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Spectral analysis of Markov kernels and application to the convergence rate of discrete random walks

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Abstract

Let $\{X_n\}_{n \in \mathbb{N}}$ be a Markov chain on a measurable space \mathbb{X} with transition kernel P and let $V : \mathbb{X} \rightarrow [1, +\infty)$. The Markov kernel P is here considered as a linear bounded operator on the weighted-supremum space \mathcal{B}_V associated with V . Then the combination of quasi-compactness arguments with precise analysis of eigen-elements of P allows us to estimate the geometric rate of convergence $\rho_V(P)$ of $\{X_n\}_{n \in \mathbb{N}}$ to its invariant probability measure in operator norm on \mathcal{B}_V . A general procedure to compute $\rho_V(P)$ for discrete Markov random walks with identically distributed bounded increments is specified.

AMS subject classification : 60J10; 47B07

Keywords : V -Geometric ergodicity, Quasi-compactness, Drift condition, Birth-and-Death Markov chains.

1 Introduction

Let $(\mathbb{X}, \mathcal{X})$ be a measurable space with a σ -field \mathcal{X} , and let $\{X_n\}_{n \geq 0}$ be a Markov chain with state space \mathbb{X} and transition kernels $\{P(x, \cdot) : x \in \mathbb{X}\}$. Let $V : \mathbb{X} \rightarrow [1, +\infty)$. Assume that $\{X_n\}_{n \geq 0}$ has an invariant probability measure π such that $\pi(V) := \int_{\mathbb{X}} V(x) \pi(dx) < \infty$. This paper is based on the connection between spectral properties of the Markov kernel P and the so-called V -geometric ergodicity [MT93] which is the following convergence property for some constants $c_\rho > 0$ and $\rho \in (0, 1)$:

$$\sup_{|f| \leq V} \sup_{x \in \mathbb{X}} \frac{|\mathbb{E}[f(X_n) | X_0 = x] - \pi(f)|}{V(x)} \leq c_\rho \rho^n. \quad (1)$$

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Let us introduce the weighted-supremum Banach space $(\mathcal{B}_V, \|\cdot\|_V)$ composed of measurable functions $f : \mathbb{X} \rightarrow \mathbb{C}$ such that

$$\|f\|_V := \sup_{x \in \mathbb{X}} \frac{|f(x)|}{V(x)} < \infty.$$

Then (1) reads as $\|P^n f - \pi(f)1_{\mathbb{X}}\|_V \leq c_\rho \rho^n$ for any $f \in \mathcal{B}_V$ such that $\|f\|_V \leq 1$, and there is a great interest in obtaining upper bounds for *the convergence rate* $\rho_V(P)$ defined by

$$\rho_V(P) := \inf \left\{ \rho \in (0, 1), \sup_{\|f\|_V \leq 1} \|P^n f - \pi(f)1_{\mathbb{X}}\|_V = O(\rho^n) \right\}. \quad (2)$$

For irreducible and aperiodic discrete Markov chains, criteria for the V -geometric ergodicity are well-known from the literature using either the equivalence between geometric ergodicity and V -geometric ergodicity of \mathbb{N} -valued Markov chains [HS92, Prop. 2.4], or the strong drift condition. For instance, when $\mathbb{X} := \mathbb{N}$ (with $\lim_n V(n) = +\infty$), the strong drift condition is

$$PV \leq \varrho V + b 1_{\{0,1,\dots,n_0\}}$$

for some $\varrho < 1, b < \infty$ and $n_0 \in \mathbb{N}$ (see [MT93]). Estimating $\rho_V(P)$ from the parameters ϱ, b, n_0 is a difficult issue. This often leads to unsatisfactory bounds, except for stochastically monotone P (see [MT94, LT96, Bax05] and the references therein).

This work presents a new procedure to study the convergence rate $\rho_V(P)$ under the following weak drift condition

$$\exists N \in \mathbb{N}^*, \exists d \in (0, +\infty), \exists \delta \in (0, 1), \quad P^N V \leq \delta^N V + d 1_{\mathbb{X}}. \quad (\mathbf{WD})$$

The V -geometric ergodicity clearly implies (\mathbf{WD}) . Conversely, such a condition with $N = 1$ was introduced in [MT93, Lem. 15.2.8] as an alternative to the drift condition [MT93, (V4)] to obtain the V -geometric ergodicity under suitable assumption on V . Note that, under Condition (\mathbf{WD}) , the following real number $\delta_V(P)$ is well defined:

$$\delta_V(P) := \inf \left\{ \delta \in [0, 1) : \exists N \in \mathbb{N}^*, \exists d \in (0, +\infty), P^N V \leq \delta^N V + d 1_{\mathbb{X}} \right\}.$$

A spectral analysis of P is presented in Section 2 using quasi-compactness. More specifically, when the Markov kernel P has an invariant probability distribution, the connection between the V -geometric ergodicity and the quasi-compactness of P is made explicit in Proposition 2.1. Namely, P is V -geometrically ergodic if and only if P is a power-bounded quasi-compact operator on \mathcal{B}_V for which $\lambda = 1$ is a simple eigenvalue and the unique eigenvalue of modulus one. In this case, if $r_{ess}(P)$ denotes the essential spectral radius of P on \mathcal{B}_V (see (5)) and if \mathcal{V} denotes the set of eigenvalues λ of P such that $r_{ess}(P) < |\lambda| < 1$, then the convergence rate $\rho_V(P)$ is given by (Proposition 2.1):

$$\rho_V(P) = r_{ess}(P) \text{ if } \mathcal{V} = \emptyset \quad \text{and} \quad \rho_V(P) = \max\{|\lambda|, \lambda \in \mathcal{V}\} \text{ if } \mathcal{V} \neq \emptyset. \quad (3)$$

Interesting bounds for generalized eigenfunctions $f \in \mathcal{B}_V \cap \text{Ker}(P - \lambda I)^p$ associated with $\lambda \in \mathcal{V}$ are presented in Proposition 2.2. Property (3) is relevant to study the convergence rate $\rho_V(P)$ provided that, first an accurate bound of $r_{ess}(P)$ is known, second the above

set \mathcal{V} is available. Bounds of $r_{ess}(P)$ related to drift conditions can be found in [Wu04] and [HL14] under various assumptions (see Subsection 2.1). In view of our applications, let us just mention that $r_{ess}(P) = \delta_V(P)$ in case $\mathbb{X} := \mathbb{N}$ and $\lim_n V(n) = +\infty$ (see Proposition 3.1). However, even if the state space is discrete, finding the above set \mathcal{V} is difficult.

In Section 3, the above spectral analysis is applied to compute the rate of convergence $\rho_V(P)$ of discrete Random Walks (RW). In particular, a complete solution is presented for RWs with identically distributed (i.d.) bounded increments. In fact, Proposition 3.4 allows us to formulate an algebraic procedure based on polynomial eliminations providing $\rho_V(P)$ (see Corollary 4.1). To the best of our knowledge, this general result is new. Note that it requires neither reversibility nor stochastic monotonicity of P .

This procedure is illustrated in Section 4. First we consider the case of birth-and-death Markov kernel P defined by $P(0,0) := a$ and $P(0,1) := 1 - a$ for some $a \in (0,1)$ and by

$$\forall n \geq 1, P(n, n-1) := p, \quad P(n, n) := r, \quad P(n, n+1) := q,$$

where $p, q, r \in [0,1]$ are such that $p + r + q = 1$, $p > q > 0$. Explicit formula for $\rho_V(P)$ with respect to $V := \{(p/q)^{n/2}\}_{n \in \mathbb{N}}$ is given in Proposition 4.1. When $r := 0$, such a result has been obtained for $a < p$ in [RT99] and [Bax05, Ex. 8.4] using Kendall's theorem, and for $a \geq p$ in [LT96] using the stochastic monotony of P . Our method gives a unified and simpler computation of $\rho_V(P)$ which moreover encompasses the case $r \neq 0$. For general RWs with i.d. bounded increments, the elimination procedure requires to use symbolic computations. The second example illustrates this point with the non reversible RW defined by

$$\forall n \geq 2, P(n, n-2) = a_{-2}, P(n, n-1) = a_{-1}, P(n, n) = a_0, P(n, n+1) = a_1$$

for any nonnegative a_i satisfying $a_{-2} + a_{-1} + a_0 + a_1 = 1$, $a_{-2} > 0$, $2a_{-2} + a_{-1} > a_1 > 0$, and for any finitely many boundary transition probabilities. In Section 5, specific examples of RWs on $\mathbb{X} := \mathbb{N}$ with unbounded increments considered in the literature are investigated.

To conclude this introduction, we mention a point which can be source of confusion in a first reading. In this paper, we are concerned with the convergence rate (2) with respect to some weighted-supremum Banach space \mathcal{B}_V . Thus, we do not consider here the decay parameter or the convergence rate of ergodic Markov chains in the usual Hilbert space $\mathbb{L}^2(\pi)$ which is related to spectral properties of the transition kernel with respect to this space. In particular, for Birth-and-Death Markov chains, we can not compare our results with those of [vDS95] on the $\ell^2(\pi)$ -spectral gap and the decay parameter. A detailed discussion is provided in Remark 4.2.

2 Quasi-compactness on \mathcal{B}_V and V -geometric ergodicity

We assume that P satisfies **(WD)**. Then P continuously acts on \mathcal{B}_V , and iterating **(WD)** shows that P is power-bounded on \mathcal{B}_V , namely $\sup_{n \geq 1} \|P^n\|_V < \infty$, where $\|\cdot\|_V$ also stands for the operator norm on \mathcal{B}_V . Thus we have $r(P) := \lim_n \|P^n\|_V^{1/n} = 1$ since P is Markov.

2.1 From quasi-compactness on \mathcal{B}_V to V -geometric ergodicity

Let I denote the identity operator on \mathcal{B}_V . Recall that P is said to be quasi-compact on \mathcal{B}_V if there exist $r_0 \in (0, 1)$ and $m \in \mathbb{N}^*$, $\lambda_i \in \mathbb{C}$, $p_i \in \mathbb{N}^*$ ($i = 1, \dots, m$) such that:

$$\mathcal{B}_V = \bigoplus_{i=1}^m \text{Ker}(P - \lambda_i I)^{p_i} \oplus H, \quad (4a)$$

where the λ_i 's are such that

$$|\lambda_i| \geq r_0 \quad \text{and} \quad 1 \leq \dim \text{Ker}(P - \lambda_i I)^{p_i} < \infty, \quad (4b)$$

and H is a closed P -invariant subspace such that

$$\inf_{n \geq 1} \left(\sup_{h \in H, \|h\| \leq 1} \|P^n h\| \right)^{1/n} < r_0. \quad (4c)$$

Concerning the essential spectral radius of P , denoted by $r_{ess}(P)$, here it is enough to have in mind that, if P is quasi-compact on \mathcal{B}_V , then we have (see for instance [Hen93])

$$r_{ess}(P) := \inf \{ r_0 \in (0, 1) \text{ such that (4a)-(4c) hold} \}. \quad (5)$$

As mentioned in Introduction, the essential spectral radius of Markov kernels acting on \mathcal{B}_V is studied in [Wu04, HL14]. For instance, under Condition **(WD)**, the following result is proved in [HL14]: if P^ℓ is compact from \mathcal{B}_0 to \mathcal{B}_V for some $\ell \geq 1$, where $(\mathcal{B}_0, \|\cdot\|_0)$ is the Banach space composed of bounded measurable functions $f : \mathbb{X} \rightarrow \mathbb{C}$ equipped with the supremum norm $\|f\|_0 := \sup_{x \in \mathbb{X}} |f(x)|$, then P is quasi-compact on \mathcal{B}_V with

$$r_{ess}(P) \leq \delta_V(P).$$

Moreover, equality $r_{ess}(P) = \delta_V(P)$ holds in many situations, in particular in the discrete state case with $V(n) \rightarrow \infty$ (see Proposition 3.1).

