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# ADMISSIBLE COVERS AND THE ELSV COMPACTIFICATION

B. DUDIN

ABSTRACT. We revisit Ekedahl, Lando, Shapiro and Vainshtein's compactification of the stack of Hurwitz covers. By drawing a connection with the Harris and Mumford stack of admissible covers we give a new geometric interpretation of boundary points of the ELSV compactification. As a byproduct we establish that this compactification holds for any algebraically closed field of sufficiently high characteristic.

## INTRODUCTION

Hurwitz covers of genus  $g$  and type  $(k_1, \dots, k_n) \in (\mathbb{N}^*)^n$  are covers of smooth rational single marked curves  $\phi : (C; p_1, \dots, p_n) \rightarrow (R, \infty)$  for a smooth genus  $g$   $n$ -marked curve  $C$  such that  $\phi$  is simply ramified away from  $\infty$  and satisfies scheme theoretically  $\phi^{-1}\infty = \sum_{i=1}^n k_i p_i$ . A cover  $\psi : (B, q_1, \dots, q_n) \rightarrow (T, \infty)$  is said to be isomorphic to  $\phi$  if there is a commutative diagram

$$\begin{array}{ccc} (C, p_1, \dots, p_n) & \longrightarrow & (R, \infty) \\ \downarrow & & \downarrow \\ (B, q_1, \dots, q_n) & \longrightarrow & (T, \infty) \end{array}$$

where vertical arrows are isomorphisms of marked curves. The number of isomorphism classes of Hurwitz covers of a given genus and type having a fixed branch locus are called Hurwitz numbers. When the base field is  $\mathbb{C}$  the celebrated ELSV formula expresses these numbers in terms of tautological intersection numbers on the stack of stable marked curves. To prove this formula Ekedahl, Lando, Shapiro and Vainshtein introduced a compactification of the stack of Hurwitz covers  $\mathcal{H}_{g, \vec{k}}$  which has a projective stack cone structure over  $\overline{\mathfrak{M}}_{g, n}$ . Let's unravel how this cone structure appears in the case of  $\mathcal{H}_{g, \vec{k}}$  over  $\mathfrak{M}_{g, n}$ . Take an isomorphism  $\alpha : (R, \infty) \simeq (\mathbb{P}^1, \infty)$ , the composition  $\alpha \circ \phi$  is given by a global non zero section  $\zeta_\alpha$  in  $H^0(\mathcal{O}_C(\sum_{i=1}^n k_i p_i))$ . Write  $\mathbf{1}$  for the section  $\mathcal{O}_C \rightarrow \mathcal{O}_C(\sum_{i=1}^n k_i p_i)$  and let  $\zeta_\beta \in H^0(\mathcal{O}_C(\sum_{i=1}^n k_i p_i))$  correspond to  $\beta : (R, \infty) \simeq (\mathbb{P}^1, \infty)$ . There exists a couple  $(\lambda, \mu) \in \mathbb{G}_a \times \mathbb{G}_m = \text{Aut}(\mathbb{P}^1, \infty)$  such that  $\zeta_\alpha = \lambda \zeta_\beta + \mu \mathbf{1}$ . One gets that the image  $[\zeta_\alpha]$  of  $\zeta_\alpha$  in  $\mathbb{P}H^0(\mathcal{O}_C(\sum_{i=1}^n k_i p_i) / \mathcal{O}_C)$  does only depend on  $\phi$ . Covers automorphic to  $\phi$  can be recovered out of lifts of  $[\zeta_\alpha]$  to  $\mathbb{P}H^0(\mathcal{O}_C(\sum_{i=1}^n k_i p_i))$ . A unique choice of a lift can be made by choosing the one having 0 sum of branch points away from  $\infty$ . Following this procedure one gets a locally closed embedding of  $\mathcal{H}_{g, \vec{k}}$  in the projective bundle over  $\mathfrak{M}_{g, n}$  having fiber  $\mathbb{P}H^0(\mathcal{O}_C(\sum_{i=1}^n k_i p_i) / \mathcal{O}_C)$  over  $(C, p_1, \dots, p_n)$ .

To compactify  $\mathcal{H}_{g, \vec{k}}$  following [ELSV01] one starts by extending underlying curves to the case of stable marked ones and replaces sections of  $\mathcal{O}_-(\sum_{i=1}^n k_i p_i)$  by sections of a more sophisticated coherent sheaf called the sheaf of generalized polar parts. This sheaf is constructed by giving local explicit descriptions which glue together over the whole of  $\overline{\mathfrak{M}}_{g, n}$ . The projective stack cone attached to the sheaf of generalized polar

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parts has a closed substack of a quotient of the stack of stable maps by  $\mathbb{G}_m$ . The ELSV compactification is the closure  $\overline{\mathcal{H}}_{g,\vec{k}}$  of  $\mathcal{H}_{g,\vec{k}}$  in this closed substack. One can raise two questions:

- Is there a construction of the stack of generalized polar parts over  $\overline{\mathcal{M}}_{g,n}$  which shortcuts the previously mentioned local descriptions?
- Can one have a finer understanding of boundary points of  $\overline{\mathcal{H}}_{g,\vec{k}}$ ?

To answer the first question we choose to keep global sections of  $\mathcal{O}_{-(\sum_{i=1}^n k_i p_i)}/\mathcal{O}_-$  rather than working with generalized polar parts over stable curves. We extend however underlying curves to specific prestable marked curves called bubbly curves (definition 1.2). These enriched bubbly curves are called polar curves (definition 1.4). The forgetful morphism that sends such data to the underlying bubbly curve and stabilizes it gives the stack of polar curves the structure of a stack cone over  $\overline{\mathcal{M}}_{g,n}$ . We prove this cone is isomorphic to the stack of generalized polar parts. In the process we check that the stack of polar parts gives birth to a tame stack in the sense of [AOV08] over any algebraically closed field. The latter contains a closed substack in which  $\mathcal{H}_{g,\vec{k}}$  lies as an open substack. One can define the ELSV compactification over any algebraically closed base field as the closure  $\overline{\mathcal{H}}_{g,\vec{k}}$  of  $\mathcal{H}_{g,\vec{k}}$  in this closed substack. In general  $\mathcal{H}_{g,\vec{k}}$  is badly behaved and we have to restrict our attention to the case of base fields of sufficiently high characteristic.

Assume the characteristic of the base field is either 0 or greater than the degree of the Hurwitz covers we're interested in, as well as greater than the number of simple ramification points away from  $\infty$ . Under this assumption we build a natural surjective morphism from Harris Mumford's stack of admissible covers to a quotient of the stack of stable maps by  $\mathbb{G}_m$ . Using the generalized branch divisor introduced by Fantechi and Pandharipande we identify the ELSV compactification with the image of this morphism. This answers the second question. The previously mentioned morphism gives combinatorial restrictions on the modular graphs of curves underlying  $\mathcal{H}_{g,\vec{k}}$ . A study of these combinatorial restrictions will appear in a forthcoming paper.

**A word about notation.** Throughout this paper we fix a natural number  $g$  and a positive integer  $n$ . All of our prestable curves shall be of genus  $g$  and marked by  $n$  points. Unless otherwise stated we will assume  $(g, n)$  is different from  $(0, 1)$  and  $(0, 2)$ . We also fix an  $n$ -tuple of positive integers  $(k_1, \dots, k_n)$  written  $\vec{k}$ . In what follows, without further notification,  $\mathbb{k}$  is an algebraically closed field of any characteristic  $p$ , and a scheme is a  $\mathbb{k}$ -scheme, that belongs to the category  $Sch/\mathbb{k}$ .

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## 1. THE STACK OF POLAR CURVES

**1.1. Stable maps of profile  $\vec{k}$ .** For an exhaustive study of marked stable curves and related stacks and moduli problems, see [ACG11]. Let  $S \in Sch/\mathbb{k}$ , and let  $\pi : \mathcal{C} \rightarrow S$  be a prestable  $S$ -curve marked by the  $S$ -points  $\sigma_1, \dots, \sigma_n$ . Recall that this means that  $\pi$  is a proper and flat  $S$ -scheme  $\pi : \mathcal{C} \rightarrow S$  whose fibers are connected reduced 1-dimensional  $\mathbb{k}$ -schemes having at most nodes for singularities. In addition we are given  $n$  disjoint sections  $(\sigma_1, \dots, \sigma_n)$  of the projection  $\pi : \mathcal{C} \rightarrow S$  whose images  $\mathcal{D}_1, \dots, \mathcal{D}_n$  are in the smooth locus of  $\pi$ . The divisor  $\sum_{i=1}^n k_i \mathcal{D}_i$  is a relative Cartier divisor on  $\mathcal{C}$ . The above

data is written  $(\mathcal{C}/S, \vec{\sigma})$ . An automorphism of  $(\mathcal{C}/S, \vec{\sigma})$  is an automorphism of  $\mathcal{C}/S$  fixing the  $S$ -sections. The  $n$ -marked  $S$ -curve  $\mathcal{C}$  is said to be stable if the automorphism groups of its fibers are all finite. The moduli stack of stable  $n$ -marked curves is written  $\overline{\mathfrak{M}}_{g,n}$ .

In a similar fashion we can define the moduli space of stable maps into a fixed target (see [FP97] for a convenient reference). In this paper we'll be interested in the case where this target is  $\mathbb{P}^1$ . A stable  $S$ -map with target  $\mathbb{P}^1$  and degree  $d$  is a degree  $d$  projective  $S$ -morphism from a prestable  $n$ -marked  $S$ -curve into  $\mathbb{P}^1$  whose fibers have finite automorphism groups, equivalently, each rational contracted component of the domain has at least 3 special points, that are either marked or nodal. The moduli space of stable maps is written  $\overline{\mathfrak{M}}_{g,n}(\mathbb{P}^1, d)$ . We have on  $\overline{\mathfrak{M}}_{g,n}(\mathbb{P}^1, d)$  evaluations maps  $ev_1, \dots, ev_n$  which send a stable  $\mathbb{k}$ -map  $\phi$  on its value at each marked point. Details about the moduli space of stable maps can be found in [FP97]. We write  $\overline{\mathfrak{M}}_{g,n}(\mathbb{P}^1, d, \infty)$  for the closed substack of  $\overline{\mathfrak{M}}_{g,n}(\mathbb{P}^1, d)$  corresponding to the locus

$$\left( \prod_{i=1}^n ev_i \right)^{-1} \underbrace{(\infty, \dots, \infty)}_{\in (\mathbb{P}^1)^n}.$$

The group scheme of automorphisms of  $\mathbb{P}^1$  fixing the point at infinity acts naturally on  $\overline{\mathfrak{M}}_{g,n}(\mathbb{P}^1, d, \infty)$ . This group is isomorphic to  $\mathbb{G}_a \times \mathbb{G}_m$  and  $(\lambda, \alpha) \in \mathcal{O}_S^* \times \mathcal{O}_S$  acts on  $\mathbb{P}_S^1$  by sending  $[t_0; t_\infty]$  on  $[\lambda t_0 + \alpha t_\infty; t_\infty]$ . The induced action on  $\overline{\mathfrak{M}}_{g,n}(\mathbb{P}^1, d, \infty)$  is given with the obvious notation by sending an  $S$ -map  $\phi: \mathcal{C} \rightarrow \mathbb{P}_S^1$  on  $\lambda\phi + \alpha$ .

**Definition 1.1.** We set  $\overline{\mathfrak{M}}_{g,n}(\mathbb{P}^1, \vec{k})$  for the category fibered in groupoids given for  $S \in \text{Sch}/\mathbb{k}$  by  $S$ -maps  $\phi: (\mathcal{C}/S, \vec{\sigma}) \rightarrow \mathbb{P}_S^1$  in  $\overline{\mathfrak{M}}_{g,n}(\mathbb{P}^1, d, \infty)$  satisfying scheme theoretically

$$(1.1) \quad \phi^{-1}\infty = \sum_{i=1}^n k_i \mathcal{D}_i$$

where  $\infty$  is the  $S$ -section at infinity of  $\mathbb{P}_S^1$ .

This category fibered in groupoids is indeed a substack of  $\overline{\mathfrak{M}}_{g,n}(\mathbb{P}^1, d, \infty)$ . The condition 1.1 expressed previously is obviously local, stable under any base change and étale descent. It is stable as well under the action of  $\mathbb{G}_a \times \mathbb{G}_m$  on  $\overline{\mathfrak{M}}_{g,n}(\mathbb{P}^1, d, \infty)$ .

**Proposition 1.1.** *The substack  $\overline{\mathfrak{M}}_{g,n}(\mathbb{P}^1, \vec{k})$  is a locally closed substack in  $\overline{\mathfrak{M}}_{g,n}(\mathbb{P}^1, d, \infty)$ . In particular, it is an algebraic stack.*

*Proof.* Let  $\phi: \mathcal{C} \rightarrow \mathbb{P}_S^1$  be an  $S$ -object in  $\overline{\mathfrak{M}}_{g,n}(\mathbb{P}^1, d, \infty)$ . Call  $\pi: \mathcal{C} \rightarrow S$  the projection on  $S$ . We are going to show that the locus of  $S$  given by points having fibers in  $\overline{\mathfrak{M}}_{g,n}(\mathbb{P}^1, \vec{k})$  is locally closed. The condition  $\sum k_i \mathcal{D}_i \subset \phi^{-1}(\infty)$  is closed. Indeed we can see  $\phi$  as given by a surjective map  $(s_0, s_\infty): \mathcal{O}_{\mathcal{C}}^2 \rightarrow \mathcal{M}$ , then the condition is equivalent to  $s_\infty = 0$ . Next the condition for  $\phi^{-1}(\infty) \rightarrow S$  to be quasi-finite, then finite, is open on  $S$ , as a consequence of the Zariski main theorem. Finally assuming  $\phi^{-1}(\infty) \rightarrow S$  finite, the condition  $\phi^{-1}(\infty) = \sum_i k_i \mathcal{D}_i$  is closed on the base.  $\square$

**1.2. Bubbly curves.** We have a natural forgetful morphism from  $\overline{\mathfrak{M}}_{g,n}(\mathbb{P}^1, \vec{k})$  to the stack of prestable  $n$ -marked curves  $\mathfrak{M}_{g,n}^{pre}$  that sends an  $S$ -map in  $\overline{\mathfrak{M}}_{g,n}(\mathbb{P}^1, \vec{k})$  on the source  $S$ -curve together with its marked points. This section is devoted to the study of the prestable curves appearing in the image of this forgetful morphism.

We shall write  $(C, \vec{p})$  for a prestable  $\mathbb{k}$ -curve  $C$  marked by  $p_1, \dots, p_n$ . An irreducible component  $E$  of  $C$  has a natural marking induced by  $(C, \vec{p})$ : points of  $\vec{p}$  that lie in  $E$  and branches of the nodes of  $C$  lying on  $E$ . Call these points special points of  $S$ . Such data is

called an irreducible component of  $(C, \vec{p})$ . An irreducible component of  $(C, \vec{p})$  is said to be stable if it is a stable marked curve with respect to this induced marking. It is called unstable when this is not the case.

**Proposition 1.2.** *Let  $\phi : C \rightarrow \mathbb{P}^1$  be a stable map of profile  $\vec{k}$  over  $\mathbb{k}$ . Then  $(C, \vec{p})$  is a prestable curve that has only stable irreducible components except for rational components having two special points only one of which is a marking.*

*Proof.* Since our curves are marked by at least one point ( $n > 0$ ) the only unstable irreducible components of  $(C, \vec{p})$  are rational components having less than 3 special points. If such a rational component has no marked points this in turn means that  $\phi$  avoids  $\infty$  on this component and thus contracts it. We can therefore assume such rational components have at least one marked point. But then such a component can't have two marked points. Indeed, because of connectedness it would mean this component is the whole of  $C$ , i.e.  $C$  is a genus 0 curve with only 2 marked points.  $\square$

**Definition 1.2.** A prestable  $n$ -marked  $\mathbb{k}$ -curve is called a bubbly  $n$ -marked  $\mathbb{k}$ -curve if it has only stable components except for bubble components, i.e., rational components having two special points only one of which is a marked point. By extension we say that a prestable  $n$ -marked  $S$ -curve is a bubbly  $n$ -marked  $S$ -curve if each of its fibers is a bubbly  $n$ -marked  $\mathbb{k}$ -curve.

Thus bubbly  $n$ -marked  $S$ -curves are the prestable marked curves underlying a stable  $S$ -map of profile  $\vec{k}$ .

**Proposition 1.3.** *Let  $(C, \vec{p})$  be a prestable  $\mathbb{k}$ -curve. We have equivalence between the following*

- (1)  $(C, \vec{p})$  is a bubbly curve,
- (2)  $\omega_C(\sum_{i=1}^n 2p_i)$  is ample,
- (3) there exists integers  $m_1, \dots, m_n$  such that each  $m_\ell \geq 2$  and  $\omega_C(\sum_{i=1}^n m_i p_i)$  is ample.

*Proof.* It is clear that (2)  $\Rightarrow$  (3). We are going to show that (3)  $\Rightarrow$  (1)  $\Rightarrow$  (2) and this will end the proof. Let  $\{C_\ell \mid 1 \leq \ell \leq v\}$  be the set of irreducible components of  $C$  and write  $\omega_\ell(\sum_{i=1}^n m_i p_i)$  for the restriction of  $\omega_C(\sum_{i=1}^n m_i p_i)$  to  $C_\ell$ . The degree of  $\omega_\ell(\sum_{i=1}^n m_i p_i)$  is given by

$$(1.2) \quad \deg\left(\omega_\ell\left(\sum_{i=1}^n m_i p_i\right)\right) = 2g_{C_\ell} - 2 + \left(\sum_{\{i \mid p_i \in C_\ell\}} m_i\right) + \#\{e \in C_\ell \mid e \text{ is a nodal point of } C\}.$$

Assume  $\omega_C(\sum_{i=1}^n m_i p_i)$  is ample. Each of its restrictions  $\omega_\ell(\sum_{i=1}^n m_i p_i)$  is also ample. This means that  $\omega_\ell(\sum_{i=1}^n m_i p_i)$  has positive degree for each  $\ell$  ranging in  $\{1, \dots, v\}$ . But equation 1.2 shows it is exactly the case for bubbly  $n$ -marked  $\mathbb{k}$ -curves. Now assume  $(C, \vec{p})$  is a bubbly curve and assume that  $m_1 = \dots = m_n = 2$  in equation 1.2. As soon as the genus of  $C_\ell$  is non-zero the degree given by the left hand side of this equation is positive. If  $C_\ell$  is rational it must have at least two special points one of which is a marking. The marking contributes for a 2 and the other special point for 1. This guarantees the positivity of the degree on  $C_\ell$  in any case and  $\omega_C(\sum_{i=1}^n m_i p_i)$  is indeed ample.  $\square$

“Being a bubbly curve” is stable under any base change. This is due to the fact that  $\omega_{\mathcal{C}/S}(\sum_{i=1}^n 2\mathcal{D}_i)$  commutes with any base change.

*Notation.* Following the previous remark we call  $\mathfrak{B}_{g,n}$  the stack of bubbly  $n$ -marked curves.

**Proposition 1.4.** *The stack  $\mathfrak{B}_{g,n}$  is an open substack of  $\mathfrak{M}_{g,n}^{pre}$  and hence is an algebraic stack.*

*Proof.* If  $(\mathcal{C}/S, \vec{\sigma})$  is a  $\mathbb{k}$ -point of  $\mathfrak{B}_{g,n}$ , a local chart around this point in the stack  $\mathfrak{M}_{g,n}^{pre}$  is provided by the versal formal deformation. We know from the Grothendieck existence theorem that this formal deformation can be algebraized and that the corresponding line bundle  $\omega(\sum \mathcal{D}_i)$  is ample. Said differently the relative ampleness at a point is indeed true on a neighborhood of this point.  $\square$

We have a forgetful morphism of algebraic stacks  $\overline{\mathfrak{M}}_{g,n}(\mathbb{P}^1, \vec{k}) \rightarrow \mathfrak{B}_{g,n}$  that sends an  $S$ -map in  $\overline{\mathfrak{M}}_{g,n}(\mathbb{P}^1, \vec{k})$  to its underlying source marked curve.

