:

$$\{\Omega_{\tilde{Q}}, \Omega_{\tilde{C}}\}^{\sim} = \Omega_{\widetilde{d_{Ha}C}}.$$

**Proposition 5.7.**

Let  $(V, \{ , \}, \Omega_Q)$  be a commutative Pinczon algebra, with  $\Omega_Q$  trilinear. Then the restriction of the Pinczon cohomology

$$\Omega \mapsto \{\Omega_{\tilde{Q}}, \Omega_{\tilde{C}}\}^{\sim}$$

to the cyclic, vanishing on shuffle products forms  $\Omega_{\tilde{C}}$  coming from multilinear mappings  $C$  from  $V^k$  into  $M$  is the Harrison cohomology of the  $(V, q)$  bimodule  $M$ .

6. PINCZON LIE ALGEBRAS

6.1. Quadratic Lie algebras.

In this section, we consider the Lie algebra structure. Suppose  $(V, q)$  is a (graded) Lie algebra, with  $|q| = 0$ , then the corresponding Bar resolution consists in replacing the space  $\otimes^+ V[1]$  of tensors by the subspace  $S^+(V[1])$  of totally symmetric tensors, spanned by the symmetric products:

$$x_1 \cdot \dots \cdot x_k = \sum_{\sigma \in \mathfrak{S}_k} x_{\sigma^{-1}(1)} \otimes \dots \otimes x_{\sigma^{-1}(k)}.$$

Then, we consider the natural comultiplication  $\Delta$  on  $S^+(V[1])$ :

$$\Delta(x_1 \cdot \dots \cdot x_k) = \sum_{\substack{I \sqcup J = [1, k] \\ 0 < \#I < k}} x_{.I} \otimes x_{.J},$$

where, if the subset  $I$  is  $I = \{i_1 < \dots < i_r\}$ ,  $x_{.I}$  means  $x_{i_1} \cdot \dots \cdot x_{i_r}$ .

As above, any coderivation  $Q$  of the comultiplication  $\Delta$  is characterized by its Taylor coefficients  $Q_k$ , totally symmetric  $k$ -linear maps from  $V[1]^k$  into  $V[1]$ , we see these maps as linear map from  $S^k(V[1])$  into  $V[1]$ .

The bracket of two such coderivations  $Q, Q'$  becomes:

$$[Q, Q'](x_1 \cdot \dots \cdot x_{k+k'-1}) = \sum_{\substack{I \sqcup J = [1, k+k'-1] \\ \#J = k'}} Q(Q'(x_{.J}) \cdot x_{.I}) - \sum_{\substack{I \sqcup J = [1, k+k'-1] \\ \#I = k}} Q'(Q(x_{.I}) \cdot x_{.J}),$$

Now to the Lie structure  $q$  on  $V$ , there corresponds a totally symmetric bilinear map  $Q : S^2(V[1]) \rightarrow V[1]$ , thus a degree 1 coderivation, still denoted  $Q$ , of  $\Delta$  and the Jacobi identity for  $q$  is equivalent to the relation  $[Q, Q] = 0$ . The corresponding cohomology is the Chevalley cohomology.

For any  $k$ , we consider  $S^k(V[1])$  as a subspace of  $\otimes^k V[1]$ , with a projection (up to a numerical factor)  $Sym$  from  $\otimes^k V[1]$  onto  $S^k(V[1])$ :

$$Sym(x_1 \otimes \dots \otimes x_k) = x_1 \cdot \dots \cdot x_k.$$

Consider now the space  $L^k(V[1])$  of  $k$ -linear maps  $Q$  from  $V[1]^k$  into  $V[1]$ , or linear maps  $Q$  from  $\otimes^k V[1]$  into  $V[1]$ . By restriction to  $S^k(V[1])$ ,  $Q$  defines a totally symmetric multilinear map, we denote it  $Q^{Sym}$ :

$$Q^{Sym}(x_1 \cdot \dots \cdot x_k) = Q \circ Sym(x_1 \otimes \dots \otimes x_k).$$

Therefore, the space  $L(S^k(V[1]), V[1])$  of totally symmetric  $k$  linear maps is a quotient of  $L^k(V[1])$ .

Similarly, if  $\mathcal{C}^{k+1}$  is the space of  $(k+1)$ -linear cyclic forms  $\Omega$ , by restriction to  $S^{k+1}(V[1])$ ,  $\Omega$  defines a totally symmetric multilinear form, we denote it  $\Omega^{Sym}$ :

$$\Omega^{Sym}(x_1 \cdot \dots \cdot x_{k+1}) = \Omega \circ Sym(x_1 \otimes \dots \otimes x_{k+1}).$$

As above, the space  $\mathcal{C}|_S$  of the restrictions of cyclic multilinear forms to  $S^+(V[1])$  is a quotient of  $\mathcal{C}$ .