Next we explicit a result which makes explicit the relationship between the quasi-compactness of P and the V -geometric ergodicity of the Markov chain $\{X_n\}_{n \in \mathbb{N}}$ with transition kernel P . Moreover, we provide an explicit formula for $\rho_V(P)$ in terms of the spectral elements of P . Note that for any $r_0 \in (r_{ess}(P), 1)$, the set of all the eigenvalues of λ of P such that $r_0 \leq |\lambda| \leq 1$ is finite (use (5)).

Proposition 2.1 *Let P be a transition kernel which has an invariant probability measure π such that $\pi(V) < \infty$. The two following assertions are equivalent:*

- (a) *P is V -geometrically ergodic.*
- (b) *P is a power-bounded quasi-compact operator on \mathcal{B}_V , for which $\lambda = 1$ is a simple eigenvalue (i.e. $\text{Ker}(P - I) = \mathbb{C} \cdot \mathbf{1}_{\mathbb{X}}$) and the unique eigenvalue of modulus one.*

Under any of these conditions, we have $\rho_V(P) \geq r_{ess}(P)$. In fact, for $r_0 \in (r_{ess}(P), 1)$, denoting the set of all the eigenvalues λ of P such that $r_0 \leq |\lambda| < 1$ by \mathcal{V}_{r_0} , we have:

- either $\rho_V(P) \leq r_0$ when $\mathcal{V}_{r_0} = \emptyset$,
- or $\rho_V(P) = \max\{|\lambda|, \lambda \in \mathcal{V}_{r_0}\}$ when $\mathcal{V}_{r_0} \neq \emptyset$.

Moreover, if $\mathcal{V}_{r_0} = \emptyset$ for all $r_0 \in (r_{ess}(P), 1)$, then $\rho_V(P) = r_{ess}(P)$.

The V -geometric ergodicity of P obviously implies that P is quasi-compact on \mathcal{B}_V with $\rho_V(P) \geq r_{ess}(P)$ (see e.g. [KM03]). This follows from (5) using $H := \{f \in \mathcal{B}_V : \pi(f) = 0\}$ in (4a)-(4c). The property that P has a spectral gap on \mathcal{B}_V in the recent paper [KM12] corresponds here to the quasi-compactness of P (which is a classical terminology in spectral theory). The spectral gap in [KM12] corresponds to the value $1 - \rho_V(P)$. Then, [KM12, Prop. 1.1]) is another formulation, under ψ -irreducibility and aperiodicity assumptions, of the equivalence of properties (a) and (b) in Proposition 2.1 (see also [KM12, Lem. 2.1]). Details on the proof of Proposition 2.1 are provided in [GHL11]. For general quasi-compact Markov kernels on \mathcal{B}_V , the result [Wu04, Th. 4.6] also provides interesting additional material on peripheral eigen-elements. The next subsection completes the previous spectral description by providing bounds for the generalized eigenfunctions associated with eigenvalues λ such that $\delta \leq |\lambda| \leq 1$, with δ given in **(WD)**.

2.2 Bound on generalized eigenfunctions of P

Proposition 2.2 *Assume that the weak drift condition **(WD)** holds true. If $\lambda \in \mathbb{C}$ is such that $\delta \leq |\lambda| \leq 1$, with δ given in **(WD)**, and if $f \in \mathcal{B}_V \cap \text{Ker}(P - \lambda I)^p$ for some $p \in \mathbb{N}^*$, then there exists $c \in (0, +\infty)$ such that*

$$|f| \leq c V^{\frac{\ln|\lambda|}{\ln\delta}} (1 + \ln V)^{\frac{p(p-1)}{2}}.$$

Thus, if λ is an eigenvalue such that $|\lambda| = 1$, then any associated eigenfunction f is bounded on \mathbb{X} . By contrast, if $|\lambda|$ is close to $\delta_V(P)$, then $|f| \leq c V^{\beta(\lambda)}$ with $\beta(\lambda)$ close to 1. The proof of Proposition 2.2 is based on the following lemma.

Lemma 2.3 *Let $\lambda \in \mathbb{C}$ be such that $\delta \leq |\lambda| \leq 1$. Then*

$$\forall f \in \mathcal{B}_V, \exists c \in (0, +\infty), \forall x \in \mathbb{X}, \quad |\lambda|^{-n(x)} |(P^{n(x)} f)(x)| \leq c V(x)^{\frac{\ln|\lambda|}{\ln\delta}} \quad (6)$$

with, for any $x \in \mathbb{X}$, $n(x) := \lfloor \frac{-\ln V(x)}{\ln\delta} \rfloor$ where $\lfloor \cdot \rfloor$ denotes the integer part function.

Proof. First note that the iteration of **(WD)** gives

$$\forall k \geq 1, \quad P^{kN} V \leq \delta^{kN} V + d \left(\sum_{j=0}^{k-1} \delta^{jN} \right) \mathbf{1}_{\mathbb{X}} \leq \delta^{kN} V + \frac{d}{1 - \delta^N} \mathbf{1}_{\mathbb{X}}.$$

Let $g \in \mathcal{B}_V$ and $x \in \mathbb{X}$. Using the last inequality, the positivity of P and $|g| \leq \|g\|_V V$, we obtain with $b := d/(1 - \delta^N)$:

$$\forall k \geq 1, \quad |(P^{kN} g)(x)| \leq (P^{kN} |g|)(x) \leq \|g\|_V (P^{kN} V)(x) \leq \|g\|_V (\delta^{kN} V(x) + b). \quad (7)$$

The previous inequality is also fulfilled with $k = 0$. Next, let $f \in \mathcal{B}_V$ and $n \in \mathbb{N}$. Writing $n = kN + r$, with $k \in \mathbb{N}$ and $r \in \{0, 1, \dots, N-1\}$, and applying (7) to $g := P^r f$, we obtain with $\xi := \max_{0 \leq \ell \leq N-1} \|P^\ell f\|_V$ (use $P^n f = P^{kN}(P^r f)$):

$$|(P^n f)(x)| \leq \xi [\delta^{kN} V(x) + b] \leq \xi [\delta^{-r} (\delta^n V(x) + b)] \leq \xi \delta^{-N} (\delta^n V(x) + b). \quad (8)$$

Using the inequality

$$-\frac{\ln V(x)}{\ln \delta} - 1 \leq n(x) \leq -\frac{\ln V(x)}{\ln \delta}$$

and the fact that $\ln \delta \leq \ln |\lambda| \leq 0$, Inequality (8) with $n := n(x)$ gives:

$$\begin{aligned} |\lambda|^{-n(x)} |(P^{n(x)} f)(x)| &\leq \xi \delta^{-N} \left((\delta |\lambda|^{-1})^{n(x)} V(x) + b |\lambda|^{-n(x)} \right) \\ &= \xi \delta^{-N} \left(e^{n(x)(\ln \delta - \ln |\lambda|)} e^{\ln V(x)} + b e^{-n(x) \ln |\lambda|} \right) \\ &\leq \xi \delta^{-N} \left(e^{(\frac{\ln V(x)}{\ln \delta} + 1)(\ln |\lambda| - \ln \delta)} e^{\ln V(x)} + b e^{\frac{\ln V(x)}{\ln \delta} \ln |\lambda|} \right) \\ &= \xi \delta^{-N} \left(e^{\frac{\ln |\lambda|}{\ln \delta} \ln V(x)} e^{\ln |\lambda| - \ln \delta} + b V(x)^{\frac{\ln |\lambda|}{\ln \delta}} \right) \\ &= \xi \delta^{-N} (e^{\ln |\lambda| - \ln \delta} + b) V(x)^{\frac{\ln |\lambda|}{\ln \delta}}. \end{aligned}$$

This gives Inequality (6) with $c := \xi \delta^{-N} (e^{\ln |\lambda| - \ln \delta} + b)$. \square

Proof of Proposition 2.2. If $f \in \mathcal{B}_V \cap \text{Ker}(P - \lambda I)$, then $|\lambda|^{-n(x)} |(P^{n(x)} f)(x)| = |f(x)|$, so that (6) gives the expected conclusion when $p = 1$. Next, let us proceed by induction. Assume that the conclusion of Proposition 2.2 holds for some $p \geq 1$. Let $f \in \mathcal{B}_V \cap \text{Ker}(P - \lambda I)^{p+1}$. We can write

$$P^n f = (P - \lambda I + \lambda I)^n f = \lambda^n f + \sum_{k=1}^{\min(n,p)} \binom{n}{k} \lambda^{n-k} (P - \lambda I)^k f. \quad (9)$$

For $k \in \{1, \dots, p\}$, we have $f_k := (P - \lambda I)^k f \in \text{Ker}(P - \lambda I)^{p+1-k} \subset \text{Ker}(P - \lambda I)^p$, thus we have from the induction hypothesis :

$$\exists c' \in (0, +\infty), \forall k \in \{1, \dots, p\}, \forall x \in \mathbb{X}, |f_k(x)| \leq c' V(x)^{\frac{\ln |\lambda|}{\ln \delta}} (1 + \ln V(x))^{\frac{p(p-1)}{2}}. \quad (10)$$

Now, we obtain from (9) (with $n := n(x)$), (10) and Lemma 2.3 that for all $x \in \mathbb{X}$:

$$\begin{aligned} |f(x)| &\leq |\lambda|^{-n(x)} |(P^{n(x)} f)(x)| + c' V(x)^{\frac{\ln |\lambda|}{\ln \delta}} (1 + \ln V(x))^{\frac{p(p-1)}{2}} |\lambda|^{-\min(n,p)} \sum_{k=1}^{\min(n,p)} \binom{n(x)}{k} \\ &\leq c V(x)^{\frac{\ln |\lambda|}{\ln \delta}} + c_1 V(x)^{\frac{\ln |\lambda|}{\ln \delta}} (1 + \ln V(x))^{\frac{p(p-1)}{2}} n(x)^p \\ &\leq c_2 V(x)^{\frac{\ln |\lambda|}{\ln \delta}} (1 + \ln V(x))^{\frac{p(p-1)}{2} + p} \end{aligned}$$

with some constants $c_1, c_2 \in (0, +\infty)$ independent of x . This gives the expected result. \square

3 Spectral properties of discrete Random Walks

In the sequel, the state space \mathbb{X} is discrete. For the sake of simplicity, we assume that $\mathbb{X} := \mathbb{N}$. Let $P = (P(i, j))_{i, j \in \mathbb{N}^2}$ be a Markov kernel on \mathbb{N} . The function $V : \mathbb{N} \rightarrow [1, +\infty)$ is assumed to satisfy

$$\lim_n V(n) = +\infty \quad \text{and} \quad \sup_{n \in \mathbb{N}} \frac{(PV)(n)}{V(n)} < \infty.$$

The first focus is on the estimation of $r_{ess}(P)$ from Condition **(WD)**.

