

# Spectral analysis of Markov kernels and application to the convergence rate of discrete random walks

Loïc HERVÉ and James LEDOUX <sup>1</sup>

INSA de Rennes, F-35708, France; IRMAR CNRS-UMR 6625, F-35000, France; Université Européenne de Bretagne, France.

{Loic.Herve,James.Ledoux}@insa-rennes.fr

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## Abstract

Let  $\{X_n\}_{n \in \mathbb{N}}$  be a Markov chain on a measurable space  $\mathbb{X}$  with transition kernel  $P$  and let  $V : \mathbb{X} \rightarrow [1, +\infty)$ . The Markov kernel  $P$  is here considered as a linear bounded operator on the weighted-supremum space  $\mathcal{B}_V$  associated with  $V$ . Then the combination of quasi-compactness arguments with precise analysis of eigen-elements of  $P$  allows us to estimate the geometric rate of convergence  $\rho_V(P)$  of  $\{X_n\}_{n \in \mathbb{N}}$  to its invariant probability measure in operator norm on  $\mathcal{B}_V$ . A general procedure to compute  $\rho_V(P)$  for discrete Markov random walks with identically distributed bounded increments is specified.

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## 1 Introduction

Let  $(\mathbb{X}, \mathcal{X})$  be a measurable space with a  $\sigma$ -field  $\mathcal{X}$ , and let  $\{X_n\}_{n \geq 0}$  be a Markov chain with state space  $\mathbb{X}$  and transition kernels  $\{P(x, \cdot) : x \in \mathbb{X}\}$ . Let  $V : \mathbb{X} \rightarrow [1, +\infty)$ . Assume that  $\{X_n\}_{n \geq 0}$  has an invariant probability measure  $\pi$  such that  $\pi(V) := \int_{\mathbb{X}} V(x) \pi(dx) < \infty$ . This paper is based on the connection between spectral properties of the Markov kernel  $P$  and the so-called  $V$ -geometric ergodicity [MT93] which is the following convergence property for some constants  $c_\rho > 0$  and  $\rho \in (0, 1)$ :

$$\sup_{|f| \leq V} \sup_{x \in \mathbb{X}} \frac{|\mathbb{E}[f(X_n) | X_0 = x] - \pi(f)|}{V(x)} \leq c_\rho \rho^n. \quad (1)$$

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<sup>1</sup>postal address : INSA de Rennes, 20 avenue des Buttes de Coesmes, CS 70 839 35708 Rennes Cedex 7

Let us introduce the weighted-supremum Banach space  $(\mathcal{B}_V, \|\cdot\|_V)$  composed of measurable functions  $f : \mathbb{X} \rightarrow \mathbb{C}$  such that

$$\|f\|_V := \sup_{x \in \mathbb{X}} \frac{|f(x)|}{V(x)} < \infty.$$

Then (1) reads as  $\|P^n f - \pi(f)1_{\mathbb{X}}\|_V \leq c_\rho \rho^n$  for any  $f \in \mathcal{B}_V$  such that  $\|f\|_V \leq 1$ , and there is a great interest in obtaining upper bounds for *the convergence rate*  $\rho_V(P)$  defined by

$$\rho_V(P) := \inf \left\{ \rho \in (0, 1), \sup_{\|f\|_V \leq 1} \|P^n f - \pi(f)1_{\mathbb{X}}\|_V = O(\rho^n) \right\}. \quad (2)$$

For irreducible and aperiodic discrete Markov chains, criteria for the  $V$ -geometric ergodicity are well-known from the literature using either the equivalence between geometric ergodicity and  $V$ -geometric ergodicity of  $\mathbb{N}$ -valued Markov chains [HS92, Prop. 2.4], or the strong drift condition. For instance, when  $\mathbb{X} := \mathbb{N}$  (with  $\lim_n V(n) = +\infty$ ), the strong drift condition is

$$PV \leq \varrho V + b 1_{\{0,1,\dots,n_0\}}$$

for some  $\varrho < 1, b < \infty$  and  $n_0 \in \mathbb{N}$  (see [MT93]). Estimating  $\rho_V(P)$  from the parameters  $\varrho, b, n_0$  is a difficult issue. This often leads to unsatisfactory bounds, except for stochastically monotone  $P$  (see [MT94, LT96, Bax05] and the references therein).

This work presents a new procedure to study the convergence rate  $\rho_V(P)$  under the following weak drift condition

$$\exists N \in \mathbb{N}^*, \exists d \in (0, +\infty), \exists \delta \in (0, 1), \quad P^N V \leq \delta^N V + d 1_{\mathbb{X}}. \quad (\mathbf{WD})$$

The  $V$ -geometric ergodicity clearly implies **(WD)**. Conversely, such a condition with  $N = 1$  was introduced in [MT93, Lem. 15.2.8] as an alternative to the drift condition [MT93, (V4)] to obtain the  $V$ -geometric ergodicity under suitable assumption on  $V$ . Note that, under Condition **(WD)**, the following real number  $\delta_V(P)$  is well defined:

$$\delta_V(P) := \inf \left\{ \delta \in [0, 1) : \exists N \in \mathbb{N}^*, \exists d \in (0, +\infty), P^N V \leq \delta^N V + d 1_{\mathbb{X}} \right\}.$$

A spectral analysis of  $P$  is presented in Section 2 using quasi-compactness. More specifically, when the Markov kernel  $P$  has an invariant probability distribution, the connection between the  $V$ -geometric ergodicity and the quasi-compactness of  $P$  is made explicit in Proposition 2.1. Namely,  $P$  is  $V$ -geometrically ergodic if and only if  $P$  is a power-bounded quasi-compact operator on  $\mathcal{B}_V$  for which  $\lambda = 1$  is a simple eigenvalue and the unique eigenvalue of modulus one. In this case, if  $r_{ess}(P)$  denotes the essential spectral radius of  $P$  on  $\mathcal{B}_V$  (see (5)) and if  $\mathcal{V}$  denotes the set of eigenvalues  $\lambda$  of  $P$  such that  $r_{ess}(P) < |\lambda| < 1$ , then the convergence rate  $\rho_V(P)$  is given by (Proposition 2.1):

$$\rho_V(P) = r_{ess}(P) \text{ if } \mathcal{V} = \emptyset \quad \text{and} \quad \rho_V(P) = \max\{|\lambda|, \lambda \in \mathcal{V}\} \text{ if } \mathcal{V} \neq \emptyset. \quad (3)$$

Interesting bounds for generalized eigenfunctions  $f \in \mathcal{B}_V \cap \text{Ker}(P - \lambda I)^p$  associated with  $\lambda \in \mathcal{V}$  are presented in Proposition 2.2. Property (3) is relevant to study the convergence rate  $\rho_V(P)$  provided that, first an accurate bound of  $r_{ess}(P)$  is known, second the above

set  $\mathcal{V}$  is available. Bounds of  $r_{ess}(P)$  related to drift conditions can be found in [Wu04] and [HL14] under various assumptions (see Subsection 2.1). In view of our applications, let us just mention that  $r_{ess}(P) = \delta_V(P)$  in case  $\mathbb{X} := \mathbb{N}$  and  $\lim_n V(n) = +\infty$  (see Proposition 3.1). However, even if the state space is discrete, finding the above set  $\mathcal{V}$  is difficult.

In Section 3, the above spectral analysis is applied to compute the rate of convergence  $\rho_V(P)$  of discrete Random Walks (RW). In particular, a complete solution is presented for RWs with identically distributed (i.d.) bounded increments. In fact, Proposition 3.4 allows us to formulate an algebraic procedure based on polynomial eliminations providing  $\rho_V(P)$  (see Corollary 4.1). To the best of our knowledge, this general result is new. Note that it requires neither reversibility nor stochastic monotonicity of  $P$ .

This procedure is illustrated in Section 4. First we consider the case of birth-and-death Markov kernel  $P$  defined by  $P(0,0) := a$  and  $P(0,1) := 1 - a$  for some  $a \in (0,1)$  and by

$$\forall n \geq 1, P(n, n-1) := p, \quad P(n, n) := r, \quad P(n, n+1) := q,$$

where  $p, q, r \in [0,1]$  are such that  $p + r + q = 1$ ,  $p > q > 0$ . Explicit formula for  $\rho_V(P)$  with respect to  $V := \{(p/q)^{n/2}\}_{n \in \mathbb{N}}$  is given in Proposition 4.1. When  $r := 0$ , such a result has been obtained for  $a < p$  in [RT99] and [Bax05, Ex. 8.4] using Kendall's theorem, and for  $a \geq p$  in [LT96] using the stochastic monotony of  $P$ . Our method gives a unified and simpler computation of  $\rho_V(P)$  which moreover encompasses the case  $r \neq 0$ . For general RWs with i.d. bounded increments, the elimination procedure requires to use symbolic computations. The second example illustrates this point with the non reversible RW defined by

$$\forall n \geq 2, P(n, n-2) = a_{-2}, \quad P(n, n-1) = a_{-1}, \quad P(n, n) = a_0, \quad P(n, n+1) = a_1$$

for any nonnegative  $a_i$  satisfying  $a_{-2} + a_{-1} + a_0 + a_1 = 1$ ,  $a_{-2} > 0$ ,  $2a_{-2} + a_{-1} > a_1 > 0$ , and for any finitely many boundary transition probabilities. In Section 5, specific examples of RWs on  $\mathbb{X} := \mathbb{N}$  with unbounded increments considered in the literature are investigated.

To conclude this introduction, we mention a point which can be source of confusion in a first reading. In this paper, we are concerned with the convergence rate (2) with respect to some weighted-supremum Banach space  $\mathcal{B}_V$ . Thus, we do not consider here the decay parameter or the convergence rate of ergodic Markov chains in the usual Hilbert space  $\mathbb{L}^2(\pi)$  which is related to spectral properties of the transition kernel with respect to this space. In particular, for Birth-and-Death Markov chains, we can not compare our results with those of [vDS95] on the  $\ell^2(\pi)$ -spectral gap and the decay parameter. A detailed discussion is provided in Remark 4.2.

## 2 Quasi-compactness on $\mathcal{B}_V$ and $V$ -geometric ergodicity

We assume that  $P$  satisfies **(WD)**. Then  $P$  continuously acts on  $\mathcal{B}_V$ , and iterating **(WD)** shows that  $P$  is power-bounded on  $\mathcal{B}_V$ , namely  $\sup_{n \geq 1} \|P^n\|_V < \infty$ , where  $\|\cdot\|_V$  also stands for the operator norm on  $\mathcal{B}_V$ . Thus we have  $r(P) := \lim_n \|P^n\|_V^{1/n} = 1$  since  $P$  is Markov.

## 2.1 From quasi-compactness on $\mathcal{B}_V$ to $V$ -geometric ergodicity

Let  $I$  denote the identity operator on  $\mathcal{B}_V$ . Recall that  $P$  is said to be quasi-compact on  $\mathcal{B}_V$  if there exist  $r_0 \in (0, 1)$  and  $m \in \mathbb{N}^*$ ,  $\lambda_i \in \mathbb{C}$ ,  $p_i \in \mathbb{N}^*$  ( $i = 1, \dots, m$ ) such that:

$$\mathcal{B}_V = \bigoplus_{i=1}^m \text{Ker}(P - \lambda_i I)^{p_i} \oplus H, \quad (4a)$$

where the  $\lambda_i$ 's are such that

$$|\lambda_i| \geq r_0 \quad \text{and} \quad 1 \leq \dim \text{Ker}(P - \lambda_i I)^{p_i} < \infty, \quad (4b)$$

and  $H$  is a closed  $P$ -invariant subspace such that

$$\inf_{n \geq 1} \left( \sup_{h \in H, \|h\| \leq 1} \|P^n h\| \right)^{1/n} < r_0. \quad (4c)$$

Concerning the essential spectral radius of  $P$ , denoted by  $r_{ess}(P)$ , here it is enough to have in mind that, if  $P$  is quasi-compact on  $\mathcal{B}_V$ , then we have (see for instance [Hen93])

$$r_{ess}(P) := \inf \{ r_0 \in (0, 1) \text{ such that (4a)-(4c) hold} \}. \quad (5)$$

As mentioned in Introduction, the essential spectral radius of Markov kernels acting on  $\mathcal{B}_V$  is studied in [Wu04, HL14]. For instance, under Condition **(WD)**, the following result is proved in [HL14]: if  $P^\ell$  is compact from  $\mathcal{B}_0$  to  $\mathcal{B}_V$  for some  $\ell \geq 1$ , where  $(\mathcal{B}_0, \|\cdot\|_0)$  is the Banach space composed of bounded measurable functions  $f : \mathbb{X} \rightarrow \mathbb{C}$  equipped with the supremum norm  $\|f\|_0 := \sup_{x \in \mathbb{X}} |f(x)|$ , then  $P$  is quasi-compact on  $\mathcal{B}_V$  with

$$r_{ess}(P) \leq \delta_V(P).$$

Moreover, equality  $r_{ess}(P) = \delta_V(P)$  holds in many situations, in particular in the discrete state case with  $V(n) \rightarrow \infty$  (see Proposition 3.1).