**1.3. Polar Curves.** In this section we study the structure of bubbly  $n$ -marked  $S$ -curves  $(\mathcal{C}/S, \vec{\sigma})$  enriched with global sections of  $\mathcal{P}_{\mathcal{C}, \vec{k}} = \pi_* \mathcal{O}_{\mathcal{C}}(\sum_{i=1}^n k_i \mathcal{D}_i) / \mathcal{O}_{\mathcal{C}}$ . Unless stated otherwise, a curve means a bubbly  $n$ -marked curve.

Let's begin by giving a description of the stalks of  $\mathcal{P}_{\mathcal{C}, \vec{k}}$  at a point  $s \in S$ . The normal sheaf  $\mathcal{N}_{\sum_i k_i \mathcal{D}_i}$  is a direct sum  $\oplus_{i=1}^n \mathcal{N}_{k_i \mathcal{D}_i}$ , consequently  $\mathcal{P}_{\mathcal{C}, \vec{k}}$  is the direct sum

$$\mathcal{P}_{\mathcal{C}, \vec{k}} = \bigoplus_{i=1}^n \mathcal{P}_{\mathcal{C}, k_i}.$$

We thus limit our interest to the case where  $\vec{k}$  has only one component and drop the indices. Let  $s$  be a point in  $S$ . The stalk of  $\mathcal{O}_{\mathcal{C}}(k\mathcal{D}) / \mathcal{O}_{\mathcal{C}}$  at a point  $x \in \mathcal{C}$  over  $s$  is zero unless  $x$  is supported on  $\mathcal{D}$ . Since  $\mathcal{D}$  is the image of a section  $\sigma$  of  $\pi$  there is only one such point namely  $x = \sigma(s)$ . Let now  $f$  be an equation of  $\mathcal{D}$  in  $\mathcal{O}_x$ , since  $\mathcal{D}$  is étale over  $S$  and is the image of an  $S$ -section of  $\pi$  the module  $\mathcal{O}_x / f \mathcal{O}_x$  is naturally identified with  $\mathcal{O}_s$ . We have  $(\mathcal{N}_{k\mathcal{D}})_x = (f^{-k} \mathcal{O}_x) / \mathcal{O}_x$ . Since  $f$  is a non-zero divisor in  $\mathcal{O}_x$  the induced  $\mathcal{O}_s$ -module  $\mathcal{P}_{\mathcal{C}, k, s}$  is a direct sum

$$\mathcal{P}_{\mathcal{C}, k, s} = \left( \frac{f^{-k} \mathcal{O}_{\sigma(s)}}{\mathcal{O}_{\sigma(s)}} \right)_{\mathcal{O}_s} = \bigoplus_{\ell=1}^k f^{-\ell} \mathcal{O}_s.$$

This means that the stalk of a section  $\rho$  of  $\mathcal{P}_{\mathcal{C}, k}$  on  $U \subset S$  at a point  $s \in U$  can be uniquely written as

$$(1.3) \quad \rho_s = \frac{a_k}{f^k} + \frac{a_{k-1}}{f^{k-1}} + \cdots + \frac{a_1}{f}$$

where each  $a_\ell$  for  $\ell \in \{1, \dots, k\}$  are elements in  $\mathcal{O}_s$ . Thus each section  $\mathcal{P}_{\mathcal{C}, k_i}$  is called a polar part along  $(\mathcal{C}, \sigma_i)$  and more generally each section of  $\mathcal{P}_{\mathcal{C}, \vec{k}}$  a polar part along  $(\mathcal{C}, \vec{\sigma})$ . In the following when this brings no confusion we shall write polar part for polar part along  $(\mathcal{C}, \vec{\sigma})$ . The collection of sheaves of polar parts  $\mathcal{P}_{-, \vec{k}}$  defines obviously a locally free sheaf  $\mathcal{P}_{g, \vec{k}}$  of rank  $d$  on  $\mathfrak{B}_{g,n}$ .

The coefficients  $a_\ell$  for  $\ell \in \{1, \dots, n\}$  appearing in equation 1.3 depend on the choice of a local coordinate  $f$ . However unlike other coefficients  $a_k$  is of global nature. Indeed, consider the quotient map

$$\frac{\mathcal{O}_{\mathcal{C}}(k\mathcal{D})}{\mathcal{O}_{\mathcal{C}}} \longrightarrow \frac{\mathcal{O}_{\mathcal{C}}(k\mathcal{D})}{\mathcal{O}_{\mathcal{C}}((k-1)\mathcal{D})}$$

The right hand side is a sheaf supported on  $\mathcal{D}$  and thus can be identified with the  $\mathcal{O}_{\mathcal{D}}$ -module  $\mathcal{O}_{\mathcal{D}}(k\mathcal{D})$ , its pushforward  $\pi_* \mathcal{O}_{\mathcal{D}}(k\mathcal{D})$  is exactly  $\mathcal{L}^{-k}$  where  $\mathcal{L}$  is the tautological invertible sheaf along  $\mathcal{D}$  (i.e. the conormal sheaf along  $\mathcal{D}$ ). Since the kernel of the previous quotient map  $\mathcal{O}_{\mathcal{C}}((k-1)\mathcal{D}) / \mathcal{O}_{\mathcal{C}}$  is supported on  $\mathcal{D}$  the push forward by  $\pi$  gives a surjective morphism

$$\mathcal{P}_{\mathcal{C}, k} \xrightarrow{\partial_{\mathcal{C}, k}} \mathcal{L}^{-k}$$

It is clear that locally around  $s$  and in terms of  $f$  which trivializes  $\mathcal{L}$  the stalk of  $\vartheta_{\mathcal{C},k}(\varrho)$  at  $s$  is given by  $a_k$ . We will write  $\vartheta_{\mathcal{C},\vec{k}}$  for the direct sum  $\bigoplus_{i=1}^n \vartheta_{\mathcal{C},k_i}$ .

**Definition 1.3.** Let  $\vec{\varrho} = (\varrho_{k_1}, \dots, \varrho_{k_n})$  be a polar part in  $\mathcal{P}_{\mathcal{C},\vec{k}}$  defined on an open subset  $U$  of  $S$ . The polar part  $\vec{\varrho}$  is said to be of order  $k_i$  at  $s \in U$  along  $\mathcal{D}_i$  if  $\vartheta_{\mathcal{C},k_i}(\varrho_i)_s$  is invertible. We say that  $\vec{\varrho}$  is of order  $\vec{k}$  at  $s \in U$  if it is of order  $k_i$  at  $s$  along  $\mathcal{D}_i$  for  $i \in \{1, \dots, n\}$ . It is of order  $\vec{k}$  if it is of order  $\vec{k}$  at each point of  $U$ .

Saying that a polar part  $\vec{\varrho}$  on an open subset  $U \subset S$  is of order  $k_i$  at a point  $s \in U$  is equivalent to  $\vartheta_{\mathcal{C},k_i,s}(\varrho_i) \otimes \mathbb{k}(s)$  being nonzero in  $\mathcal{L}_i^{-k_i} \otimes \mathbb{k}(s)$ . Thus it is of order  $k_i$  if  $\vartheta_{\mathcal{C},k_i}(\varrho)$  trivializes  $\mathcal{L}_i^{-k_i}$  on  $U$ . Thus, given a morphism  $\phi \in \text{Hom}(T, S)$  and a polar part  $\vec{\varrho} \in \mathcal{P}_{\mathcal{C},\vec{k}}$  of order  $\vec{k}$  the pullback  $\phi^* \vec{\varrho}$  is a polar part of order  $\vec{k}$  as well.

One can easily check that  $\vartheta_{\mathcal{C},\vec{k}}$  commutes with base change. In particular, the collection  $\vartheta_{-, \vec{k}}$  defines a map of  $\mathcal{O}_{\mathcal{B}_{g,n}}$ -modules  $\vartheta_{g,\vec{k}}$  from  $\mathcal{P}_{g,\vec{k}}$  on  $\bigoplus_{i=1}^n \mathcal{L}_{g,i}^{-k_i}$ .

**Definition 1.4** (Polar curve). A curve marked by a collection of polar parts—in short a polar curve—over a scheme  $S \in \text{Sch}/\mathbb{k}$  is the data given by a bubbly  $n$ -marked  $S$ -curve  $(\mathcal{C}/S, \vec{\sigma})$  together with a polar part  $\vec{\varrho}$ , a global section of  $\mathcal{P}_{\mathcal{C},\vec{k}}$  of order  $\vec{k}$  along the fibers, i.e., a polar part of exact order  $k_i$  along  $\sigma_i$ . This data is written  $(\mathcal{C}/S, \vec{\sigma}, \vec{\varrho})$ .

A morphism  $(\mathcal{C}/S, \vec{\sigma}, \vec{\varrho}) \rightarrow (\mathcal{B}/S, \vec{\tau}, \vec{\chi})$  means a morphism of pointed curves preserving the polar parts along the sections. Because of the invariance under any base change of the order of a polar part, “being a polar curve” is local on the base and stable under any base change. The category of polar curves as described previously is a category fibered in groupoids over  $\text{Sch}/\mathbb{k}$  denoted  $\mathfrak{P}_{g,\vec{k}}$ . A polar  $S$ -curve is given by a couple of objects, one of which is a bubbly  $S$ -curve, the other being a global section of a locally free sheaf  $\mathcal{P}_{g,\vec{k}}$  on  $S$ . The global section in question satisfies a condition expressed in term of  $\vartheta_{g,\vec{k}}$ . This is an instance of a wider situation we shall only develop for the case  $n = 1$ , the general case is a straightforward variation of this one.

Let  $\mathfrak{M}$  be an algebraic stack together with a morphism of locally free coherent  $\mathcal{O}_{\mathfrak{M}}$ -modules  $\vartheta: \mathcal{P} \rightarrow \mathcal{F}$  where  $\mathcal{F}$  is of rank one. Let  $\varphi \in \text{Hom}(T, S)$  and  $\phi$  be a morphism from  $\xi \in \mathfrak{M}(T)$  on  $\zeta \in \mathfrak{M}(S)$  over  $\varphi$ , we write  $\alpha_\phi$  (resp.  $\beta_\phi$ ) for the induced isomorphism from  $\varphi^* \zeta^* \mathcal{P}$  on  $\xi^* \mathcal{P}$  (resp.  $\varphi^* \zeta^* \mathcal{F}$  on  $\xi^* \mathcal{F}$ ). Consider the category  $\mathfrak{P}_{\mathfrak{M}}$  whose

- objects over a scheme  $S \in \text{Sch}/\mathbb{k}$  are given by couples  $(\zeta, s)$  where  $\zeta$  is an object  $S \rightarrow \mathfrak{M}$  and  $s \in \Gamma(S, \zeta^* \mathcal{P})$  such that  $\zeta^* \vartheta(s)$  trivializes  $\zeta^* \mathcal{F}$
- morphisms of  $\mathfrak{P}_{\mathfrak{M}}$  over a morphism  $\varphi \in \text{Hom}(T, S)$  are given by morphisms  $\phi$  from  $\xi \in \mathfrak{M}(T)$  to  $\zeta \in \mathfrak{M}(S)$  such that  $\alpha_\phi(\varphi^* s) = t$ .

The morphism of  $\mathfrak{P}_{\mathfrak{M}}$  on  $\text{Sch}/\mathbb{k}$  is the obvious one. This defines a category fibered in groupoid. Indeed, let  $(\zeta, s)$  be an object of  $\mathfrak{P}_{\mathfrak{M}}$  over  $S$ ,  $\xi$  an object of  $\mathfrak{M}$  over  $T$  and  $\phi$  a morphism over  $\varphi \in \text{Hom}(T, S)$ . The  $T$ -section  $(\xi^* \vartheta) \alpha_\phi(\varphi^* s)$  trivializes  $\xi^* \mathcal{F}$  since

$$\xi^* \vartheta \circ \alpha_\phi = \beta_\phi \circ \varphi^* \zeta^* \vartheta.$$

Thus the couple  $(\xi, \alpha_\phi(\varphi^* s))$  defines a  $T$ -object of  $\mathfrak{P}_{\mathfrak{M}}$ .

**Proposition 1.5.** *The category fibered in groupoids  $\mathfrak{P}_{\mathfrak{M}}$  is an algebraic stack.*

*Proof.* Let's first show it is a stack. Let  $(\zeta, s)$  and  $(\xi, t)$  be objects of  $\mathfrak{P}_{\mathfrak{M}}$  over  $S \in \text{Sch}/\mathbb{k}$ . We have a natural forgetful morphism  $\delta: \mathbf{Isom}_S(\zeta, \xi) \rightarrow S$ . By pullback through  $\phi$  we get objects  $\delta^* \zeta$  and  $\delta^* \xi$  over  $\mathbf{Isom}_S(\zeta, \xi)$  together with sections  $\delta^* s$  and  $\delta^* t$  of the corresponding pullbacks. The sheaf  $\mathcal{S}om_S((\zeta, s), (\xi, t))$  is represented by the subscheme of  $\mathbf{Isom}_S(\zeta, \xi)$  defined by  $\delta^* s = \delta^* t$ . Now descent data along an étale covering  $\{U_\ell \rightarrow S\}_\ell$  attached to objects  $(\zeta_\ell, s_\ell) \in \mathfrak{P}_{\mathfrak{M}}$  gives naturally descent data for  $\zeta_\ell$  along  $\{U_\ell \rightarrow S\}_\ell$  for  $\mathfrak{M}$ .

By hypothesis this last descent data is effective and thus descends to an object  $\zeta \in \mathfrak{M}(S)$ . The sections  $s_\ell$  define descent data for the quasi-coherent sheaf  $\zeta^* \mathfrak{P}_{\mathfrak{M}}$ , they glue together because of effectiveness of descent for quasi-coherent sheaves to give a section  $s$  on  $\zeta^* \mathfrak{P}_{\mathfrak{M}}$ .

Now let's come to algebraicity. If we drop the last condition, what we get is an  $S$ -section of the vector bundle  $\mathbb{V}(\xi^* \mathcal{P}) = \text{Spec}(\text{Sym}(\mathcal{P}^\vee))$  over  $\mathfrak{M}$ . This vector bundle is an algebraic stack. Moreover the surjection  $\vartheta : \mathcal{P} \rightarrow \mathcal{F}$  defines a morphism of vector bundles  $\theta : \mathbb{V}(\mathcal{P}) \rightarrow \mathbb{V}(\mathcal{F})$ . The last condition simply means that the image  $\theta(s)$  is in the complement of the zero section of  $\mathcal{F}$ , which in turn yields  $\mathfrak{P}_{\mathfrak{M}} = \theta^{-1}(\mathbb{V}(\mathcal{F}) \setminus \{0\})$ .  $\square$

**Corollary 1.6.** *The category  $\mathfrak{P}_{g, \vec{k}}$  is an algebraic stack.*

*Proof.* This is a consequence of 1.5 and 1.4.  $\square$

We now come to the study of automorphism groups of polar curves.

**Proposition 1.7.** *A polar  $\mathbb{k}$ -curve  $(C, \vec{p}, \vec{\varrho})$  has a finite group of automorphism.*

*Proof.* It is enough to look at bubble components of  $(C, \vec{p}, \vec{\varrho})$ . Indeed, a bubbly curve has infinite automorphism group if and only if it contains bubble components. Let  $(E, p, e)$  be a bubble component of  $(C, \vec{p})$  where  $p$  comes from a marked point of  $C$ . Since  $E$  is rational there exists an isomorphism of marked curves from  $(E, p, e)$  on  $(\mathbb{P}^1, \infty, 0)$ . Given two coordinates  $t_0$  and  $t_\infty$  on  $\mathbb{P}^1$  an automorphism  $\phi$  of  $(\mathbb{P}^1, \infty, 0)$  is given by an element  $\lambda \in \mathbb{G}_m$ . The global polar part  $\varrho_k$  is given by its local behaviour at  $\infty$ . Using the local chart  $\mathbb{A}^1 \setminus \{0\}$  at infinity with coordinate  $t = t_\infty / t_0$  we have

$$\varrho_k = \sum_{\ell=1}^k \frac{a_\ell}{t^\ell}, \quad \text{for } a_k \neq 0.$$

Using the description of  $\mathcal{P}(\phi)(\varrho_k)$  we have that

$$\mathcal{P}(\phi)(\varrho_k) = \varrho_k \iff \forall \ell \in \{1, \dots, k\}, \quad a_\ell = a_\ell \lambda^\ell.$$

For  $\ell$  equal to  $k$  this gives  $a_k \lambda^k = a_k$ . Since  $a_k \neq 0$ ,  $\lambda$  is a root of unity and thus the group of automorphism of such a component is finite.  $\square$

*Remark 1.* Conversely marked prestable curves together with a polar part at each marked point and a finite automorphism group are precisely the polar curves. Following proposition 1.3 this is exactly the locus of points  $(C, \vec{p}, \vec{\varrho})$  in  $\mathbb{V}(\mathcal{P}_{g, \vec{k}})$  such that  $\omega_C(\sum_{i=1}^n 2p_i)$  is ample.