Proposition 3.1 *Let $\mathbb{X} := \mathbb{N}$. The two following conditions are equivalent:*

(a) *Condition **(WD)** holds with V ;*

(b) $L := \inf_{N \geq 1} (\ell_N)^{\frac{1}{N}} < 1$ where $\ell_N := \limsup_{n \rightarrow +\infty} \frac{(P^N V)(n)}{V(n)}$.

In this case, P is power-bounded and quasi-compact on \mathcal{B}_V with $r_{ess}(P) = \delta_V(P) = L$.

The proof of the equivalence (a) \Leftrightarrow (b), as well as the equality $\delta_V(P) = L$, is straightforward (see [GHL11, Cor. 4]). That P is quasi-compact on \mathcal{B}_V under **(WD)** in the discrete case, with $r_{ess}(P) \leq \delta_V(P)$, can be derived from [Wu04] or [HL14] (see Subsection 2.1 and use the fact that the injection from \mathcal{B}_0 to \mathcal{B}_V is compact when $\mathbb{X} := \mathbb{N}$ and $\lim_n V(n) = +\infty$). Equality $r_{ess}(P) = \delta_V(P)$ can be proved by combining the results [Wu04, HL14] (see [GHL11, Cor. 1] for details).

In Sections 3 and 4, sequences of the special form $V_\gamma := \{\gamma^n\}_{n \in \mathbb{N}}$ for some $\gamma \in (1, +\infty)$ will be considered. The associated weighted-supremum space $\mathcal{B}_\gamma \equiv \mathcal{B}_{V_\gamma}$ is defined by:

$$\mathcal{B}_\gamma := \left\{ \{f(n)\}_{n \in \mathbb{N}} \in \mathbb{C}^{\mathbb{N}} : \sup_{n \in \mathbb{N}} \gamma^{-n} |f(n)| < \infty \right\}.$$

3.1 Quasi-compactness of RWs with bounded state-dependent increments

Let us fix $c, g, d \in \mathbb{N}^*$, and assume that the kernel P satisfies the following conditions:

$$\forall i \in \{0, \dots, g-1\}, \quad \sum_{j=0}^c P(i, j) = 1; \tag{11a}$$

$$\forall i \geq g, \forall j \in \mathbb{N}, \quad P(i, j) = \begin{cases} a_{j-i}(i) & \text{if } i-g \leq j \leq i+d \\ 0 & \text{otherwise} \end{cases} \tag{11b}$$

where $(a_{-g}(i), \dots, a_d(i)) \in [0, 1]^{g+d+1}$ satisfies $\sum_{k=-g}^d a_k(i) = 1$ for all $i \geq g$. This kind of kernels arises, for instance, from time-discretization of Markovian queuing models. Note that more general models and their use in queuing theory are discussed in [KD06]. In particular, conditions for (non) positive recurrence are provided.

Proposition 3.2 Assume that, for every $k \in \mathbb{Z}$ such that $-g \leq k \leq d$, $\lim_n a_k(n) = a_k \in [0, 1]$, and that

$$\exists \gamma \in (1, +\infty) : \quad \phi(\gamma) := \sum_{k=-g}^d a_k \gamma^k < 1. \quad (12)$$

Then P satisfies Condition **(WD)** with $\delta = \phi(\gamma)$. Moreover P is power-bounded and quasi-compact on \mathcal{B}_γ with $r_{ess}(P) = L = \phi(\gamma)$.

Lemma 3.3 When a_{-g} and a_d are positive, Condition (12) is equivalent to

$$\sum_{k=-g}^d k a_k < 0. \quad (\mathbf{NERI})$$

Then, there exists a unique real number $\gamma_0 > 1$ such that $\phi(\gamma_0) = 1$ and

$$\forall \gamma \in (1, \gamma_0), \quad \phi(\gamma) < 1$$

and there is a unique $\hat{\gamma}$ such that

$$\hat{\delta} := \phi(\hat{\gamma}) = \min_{\gamma \in (1, \infty)} \phi(\gamma) = \min_{\gamma \in (1, \gamma_0)} \phi(\gamma) < 1.$$

Condition **(NERI)** means that the expectation of the probability distribution of the random increment is negative. Although the results of the paper on RWs with i.d. bounded increments involving Condition **(NERI)** and $a_{-g}, a_d > 0$ will be valid for $\gamma \in (1, \gamma_0)$, only this value $\hat{\gamma}$ is considered in the statements. Note that the essential spectral radius $r_{ess}(P|_{\mathcal{B}_{\hat{\gamma}}})$ of P with respect to $\mathcal{B}_{\hat{\gamma}}$, which will be denoted by $\hat{r}_{ess}(P)$ in the sequel, is the smallest value of $r_{ess}(P|_{\mathcal{B}_\gamma})$ on \mathcal{B}_γ for $\gamma \in (1, \gamma_0)$. When $\gamma \nearrow \gamma_0$, the essential spectral radius $r_{ess}(P|_{\mathcal{B}_\gamma}) \nearrow 1$ since the space \mathcal{B}_γ becomes large. When $\gamma \searrow 1$, then $r_{ess}(P|_{\mathcal{B}_\gamma}) \nearrow 1$ since \mathcal{B}_γ becomes close to the space \mathcal{B}_0 of bounded functions. In this case, the geometric ergodicity is lost since the RWs are typically not uniformly ergodic (i.e. $V \equiv 1$) due the non quasi-compactness of P on \mathcal{B}_0 .

Example 1 (State-dependent birth-and-death Markov chains) When $c = g = d := 1$ in (11a)-(11b), we obtain the standard class of state-dependent birth-and-death Markov chains:

$$\begin{aligned} P(0, 0) &:= r_0, & P(0, 1) &:= q_0 \\ \forall n \geq 1, & P(n, n-1) &:= p_n, & P(n, n) &:= r_n, & P(n, n+1) &:= q_n, \end{aligned}$$

where $(p_0, q_0) \in [0, 1]^2$, $p_0 + q_0 = 1$ and $(p_n, r_n, q_n) \in [0, 1]^3$, $p_n + r_n + q_n = 1$. Assume that:

$$\lim_n p_n := p \quad \lim_n r_n := r, \quad \lim_n q_n := q.$$

If $\gamma \in (1, +\infty)$ is such that $\phi(\gamma) := p/\gamma + r + q\gamma < 1$ then it follows from Proposition 3.2 that $r_{ess}(P) = p/\gamma + r + q\gamma$. The conditions $\gamma > 1$ and $p/\gamma + r + q\gamma < 1$ are equivalent to the following ones (use $r = 1 - p - q$ for (i)):

- (i) either $q > 0$, $q - p < 0$ (i.e. **(NERI)**) and $1 < \gamma < \gamma_0 = p/q$;
- (ii) or $q = 0$, $p > 0$ and $\gamma > 1$.

(i) When $p > q > 0$ and $1 < \gamma < \gamma_0$: P is power-bounded and quasi-compact on \mathcal{B}_γ with $r_{\text{ess}}(P) = \phi(\gamma)$. Set $\hat{\gamma} := \sqrt{\gamma_0} = \sqrt{p/q} \in (1, \gamma_0)$. Then $\min_{\gamma > 1} \phi(\gamma) = \phi(\hat{\gamma}) = r + 2\sqrt{pq}$ and the essential spectral radius $\hat{r}_{\text{ess}}(P)$ on $\mathcal{B}_{\hat{\gamma}}$ satisfies $\hat{r}_{\text{ess}}(P) = r + 2\sqrt{pq}$.

(ii) When $q := 0, p > 0$ and $\gamma > 1$: $r_{\text{ess}}(P) = \phi(\gamma) = p/\gamma + r$.

Remark 3.1 If c is allowed to be $+\infty$ in Condition (11a), that is

$$\forall i \in \{0, \dots, g-1\}, \quad \sum_{j \geq 0} P(i, j) \gamma^j < \infty, \quad (13)$$

then the conclusions of Proposition 3.2 and Example 1 are still valid under the additional Condition (13).

Proof of Proposition 3.2. Set $\phi_n(\gamma) := \sum_{k=-g}^d a_k(n) \gamma^k$. We have $(PV_\gamma)(n) = \phi_n(\gamma)V_\gamma(n)$ for each $n \geq g$. Thus $\ell_1 = \lim_n \phi_n(\gamma) = \phi(\gamma)$. Now assume that $\ell_{N-1} := \lim_n (P^{N-1}V)(n)/V(n) = \phi(\gamma)^{N-1}$ for some $N \geq 1$. Since

$$\forall i \geq Ng, \quad (P^N V)(i) = \sum_{j=-g}^d a_j(i) (P^{N-1}V)(i+j)$$

we obtain

$$\frac{(P^N V)(i)}{V(i)} = \sum_{j=-g}^d a_j(i) \gamma^j \frac{(P^{N-1}V)(i+j)}{\gamma^{i+j}} \xrightarrow{i \rightarrow +\infty} \phi(\gamma) \phi(\gamma)^{N-1}.$$

Hence $\ell_N = \phi(\gamma)^N$, and $\phi(\gamma) = L = r_{\text{ess}}(P)$ from Proposition 3.1. \square

Proof of Lemma 3.3. Since the second derivative of ϕ is positive on $(0, +\infty)$, ϕ is convex on $(0, +\infty)$. When a_{-g} and a_d are positive then $\lim_{t \rightarrow 0^+} \phi(t) = \lim_{t \rightarrow +\infty} \phi(t) = +\infty$ and, since $\phi(1) = 1$, Condition (12) is equivalent to $\phi'(1) < 0$, that is **(NERI)**. The other properties of $\phi(\cdot)$ are immediate. \square

3.2 Spectral analysis of RW with i.d. bounded increments

Let $P := (P(i, j))_{(i, j) \in \mathbb{N}^2}$ be the transition kernel of a RW with i.d. bounded increments. Specifically we assume that there exist some positive integers $c, g, d \in \mathbb{N}^*$ such that

$$\forall i \in \{0, \dots, g-1\}, \quad \sum_{j=0}^c P(i, j) = 1; \quad (14a)$$

$$\forall i \geq g, \forall j \in \mathbb{N}, \quad P(i, j) = \begin{cases} a_{j-i} & \text{if } i-g \leq j \leq i+d \\ 0 & \text{otherwise.} \end{cases} \quad (14b)$$

$$(a_{-g}, \dots, a_d) \in [0, 1]^{g+d+1} : a_{-g} > 0, a_d > 0, \quad \sum_{k=-g}^d a_k = 1. \quad (14c)$$

Let us assume that Condition **(NERI)** holds. We know from Lemma 3.3 and Proposition 3.2 that P is quasi-compact on $\mathcal{B}_{\widehat{\gamma}}$ with

$$\widehat{r}_{ess}(P) = \widehat{\delta} := \phi(\widehat{\gamma}) < 1$$

where $\phi(\cdot)$ is given by (12).