Next we explicit a result which makes explicit the relationship between the quasi-compactness of  $P$  and the  $V$ -geometric ergodicity of the Markov chain  $\{X_n\}_{n \in \mathbb{N}}$  with transition kernel  $P$ . Moreover, we provide an explicit formula for  $\rho_V(P)$  in terms of the spectral elements of  $P$ . Note that for any  $r_0 \in (r_{ess}(P), 1)$ , the set of all the eigenvalues of  $\lambda$  of  $P$  such that  $r_0 \leq |\lambda| \leq 1$  is finite (use (5)).

**Proposition 2.1** *Let  $P$  be a transition kernel which has an invariant probability measure  $\pi$  such that  $\pi(V) < \infty$ . The two following assertions are equivalent:*

- (a)  $P$  is  $V$ -geometrically ergodic.
- (b)  $P$  is a power-bounded quasi-compact operator on  $\mathcal{B}_V$ , for which  $\lambda = 1$  is a simple eigenvalue (i.e.  $\text{Ker}(P - I) = \mathbb{C} \cdot \mathbf{1}_{\mathbb{X}}$ ) and the unique eigenvalue of modulus one.

*Under any of these conditions, we have  $\rho_V(P) \geq r_{ess}(P)$ . In fact, for  $r_0 \in (r_{ess}(P), 1)$ , denoting the set of all the eigenvalues  $\lambda$  of  $P$  such that  $r_0 \leq |\lambda| < 1$  by  $\mathcal{V}_{r_0}$ , we have:*

- either  $\rho_V(P) \leq r_0$  when  $\mathcal{V}_{r_0} = \emptyset$ ,
- or  $\rho_V(P) = \max\{|\lambda|, \lambda \in \mathcal{V}_{r_0}\}$  when  $\mathcal{V}_{r_0} \neq \emptyset$ .

Moreover, if  $\mathcal{V}_{r_0} = \emptyset$  for all  $r_0 \in (r_{ess}(P), 1)$ , then  $\rho_V(P) = r_{ess}(P)$ .

The  $V$ -geometric ergodicity of  $P$  obviously implies that  $P$  is quasi-compact on  $\mathcal{B}_V$  with  $\rho_V(P) \geq r_{ess}(P)$  (see e.g. [KM03]). This follows from (5) using  $H := \{f \in \mathcal{B}_V : \pi(f) = 0\}$  in (4a)-(4c). The property that  $P$  has a spectral gap on  $\mathcal{B}_V$  in the recent paper [KM12] corresponds here to the quasi-compactness of  $P$  (which is a classical terminology in spectral theory). The spectral gap in [KM12] corresponds to the value  $1 - \rho_V(P)$ . Then, [KM12, Prop. 1.1]) is another formulation, under  $\psi$ -irreducibility and aperiodicity assumptions, of the equivalence of properties (a) and (b) in Proposition 2.1 (see also [KM12, Lem. 2.1]). Details on the proof of Proposition 2.1 are provided in [GHL11]. For general quasi-compact Markov kernels on  $\mathcal{B}_V$ , the result [Wu04, Th. 4.6] also provides interesting additional material on peripheral eigen-elements. The next subsection completes the previous spectral description by providing bounds for the generalized eigenfunctions associated with eigenvalues  $\lambda$  such that  $\delta \leq |\lambda| \leq 1$ , with  $\delta$  given in **(WD)**.

## 2.2 Bound on generalized eigenfunctions of $P$

**Proposition 2.2** *Assume that the weak drift condition **(WD)** holds true. If  $\lambda \in \mathbb{C}$  is such that  $\delta \leq |\lambda| \leq 1$ , with  $\delta$  given in **(WD)**, and if  $f \in \mathcal{B}_V \cap \text{Ker}(P - \lambda I)^p$  for some  $p \in \mathbb{N}^*$ , then there exists  $c \in (0, +\infty)$  such that*

$$|f| \leq c V^{\frac{\ln|\lambda|}{\ln\delta}} (1 + \ln V)^{\frac{p(p-1)}{2}}.$$

Thus, if  $\lambda$  is an eigenvalue such that  $|\lambda| = 1$ , then any associated eigenfunction  $f$  is bounded on  $\mathbb{X}$ . By contrast, if  $|\lambda|$  is close to  $\delta_V(P)$ , then  $|f| \leq c V^{\beta(\lambda)}$  with  $\beta(\lambda)$  close to 1. The proof of Proposition 2.2 is based on the following lemma.

**Lemma 2.3** *Let  $\lambda \in \mathbb{C}$  be such that  $\delta \leq |\lambda| \leq 1$ . Then*

$$\forall f \in \mathcal{B}_V, \exists c \in (0, +\infty), \forall x \in \mathbb{X}, \quad |\lambda|^{-n(x)} |(P^{n(x)} f)(x)| \leq c V(x)^{\frac{\ln|\lambda|}{\ln\delta}} \quad (6)$$

with, for any  $x \in \mathbb{X}$ ,  $n(x) := \lfloor \frac{-\ln V(x)}{\ln\delta} \rfloor$  where  $\lfloor \cdot \rfloor$  denotes the integer part function.

*Proof.* First note that the iteration of **(WD)** gives

$$\forall k \geq 1, \quad P^{kN} V \leq \delta^{kN} V + d \left( \sum_{j=0}^{k-1} \delta^{jN} \right) \mathbf{1}_{\mathbb{X}} \leq \delta^{kN} V + \frac{d}{1 - \delta^N} \mathbf{1}_{\mathbb{X}}.$$

Let  $g \in \mathcal{B}_V$  and  $x \in \mathbb{X}$ . Using the last inequality, the positivity of  $P$  and  $|g| \leq \|g\|_V V$ , we obtain with  $b := d/(1 - \delta^N)$ :

$$\forall k \geq 1, \quad |(P^{kN} g)(x)| \leq (P^{kN} |g|)(x) \leq \|g\|_V (P^{kN} V)(x) \leq \|g\|_V (\delta^{kN} V(x) + b). \quad (7)$$

The previous inequality is also fulfilled with  $k = 0$ . Next, let  $f \in \mathcal{B}_V$  and  $n \in \mathbb{N}$ . Writing  $n = kN + r$ , with  $k \in \mathbb{N}$  and  $r \in \{0, 1, \dots, N-1\}$ , and applying (7) to  $g := P^r f$ , we obtain with  $\xi := \max_{0 \leq \ell \leq N-1} \|P^\ell f\|_V$  (use  $P^n f = P^{kN}(P^r f)$ ):

$$|(P^n f)(x)| \leq \xi [\delta^{kN} V(x) + b] \leq \xi [\delta^{-r} (\delta^n V(x) + b)] \leq \xi \delta^{-N} (\delta^n V(x) + b). \quad (8)$$

Using the inequality

$$-\frac{\ln V(x)}{\ln \delta} - 1 \leq n(x) \leq -\frac{\ln V(x)}{\ln \delta}$$

and the fact that  $\ln \delta \leq \ln |\lambda| \leq 0$ , Inequality (8) with  $n := n(x)$  gives:

$$\begin{aligned} |\lambda|^{-n(x)} |(P^{n(x)} f)(x)| &\leq \xi \delta^{-N} \left( (\delta |\lambda|^{-1})^{n(x)} V(x) + b |\lambda|^{-n(x)} \right) \\ &= \xi \delta^{-N} \left( e^{n(x)(\ln \delta - \ln |\lambda|)} e^{\ln V(x)} + b e^{-n(x) \ln |\lambda|} \right) \\ &\leq \xi \delta^{-N} \left( e^{(\frac{\ln V(x)}{\ln \delta} + 1)(\ln |\lambda| - \ln \delta)} e^{\ln V(x)} + b e^{\frac{\ln V(x)}{\ln \delta} \ln |\lambda|} \right) \\ &= \xi \delta^{-N} \left( e^{\frac{\ln |\lambda|}{\ln \delta} \ln V(x)} e^{\ln |\lambda| - \ln \delta} + b V(x)^{\frac{\ln |\lambda|}{\ln \delta}} \right) \\ &= \xi \delta^{-N} (e^{\ln |\lambda| - \ln \delta} + b) V(x)^{\frac{\ln |\lambda|}{\ln \delta}}. \end{aligned}$$

This gives Inequality (6) with  $c := \xi \delta^{-N} (e^{\ln |\lambda| - \ln \delta} + b)$ .  $\square$

*Proof of Proposition 2.2.* If  $f \in \mathcal{B}_V \cap \text{Ker}(P - \lambda I)$ , then  $|\lambda|^{-n(x)} |(P^{n(x)} f)(x)| = |f(x)|$ , so that (6) gives the expected conclusion when  $p = 1$ . Next, let us proceed by induction. Assume that the conclusion of Proposition 2.2 holds for some  $p \geq 1$ . Let  $f \in \mathcal{B}_V \cap \text{Ker}(P - \lambda I)^{p+1}$ . We can write

$$P^n f = (P - \lambda I + \lambda I)^n f = \lambda^n f + \sum_{k=1}^{\min(n,p)} \binom{n}{k} \lambda^{n-k} (P - \lambda I)^k f. \quad (9)$$

For  $k \in \{1, \dots, p\}$ , we have  $f_k := (P - \lambda I)^k f \in \text{Ker}(P - \lambda I)^{p+1-k} \subset \text{Ker}(P - \lambda I)^p$ , thus we have from the induction hypothesis :

$$\exists c' \in (0, +\infty), \forall k \in \{1, \dots, p\}, \forall x \in \mathbb{X}, |f_k(x)| \leq c' V(x)^{\frac{\ln |\lambda|}{\ln \delta}} (1 + \ln V(x))^{\frac{p(p-1)}{2}}. \quad (10)$$

Now, we obtain from (9) (with  $n := n(x)$ ), (10) and Lemma 2.3 that for all  $x \in \mathbb{X}$ :

$$\begin{aligned} |f(x)| &\leq |\lambda|^{-n(x)} |(P^{n(x)} f)(x)| + c' V(x)^{\frac{\ln |\lambda|}{\ln \delta}} (1 + \ln V(x))^{\frac{p(p-1)}{2}} |\lambda|^{-\min(n,p)} \sum_{k=1}^{\min(n,p)} \binom{n(x)}{k} \\ &\leq c V(x)^{\frac{\ln |\lambda|}{\ln \delta}} + c_1 V(x)^{\frac{\ln |\lambda|}{\ln \delta}} (1 + \ln V(x))^{\frac{p(p-1)}{2}} n(x)^p \\ &\leq c_2 V(x)^{\frac{\ln |\lambda|}{\ln \delta}} (1 + \ln V(x))^{\frac{p(p-1)}{2} + p} \end{aligned}$$

with some constants  $c_1, c_2 \in (0, +\infty)$  independent of  $x$ . This gives the expected result.  $\square$

### 3 Spectral properties of discrete Random Walks

In the sequel, the state space  $\mathbb{X}$  is discrete. For the sake of simplicity, we assume that  $\mathbb{X} := \mathbb{N}$ . Let  $P = (P(i, j))_{i, j \in \mathbb{N}^2}$  be a Markov kernel on  $\mathbb{N}$ . The function  $V : \mathbb{N} \rightarrow [1, +\infty)$  is assumed to satisfy

$$\lim_n V(n) = +\infty \quad \text{and} \quad \sup_{n \in \mathbb{N}} \frac{(PV)(n)}{V(n)} < \infty.$$

The first focus is on the estimation of  $r_{ess}(P)$  from Condition **(WD)**.