**Proposition 1.8.** *The stack  $\mathfrak{P}_{g, \vec{k}}$  is an algebraic tame stack in the sense of [AOV08]. It is Deligne–Mumford if  $\mathbf{p}$  doesn't divide  $k_i$  for each  $i \in \{1, \dots, n\}$ .*

*Proof.* The stack  $\mathfrak{P}_{g, \vec{k}}$  is obtained as a fiber product of algebraic stacks. It is therefore algebraic. For  $\mathfrak{P}_{g, \vec{k}}$  to be a Deligne–Mumford stack it is equivalent to show that its diagonal morphism is formally unramified [DM69, §4]. Since  $\mathfrak{B}_{g, n}$  is locally of finite presentation and  $\mathfrak{P}_{g, \vec{k}} \rightarrow \mathfrak{B}_{g, n}$  as well, so is the case  $\mathfrak{P}_{g, \vec{k}}$ . Hence, it is enough to show that closed fibers of the diagonal are discrete and reduced. As these fibers are given by isomorphism groups of objects of  $\mathfrak{P}_{g, \vec{k}}$ , they are quasi-finite. To show they're reduced it is enough to check that automorphism groups of objects of  $\mathfrak{P}_{g, \vec{k}}$  over  $\text{Spec } \mathbb{k}[\epsilon]/(\epsilon^2)$  are trivial. Let  $(\mathcal{C}_\epsilon, \vec{\sigma}, \vec{\varrho})$  be an object of  $\mathfrak{P}_{g, \vec{k}}$  over  $\text{Spec } \mathbb{k}[\epsilon]/(\epsilon^2)$ . An automorphism of  $(\mathcal{C}_\epsilon, \vec{\sigma})$  is equivalently given by an infinitesimal vector field fixing  $\vec{\sigma}$ . It is well known that stable curves have no such vector fields ([DM69, 1.4]). Hence it is enough to check the case of rational unstable components of  $(\mathcal{C}_\epsilon, \vec{\sigma})$ . Let  $(E, p, e)$  be such a component. The space of infinitesimal automorphisms of this data is equal to  $H^0(T_E(-p - e))$ ; its dimension is

1. Fix an isomorphism of  $(E, p, e)$  with  $(\mathbb{P}^1, \infty, 0)$  and let  $t$  be a local parameter for  $p$  in  $\mathbb{P}^1$ . Let  $\partial$  be a vector field on  $(E, p, e)$ , up to multiplication it is given by the vector field  $t \frac{\partial}{\partial t}$  on  $\mathbb{A}_t^1$ . Let  $\lambda \in \mathbb{k}$  be such that  $\partial = \lambda t \frac{\partial}{\partial t}$ . The condition on  $\partial$  relative to  $\varrho_{k_p}$  can be tested locally around  $p$ . The polar part  $\varrho_{k_p}$  is equal in the  $\mathbb{A}_t^1 = \text{Spec } \mathbb{k}[t]$  neighborhood of  $p$  to

$$\varrho_{k_p} = \sum_{\ell=1}^{k_p} \frac{a_\ell}{t^\ell}.$$

The restriction of  $\partial$  to this neighborhood gives an automorphism  $\phi = \text{id} + \epsilon \partial$  of  $(\mathbb{A}_t^1, p)$ . The restriction on  $\partial$  is equivalent to the fact that this local automorphism should fix  $\varrho_{k_p}$ . Explicitly, this is written

$$\sum_{\ell=1}^{k_p} \frac{a_\ell}{t^\ell} = \sum_{\ell=1}^{k_p} \frac{a_\ell}{t^\ell + \epsilon \partial(t^\ell)} = \sum_{\ell=1}^{k_p} \frac{a_\ell}{t^\ell (1 + \lambda \ell \epsilon)}$$

which gives the relation

$$\sum_{\ell=1}^{k_p} \frac{a_\ell}{t^\ell} = \sum_{\ell=1}^{k_p} \frac{a_\ell}{t^\ell} - \lambda \epsilon \sum_{\ell=1}^{k_p} \frac{\ell a_\ell}{t^\ell}.$$

Now we can see that  $\partial = 0$  is equivalent to the fact that there exists a non-zero  $a_\ell$  for an  $\ell$  which is not a multiple of  $p$ . Equivalently  $\partial \neq 0$  if and only if  $\ell$  is a multiple of  $p$  for each non-zero  $a_\ell$ , i.e.  $\varrho$  is a power of  $p$ . In this worst case scenario the automorphism group of  $\varrho$  is isomorphic to some  $\mu_{p^m}$ , which is a linearly reductive group scheme and the result follows from [AOV08, 3.2].

Since  $a_{k_p}$  is never 0 following what has been said, it is enough for  $\partial$  to be zero to have that  $k_p$  is not a multiple of  $p$ . In this case we get a Deligne–Mumford stack.  $\square$

**1.4. Stack Cone Structure of  $\mathfrak{B}_{g, \vec{k}}$  over  $\overline{\mathfrak{M}}_{g, n}$ .** Given a prestable marked curve there is a contraction sending it to its marked stable model (see [Knu83, 1.6]). Restriction to bubbly curves gives a morphism  $\mathfrak{c} : \mathfrak{B}_{g, n} \rightarrow \overline{\mathfrak{M}}_{g, n}$ . On the level of bubbly  $\mathbb{k}$ -curves it contracts an unstable component on a point and keeps track of this point. Contraction induces a  $\mathbb{G}_m$ -invariant morphism  $\mathfrak{B}_{g, \vec{k}} \rightarrow \overline{\mathfrak{M}}_{g, n}$ , we aim at understanding the local structure of this last morphism. It is clear that a good understanding of  $\mathfrak{c}$  is needed for our purpose. We'll be thus looking at fibers of  $\mathfrak{c}$  for objects in  $\overline{\mathfrak{M}}_{g, n}$  over relative schemes in the topology of  $\overline{\mathfrak{M}}_{g, n}$ . In our case Zariski open subsets will be fine.

**Lemma 1.9.** *Let  $(\mathcal{C}/S, \vec{\sigma})$  be an  $n$ -bubbly curve and let  $(\mathcal{C}^*/S, \vec{\sigma}^*)$  be its image by  $\mathfrak{c}$ . The contraction  $\mathfrak{c}$  induces a natural morphism  $\mathcal{N}_{k_i \mathcal{D}_i} \rightarrow \mathcal{N}_{k_i \mathcal{D}_i^*}$  for each  $i \in \{1, \dots, n\}$  and thus by direct sum a morphism  $\mathcal{N}_{\sum_{i=1}^n k_i \mathcal{D}_i} \rightarrow \mathcal{N}_{\sum_{i=1}^n k_i \mathcal{D}_i^*}$ .*

*Proof.* This is just a refinement of the case of a morphism of marked curves. We start by studying the single marked case. Since  $\mathfrak{c}$  commutes with  $\mathcal{D}$  and  $\mathcal{D}^*$  we have  $\mathcal{D} \subset \mathfrak{c}^{-1} \mathcal{D}^*$  and thus  $\mathfrak{c}$  defines a morphism of  $\mathcal{O}_{\mathcal{B}}$ -modules  $\mathfrak{c}^* \mathcal{O}_{\mathcal{C}^*}(-k \mathcal{D}^*) \rightarrow \mathcal{O}_{\mathcal{C}}(-k \mathcal{D})$ . Dualizing we get a morphism

$$\mathcal{O}_{\mathcal{C}}(k \mathcal{D}) \longrightarrow \mathcal{H}om_{\mathcal{O}_{\mathcal{C}}}(\mathfrak{c}^* \mathcal{O}_{\mathcal{C}^*}(-k \mathcal{D}^*), \mathcal{O}_{\mathcal{C}}).$$

which by adjunction gives a morphism

$$\mathfrak{c}_* \mathcal{O}_{\mathcal{C}}(k \mathcal{D}) \longrightarrow \mathcal{H}om_{\mathcal{O}_{\mathcal{C}^*}}(\mathcal{O}_{\mathcal{C}^*}(-k \mathcal{D}^*), \mathfrak{c}_* \mathcal{O}_{\mathcal{C}}).$$

The right hand side is isomorphic to  $\mathcal{O}_{\mathcal{C}^*}(k \mathcal{D}^*)$  for  $\mathfrak{c}_* \mathcal{O}_{\mathcal{C}} = \mathcal{O}_{\mathcal{C}^*}$ . We therefore get the desired morphism from  $\mathfrak{c}_* \mathcal{N}_{k \mathcal{D}}$  on  $\mathcal{N}_{k \mathcal{D}^*}$ . The general case is now straightforward.  $\square$

Let  $(\mathcal{C}/S, \vec{\sigma})$  be an  $n$ -bubbly marked curve and let  $(\mathcal{C}^*/S, \vec{\sigma}^*)$  be the stabilisation of  $(\mathcal{C}/S, \vec{\sigma})$ . We write  $\mathcal{D}_i$  and  $\mathcal{D}_i^*$  for the respective images of the  $i^{\text{th}}$  components of  $\vec{\sigma}$  and  $\vec{\sigma}^*$ . The bubbly curve  $(\mathcal{C}/S, \vec{\sigma})$  can be reconstructed as  $\mathbb{P}(\mathbf{c}_* \mathcal{O}_{\mathcal{C}}(\sum_{i=1}^n \mathcal{D}_i))$ . The section defining  $\mathcal{D}_i$  is given by the surjective morphism  $(\sigma^*)^* \mathbf{c}_* \mathcal{O}_{\mathcal{C}}(\mathcal{D}_i) \rightarrow \mathbf{c}_* \mathcal{N}_{\mathcal{D}_i}$ . It makes sense since the push forward  $\mathbf{c}_* \mathcal{N}_{\mathcal{D}_i}$  has support on  $\mathcal{D}_i^*$  and coincides with  $\pi_* \mathcal{N}_{\mathcal{D}_i}$  hence is invertible. These facts, including the surjectivity of the last morphism are treated in [Knu83, §2] and specifically in [Knu83, proof 2.4 case 2]. Using the proof of 1.9 we have a commutative diagram

$$(1.4) \quad \begin{array}{ccccccc} 0 & \longrightarrow & \mathcal{O}_{\mathcal{C}^*} & \longrightarrow & \mathbf{c}_* \mathcal{O}_{\mathcal{C}}(\sum_{i=1}^n \mathcal{D}_i) & \longrightarrow & \bigoplus_{i=1}^n \mathbf{c}_* \mathcal{N}_{\mathcal{D}_i} \longrightarrow 0 \\ & & \parallel & & \downarrow \hat{\alpha} & & \downarrow \alpha \\ 0 & \longrightarrow & \mathcal{O}_{\mathcal{C}^*} & \longrightarrow & \mathcal{O}_{\mathcal{C}^*}(\sum_{i=1}^n \mathcal{D}_i^*) & \longrightarrow & \bigoplus_{i=1}^n \mathcal{N}_{\mathcal{D}_i^*} \longrightarrow 0 \end{array}$$

where  $\hat{\alpha}$  and  $\alpha$  are the morphisms induced by  $\mathbf{c}$  in the proof of 1.9. The right exactness of the first row is due to the fact  $R^1 \mathbf{c}_* \mathcal{O}_{\mathcal{C}} = 0$ . Indeed,  $\mathbf{c}$  is a proper morphism such that  $\mathbf{c}_* \mathcal{O}_{\mathcal{C}} = \mathcal{O}_{\mathcal{C}^*}$  and has at most 1-dimensional rational fibers. Hence  $H^1(\mathbf{c}^{-1}(x), \mathcal{O}_{\mathbf{c}^{-1}(x)})$  is zero at any point  $x \in \mathcal{C}^*$  and using corollary 1.5 of [Knu83] we get  $R^1 \mathbf{c}_* \mathcal{O}_{\mathcal{C}} = 0$ . Diagram 1.4 is characterized by  $\alpha$ , universality of pullback of Yoneda extensions gives a canonical isomorphism of  $\mathbf{c}_* \mathcal{O}_{\mathcal{C}}(\mathcal{D})$  with  $\mathcal{O}_{\mathcal{C}^*}(\mathcal{D}^*) \times_{\mathcal{N}_{\mathcal{D}^*}} \mathbf{c}_* \mathcal{N}_{\mathcal{D}}$  commuting with  $\hat{\alpha}$ . The fiber of  $\mathbf{c}_*$  over  $(\mathcal{C}^*/S, \vec{\sigma}^*)$  is described by a stack whose objects are morphisms such as  $\alpha$ . This is the content of proposition 1.11.

*Notation.* Let  $(\mathcal{C}^*/S, \vec{\sigma}^*)$  be a stable marked curve. Given a map  $\alpha : \mathcal{R} \rightarrow \mathcal{N}_{\sum_{i=1}^n \mathcal{D}_i^*}$  of  $\mathcal{O}_{\mathcal{C}^*}$ -modules we write  $\mathcal{E}_\alpha$  for the fiber product

$$\mathcal{O}_{\mathcal{C}^*}(\sum_{i=1}^n \mathcal{D}_i^*) \times_{\mathcal{N}_{\sum_{i=1}^n \mathcal{D}_i^*}} \mathcal{R},$$

$\vec{\sigma}_\alpha$  for the S-sections of  $\mathbb{P}(\mathcal{E}_\alpha)$  induced by

$$((\sigma^*)^* \mathcal{E}_\alpha \rightarrow \mathcal{N}_{\sum_{i=1}^n \mathcal{D}_i^*})_{i=1}^n$$

and  $\mathcal{D}_{\alpha,i}$  for the image of each component of  $\vec{\sigma}_\alpha^*$ .

**Lemma 1.10.** *Let  $(\mathcal{C}^*/U, \sigma^*)$  be a single stable marked curve over an affine scheme  $U = \text{Spec}(\mathbf{A})$  and write  $\mathcal{D}^*$  for the image of  $\sigma$ . Consider a morphism  $\alpha : \mathcal{R} \rightarrow \mathcal{N}_{\mathcal{D}^*}$  where  $\mathcal{R}$  is a locally invertible sheaf and let  $\mathcal{E}_\alpha$  be the product  $\mathcal{O}_{\mathcal{C}^*}(\mathcal{D}) \times_{\mathcal{N}_{\mathcal{D}^*}} \mathcal{R}$ . Let  $f$  be a local equation for  $\mathcal{D}^*$  at the neighborhood  $V = \text{Spec}(\mathbf{B})$  of a point in the support of  $\mathcal{D}^*$ . Since  $\mathcal{D}^*$  is étale on  $U$  we identify  $\mathbf{B}/f\mathbf{B}$  with  $\mathbf{A}$  and write  $c_0(-)$  for the quotient  $\mathbf{B} \rightarrow \mathbf{A}$ . Then the restriction of  $\mathcal{E}_\alpha$  to  $V$ —written  $\mathbf{E}_\alpha$ —has a presentation*

$$0 \longrightarrow \mathbf{B} \xrightarrow{\begin{pmatrix} -\alpha \\ f \end{pmatrix}} \mathbf{B} \oplus \mathbf{B} \xrightarrow{\varphi} \mathbf{E}_\alpha \longrightarrow 0$$

where  $\varphi(b, b') = (fb + \alpha b', c_0(b'))$ . Thus we have

$$\text{Sym}_{\mathbf{B}} \mathbf{E}_\alpha = \frac{\mathbf{B}[x, y]}{(-\alpha x + f y)}.$$

*Proof.* The morphism  $\alpha$  is given in the local situation by an element in  $\mathbf{A}$  also written  $\alpha$ . By definition  $\mathbf{E}_\alpha$  is equal to  $\mathbf{B}f^{-1} \times_{\mathbf{B}} \mathbf{A}$ . This is exactly given by

$$\mathbf{E}_\alpha = \{(bf^{-1}, a) \in \mathbf{B}f^{-1} \times \mathbf{A} \mid c_0(b) = \alpha a\} \simeq \{(b, a) \in \mathbf{B} \times \mathbf{A} \mid c_0(b) = \alpha a\}.$$

Any element  $(b, a) \in E_\alpha$  can thus be written as  $(\alpha a + f b', a)$  for  $b' \in B$ , hence  $(b, a) = a(\alpha, 1) + b'(f, 0)$ . Since both  $(\alpha, 1)$  and  $(f, 0)$  are elements in  $E_\alpha$  they generate  $E_\alpha$  and define the above onto map  $\varphi$ . The kernel of  $\varphi$  is given by couples  $(b, b')$  such that  $b' = f b''$  and  $f b + \alpha f b'' = 0$  hence  $b = -\alpha b''$ . We finally get  $\text{Ker}(\varphi) = B(-\alpha, f)$ . The presentation of  $\text{Sym}_B E_\alpha$  is obtained by taking  $x = (1, 0)$  and  $y = (0, 1)$  thus  $\varphi(x) = (f, 0)$  and  $\varphi(y) = (\alpha, 1)$ .  $\square$

**Proposition 1.11.** *The fiber of  $\mathfrak{c} : \mathfrak{B}_{g,n} \rightarrow \overline{\mathfrak{M}}_{g,n}$  over an object  $(\mathcal{C}^* \xrightarrow{\pi^*} S, \bar{\sigma}^*)$  of  $\overline{\mathfrak{M}}_{g,n}$  is isomorphic to the stack whose*

- *objects over  $T \in \text{Sch}/S$  are given by  $n$ -tuples of morphisms  $(\alpha_i : \mathcal{R}_i \rightarrow \mathcal{N}_{\mathcal{D}_i^*})_{i=1}^n$  where each  $\mathcal{R}_i$  is a locally invertible sheaf on  $T$ ,*
- *morphisms over  $\phi \in \text{Hom}_S(T, T')$  are given for objects  $(\alpha_i : \mathcal{R} \rightarrow \mathcal{N}_{\mathcal{D}_i^*})_{i=1}^n$  and  $(\beta_i : \mathcal{S}_i \rightarrow \mathcal{N}_{\mathcal{D}_i^*})_{i=1}^n$  by  $n$  isomorphisms  $\mathcal{R}_i \simeq \phi^* \mathcal{S}_i$  commuting with  $\alpha_i$  and  $\beta_i$  for  $i \in \{1, \dots, n\}$ .*

*Proof.* Let's start with the single marked case, the general case will be straightforward once the single marked one is proved. The discussion preceding lemma 1.10 defines on the level of objects a functor of fibered categories  $F$  which sends a bubbly curve  $(\mathcal{C}/T, \sigma)$  for  $T \in \text{Sch}/S$  on a morphism  $F(\mathcal{C}, \sigma) = \alpha_\mathcal{C} : \mathfrak{c}_* \mathcal{N}_\mathcal{C} \rightarrow \mathcal{N}_{\mathcal{D}^*}$ . Let's define  $F$  on morphisms. We keep the notation introduced in the discussion preceding lemma 1.10. Let  $T$  be a scheme in  $\text{Sch}/S$  and let  $(\mathcal{C}/T, \sigma)$  and  $(\mathcal{B}/T, \tau)$  be two bubbly curves in the preimage  $\mathfrak{c}^{-1}(\mathcal{C}^*/S, \sigma^*)$ . These bubbly curves give rise to a map of  $\mathcal{O}_T$ -modules  $\alpha_\mathcal{C}$  and  $\alpha_\mathcal{B}$ . Looking back at the proof of lemma 1.9 it is clear an isomorphism of  $(\mathcal{C}/T, \sigma)$  on  $(\mathcal{B}/T, \tau)$  gives rise to an isomorphism of  $\mathfrak{c}_* \mathcal{N}_\mathcal{C}$  on  $\mathfrak{c}_* \mathcal{N}_\mathcal{B}$  commuting with  $\alpha_\mathcal{C}$  and  $\alpha_\mathcal{B}$ . Now take  $\phi \in \text{Hom}_S(T, T')$ , two bubbly curves  $(\mathcal{C}/T, \sigma)$  and  $(\mathcal{B}/T', \tau)$  and a morphism  $\Phi \in \text{Hom}((\mathcal{C}/T, \sigma), (\mathcal{B}/T', \tau))$  over  $\phi$ . By definition  $\Phi$  defines an isomorphism from  $(\mathcal{C}/T, \sigma)$  on  $(\phi^{-1}(\mathcal{B})/T, \phi^{-1}\tau)$ . We thus have an isomorphism of  $\mathfrak{c}_* \mathcal{N}_\mathcal{C}$  on  $\mathfrak{c}_* \mathcal{N}_{\phi^{-1}\mathcal{B}}$ . But following the discussion preceding lemma 1.10 the formation of  $\mathfrak{c}_* \mathcal{N}_{\phi^{-1}\mathcal{B}}$  commutes with any base change. We thus get an isomorphism  $\mu_\Phi$  of  $\mathfrak{c}_* \mathcal{N}_\mathcal{C}$  on  $\mathfrak{c}_* \mathcal{N}_{\phi^{-1}\mathcal{B}}$  commuting with  $\alpha_\mathcal{C}$  and  $\phi^* \alpha_\mathcal{B}$ . By writing  $F(\Phi) = \mu_\Phi$  we have defined a morphism from  $\mathfrak{c}^{-1}(\mathcal{C}^*/S, \sigma^*)$  on the category fibered in groupoids over  $\text{Sch}/S$  described in our assertion. It is straightforward to see that  $F$  is fully faithful since conversely a morphism from  $\alpha_\mathcal{C}$  to  $\alpha_\mathcal{B}$  over  $\phi$  defines a morphism from  $\mathfrak{c}_* \mathcal{O}_\mathcal{C}(\mathcal{D})$  on  $\mathfrak{c}_* \phi^* \mathcal{O}_\mathcal{B}(\mathcal{E}) = \phi^* \mathfrak{c}_* \mathcal{O}_\mathcal{B}(\mathcal{E})$  hence a  $\mathcal{C}^*$ -morphism from  $\mathbb{P}(\mathfrak{c}_* \mathcal{O}_\mathcal{C}(\mathcal{D}))$  on  $\phi^{-1}\mathbb{P}(\mathfrak{c}_* \mathcal{O}_\mathcal{B}(\mathcal{E}))$ .

