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A NOTE ON POSITIVE EIGENFUNCTIONS AND HIDDEN CONVEXITY

LORENZO BRASCO AND GIOVANNI FRANZINA

ABSTRACT. We give a simple convexity-based proof of the following fact: the only eigenfunction of the p -Laplacian that does not change sign is the first one. The method of proof covers as well more general nonlinear eigenvalue problems.

1. INTRODUCTION

Given a connected open set $\Omega \subset \mathbb{R}^N$ and $1 < p < \infty$, we recall that a (positive) number λ is said to be a Dirichlet *eigenvalue* of the p -Laplace operator, if there holds

$$(1.1) \quad -\operatorname{div}(|\nabla u|^{p-2} \nabla u) = \lambda |u|^{p-2} u, \quad \text{in } \Omega,$$

for some nontrivial function $u \in W_0^{1,p}(\Omega)$: correspondingly, such a function u is called *eigenfunction*. Here solutions to (1.1) are always intended in a weak sense. Observe that eigenvalues can be characterized as critical values of the nonlinear Dirichlet integral $\int_{\Omega} |\nabla u|^p$, restricted to the manifold

$$S_p = \{u \in W_0^{1,p}(\Omega) : \|u\|_{L^p(\Omega)} = 1\}.$$

The corresponding critical points are of course the eigenfunctions, normalized by the constraint on the L^p norm. The first eigenvalue $\lambda_1(\Omega)$ plays a distinguished role, since it corresponds to the global minimum of the Dirichlet integral on S_p : then up to the choice of the sign, the first (normalized) eigenfunction is unique (see [3]).

The aim of this short note is to show how a subtle form of *hidden convexity*¹ is responsible for the well-known result, asserting that only eigenfunctions relative to $\lambda_1(\Omega)$ can have constant sign. This fact has been derived in various places, under different assumptions on the regularity of Ω (see [1, 4, 6] and [7] for example): at present, we believe that the most simple and direct proof of this fact, was the one by Kawohl and Lindqvist ([4]), in turn inspired to that by Ôtani and Teshima ([7]).

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¹We owe this terminology to Bernd Kawohl.

The proof in [4] is based on a clever use of the equation, but it does not clearly display – at least, not at an explicit level – the reason behind such a remarkable result: as we will show, it is just a matter of convexity of the energy functional $\int_{\Omega} |\nabla u|^p$. More precisely, $\int_{\Omega} |\nabla u|^p$ enjoys a sort of geodesic convexity on the intersection between the cone of positive functions and the manifold S_p . Clearly, this permits to conclude that on this space the global analysis of $\int_{\Omega} |\nabla u|^p$ is trivial: convexity implies that *there can not be any critical point, except for the global minimizer*, which as already said is unique (up to the sign). In the end, we believe this to be the real motivation which neatly explains why it is forbidden to have positive eigenfunctions, corresponding to a λ strictly greater than $\lambda_1(\Omega)$.

The plan of this small note is as follows: in the next section, we prove a convexity property of variational integrals, whose Lagrangians depend homogeneously on the gradient. This is a variation on the convexity principle employed by Belloni and Kawohl in [3] (see also [2]), in order to prove the uniqueness of the first (normalized) eigenfunction of the p -Laplacian. Then, Section 3 shows how to use this convexity, so as to derive the above claimed result, about positive eigenfunctions. For the sake of generality – and since this does not require any additional efforts – we will give the result in a slightly general version (see Theorem 3.1), suitable to be applied to more general nonlinear eigenvalue problems, like for example

$$(1.2) \quad -\operatorname{div}(\nabla H(x, \nabla u)) = b(x) \lambda |u|^{p-2} u,$$

where $H : \Omega \times \mathbb{R}^N \rightarrow \mathbb{R}$ is C^1 convex and p -positively homogeneous in the gradient variable and $b \in L^\infty(\Omega)$, with $b \geq 0$.

2. THE HIDDEN CONVEXITY LEMMA

The main tool of our proof is the following convexity principle, used by Belloni and Kawohl in [3] for the functional $\int |\nabla u|^p$. To make the paper self-contained, we repeat here the proof: the statement is slightly more general, in order to include a wider list of functionals. Also, we relax the requirement on the strict positivity of the functions.