For any $\lambda \in \mathbb{C}$, we denote by $E_\lambda(\cdot)$ the following polynomial of degree $N := d + g$

$$\forall z \in \mathbb{C}, \quad E_\lambda(z) := z^g (\phi(z) - \lambda) = \sum_{k=-g}^d a_k z^{g+k} - \lambda z^g,$$

and by \mathcal{E}_λ the set of complex roots of $E_\lambda(\cdot)$. Since $E_\lambda(0) = a_{-g} > 0$, we have for any $\lambda \in \mathbb{C}$:

$$z \in \mathcal{E}_\lambda \iff \mathbb{E}_\lambda(z) = 0 \iff \lambda = \phi(z).$$

The next proposition investigates the eigenvalues of P on $\mathcal{B}_{\widehat{\gamma}}$ which belong to the annulus

$$\Lambda := \{\lambda \in \mathbb{C} : \widehat{\delta} < |\lambda| < 1\}.$$

To that effect, for any $\lambda \in \Lambda$, we introduce the following subset \mathcal{E}_λ^- of \mathcal{E}_λ

$$\mathcal{E}_\lambda^- := \{z \in \mathbb{C} : E_\lambda(z) = 0, |z| < \widehat{\gamma}\}.$$

If $\mathcal{E}_\lambda^- = \emptyset$, we set $N(\lambda) := 0$. If $\mathcal{E}_\lambda^- \neq \emptyset$, then $N(\lambda)$ is defined as

$$N(\lambda) := \sum_{z \in \mathcal{E}_\lambda^-} m_z,$$

where m_z denotes the multiplicity of z as root of $E_\lambda(\cdot)$. Finally, for any $z \in \mathbb{C}$, we set $z^{(1)} := \{z^n\}_{n \in \mathbb{N}}$, and for any $k \geq 2$, $z^{(k)} \in \mathbb{C}^{\mathbb{N}}$ is defined by:

$$\forall n \in \mathbb{N}, \quad z^{(k)}(n) := n(n-1) \cdots (n-k+2) z^{n-k+1}.$$

Proposition 3.4 *Assume that Assumptions (14a)-(14c) and **(NERI)** hold true. Then*

$$\exists \eta \geq 1, \forall \lambda \in \Lambda, \quad N(\lambda) = \eta.$$

Moreover the two following assertions are equivalent:

- (i) $\lambda \in \Lambda$ is an eigenvalue of P on $\mathcal{B}_{\widehat{\gamma}}$.
- (ii) There exists a nonzero $\{\alpha_{\lambda,z,k}\}_{z \in \mathcal{E}_\lambda^-, 1 \leq k \leq m_z} \in \mathbb{C}^\eta$ such that

$$f := \sum_{z \in \mathcal{E}_\lambda^-} \sum_{k=1}^{m_z} \alpha_{\lambda,z,k} z^{(k)} \in \mathbb{C}^{\mathbb{N}} \tag{15}$$

satisfies the boundary equations: $\forall i = 0, \dots, g-1, \lambda f(i) = (Pf)(i)$.

The first step of the elimination procedure of Section 4 is to plug f of the form (15) in the boundary equations. This gives a linear system in $\alpha_{\lambda,z,k}$. Since Λ is infinite, that $N(\lambda)$ does not depend on λ is crucial to initialize this procedure. To specify the value of η , it is sufficient to compute $N(\lambda)$ for some (any) $\lambda \in \Lambda$.

Remark 3.2 Under Condition **(NERI)**, $\phi(\cdot)$ is strictly decreasing from $(1, \widehat{\gamma})$ to $(\widehat{\delta}, 1)$, so that we have: $\forall \lambda \in (\widehat{\delta}, 1)$, $\phi^{-1}(\lambda) \in (1, \widehat{\gamma})$. Since $\phi^{-1}(\lambda) \in \mathcal{E}_\lambda$, we obtain

$$\forall \lambda \in (\widehat{\delta}, 1), \quad N(\lambda) \geq 1. \quad (16)$$

Remark 3.3 Let Condition **(NERI)** be satisfied. Set $\mathcal{E}_\lambda^+ := \{z \in \mathbb{C} : E_\lambda(z) = 0, |z| > \widehat{\gamma}\}$. Then

$$\forall \lambda \in \Lambda, \quad \mathcal{E}_\lambda = \mathcal{E}_\lambda^- \sqcup \mathcal{E}_\lambda^+.$$

In other words, for any $\lambda \in \Lambda$, $E_\lambda(\cdot)$ has no root of modulus $\widehat{\gamma}$. Indeed, consider $\lambda \in \Lambda$, $z \in \mathcal{E}_\lambda$, and assume that $|z| = \widehat{\gamma}$. Since $\lambda = \phi(z)$, we obtain the inequality $|\lambda| \leq \phi(|z|) = \phi(\widehat{\gamma})$ which is impossible since $\phi(\widehat{\gamma}) = \widehat{\delta}$ and $\lambda \in \Lambda$.

Remark 3.4 Assertion (ii) of Proposition 3.4 does not mean that the dimension of the eigenspace $\text{Ker}(P - \lambda I)$ associated with λ is η . We shall see in Subsection 4.2 that we can have $\eta = 2$ when $g = 2$, $d = 1$ and $c = 2$ in (14a)-(14c), while $\dim \text{Ker}(P - \lambda I) \leq 1$ since $Pf = \lambda f$ and $f(0) = 0$ clearly imply $f = 0$ (by induction).

The following surprising lemma, based on Remark 3.3, is used to derive Proposition 3.4.

Lemma 3.5 Under Condition **(NERI)**, the function $N(\cdot)$ is constant on Λ .

Proof. Since Λ is connected and $N(\cdot)$ is \mathbb{N} -valued, it suffices to prove that $N(\cdot)$ is continuous on Λ . Note that the set $\cup_{\lambda \in \Lambda} \mathcal{E}_\lambda$ is bounded in \mathbb{C} since the coefficients of $E_\lambda(\cdot)$ are obviously uniformly bounded in $\lambda \in \Lambda$. Now let $\lambda \in \Lambda$ and assume that $N(\cdot)$ is not continuous at λ . Then there exists a sequence $\{\lambda_n\}_{n \in \mathbb{N}} \in \Lambda^\mathbb{N}$ such that $\lim_n \lambda_n = \lambda$ and

(a) either: $\forall n \geq 0, N(\lambda_n) \geq N(\lambda) + 1$,

(b) or: $\forall n \geq 0, N(\lambda_n) \leq N(\lambda) - 1$.

For any $n \geq 0$, let us denote the roots of $E_{\lambda_n}(\cdot)$ by $z_1(\lambda_n), \dots, z_N(\lambda_n)$, and suppose for convenience that they are listed by increasing modulus, and by increasing argument when they have the same modulus. Applying Remark 3.3 to λ_n , we obtain:

$$\forall i \in \{1, \dots, N(\lambda_n)\}, |z_i(\lambda_n)| < \widehat{\gamma} \quad \text{and} \quad \forall i \in \{N(\lambda_n) + 1, \dots, N\}, |z_i(\lambda_n)| > \widehat{\gamma}.$$

Up to consider a subsequence, we may suppose that, for every $1 \leq i \leq N$, the sequence $\{z_i(\lambda_n)\}_{n \in \mathbb{N}}$ converges to some $z_i \in \mathbb{C}$. Note that

$$\mathcal{E}_\lambda = \{z_1, z_2, \dots, z_N\}$$

where z_i is repeated in this list with respect to its multiplicity m_{z_i} , since

$$\forall z \in \mathbb{C}, \quad E_\lambda(z) = \lim_n E_{\lambda_n}(z) = \lim_n a_d \prod_{i=1}^N (z - z_i(\lambda_n)) = a_d \prod_{i=1}^N (z - z_i).$$

In case (a), we have

$$\forall n \geq 0, \quad |z_1(\lambda_n)| < \widehat{\gamma}, \dots, |z_{N(\lambda)+1}(\lambda_n)| < \widehat{\gamma}.$$

When $n \rightarrow +\infty$, this gives using Remark 3.3:

$$|z_1| < \widehat{\gamma}, \dots, |z_{N(\lambda)+1}| < \widehat{\gamma}.$$

Thus at least $N(\lambda) + 1$ roots of $E_\lambda(\cdot)$ (counted with their multiplicity) are of modulus strictly less than $\widehat{\gamma}$: this contradicts the definition of $N(\lambda)$.

In case (b), we have

$$\forall n \geq 0, \quad |z_{N(\lambda)}(\lambda_n)| > \widehat{\gamma}, |z_{N(\lambda)+1}(\lambda_n)| > \widehat{\gamma}, \dots, |z_N(\lambda_n)| > \widehat{\gamma},$$

and this gives similarly when $n \rightarrow +\infty$

$$|z_{N(\lambda)}| > \widehat{\gamma}, |z_{N(\lambda)+1}| > \widehat{\gamma}, \dots, |z_N| > \widehat{\gamma}.$$

Thus at least $N - N(\lambda) + 1$ roots of $E_\lambda(\cdot)$ (counted with their multiplicity) are of modulus strictly larger than $\widehat{\gamma}$. This contradicts the definition of $N(\lambda)$. \square

Proof of Proposition 3.4. From Lemma 3.5 and (16), we obtain: $\forall \lambda \in \Lambda$, $N(\lambda) = \eta$ for some $\eta \geq 1$. Now we prove the implication (i) \Rightarrow (ii). Let $\lambda \in \Lambda$ be any eigenvalue of P on $\mathcal{B}_{\widehat{\gamma}}$ and let $f := \{f(n)\}_{n \in \mathbb{N}}$ be a nonzero sequence in $\mathcal{B}_{\widehat{\gamma}}$ satisfying $Pf = \lambda f$. In particular f satisfies the following equalities

$$\forall i \geq g, \quad \lambda f(i) = \sum_{j=i-g}^{i+g} a_{j-i} f(j). \quad (17)$$

Since the characteristic polynomial associated with these recursive formulas is $E_\lambda(\cdot)$, there exists $\{\alpha_{\lambda,z,k}\}_{z \in \mathcal{E}_\lambda, 1 \leq k \leq m_z} \in \mathbb{C}^\eta$ such that

$$f = \sum_{z \in \mathcal{E}_\lambda} \sum_{k=1}^{m_z} \alpha_{\lambda,z,k} z^{(k)} \in \mathbb{C}^\mathbb{N}$$

where m_z denotes the multiplicity of $z \in \mathcal{E}_\lambda$. Next, since $|f| \leq C V_{\widehat{\gamma}}$ for some $C > 0$ (i.e. $f \in \mathcal{B}_{\widehat{\gamma}}$), it can be easily seen that $\alpha_{\lambda,z,k} = 0$ for every $z \in \mathcal{E}_\lambda$ such that $|z| > \widehat{\gamma}$ and for every $k = 1, \dots, m_z$: first delete α_{λ,z,m_z} for z of maximum modulus and for m_z maximal if there are several z of maximal modulus (to that effect, divide f by $n(n-1) \cdots (n-m_z+2) z^{n-m_z+1}$ and use $|f| \leq C V_{\widehat{\gamma}}$). Therefore f is of the form (15), and it satisfies the boundary equations in (ii) since $Pf = \lambda f$ by hypothesis.