**Proposition 3.1** *Let  $\mathbb{X} := \mathbb{N}$ . The two following conditions are equivalent:*

(a) *Condition **(WD)** holds with  $V$ ;*

(b)  $L := \inf_{N \geq 1} (\ell_N)^{\frac{1}{N}} < 1$  where  $\ell_N := \limsup_{n \rightarrow +\infty} \frac{(P^N V)(n)}{V(n)}$ .

*In this case,  $P$  is power-bounded and quasi-compact on  $\mathcal{B}_V$  with  $r_{ess}(P) = \delta_V(P) = L$ .*

The proof of the equivalence (a)  $\Leftrightarrow$  (b), as well as the equality  $\delta_V(P) = L$ , is straightforward (see [GHL11, Cor. 4]). That  $P$  is quasi-compact on  $\mathcal{B}_V$  under **(WD)** in the discrete case, with  $r_{ess}(P) \leq \delta_V(P)$ , can be derived from [Wu04] or [HL14] (see Subsection 2.1 and use the fact that the injection from  $\mathcal{B}_0$  to  $\mathcal{B}_V$  is compact when  $\mathbb{X} := \mathbb{N}$  and  $\lim_n V(n) = +\infty$ ). Equality  $r_{ess}(P) = \delta_V(P)$  can be proved by combining the results [Wu04, HL14] (see [GHL11, Cor. 1] for details).

In Sections 3 and 4, sequences of the special form  $V_\gamma := \{\gamma^n\}_{n \in \mathbb{N}}$  for some  $\gamma \in (1, +\infty)$  will be considered. The associated weighted-supremum space  $\mathcal{B}_\gamma \equiv \mathcal{B}_{V_\gamma}$  is defined by:

$$\mathcal{B}_\gamma := \left\{ \{f(n)\}_{n \in \mathbb{N}} \in \mathbb{C}^{\mathbb{N}} : \sup_{n \in \mathbb{N}} \gamma^{-n} |f(n)| < \infty \right\}.$$

#### 3.1 Quasi-compactness of RWs with bounded state-dependent increments

Let us fix  $c, g, d \in \mathbb{N}^*$ , and assume that the kernel  $P$  satisfies the following conditions:

$$\forall i \in \{0, \dots, g-1\}, \quad \sum_{j=0}^c P(i, j) = 1; \tag{11a}$$

$$\forall i \geq g, \forall j \in \mathbb{N}, \quad P(i, j) = \begin{cases} a_{j-i}(i) & \text{if } i-g \leq j \leq i+d \\ 0 & \text{otherwise} \end{cases} \tag{11b}$$

where  $(a_{-g}(i), \dots, a_d(i)) \in [0, 1]^{g+d+1}$  satisfies  $\sum_{k=-g}^d a_k(i) = 1$  for all  $i \geq g$ . This kind of kernels arises, for instance, from time-discretization of Markovian queuing models. Note that more general models and their use in queuing theory are discussed in [KD06]. In particular, conditions for (non) positive recurrence are provided.

**Proposition 3.2** Assume that, for every  $k \in \mathbb{Z}$  such that  $-g \leq k \leq d$ ,  $\lim_n a_k(n) = a_k \in [0, 1]$ , and that

$$\exists \gamma \in (1, +\infty) : \quad \phi(\gamma) := \sum_{k=-g}^d a_k \gamma^k < 1. \quad (12)$$

Then  $P$  satisfies Condition **(WD)** with  $\delta = \phi(\gamma)$ . Moreover  $P$  is power-bounded and quasi-compact on  $\mathcal{B}_\gamma$  with  $r_{ess}(P) = L = \phi(\gamma)$ .

**Lemma 3.3** When  $a_{-g}$  and  $a_d$  are positive, Condition (12) is equivalent to

$$\sum_{k=-g}^d k a_k < 0. \quad (\mathbf{NERI})$$

Then, there exists a unique real number  $\gamma_0 > 1$  such that  $\phi(\gamma_0) = 1$  and

$$\forall \gamma \in (1, \gamma_0), \quad \phi(\gamma) < 1$$

and there is a unique  $\hat{\gamma}$  such that

$$\hat{\delta} := \phi(\hat{\gamma}) = \min_{\gamma \in (1, \infty)} \phi(\gamma) = \min_{\gamma \in (1, \gamma_0)} \phi(\gamma) < 1.$$

Condition **(NERI)** means that the expectation of the probability distribution of the random increment is negative. Although the results of the paper on RWs with i.d. bounded increments involving Condition **(NERI)** and  $a_{-g}, a_d > 0$  will be valid for  $\gamma \in (1, \gamma_0)$ , only this value  $\hat{\gamma}$  is considered in the statements. Note that the essential spectral radius  $r_{ess}(P|_{\mathcal{B}_{\hat{\gamma}}})$  of  $P$  with respect to  $\mathcal{B}_{\hat{\gamma}}$ , which will be denoted by  $\hat{r}_{ess}(P)$  in the sequel, is the smallest value of  $r_{ess}(P|_{\mathcal{B}_\gamma})$  on  $\mathcal{B}_\gamma$  for  $\gamma \in (1, \gamma_0)$ . When  $\gamma \nearrow \gamma_0$ , the essential spectral radius  $r_{ess}(P|_{\mathcal{B}_\gamma}) \nearrow 1$  since the space  $\mathcal{B}_\gamma$  becomes large. When  $\gamma \searrow 1$ , then  $r_{ess}(P|_{\mathcal{B}_\gamma}) \nearrow 1$  since  $\mathcal{B}_\gamma$  becomes close to the space  $\mathcal{B}_0$  of bounded functions. In this case, the geometric ergodicity is lost since the RWs are typically not uniformly ergodic (i.e.  $V \equiv 1$ ) due the non quasi-compactness of  $P$  on  $\mathcal{B}_0$ .

**Example 1 (State-dependent birth-and-death Markov chains)** When  $c = g = d := 1$  in (11a)-(11b), we obtain the standard class of state-dependent birth-and-death Markov chains:

$$\begin{aligned} P(0, 0) &:= r_0, & P(0, 1) &:= q_0 \\ \forall n \geq 1, & P(n, n-1) &:= p_n, & P(n, n) &:= r_n, & P(n, n+1) &:= q_n, \end{aligned}$$

where  $(p_0, q_0) \in [0, 1]^2$ ,  $p_0 + q_0 = 1$  and  $(p_n, r_n, q_n) \in [0, 1]^3$ ,  $p_n + r_n + q_n = 1$ . Assume that:

$$\lim_n p_n := p \quad \lim_n r_n := r, \quad \lim_n q_n := q.$$

If  $\gamma \in (1, +\infty)$  is such that  $\phi(\gamma) := p/\gamma + r + q\gamma < 1$  then it follows from Proposition 3.2 that  $r_{ess}(P) = p/\gamma + r + q\gamma$ . The conditions  $\gamma > 1$  and  $p/\gamma + r + q\gamma < 1$  are equivalent to the following ones (use  $r = 1 - p - q$  for (i)):



- (i) either  $q > 0$ ,  $q - p < 0$  (i.e. **(NERI)**) and  $1 < \gamma < \gamma_0 = p/q$ ;
- (ii) or  $q = 0$ ,  $p > 0$  and  $\gamma > 1$ .

(i) When  $p > q > 0$  and  $1 < \gamma < \gamma_0$ :  $P$  is power-bounded and quasi-compact on  $\mathcal{B}_\gamma$  with  $r_{\text{ess}}(P) = \phi(\gamma)$ . Set  $\hat{\gamma} := \sqrt{\gamma_0} = \sqrt{p/q} \in (1, \gamma_0)$ . Then  $\min_{\gamma > 1} \phi(\gamma) = \phi(\hat{\gamma}) = r + 2\sqrt{pq}$  and the essential spectral radius  $\hat{r}_{\text{ess}}(P)$  on  $\mathcal{B}_{\hat{\gamma}}$  satisfies  $\hat{r}_{\text{ess}}(P) = r + 2\sqrt{pq}$ .

(ii) When  $q := 0, p > 0$  and  $\gamma > 1$ :  $r_{\text{ess}}(P) = \phi(\gamma) = p/\gamma + r$ .

**Remark 3.1** If  $c$  is allowed to be  $+\infty$  in Condition (11a), that is

$$\forall i \in \{0, \dots, g-1\}, \quad \sum_{j \geq 0} P(i, j) \gamma^j < \infty, \quad (13)$$

then the conclusions of Proposition 3.2 and Example 1 are still valid under the additional Condition (13).

*Proof of Proposition 3.2.* Set  $\phi_n(\gamma) := \sum_{k=-g}^d a_k(n) \gamma^k$ . We have  $(PV_\gamma)(n) = \phi_n(\gamma)V_\gamma(n)$  for each  $n \geq g$ . Thus  $\ell_1 = \lim_n \phi_n(\gamma) = \phi(\gamma)$ . Now assume that  $\ell_{N-1} := \lim_n (P^{N-1}V)(n)/V(n) = \phi(\gamma)^{N-1}$  for some  $N \geq 1$ . Since

$$\forall i \geq Ng, \quad (P^N V)(i) = \sum_{j=-g}^d a_j(i) (P^{N-1}V)(i+j)$$

we obtain

$$\frac{(P^N V)(i)}{V(i)} = \sum_{j=-g}^d a_j(i) \gamma^j \frac{(P^{N-1}V)(i+j)}{\gamma^{i+j}} \xrightarrow{i \rightarrow +\infty} \phi(\gamma) \phi(\gamma)^{N-1}.$$

Hence  $\ell_N = \phi(\gamma)^N$ , and  $\phi(\gamma) = L = r_{\text{ess}}(P)$  from Proposition 3.1.  $\square$

*Proof of Lemma 3.3.* Since the second derivative of  $\phi$  is positive on  $(0, +\infty)$ ,  $\phi$  is convex on  $(0, +\infty)$ . When  $a_{-g}$  and  $a_d$  are positive then  $\lim_{t \rightarrow 0^+} \phi(t) = \lim_{t \rightarrow +\infty} \phi(t) = +\infty$  and, since  $\phi(1) = 1$ , Condition (12) is equivalent to  $\phi'(1) < 0$ , that is **(NERI)**. The other properties of  $\phi(\cdot)$  are immediate.  $\square$

### 3.2 Spectral analysis of RW with i.d. bounded increments

Let  $P := (P(i, j))_{(i, j) \in \mathbb{N}^2}$  be the transition kernel of a RW with i.d. bounded increments. Specifically we assume that there exist some positive integers  $c, g, d \in \mathbb{N}^*$  such that

$$\forall i \in \{0, \dots, g-1\}, \quad \sum_{j=0}^c P(i, j) = 1; \quad (14a)$$

$$\forall i \geq g, \forall j \in \mathbb{N}, \quad P(i, j) = \begin{cases} a_{j-i} & \text{if } i-g \leq j \leq i+d \\ 0 & \text{otherwise.} \end{cases} \quad (14b)$$

$$(a_{-g}, \dots, a_d) \in [0, 1]^{g+d+1} : a_{-g} > 0, a_d > 0, \quad \sum_{k=-g}^d a_k = 1. \quad (14c)$$

Let us assume that Condition **(NERI)** holds. We know from Lemma 3.3 and Proposition 3.2 that  $P$  is quasi-compact on  $\mathcal{B}_{\widehat{\gamma}}$  with

$$\widehat{r}_{ess}(P) = \widehat{\delta} := \phi(\widehat{\gamma}) < 1$$

where  $\phi(\cdot)$  is given by (12).