It remains to show that each morphism of  $\mathcal{O}_T$ -modules  $\alpha : \mathcal{R} \rightarrow \mathcal{N}_{\mathcal{D}^*}$  gives rise to a bubbly  $T$ -curve in the preimage of  $(\mathcal{C}^*, \sigma^*)$  by  $\mathfrak{c}$ . Write  $(\mathcal{C}/T, \sigma)$  for  $(\mathbb{P}(\mathcal{E}_\alpha), \sigma_\alpha)$ . We need to show that  $\mathcal{E}_\alpha$  is isomorphic to  $\mathfrak{c}_* \mathcal{O}_\mathcal{C}(\mathcal{D})$  where  $\mathcal{D}$  is the image of  $\sigma$ . This is true if and only if  $\mathcal{R}$  is isomorphic to  $\mathfrak{c}_* \mathcal{N}_\mathcal{D}$ . Since  $\mathfrak{c}_* \mathcal{N}_\mathcal{D}$  is isomorphic to  $\sigma^* \mathcal{O}_\mathcal{C}(1)$  it is enough to show that  $\mathcal{O}_\mathcal{C}(1)$  is isomorphic to  $\mathcal{O}_\mathcal{C}(\mathcal{D})$ . The equation of  $\mathcal{D}$  is given by the global section  $\mathcal{O}_\mathcal{C} \rightarrow \mathfrak{c}^* \mathcal{E}_\alpha$  obtained out of the given one  $\mathcal{O}_{\mathcal{C}^*} \rightarrow \mathcal{E}_\alpha = \mathcal{O}_{\mathcal{C}^*} \times_{\mathcal{N}_{\mathcal{D}^*}} \mathcal{R}$ . The former gives naturally rise to a morphism  $\mathcal{O}_\mathcal{C} \rightarrow \mathcal{O}_\mathcal{C}(1)$  giving the section  $\mathcal{D}$ . We need to show that this last global section coincides with the canonical morphism  $\mathcal{O}_\mathcal{C} \rightarrow \mathcal{O}_\mathcal{C}(1)$ . This is local on both  $S$  and  $\mathcal{C}$ . Using notation of lemma 1.10 locally  $\mathcal{D}$  is contained in the local chart  $D_+(y) = \text{Spec}(\mathbf{B}[t]/(-\alpha t + f))$  for  $t = \frac{x}{y}$ . In this local chart the morphism  $\mathcal{O}_\mathcal{C} \rightarrow \mathfrak{c}^* \mathcal{E}_\alpha$  is given by  $\varphi(x)$  thus its image in  $\mathcal{O}_\mathcal{C}(1)$  is exactly  $t = \frac{x}{y}$  which is what we need.  $\square$

**Corollary 1.12.** *Keep the previous notations. We have an isomorphism*

$$\mathfrak{c}^{-1}(\mathcal{C}^* \xrightarrow{\pi^*} S, \bar{\sigma}^*) \simeq \prod_{i=1}^n \left[ \mathbb{V}_S(\mathcal{N}_{\mathcal{D}_i^*}) / \mathbb{G}_m \right]$$

*Proof.* It is standard that objects and morphisms described proposition 1.11 are those of this category fibered in groupoids.  $\square$

**Corollary 1.13.** *Let  $U$  be an affine  $\mathbb{k}$ -scheme  $U = \text{Spec } \mathbf{A}$  for a local ring  $\mathbf{A}$  and let  $(\mathcal{C}^*/U, \bar{\sigma}^*)$  be a  $n$ -marked stable curve. The fiber  $c^{-1}(\mathcal{C}^*/U, \bar{\sigma}^*)$  is isomorphic to the quotient stack*

$$\prod_{i=1}^n [\mathbb{A}_{\mathbf{A}}^1 / \mathbb{G}_m]$$

where each  $\mathbb{G}_m$  action on  $\mathbb{A}_{\mathbf{A}}^1$  is of weight one.

Keep the notation of corollary 1.13. The fiber over a stable marked curve  $(\mathcal{C}^*/S, \bar{\sigma}^*)$  of  $\mathfrak{P}_{g, \bar{k}} \rightarrow \overline{\mathfrak{M}}_{g, n}$  is now known to be isomorphic to the stack

$$\mathfrak{P}_{g, \bar{k}} \times_{\mathfrak{B}_{g, n}} \prod_{i=1}^n [\mathbb{V}_S(\mathcal{N}_{\mathcal{D}_i^*}) / \mathbb{G}_m] = \prod_{i=1}^n \left( \mathfrak{P}_{g, \bar{k}} \times_{\mathfrak{B}_{g, n}} [\mathbb{V}_S(\mathcal{N}_{\mathcal{D}_i^*}) / \mathbb{G}_m] \right).$$

It shows we only need to study the single marked case in order to understand the local behaviour of  $\mathfrak{P}_{g, \bar{k}} \rightarrow \overline{\mathfrak{M}}_{g, n}$ .

**Proposition 1.14.** *Keep the notation of 1.13. The fiber of  $\mathfrak{P}_{g, k} \rightarrow \overline{\mathfrak{M}}_{g, n}$  over  $(\mathcal{C}^*/U, \sigma^*)$  is isomorphic to the quotient stack*

$$[\mathbb{A}_{\mathbf{A}}^k / \boldsymbol{\mu}_k]$$

where the action of  $\boldsymbol{\mu}_k$  is given by weights  $(1, k-1, k-2, \dots, 1)$ .

*Proof.* Let's fix some notation first. We write  $v$  for the standard atlas  $V = \mathbb{V}_S(\mathcal{N}_{\mathcal{C}D^*}) \rightarrow [\mathbb{V}_S(\mathcal{N}_{\mathcal{D}}) / \mathbb{G}_m]$  and identify any morphism  $\alpha$  from a  $V$ -scheme  $T$  to  $[V / \mathbb{G}_m]$  with its corresponding  $\mathcal{O}_T$ -morphism from an invertible sheaf on  $\mathcal{N}_{\mathcal{D}}$  (see proposition 1.11). The fiber product  $\mathfrak{P}_{g, k} \times_{\mathfrak{B}_{g, n}} [V / \mathbb{G}_m]$  has objects over  $T \in \text{Sch}/V$  given by triples

$$\left( T \xrightarrow{\alpha} [V / \mathbb{G}_m], (\mathcal{C}/T, \sigma, \varrho_k) \in \mathfrak{P}_{g, k}(T), \Phi : (\mathbb{P}(\mathcal{E}_{\alpha}), \sigma_{\alpha}) \simeq (\mathcal{C}/T, \sigma) \right).$$

It is straightforward to see that this fiber product has objects over  $T$  that correspond to objects of the type

$$\left( T \xrightarrow{\alpha} [V / \mathbb{G}_m], \varrho_k \in H^0(\mathcal{P}_{\mathbb{P}(\mathcal{E}_v), k}) \text{ of order } k \right).$$

and whose morphisms over  $\phi \in \text{Hom}_V(T, T')$  correspond to those of polar curves from  $(\mathbb{P}(\mathcal{E}_{\alpha}), \sigma_{\alpha}, (T \rightarrow V)^* \varrho_k)$  on  $(\mathbb{P}(\mathcal{E}_{\beta}), \sigma_{\beta}, (T' \rightarrow V)^* \varrho_k)$ . A morphism  $\Phi$  of marked curves from  $(\mathbb{P}(\mathcal{E}_{\alpha}), \sigma_{\alpha})$  on  $(\mathbb{P}(\mathcal{E}_{\beta}), \sigma_{\beta})$  over  $\phi$  corresponds to a morphism  $\mu_{\Phi}$  from  $\alpha$  on  $\phi^* \beta'$  (see proof of 1.11). The morphism  $\mu_{\Phi}$  induces a natural morphism on the level of polar parts. Indeed following the proof of proposition 1.11 we know that for any  $\gamma : R \rightarrow [V / \mathbb{G}_m]$  we have  $\mathcal{O}_{\mathbb{P}(\mathcal{E}_{\gamma})}(1)$  is equal to  $\mathcal{O}_{\mathbb{P}(\mathcal{E}_{\gamma})}(\mathcal{D}_{\gamma})$ . Thus  $c_* \mathcal{O}_{\mathbb{P}(\mathcal{E}_{\gamma})}(k \mathcal{D}_{\gamma})$  is given by  $\text{Sym}^k \mathcal{E}_{\gamma}$  and we can see that as  $\mathcal{O}_R$ -modules we have

$$c_* \left( \frac{\mathcal{O}_{\mathbb{P}(\mathcal{E}_{\gamma})}(k \mathcal{D}_{\gamma})}{\mathcal{O}_{\mathbb{P}(\mathcal{E}_{\gamma})}} \right) = \mathcal{P}_{\mathbb{P}(\mathcal{E}_{\gamma}), k} = \frac{\text{Sym}^k(\mathcal{E}_{\gamma})}{\mathcal{O}\zeta^k}$$

where  $\zeta$  is the global section given by the inclusion of  $\mathcal{O}_{\mathcal{C}^* \times_U R} \rightarrow \mathcal{E}_{\gamma}$ . Let's go back to  $\Phi$ . The map  $\mu_{\Phi}$  induces by pull back of extensions a morphism from  $\mathcal{E}_{\alpha}$  on  $\mathcal{E}_{\beta}$  the morphism  $\text{Sym}^k(\mu_{\Phi})$  is the one we're looking for. The induced morphism from  $\text{Sym}^k(\mathcal{E}_{\alpha}) / \mathcal{O}\zeta_{\alpha}^k$  on  $\text{Sym}^k(\mathcal{E}_{\beta}) / \mathcal{O}\zeta_{\beta}^k$  is clearly seen to be  $\mathcal{P}(\Phi)$ . We can now say that  $\mathfrak{P}_{g, k} \times_{\mathfrak{B}_{g, n}} [V / \mathbb{G}_m]$  is isomorphic to the stack whose objects over  $T \in \text{Sch}/T$  are

$$\left( T \xrightarrow{\alpha} [V / \mathbb{G}_m], \varrho_k \in H^0(\mathcal{P}_{\mathbb{P}(\mathcal{E}_v), k}) \text{ of order } k \right).$$

and whose morphisms are given by morphisms  $\mu$  in  $[V/\mathbb{G}_m]$  such that  $\mathcal{P}(\Phi_\mu)$ —with the obvious notation—commutes with the polar parts. Now come to the local case. We use the notation of corollary 1.13. Write  $u$  for the standard atlas  $U \rightarrow [\mathbb{A}_A^1/\mathbb{G}_m]$ . Following lemma 1.10  $\mathcal{D}_u$  is contained in the chart  $D_+(y)$ . A local equation for  $\mathcal{D}_u$  at the unique closed point in the preimage of  $(\pi^{-1}U) \cap \mathcal{D}_u$  is given by  $t = \frac{x}{y}$  (see end of proof of 1.11). In this local chart a global section of  $\mathcal{P}_{\mathbb{P}(\mathcal{O}_u),k}$  is written as

$$\varrho_k = \sum_{\ell=1}^k \frac{a_\ell}{t^\ell}, \quad \text{for } a_\ell \in A,$$

this section is of order  $k$  if  $a_k$  is in  $A^*$ . Looking at relations in  $E_u$  we have  $ut = f$  where  $f$  is the corresponding local equation for  $\mathcal{D}^*$ . We can write the previous polar part as

$$(1.5) \quad \varrho_k = \sum_{\ell=1}^k a_\ell \left( \frac{u}{f} \right)^\ell, \quad \text{for } a_\ell \in A.$$

Let  $\lambda$  be an automorphism of  $u$ . The action of  $\mathcal{P}(\Phi_\lambda)$  on such a polar part is given by

$$(1.6) \quad \mathcal{P}(\Phi_\lambda)(\varrho_k) = \sum_{\ell=1}^k \lambda^\ell a_\ell \left( \frac{u}{f} \right)^\ell = \sum_{\ell=1}^k a_\ell \left( \frac{\lambda u}{f} \right)^\ell,$$

where  $\lambda$  is identified with an element of  $H^0(\mathcal{O}_A^*) = A^*$ . Now each element  $a_\ell$  for  $\ell \in \{1, \dots, n\}$  is given in this local context by a section of  $\mathcal{L}^{-\ell}$ . Thus a polar part given locally by sections  $(a_k, a_{k-1}, \dots, a_1)$  is equivalently given by a map  $\text{Spec}(\oplus_{\ell \geq 0} \mathcal{L}^{-\ell}) \rightarrow \mathbb{A}_A^k$  which is equivariant for the action of  $\mathbb{G}_m$  on  $\mathbb{A}_A^k$  by weights  $(k, k-1, \dots, 1)$ . Such a polar part is of order  $k$  if the projection on the first coefficient given by  $a_k$  is invertible or, equivalently, defines a trivialization of  $\mathcal{L}^{-1}$  on  $U$ . The fiber product  $\mathfrak{Y}_{g,k} \times_{\mathfrak{B}_{g,n}} [\mathbb{A}_A^1/\mathbb{G}_m]$  is—using the local description  $(\alpha, a_k, a_{k-1}, \dots, a_1)$ —isomorphic to the open substack of  $[\mathbb{A}_A^{k+1}/\mathbb{G}_m]$  given by  $a_k \neq 0$  and with  $\mathbb{G}_m$  weights  $(1, k, k-1, \dots, 1)$ . This is a consequence of 1.6 since the  $\mathbb{G}_m$  action on  $u$  is precisely the one giving the action of  $\mathbb{G}_m$  on  $(a_k, a_{k-1}, \dots, a_1)$ . But this open substack is precisely the quotient stack

$$\left[ \mathbb{A}_A^k / \mu_k \right]$$

given by coordinates  $(u, \frac{a_{k-1}}{a_k}, \frac{a_{k-2}}{a_k}, \dots, \frac{a_1}{a_k})$  and where  $\mu_k$  acts with weights  $(1, k-1, k-2, \dots, 1)$ .  $\square$

**Corollary 1.15.** *The stack  $\mathfrak{Y}_{g,\bar{k}}$  is smooth and irreducible stack cone in the sense of [BF97, 1.8] over  $\overline{\mathfrak{M}}_{g,n}$ .*

*Proof.* The fact it is a stack cone in the sense of [BF97] is obvious. Smoothness and irreducibility are consequences of the previous proposition together with the fact that this is the case for  $\overline{\mathfrak{M}}_{g,n}$ .  $\square$

**1.5. Generalized Polar Parts.** We give an alternative description of the cone  $\mathfrak{Y}_{g,n}$  over  $\overline{\mathfrak{M}}_{g,n}$  which is closer to the original cone built in [ELSV01]. This cone was built by extending the notion of a polar part on a stable curve to what is called a generalized polar part. No full proofs of our claims will be given in this section. One can refer to *loc. cit.* if needed. We will restrict our attention to the single marked case ( $n = 1$ ), the general case reduces to this one.

Let  $(\pi : \mathcal{C} \rightarrow S, \sigma)$  be a stable single marked  $S$ -curve. Let  $p$  be a point in the support of  $\mathcal{D} = \text{Im}(\sigma)$  and write  $s \in S$  for its image by  $\pi$ . Write  $\mathcal{O}_s$  (resp.  $\mathcal{O}_p$ ) for the local rings of  $S$  at  $s$  (resp.  $\mathcal{C}$  at  $p$ ). Choose a local coordinate  $t \in \mathfrak{m}_p$  at  $p$ , since  $p$  is given by a section  $\sigma$  of  $\pi$  we identify  $\mathcal{O}_s$  to  $\mathcal{O}_p/(t)$  using  $\sigma$ . Let  $\widehat{\mathcal{U}}_s$  be the group of  $\widehat{\mathcal{O}}_s$ -automorphisms of the

completion  $\widehat{\mathcal{O}}_p$  of  $\mathcal{O}_p$  with respect to its maximal ideal. There is a natural action of  $\widehat{\mathcal{U}}_s$  on the quotient  $\widehat{\mathcal{O}}_s$ -module  $\widehat{\mathcal{O}}_p/\widehat{\mathcal{O}}_p(-k+1)p$ . The image of this group in the group of  $\widehat{\mathcal{O}}_s$ -automorphisms of  $\widehat{\mathcal{O}}_p/\widehat{\mathcal{O}}_p(-k+1)p$  is written  $\widehat{\mathcal{U}}_{k,s}$ . Notice that  $\widehat{\mathcal{O}}_p/\widehat{\mathcal{O}}_p(-k+1)p$  is an  $\mathcal{O}_s$ -module which is isomorphic to  $\mathcal{O}_p/\mathcal{O}_p(-k+1)p$ . We thus drop the *hat* symbol in the previous notation and write  $\mathcal{U}_{k,s}$  for the subgroup of  $\mathcal{O}_s$ -automorphisms of  $\mathcal{O}_p/\mathcal{O}_p(-k+1)p$ . In terms of the coordinate  $t$  this is just given by polynomial expressions  $\phi$  of the form

$$\phi(t) = ut + \lambda_1 t^2 + \lambda_2 t^3 + \dots + \lambda_{k-1} t^k, \quad \lambda_1, \dots, \lambda_{k-1} \in \mathcal{O}_s, u \in \mathcal{O}_s^*.$$

The group law in  $\mathcal{U}_k$  is the one given by a change of coordinate, i.e., for  $\psi \in \mathcal{U}_k$  we  $(\psi\phi)(t) = \phi(\psi(t))$ .