To prove the implication (ii) \Rightarrow (i), note that any $f := \{f(n)\}_{n \in \mathbb{N}}$ of the form (15) belongs to $\mathcal{B}_{\widehat{\gamma}}$ and satisfies (17) since $\mathcal{E}_\lambda^- \subset \mathcal{E}_\lambda$. If moreover f is non zero and satisfies the boundary equations, then $Pf = \lambda f$. This gives (i). \square

We conclude this study with an additional refinement of Proposition 3.4. For any $\lambda \in \Lambda$, let us define the set $\mathcal{E}_{\lambda, \tau}^-$ as follows:

$$\mathcal{E}_{\lambda, \tau}^- := \{z \in \mathbb{C} : E_\lambda(z) = 0, |z| < \widehat{\gamma}^\tau\} \quad \text{with} \quad \tau \equiv \tau(\lambda) := \frac{\ln |\lambda|}{\ln \widehat{\delta}}.$$

Moreover define the associated function $N'(\cdot)$ by

$$N'(\lambda) := \sum_{z \in \mathcal{E}_{\lambda, \tau}^-} m_z,$$

where m_z is the multiplicity of z as root of $E_\lambda(\cdot)$ (with the convention $N'(\lambda) = 0$ if $\mathcal{E}_{\lambda, \tau}^- = \emptyset$).

Lemma 3.6 *Assume that $P := (P(i, j))_{(i, j) \in \mathbb{N}^2}$ satisfies Conditions (14a)-(14c) and **(NERI)**. Moreover assume that*

$$\forall t \in (1, \widehat{\gamma}), \quad \phi(t) < t^{\ln \widehat{\delta} / \ln \widehat{\gamma}} \quad (18)$$

Then the function $N'(\cdot)$ is constant on Λ : $\exists \eta' \geq 1, \forall \lambda \in \Lambda, N'(\lambda) = \eta'$.

From Lemma 3.6, all the assertions of Proposition 3.4 are still valid when η and \mathcal{E}_λ^- are replaced with η' and $\mathcal{E}_{\lambda, \tau}^-$ respectively. That \mathcal{E}_λ^- may be replaced with $\mathcal{E}_{\lambda, \tau}^-$ in (15) follows from Proposition 2.2. Consequently, under the additional condition $\eta' \leq g$, the elimination procedure of Section 4 may be adapted by using Lemma 3.6. Since $\eta' \leq \eta$, the resulting procedure is computationally interesting when g or d are large.

Remark 3.5 *Condition (18) is the additional assumption in Lemma 3.6 with respect to Lemma 3.5. Since ϕ is strictly decreasing on $(1, \widehat{\gamma})$ under Condition **(NERI)**, Condition (18) is equivalent to the following one*

$$\forall z \in (1, \widehat{\gamma}), \quad z < \widehat{\gamma}^{\ln \phi(z) / \ln \widehat{\delta}}. \quad (19)$$

Indeed, for every $t \in (1, \widehat{\gamma})$, we have $u := t^{\ln \widehat{\delta} / \ln \widehat{\gamma}} \in (\widehat{\delta}, 1)$ and $z := \phi^{-1}(u) \in (1, \widehat{\gamma})$. Hence

$$(18) \iff \forall u \in (\widehat{\delta}, 1), \phi(\widehat{\gamma}^{\ln u / \ln \widehat{\delta}}) < u \iff (19). \quad (20)$$

Therefore, under Condition (18), for any $\lambda \in (\widehat{\delta}, 1)$ we have $\mathcal{E}_{\lambda, \tau}^- \neq \emptyset$ since $z = \phi^{-1}(\lambda)$ satisfies $z < \widehat{\gamma}^{\tau(\lambda)}$ from (19).

Proof of Lemma 3.6. The proof is similar to that of Lemma 3.5. Under Condition (18), Remark 3.3 extends as follows:

$$\mathcal{E}_\lambda = \mathcal{E}_{\lambda, \tau}^- \sqcup (\mathcal{E}_\lambda \cap \{z \in \mathbb{C} : |z| > \widehat{\gamma}^\tau\}). \quad (21)$$

Indeed, consider $\lambda \in \Lambda$ and $z \in \mathcal{E}_\lambda$ such that $|z| = \widehat{\gamma}^\tau$. Since $\lambda = \phi(z)$, we have $|\lambda| \leq \phi(|z|)$, thus $|\lambda| \leq \phi(\widehat{\gamma}^\tau)$. This inequality contradicts Condition (18) (use the definition of τ and the second equivalence in (20) with $u := |\lambda|$). Next, using (21) and the continuity of $\tau(\cdot)$, Lemma 3.5 easily extends to the function $N'(\cdot)$. \square

4 Convergence rate for RWs with i.d. bounded increments

Let us recall that any RW with i.d. bounded increments defined by (14a)-(14c) and satisfying **(NERI)**, has an invariant probability measure π on \mathbb{N} such $\pi(V_{\hat{\gamma}}) < \infty$ where $V_{\hat{\gamma}} := \{\hat{\gamma}^n\}_{n \in \mathbb{N}}$ and $\hat{\gamma}$ is defined in Lemma 3.3. Indeed $\hat{\delta} := \phi(\hat{\gamma}) < 1$ so that Condition **(WD)** holds with $V_{\hat{\gamma}}$ from Proposition 3.2. The expected conclusions on π can be deduced from the first statement of [GHL11, Cor 5]. Note that, from Lemma 3.3, the previous fact is valid for any $\gamma \in (1, \gamma_0)$ in place of $\hat{\gamma}$.

The $V_{\hat{\gamma}}$ -geometric ergodicity of the RW may be studied using Proposition 2.1. Next we can derive from Proposition 3.4 an effective procedure to compute the rate of convergence with respect to $\mathcal{B}_{\hat{\gamma}}$ (see (2)), that is denoted by $\hat{\rho}(P)$. The most favorable case for initializing the procedure (see (24) and (26)) is to assume that for some (any) $\lambda \in \Lambda$

$$\eta := N(\lambda) \leq g. \quad (22)$$

- *First step: checking Condition (22).* From Lemma 3.5, computing η and testing $\eta \leq g$ of Assumption (22) can be done by analyzing the roots of $E_{\lambda}(\cdot)$ for some (any) $\lambda \in \Lambda$.
- *Second step: linear and polynomial eliminations.* This second step consists in applying some linear and (successive) polynomial eliminations in order to find a finite set $\mathcal{Z} \subset \Lambda$ containing all the eigenvalues of P on $\mathcal{B}_{\hat{\gamma}}$ in Λ . Conversely, the elements of \mathcal{Z} providing eigenvalues of P on $\mathcal{B}_{\hat{\gamma}}$ can be identified using Condition (ii) of Proposition 3.4. Note that the explicit computation of the roots of $E_{\lambda}(\cdot)$ is only required for the elements λ of the finite set \mathcal{Z} . This is detailed in Corollary 4.1.

Under the assumptions of Proposition 3.4, we define the set

$$\mathcal{M} := \{(m_1, \dots, m_s) \in \{1, \dots, s\}^s : s \in \{1, \dots, \eta\}, m_1 \leq \dots \leq m_s \text{ and } \sum_{i=1}^s m_i = \eta\}.$$

Note that \mathcal{M} is a finite set and that, for every $\lambda \in \Lambda$, there exists a unique $\mu \in \mathcal{M}$ such that the set \mathcal{E}_{λ}^- is composed of s distinct roots of $E_{\lambda}(\cdot)$ with multiplicity m_1, \dots, m_s respectively.

Corollary 4.1 *Assume that Assumptions (14a)-(14c) and **(NERI)** hold true. Set $\ell := \binom{g}{\eta}$. Then there exist a family of polynomial functions $\{\mathcal{R}_{\mu,k}, \mu \in \mathcal{M}, 1 \leq k \leq \ell\}$, with coefficients only depending on μ and on the transition probabilities $P(i,j)$, such that the following assertions hold true for any $\mu \in \mathcal{M}$.*

- (i) *Let $\lambda \in \Lambda$ be an eigenvalue of P on $\mathcal{B}_{\hat{\gamma}}$ such that, for some $s \in \{1, \dots, \eta\}$, the set \mathcal{E}_{λ}^- is composed of s roots of $E_{\lambda}(\cdot)$ with multiplicity m_1, \dots, m_s respectively. Then*

$$\mathcal{R}_{\mu,1}(\lambda) = 0, \dots, \mathcal{R}_{\mu,\ell}(\lambda) = 0. \quad (23)$$

- (ii) *Conversely, let $\lambda \in \Lambda$ satisfying (23) such that, for some $s \in \{1, \dots, \eta\}$, the set \mathcal{E}_{λ}^- is composed of s roots of $E_{\lambda}(\cdot)$ with multiplicity m_1, \dots, m_s respectively. Then a necessary and sufficient condition for λ to be an eigenvalue of P on $\mathcal{B}_{\hat{\gamma}}$ is that λ satisfies Condition (ii) of Proposition 3.4.*

Proof. Assertion (ii) follows from Proposition 3.4. To prove (i), first assume for convenience that $\eta = g$ and that $\lambda \in \Lambda$ is an eigenvalue of P on $\mathcal{B}_{\hat{\gamma}}$ such that the associated set $\mathcal{E}_{\lambda}^{-}$ contains η distinct roots z_1, \dots, z_{η} of $E_{\lambda}(\cdot)$ with multiplicity one. We know from Proposition 3.4 that there exists $f := \{f(n)\}_{n \in \mathbb{N}} \neq 0$ of the form

$$f = \sum_{i=1}^{\eta} \alpha_i z_i^{(1)}$$

which satisfies the $g = \eta$ boundary equations: $\forall i = 0, \dots, \eta - 1, \lambda f(i) = (Pf)(i)$. In other words the linear system provided by these η equations has a nonzero solution $(\alpha_i)_{1 \leq i \leq \eta} \in \mathbb{C}^{\eta}$. Therefore the associated determinant is zero: this leads to a polynomial equation of the form

$$P_{0,1}(\lambda, z_1, \dots, z_{\eta}) = 0. \quad (24)$$

Since this polynomial is divisible by $\prod_{i \neq j} (z_i - z_j)$, Equation (24) is equivalent to

$$P_0(\lambda, z_1, \dots, z_{\eta}) = 0 \quad \text{with } P_0(\lambda, z_1, \dots, z_{\eta}) = \frac{P_{0,1}(\lambda, z_1, \dots, z_{\eta})}{\prod_{i \neq j} (z_i - z_j)}. \quad (25)$$

Note that the coefficients of P_0 only depend on the $P(i, j)$'s.

Next, z_{η} is a common root of the polynomials $P_0(\lambda, z_1, \dots, z_{\eta-1}, z)$ and $E_{\lambda}(z)$ with respect to the variable z : this leads to the following necessary condition

$$P_1(\lambda, z_1, \dots, z_{\eta-1}) := \text{Res}_{z_{\eta}}(P_0, E_{\lambda}) = 0$$

where $\text{Res}_{z_{\eta}}(P_0, E_{\lambda})$ denotes the resultant of the two polynomials P_0 and E_{λ} corresponding to the elimination of the variable z_{η} . Again the coefficients of P_1 only depend on the $P(i, j)$'s. Next, considering the common root $z_{\eta-1}$ of the polynomials $P_1(\lambda, z_1, \dots, z_{\eta-2}, z)$ and $E_{\lambda}(z)$ leads to the elimination of the variable $z_{\eta-1}$

$$P_2(\lambda, z_1, \dots, z_{\eta-2}) := \text{Res}_{z_{\eta-1}}(P_1, E_{\lambda}) = 0.$$

Repeating this method, we obtain that a necessary condition for λ to be an eigenvalue of P is $\mathcal{R}(\lambda) = 0$ where \mathcal{R} is some polynomial with coefficients only depending on the $P(i, j)$'s.