Suppose there is a Pinczon bracket  $\{ , \}$  on  $\mathcal{C}$ , then we can write  $\Omega$  as  $\Omega = \Omega_Q$ , with  $Q$   $k$ -linear and cyclic. But, any element  $\sigma$  in  $\mathfrak{S}_{k+1}$  can be written in an unique way as a product  $\tau \circ \rho$  where  $\tau$  is in  $\mathfrak{S}_k$ , viewed as a subgroup of  $\mathfrak{S}_{k+1}$  and  $\rho$  in  $Cycl$ . Therefore, with our notation, we get:

$$\Omega^{Sym} = (\Omega_Q)^{Sym} = (\Omega_{Q^{Sym}})^{Cycl} = (k+1)\Omega_{Q^{Sym}}.$$

**Proposition 6.1.**

*The bracket defined on  $L(\otimes^+ V[1], V[1])$  by the commutator of coderivations in  $(\otimes^+ V[1], \Delta)$ , induces a well defined bracket on the quotient  $L(S^+(V[1]), V[1])$ , this bracket is the above commutator of coderivations in  $(S^+(V[1]), \Delta)$ :*

$$[Q^{Sym}, (Q')^{Sym}] = [Q, Q']^{Sym}.$$

Any Pinczon bracket  $\{ , \}$  on the space  $\mathcal{C}$  of multilinear cyclic forms on  $V[1]$  induces a well defined bracket on the quotient  $\mathcal{C}|_S$ , this bracket denoted  $\{ , \}|$  is:

$$\{\Omega^{Sym}, (\Omega')^{Sym}\}| = (\{\Omega, \Omega'\})^{Sym} = \frac{k+k'}{(k+1)!(k'+1)!} \sum_i \iota_{e_i} (\Omega^{Sym}) \cdot \iota_{e'_i} ((\Omega')^{Sym}).$$

*Proof.* The first assertion is a simple computation. Put

$$\begin{aligned} (Q \circ Q')(x_1, \dots, x_{k+k'-1}) &= \sum_{r=1}^{k-1} Q(x_1, \dots, x_{r-1}, Q'(x_r, \dots, x_{r+k'-1}), x_{r+k'}, \dots, x_{k+k'-1}) \\ (Q \circ Q')(x_{[1, k+k'-1]}) &= \sum_{r=1}^{k-1} Q(x_{[1, r-1]} \otimes Q'(x_{[r, r+k'-1]}) \otimes x_{[r+k', k+k'-1]}). \end{aligned}$$

Then, if we abusively note  $x_{\sigma^{-1}([a, b])}$  the quantity  $x_{\sigma^{-1}(a)} \otimes \dots \otimes x_{\sigma^{-1}(b)}$ ,

$$(Q \circ Q')^{Sym}(x_{[1, k+k'-1]}) = \sum_{r=1}^{k-1} \sum_{\sigma \in \mathfrak{S}_{k+k'-1}} Q(x_{\sigma^{-1}([1, r-1])}, Q'(x_{\sigma^{-1}([r, r+k'-1])}), x_{\sigma^{-1}([r+k', k+k'-1])})$$

For any subset  $J$  in  $[1, k+k'-1]$ , with  $\#J = k'$ , for any  $r$ , we put  $I = [1, k+k'-1] \setminus J$  and, for any term in the above sum for which  $\sigma^{-1}([r, r+k'-1]) = J$ , we define two permutaions  $\tau \in \mathfrak{S}_{k'}$ , and  $\rho \in \mathfrak{S}_k$  as follows:

write  $J = \{j_1 < \dots < j_{k'}\}$ , then put

$$j_{\tau^{-1}(t)} = \sigma^{-1}(r+t-1) \quad (1 \leq t \leq k'),$$

write  $\{0\} \cup I = \{i_1 < \dots < i_k\}$ , then put

$$i_{\rho^{-1}(t)} = \sigma^{-1}(t) \quad (1 \leq t \leq r-1), \quad i_{\rho^{-1}(r)} = 0, \quad i_{\rho^{-1}(t)} = \sigma^{-1}(t-1) \quad (r+1 \leq t \leq k).$$

The correspondence  $(r, \sigma) \mapsto (r, \tau, \rho)$  is one-to-one and, suming up, we get:

$$(Q \circ Q')^{Sym}(x_{[1, k+k'-1]}) = \sum_{\substack{I \sqcup J = [1, k+k'-1] \\ \#J = k'}} Q^{Sym}((Q')^{Sym}(x_J) \cdot x_I).$$

Therefore, the quotient bracket is well defined and is the bracket of coderivation of  $(S^+(V[1]), \Delta)$ :

$$[Q, Q']^{Sym} = (Q \circ Q')^{Sym} - (Q' \circ Q)^{Sym} = [Q^{Sym}, (Q')^{Sym}].$$