Lemma 2.1. *Given $\Omega \subset \mathbb{R}^N$ an open set, let $p \geq 1$ and let $H : \Omega \times \mathbb{R}^N \rightarrow \mathbb{R}_+$ be a measurable functions, such that*

$$(2.1) \quad z \mapsto H(x, z) \quad \text{is convex and } p\text{-positively homogeneous, for every } x \in \Omega.$$

For every $u_0, u_1 \in W^{1,p}(\Omega)$ such that $u_0, u_1 \geq 0$ on Ω and

$$\int_{\Omega} H(x, \nabla u_i(x)) dx < +\infty, \quad i = 0, 1,$$

we define

$$\sigma_t(x) = \left((1-t)u_0(x)^p + tu_1(x)^p \right)^{\frac{1}{p}}, \quad t \in [0, 1], x \in \Omega.$$

Then

$$(2.2) \quad t \mapsto \int_{\Omega} H(x, \nabla \sigma_t(x)) dx \quad \text{is convex on } [0, 1].$$

Proof. First of all, we observe that in order to prove (2.2), it suffices to show that

$$(2.3) \quad \int_{\Omega} H(x, \nabla \sigma_t(x)) dx \leq (1-t) \int_{\Omega} H(x, \nabla u_0(x)) dx \\ + t \int_{\Omega} H(x, \nabla u_1(x)) dx, \quad t \in [0, 1],$$

It is easily seen that for every $t \in [0, 1]$, σ_t defines an element of $W^{1,p}(\Omega)$, since this is nothing but the composition of the vector-valued Sobolev map

$$\left((1-t)^{\frac{1}{p}} u_0, t^{\frac{1}{p}} u_1 \right) \in W^{1,p}(\Omega; \mathbb{R}^2),$$

with the ℓ_p norm, i.e. $\|(x, y)\|_{\ell_p} = (|x|^p + |y|^p)^{1/p}$. Moreover, the latter is a C^1 function outside the origin and ∇u_i vanishes almost everywhere on the set $u_i^{-1}(\{0\})$, $i = 0, 1$: then we have the usual chain rule formula, i.e. we obtain

$$\nabla \sigma_t = \sigma_t^{1-p} \left[(1-t) \nabla u_0(x) u_0^{p-1} + t \nabla u_1(x) u_1^{p-1} \right] \\ = \sigma_t \left[\frac{(1-t) u_0^p}{\sigma_t^p} \frac{\nabla u_0}{u_0} + \frac{t u_1^p}{\sigma_t^p} \frac{\nabla u_1}{u_1} \right],$$

where the previous expression has to be intended equal to 0, as soon as u_0 and u_1 both vanish. Observe that inside the square brackets we have a convex combination of $\nabla u_0/u_0$ and $\nabla u_1/u_1$: using the convexity and homogeneity of H in the gradient variable, we then get

$$H(x, \nabla \sigma_t) \leq \sigma_t^p \left[\frac{(1-t) u_0^p}{\sigma_t^p} H \left(x, \frac{\nabla u_0}{u_0} \right) + \frac{t u_1^p}{\sigma_t^p} H \left(x, \frac{\nabla u_1}{u_1} \right) \right] \\ = (1-t) u_0^p H \left(x, \frac{\nabla u_0}{u_0} \right) + t u_1^p H \left(x, \frac{\nabla u_1}{u_1} \right) \\ = (1-t) H(x, \nabla u_0) + t H(x, \nabla u_1).$$

By integrating over Ω , we finally get (2.3) and thus the thesis. \square

Remark 2.2. Differently from the scalar case, it is useful to recall that in general the chain rule formula is not true, for the composition between a Lipschitz function and a vector-valued Sobolev function. Necessary and sufficient conditions for this to hold are described in [5].

Remark 2.3. We like to point out that the curves of the form

$$\sigma_t(x) = \left((1-t)u_0(x)^p + tu_1(x)^p \right)^{\frac{1}{p}}, \quad t \in [0, 1],$$

are the constant speed geodesics of the cone $C_p = \{u \in L^p(\Omega) : u \geq 0\}$, endowed with the metric

$$d_p(u_0, u_1) = \left(\int_{\Omega} |u_0(x)^p - u_1(x)^p| dx \right)^{\frac{1}{p}} = \|u_0^p - u_1^p\|_{L^1(\Omega)}, \quad u_0, u_1 \in C_p.$$

Indeed, we have

$$\begin{aligned} d_p(\sigma_t, \sigma_s) &= \left(\int_{\Omega} |\sigma_t(x)^p - \sigma_s(x)^p| dx \right)^{\frac{1}{p}} \\ &= |t - s| \left(\int_{\Omega} |u_0(x)^p - u_1(x)^p| dx \right)^{\frac{1}{p}} = |t - s| d(u_0, u_1), \quad s, t \in [0, 1]. \end{aligned}$$

3. THE MAIN RESULT

We are going to prove the main result of this note: the argument is very simple and just based on the convexity principle of Lemma 2.1, but we have to go on through a mild approximation argument. Also, since this is essentially a *uniqueness result*, we do not insist on the sharp hypotheses, needed to obtain existence for the variational problem defining $\lambda_1(\Omega)$ below. Rather, we will directly assume that this is well-defined: in what follows, for simplicity with $\nabla H(x, z)$ we will indicate the gradient of H with respect to the gradient variable.