Now let us consider the case when $\eta < g$, $s \in \{1, \dots, \eta\}$, and $\lambda \in \Lambda$ is assumed to be an eigenvalue of P on $\mathcal{B}_{\hat{\gamma}}$ such that the associated set $\mathcal{E}_{\lambda}^{-}$ contains s distinct roots of $E_{\lambda}(\cdot)$ with respective multiplicity m_1, \dots, m_s satisfying $\sum_{i=1}^s m_i = \eta$. Then the elimination (by using determinants) of $(\alpha_{\lambda, z, \ell}) \in \mathbb{C}^{\eta}$ provided by the linear system of Proposition 3.4, leads to $\ell := \binom{g}{\eta}$ polynomial equations

$$P_{0,\mu,1}(\lambda, z_1, \dots, z_{\eta}) = 0, \dots, P_{0,\mu,\ell}(\lambda, z_1, \dots, z_{\eta}) = 0. \quad (26)$$

As in the case $\eta = g$, these polynomials are replaced in the sequel by the polynomials obtained by division of the $P_{0,\mu,k}$'s by $\prod_{i \neq j} (z_i - z_j)^{n_{i,j}}$ where $n_{i,j} := \min(m_i, m_j)$.

The successive polynomial eliminations of z_{η}, \dots, z_1 can be derived as above from each polynomial equation $P_{0,\mu,k}(\lambda, z_1, \dots, z_{\eta}) = 0$. This gives ℓ polynomial equations

$$\mathcal{R}_{\mu,1}(\lambda) = 0, \dots, \mathcal{R}_{\mu,\ell}(\lambda) = 0.$$

Satisfying this set of polynomial equations is a necessary condition for λ to be an eigenvalue of P on $\mathcal{B}_{\widehat{\gamma}}$. Finally the polynomial functions $\mathcal{R}_{\mu,1}, \dots, \mathcal{R}_{\mu,\ell}$ depend on the $P(i,j)$'s and also on (m_1, \dots, m_s) , since the linear system used to eliminate $(\alpha_{\lambda,k,\ell}) \in \mathbb{C}^\eta$ involves coefficients $i(i-1)\cdots(i-k+1)$ for some finitely many integers i and for $k = 1, \dots, m_i$ ($i = 1, \dots, s$). \square

To compute $\widehat{\rho}(P)$, we define the following (finite and possibly empty) sets:

$$\forall \mu \in \mathcal{M}, \quad \Lambda_\mu := \{\lambda \in \Lambda : \mathcal{R}_{\mu,1}(\lambda) = 0, \dots, \mathcal{R}_{\mu,\ell}(\lambda) = 0\}.$$

Let us denote by \mathcal{Z} the (finite and possibly empty) set composed of all the complex numbers $\lambda \in \cup_{\mu \in \mathcal{M}} \Lambda_\mu$ such that Condition (ii) of Proposition 3.4 holds true.

Corollary 4.2 *Assume that Assumptions (14a)-(14c) and (NERI) hold true and that P is irreducible and aperiodic. Then*

$$\widehat{\rho}(P) = \max(\widehat{\delta}, \max\{|\lambda|, \lambda \in \mathcal{Z}\}) \quad \text{where } \widehat{\delta} := \phi(\widehat{\gamma}).$$

Proof. Under the assumptions on P , we know from Proposition 2.1 that the RW is $V_{\widehat{\gamma}}$ -geometrically ergodic. Since $\widehat{r}_{ess}(P) = \widehat{\delta}$ from Proposition 3.2, the corollary follows from Corollary 4.1 and from Proposition 2.1 applied either with any r_0 such that $\widehat{\delta} < r_0 < \min\{|\lambda|, \lambda \in \mathcal{Z}\}$ if $\mathcal{Z} \neq \emptyset$, or with any r_0 such that $\widehat{\delta} < r_0 < 1$ if $\mathcal{Z} = \emptyset$. \square

Remark 4.1 *When $\eta \geq 2$ and $\mu := (m_1, \dots, m_s)$ with $s < \eta$, the set Λ_μ used in Corollary 4.2 may be reduced. For the sake of simplicity, this fact has been omitted in Corollary 4.2, but it is relevant in practice. Actually, when $s < \eta$, the part (ii) of Corollary 4.1 can be specified since it requires that $E_\lambda(\cdot)$ admits roots of multiplicity ≥ 2 . This involves some additional necessary conditions on λ derived from some polynomial eliminations with respect to the derivatives of $E_\lambda(\cdot)$.*

For instance, in case $g = 2, \eta = 2, s = 1$ (thus $\mu := (2)$), a necessary condition on λ for $E_\lambda(\cdot)$ to have a double root is that $E_\lambda(\cdot)$ and $E'_\lambda(\cdot)$ admits a common root. This leads to

$$Q(\lambda) := \text{Res}_z(E_\lambda, E'_\lambda) = 0.$$

Consequently, if $g = 2$ and $\eta = 2$ (thus $\ell := 1$), then Condition (ii) of Proposition 3.4 can be tested in case $s = 1$ by using the following finite set

$$\Lambda'_\mu := \Lambda_\mu \cap \{\lambda \in \Lambda : Q(\lambda) = 0\}.$$

In general Λ'_μ is strictly contained in Λ_μ . Even Λ'_μ may be empty while Λ_μ is not (see Subsection 4.2).

Proposition 3.4 and the above elimination procedure obviously extend to any $\gamma \in (1, \gamma_0)$ in place of $\widehat{\gamma}$, where γ_0 is given in Lemma 3.3. Of course $\widehat{\delta} = \phi(\widehat{\gamma})$ is then replaced by $\delta = \phi(\gamma)$.

4.1 RWs with $g = d := 1$: birth-and-death Markov chains

Let $p, q, r \in [0, 1]$ be such that $p + r + q = 1$, and let P be defined by

$$\begin{aligned} P(0, 0) &\in (0, 1), P(0, 1) = 1 - P(0, 0) \\ \forall n \geq 1, P(n, n-1) &:= p, \quad P(n, n) := r, \quad P(n, n+1) := q \quad \text{with } 0 < q < p. \end{aligned} \quad (27)$$

Note that $a_{-1} := p, a_1 := q > 0$ and **(NERI)** holds true. We have $\gamma_0 = p/q \in (1, +\infty)$ and $\hat{\gamma} := \sqrt{p/q} \in (1, +\infty)$ is such that $\hat{\delta} := \min_{\gamma > 1} \phi(\gamma) = \phi(\hat{\gamma}) < 1$ (see Lemma 3.3). Let $V_{\hat{\gamma}} := \{\hat{\gamma}^n\}_{n \in \mathbb{N}}$ and its associated weighted-supremum space $\mathcal{B}_{\hat{\gamma}}$. Here we have

$$\hat{r}_{ess}(P) = \hat{\delta} = r + 2\sqrt{pq}.$$

Proposition 4.1 *Let P be defined by Conditions (27). The boundary transition probabilities are denoted by $P(0, 0) := a, P(0, 1) := 1 - a$ for some $a \in (0, 1)$. Then P is $V_{\hat{\gamma}}$ -geometrically ergodic. Furthermore, defining $a_0 := 1 - q - \sqrt{pq}$, the convergence rate $\hat{\rho}(P)$ of P with respect to $\mathcal{B}_{\hat{\gamma}}$ is given by:*

- when $a \in (a_0, 1)$:

$$\hat{\rho}(P) = r + 2\sqrt{pq}; \quad (28)$$

- when $a \in (0, a_0]$:

(a) in case $2p \leq (1 - q + \sqrt{pq})^2$:

$$\hat{\rho}(P) = r + 2\sqrt{pq}; \quad (29)$$

(b) in case $2p > (1 - q + \sqrt{pq})^2$, set $a_1 := p - \sqrt{pq} - \sqrt{r(r + 2\sqrt{pq})}$:

$$\hat{\rho}(P) = \left| a + \frac{p(1-a)}{a-1+q} \right| \quad \text{when } a \in (0, a_1] \quad (30a)$$

$$\hat{\rho}(P) = r + 2\sqrt{pq} \quad \text{when } a \in [a_1, a_0]. \quad (30b)$$

When $r := 0$, such results have been obtained in [RT99, Bax05, LT96] by using various methods involving conditions on a (see the end of Introduction). Let us specify the above formulas in case $r := 0$. We have $a_0 = a_1 = p - \sqrt{pq} = (p - q)/(1 + \sqrt{q/p})$, and it can be easily checked that $2p > (1 - q + \sqrt{pq})^2$. Then the properties (28), (30a), (30b) then rewrite as: $\hat{\rho}(P) = (pq + (a - p)^2)/|a - p|$ when $a \in (0, a_0]$, and $\hat{\rho}(P) = 2\sqrt{pq}$ when $a \in (a_0, 1)$.

Proof. We apply the elimination procedure of Section 4. Then $\Lambda := \{\lambda \in \mathbb{C} : \hat{\delta} < |\lambda| < 1\}$ with $\hat{\delta} := r + 2\sqrt{pq}$. The characteristic polynomial $E_\lambda(\cdot)$ is

$$E_\lambda(z) := qz^2 + (r - \lambda)z + p.$$

A simple study of the graph of $\phi(t) := p/t + r + qt$ on $\mathbb{R} \setminus \{0\}$ shows that, for any $\lambda \in (\hat{\delta}, 1)$, the equation $\phi(z) = \lambda$ (ie. $E_\lambda(z) = 0$) admits a solution in $(1, \hat{\gamma})$ and another one in $(\hat{\gamma}, +\infty)$, so that $N(\lambda) = 1$. It follows from Proposition 3.4 that $\eta = 1$. Thus the linear elimination

used in Corollary 4.1 is here trivial. Indeed, a necessary condition for $f := \{z^n\}_{n \in \mathbb{N}}$ to satisfy $Pf = \lambda f$ is obtained by eliminating the variable z with respect to the boundary equation $(Pf)(0) = \lambda f(0)$, namely $P_0(\lambda, z) := a + (1-a)z = \lambda$, and Equation $E_\lambda(z) = 0$. This leads to

$$P_1(\lambda, z) := \text{Res}_z(P_0, E_\lambda) = (1-\lambda)[(\lambda-a)(1-a-q) + p(1-a)]. \quad (31)$$

In the special case $a = 1 - q$, the only solution of (31) is $\lambda = 1$. Corollary 4.2 then gives $\widehat{\rho}(P) = r + 2\sqrt{pq}$.

Now assume that $a \neq 1 - q$. Then $\lambda = 1$ is a solution of (31) and the other solution of (31), say $\lambda(a)$, and the associated complex number, say $z(a)$, are given by the following formulas (use $a + (1-a)z = \lambda$ to obtain $z(a)$):

$$\lambda(a) := a + \frac{p(1-a)}{a-1+q} \in \mathbb{R} \quad \text{and} \quad z(a) := \frac{p}{a+q-1} \in \mathbb{R}.$$

To apply Corollary 4.2 we must find the values $a \in (0, 1)$ for which both conditions $\widehat{\delta} < |\lambda(a)| < 1$ and $|z(a)| \leq \widehat{\gamma}$ hold. Observe that

$$|z(a)| \leq \widehat{\gamma} \Leftrightarrow |a-1+q| \geq \sqrt{pq}.$$

Hence, if $a \in (a_0, 1)$ (recall that $a_0 := 1 - q - \sqrt{pq}$), then $|z(a)| > \widehat{\gamma}$. This gives (28).