Let now  $\{ , \}$  be a Pinczon bracket on the space  $\mathcal{C}$  of cyclic multilinear forms on  $V[1]$ . Thus there is a symplectic form  $B$  on  $V[1]$  and we can write any cyclic form  $\Omega$  as  $\Omega_Q$ . We get:

$$\{\Omega, \Omega'\}^{Sym} = \{\Omega_Q, \Omega_{Q'}\}^{Sym} = \Omega_{[Q, Q']}^{Sym} = (k+k')\Omega_{[Q, Q']^{Sym}} = (k+k')\Omega_{[Q^{Sym}, (Q')^{Sym}]}.$$

The last bracket is the commutator of coderivations in  $S^+(V[1])$ , thus

$$\begin{aligned} \{\Omega, \Omega'\}^{Sym}(x_{\cdot[1, k+k']}) &= (k+k') \sum_{\substack{I \sqcup J = [1, k+k'-1] \\ \#J = k'}} B(Q^{Sym}((Q')^{Sym}(x_{\cdot J}) \cdot x_{\cdot I}), x_{k+k'}) - \\ &\quad - (k+k') \sum_{\substack{I \sqcup J = [1, k+k'-1] \\ \#I = k}} B((Q')^{Sym}(Q^{Sym}(x_{\cdot I}) \cdot x_{\cdot J}), x_{k+k'}) \\ &= (k+k') \sum_{\substack{I \sqcup J = [1, k+k'] \\ \#J = k'}} B(Q^{Sym}(x_{\cdot I}), (Q')^{Sym}(x_{\cdot J})). \end{aligned}$$

On the other hand,

$$\iota_{e_i} \left( \Omega_Q^{Sym} \right) (x_{\cdot[1, k]}) = (k+1) B(Q^{Sym}(x_{\cdot[1, k]}), e_i) = (k+1) (\iota_{e_i} \Omega)^{Sym} (x_{\cdot[1, k]}).$$

Therefore

$$\sum_i \iota_{e_i} \left( \Omega_Q^{Sym} \right) \otimes \iota_{e'_i} \left( \Omega_{Q'}^{Sym} \right) (x_{\cdot[1, k+k']}) = (k+1)(k'+1) B(Q^{Sym}(x_{\cdot[1, k]}), (Q')^{Sym}(x_{\cdot[k, k+k']})),$$

and

$$\begin{aligned} \sum_i \iota_{e_i} \left( \Omega_Q^{Sym} \right) \cdot \iota_{e'_i} \left( \Omega_{Q'}^{Sym} \right) (x_{\cdot[1, k+k']}) &= \sum_i \left( \iota_{e_i} \left( \Omega_Q^{Sym} \right) \otimes \iota_{e'_i} \left( \Omega_{Q'}^{Sym} \right) \right)^{Sym} (x_{\cdot[1, k+k']}) \\ &= (k+1)!(k'+1)! \sum_{\substack{I \sqcup J = [1, k+k'] \\ \#J = k'}} B(Q^{Sym}(x_{\cdot I}), (Q')^{Sym}(x_{\cdot J})) \end{aligned}$$

This proves that the Pinczon bracket induces a well defined bracket on the quotient  $\mathcal{C}$ , and gives our last assertion.  $\square$

Explicitly, if we suppose the forms  $\Omega$  and  $\Omega'$  completely symmetric ( $\Omega, \Omega'$  belongs to  $\mathcal{S}$ ),  $\Omega^{Sym} = (k+1)!\Omega$  and the Pinczon bracket on the quotient becomes:

$$\{\Omega, \Omega'\} = (k+k') \sum_i \iota_{e_i} \Omega \cdot \iota_{e'_i} \Omega'.$$

We rekind the expression given in [PU, DPU].

## 6.2. Quadratic $L_\infty$ algebras.

We are now in position to define the Pinczon Lie algebras:

### Definition 6.2.

We call Pinczon Lie algebra any vector space  $V$ , such that  $\otimes(V[1])^*$  is equipped with a Pinczon bracket  $\{ , \}$ , and a symmetric multilinear form  $\Omega$ , with degree 3,

such that  $\{\Omega, \Omega\} = 0$ .

Recall that a structure of  $L_\infty$  algebra (or Lie algebra up to homotopy) on  $V$  is a degree 1 coderivation  $Q$  of  $(S^+(V[1]), \Delta)$ , such that the commutator  $[Q, Q]$  vanishes. With the preceding computations, we see that a Pinczon Lie algebra is in fact a quadratic  $L_\infty$  algebra.