For any  $\lambda \in \mathbb{C}$ , we denote by  $E_\lambda(\cdot)$  the following polynomial of degree  $N := d + g$

$$\forall z \in \mathbb{C}, \quad E_\lambda(z) := z^g (\phi(z) - \lambda) = \sum_{k=-g}^d a_k z^{g+k} - \lambda z^g,$$

and by  $\mathcal{E}_\lambda$  the set of complex roots of  $E_\lambda(\cdot)$ . Since  $E_\lambda(0) = a_{-g} > 0$ , we have for any  $\lambda \in \mathbb{C}$ :

$$z \in \mathcal{E}_\lambda \iff \mathbb{E}_\lambda(z) = 0 \iff \lambda = \phi(z).$$

The next proposition investigates the eigenvalues of  $P$  on  $\mathcal{B}_{\widehat{\gamma}}$  which belong to the annulus

$$\Lambda := \{\lambda \in \mathbb{C} : \widehat{\delta} < |\lambda| < 1\}.$$

To that effect, for any  $\lambda \in \Lambda$ , we introduce the following subset  $\mathcal{E}_\lambda^-$  of  $\mathcal{E}_\lambda$

$$\mathcal{E}_\lambda^- := \{z \in \mathbb{C} : E_\lambda(z) = 0, |z| < \widehat{\gamma}\}.$$

If  $\mathcal{E}_\lambda^- = \emptyset$ , we set  $N(\lambda) := 0$ . If  $\mathcal{E}_\lambda^- \neq \emptyset$ , then  $N(\lambda)$  is defined as

$$N(\lambda) := \sum_{z \in \mathcal{E}_\lambda^-} m_z,$$

where  $m_z$  denotes the multiplicity of  $z$  as root of  $E_\lambda(\cdot)$ . Finally, for any  $z \in \mathbb{C}$ , we set  $z^{(1)} := \{z^n\}_{n \in \mathbb{N}}$ , and for any  $k \geq 2$ ,  $z^{(k)} \in \mathbb{C}^{\mathbb{N}}$  is defined by:

$$\forall n \in \mathbb{N}, \quad z^{(k)}(n) := n(n-1) \cdots (n-k+2) z^{n-k+1}.$$

**Proposition 3.4** *Assume that Assumptions (14a)-(14c) and **(NERI)** hold true. Then*

$$\exists \eta \geq 1, \forall \lambda \in \Lambda, \quad N(\lambda) = \eta.$$

Moreover the two following assertions are equivalent:

- (i)  $\lambda \in \Lambda$  is an eigenvalue of  $P$  on  $\mathcal{B}_{\widehat{\gamma}}$ .
- (ii) There exists a nonzero  $\{\alpha_{\lambda,z,k}\}_{z \in \mathcal{E}_\lambda^-, 1 \leq k \leq m_z} \in \mathbb{C}^\eta$  such that

$$f := \sum_{z \in \mathcal{E}_\lambda^-} \sum_{k=1}^{m_z} \alpha_{\lambda,z,k} z^{(k)} \in \mathbb{C}^{\mathbb{N}} \tag{15}$$

satisfies the boundary equations:  $\forall i = 0, \dots, g-1, \lambda f(i) = (Pf)(i)$ .

The first step of the elimination procedure of Section 4 is to plug  $f$  of the form (15) in the boundary equations. This gives a linear system in  $\alpha_{\lambda,z,k}$ . Since  $\Lambda$  is infinite, that  $N(\lambda)$  does not depend on  $\lambda$  is crucial to initialize this procedure. To specify the value of  $\eta$ , it is sufficient to compute  $N(\lambda)$  for some (any)  $\lambda \in \Lambda$ .

**Remark 3.2** Under Condition **(NERI)**,  $\phi(\cdot)$  is strictly decreasing from  $(1, \widehat{\gamma})$  to  $(\widehat{\delta}, 1)$ , so that we have:  $\forall \lambda \in (\widehat{\delta}, 1)$ ,  $\phi^{-1}(\lambda) \in (1, \widehat{\gamma})$ . Since  $\phi^{-1}(\lambda) \in \mathcal{E}_\lambda$ , we obtain

$$\forall \lambda \in (\widehat{\delta}, 1), \quad N(\lambda) \geq 1. \quad (16)$$

**Remark 3.3** Let Condition **(NERI)** be satisfied. Set  $\mathcal{E}_\lambda^+ := \{z \in \mathbb{C} : E_\lambda(z) = 0, |z| > \widehat{\gamma}\}$ . Then

$$\forall \lambda \in \Lambda, \quad \mathcal{E}_\lambda = \mathcal{E}_\lambda^- \sqcup \mathcal{E}_\lambda^+.$$

In other words, for any  $\lambda \in \Lambda$ ,  $E_\lambda(\cdot)$  has no root of modulus  $\widehat{\gamma}$ . Indeed, consider  $\lambda \in \Lambda$ ,  $z \in \mathcal{E}_\lambda$ , and assume that  $|z| = \widehat{\gamma}$ . Since  $\lambda = \phi(z)$ , we obtain the inequality  $|\lambda| \leq \phi(|z|) = \phi(\widehat{\gamma})$  which is impossible since  $\phi(\widehat{\gamma}) = \widehat{\delta}$  and  $\lambda \in \Lambda$ .

**Remark 3.4** Assertion (ii) of Proposition 3.4 does not mean that the dimension of the eigenspace  $\text{Ker}(P - \lambda I)$  associated with  $\lambda$  is  $\eta$ . We shall see in Subsection 4.2 that we can have  $\eta = 2$  when  $g = 2$ ,  $d = 1$  and  $c = 2$  in (14a)-(14c), while  $\dim \text{Ker}(P - \lambda I) \leq 1$  since  $Pf = \lambda f$  and  $f(0) = 0$  clearly imply  $f = 0$  (by induction).

The following surprising lemma, based on Remark 3.3, is used to derive Proposition 3.4.

**Lemma 3.5** Under Condition **(NERI)**, the function  $N(\cdot)$  is constant on  $\Lambda$ .

*Proof.* Since  $\Lambda$  is connected and  $N(\cdot)$  is  $\mathbb{N}$ -valued, it suffices to prove that  $N(\cdot)$  is continuous on  $\Lambda$ . Note that the set  $\cup_{\lambda \in \Lambda} \mathcal{E}_\lambda$  is bounded in  $\mathbb{C}$  since the coefficients of  $E_\lambda(\cdot)$  are obviously uniformly bounded in  $\lambda \in \Lambda$ . Now let  $\lambda \in \Lambda$  and assume that  $N(\cdot)$  is not continuous at  $\lambda$ . Then there exists a sequence  $\{\lambda_n\}_{n \in \mathbb{N}} \in \Lambda^\mathbb{N}$  such that  $\lim_n \lambda_n = \lambda$  and

$$(a) \text{ either: } \forall n \geq 0, \quad N(\lambda_n) \geq N(\lambda) + 1,$$

$$(b) \text{ or: } \forall n \geq 0, \quad N(\lambda_n) \leq N(\lambda) - 1.$$

For any  $n \geq 0$ , let us denote the roots of  $E_{\lambda_n}(\cdot)$  by  $z_1(\lambda_n), \dots, z_N(\lambda_n)$ , and suppose for convenience that they are listed by increasing modulus, and by increasing argument when they have the same modulus. Applying Remark 3.3 to  $\lambda_n$ , we obtain:

$$\forall i \in \{1, \dots, N(\lambda_n)\}, \quad |z_i(\lambda_n)| < \widehat{\gamma} \quad \text{and} \quad \forall i \in \{N(\lambda_n) + 1, \dots, N\}, \quad |z_i(\lambda_n)| > \widehat{\gamma}.$$

Up to consider a subsequence, we may suppose that, for every  $1 \leq i \leq N$ , the sequence  $\{z_i(\lambda_n)\}_{n \in \mathbb{N}}$  converges to some  $z_i \in \mathbb{C}$ . Note that

$$\mathcal{E}_\lambda = \{z_1, z_2, \dots, z_N\}$$

where  $z_i$  is repeated in this list with respect to its multiplicity  $m_{z_i}$ , since

$$\forall z \in \mathbb{C}, \quad E_\lambda(z) = \lim_n E_{\lambda_n}(z) = \lim_n a_d \prod_{i=1}^N (z - z_i(\lambda_n)) = a_d \prod_{i=1}^N (z - z_i).$$

In case (a), we have

$$\forall n \geq 0, \quad |z_1(\lambda_n)| < \widehat{\gamma}, \dots, |z_{N(\lambda)+1}(\lambda_n)| < \widehat{\gamma}.$$

When  $n \rightarrow +\infty$ , this gives using Remark 3.3:

$$|z_1| < \widehat{\gamma}, \dots, |z_{N(\lambda)+1}| < \widehat{\gamma}.$$

Thus at least  $N(\lambda) + 1$  roots of  $E_\lambda(\cdot)$  (counted with their multiplicity) are of modulus strictly less than  $\widehat{\gamma}$ : this contradicts the definition of  $N(\lambda)$ .

In case (b), we have

$$\forall n \geq 0, \quad |z_{N(\lambda)}(\lambda_n)| > \widehat{\gamma}, |z_{N(\lambda)+1}(\lambda_n)| > \widehat{\gamma}, \dots, |z_N(\lambda_n)| > \widehat{\gamma},$$

and this gives similarly when  $n \rightarrow +\infty$

$$|z_{N(\lambda)}| > \widehat{\gamma}, |z_{N(\lambda)+1}| > \widehat{\gamma}, \dots, |z_N| > \widehat{\gamma}.$$

Thus at least  $N - N(\lambda) + 1$  roots of  $E_\lambda(\cdot)$  (counted with their multiplicity) are of modulus strictly larger than  $\widehat{\gamma}$ . This contradicts the definition of  $N(\lambda)$ .  $\square$

*Proof of Proposition 3.4.* From Lemma 3.5 and (16), we obtain:  $\forall \lambda \in \Lambda$ ,  $N(\lambda) = \eta$  for some  $\eta \geq 1$ . Now we prove the implication (i)  $\Rightarrow$  (ii). Let  $\lambda \in \Lambda$  be any eigenvalue of  $P$  on  $\mathcal{B}_{\widehat{\gamma}}$  and let  $f := \{f(n)\}_{n \in \mathbb{N}}$  be a nonzero sequence in  $\mathcal{B}_{\widehat{\gamma}}$  satisfying  $Pf = \lambda f$ . In particular  $f$  satisfies the following equalities

$$\forall i \geq g, \quad \lambda f(i) = \sum_{j=i-g}^{i+g} a_{j-i} f(j). \quad (17)$$

Since the characteristic polynomial associated with these recursive formulas is  $E_\lambda(\cdot)$ , there exists  $\{\alpha_{\lambda,z,k}\}_{z \in \mathcal{E}_\lambda, 1 \leq k \leq m_z} \in \mathbb{C}^\eta$  such that

$$f = \sum_{z \in \mathcal{E}_\lambda} \sum_{k=1}^{m_z} \alpha_{\lambda,z,k} z^{(k)} \in \mathbb{C}^\mathbb{N}$$

where  $m_z$  denotes the multiplicity of  $z \in \mathcal{E}_\lambda$ . Next, since  $|f| \leq C V_{\widehat{\gamma}}$  for some  $C > 0$  (i.e.  $f \in \mathcal{B}_{\widehat{\gamma}}$ ), it can be easily seen that  $\alpha_{\lambda,z,k} = 0$  for every  $z \in \mathcal{E}_\lambda$  such that  $|z| > \widehat{\gamma}$  and for every  $k = 1, \dots, m_z$ : first delete  $\alpha_{\lambda,z,m_z}$  for  $z$  of maximum modulus and for  $m_z$  maximal if there are several  $z$  of maximal modulus (to that effect, divide  $f$  by  $n(n-1) \cdots (n-m_z+2) z^{n-m_z+1}$  and use  $|f| \leq C V_{\widehat{\gamma}}$ ). Therefore  $f$  is of the form (15), and it satisfies the boundary equations in (ii) since  $Pf = \lambda f$  by hypothesis.