Write  $(\mathcal{E}^s, p)$  for the stable marked curve obtained over  $\text{Spec}(\mathcal{O}_s)$  obtained by base change from the natural morphism  $\text{Spec}(\mathcal{O}_s) \rightarrow S$ . We have a natural action from  $\mathcal{U}_{k,s}$  on the sheaf of polar parts of order  $k$  centered at  $p$ . If  $\varrho = \sum_{j=1}^k a_j t^{-j}$  for  $a_1, \dots, a_k \in \mathcal{O}_s$  is a polar part centered at  $p$  this action can be written as

$$\phi \dots \varrho = \sum_{j=1}^k \frac{a_j}{\phi(t)^j}.$$

This gives a right action of  $\mathcal{U}_{k,s}$  on  $\Gamma(\mathcal{P}_{\mathcal{E}^s, k})$ . It is elementary to check that this action of  $\mathcal{U}_{k,s}$  on  $\Gamma(\mathcal{P}_{\mathcal{E}^s, k})$  is transitive. The stabilizer of a section in  $\Gamma(\mathcal{P}_{\mathcal{E}^s, k})$  is equal to  $\mu_k(\mathcal{O}_s)$ ; this can be easily computed for  $(1/t)^k$ . This means that we have an isomorphism between the groupoids given by on one side polar parts on  $\mathcal{E}^s$  centered at  $p$  and on the other side the groupoid given by sections of  $[\mathcal{U}_{k,s}/\mu_k(\mathcal{O}_s)]$ . Isomorphisms in both cases are given by automorphisms induced by those of  $(\mathcal{E}^s, p)$ . Thus building an affine cone on  $\overline{\mathfrak{M}}_{g,n}(\text{Spec}(\mathcal{O}_s))$  extending polar parts can be reinterpreted by finding affine embeddings of  $\mathcal{U}_{k,s}$ . Now notice that we have a natural surjective morphism  $\kappa: \mathcal{U}_{k,s} \rightarrow \mathbb{G}_m(\mathcal{O}_s)$  coming from the natural quotient morphism  $\mathcal{O}_p(-p)/\mathcal{O}_p(-k+1)p$  on  $\mathcal{O}_p(-p)/\mathcal{O}_p(-2p) = N_p^\vee$ . It gives an exact sequence of groups

$$(1.7) \quad 0 \rightarrow \mathcal{A}_{k,s} \rightarrow \mathcal{U}_{k,s} \xrightarrow{\kappa} \mathbb{G}_m(\mathcal{O}_s) \rightarrow 0,$$

where  $\mathcal{A}_{k,s}$  is a unipotent distinguished subgroup of  $\mathcal{U}_{k,s}$  given by change of variables of the form  $t + O(t^2)$ . The quotient  $\kappa$  is simply given by the map sending  $\phi$  on its first coefficient  $u \in \mathcal{O}_s^*$ . The choice of the coordinate  $t$  gives a section of  $\kappa$  and thus a decomposition  $\mathcal{U}_{k,s} = \mathcal{A}_{k,s} \times \mathbb{G}_m(\mathcal{O}_s)$ . Explicitly, we can write  $\phi$  as

$$(1.8) \quad \phi(t) = \underbrace{(ut)}_{\in \mathbb{G}_m(\mathcal{O}_s)} \underbrace{(t + u^{-2}\lambda_1 t^2 + u^{-3}\lambda_2 t^3 + \dots + u^{-k}\lambda_{k-1} t^k)}_{\in \mathcal{A}_{k,s}}$$

For short we write  $\phi = u\tau_{1, \lambda_1, \dots, \lambda_{k-1}}$ . The previous semi-direct product means  $\mathcal{U}_{k,s}$  is a solvable group and the image of the section of  $\kappa$  defines a maximal torus  $\mathbb{T}$  in  $\mathcal{U}_{k,s}$ . Using techniques due to [KKMSD73] one can build affine equivariant embeddings of  $\mathcal{U}_{k,s}$ . The construction goes as follows : Take an affine embedding  $\mathbb{T} \rightarrow X_\sigma$  for a cone  $\sigma$  in the group of characters of  $\mathbb{T}$  (there are two such embeddings). Given this affine embedding one can build an equivariant embedding

$$Y_\sigma = \mathcal{U}_{k,s} \times^{\mathbb{T}} X_\sigma.$$

In the case at hand we want for the action of  $\mathcal{U}_{k,s}$  on polar parts of order  $k$  to be compatible with its embedding in  $Y_\sigma$ . Writing this down shows that the affine embedding we're looking for is given by  $\sigma = ] - \infty, 0[$  for the chosen coordinate  $t$ . We write  $\mathcal{Z}$  for the attached toroidal affine embedding. Now this construction boils down to the following

local description. Locally  $\mathcal{Z}$  is isomorphic to the affine space  $\mathbb{A}(\alpha, \lambda_1, \dots, \lambda_{k-1})$  where appearing coordinates correspond to those of 1.8. Let  $\psi(t)$  for  $\psi \in \mathcal{U}_{k,S}$  be another coordinate at the neighborhood of  $p$ . Assume  $\psi = v\tau_{1,\nu_1,\dots,\nu_{k-1}}$ . Transition functions relating the corresponding coordinates of  $\mathbb{A}(u, \lambda_i)$  and  $\mathbb{A}(u^*, \lambda_i^*)$  are given by

$$(1.9) \quad u = u^* \beta^{-1}, \quad \lambda_i = \sum_{\ell+j=i} \lambda_\ell^* u^{*i} Q_{k-\ell,j}(\nu_1, \dots, \nu_j),$$

where  $Q_{i,j}(X_1, \dots, X_j)$  are the polynomials given by the relation

$$\frac{1}{(\sum_{j \geq 0} X_j Y^j)^i} = \sum_{j \geq 0} Q_{i,j}(X_1, \dots, X_j) Y^j.$$

One can get such relations by looking at the polar parts represented in terms of each coordinates and writing the  $\psi(t)$  polar part in terms of negative powers of  $t$ . These local coordinates are what enables us to make sense of the relative case. We give the main definitions and relation to  $\mathfrak{P}_{g,k}$ .

**Definition 1.5.** Let  $(\pi : \mathcal{C} \rightarrow S, \sigma)$  be a stable single marked  $S$ -curve. Write  $\mathcal{D}^{(k)}$  for the subscheme of  $\mathcal{C}$  given by the sheaf of ideals  $\mathcal{O}_{\mathcal{C}}(-(k+1)\mathcal{D})$ . We call  $\mathcal{U}_{k,S}$  the group  $\text{Aut}_{\mathcal{O}_S}(\pi_* \mathcal{O}_{\mathcal{D}^{(k)}})$ .

It is easy to check that the  $S$ -group  $\mathcal{U}_{k,S}$  defined in this way is a smooth  $S$ -group which is solvable, of relative dimension  $k$  and an extension of a torus by a unipotent subgroup. This is an isotrivial group with unipotent fibers equal to  $\mathcal{U}_{k,s}^k$  over  $s \in S$ . Using the previous procedure we can build étale locally affine equivariant embeddings of such an  $S$ -group. These data glue to give an embedding globally over  $S$  and by extension over  $\overline{\mathfrak{M}}_{g,1}$  (refer to [KKMSD73] for more details). We call  $\mathcal{Z}_{g,k}$  such an embedding.

**Definition 1.6.** A generalized polar part of  $(\mathcal{C}/S, \sigma)$  along  $\mathcal{D}$  is an  $S$ -object of  $[\mathcal{Z}_{g,k}/\mu_k]$ .

The following proposition gives the link between our construction and the one introduced in [ELSV01].

**Proposition 1.16.** *The stack of generalized polar parts over  $\overline{\mathfrak{M}}_{g,n}$  is isomorphic to  $\mathfrak{P}_{g,n}$ .*

*Proof.* In proof of proposition 1.14 we proved we had an isomorphism of stacks from  $\mathfrak{P}_{g,n}$  on the stack whose objects were given by triples  $(\pi : \mathcal{C}^* \rightarrow S, \sigma^*, \rho \in \text{Sym}^k(\mathcal{E}_\alpha)/\mathcal{O}_{C^*}\zeta^k)$  with the corresponding notation. To prove our claim it is therefore enough to show that  $\text{Sym}^k(\mathcal{E}_\alpha)/\mathcal{O}\zeta^k$  is isomorphic to the sheaf of generalized polar parts of order  $k$ . Using 1.10  $\text{Sym}^k(\mathcal{E}_\alpha)/\mathcal{O}\zeta^k$  is locally equal to

$$\frac{\mathbf{B}[x, y]_k}{\mathbf{B}[x, y]_{k-1}(-\alpha x + f y) + Bx^k}$$

with notation of lemma 1.10. One can show that this is isomorphic to the direct sum

$$\bigoplus_{j=0}^{k-1} Ax^j y^{k-j}.$$

There is a natural action of  $\mathcal{U}_k$  on this  $A$ -module. Let  $\phi = (uf)\tau_{1,\lambda_1,\dots,\lambda_{k-1}} \in \mathcal{U}_k$ . The automorphism attached to  $g$  for this action is defined by  $(x \mapsto x, f \mapsto g(f), y \mapsto \alpha\tau_{1,\lambda_1,\dots,\lambda_{k-1}})$ . The induced relations on coefficients of an element in  $\bigoplus_{j=0}^{k-1} Ax^j y^{k-j}$  after the action of  $\phi$  are exactly those given by relations 1.9. This ends the proof.  $\square$

## 2. ELSV CONE

We begin this section by giving a closed substack  $\mathfrak{Z}_{g,\vec{k}}$  of  $\mathfrak{P}_{g,\vec{k}}$  firstly introduced in [ELSV01] in the case  $\mathbb{k} = \mathbb{C}$ . This substack is isomorphic to the quotient  $[\overline{\mathfrak{M}}_{g,n}(\mathbb{P}^1, \vec{k})/\mathbb{G}_a]$  where the  $\mathbb{G}_a$  action on  $\overline{\mathfrak{M}}_{g,n}(\mathbb{P}^1, \vec{k})$  is the standard one. Under a tame condition on the base field, namely that the characteristic  $p$  is zero or greater than  $d$  and doesn't divide  $2g - 2 + d + n$ , we can build a section of the quotient morphism  $\overline{\mathfrak{M}}_{g,n}(\mathbb{P}^1, \vec{k})$  on  $[\overline{\mathfrak{M}}_{g,n}(\mathbb{P}^1, \vec{k})/\mathbb{G}_a]$ . It does enable us to embed  $\mathfrak{Z}_{g,\vec{k}}$  as a closed substack of  $\overline{\mathfrak{M}}_{g,n}(\mathbb{P}^1, \vec{k})$ . This is made possible by using the notion of generalized branch divisor built in [FP02].

*Notation.* For further use we write  $r$  for the natural number  $2g - 2 + d + n$ .

**2.1. The ELSV Cone  $\mathfrak{Z}_{g,\vec{k}}$ .** Let  $(\pi : \mathcal{C} \rightarrow S, \vec{\sigma}, \vec{\rho})$  be a stable polar curve in  $\mathfrak{P}_{g,\vec{k}}$ . There is a long exact sequence in cohomology attached to this polar curve given by

(2.1)

$$0 \rightarrow \mathcal{O}_S \rightarrow \pi_* \mathcal{O}_{\mathcal{C}}(\sum_{i=1}^n k_i \mathcal{D}_i) \rightarrow \mathcal{P}_{\mathcal{C},\vec{k}}^{\nabla_{\mathcal{C}}(\vec{k})} \rightarrow R^1 \pi_* \mathcal{O}_{\mathcal{C}} \rightarrow R^1 \pi_* \mathcal{O}_{\mathcal{C}}(\sum_{i=1}^n k_i \mathcal{D}_i) \rightarrow 0.$$

The theory of base change in cohomology for proper flat morphisms implies that  $\mathcal{P}_{\mathcal{C},\vec{k}}$ ,  $R^1 \pi_* \mathcal{O}_{\mathcal{C}}$  and  $R^1 \pi_* \mathcal{O}_{\mathcal{C}}(\sum_{i=1}^n k_i \mathcal{D}_i)$  commute with any base change and are locally free. This is enough to ensure that  $\nabla_{\mathcal{C}}(\vec{k})$  commutes with any base change as well. Thus to each  $S$ -object  $(\mathcal{C}/S, \vec{\rho})$  in  $\mathfrak{P}_{g,n}$  we get a global section  $\nabla_{\mathcal{C}}(\vec{k})(\vec{\rho})$  of  $R^1 \pi_* \mathcal{O}_{\mathcal{C}}$ . This is precisely giving a section  $\nabla(\vec{k})$  of the vector bundle defined over  $\mathfrak{P}_{g,\vec{k}}$  by  $R^1 \pi_* \mathcal{O}_{\mathcal{C}}$ . Now following Grothendieck-Serre duality we have a canonical isomorphism  $R^1 \pi_* \mathcal{O}_{\mathcal{C}} \simeq \pi_* \omega_{\mathcal{C}}^{\vee}$ . Following results of [Har66] (see also [Con00]) this isomorphism commutes with any base change. The section  $\nabla(\vec{k})$  defines using this isomorphism a natural section  $\nabla(\vec{k})^{\vee}$  of the pullback of the dual of the Hodge bundle on  $\overline{\mathfrak{M}}_{g,n}$  to  $\mathfrak{P}_{g,\vec{k}}$ . We shall loosely denote this bundle by  $\mathbb{E}$  as is the case for the Hodge bundle on  $\overline{\mathfrak{M}}_{g,n}$ .

**Definition 2.1.** Let  $\nabla(\vec{k})^{\vee}$  be the natural section of  $\mathbb{E}^{\vee}$  given by 2.1. The ELSV cone  $\mathfrak{Z}_{g,\vec{k}}$  is defined as the zero locus of  $\nabla(\vec{k})^{\vee}$ . This is a closed substack of  $\mathfrak{P}_{g,\vec{k}}$ .

let  $(\pi : \mathcal{C} \rightarrow S, \vec{\sigma}, \vec{\rho})$  be a polar  $S$ -curve in  $\mathfrak{Z}_{g,\vec{k}}$ , let  $\{U_{\ell} \rightarrow S\}_{\ell}$  be an open affine cover of  $S$  and write  $\mathcal{C}_{\ell}$  for the restriction of  $\mathcal{C}/S$  to  $U_{\ell} = \text{Spec}(\mathcal{A}_{\ell})$ . If we pull back exact sequence 2.1 on  $\mathcal{C}_{\ell}$  we get as part of the long exact sequence in cohomology

$$0 \rightarrow \mathcal{A}_{\ell} \rightarrow H^0(\mathcal{O}_{\mathcal{C}_{\ell}}(\sum_{i=1}^n k_i \mathcal{D}_{\ell,i})) \rightarrow H^0(\mathcal{P}_{\mathcal{C}_{\ell},\vec{k}}) \rightarrow H^1(\mathcal{O}_{\mathcal{C}_{\ell}}).$$

Thus  $\vec{\rho}_{\ell}$  can be lifted to a section in  $H^0(\mathcal{O}_{\mathcal{C}_{\ell}}(\sum_{i=1}^n k_i \mathcal{D}_{\ell,i}))$  meaning we have local lifts of  $\vec{\rho}$  to  $\pi_* \mathcal{O}_{\mathcal{C}}(\sum_{i=1}^n k_i \mathcal{D}_i)$ . Let's focus for the time being on the local case at hand. We shall drop the indices and write  $(\mathcal{C}_U, \vec{\sigma}, \vec{\rho})$  for the restriction of our initial polar curve to  $U = \text{Spec}(\mathcal{A})$ . Assume for simplicity that  $\mathcal{A}$  is local of maximal ideal  $\mathfrak{m}$  and residue field  $\mathbb{k}$ . Write  $U = \text{Spec}(\mathcal{A})$  for the spectrum of  $\mathcal{A}$  and let  $(\mathcal{C}_U, \vec{\sigma}, \vec{\rho})$  be a polar  $U$ -curve in  $\mathfrak{Z}_{g,\vec{k}}$ . Write  $\mathbf{1}$  for the canonical section of  $\mathcal{O}_{\mathcal{C}_U}(\sum_{i=1}^n k_i \mathcal{D}_i)$ . The set of lifts of  $\vec{\rho}$  to  $H^0(\mathcal{O}_{\mathcal{C}_U}(\sum_{i=1}^n k_i \mathcal{D}_i))$  is a torsor under the additive action of  $\mathcal{A}$  along  $\mathbf{1}$ . This is indeed a torsor because a lift of  $\vec{\rho}$  can never be colinear to  $\mathbf{1}$  because of the hypothesis on the order of  $\vec{\rho}$ . Now fix a lift  $\zeta$  of  $\vec{\rho}$ . Each lift  $\zeta + \alpha \mathbf{1}$  for  $\alpha \in \mathcal{A}$  defines a surjective morphism

$$\mathcal{O}_{\mathcal{C}_U}^{\oplus 2} \longrightarrow \mathcal{O}_{\mathcal{C}_U}(\sum_{i=1}^n k_i \mathcal{D}_i) \longrightarrow 0.$$

attached to the sections  $\{\mathbf{1}, \zeta + \alpha \mathbf{1}\}$ . These give morphisms  $\Phi_{\zeta + \alpha \mathbf{1}}$  from  $\mathcal{C}_U$  on  $\mathbb{P}_{\mathcal{A}}^1$  sending markings  $\mathcal{D}_1, \dots, \mathcal{D}_n$  on the  $\infty$  section of  $\mathbb{P}_{\mathcal{A}}^1$ . Locally at a point along  $\mathcal{D}_i$  the map  $\Phi_{\zeta + \alpha \mathbf{1}}$  is given by the regular function  $(\zeta + \alpha \mathbf{1})^{-1}$  and away from infinity it is locally given by

$\zeta + \alpha \mathbf{1}$ . This makes it clear that the  $\mathbb{G}_a$  action on lifts of  $\vec{\rho}$  corresponds to the  $\mathbb{G}_a$  action on attached maps to  $\mathbb{P}_A^1$  fixing the section at  $\infty$ .

**Lemma 2.1.** *The maps  $\Phi_{\zeta+\alpha \mathbf{1}}$  for  $\alpha \in A$  are stable maps of degree  $d$ .*

*Proof.* By construction the restriction of  $\Phi_{\zeta+\alpha \mathbf{1}}$  on the fiber  $C$  of the closed point of  $A$  is constant on irreducible components of  $C$  containing no marked points and dominant on those containing any of the marked points. Since each unstable component of  $C$  contains at least a marked point  $\Phi_{\zeta+\alpha \mathbf{1}} \otimes \mathbf{k}$  can't be unstable. The degree of  $\Phi_{\zeta+\alpha \mathbf{1}}$  is equal to the dominant coefficient in the Hilbert polynomial of  $\Phi_{\zeta+\alpha \mathbf{1}}^*(\mathcal{O}_{\mathbb{P}_A^1}(1))$  which is written  $(\Phi_{\zeta+\alpha \mathbf{1}}^*(\mathcal{O}_{\mathbb{P}_A^1}(1)))$ . By definition we have

$$\chi(\Phi_{\zeta+\alpha \mathbf{1}}^* \mathcal{O}_{\mathbb{P}_A^1}(\ell)) = \ell(\Phi_{\zeta+\alpha \mathbf{1}}^*(\mathcal{O}_{\mathbb{P}_A^1}(1))) + \chi(\mathcal{O}_{\mathcal{C}}).$$

Thus the degree is constant along the fibers of  $\pi$ . On a fiber  $C$  of  $\pi$  the previous degree is equal to the sum of the degrees of restrictions of  $(\Phi_{\zeta+\alpha \mathbf{1}} \otimes \mathbf{k})^*(\mathcal{O}_{\mathbb{P}^1}(1))$  to each irreducible component of  $C$ . Since this degree is zero on contracted components the sum takes place only on components of  $C$  having a marked point. This means the degree is equal to  $\sum_{i=1}^n k_i = d$ .  $\square$

Let  $\phi: \mathcal{C} \rightarrow \mathbb{P}_S^1$  be an  $S$ -map in  $\overline{\mathfrak{M}}_{g,n}(\mathbb{P}^1, \vec{\mathbf{k}})$ . This map is given by a unique meromorphic function  $\zeta_\phi$  in  $\mathcal{M}_{\mathcal{C}/S}$ . This meromorphic function is the extension to the whole of  $\mathcal{C}$  of the regular function on  $\mathcal{C} \setminus \text{Supp}(\sum_{i=1}^n \mathcal{D}_i)$  defining  $\phi$ . In terms of  $\zeta_\phi$  we recover  $\phi$  by looking at the morphism from  $\mathcal{O}_{\mathcal{C}}^{\oplus 2}$  to  $\mathcal{O}_{\mathcal{C}}(\sum_{i=1}^n k_i \mathcal{D}_i) = \phi^* \mathcal{O}_{\mathbb{P}_S^1}(1)$  given by the collection of sections  $\{\mathbf{1}, \zeta_\phi\}$ . This means we get a morphism of stacks from  $\overline{\mathfrak{M}}_{g,n}(\mathbb{P}^1, \vec{\mathbf{k}})$  on  $\mathfrak{Z}_{g,\vec{\mathbf{k}}}$  by sending  $\phi$  on the polar curve  $(\mathcal{C}, \vec{\sigma}, \vec{\zeta}_\phi)$  where  $\vec{\zeta}_\phi$  is the image of  $\zeta_\phi$  in  $\Gamma(\mathcal{P}_{\mathcal{C},\vec{\mathbf{k}}})$ .