Theorem 3.1. *Let $\Omega \subset \mathbb{R}^N$ be an open set, having finite measure. Given $p > 1$, we consider $H : \Omega \times \mathbb{R}^N \rightarrow \mathbb{R}_+$ to be a C^1 function satisfying (2.1) and we suppose that the variational problem*

$$\lambda_1(\Omega) = \min_{u \in W_0^{1,p}(\Omega)} \left\{ \int_{\Omega} H(x, \nabla u) dx : \|u\|_{L^p(\Omega)} = 1 \right\},$$

is well-posed, i.e. there exists at least a solution. Assume that λ is such that

$$(3.1) \quad \frac{1}{p} \int_{\Omega} \langle \nabla H(x, \nabla v(x)), \nabla \varphi(x) \rangle dx = \lambda \int_{\Omega} |v(x)|^{p-2} v(x) \varphi(x) dx, \quad \varphi \in W_0^{1,p}(\Omega),$$

for some strictly positive $v \in W_0^{1,p}(\Omega) \setminus \{0\}$. Then necessarily

$$(3.2) \quad \lambda = \lambda_1(\Omega).$$

Proof. First of all, observe that we can always assume that $\|v\|_{L^p(\Omega)} = 1$, since equation (3.1) is $(p-1)$ -homogeneous and $v \not\equiv 0$. We also observe that by testing the equation with $\varphi = v$ and by homogeneity of H , we get

$$\int_{\Omega} H(x, \nabla v(x)) dx = \frac{1}{p} \int_{\Omega} \langle \nabla H(x, \nabla v(x)), \nabla v(x) \rangle dx = \lambda \geq \lambda_1(\Omega),$$

since v is admissible for the problem defining $\lambda_1(\Omega)$. Let us assume by contradiction that (3.2) is not true, this means that we have

$$(3.3) \quad \lambda_1(\Omega) - \lambda < 0.$$

Then we take $u \in W_0^{1,p}(\Omega)$ a minimizer for (3.2) and for every $\varepsilon \ll 1$, we set for simplicity

$$u_{\varepsilon} = u + \varepsilon \quad \text{and} \quad v_{\varepsilon} = v + \varepsilon.$$

We now want to use Lemma 2.1: at this aim, we define as before the following curve of functions

$$\sigma_t(x) = [(1-t)v_{\varepsilon}(x)^p + tu_{\varepsilon}(x)^p]^{\frac{1}{p}}, \quad x \in \Omega, t \in [0, 1],$$

connecting v_{ε} and u_{ε} . Thanks to Lemma 2.1, we can infer that

$$\begin{aligned} \int_{\Omega} H(x, \nabla \sigma_t(x)) dx &\leq (1-t) \int_{\Omega} H(x, \nabla v_{\varepsilon}(x)) dx + t \int_{\Omega} H(x, \nabla u_{\varepsilon}(x)) dx \\ &= t \left[\int_{\Omega} H(x, \nabla u(x)) dx - \int_{\Omega} H(x, \nabla v(x)) dx \right] \\ &\quad + \int_{\Omega} H(x, \nabla v_{\varepsilon}(x)) dx, \quad t \in [0, 1], \end{aligned}$$

where we used that $\nabla u_{\varepsilon} = \nabla u$ and $\nabla v_{\varepsilon} = \nabla v$. From the previous, we then arrive at

$$\begin{aligned} \int_{\Omega} \frac{H(x, \nabla \sigma_t(x)) - H(x, \nabla \sigma_0(x))}{t} dx &\leq \int_{\Omega} H(x, \nabla u(x)) dx - \int_{\Omega} H(x, \nabla v(x)) dx \\ &= \lambda_1(\Omega) - \lambda, \quad t \in (0, 1], \end{aligned}$$

and we recall that $\sigma_0 = v_{\varepsilon}$, by construction. We then use the (standard) convexity of H in the left-hand side: thus, we can infer

$$\int_{\Omega} \left\langle \nabla H(x, \nabla \sigma_0(x)), \frac{\nabla \sigma_t(x) - \nabla \sigma_0(x)}{t} \right\rangle dx \leq \lambda_1(\Omega) - \lambda.$$