To prove the implication (ii)  $\Rightarrow$  (i), note that any  $f := \{f(n)\}_{n \in \mathbb{N}}$  of the form (15) belongs to  $\mathcal{B}_{\widehat{\gamma}}$  and satisfies (17) since  $\mathcal{E}_\lambda^- \subset \mathcal{E}_\lambda$ . If moreover  $f$  is non zero and satisfies the boundary equations, then  $Pf = \lambda f$ . This gives (i).  $\square$

We conclude this study with an additional refinement of Proposition 3.4. For any  $\lambda \in \Lambda$ , let us define the set  $\mathcal{E}_{\lambda, \tau}^-$  as follows:

$$\mathcal{E}_{\lambda, \tau}^- := \{z \in \mathbb{C} : E_\lambda(z) = 0, |z| < \widehat{\gamma}^\tau\} \quad \text{with} \quad \tau \equiv \tau(\lambda) := \frac{\ln |\lambda|}{\ln \widehat{\delta}}.$$

Moreover define the associated function  $N'(\cdot)$  by

$$N'(\lambda) := \sum_{z \in \mathcal{E}_{\lambda, \tau}^-} m_z,$$

where  $m_z$  is the multiplicity of  $z$  as root of  $E_\lambda(\cdot)$  (with the convention  $N'(\lambda) = 0$  if  $\mathcal{E}_{\lambda, \tau}^- = \emptyset$ ).

**Lemma 3.6** *Assume that  $P := (P(i, j))_{(i, j) \in \mathbb{N}^2}$  satisfies Conditions (14a)-(14c) and **(NERI)**. Moreover assume that*

$$\forall t \in (1, \widehat{\gamma}), \quad \phi(t) < t^{\ln \widehat{\delta} / \ln \widehat{\gamma}} \quad (18)$$

*Then the function  $N'(\cdot)$  is constant on  $\Lambda$ :  $\exists \eta' \geq 1, \forall \lambda \in \Lambda, N'(\lambda) = \eta'$ .*

From Lemma 3.6, all the assertions of Proposition 3.4 are still valid when  $\eta$  and  $\mathcal{E}_\lambda^-$  are replaced with  $\eta'$  and  $\mathcal{E}_{\lambda, \tau}^-$  respectively. That  $\mathcal{E}_\lambda^-$  may be replaced with  $\mathcal{E}_{\lambda, \tau}^-$  in (15) follows from Proposition 2.2. Consequently, under the additional condition  $\eta' \leq g$ , the elimination procedure of Section 4 may be adapted by using Lemma 3.6. Since  $\eta' \leq \eta$ , the resulting procedure is computationally interesting when  $g$  or  $d$  are large.

**Remark 3.5** *Condition (18) is the additional assumption in Lemma 3.6 with respect to Lemma 3.5. Since  $\phi$  is strictly decreasing on  $(1, \widehat{\gamma})$  under Condition **(NERI)**, Condition (18) is equivalent to the following one*

$$\forall z \in (1, \widehat{\gamma}), \quad z < \widehat{\gamma}^{\ln \phi(z) / \ln \widehat{\delta}}. \quad (19)$$

*Indeed, for every  $t \in (1, \widehat{\gamma})$ , we have  $u := t^{\ln \widehat{\delta} / \ln \widehat{\gamma}} \in (\widehat{\delta}, 1)$  and  $z := \phi^{-1}(u) \in (1, \widehat{\gamma})$ . Hence*

$$(18) \iff \forall u \in (\widehat{\delta}, 1), \phi(\widehat{\gamma}^{\ln u / \ln \widehat{\delta}}) < u \iff (19). \quad (20)$$

*Therefore, under Condition (18), for any  $\lambda \in (\widehat{\delta}, 1)$  we have  $\mathcal{E}_{\lambda, \tau}^- \neq \emptyset$  since  $z = \phi^{-1}(\lambda)$  satisfies  $z < \widehat{\gamma}^{\tau(\lambda)}$  from (19).*

*Proof of Lemma 3.6.* The proof is similar to that of Lemma 3.5. Under Condition (18), Remark 3.3 extends as follows:

$$\mathcal{E}_\lambda = \mathcal{E}_{\lambda, \tau}^- \sqcup (\mathcal{E}_\lambda \cap \{z \in \mathbb{C} : |z| > \widehat{\gamma}^\tau\}). \quad (21)$$

Indeed, consider  $\lambda \in \Lambda$  and  $z \in \mathcal{E}_\lambda$  such that  $|z| = \widehat{\gamma}^\tau$ . Since  $\lambda = \phi(z)$ , we have  $|\lambda| \leq \phi(|z|)$ , thus  $|\lambda| \leq \phi(\widehat{\gamma}^\tau)$ . This inequality contradicts Condition (18) (use the definition of  $\tau$  and the second equivalence in (20) with  $u := |\lambda|$ ). Next, using (21) and the continuity of  $\tau(\cdot)$ , Lemma 3.5 easily extends to the function  $N'(\cdot)$ .  $\square$

## 4 Convergence rate for RWs with i.d. bounded increments

Let us recall that any RW with i.d. bounded increments defined by (14a)-(14c) and satisfying **(NERI)**, has an invariant probability measure  $\pi$  on  $\mathbb{N}$  such  $\pi(V_{\hat{\gamma}}) < \infty$  where  $V_{\hat{\gamma}} := \{\hat{\gamma}^n\}_{n \in \mathbb{N}}$  and  $\hat{\gamma}$  is defined in Lemma 3.3. Indeed  $\hat{\delta} := \phi(\hat{\gamma}) < 1$  so that Condition **(WD)** holds with  $V_{\hat{\gamma}}$  from Proposition 3.2. The expected conclusions on  $\pi$  can be deduced from the first statement of [GHL11, Cor 5]. Note that, from Lemma 3.3, the previous fact is valid for any  $\gamma \in (1, \gamma_0)$  in place of  $\hat{\gamma}$ .

The  $V_{\hat{\gamma}}$ -geometric ergodicity of the RW may be studied using Proposition 2.1. Next we can derive from Proposition 3.4 an effective procedure to compute the rate of convergence with respect to  $\mathcal{B}_{\hat{\gamma}}$  (see (2)), that is denoted by  $\hat{\rho}(P)$ . The most favorable case for initializing the procedure (see (24) and (26)) is to assume that for some (any)  $\lambda \in \Lambda$

$$\eta := N(\lambda) \leq g. \quad (22)$$

- *First step: checking Condition (22).* From Lemma 3.5, computing  $\eta$  and testing  $\eta \leq g$  of Assumption (22) can be done by analyzing the roots of  $E_{\lambda}(\cdot)$  for some (any)  $\lambda \in \Lambda$ .
- *Second step: linear and polynomial eliminations.* This second step consists in applying some linear and (successive) polynomial eliminations in order to find a finite set  $\mathcal{Z} \subset \Lambda$  containing all the eigenvalues of  $P$  on  $\mathcal{B}_{\hat{\gamma}}$  in  $\Lambda$ . Conversely, the elements of  $\mathcal{Z}$  providing eigenvalues of  $P$  on  $\mathcal{B}_{\hat{\gamma}}$  can be identified using Condition (ii) of Proposition 3.4. Note that the explicit computation of the roots of  $E_{\lambda}(\cdot)$  is only required for the elements  $\lambda$  of the finite set  $\mathcal{Z}$ . This is detailed in Corollary 4.1.

Under the assumptions of Proposition 3.4, we define the set

$$\mathcal{M} := \{(m_1, \dots, m_s) \in \{1, \dots, s\}^s : s \in \{1, \dots, \eta\}, m_1 \leq \dots \leq m_s \text{ and } \sum_{i=1}^s m_i = \eta\}.$$

Note that  $\mathcal{M}$  is a finite set and that, for every  $\lambda \in \Lambda$ , there exists a unique  $\mu \in \mathcal{M}$  such that the set  $\mathcal{E}_{\lambda}^-$  is composed of  $s$  distinct roots of  $E_{\lambda}(\cdot)$  with multiplicity  $m_1, \dots, m_s$  respectively.

**Corollary 4.1** *Assume that Assumptions (14a)-(14c) and **(NERI)** hold true. Set  $\ell := \binom{g}{\eta}$ . Then there exist a family of polynomial functions  $\{\mathcal{R}_{\mu,k}, \mu \in \mathcal{M}, 1 \leq k \leq \ell\}$ , with coefficients only depending on  $\mu$  and on the transition probabilities  $P(i,j)$ , such that the following assertions hold true for any  $\mu \in \mathcal{M}$ .*

- (i) *Let  $\lambda \in \Lambda$  be an eigenvalue of  $P$  on  $\mathcal{B}_{\hat{\gamma}}$  such that, for some  $s \in \{1, \dots, \eta\}$ , the set  $\mathcal{E}_{\lambda}^-$  is composed of  $s$  roots of  $E_{\lambda}(\cdot)$  with multiplicity  $m_1, \dots, m_s$  respectively. Then*

$$\mathcal{R}_{\mu,1}(\lambda) = 0, \dots, \mathcal{R}_{\mu,\ell}(\lambda) = 0. \quad (23)$$

- (ii) *Conversely, let  $\lambda \in \Lambda$  satisfying (23) such that, for some  $s \in \{1, \dots, \eta\}$ , the set  $\mathcal{E}_{\lambda}^-$  is composed of  $s$  roots of  $E_{\lambda}(\cdot)$  with multiplicity  $m_1, \dots, m_s$  respectively. Then a necessary and sufficient condition for  $\lambda$  to be an eigenvalue of  $P$  on  $\mathcal{B}_{\hat{\gamma}}$  is that  $\lambda$  satisfies Condition (ii) of Proposition 3.4.*

*Proof.* Assertion (ii) follows from Proposition 3.4. To prove (i), first assume for convenience that  $\eta = g$  and that  $\lambda \in \Lambda$  is an eigenvalue of  $P$  on  $\mathcal{B}_{\hat{\gamma}}$  such that the associated set  $\mathcal{E}_{\lambda}^-$  contains  $\eta$  distinct roots  $z_1, \dots, z_{\eta}$  of  $E_{\lambda}(\cdot)$  with multiplicity one. We know from Proposition 3.4 that there exists  $f := \{f(n)\}_{n \in \mathbb{N}} \neq 0$  of the form

$$f = \sum_{i=1}^{\eta} \alpha_i z_i^{(1)}$$

which satisfies the  $g = \eta$  boundary equations:  $\forall i = 0, \dots, \eta - 1, \lambda f(i) = (Pf)(i)$ . In other words the linear system provided by these  $\eta$  equations has a nonzero solution  $(\alpha_i)_{1 \leq i \leq \eta} \in \mathbb{C}^{\eta}$ . Therefore the associated determinant is zero: this leads to a polynomial equation of the form

$$P_{0,1}(\lambda, z_1, \dots, z_{\eta}) = 0. \quad (24)$$

Since this polynomial is divisible by  $\prod_{i \neq j} (z_i - z_j)$ , Equation (24) is equivalent to

$$P_0(\lambda, z_1, \dots, z_{\eta}) = 0 \quad \text{with } P_0(\lambda, z_1, \dots, z_{\eta}) = \frac{P_{0,1}(\lambda, z_1, \dots, z_{\eta})}{\prod_{i \neq j} (z_i - z_j)}. \quad (25)$$

Note that the coefficients of  $P_0$  only depend on the  $P(i, j)$ 's.

Next,  $z_{\eta}$  is a common root of the polynomials  $P_0(\lambda, z_1, \dots, z_{\eta-1}, z)$  and  $E_{\lambda}(z)$  with respect to the variable  $z$ : this leads to the following necessary condition

$$P_1(\lambda, z_1, \dots, z_{\eta-1}) := \text{Res}_{z_{\eta}}(P_0, E_{\lambda}) = 0$$

where  $\text{Res}_{z_{\eta}}(P_0, E_{\lambda})$  denotes the resultant of the two polynomials  $P_0$  and  $E_{\lambda}$  corresponding to the elimination of the variable  $z_{\eta}$ . Again the coefficients of  $P_1$  only depend on the  $P(i, j)$ 's. Next, considering the common root  $z_{\eta-1}$  of the polynomials  $P_1(\lambda, z_1, \dots, z_{\eta-2}, z)$  and  $E_{\lambda}(z)$  leads to the elimination of the variable  $z_{\eta-1}$

$$P_2(\lambda, z_1, \dots, z_{\eta-2}) := \text{Res}_{z_{\eta-1}}(P_1, E_{\lambda}) = 0.$$

Repeating this method, we obtain that a necessary condition for  $\lambda$  to be an eigenvalue of  $P$  is  $\mathcal{R}(\lambda) = 0$  where  $\mathcal{R}$  is some polynomial with coefficients only depending on the  $P(i, j)$ 's.