**Proposition 2.2.** *The previous morphism  $\overline{\mathfrak{M}}_{g,n}(\mathbb{P}^1, \vec{\mathbf{k}}) \rightarrow \mathfrak{Z}_{g,\vec{\mathbf{k}}}$  is a  $\mathbb{G}_a$ -torsor. In particular*

$$[\overline{\mathfrak{M}}_{g,n}(\mathbb{P}^1, \vec{\mathbf{k}}) / \mathbb{G}_a] \simeq \mathfrak{Z}_{g,\vec{\mathbf{k}}}.$$

*Proof.* This is a straightforward consequence of discussion following definition 2.1 and lemma 2.1.  $\square$

**Corollary 2.3.** *The quotient*

$$[\overline{\mathfrak{M}}_{g,n}(\mathbb{P}^1, \vec{\mathbf{k}}) / (\mathbb{G}_m \times \mathbb{G}_a)]$$

*is proper.*

*Proof.* This follows from the fact that

$$[\overline{\mathfrak{M}}_{g,n}(\mathbb{P}^1, \vec{\mathbf{k}}) / (\mathbb{G}_m \times \mathbb{G}_a)] \simeq \mathbb{P} \mathfrak{Z}_{g,\vec{\mathbf{k}}}.$$

$\square$

**2.2. Branch Morphism and Modular Interpretation of  $\mathfrak{Z}_{g,\vec{\mathbf{k}}}$ .** There is a more convenient way of looking at objects of  $\mathfrak{Z}_{g,\vec{\mathbf{k}}}$ . It does however need a restriction on the characteristic of the base field.

**Tame Assumption 2.1.** For the sequel of the paper we assume that the characteristic of the base field  $\mathbf{p}$  is zero or greater than  $d$  and doesn't divide  $r$ .

Under this assumption there is a natural way of attaching a stable  $S$ -map to an  $S$ -object of  $\mathfrak{Z}_{g,\vec{\mathbf{k}}}$ . This gives a closed embedding of  $\mathfrak{Z}_{g,\vec{\mathbf{k}}}$  into  $\overline{\mathfrak{M}}_{g,n}(\mathbb{P}^1, \vec{\mathbf{k}})$ .

This embedding is given by a section of the forgetful morphism  $\overline{\mathfrak{M}}_{g,n}(\mathbb{P}^1, \vec{\mathbf{k}}) \rightarrow \mathfrak{Z}_{g,\vec{\mathbf{k}}}$  constructed with the help of the generalized branch morphism due to [FP02]. It is to

mention that results in [FP02] are stated in the case of a characteristic 0 base field. Following a remark at the beginning of section [FP02, 1.1] this condition is only needed (at least for the first three sections) in [FP02, 9] to prove that given a stable  $S$ -map  $F: X \rightarrow Y$  the complex

$$RF_*[F^*\omega_{Y/S} \rightarrow \omega_{X/S}]$$

is a perfect torsion complex on  $D_{coh}(Y)$ . One can lighten the previous hypothesis on  $\mathbf{p}$ . Indeed using lemmas [FP02, 5 & 7] (which do not depend on the characteristic see [FP02, 1.1]) it is enough to prove our claim for moduli points. Following the proof of [FP02, 9] it is obvious that what is needed is generic smoothness of  $F$  on non-contracted components of  $X$ . This is true since  $F$  is finite on such components as soon as  $F$  restricted to each of these components is separable. To ensure this is always the case we need either  $\mathbf{p} = 0$  or  $\mathbf{p} > d$  and both are guaranteed by the assumption 2.1. We shall therefore use needed results of [FP02] in this generality.

Following [FP02] we have a morphism

$$\overline{\mathfrak{M}}_{g,n}(\mathbb{P}^1, d, \infty) \xrightarrow{\text{Br}} \text{Div}^{2g-2+2d} \mathbb{P}^1$$

which sends a stable map in  $\overline{\mathfrak{M}}_{g,n}(\mathbb{P}^1, d, \infty)$  on an effective Cartier divisor on  $\mathbb{P}^1$ . It does coincide on the locus of  $\overline{\mathfrak{M}}_{g,n}(\mathbb{P}^1, d, \infty)$  given by dominant smooth maps with the classical branch morphism.

Let  $(\mathcal{C}_U, \vec{\sigma}, \vec{\rho})$  be a polar  $U$ -curve in  $\mathfrak{Z}_{g,\mathbf{k}}$  for an affine scheme  $U$  that is the spectrum of a local ring  $\mathbf{A}$  of maximal ideal  $\mathfrak{m}$  and residue field  $\mathbf{k}$ .

**Lemma 2.4.** *Fix a lift  $\zeta$  of  $\vec{\rho}$  to  $H^0(\mathcal{C}_U, \sum_{i=1}^n k_i \mathcal{D}_i)$ . The restriction  $\text{Br}_0(\Phi_\zeta)$  of  $\text{Br}(\Phi_\zeta)$  to the affine chart  $\mathbb{A}_U^1$  of  $\mathbb{P}_U^1$  given by the complement of the section at infinity is a relative effective Cartier divisor of degree  $r = 2g - 2 + d + n$ .*

*Proof.* Since  $\text{Br}(\Phi_\zeta)$  is effective, its restriction to an open subset of  $\mathbb{P}^1$  is so as well. To show that  $\text{Br}_0(\Phi_\zeta)$  is relative Cartier it is enough to check it on the closed fiber of  $\mathcal{C}_U/U$ . The map  $\Phi_\zeta \otimes \mathbf{k}$  is a stable map attached to the polar  $\mathbb{k}$ -curve  $(C, \vec{\rho}, \vec{\rho} \otimes \mathbf{k})$  obtained by base change. Write  $\Phi_\ell$  for the restriction of  $\Phi_\zeta \otimes \mathbf{k}$  to an irreducible component  $C_\ell$  of  $C$  and write  $\mathfrak{N}$  for the Weil divisor of nodal points of  $C$ . Following [FP02, 3.1] the branch divisor of  $\Phi_\zeta \otimes \mathbf{k}$  is given by

$$\text{Br}(\Phi_\zeta \otimes \mathbf{k}) = \left( \sum_{\{\ell \mid \Phi_\ell \text{ is constant}\}} \text{Br}(\Phi_\ell) \right) + 2\Phi_{\zeta*} \mathfrak{N} + \left( \sum_{\{\ell \mid \Phi_\ell \text{ is dominant}\}} \text{Br}(\Phi_\ell) \right).$$

For an irreducible component  $C_\ell$  such that  $\Phi_\ell$  is dominant the branch divisor  $\text{Br}(\Phi_\ell)$  is the classical branch divisor. Thus for such a component  $\text{Br}(\Phi_\ell)$  contains the Cartier divisor  $(\sum_{\{i \mid p_i \in C_\ell\}} (k_i - 1))\infty$ . By summing up over all non contracted components  $\text{Br}(\Phi_\zeta \otimes \mathbf{k})$  always contains the Cartier divisor  $(d - n)\infty$ . Which finally gives that  $\text{Br}(\Phi_\zeta \otimes \mathbf{k}) - (d - n)\infty$  is an effective Cartier divisor of degree  $2g - 2 + d + n$  equal to  $\text{Br}_0(\Phi_\zeta \otimes \mathbf{k})$ .  $\square$

**Lemma 2.5.** *Let  $\mathbf{I}$  be an ideal in  $\mathbf{A}[t]$  defining a relative Cartier divisor, i.e.  $\mathbf{A}[t]/\mathbf{I}$  is flat over  $\mathbf{A}$ , then  $\mathbf{I}$  is principal.*

*Proof.* Write  $S = \text{Spec}(\mathbf{A})$  and let  $D$  be the Cartier divisor with defining ideal  $\mathbf{I}$ . It is well known that the Picard group of  $\mathbb{A}_S^1$  is trivial since  $\mathbf{A}$  is a local ring, which in turn gives the result.  $\square$

**Proposition 2.6.** *Keep notation of lemmas 2.4 and 2.5. There is a unique lift  $\zeta(\vec{\rho})$  of  $\vec{\rho}$  to  $H^0(\mathcal{C}_{\mathcal{C}_U}, \sum_{i=1}^n k_i \mathcal{D}_i)$  such that  $\text{Br}_0(\Phi_{\zeta(\vec{\rho})})$  is given by a polynomial in  $\mathbf{A}[t]$  having a zero coefficient in degree  $r - 1$ .*

*Proof.* Let  $\zeta$  be any lift of  $\vec{\rho}$ . Since invertible elements in  $\mathbf{A}[t]$  are those in  $\mathbf{A}^*$  the condition appearing in the lemma on the vanishing of the coefficient of a polynomial defining  $\text{Br}_0(\Phi_\zeta)$  doesn't depend on the choice of this polynomial. Let  $\alpha$  be an element in  $\mathbf{A}$  and write  $\tau_\alpha$  for the automorphism of  $\mathbb{P}_A^1$  such that  $\tau_\alpha(t) = t + \alpha$  where  $t$  is the coordinate of  $\mathbb{P}_A^1$  centered at 0. By definition

$$\text{Br}(\tau_\alpha \circ \Phi_\zeta) = \text{Div} \left( R(\tau_\alpha \circ \Phi_\zeta)_* [(\tau_\alpha \circ \Phi_\zeta)^* \omega_{\mathbb{P}_A^1/U} \rightarrow \omega_{\mathcal{C}/U}] \right).$$

Since the complexes  $[\tau_\alpha \circ \Phi_\zeta^* \omega_{\mathbb{P}_A^1/U} \rightarrow \omega_{\mathcal{C}/U}]$  and  $[\Phi_\zeta^* \omega_{\mathbb{P}_A^1/U} \rightarrow \omega_{\mathcal{C}/U}]$  are isomorphic and  $\tau_\alpha$  is an automorphism we get that

$$\begin{aligned} \text{Br}(\tau_\alpha \circ \Phi_\zeta) &= \text{Div} \left( \tau_{\alpha*} \left( R\Phi_{\zeta*} [(\Phi_\zeta)^* \omega_{\mathbb{P}_A^1/U} \rightarrow \omega_{\mathcal{C}/U}] \right) \right), \\ &= \tau_{\alpha*} \text{Div} \left( R\Phi_{\zeta*} [(\Phi_\zeta)^* \omega_{\mathbb{P}_A^1/U} \rightarrow \omega_{\mathcal{C}/U}] \right), \\ &= \tau_{\alpha*} \text{Br}(\Phi_\zeta). \end{aligned}$$

Following lemma 2.5  $\text{Br}_0(\Phi_\zeta)$  is generated by a single polynomial in  $\mathbf{A}[t]$ . Thus if  $P(t) \in \mathbf{A}[t]$  is an equation for  $\text{Br}_0(\Phi_\zeta)$  then  $P(t + \alpha)$  is an equation for  $\text{Br}_0(\Phi_{\zeta+\alpha\mathbf{1}})$ . The degree of  $P(t)$  is that of  $\text{Br}_0(\Phi_\zeta)$  and that is  $r$ . Write  $P(t)$  as

$$P(t) = \sum_{\ell=0}^r \beta_\ell t^\ell, \quad \beta_r \in \mathbf{A}^*.$$

The branch divisor of  $\Phi_{\zeta+\alpha\mathbf{1}}$  is written

$$P(t + \alpha) = \sum_{\ell=0}^r \beta_\ell (t + \alpha)^\ell = \sum_{\ell=0}^r \sum_{q=0}^{\ell} \binom{\ell}{q} \beta_\ell t^q \alpha^{\ell-q} = \sum_{\ell=0}^r \left( \sum_{q=\ell}^r \binom{q}{\ell} \beta_q \alpha^{q-\ell} \right) t^\ell.$$

In particular the coefficient of degree  $r - 1$  is given by  $\beta_{r-1} + r\beta_r\alpha$ . Since  $r$  and  $\beta_r$  are invertible in  $\mathbf{A}^*$  we get that there is a unique  $\alpha$  for which the branch divisor of  $\Phi_{\zeta+\alpha\mathbf{1}}$  has an equation with zero  $r - 1$  coefficient.  $\square$

This justifies the following definition

**Definition 2.2.** Let  $\phi : \mathcal{C} \rightarrow \mathbb{P}_S^1$  be a stable  $S$ -map of degree  $d$ . We say that  $\phi$  is normalized if locally any equation of  $\text{Br}_0(\phi)$  has zero coefficient in degree  $r - 1$ .

**Corollary 2.7.** Given a polar  $S$ -curve  $(\mathcal{C}, \vec{\sigma}, \vec{\rho})$  in  $\mathfrak{Z}_{g, \vec{k}}$  there is a unique lift  $\zeta(\vec{\rho})$  of  $\vec{\rho}$  to  $H^0(\mathcal{O}_{\mathcal{C}}(\sum_{i=1}^n k_i \mathcal{D}_i))$  that is normalized.

*Proof.* We only need to justify that the condition expressed in 2.6 is local on the base and stable under base change. It is indeed local on the base. Being stable under base change is a consequence of the fact that the branch divisor is so. Indeed, keep notation introduced in the proof of 2.6 and let  $\psi : \mathbf{B} \rightarrow \mathbf{A}$  be a morphism of  $\mathbb{k}$ -algebras. Since  $\text{Br}$  commutes with any base change the polynomial giving the branch morphism of  $\psi^* \zeta$  is given by  $P(t) \otimes 1_{\mathbf{B}}$ . And this last polynomial has a zero degree  $r - 1$  coefficient.  $\square$

**Corollary 2.8.** We have a natural closed embedding of  $\mathfrak{Z}_{g, \vec{k}}$  into the space of stable maps of profile  $\vec{k}$

*Proof.* This embedding is given by the section of  $\overline{\mathfrak{M}}_{g, n}(\mathbb{P}^1, \vec{k}) \rightarrow \mathfrak{Z}_{g, \vec{k}}$  sending a polar  $S$ -curve in  $(\mathcal{C}, \vec{\sigma}, \vec{\rho})$  on the  $S$ -map  $\Phi_{\zeta(\vec{\rho})}$ .  $\square$

**Corollary 2.9.** We have a branch morphism  $\text{Br}_0 : \mathfrak{Z}_{g, \vec{k}} \rightarrow \text{Div}^{2g-2+d+n} \mathbb{A}^1$  that sends an object in  $\mathfrak{Z}_{g, \vec{k}}$  on an effective relative Cartier divisor whose support is in  $\mathbb{P}^1 \setminus \{\infty\}$ .

*Proof.* This is the composition of the previous embedding with the generalized branch morphism following [FP02].  $\square$

*Remark 2.* The previous branch morphism should be thought of as the  $\mathcal{L}\mathcal{L}$  mapping introduced in [ELSV01]. The use of the generalized branch morphism in the sense of [FP02] avoids showing that the notion of finite critical values of a map from a smooth curve to  $\mathbb{P}^1$  extends to the case of a nodal domain curve.

**2.3. Natural Strata of  $\mathfrak{Z}_{g,\bar{k}}$ .** Let  $\Gamma$  be a modular graph in the sense of [BM96] that corresponds to a bubbly curve. There is a natural stratification of  $\mathfrak{B}_{g,n}$  by locally closed substacks  $\overline{\mathfrak{M}}_g(\Gamma)$  given by curves of type  $\Gamma$  (see [BM96]). Taking the fiber product

$$\mathfrak{Z}_g(\Gamma) = \overline{\mathfrak{M}}_g(\Gamma) \times_{\mathfrak{B}_{g,n}} \mathfrak{Z}_{g,\bar{k}}$$

we get a natural stratification on  $\mathfrak{Z}_{g,\bar{k}}$ . Our aim is to give estimates on the dimension of these strata. For this purpose we introduce slight variants of the stack  $\mathfrak{Z}_g(\Gamma)$ . Let  $L_\Gamma$  be a choice of extra legs for  $\Gamma$ . Write  $\Gamma(L_\Gamma)$  for the resulting modular graph and write  $\overline{\mathfrak{M}}_g(\Gamma, L_\Gamma)$  for the stack of curves of types  $\Gamma(L_\Gamma)$  together with the subset  $L_\Gamma$  of legs of  $\Gamma(L_\Gamma)$ . Contracting these legs and stabilizing unstable rational components having no marked points in  $L_\Gamma$  we get a forgetful morphism  $\overline{\mathfrak{M}}_g(\Gamma, L_\Gamma) \rightarrow \overline{\mathfrak{M}}_g(\Gamma)$ . This is a variant of the stabilization morphism where an  $S$ -curve  $(\pi : \mathcal{C} \rightarrow S, \mathcal{D}_1, \dots, \mathcal{D}_n, \mathcal{E}_1, \dots, \mathcal{E}_m) \in \overline{\mathfrak{M}}_g(\Gamma, L_\Gamma)$  is sent on the **Proj** of

$$\pi_* \omega_{\mathcal{C}/S} \left( \sum_{i=1}^n 2\mathcal{D}_i + \sum_{j=1}^m \mathcal{E}_j \right).$$

This variant does not contract bubbles of  $C$ . We call  $\mathfrak{Z}_g(\Gamma, L_\Gamma)$  the fiber product  $\overline{\mathfrak{M}}_g(\Gamma, L_\Gamma) \times_{\overline{\mathfrak{M}}_g(\Gamma)} \mathfrak{Z}_g(\Gamma)$ . If we look at the special case of a one vertex modular graph having  $n + m$  half legs we get objects of  $\mathfrak{Z}_{g,n}$  having smooth underlying curves and  $m$  extra markings. In this case such a stack will be simply written  $\mathfrak{Z}_g(-, n; m)$ . All of the previous stacks are defined as fiber products of algebraic stacks, they're algebraic as well.

*Notation.* Given a relative curve  $\mathcal{C}/S$  denote  $\mathfrak{A}_{\mathcal{C}}$  the set of relative maps  $\mathcal{C} \rightarrow \mathbb{P}_S^1$  up to the  $\mathbb{G}_a$  action on  $\mathbb{P}_S^1$  fixing the section at  $\infty$ .