**Proposition 6.3.**

*Let  $V, \{ , \}, \Omega$  be a Pinczon Lie algebra, then there is an unique  $B$  symplectic on  $V[1]$  and an unique coderivation  $Q$  of  $(S^+(V[1]), \Delta)$ , which is cyclic with respect to  $B$ , such that  $\Omega = \Omega_Q$ .*

*Conversely, if  $(V, b)$  is a vector space equipped with a non degenerated symmetric bilinear form, any structure  $Q$  of  $L_\infty$  algebra which is cyclic defines an unique Pinczon Lie algebra  $(V, \{ , \}, \Omega_Q)$ .*

**6.3. Modules and Chevalley cohomology.**

Suppose in this section that  $(V, b, q)$  is a quadratic Lie Algebra, and  $M$  a finite dimensional  $(V, q)$ -module. To refine the Chevalley cohomology, we first build (see [BB, MR]), as above, the double semidirect product of  $V$  by  $M$ .

First, we consider the semidirect product  $W = V \rtimes M$ , that is the vector space  $V \oplus M$ , equipped with the Lie bracket:

$$q_W(x + a, y + b) = q(x, y) + x \cdot b - y \cdot a,$$

then  $V$  has a natural  $(W, q_W)$  module structure, namely:

$$(x + a) \cdot y = q(x, y).$$

Now, since  $W$  is a  $(W, q_W)$  bimodule, its dual  $W^*$  is also a  $(W, q_W)$ -module, with:

$$((x + a) \cdot f)(z + c) = f(q_W(z + c, x + a)).$$

finally, there is a natural mapping  $\varphi$  from  $V \times V$  into  $W^*$ , namely:

$$\varphi(y, y')(z + c) = b(q(z, y), y') = b(q(y, y'), z).$$

With these objects, we define a structure of Lie algebra on the space  $\tilde{V} = W \times V \times W^*$ , by putting:

$$\begin{aligned} \tilde{q}((x + a) + y + f, (x' + a') + y' + f') &= (q(x, x') + x \cdot a' - x' \cdot a) + \\ &+ (q(y, y') + q(x, y') + q(y, x')) + ((x + a) \cdot f' - (x' + a') \cdot f + \varphi(y, y')), \end{aligned}$$

and a non degenerated symmetric bilinear form  $\tilde{b}$ , by

$$\tilde{b}((x + a) + y + f, (x' + a') + y' + f') = b(y, y') + f(x' + a') + f'(x + a).$$

A direct computation (see [BB]) shows that  $(\tilde{V}, \tilde{b}, \tilde{q})$  is a quadratic Lie algebra, we call it the double extension of  $(V, b, q)$  by the bimodule  $M$ .

**Definition 6.4.**

Let  $(V, b, q)$  be a quadratic Lie algebra and  $M$  a module, the quadratic Lie algebra:

$$(\tilde{V}, \tilde{b}, \tilde{q})$$

is the double semi-direct product of  $V$  by  $M$ .

Similarly, the Pinczon Lie algebra  $(\tilde{V}[1], \{ , \} | \tilde{\cdot}, \Omega_{\tilde{Q}})$  associated to  $(\tilde{V}, \tilde{b}, \tilde{q})$  is the Pinczon Lie algebra double semi-direct product of  $(V[1], \{ , \} | \cdot, \Omega_Q)$  by  $M$ .

We look now for a  $k$  linear mapping  $C$  from  $V^k$  into  $M$ , completely symmetric, and consider it as a mapping from  $\tilde{V}^k$  into  $\tilde{V}$ , as follows:

$$\tilde{C}((x_1 + a_1) + y_1 + f_1, \dots, (x_k + a_k) + y_k + f_k) = (0 + C(x_1, \dots, x_k)) + 0 + 0.$$

Then, if  $\tilde{Q}$  is associated to  $\tilde{q}$  as above, we get:

$$[\tilde{Q}, \tilde{C}] = \widetilde{d_{Ch}C},$$

where  $d_{Ch}$  is the Chevalley coboundary operator on the module  $M$ , and similiary:

$$\frac{1}{2+k} \{\Omega_{\tilde{Q}}, \Omega_{\tilde{C}}\} | \tilde{\cdot} = \Omega_{\widetilde{d_{Ch}C}}.$$

**Proposition 6.5.**

Let  $(V, \{ , \}, \Omega_Q)$  be a Pinczon Lie algebra, with  $\Omega_Q$  trilinear. Then the restriction of the Pinczon cohomology

$$\Omega \mapsto \{\Omega_{\tilde{Q}}, \Omega_{\tilde{C}}\} | \tilde{\cdot}$$

to the completely symmetric forms  $\Omega_{\tilde{C}}$  coming from multilinear mappings  $C$  from  $V^k$  into  $M$  is the Chevalley cohomology of the  $(V, q)$ -module  $M$ .

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