Now let us consider the case when  $\eta < g$ ,  $s \in \{1, \dots, \eta\}$ , and  $\lambda \in \Lambda$  is assumed to be an eigenvalue of  $P$  on  $\mathcal{B}_{\hat{\gamma}}$  such that the associated set  $\mathcal{E}_{\lambda}^-$  contains  $s$  distinct roots of  $E_{\lambda}(\cdot)$  with respective multiplicity  $m_1, \dots, m_s$  satisfying  $\sum_{i=1}^s m_i = \eta$ . Then the elimination (by using determinants) of  $(\alpha_{\lambda, z, \ell}) \in \mathbb{C}^{\eta}$  provided by the linear system of Proposition 3.4, leads to  $\ell := \binom{g}{\eta}$  polynomial equations

$$P_{0,\mu,1}(\lambda, z_1, \dots, z_{\eta}) = 0, \dots, P_{0,\mu,\ell}(\lambda, z_1, \dots, z_{\eta}) = 0. \quad (26)$$

As in the case  $\eta = g$ , these polynomials are replaced in the sequel by the polynomials obtained by division of the  $P_{0,\mu,k}$ 's by  $\prod_{i \neq j} (z_i - z_j)^{n_{i,j}}$  where  $n_{i,j} := \min(m_i, m_j)$ .

The successive polynomial eliminations of  $z_{\eta}, \dots, z_1$  can be derived as above from each polynomial equation  $P_{0,\mu,k}(\lambda, z_1, \dots, z_{\eta}) = 0$ . This gives  $\ell$  polynomial equations

$$\mathcal{R}_{\mu,1}(\lambda) = 0, \dots, \mathcal{R}_{\mu,\ell}(\lambda) = 0.$$

Satisfying this set of polynomial equations is a necessary condition for  $\lambda$  to be an eigenvalue of  $P$  on  $\mathcal{B}_{\widehat{\gamma}}$ . Finally the polynomial functions  $\mathcal{R}_{\mu,1}, \dots, \mathcal{R}_{\mu,\ell}$  depend on the  $P(i,j)$ 's and also on  $(m_1, \dots, m_s)$ , since the linear system used to eliminate  $(\alpha_{\lambda,k,\ell}) \in \mathbb{C}^\eta$  involves coefficients  $i(i-1)\cdots(i-k+1)$  for some finitely many integers  $i$  and for  $k = 1, \dots, m_i$  ( $i = 1, \dots, s$ ).  $\square$

To compute  $\widehat{\rho}(P)$ , we define the following (finite and possibly empty) sets:

$$\forall \mu \in \mathcal{M}, \quad \Lambda_\mu := \{\lambda \in \Lambda : \mathcal{R}_{\mu,1}(\lambda) = 0, \dots, \mathcal{R}_{\mu,\ell}(\lambda) = 0\}.$$

Let us denote by  $\mathcal{Z}$  the (finite and possibly empty) set composed of all the complex numbers  $\lambda \in \cup_{\mu \in \mathcal{M}} \Lambda_\mu$  such that Condition (ii) of Proposition 3.4 holds true.

**Corollary 4.2** *Assume that Assumptions (14a)-(14c) and (NERI) hold true and that  $P$  is irreducible and aperiodic. Then*

$$\widehat{\rho}(P) = \max(\widehat{\delta}, \max\{|\lambda|, \lambda \in \mathcal{Z}\}) \quad \text{where } \widehat{\delta} := \phi(\widehat{\gamma}).$$

*Proof.* Under the assumptions on  $P$ , we know from Proposition 2.1 that the RW is  $V_{\widehat{\gamma}}$ -geometrically ergodic. Since  $\widehat{r}_{ess}(P) = \widehat{\delta}$  from Proposition 3.2, the corollary follows from Corollary 4.1 and from Proposition 2.1 applied either with any  $r_0$  such that  $\widehat{\delta} < r_0 < \min\{|\lambda|, \lambda \in \mathcal{Z}\}$  if  $\mathcal{Z} \neq \emptyset$ , or with any  $r_0$  such that  $\widehat{\delta} < r_0 < 1$  if  $\mathcal{Z} = \emptyset$ .  $\square$

**Remark 4.1** *When  $\eta \geq 2$  and  $\mu := (m_1, \dots, m_s)$  with  $s < \eta$ , the set  $\Lambda_\mu$  used in Corollary 4.2 may be reduced. For the sake of simplicity, this fact has been omitted in Corollary 4.2, but it is relevant in practice. Actually, when  $s < \eta$ , the part (ii) of Corollary 4.1 can be specified since it requires that  $E_\lambda(\cdot)$  admits roots of multiplicity  $\geq 2$ . This involves some additional necessary conditions on  $\lambda$  derived from some polynomial eliminations with respect to the derivatives of  $E_\lambda(\cdot)$ .*

*For instance, in case  $g = 2, \eta = 2, s = 1$  (thus  $\mu := (2)$ ), a necessary condition on  $\lambda$  for  $E_\lambda(\cdot)$  to have a double root is that  $E_\lambda(\cdot)$  and  $E'_\lambda(\cdot)$  admits a common root. This leads to*

$$Q(\lambda) := \text{Res}_z(E_\lambda, E'_\lambda) = 0.$$

*Consequently, if  $g = 2$  and  $\eta = 2$  (thus  $\ell := 1$ ), then Condition (ii) of Proposition 3.4 can be tested in case  $s = 1$  by using the following finite set*

$$\Lambda'_\mu := \Lambda_\mu \cap \{\lambda \in \Lambda : Q(\lambda) = 0\}.$$

*In general  $\Lambda'_\mu$  is strictly contained in  $\Lambda_\mu$ . Even  $\Lambda'_\mu$  may be empty while  $\Lambda_\mu$  is not (see Subsection 4.2).*

Proposition 3.4 and the above elimination procedure obviously extend to any  $\gamma \in (1, \gamma_0)$  in place of  $\widehat{\gamma}$ , where  $\gamma_0$  is given in Lemma 3.3. Of course  $\widehat{\delta} = \phi(\widehat{\gamma})$  is then replaced by  $\delta = \phi(\gamma)$ .



#### 4.1 RWs with $g = d := 1$ : birth-and-death Markov chains

Let  $p, q, r \in [0, 1]$  be such that  $p + r + q = 1$ , and let  $P$  be defined by

$$\begin{aligned} P(0, 0) &\in (0, 1), P(0, 1) = 1 - P(0, 0) \\ \forall n \geq 1, P(n, n-1) &:= p, \quad P(n, n) := r, \quad P(n, n+1) := q \quad \text{with } 0 < q < p. \end{aligned} \quad (27)$$

Note that  $a_{-1} := p, a_1 := q > 0$  and **(NERI)** holds true. We have  $\gamma_0 = p/q \in (1, +\infty)$  and  $\hat{\gamma} := \sqrt{p/q} \in (1, +\infty)$  is such that  $\hat{\delta} := \min_{\gamma > 1} \phi(\gamma) = \phi(\hat{\gamma}) < 1$  (see Lemma 3.3). Let  $V_{\hat{\gamma}} := \{\hat{\gamma}^n\}_{n \in \mathbb{N}}$  and its associated weighted-supremum space  $\mathcal{B}_{\hat{\gamma}}$ . Here we have

$$\hat{r}_{ess}(P) = \hat{\delta} = r + 2\sqrt{pq}.$$

**Proposition 4.1** *Let  $P$  be defined by Conditions (27). The boundary transition probabilities are denoted by  $P(0, 0) := a, P(0, 1) := 1 - a$  for some  $a \in (0, 1)$ . Then  $P$  is  $V_{\hat{\gamma}}$ -geometrically ergodic. Furthermore, defining  $a_0 := 1 - q - \sqrt{pq}$ , the convergence rate  $\hat{\rho}(P)$  of  $P$  with respect to  $\mathcal{B}_{\hat{\gamma}}$  is given by:*

- when  $a \in (a_0, 1)$ :

$$\hat{\rho}(P) = r + 2\sqrt{pq}; \quad (28)$$

- when  $a \in (0, a_0]$ :

(a) in case  $2p \leq (1 - q + \sqrt{pq})^2$ :

$$\hat{\rho}(P) = r + 2\sqrt{pq}; \quad (29)$$

(b) in case  $2p > (1 - q + \sqrt{pq})^2$ , set  $a_1 := p - \sqrt{pq} - \sqrt{r(r + 2\sqrt{pq})}$ :

$$\hat{\rho}(P) = \left| a + \frac{p(1-a)}{a-1+q} \right| \quad \text{when } a \in (0, a_1] \quad (30a)$$

$$\hat{\rho}(P) = r + 2\sqrt{pq} \quad \text{when } a \in [a_1, a_0]. \quad (30b)$$

When  $r := 0$ , such results have been obtained in [RT99, Bax05, LT96] by using various methods involving conditions on  $a$  (see the end of Introduction). Let us specify the above formulas in case  $r := 0$ . We have  $a_0 = a_1 = p - \sqrt{pq} = (p - q)/(1 + \sqrt{q/p})$ , and it can be easily checked that  $2p > (1 - q + \sqrt{pq})^2$ . Then the properties (28), (30a), (30b) then rewrite as:  $\hat{\rho}(P) = (pq + (a - p)^2)/|a - p|$  when  $a \in (0, a_0]$ , and  $\hat{\rho}(P) = 2\sqrt{pq}$  when  $a \in (a_0, 1)$ .

*Proof.* We apply the elimination procedure of Section 4. Then  $\Lambda := \{\lambda \in \mathbb{C} : \hat{\delta} < |\lambda| < 1\}$  with  $\hat{\delta} := r + 2\sqrt{pq}$ . The characteristic polynomial  $E_\lambda(\cdot)$  is

$$E_\lambda(z) := qz^2 + (r - \lambda)z + p.$$

A simple study of the graph of  $\phi(t) := p/t + r + qt$  on  $\mathbb{R} \setminus \{0\}$  shows that, for any  $\lambda \in (\hat{\delta}, 1)$ , the equation  $\phi(z) = \lambda$  (ie.  $E_\lambda(z) = 0$ ) admits a solution in  $(1, \hat{\gamma})$  and another one in  $(\hat{\gamma}, +\infty)$ , so that  $N(\lambda) = 1$ . It follows from Proposition 3.4 that  $\eta = 1$ . Thus the linear elimination

used in Corollary 4.1 is here trivial. Indeed, a necessary condition for  $f := \{z^n\}_{n \in \mathbb{N}}$  to satisfy  $Pf = \lambda f$  is obtained by eliminating the variable  $z$  with respect to the boundary equation  $(Pf)(0) = \lambda f(0)$ , namely  $P_0(\lambda, z) := a + (1-a)z = \lambda$ , and Equation  $E_\lambda(z) = 0$ . This leads to

$$P_1(\lambda, z) := \text{Res}_z(P_0, E_\lambda) = (1-\lambda)[(\lambda-a)(1-a-q) + p(1-a)]. \quad (31)$$

In the special case  $a = 1 - q$ , the only solution of (31) is  $\lambda = 1$ . Corollary 4.2 then gives  $\widehat{\rho}(P) = r + 2\sqrt{pq}$ .