**Lemma 2.10.** *Let  $\Delta$  be a modular tree. Take a point  $(C, \vec{p})$  in  $\overline{\mathfrak{M}}_g(\Delta)$  and assume we have on each irreducible component  $C_v$  an element  $f_v \in \mathfrak{A}_{C_v}$ . Then there is a unique element  $f \in \mathfrak{A}_C$  whose restriction to each component  $C_v$  is equal to  $f_v$ .*

*Proof.* Make a choice of representatives  $\hat{f}_v$  for each  $f_v \in \mathfrak{A}_{C_v}$ . If  $e$  is an edge of  $\Delta$  we write  $\{v_+(e), v_-(e)\}$  for its set of adjacent vertices. We denote  $x_e$  the nodal point of  $C$  attached to the edge  $e$ . Let  $E$  be the set of edges  $e$  in  $\Delta$  such that  $\hat{f}_{v_+(e)}(x_e)$  and  $\hat{f}_{v_-(e)}(x_e)$  do not coincide. We're going to show that one can always decrease  $\#E$  by choosing different representatives of  $f_v$ . Take a leaf  $v_0 \in \Delta$ . If  $E \neq \emptyset$  then there exists a vertex  $v_{m+1}$  and a unique shortest path  $v_0, e_0, v_1, e_1, \dots, e_m, v_{m+1}$  of length  $m + 1$  such that  $\hat{f}_{v_j}(x_{e_j}) \neq \hat{f}_{v_{j+1}}(x_{e_j})$  only if  $j = m$ . Let  $\Delta(v_0, v_{m+1})$  be the subgraph of  $\Delta$  generated by the vertices in the connected component of  $\Delta \setminus \{v_{m+1}\}$  containing  $v_0$ . Exchange each representative  $\hat{f}_v$  of  $f_v$  for  $v$  a vertex of  $\Delta(v_0, v_{m+1})$  by  $\tilde{f}_v = \hat{f}_v + \hat{f}_{v_{m+1}}(x_{e_m}) - \hat{f}_{v_{m+1}}(x_{e_m})$ . For such a choice  $\tilde{f}_{v_m}(x_{e_m}) = \hat{f}_{v_{m+1}}(x_{e_m})$  and we do not change coincidence relations at edges of  $\Delta(v_0, v_{m+1})$ . In such a way we have reduced  $\#E$  by 1. Now if we take two representatives of extensions of  $\{f_v\}_v$  to the whole of  $C$ , their difference on each component  $C_v$  is a constant map to  $\mathbb{P}^1$ . The condition on edges forces this map to be constant on the whole of  $C$  which ends the proof.  $\square$

Let  $\Delta$  be a subgraph of  $\Gamma$  and let  $L_\Delta$  be the set of legs obtained out of edges in  $\Gamma \setminus \Delta$ . Take an  $S$ -object  $(\mathcal{C}/S, \vec{\rho})$  in  $\mathfrak{Z}_g(\Delta, L_\Delta)$  and let  $e_+$  and  $e_-$  be two legs in  $L_\Delta$  coming from the same edge in  $\Gamma \setminus \Delta$ . Let  $f$  be a morphism to  $\mathbb{P}_S^1$  coming from a lift of  $\vec{\rho}$  to a global section of  $\mathcal{O}_{\mathcal{C}}(\sum_i k_i \mathcal{D}_i)$ . It is clear that the  $S$ -section of  $\mathbb{P}_S^1$  given by  $f \circ \tau_{e_+} - f \circ \tau_{e_-}$  where  $\tau_{e_+}$  and  $\tau_{e_-}$  are the  $S$ -markings attached to  $e_+$  and  $e_-$  does not depend on the lift  $f$ . Hence, once we've numbered legs of  $L_\Delta$  by  $e_{1,+}, e_{1,-}, \dots, e_{\ell_\Delta,+}, e_{\ell_\Delta,-}$  we get a well defined evaluation morphism  $ev_{L_\Delta} : \mathfrak{Z}_g(\Delta, L_\Delta) \rightarrow (\mathbb{P}^1)^{\ell_\Delta}$ . Take a look at the fiber  $ev_{L_\Delta}^{-1}\{(0, \dots, 0)\}$ . Its objects are precisely those of  $\mathfrak{Z}_g(\Gamma)$  but a point in  $ev_{L_\Delta}^{-1}\{(0, \dots, 0)\}$  has in general less automorphisms than a point in  $\mathfrak{Z}_g(\Gamma)$ . However in both cases automorphisms groups of points are finite thus dimension of  $ev_{L_\Delta}^{-1}\{(0, \dots, 0)\}$  and  $\mathfrak{Z}_g(\Gamma)$  are equal. Thus if  $\Delta$  is a maximal tree in  $\Gamma$  the stratum  $\mathfrak{Z}_g(\Gamma)$  can be embedded as a closed substack of  $\mathfrak{Z}_g(\Delta, L_\Delta)$  of codimension at most  $\ell_\Delta = 1 + e(\Gamma) - \nu(\Gamma)$ . Estimates on the dimension of  $\mathfrak{Z}_g(\Gamma)$  are thus given by estimates on the dimension of  $\mathfrak{Z}_g(\Delta, L_\Delta)$  where  $\Delta$  is a tree and  $L_\Delta$  is a set of extra legs.

*Remark 3.* For the following proposition to be meaningful it is necessary to point out that we can always make sense of the stack  $\mathfrak{Z}_0(\{pt\}, 1; 1)$  even though  $(g, n)$  is equal to  $(0, 1)$ . Ignoring the case of  $(g, n) = (0, 1)$  or  $(0, 2)$  was to keep homogenous the treatment of the stack cone structure over  $\mathfrak{M}_{g,n}$  that is void for these entries. If we consider such a question irrelevant, disqualifying the single marked case of genus 0 was due to the fact prestable rational curves having only 1 polar part could have an infinite automorphism group. However, in the case of  $\mathfrak{Z}_0(\{pt\}, 1; 1)$  we're looking at rational curves having two marked points one of which has a polar part. The automorphism group of such an object is following proof of 1.7 finite. We get thus an algebraic stack that is obviously of dimension  $k_1$ .

**Proposition 2.11.** *Let  $\Delta$  be a modular tree having only  $n$  ordered legs and write  $\mathfrak{V}$  for the set of vertices of  $\Delta$  containing at least one leg. Take an extra set of legs  $L_\Delta$  for  $\Delta$ . We have a finite morphism*

$$(2.2) \quad \prod_{v \in \mathfrak{V}} \mathfrak{Z}_{g_v}(v, n_v; \ell_v) \times \prod_{v \in \mathfrak{C}\mathfrak{V}} \mathfrak{M}_{g_v, \ell_v} \longrightarrow \mathfrak{Z}_g(\Delta, L_\Delta)$$

where  $g_v$  is the genus attached to the vertex  $v$ ,  $n_v$  is the number of legs incident to  $v$  and  $\ell_v$  the number of edges and of legs of  $L_\Delta$  out of  $v$ .

*Proof.* Consider the natural clutching morphism attached to  $\Delta$

$$\prod_{v \in \mathfrak{V}} \mathfrak{M}_{g_v, n_v + \ell_v} \times \prod_{v \in \mathfrak{C}\mathfrak{V}} \mathfrak{M}_{g_v, \ell_v} \longrightarrow \overline{\mathfrak{M}}_g(\Delta, L_\Delta).$$

A collection of marked curves in the left hand side are attached along half edges in a manner respecting incidence relations in  $\Delta$ . This clutching morphism is representable and finite. Details can be found in [ACG11, 12.10]. Let  $\mathfrak{X}$  be the fiber product obtained out of the previous morphism and of the forgetful one  $\mathfrak{Z}_g(\Delta, L_\Delta) \rightarrow \overline{\mathfrak{M}}_g(\Delta, L_\Delta)$ . We have a natural finite morphism  $\mathfrak{X} \rightarrow \mathfrak{Z}_g(\Delta, L_\Delta)$ . It is enough to show that  $\mathfrak{X}$  is isomorphic to the left hand side of 2.2 to prove our claim. Take an  $S$ -object

$$(\{\pi_v : \mathcal{C}_v \rightarrow S, \vec{\rho}_v\}_{v \in \mathfrak{V}}, \{\pi_v : \mathcal{C}_v \rightarrow S\}_{v \in \mathfrak{C}\mathfrak{V}})$$

in the left hand side of 2.2. This  $S$ -object naturally induces data  $(\pi : \mathcal{C} \rightarrow S, \{\vec{\rho}_v\}_{v \in \mathfrak{V}})$  where each  $\vec{\rho}_v$  defines an element in  $\mathfrak{A}_{\mathcal{C}_v}$ . Notice that we can attach a constant map in  $\mathfrak{A}_{\mathcal{C}_v}$  to each component  $\mathcal{C}_v$  for  $v \in \mathfrak{C}\mathfrak{V}$  by choosing an  $S$ -section of  $\mathbb{P}_S^1$ . Thus we do have a collection of elements in  $\mathfrak{A}_{\mathcal{C}_v}$  for each  $v \in V(\Gamma)$ . Proof of lemma 2.10 obviously generalizes to this relative case. Hence we get out of the latter data a unique element in

$\mathfrak{A}_{\mathcal{C}}$  whose restriction to each  $\mathcal{C}_v$  gives back  $\tilde{\rho}_v$ . As soon as we unravel the definition of  $\mathfrak{X}$  as a fiber product we see this correspondance defines an isomorphism from the left hand side of 2.2 on  $\mathfrak{X}$ .  $\square$

The coming corollary uses results of section 3.2.

**Corollary 2.12.** *Let  $\Delta$  be a maximal tree in  $\Gamma$  and write  $L_\Delta$  for the set of halfedges in  $\Gamma \setminus \Delta$ . Then we have*

$$\dim \mathfrak{Z}_g(\Delta, L_\Delta) = (3g_\Delta - 3 + n + d) - g_\Delta - e(\Delta) + \sum_{v \in \mathbb{L}\mathfrak{V}} g_v.$$

In particular we get that

$$\dim \mathfrak{Z}_g(\Gamma) \geq (3g_\Gamma - 3 + n + d) - g_\Gamma - e(\Gamma) + \sum_{v \in \mathbb{L}\mathfrak{V}} g_v.$$

*Proof.* The second inequality is a consequence of the fact that  $\dim \mathfrak{Z}_g(\Gamma) \geq \dim \mathfrak{Z}_g(\Delta, L_\Delta) - 1 - \nu(\Gamma) + e(\Gamma)$ . It is therefore enough to compute the dimension of  $\mathfrak{Z}_g(\Delta, L_\Delta)$ . Following proposition 2.11 we have

$$\dim \mathfrak{Z}_g(\Delta, L_\Delta) = \sum_{v \in \mathfrak{V}} \dim \mathfrak{Z}_g(v, n_v; l_v) + \sum_{v \in \mathbb{L}\mathfrak{V}} 3g_v - 3 + l_v.$$

It is a consequence of propositions 3.8 and 3.12 that  $\dim \mathfrak{Z}_g(v, n_v; l_v) = (3g_v - 3 + n_v + l_v + d_v) - g_v$  where  $d_v$  is the sum of  $k_i$  for  $i$  a marking on  $v$ . We therefore have

$$\begin{aligned} \dim \mathfrak{Z}_g(\Delta, L_\Delta) &= 3g_\Delta - 3\nu(\Gamma) + n + d + \#L_\Delta + 2e(\Gamma) - \sum_{v \in \mathfrak{V}} g_v, \\ &= (3g_\Gamma - 3 + n + d) + 3(g_\Delta - g_\Gamma) - 2(\nu(\Gamma) - e(\Gamma) - 1) + 1 - \nu(\Gamma) - \sum_{v \in \mathfrak{V}} g_v + \#L_\Delta, \end{aligned}$$

since  $\nu(\Gamma) - e(\Gamma) - 1 = g_\Delta - g_\Gamma$  we get

$$\dim \mathfrak{Z}_g(\Delta, L_\Delta) = (3g_\Gamma - 3 + n + d) + g_\Delta - g_\Gamma + 1 - \nu(\Gamma) - \sum_{v \in \mathfrak{V}} g_v + \#L_\Delta,$$

but  $\#L_\Delta = 2(1 - \nu(\Gamma) + e(\Gamma))$ , hence

$$\dim \mathfrak{Z}_g(\Delta, L_\Delta) = (3g_\Gamma - 3 + n + d) - g_\Gamma - e(\Gamma) + \sum_{v \in \mathbb{L}\mathfrak{V}} g_v + (1 - \nu(\Gamma) + e(\Gamma)),$$

which is what we need.  $\square$

### 3. BOUNDARY POINTS OF THE ELSV COMPACTIFICATION

We build a contraction from the stack of admissible covers of rational curves by genus  $g$  curves on  $\mathbb{P}^3_{g, \vec{k}}$ . Points in the closure  $\overline{\mathcal{H}}_{g, \vec{k}}$  of  $\mathcal{H}_{g, \vec{k}}$  in  $\mathbb{P}^3_{g, \vec{k}}$  are those in the image of this contraction.

**3.1. Rigidified admissible covers.** We shall only be concerned by covers having genus 0 target curves and specific ramification behaviour. Details about the stack of admissible covers can be found for instance in [HM82] and [BR11]. We shall fix here the type of admissible covers we're interested in.

Let  $\tau_j$  for  $j \in \{1, \dots, r\}$  be partitions of  $d$ . Write  $\overline{\mathfrak{H}}(\vec{k})_{g, \tau_1, \dots, \tau_r}$  for the stack of admissible covers having closed points given by finite surjective maps of marked nodal curves  $\phi: C \rightarrow R$  satisfying

- $\phi^{-1}(\text{Sing}_R) = \text{Sing}_C$ ,
- let  $\tilde{C}$  and  $\tilde{R}$  be the respective normalizations of  $C$  and  $R$ . Write  $\tilde{\phi}: \tilde{C} \rightarrow \tilde{R}$  for the morphism induced by  $\phi$ . Ramification indexes of  $\tilde{\phi}$  at points lying over the same nodal point of  $C$  are equal,

- $R$  is a genus 0 curve marked by  $\infty, q_1, \dots, q_r$  and  $C$  is a genus  $g$  curve marked by  $p_1, \dots, p_n$  and points in the fibers of each  $q_1, \dots, q_r$ . Marked points over each  $q_1, \dots, q_r$  not being ordered.
- both  $R$  and  $C$  are stable with respect to the previous marking,
- $\phi$  is étale away from  $\infty, q_1, \dots, q_r$ , has a profile given by  $\tau_j$  over  $q_j$  for each  $j \in \{1, \dots, r\}$  and satisfies scheme theoretically

$$\phi^{-1}\infty = \sum_{i=1}^n k_i p_i.$$

This last condition implies  $\phi$  has a profile given by  $k_1, \dots, k_n$  over  $\infty$ .

The stack  $\overline{\mathfrak{H}}(\vec{k})_{g, \tau_1, \dots, \tau_r}$  is a compactification of the classical Hurwitz stack  $\mathfrak{H}(\vec{k})_{g, \tau_1, \dots, \tau_r}$  whose points are admissible covers of  $\mathfrak{H}(\vec{k})_{g, \tau_1, \dots, \tau_r}$  having smooth source curve. The stack  $\overline{\mathfrak{H}}(\vec{k})_{g, \tau_1, \dots, \tau_r}$  contains  $\mathfrak{H}(\vec{k})_g$  as a dense open substack. This means every admissible cover is smoothable. Following the description of [HM82] of the versal deformation of an admissible cover one can check that  $\overline{\mathfrak{H}}(\vec{k})_{g, \tau_1, \dots, \tau_r}$  is a reduced stack. On the bad side  $\overline{\mathfrak{H}}(\vec{k})_{g, \tau_1, \dots, \tau_r}$  is in general a singular stack that is not even normal. Its normalization is however a smooth stack. There were different modular descriptions of this normalization, see for instance [Moc95], [ACV03] and [BR11].

We write  $\overline{\mathfrak{H}}(\vec{k})_g$  for the simply ramified case away from  $\infty$ . A result of [Waj96] states that  $\overline{\mathfrak{H}}(\vec{k})_g$  is irreducible. We denote  $\overline{\mathfrak{H}}(\vec{k})_{g, \bullet}$  the disjoint union

$$\overline{\mathfrak{H}}(\vec{k})_{g, \bullet} = \bigsqcup_{(\tau_1, \dots, \tau_r)} \mathfrak{H}(\vec{k})_{g, \tau_1, \dots, \tau_r},$$

where the disjoint union takes place over all  $r$ -tuples of partitions of  $d$ .

In the following, curves in  $\overline{\mathfrak{M}}_{0, r+1}$  are marked by  $r+1$  sections one of which is special and thus indexed by  $\infty$ . Let  $(R, \vec{\eta})$  be an  $r+1$ -marked stable genus 0 curve and write  $R_1, \dots, R_\nu$  for irreducible components of  $R$ . The nodal curve defined by  $R$  is a tree of genus 0 smooth curves. Following [OS79, 10.2] we have isomorphisms

$$\text{Pic}(R) \simeq \prod_{j=1}^{\nu} \text{Pic}(R_j) \simeq \mathbb{Z}^{\nu}.$$

An invertible sheaf  $\mathcal{F}$  on  $R$  is thus given by the  $\nu$ -tuple of its degrees on  $R_1, \dots, R_\nu$ , we call it the multidegree of  $\mathcal{F}$  and write it  $\text{deg}_m(\mathcal{F})$ . The invertible sheaf on  $R$  of multidegree  $\vec{d}$  is written  $\mathcal{O}_R(\vec{d})$ . For example the canonical line bundle on  $R$  is given by  $(-2 + n(R_1), \dots, -2 + n(R_\nu))$  where  $n(R_i)$  is the number of special nodal points on  $R_i$ .

**Lemma 3.1.** *Keep previous notation. Assume  $\vec{d}$  is an  $\nu$ -tuple of natural numbers. We have  $h^1(\mathcal{O}_R(\vec{d})) = 0$  and  $h^0(\mathcal{O}_R(\vec{d})) = \|\vec{d}\| + 1$ .*

*Proof.* Using Serre duality  $h^1(\mathcal{O}_R(\vec{d})) = h^0(\omega_R(-\vec{d}))$ . If  $R$  is smooth  $\omega_R(-\vec{d})$  is of negative degree and thus  $H^0(\omega_R(-\vec{d})) = 0$ . When this is not the case take a component  $R_j$  of  $R$  that is a leaf of its dual graph. Since  $R_j$  is a leaf it can only have one nodal point. This means the restriction of  $\omega_R(-\vec{d})$  on  $R_j$  is still of negative degree and any section in  $H^0(\omega_R(-\vec{d}))$  is zero on  $R_j$ . Let  $R_j^*$  be the closure of  $R \setminus R_j$  in  $R$ , we thus get an isomorphism of  $H^0(\omega_R(-\vec{d}))$  with  $H^0(\omega_R(-\vec{d})|_{R_j^*})$ . Since  $R_j^*$  is a nodal genus 0 curve having  $\nu - 1$  irreducible components we conclude by induction. The expression for  $h^0(\mathcal{O}_R(\vec{d}))$  is a direct consequence of [OS79, 10.4].  $\square$

Let  $q_1, \dots, q_r, q_\infty$  be the images of  $\eta_1, \dots, \eta_r, \eta_\infty$ . Write  $R_\infty$  for the irreducible component of  $R$  containing  $\eta_\infty$ . Consider the case of the invertible sheaf  $\mathcal{O}_R(q_\infty)$ . Since

restriction to  $R_\infty$  gives an isomorphism  $H^0(\mathcal{O}_R(q_\infty)) = H^0(\mathcal{O}_{R_\infty}(q_\infty))$  the sheaf  $\mathcal{O}_R(q_\infty)$  is base point free. Following 3.1  $h^0(\mathcal{O}_{R_\infty}(q_\infty)) = 2$  and

$$\oplus_{\ell \geq 0} H^0(\mathcal{O}_R(\ell q_\infty)) \simeq \text{Sym } H^0(\mathcal{O}_R(q_\infty)).$$

This induces a morphism  $R \rightarrow \text{Proj } H^0(\mathcal{O}_R(q_\infty))$ . The choice of a basis of  $H^0(\mathcal{O}_R(q_\infty))$  gives a morphism from  $R$  to  $\mathbb{P}^1$ . Each moduli point  $\phi : C \rightarrow R$  in  $\overline{\mathfrak{H}}(\vec{\mathbf{k}})_{g,\bullet}$  has target that is a genus 0  $r+1$ -marked stable marked curve  $(R, \vec{\eta})$ . If we choose a basis of  $H^0(\mathcal{O}_R(q_\infty))$  following the previous steps we can get out of  $\phi$  a morphism  $\varphi : C \rightarrow \mathbb{P}^1$ . This is a complete intersection morphism having smooth target. In a fashion similar to what has been done in 2.2 we're going to impose a condition on admissible choices of a basis of  $H^0(\mathcal{O}_R(q_\infty))$ . Once this is done we built a stack whose moduli points are couples of covers in  $\overline{\mathfrak{H}}(\vec{\mathbf{k}})_{g,\bullet}$  together with a good choice of a basis of  $H^0(\mathcal{O}_R(q_\infty))$ .