Now let $a \in (0, a_0]$. Then $|z(a)| \leq \widehat{\gamma}$. Let us study $\lambda(a)$. We have $\lambda'(a) = 1 - pq/(a-1+q)^2$, so that $a \mapsto \lambda(a)$ is increasing on $(-\infty, a_0]$ from $-\infty$ to $\lambda(a_0) = r - 2\sqrt{pq}$. Thus

$$\forall a \in (0, a_0], \quad \lambda(a) \leq r - 2\sqrt{pq} < r + 2\sqrt{pq}.$$

and the equation $\lambda(a) = -(r + 2\sqrt{pq})$ has a unique solution $a_1 \in (-\infty, a_0)$. Note that $a_1 < a_0$ and $\lambda(a_1) = -(r + 2\sqrt{pq})$, that $\lambda(0) = p/(q-1) \in [-1, 0)$ and finally that

$$\lambda(0) - \lambda(a_1) = \frac{p}{q-1} + r + 2\sqrt{pq} = \frac{(q - \sqrt{pq} - 1)^2 - 2p}{1-q}.$$

When $2p \leq (1 - q + \sqrt{pq})^2$, we obtain (29). Indeed $|\lambda(a)| < r + 2\sqrt{pq}$ since

$$\forall a \in (0, a_0], \quad -(r + 2\sqrt{pq}) = \lambda(a_1) \leq \lambda(0) < \lambda(a) < r + 2\sqrt{pq}.$$

When $2p > (1 - q + \sqrt{pq})^2$, we have $a_1 \in (0, a_0]$ and:

- if $a \in (0, a_1)$, then (30a) holds. Indeed $r + 2\sqrt{pq} < |\lambda(a)| < 1$ since

$$\forall a \in (0, a_1], \quad -1 \leq \lambda(0) < \lambda(a) < \lambda(a_1) = -(r + 2\sqrt{pq});$$

- if $a \in [a_1, a_0]$, then (30b) holds. Indeed $|\lambda(a)| < r + 2\sqrt{pq}$ since

$$-(r + 2\sqrt{pq}) = \lambda(a_1) \leq \lambda(a) < r + 2\sqrt{pq}.$$

□

Remark 4.2 (Discussion on the $\ell^2(\pi)$ -spectral gap and the decay parameter)

As mentioned in the introduction, we are not concerned with the usual $\ell^2(\pi)$ spectral gap $\rho_2(P)$ for Birth-and-Death Markov Chains (BDMC). In particular, we can not compare our results with that of [vDS95]. To give a comprehensive discussion on [vDS95], let P be a kernel of an BDMC defined by (27) with invariant probability measure π . P is reversible with respect to π . It can be proved that the decay parameter of P , denoted by γ in [vDS95] but by γ_{DS} here to avoid confusion with our parameter γ , is also the rate of convergence $\rho_2(P)$:

$$\gamma_{DS} = \rho_2(P) := \lim_n \|P^n - \Pi\|_2^{\frac{1}{n}},$$

where $\Pi f := \pi(f)\mathbf{1}$ and $\|\cdot\|_2$ denotes the operator norm on $\ell^2(\pi)$. When P is assumed to be $V_{\hat{\gamma}}$ -geometrically ergodic with $V := \{\hat{\gamma}^n\}_{n \in \mathbb{N}}$, it follows from [Bax05, Th. 6.1], that

$$\gamma_{SD} \leq \hat{\rho}(P).$$

Consequently the bounds of the decay parameter γ_{DS} given in [vDS95] cannot provide bounds for $\hat{\rho}(P)$ since the converse inequality $\hat{\rho}(P) \leq \gamma_{DS}$ is not known to the best of our knowledge. Moreover, even if the equality $\gamma_{DS} = \hat{\rho}(P)$ was true, the bounds obtained in our Proposition 4.1 could be derived from [vDS95] only for some specific values of $P(0,0)$. Indeed the difficulty in [vDS95, p. 139-140] to cover all the values $P(0,0) \in (0,1)$ is that the spectral measure associated with Karlin and McGregor polynomials cannot be easily computed, except for some specific values of $P(0,0)$ (see [Kov09] for a recent contribution).

4.2 A non-reversible case : RWs with $g = 2$ and $d = 1$

Let $P := (P(i,j))_{(i,j) \in \mathbb{N}^2}$ be defined by

$$P(0,0) = a \in (0,1), \quad P(0,1) = 1 - a, \quad P(1,0) = b \in (0,1), \quad P(1,2) = 1 - b \quad (32)$$

$$\forall n \geq 2, \quad P(n, n-2) = a_{-2} > 0, \quad P(n, n-1) = a_{-1}, \quad P(n, n) = a_0, \quad P(n, n+1) = a_1 > 0.$$

The form of boundary probabilities in (32) is chosen for convenience. Other (finitely many) boundary probabilities could be considered provided that P is irreducible and aperiodic. To illustrate the procedure proposed in Section 4 for this class of RWs, we also specify the numerical values

$$a_{-2} := 1/2, \quad a_{-1} := 1/3, \quad a_0 = 0, \quad a_1 := 1/6.$$

The procedure could be developed in the same way for any other values of $(a_{-2}, a_{-1}, a_0, a_1)$ satisfying $a_{-2}, a_1 > 0$ and Condition **(NERI)** i.e. $a_1 < 2a_{-2} + a_{-1}$. Here we have

$$\phi(t) := \frac{1}{2t^2} + \frac{1}{3t} + \frac{t}{6} = 1 + \frac{1}{6t^2}(t-1)(t^2 - 5t - 3).$$

Function $\phi(\cdot)$ has a minimum over $(1, +\infty)$ at $\hat{\gamma} \approx 2.18$, with $\hat{\delta} := \phi(\hat{\gamma}) \approx 0.621$. Let $V_{\hat{\gamma}} := \{\hat{\gamma}^n\}_{n \in \mathbb{N}}$ and let $\mathcal{B}_{\hat{\gamma}}$ be the associated weighted space. We know from Proposition 3.2 and from irreducibility and aperiodicity properties that $\hat{r}_{ess}(P) = \hat{\delta}$ and P is $V_{\hat{\gamma}}$ -geometrically ergodic (see Proposition 2.1). The polynomial $E_{\lambda}(\cdot)$ is

$$\forall z \in \mathbb{C}, \quad E_{\lambda}(z) := \frac{z^3}{6} - \lambda z^2 + \frac{z}{3} + \frac{1}{2}.$$

A simple examination of the graph of $\phi(\cdot)$ shows that $\eta = 2$. Thus the set \mathcal{M} of Corollary 4.2 is $\mathcal{M} := \{(1,1), (2)\}$. Next, the constructive proof of Corollary 4.1 provides the following procedure to compute $\widehat{\rho}(P)$ (see also Remark 4.1 in the second case). Recall that $\Lambda := \{\lambda \in \mathbb{C} : \widehat{\delta} < |\lambda| < 1\}$.

First case: $\mu = (1, 1)$

- (a) When $\lambda \in \Lambda$ is such that \mathcal{E}_λ^- is composed of 2 simple roots of $E_\lambda(\cdot)$, a necessary condition for λ to be an eigenvalue of P on $\mathcal{B}_{\widehat{\gamma}}$ is that

$$R_1(\lambda) := \text{Res}_{z_1}(P_1, E_\lambda) = 0,$$

where

$$P_1(\lambda, z_1) := \text{Res}_{z_2}(P_0, E_\lambda) = \begin{vmatrix} 1/6 & 0 & A(\lambda, z_1) & 0 & 0 \\ -\lambda & 1/6 & B(\lambda, z_1) & A(\lambda, z_1) & 0 \\ 1/3 & -\lambda & C(\lambda, z_1) & B(\lambda, z_1) & A(\lambda, z_1) \\ 1/2 & 1/3 & 0 & C(\lambda, z_1) & B(\lambda, z_1) \\ 0 & 1/2 & 0 & 0 & C(\lambda, z_1) \end{vmatrix}.$$

and $P_0(\lambda, z_1, z_2) := A(\lambda, z_1)z_2^2 + B(\lambda, z_1)z_2 + C(\lambda, z_1)$ is given by

$$P_0(\lambda, z_1, z_2) := \begin{vmatrix} (1-a) & a + (1-a)z_2 - \lambda \\ (1-b)(z_1 + z_2) - \lambda & b + (1-b)z_2^2 - \lambda z_2 \end{vmatrix}. \quad (33)$$

$P_0(\lambda, z_1, z_2)$ is derived using (25) from

$$P_{0,1}(\lambda, z_1, z_2) := \begin{vmatrix} a + (1-a)z_1 - \lambda & a + (1-a)z_2 - \lambda \\ b + (1-b)z_1^2 - \lambda z_1 & b + (1-b)z_2^2 - \lambda z_2 \end{vmatrix} = (z_1 - z_2)P_0(\lambda, z_1, z_2).$$

- (b) *Sufficient part.* Consider

$$\Lambda_{(1,1)} = \text{Root}(R_1) \cap \Lambda = \text{Root}(R_1) \cap \{\lambda \in \mathbb{C} : 0.621 \approx \widehat{\delta} < |\lambda| < 1\}.$$

For every $\lambda \in \Lambda_{(1,1)}$:

- (i) Check that $E_\lambda(z) = 0$ has two simple roots z_1 and z_2 such that $|z_i| < \widehat{\gamma} \approx 2.18$.
 - (ii) If (i) is OK, then test if $P_0(\lambda, z_1, z_2) = 0$ with P_0 given in (33).
- If (i) and (ii) are OK, then λ is an eigenvalue of P on $\mathcal{B}_{\widehat{\gamma}}$.

Second case: $\mu = (2)$.