Now assume that  $a \neq 1 - q$ . Then  $\lambda = 1$  is a solution of (31) and the other solution of (31), say  $\lambda(a)$ , and the associated complex number, say  $z(a)$ , are given by the following formulas (use  $a + (1-a)z = \lambda$  to obtain  $z(a)$ ):

$$\lambda(a) := a + \frac{p(1-a)}{a-1+q} \in \mathbb{R} \quad \text{and} \quad z(a) := \frac{p}{a+q-1} \in \mathbb{R}.$$

To apply Corollary 4.2 we must find the values  $a \in (0, 1)$  for which both conditions  $\widehat{\delta} < |\lambda(a)| < 1$  and  $|z(a)| \leq \widehat{\gamma}$  hold. Observe that

$$|z(a)| \leq \widehat{\gamma} \Leftrightarrow |a-1+q| \geq \sqrt{pq}.$$

Hence, if  $a \in (a_0, 1)$  (recall that  $a_0 := 1 - q - \sqrt{pq}$ ), then  $|z(a)| > \widehat{\gamma}$ . This gives (28).

Now let  $a \in (0, a_0]$ . Then  $|z(a)| \leq \widehat{\gamma}$ . Let us study  $\lambda(a)$ . We have  $\lambda'(a) = 1 - pq/(a-1+q)^2$ , so that  $a \mapsto \lambda(a)$  is increasing on  $(-\infty, a_0]$  from  $-\infty$  to  $\lambda(a_0) = r - 2\sqrt{pq}$ . Thus

$$\forall a \in (0, a_0], \quad \lambda(a) \leq r - 2\sqrt{pq} < r + 2\sqrt{pq}.$$

and the equation  $\lambda(a) = -(r + 2\sqrt{pq})$  has a unique solution  $a_1 \in (-\infty, a_0)$ . Note that  $a_1 < a_0$  and  $\lambda(a_1) = -(r + 2\sqrt{pq})$ , that  $\lambda(0) = p/(q-1) \in [-1, 0)$  and finally that

$$\lambda(0) - \lambda(a_1) = \frac{p}{q-1} + r + 2\sqrt{pq} = \frac{(q - \sqrt{pq} - 1)^2 - 2p}{1-q}.$$

When  $2p \leq (1 - q + \sqrt{pq})^2$ , we obtain (29). Indeed  $|\lambda(a)| < r + 2\sqrt{pq}$  since

$$\forall a \in (0, a_0], \quad -(r + 2\sqrt{pq}) = \lambda(a_1) \leq \lambda(0) < \lambda(a) < r + 2\sqrt{pq}.$$

When  $2p > (1 - q + \sqrt{pq})^2$ , we have  $a_1 \in (0, a_0]$  and:

- if  $a \in (0, a_1)$ , then (30a) holds. Indeed  $r + 2\sqrt{pq} < |\lambda(a)| < 1$  since

$$\forall a \in (0, a_1], \quad -1 \leq \lambda(0) < \lambda(a) < \lambda(a_1) = -(r + 2\sqrt{pq});$$

- if  $a \in [a_1, a_0]$ , then (30b) holds. Indeed  $|\lambda(a)| < r + 2\sqrt{pq}$  since

$$-(r + 2\sqrt{pq}) = \lambda(a_1) \leq \lambda(a) < r + 2\sqrt{pq}.$$

□

**Remark 4.2 (Discussion on the  $\ell^2(\pi)$ -spectral gap and the decay parameter)**

As mentioned in the introduction, we are not concerned with the usual  $\ell^2(\pi)$  spectral gap  $\rho_2(P)$  for Birth-and-Death Markov Chains (BDMC). In particular, we can not compare our results with that of [vDS95]. To give a comprehensive discussion on [vDS95], let  $P$  be a kernel of an BDMC defined by (27) with invariant probability measure  $\pi$ .  $P$  is reversible with respect to  $\pi$ . It can be proved that the decay parameter of  $P$ , denoted by  $\gamma$  in [vDS95] but by  $\gamma_{DS}$  here to avoid confusion with our parameter  $\gamma$ , is also the rate of convergence  $\rho_2(P)$ :

$$\gamma_{DS} = \rho_2(P) := \lim_n \|P^n - \Pi\|_2^{\frac{1}{n}},$$

where  $\Pi f := \pi(f)\mathbf{1}$  and  $\|\cdot\|_2$  denotes the operator norm on  $\ell^2(\pi)$ . When  $P$  is assumed to be  $V_{\hat{\gamma}}$ -geometrically ergodic with  $V := \{\hat{\gamma}^n\}_{n \in \mathbb{N}}$ , it follows from [Bax05, Th. 6.1], that

$$\gamma_{SD} \leq \hat{\rho}(P).$$

Consequently the bounds of the decay parameter  $\gamma_{DS}$  given in [vDS95] cannot provide bounds for  $\hat{\rho}(P)$  since the converse inequality  $\hat{\rho}(P) \leq \gamma_{DS}$  is not known to the best of our knowledge. Moreover, even if the equality  $\gamma_{DS} = \hat{\rho}(P)$  was true, the bounds obtained in our Proposition 4.1 could be derived from [vDS95] only for some specific values of  $P(0,0)$ . Indeed the difficulty in [vDS95, p. 139-140] to cover all the values  $P(0,0) \in (0,1)$  is that the spectral measure associated with Karlin and McGregor polynomials cannot be easily computed, except for some specific values of  $P(0,0)$  (see [Kov09] for a recent contribution).

**4.2 A non-reversible case : RWs with  $g = 2$  and  $d = 1$** 

Let  $P := (P(i,j))_{(i,j) \in \mathbb{N}^2}$  be defined by

$$P(0,0) = a \in (0,1), \quad P(0,1) = 1 - a, \quad P(1,0) = b \in (0,1), \quad P(1,2) = 1 - b \quad (32)$$

$$\forall n \geq 2, \quad P(n, n-2) = a_{-2} > 0, \quad P(n, n-1) = a_{-1}, \quad P(n, n) = a_0, \quad P(n, n+1) = a_1 > 0.$$

The form of boundary probabilities in (32) is chosen for convenience. Other (finitely many) boundary probabilities could be considered provided that  $P$  is irreducible and aperiodic. To illustrate the procedure proposed in Section 4 for this class of RWs, we also specify the numerical values

$$a_{-2} := 1/2, \quad a_{-1} := 1/3, \quad a_0 = 0, \quad a_1 := 1/6.$$

The procedure could be developed in the same way for any other values of  $(a_{-2}, a_{-1}, a_0, a_1)$  satisfying  $a_{-2}, a_1 > 0$  and Condition **(NERI)** i.e.  $a_1 < 2a_{-2} + a_{-1}$ . Here we have

$$\phi(t) := \frac{1}{2t^2} + \frac{1}{3t} + \frac{t}{6} = 1 + \frac{1}{6t^2}(t-1)(t^2 - 5t - 3).$$

Function  $\phi(\cdot)$  has a minimum over  $(1, +\infty)$  at  $\hat{\gamma} \approx 2.18$ , with  $\hat{\delta} := \phi(\hat{\gamma}) \approx 0.621$ . Let  $V_{\hat{\gamma}} := \{\hat{\gamma}^n\}_{n \in \mathbb{N}}$  and let  $\mathcal{B}_{\hat{\gamma}}$  be the associated weighted space. We know from Proposition 3.2 and from irreducibility and aperiodicity properties that  $\hat{r}_{ess}(P) = \hat{\delta}$  and  $P$  is  $V_{\hat{\gamma}}$ -geometrically ergodic (see Proposition 2.1). The polynomial  $E_{\lambda}(\cdot)$  is

$$\forall z \in \mathbb{C}, \quad E_{\lambda}(z) := \frac{z^3}{6} - \lambda z^2 + \frac{z}{3} + \frac{1}{2}.$$

A simple examination of the graph of  $\phi(\cdot)$  shows that  $\eta = 2$ . Thus the set  $\mathcal{M}$  of Corollary 4.2 is  $\mathcal{M} := \{(1,1), (2)\}$ . Next, the constructive proof of Corollary 4.1 provides the following procedure to compute  $\widehat{\rho}(P)$  (see also Remark 4.1 in the second case). Recall that  $\Lambda := \{\lambda \in \mathbb{C} : \widehat{\delta} < |\lambda| < 1\}$ .

**First case:**  $\mu = (1, 1)$

- (a) When  $\lambda \in \Lambda$  is such that  $\mathcal{E}_\lambda^-$  is composed of 2 simple roots of  $E_\lambda(\cdot)$ , a necessary condition for  $\lambda$  to be an eigenvalue of  $P$  on  $\mathcal{B}_{\widehat{\gamma}}$  is that

$$R_1(\lambda) := \text{Res}_{z_1}(P_1, E_\lambda) = 0,$$

where

$$P_1(\lambda, z_1) := \text{Res}_{z_2}(P_0, E_\lambda) = \begin{vmatrix} 1/6 & 0 & A(\lambda, z_1) & 0 & 0 \\ -\lambda & 1/6 & B(\lambda, z_1) & A(\lambda, z_1) & 0 \\ 1/3 & -\lambda & C(\lambda, z_1) & B(\lambda, z_1) & A(\lambda, z_1) \\ 1/2 & 1/3 & 0 & C(\lambda, z_1) & B(\lambda, z_1) \\ 0 & 1/2 & 0 & 0 & C(\lambda, z_1) \end{vmatrix}.$$

and  $P_0(\lambda, z_1, z_2) := A(\lambda, z_1)z_2^2 + B(\lambda, z_1)z_2 + C(\lambda, z_1)$  is given by

$$P_0(\lambda, z_1, z_2) := \begin{vmatrix} (1-a) & a + (1-a)z_2 - \lambda \\ (1-b)(z_1 + z_2) - \lambda & b + (1-b)z_2^2 - \lambda z_2 \end{vmatrix}. \quad (33)$$

$P_0(\lambda, z_1, z_2)$  is derived using (25) from

$$P_{0,1}(\lambda, z_1, z_2) := \begin{vmatrix} a + (1-a)z_1 - \lambda & a + (1-a)z_2 - \lambda \\ b + (1-b)z_1^2 - \lambda z_1 & b + (1-b)z_2^2 - \lambda z_2 \end{vmatrix} = (z_1 - z_2)P_0(\lambda, z_1, z_2).$$

- (b) *Sufficient part.* Consider

$$\Lambda_{(1,1)} = \text{Root}(R_1) \cap \Lambda = \text{Root}(R_1) \cap \{\lambda \in \mathbb{C} : 0.621 \approx \widehat{\delta} < |\lambda| < 1\}.$$

For every  $\lambda \in \Lambda_{(1,1)}$ :

- (i) Check that  $E_\lambda(z) = 0$  has two simple roots  $z_1$  and  $z_2$  such that  $|z_i| < \widehat{\gamma} \approx 2.18$ .
  - (ii) If (i) is OK, then test if  $P_0(\lambda, z_1, z_2) = 0$  with  $P_0$  given in (33).
- If (i) and (ii) are OK, then  $\lambda$  is an eigenvalue of  $P$  on  $\mathcal{B}_{\widehat{\gamma}}$ .