Since $\nabla \sigma_0 = \nabla v_{\varepsilon} = \nabla v$, we can use the equation (3.1), by testing it with the function $\sigma_t - \sigma_0 \in W_0^{1,p}(\Omega)$: this yields

$$\lambda \int_{\Omega} v(x)^{p-1} \frac{\sigma_t(x) - \sigma_0(x)}{t} dt \leq \lambda_1(\Omega) - \lambda, \quad \text{for every } \varepsilon \ll 1, t \in (0, 1].$$

Observe that there holds

$$\left| \frac{\sigma_t(x) - \sigma_0(x)}{t} \right| \leq \frac{1}{p} \varepsilon^{1-p} |u_\varepsilon^p - v_\varepsilon^p| \in L^1(\Omega), \quad \text{for every } t \in (0, 1],$$

then by Lebesgue Dominated Convergence, the limit as t goes to 0 takes us to

$$\frac{\lambda}{p} \int_{\Omega} \left(\frac{v(x)}{v_\varepsilon(x)} \right)^{p-1} [u_\varepsilon(x)^p - v_\varepsilon(x)^p] dx \leq \lambda_1(\Omega) - \lambda, \quad \text{for every } \varepsilon \ll 1.$$

By passing to the limit as ε converges to 0 in the previous – and keeping in mind that u and v have the same L^p norm – we finally end up with

$$0 = \frac{\lambda}{p} \left[\int_{\Omega} u(x)^p dx - \int_{\Omega} v(x)^p dx \right] \leq \lambda_1(\Omega) - \lambda,$$

where we used that $v > 0$ on Ω . This gives the desired contradiction, thanks to assumption (3.3). \square

Remark 3.2. Observe that we required the solution v to (3.1) to be *strictly* positive on Ω : this is not a big deal, since in most of the situation of interest, the Harnack inequality holds true and guarantees that positive solutions of (3.1) does not vanish in the interior on Ω . This is the case for example of $H : \Omega \times \mathbb{R}^n \rightarrow \mathbb{R}_+$ satisfying (2.1) and the growth conditions

$$c_1 |z|^p \leq |H(x, z)| \leq c_2 |z|^p, \quad (x, z) \in \Omega \times \mathbb{R}^N,$$

for two positive constants $c_1 \geq c_2 > 0$.

The uniqueness of positive eigenfunctions of the p -Laplacian is now an easy consequence of Theorem 3.1.

Corollary 3.3. *Let $\Omega \subset \mathbb{R}^N$ be a connected open set, having finite measure. Then the only Dirichlet eigenfunctions of the p -Laplacian that do not change sign, are those corresponding to the first eigenvalue, defined by*

$$\lambda_1(\Omega) = \min_{u \in W_0^{1,p}(\Omega)} \left\{ \int_{\Omega} |\nabla u(x)|^p dx : \|u\|_{L^p(\Omega)} = 1 \right\}.$$

Remark 3.4. The same conclusions can be drawn for the Dirichlet eigenfunctions of the so called *pseudo p -Laplacian* $\tilde{\Delta}_p$, defined by

$$\tilde{\Delta}_p u := \sum_{i=1}^N \partial_{x_i} (|\partial_{x_i} u|^{p-2} \partial_{x_i} u).$$

Here of course the eigenvalue problem – introduced in [2] – consists in finding the positive numbers $\lambda > 0$, such that the equation

$$-\tilde{\Delta}_p u = \lambda |u|^{p-2} u,$$

has nontrivial solutions in $W_0^{1,p}(\Omega)$. The proof amounts to applying again Theorem 3.1, now with the variational integral

$$\sum_{i=1}^N \int_{\Omega} |\partial_{x_i} u(x)|^p dx,$$

which satisfies the hypotheses of Lemma 2.1.

Remark 3.5. We explicitly observe that the statement of Theorem 3.1 still holds – and the proof is exactly the same – if we replace the L^p constraint $\|u\|_{L^p(\Omega)} = 1$, by the following one

$$\int_{\Omega} b(x) |u(x)|^p dx = 1,$$

i.e. if we look at the eigenvalue problem

$$-\operatorname{div}(\nabla H(x, \nabla u(x))) = \lambda b(x) |u(x)|^{p-2} u(x).$$

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