Take an  $S$ -object  $\phi : \mathcal{C} \rightarrow \mathcal{R}$  and let  $(\pi : \mathcal{R} \rightarrow S, \vec{\eta})$  be its target in  $\mathfrak{M}_{0,r+1}$ . Let  $\mathcal{Q}_1, \dots, \mathcal{Q}_\infty$  be the images of the sections given by  $\vec{\eta}$ . Let  $\mathcal{E}_\infty$  be the direct image  $\pi_* \mathcal{O}(\mathcal{Q}_\infty)$ . Following lemma 3.1 we have  $h^0(\mathcal{O}_{\mathcal{R}_s}(\mathcal{Q}_{s,\infty}))$  is 2 and  $h^1(\mathcal{O}_{\mathcal{R}_s}(\mathcal{Q}_\infty)) = 0$ . By [Knu83, 1.5] we get that  $\mathcal{E}_\infty$  is a locally free sheaf of rank 2 that commutes with any base change. Attached to  $(\mathcal{R}, \vec{\eta})$  we have by push forward an exact sequence

$$(3.1) \quad 0 \longrightarrow \mathcal{O}_S \longrightarrow \mathcal{E}_\infty \xrightarrow{\kappa} \mathcal{L}_\infty^{-1} \longrightarrow 0.$$

We write  $\mathbf{1} \in \Gamma(\mathcal{E}_\infty)$  for the image of the canonical unity section of  $\mathcal{O}_S$ . This is a nowhere vanishing section of  $\Gamma(\mathcal{E}_\infty)$ .

**Definition 3.1.** A rigidification of  $\phi$  is given by the choice of a global section  $\xi$  of  $\mathcal{E}_\infty$  such that  $\{\mathbf{1}, \xi\}$  trivializes  $\mathcal{E}_\infty$ .

A rigidification is precisely given by the choice of a global non-zero section of  $\mathcal{L}_\infty^{-1}$  and a section of  $\kappa$ . Two sections of  $\kappa$  differ by an element of the form  $\mu \mathbf{1}$  for  $\mu \in \Gamma(S, \mathcal{O}_S)$  and two global sections trivializing  $\mathcal{L}_\infty^{-1}$  by  $\lambda \in \Gamma(S, \mathcal{O}_S^*)$ . This means that given a rigidification  $\xi \in \Gamma(\mathcal{E}_\infty)$  any other rigidification is of the form  $\lambda \xi + \mu \mathbf{1}$  for  $(\lambda, \mu) \in \Gamma(\mathcal{O}_S^*) \times \Gamma(\mathcal{O}_S)$ . Equivalently, the set of rigidifications of  $\phi$  is a torsor under  $\mathbb{G}_a \rtimes \mathbb{G}_m$ . Let  $\varphi(\xi) : \mathcal{C} \rightarrow \mathbb{P}_S^1$  be the morphism attached to  $\xi$ , the morphism  $\varphi(\lambda \xi + \mu)$  attached to  $\lambda \xi + \mu$  is obviously obtained from  $\varphi(\xi)$  composing with the automorphism of  $(\mathbb{P}^1, \infty)$  given by  $(\lambda, \mu) \in \mathbb{G}_m \times \mathbb{G}_a$ .

**Lemma 3.2.** *Keep previous notation. Let  $\xi$  be a rigidification of  $\phi$ . There is a unique rigidification  $\hat{\xi}$  lift  $\kappa(\xi)$  such that the induced morphism  $R \rightarrow \mathbb{P}^1$  is normalized in the sense of definition 2.2.*

*Proof.* The morphism  $\varphi$  attached to  $\phi$  has relative complete intersection source and relative smooth target. In our case this is the only needed assumptions to define the branch divisor in the sense of [FP02]. Following the proof of proposition 2.6 step by step we get the result.  $\square$

**Definition 3.2.** A rigidification of  $\phi$  satisfying such that its induced morphism  $R \rightarrow \mathbb{P}^1$  is normalized is said to be normalized.

**Definition 3.3.** Let  $\overline{\mathfrak{H}}(\vec{\mathbf{k}})_{g,\bullet}^{rig}$  be the stack whose  $S$ -objects are given by an  $S$ -object in  $\overline{\mathfrak{H}}(\vec{\mathbf{k}})_{g,\bullet}$  together with a normalized rigidification. Morphisms of  $\overline{\mathfrak{H}}(\vec{\mathbf{k}})_{g,\bullet}^{rig}$  are those of  $\overline{\mathfrak{H}}(\vec{\mathbf{k}})_{g,\bullet}$  fixing the rigidification. We write  $\overline{\mathfrak{H}}(\vec{\mathbf{k}})_{g,\tau_1,\dots,\tau_r}^{rig}$  and  $\overline{\mathfrak{H}}(\vec{\mathbf{k}})_g^{rig}$  for the obvious variants of  $\overline{\mathfrak{H}}(\vec{\mathbf{k}})_{g,\bullet}^{rig}$ .

It is obvious to see that the set of normalized rigidifications of  $\phi$  is a  $\mathbb{G}_m$ -torsor. This says that the forgetful morphism  $\overline{\mathfrak{H}}(\vec{\mathbf{k}})_{g,\bullet}^{rig} \rightarrow \overline{\mathfrak{H}}(\vec{\mathbf{k}})_{g,\bullet}$  makes of  $\overline{\mathfrak{H}}(\vec{\mathbf{k}})_{g,\bullet}^{rig}$  a  $\mathbb{G}_m$ -torsor over  $\overline{\mathfrak{H}}(\vec{\mathbf{k}})_{g,\bullet}$ . Since  $\overline{\mathfrak{H}}(\vec{\mathbf{k}})_{g,\tau_1,\dots,\tau_r}$  is of dimension  $r - 2$  the dimension of  $\overline{\mathfrak{H}}(\vec{\mathbf{k}})_{g,\tau_1,\dots,\tau_r}^{rig}$  is  $r - 1$ .

**Proposition 3.3.** *The stack  $\overline{\mathfrak{H}}(\vec{\mathbf{k}})_{g,\bullet}^{rig}$  is a  $\mathbb{G}_m$ -torsor over  $\overline{\mathfrak{H}}(\vec{\mathbf{k}})_{g,\bullet}$ . It is the one defined by the line bundle  $\mathcal{L}_\infty$ .*

*Proof.* A normalized rigidification of an  $S$ -object  $\phi$  in  $\overline{\mathfrak{H}}(\vec{\mathbf{k}})_{g,\bullet}^{rig}$  is given by the choice of a global nonvanishing  $S$ -section of  $\mathcal{L}_\infty^{-1}$ . Equivalently an  $S$ -section of  $\mathbf{Spec}_S(\oplus_{\ell \in \mathbb{Z}} \mathcal{L}_\infty^\ell)$ .  $\square$

**Corollary 3.4.** *We have an isomorphism  $\overline{\mathfrak{H}}(\vec{\mathbf{k}})_{g,\bullet} \simeq [\overline{\mathfrak{H}}(\vec{\mathbf{k}})_{g,\bullet}^{rig} / \mathbb{G}_m]$ .*

**3.2. Hurwitz Projection.** Let  $(\phi : \mathcal{C} \rightarrow \mathcal{R}, \xi)$  be an admissible cover in  $\overline{\mathfrak{H}}(\vec{\mathbf{k}})_{g,\bullet}^{rig}$  over a scheme  $S$ . Rigidification defines a morphism of marked curves from  $\mathcal{R}$  onto  $\mathbb{P}_S^1$ . Composing with  $\phi$  we get a morphism  $\varphi : \mathcal{C} \rightarrow \mathbb{P}_S^1$ .

**Lemma 3.5.** *The map  $\varphi$  is of degree  $d$ .*

*Proof.* Following the proof of lemma 2.1 it is enough to look at closed fibers  $\varphi_s$ . In this case  $\mathcal{R}_s \rightarrow \mathbb{P}^1$  is an isomorphism on  $\mathcal{R}_{s,\infty}$  and contracts any other component on a point. This implies the degree of  $\varphi_s : \mathcal{C}_s \rightarrow \mathbb{P}^1$  is equal to the degree of  $\phi_{s|\mathcal{R}_{s,\infty}}$  which is, in turn, equal to the degree of  $\phi_s$  on any other component for  $\phi_s$  is admissible. Hence  $\varphi_s$  is of degree  $d$ .  $\square$

Keep previous notation. The map  $\varphi$  is a map between stable marked curves. There is a forgetful morphism that only keeps on mind the marking of  $\mathbb{P}^1$  at  $\infty$  and markings of  $\mathcal{C}$  given by the sections  $\sigma_1, \dots, \sigma_n$  going on  $\infty$ . This forgetful morphism can destroy stability of the map  $\varphi$ . Let  $\psi : (\mathcal{C}, \vec{\sigma}) \rightarrow (\mathbb{P}_S^1, \infty)$  be the stabilization of  $\varphi$ .

**Lemma 3.6.**  *$\psi$  is a stable map of profile  $\vec{\mathbf{k}}$ .*

*Proof.* This is a statement on closed fibers. Let  $s$  be a point in  $S$ . We need to show that locally around each point  $\sigma_{s,\ell}$  the map  $\psi_s$  is of profile  $k_\ell$ . But each such point comes from a point on  $\mathcal{C}_s$  that lie on a component going on  $\mathcal{R}_\infty$ , i.e. a component that is not contracted on  $\mathbb{P}^1$ . Hence the local behaviour of  $\psi_s$  at  $\sigma_{s,\ell}$  is the one of  $\phi_s$  at the neighborhood of this same point and this ends the proof.  $\square$

**Definition 3.4.** Let  $\rho_{\vec{\tau}}$  be the functor from  $\overline{\mathfrak{H}}(\vec{\mathbf{k}})_{g,\tau_1,\dots,\tau_r}^{rig}$  on  $\overline{\mathfrak{M}}_{g,n}(\mathbb{P}^1, \vec{\mathbf{k}})$  defined in the previous fashion. Write  $\rho$  for the case of  $\overline{\mathfrak{H}}(\vec{\mathbf{k}})_g^{rig}$  and  $\rho_\bullet$  for  $\overline{\mathfrak{H}}(\vec{\mathbf{k}})_{g,\bullet}^{rig}$ . This last morphism is called the Hurwitz projection.

**Proposition 3.7.** *The projection  $\rho_\bullet$  factors through the embedding of  $\mathfrak{Z}_{g,\vec{\mathbf{k}}}$  as a closed substack of  $\overline{\mathfrak{M}}_{g,n}(\mathbb{P}^1, \vec{\mathbf{k}})$  built in section 2.2.*

*Proof.* Let  $\hat{\rho}_\bullet$  be the composition of  $\rho_\bullet$  with the natural quotient  $\overline{\mathfrak{M}}_{g,n}(\mathbb{P}^1, \vec{\mathbf{k}}) \rightarrow \mathfrak{Z}_{g,\vec{\mathbf{k}}}$  followed by the its section built in section 2.2. By definition of rigidification it is obvious that  $\rho_\bullet$  and  $\hat{\rho}_\bullet$  coincide on  $\mathfrak{H}(\vec{\mathbf{k}})_{g,\bullet}^{rig}$ . This means they're equal on an open dense substack of  $\overline{\mathfrak{H}}(\vec{\mathbf{k}})_{g,\bullet}^{rig}$ . Since  $\overline{\mathfrak{M}}_{g,n}(\mathbb{P}^1, \vec{\mathbf{k}})$  is separated, the locus where  $\rho_\bullet$  and  $\hat{\rho}_\bullet$  are equal is closed in  $\overline{\mathfrak{H}}(\vec{\mathbf{k}})_{g,\bullet}^{rig}$ . But since  $\overline{\mathfrak{H}}(\vec{\mathbf{k}})_{g,\bullet}^{rig}$  is reduced such a substack is the whole of  $\overline{\mathfrak{H}}(\vec{\mathbf{k}})_{g,\bullet}^{rig}$  (see [The, 050A]).  $\square$

Obviously  $\rho_\bullet$  is  $\mathbb{G}_m$ -equivariant for the action of  $\mathbb{G}_m$  on the rigidification of an object in  $\overline{\mathfrak{H}}(\vec{k})_{g,\bullet}^{rig}$ . Thus  $\rho_\bullet$  defines a morphism  $\rho_\bullet : \overline{\mathfrak{H}}(\vec{k})_g \rightarrow \mathbb{P}\mathfrak{Z}_{g,\vec{k}}$ .

Let  $H_{g,\vec{k},\bullet}$  be the open substack of  $\mathfrak{Z}_{g,\vec{k}}$  given by objects having smooth underlying curves and write  $\overline{H}_{g,\vec{k},\bullet}$  for the closure of  $H_{g,\vec{k},\bullet}$  in  $\mathfrak{Z}_{g,\vec{k}}$ . Write  $H_{g,\vec{k}}$  be the open substack of  $H_{g,\vec{k},\bullet}$  corresponding to points of  $\mathfrak{Z}_{g,\vec{k}}$  having simple ramification away from  $\infty$ . One can check that the quotient of  $H_{g,\vec{k}}$  by the natural  $\mathbb{G}_m$ -action written  $\mathbb{P}H_{g,\vec{k}}$  is the stack  $\mathcal{H}_{g,\vec{k}}$  mentioned in the introduction to this paper. The ELSV compactification  $\overline{\mathcal{H}}_{g,\vec{k}}$  is equal to the quotient  $\mathbb{P}\overline{H}_{g,\vec{k}}$ .

**Proposition 3.8.** *The restriction of  $\rho_\bullet$  to  $\mathfrak{H}(\vec{k})_{g,\bullet}^{rig}$  is a trivial  $\mathfrak{S}_r$ -torsor over  $H_{g,\vec{k},\bullet}$ .*

*Proof.* Taking the image of an  $S$ -object  $(\phi : \mathcal{C} \rightarrow D, \xi) \in \mathfrak{H}(\vec{k})_{g,\bullet}^{rig}$  involves no contraction on the target curve and thus no contraction on the source. The choice of  $\xi$  makes sure the resulting stable map  $\rho_\bullet(\phi)$  is in the embedding of  $\mathfrak{Z}_{g,\vec{k}}$  in  $\overline{\mathfrak{M}}_{g,n}(\mathbb{P}^1, \vec{k})$  described in section 2.2. Now take an  $S$ -object  $\psi : \mathcal{C} \rightarrow \mathbb{P}^1$  in  $H_{g,\vec{k},\bullet}$  and choose an order on its branch points outside  $\infty$ . This data defines a unique  $S$ -object in  $\mathfrak{H}(\vec{k})_{g,\bullet}^{rig}$  by taking for rigidification the one defined by  $x_\infty$  where  $x_\infty$  is the coordinate in the neighborhood of  $\infty$ .  $\square$

**Corollary 3.9.** *The open substack of  $\mathfrak{Z}_{g,\vec{k}}$  given by  $H_{g,\vec{k},\bullet}$  is smooth. In particular it is a local complete intersection in  $\mathfrak{X}_{g,\vec{k}}$ .*

*Proof.* This is the case for  $\mathfrak{H}(\vec{k})_{g,\bullet}^{rig}$ .  $\square$

**Lemma 3.10.** *The morphism  $\rho_\bullet$  is surjective on  $\overline{H}_{g,\vec{k},\bullet}$ .*

*Proof.* It is enough to show this on moduli points of  $\overline{H}_{g,\vec{k},\bullet}$ . Any point  $\phi : C \rightarrow \mathbb{P}^1$  living in  $\overline{H}_{g,\vec{k},\bullet}$  has got—by definition—a smoothing family  $\phi_A : \mathcal{C}_A \rightarrow \mathbb{P}_A^1$  over a discrete valuation ring  $A$ . This means the central fiber of  $\phi_A$  is equal to  $\phi$  and that  $\phi_A$  is a finite morphism of smooth curves on the generic fiber. Now the generic fiber is precisely an object in  $\mathfrak{H}(\vec{k})_{g,\bullet}$ . Since  $\mathfrak{H}(\vec{k})_{g,\bullet}$  is proper it can be extended uniquely to an  $A$ -object of  $\overline{\mathfrak{H}}(\vec{k})_{g,\bullet}$ . Because  $\mathfrak{Z}_{g,\vec{k}}$  is separated the image of this extension under  $\rho_\bullet$  gives precisely on the central fiber  $\phi$ .  $\square$

**Proposition 3.11.** *The restriction  $\rho$  of  $\rho_\bullet$  to  $\overline{\mathfrak{H}}(\vec{k})_g^{rig}$  is surjective on  $\overline{H}_{g,\vec{k},\bullet}$ .*

*Proof.* This is as well a claim on moduli points. Following proposition 3.10 it is enough to show that any point  $\phi : C \rightarrow D$  in  $\overline{\mathfrak{H}}(\vec{k})_{g,r_1,\dots,r_r}^{rig}$  can be obtained from a point  $\psi : B \rightarrow E$  by contracting rational components on the base and target of  $\psi$ . Take a branch point  $q$  of  $D$  that is not simple. It's fiber is given by  $m$  points  $q_1, \dots, q_m$  with multiplicities  $m_1, \dots, m_m$ . Attach a rational curve at each of the points  $q, q_1, \dots, q_m$ . Write  $E$  for the rational curve at  $q$  and  $D_\ell$  for the rational curve at  $q_\ell$ . Take on each  $D_\ell$  a simply ramified map to  $\mathbb{P}^1$  except at the point  $q_\ell$  where it has ramification of order  $m_\ell$ . Making such choices for each branch point of  $D$  gives a simply ramified admissible cover whose image by  $\rho$  gives back  $\phi$ .  $\square$

Using the previous proof one can see that  $\rho$  is surjective on  $\overline{H}_{g,\vec{k}}$  as well. This means that both closures  $\overline{H}_{g,\vec{k},\bullet}$  and  $\overline{H}_{g,\vec{k}}$  are equal. In particular  $H_{g,\vec{k}}$  is dense in  $\overline{H}_{g,\vec{k},\bullet}$ . In particular one gets that  $\mathcal{H}_{g,\vec{k}}$  is dense in  $\mathbb{P}\overline{H}_{g,\vec{k},\bullet}$ . This means that there are no restrictions on the branch divisor of boundary points of the ELSV compactification.

**Corollary 3.12.**  *$\overline{H}_{g,\vec{k},\bullet}$  is irreducible of codimension  $g$  in  $\mathfrak{X}_{g,\vec{k}}$ .*

*Proof.* It is well known that  $\overline{\mathfrak{H}}(\vec{k})_g^{rig}$  is irreducible. We've just seen that  $H_{g,\vec{k}}$  is dense in  $\overline{H}_{g,\vec{k},\bullet}$ . The dimension of  $H_{g,\vec{k}}$  is equal to the one of  $\mathfrak{H}(\vec{k})_g^{rig}$  which is  $r - 1 = 2g - 2 + n + d - 1 = (3g - 3 + n + d) - g$ .  $\square$

It is natural to wonder whether  $\overline{H}_{g,\vec{k},\bullet}$  is the whole of  $\mathfrak{Z}_{g,\vec{k}}$  or not. The answer is no. A stratum  $\mathfrak{Z}_g(\Gamma)$  for a modular graph  $\Gamma$  cannot be a substack of  $\mathfrak{Z}_g(\Gamma)$  unless its dimension is less than the dimension of  $\overline{H}_{g,\vec{k},\bullet}$  given by maps to  $\mathbb{P}^1$  of source curve of type  $\Gamma$ . This is the image of the corresponding stratum in  $\overline{\mathfrak{H}}(\vec{k})_g^{rig}$  and is thus of dimension  $3g - 3 + n + d - g - e(\Gamma)$ . Using 2.12 we get that for  $\mathfrak{Z}_g(\Gamma)$  to be included in  $\overline{H}_{g,\vec{k},\bullet}$

$$(3g - 3 + n + d) - g - e(\Gamma) + \sum_{v \in \mathcal{CV}} g_v \leq \dim \mathfrak{Z}_g(\Gamma) \leq (3g - 3 + n + d) - g - e(\Gamma).$$

This implies any point in such  $\mathfrak{Z}_g(\Gamma)$  must contract only rational curves and this is not the case of all points in  $\mathfrak{Z}_{g,\vec{k}}$  as soon as  $g \neq 0$ .

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