**Second case:**  $\mu = (2)$ .

- (a) When  $\lambda \in \Lambda$  is such that  $\mathcal{E}_\lambda^-$  is composed of a double root of  $E_\lambda(\cdot)$ , a necessary condition for  $\lambda$  to be an eigenvalue of  $P$  on  $\mathcal{B}_{\widehat{\gamma}}$  is that (see Remark 4.1)

$$Q(\lambda) = 0 \quad \text{and} \quad R_2(\lambda) := \text{Res}_{z_1}(P_1, E_\lambda) = 0,$$

where

$$Q(\lambda) := \begin{vmatrix} 1/6 & 0 & 1/2 & 0 & 0 \\ -\lambda & 1/6 & -2\lambda & 1/2 & 0 \\ 1/3 & -\lambda & 1/3 & -2\lambda & 1/2 \\ 1/2 & 1/3 & 0 & 1/3 & -2\lambda \\ 0 & 1/2 & 0 & 0 & 1/3 \end{vmatrix}$$

and

$$P_1(\lambda) := \text{Res}_{z_1}(P_0, E_\lambda) = \begin{vmatrix} 1/6 & 0 & A(\lambda) & 0 & 0 \\ -\lambda & 1/6 & B(\lambda) & A(\lambda) & 0 \\ 1/3 & -\lambda & C(\lambda) & B(\lambda) & A(\lambda) \\ 1/2 & 1/3 & 0 & C(\lambda) & B(\lambda) \\ 0 & 1/2 & 0 & 0 & C(\lambda) \end{vmatrix}.$$

where  $P_0(\lambda, z_1) := A(\lambda)z_1^2 + B(\lambda)z_1 + C(\lambda)$  is given by

$$P_0(\lambda, z_1) := \begin{vmatrix} a + (1-a)z_1 - \lambda & 1-a \\ b + (1-b)z_1^2 - \lambda z_1 & 2(1-b)z_1 - \lambda \end{vmatrix}. \quad (34)$$

(b) *Sufficient part.* Consider

$$\Lambda'_{(2)} = \text{Root}(Q) \cap \Lambda_{(2)} = \text{Root}(Q) \cap \text{Root}(R_2) \cap \{\lambda \in \mathbb{C} : 0.621 \approx \widehat{\delta} < |\lambda| < 1\}.$$

For every  $\lambda \in \Lambda'_{(2)}$ :

- (i) Check that Equation  $E_\lambda(z) = 0$  has a double root  $z_1$  such that  $|z_1| < \widehat{\gamma} \approx 2.18$ .
  - (ii) If (i) is OK, then test if  $P_0(\lambda, z_1) = 0$  with  $P_0$  given in (34).
- If (i) and (ii) are OK, then  $\lambda$  is an eigenvalue of  $P$  on  $\mathcal{B}_{\widehat{\gamma}}$ .

**Final results** Define  $\mathcal{Z}_{(1,1)}$  as the set of all the  $\lambda \in \Lambda_{(1,1)}$  satisfying (i)-(ii) in the first case, and  $\mathcal{Z}_{(2)}$  as the set of all the  $\lambda \in \Lambda'_{(2)}$  satisfying (i)-(ii) in the second one. Finally set  $\mathcal{Z} := \mathcal{Z}_{(1,1)} \cup \mathcal{Z}_{(2)}$ . Then

$$\widehat{\rho}(P) = \max(\widehat{\delta}, \max\{|\lambda|, \lambda \in \mathcal{Z}\}).$$

The results (using Maple computation engine) for different instances of the values of boundary transition probabilities are reported in Table 1. In these specific examples, note that  $\Lambda'_{(2)}$  is always the empty set. As expected, we obtain that  $\rho_{\widehat{\gamma}}(P) \nearrow 1$  when  $(a, b) \rightarrow (0, 0)$ .

## 5 Convergence rate for RWs with unbounded increments

In this subsection, we propose two instances of RW on  $\mathbb{X} := \mathbb{N}$  with unbounded increments for which estimate of the convergence rate with respect to some weighted-supremum space  $\mathcal{B}_V$  can be obtained using Proposition 3.1 and Proposition 2.1. The first example is from [MS95]. The second one is a reversible transition kernel  $P$  inspired from the “infinite star” example in [Ros96]. Note that using a result of [Bax05] (see Remark 4.2), estimates of  $\rho_V(P)$  with respect to  $\mathcal{B}_V$  may be useful to obtain estimates on the usual spectral gap  $\rho_2(P)$  with respect to Lebesgue’s space  $\ell^2(\pi)$ . Recall that the converse is not true in general.

$(a, b)$	$\Lambda_{(1,1)}$	$\mathcal{Z}_{(1,1)}$	$\Lambda'_{(2)}$	$\mathcal{Z}_{(2)}$	$\widehat{\delta}$	$\widehat{\rho}(P)$
$(1/2, 1/2)$	$-0.625 \pm 0.466i,$ $-0.798, 0.804$	$\emptyset$	$\emptyset$	$\emptyset$	0.621	0.621
$(1/10, 1/10)$	$-0.681 \pm 0.610i$ $-0.466 \pm -0.506i$ $-0.384 \pm 0.555i$	$\{-0.466 \pm 0.506i\}$	$\emptyset$	$\emptyset$	0.621	0.688
$(1/50, 1/50)$	$-0.598 \pm 0.614i$ $-0.383 \pm 0.542i$ $-0.493 \pm 0.574i$ $-0.477 \pm 0.584i$ 0.994	$\{-0.493 \pm 0.574i\}$	$\emptyset$	$\emptyset$	0.621	0.757

Table 1: Convergence rate with different values of boundary transition probabilities  $(a, b)$

### 5.1 A non-reversible RW with unbounded increments [MS95]

Let  $P$  be defined by

$$\forall n \geq 1, P(0, n) := q_n, \quad \forall n \geq 1, P(n, 0) := p, \quad P(n, n+1) := q = 1 - p,$$

with  $p \in (0, 1)$  and  $q_n \in [0, 1]$  such that  $\sum_{n \geq 1} q_n = 1$ .

**Proposition 5.1** *Assume that  $\gamma \in (1, 1/q)$  is such that  $\sum_{n \geq 1} q_n \gamma^n < \infty$ . Then  $r_{ess}(P) \leq q\gamma$ . Moreover  $P$  is  $V_\gamma$ -geometrically ergodic with convergence rate  $\rho_{V_\gamma}(P) \leq \max(q\gamma, p)$ .*

*Proof.* We have:  $\forall n \geq 1, (PV_\gamma)(n) = q\gamma^{n+1} + p$ . Thus, if  $\gamma \in (1, 1/q)$  and  $\sum_{n \geq 1} q_n \gamma^n < \infty$ , then Condition **(WD)** holds with  $V_\gamma$ , and we have  $\delta_{V_\gamma}(P) \leq q\gamma$ . Therefore it follows from Proposition 3.1 that  $r_{ess}(P) \leq q\gamma$ . Now Proposition 2.1 is applied with any  $r_0 > \max(q\gamma, p)$ . Let  $\lambda \in \mathbb{C}$  be such that  $\max(q\gamma, p) < |\lambda| \leq 1$ , and let  $f \in \mathcal{B}_\gamma, f \neq 0$ , be such that  $Pf = \lambda f$ . We obtain  $f(n) = (\lambda/q)f(n-1) - pf(0)/q$  for any  $n \geq 2$ , so that

$$\forall n \geq 2, \quad f(n) = \left(\frac{\lambda}{q}\right)^{n-1} \left(f(1) - \frac{pf(0)}{\lambda - q}\right) + \frac{pf(0)}{\lambda - q}.$$

Since  $f \in \mathcal{B}_{V_\gamma}$  and  $|\lambda|/q > \gamma$ , we obtain  $f(1) = pf(0)/(\lambda - q)$ , and consequently:  $\forall n \geq 1, f(n) = pf(0)/(\lambda - q)$ . Next the equality  $\lambda f(0) = (Pf)(0) = \sum_{n \geq 1} q_n f(n)$  gives:  $\lambda f(0) = pf(0)/(\lambda - q)$  since  $\sum_{n \geq 1} q_n = 1$ . We have  $f(0) \neq 0$  since we look for  $f \neq 0$ . Thus  $\lambda$  satisfies  $\lambda^2 - q\lambda - p = 0$ , that is:  $\lambda = 1$  or  $\lambda = -p$ . The case  $\lambda = -p$  has not to be considered since  $|\lambda| > \max(q\gamma, p)$ . If  $\lambda = 1$ , then  $f(n) = f(0)$  for any  $n \in \mathbb{N}$ , so that  $\lambda = 1$  is a simple eigenvalue of  $P$  on  $\mathcal{B}_\gamma$  and is the only eigenvalue such that  $\max(q\gamma, p) < |\lambda| \leq 1$ . Then Proposition 2.1 gives the second conclusion of Proposition 5.1.  $\square$

Note that  $p$  cannot be dropped in the inequality  $\rho_{V_\gamma}(P) \leq \max(q\gamma, p)$  since  $\lambda = -p$  is an eigenvalue of  $P$  on  $\mathcal{B}_\gamma$  with corresponding eigenvector  $f_p := (1, -p, -p, \dots)$ .

## 5.2 A reversible RW inspired from [Ros96]

Let  $\{\pi_n\}_{n \in \mathbb{N}}$  be a probability distribution (with  $\pi_n > 0$  for every  $n \in \mathbb{N}$ ) and  $P$  be defined by

$$\forall n \in \mathbb{N}, P(0, n) = \pi_n \quad \text{and} \quad \forall n \geq 1, P(n, 0) = \pi_0, P(n, n) = 1 - \pi_0.$$

It is easily checked that  $P$  is reversible with respect to  $\{\pi_n\}_{n \in \mathbb{N}}$ , so that  $\{\pi_n\}_{n \in \mathbb{N}}$  is an invariant probability distribution of  $P$ .

**Proposition 5.2** *Assume that there exists  $V \in [1, +\infty)^{\mathbb{N}}$  such that  $V(0) = 1$ ,  $V(n) \rightarrow +\infty$  as  $n \rightarrow +\infty$  and  $\pi(V) := \sum_{n \geq 0} \pi_n V(n) < \infty$ . Then  $P$  is  $V$ -geometrically ergodic with  $\rho_V(P) \leq 1 - \pi_0$ .*

It can be checked that  $P$  is not stochastically monotone so that the estimate  $\rho_V \leq 1 - \pi_0$  cannot be directly deduced from [LT96].

*Proof.* From  $(PV)(0) = \pi(V)$  and  $\forall n \geq 1$ ,  $(PV)(n) = \pi_0 V(0) + (1 - \pi_0)V(n)$ , it follows that

$$PV \leq (1 - \pi_0)V + (\pi(V) + \pi_0) \mathbf{1}_{\mathbb{X}}.$$

That is, Condition **(WD)** holds true with  $N := 1$ ,  $\delta := 1 - \pi_0$  and  $d := \pi(V) + \pi_0$ . The inequality  $r_{ess}(P) \leq 1 - \pi_0$  is deduced from Proposition 3.1.

Let  $\lambda \in \mathbb{C}$  be an eigenvalue of  $P$  and  $f := \{f(n)\}_{n \in \mathbb{N}}$  be a non trivial associated eigenvector. Then

$$\lambda f(0) = \sum_{n=0}^{+\infty} \pi_n f(n) \quad \text{and} \quad \forall n \geq 1, \quad \lambda f(n) = \pi_0 f(0) + (1 - \pi_0)f(n). \quad (35)$$

This gives:  $\forall n \geq 1$ ,  $f(n) = f(0)\pi_0/(\lambda - 1 + \pi_0)$ . Since  $f \neq 0$ , it follows from the first equality in (35) that

$$\lambda = \pi_0 + \frac{\pi_0}{\lambda - 1 + \pi_0}(1 - \pi_0),$$

which is equivalent to  $\lambda^2 - \lambda = 0$ . Thus,  $\lambda = 1$  or  $0$ . That  $1$  is a simple eigenvalue is standard from the irreducibility of  $P$ . The result follows from Proposition 2.1.  $\square$

A specific instance of this model is considered in [Ros96, p. 68]. Let  $\{w_n\}_{n \geq 1}$  be a sequence of positive scalars such that  $\sum_{n \geq 1} w_n = 1/2$ . Then  $P$  is given by

$$\forall n \in \mathbb{N}, \quad P(n, n) = 1/2 \quad \text{and} \quad \forall n \geq 1, P(0, n) = w_n, P(n, 0) = 1/2$$

which is reversible with respect to its invariant probability distribution  $\pi$  defined by  $\pi_0 := 1/2$  and  $\pi_n := w_n$  for  $n \geq 1$ . It has been proved in [Ros96, p. 68] that, for any  $X_0 \sim \alpha \in \ell^2(1/\pi)$ , there exists a constant  $C_{\alpha, \pi} > 0$  such that

$$\|\alpha P^n - \pi\|_{TV} \leq C_{\alpha, \pi} (3/4)^n \quad (36)$$

where  $\|\cdot\|_{TV}$  is the total variation distance. Since we know that  $\rho_2(P) \leq \rho_V(P)$  from [Bax05] and  $\rho_V(P) \leq 1/2$  from Proposition 5.2, the rate of convergence in (36) is improved.

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