- (a) When $\lambda \in \Lambda$ is such that \mathcal{E}_λ^- is composed of a double root of $E_\lambda(\cdot)$, a necessary condition for λ to be an eigenvalue of P on $\mathcal{B}_{\widehat{\gamma}}$ is that (see Remark 4.1)

$$Q(\lambda) = 0 \quad \text{and} \quad R_2(\lambda) := \text{Res}_{z_1}(P_1, E_\lambda) = 0,$$

where

$$Q(\lambda) := \begin{vmatrix} 1/6 & 0 & 1/2 & 0 & 0 \\ -\lambda & 1/6 & -2\lambda & 1/2 & 0 \\ 1/3 & -\lambda & 1/3 & -2\lambda & 1/2 \\ 1/2 & 1/3 & 0 & 1/3 & -2\lambda \\ 0 & 1/2 & 0 & 0 & 1/3 \end{vmatrix}$$

and

$$P_1(\lambda) := \text{Res}_{z_1}(P_0, E_\lambda) = \begin{vmatrix} 1/6 & 0 & A(\lambda) & 0 & 0 \\ -\lambda & 1/6 & B(\lambda) & A(\lambda) & 0 \\ 1/3 & -\lambda & C(\lambda) & B(\lambda) & A(\lambda) \\ 1/2 & 1/3 & 0 & C(\lambda) & B(\lambda) \\ 0 & 1/2 & 0 & 0 & C(\lambda) \end{vmatrix}.$$

where $P_0(\lambda, z_1) := A(\lambda)z_1^2 + B(\lambda)z_1 + C(\lambda)$ is given by

$$P_0(\lambda, z_1) := \begin{vmatrix} a + (1-a)z_1 - \lambda & 1-a \\ b + (1-b)z_1^2 - \lambda z_1 & 2(1-b)z_1 - \lambda \end{vmatrix}. \quad (34)$$

(b) *Sufficient part.* Consider

$$\Lambda'_{(2)} = \text{Root}(Q) \cap \Lambda_{(2)} = \text{Root}(Q) \cap \text{Root}(R_2) \cap \{\lambda \in \mathbb{C} : 0.621 \approx \widehat{\delta} < |\lambda| < 1\}.$$

For every $\lambda \in \Lambda'_{(2)}$:

- (i) Check that Equation $E_\lambda(z) = 0$ has a double root z_1 such that $|z_1| < \widehat{\gamma} \approx 2.18$.
 - (ii) If (i) is OK, then test if $P_0(\lambda, z_1) = 0$ with P_0 given in (34).
- If (i) and (ii) are OK, then λ is an eigenvalue of P on $\mathcal{B}_{\widehat{\gamma}}$.

Final results Define $\mathcal{Z}_{(1,1)}$ as the set of all the $\lambda \in \Lambda_{(1,1)}$ satisfying (i)-(ii) in the first case, and $\mathcal{Z}_{(2)}$ as the set of all the $\lambda \in \Lambda'_{(2)}$ satisfying (i)-(ii) in the second one. Finally set $\mathcal{Z} := \mathcal{Z}_{(1,1)} \cup \mathcal{Z}_{(2)}$. Then

$$\widehat{\rho}(P) = \max(\widehat{\delta}, \max\{|\lambda|, \lambda \in \mathcal{Z}\}).$$

The results (using Maple computation engine) for different instances of the values of boundary transition probabilities are reported in Table 1. In these specific examples, note that $\Lambda'_{(2)}$ is always the empty set. As expected, we obtain that $\rho_{\widehat{\gamma}}(P) \nearrow 1$ when $(a, b) \rightarrow (0, 0)$.

5 Convergence rate for RWs with unbounded increments

In this subsection, we propose two instances of RW on $\mathbb{X} := \mathbb{N}$ with unbounded increments for which estimate of the convergence rate with respect to some weighted-supremum space \mathcal{B}_V can be obtained using Proposition 3.1 and Proposition 2.1. The first example is from [MS95]. The second one is a reversible transition kernel P inspired from the ‘‘infinite star’’ example in [Ros96]. Note that using a result of [Bax05] (see Remark 4.2), estimates of $\rho_V(P)$ with respect to \mathcal{B}_V may be useful to obtain estimates on the usual spectral gap $\rho_2(P)$ with respect to Lebesgue’s space $\ell^2(\pi)$. Recall that the converse is not true in general.

(a, b)	$\Lambda_{(1,1)}$	$\mathcal{Z}_{(1,1)}$	$\Lambda'_{(2)}$	$\mathcal{Z}_{(2)}$	$\widehat{\delta}$	$\widehat{\rho}(P)$
$(1/2, 1/2)$	$-0.625 \pm 0.466i,$ $-0.798, 0.804$	\emptyset	\emptyset	\emptyset	0.621	0.621
$(1/10, 1/10)$	$-0.681 \pm 0.610i$ $-0.466 \pm -0.506i$ $-0.384 \pm 0.555i$	$\{-0.466 \pm 0.506i\}$	\emptyset	\emptyset	0.621	0.688
$(1/50, 1/50)$	$-0.598 \pm 0.614i$ $-0.383 \pm 0.542i$ $-0.493 \pm 0.574i$ $-0.477 \pm 0.584i$ 0.994	$\{-0.493 \pm 0.574i\}$	\emptyset	\emptyset	0.621	0.757

Table 1: Convergence rate with different values of boundary transition probabilities (a, b)

5.1 A non-reversible RW with unbounded increments [MS95]

Let P be defined by

$$\forall n \geq 1, P(0, n) := q_n, \quad \forall n \geq 1, P(n, 0) := p, \quad P(n, n+1) := q = 1 - p,$$

with $p \in (0, 1)$ and $q_n \in [0, 1]$ such that $\sum_{n \geq 1} q_n = 1$.

Proposition 5.1 *Assume that $\gamma \in (1, 1/q)$ is such that $\sum_{n \geq 1} q_n \gamma^n < \infty$. Then $r_{ess}(P) \leq q\gamma$. Moreover P is V_γ -geometrically ergodic with convergence rate $\rho_{V_\gamma}(P) \leq \max(q\gamma, p)$.*

Proof. We have: $\forall n \geq 1, (PV_\gamma)(n) = q\gamma^{n+1} + p$. Thus, if $\gamma \in (1, 1/q)$ and $\sum_{n \geq 1} q_n \gamma^n < \infty$, then Condition **(WD)** holds with V_γ , and we have $\delta_{V_\gamma}(P) \leq q\gamma$. Therefore it follows from Proposition 3.1 that $r_{ess}(P) \leq q\gamma$. Now Proposition 2.1 is applied with any $r_0 > \max(q\gamma, p)$. Let $\lambda \in \mathbb{C}$ be such that $\max(q\gamma, p) < |\lambda| \leq 1$, and let $f \in \mathcal{B}_\gamma, f \neq 0$, be such that $Pf = \lambda f$. We obtain $f(n) = (\lambda/q)f(n-1) - pf(0)/q$ for any $n \geq 2$, so that

$$\forall n \geq 2, \quad f(n) = \left(\frac{\lambda}{q}\right)^{n-1} \left(f(1) - \frac{pf(0)}{\lambda - q}\right) + \frac{pf(0)}{\lambda - q}.$$

Since $f \in \mathcal{B}_{V_\gamma}$ and $|\lambda|/q > \gamma$, we obtain $f(1) = pf(0)/(\lambda - q)$, and consequently: $\forall n \geq 1, f(n) = pf(0)/(\lambda - q)$. Next the equality $\lambda f(0) = (Pf)(0) = \sum_{n \geq 1} q_n f(n)$ gives: $\lambda f(0) = pf(0)/(\lambda - q)$ since $\sum_{n \geq 1} q_n = 1$. We have $f(0) \neq 0$ since we look for $f \neq 0$. Thus λ satisfies $\lambda^2 - q\lambda - p = 0$, that is: $\lambda = 1$ or $\lambda = -p$. The case $\lambda = -p$ has not to be considered since $|\lambda| > \max(q\gamma, p)$. If $\lambda = 1$, then $f(n) = f(0)$ for any $n \in \mathbb{N}$, so that $\lambda = 1$ is a simple eigenvalue of P on \mathcal{B}_γ and is the only eigenvalue such that $\max(q\gamma, p) < |\lambda| \leq 1$. Then Proposition 2.1 gives the second conclusion of Proposition 5.1. \square

Note that p cannot be dropped in the inequality $\rho_{V_\gamma}(P) \leq \max(q\gamma, p)$ since $\lambda = -p$ is an eigenvalue of P on \mathcal{B}_γ with corresponding eigenvector $f_p := (1, -p, -p, \dots)$.

5.2 A reversible RW inspired from [Ros96]

Let $\{\pi_n\}_{n \in \mathbb{N}}$ be a probability distribution (with $\pi_n > 0$ for every $n \in \mathbb{N}$) and P be defined by

$$\forall n \in \mathbb{N}, P(0, n) = \pi_n \quad \text{and} \quad \forall n \geq 1, P(n, 0) = \pi_0, P(n, n) = 1 - \pi_0.$$

It is easily checked that P is reversible with respect to $\{\pi_n\}_{n \in \mathbb{N}}$, so that $\{\pi_n\}_{n \in \mathbb{N}}$ is an invariant probability distribution of P .

Proposition 5.2 *Assume that there exists $V \in [1, +\infty)^{\mathbb{N}}$ such that $V(0) = 1$, $V(n) \rightarrow +\infty$ as $n \rightarrow +\infty$ and $\pi(V) := \sum_{n \geq 0} \pi_n V(n) < \infty$. Then P is V -geometrically ergodic with $\rho_V(P) \leq 1 - \pi_0$.*

It can be checked that P is not stochastically monotone so that the estimate $\rho_V \leq 1 - \pi_0$ cannot be directly deduced from [LT96].

Proof. From $(PV)(0) = \pi(V)$ and $\forall n \geq 1, (PV)(n) = \pi_0 V(0) + (1 - \pi_0)V(n)$, it follows that

$$PV \leq (1 - \pi_0)V + (\pi(V) + \pi_0) \mathbf{1}_{\mathbb{X}}.$$

That is, Condition **(WD)** holds true with $N := 1$, $\delta := 1 - \pi_0$ and $d := \pi(V) + \pi_0$. The inequality $r_{ess}(P) \leq 1 - \pi_0$ is deduced from Proposition 3.1.

Let $\lambda \in \mathbb{C}$ be an eigenvalue of P and $f := \{f(n)\}_{n \in \mathbb{N}}$ be a non trivial associated eigenvector. Then

$$\lambda f(0) = \sum_{n=0}^{+\infty} \pi_n f(n) \quad \text{and} \quad \forall n \geq 1, \quad \lambda f(n) = \pi_0 f(0) + (1 - \pi_0)f(n). \quad (35)$$

This gives: $\forall n \geq 1, f(n) = f(0)\pi_0/(\lambda - 1 + \pi_0)$. Since $f \neq 0$, it follows from the first equality in (35) that

$$\lambda = \pi_0 + \frac{\pi_0}{\lambda - 1 + \pi_0}(1 - \pi_0),$$

which is equivalent to $\lambda^2 - \lambda = 0$. Thus, $\lambda = 1$ or 0 . That 1 is a simple eigenvalue is standard from the irreducibility of P . The result follows from Proposition 2.1. \square

A specific instance of this model is considered in [Ros96, p. 68]. Let $\{w_n\}_{n \geq 1}$ be a sequence of positive scalars such that $\sum_{n \geq 1} w_n = 1/2$. Then P is given by

$$\forall n \in \mathbb{N}, P(n, n) = 1/2 \quad \text{and} \quad \forall n \geq 1, P(0, n) = w_n, P(n, 0) = 1/2$$

which is reversible with respect to its invariant probability distribution π defined by $\pi_0 := 1/2$ and $\pi_n := w_n$ for $n \geq 1$. It has been proved in [Ros96, p. 68] that, for any $X_0 \sim \alpha \in \ell^2(1/\pi)$, there exists a constant $C_{\alpha, \pi} > 0$ such that

$$\|\alpha P^n - \pi\|_{TV} \leq C_{\alpha, \pi} (3/4)^n \quad (36)$$

where $\|\cdot\|_{TV}$ is the total variation distance. Since we know that $\rho_2(P) \leq \rho_V(P)$ from [Bax05] and $\rho_V(P) \leq 1/2$ from Proposition 5.2, the rate of convergence in (36) is improved.

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