Sensitivity and importance analysis of Markov models using perturbation analysis: Applications in reliability studies

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ABSTRACT: Sensitivity (or importance analysis) has been first defined for "static systems", i.e. systems described by combinatorial reliability models (fault or event trees) and several measures, both structural and probabilistic, have been proposed to assess component importance. For dynamic systems including inter-component and functional dependencies (cold spare, shared load, shared ressources, ...), and described by Markov models or, more generally, by discrete events dynamic systems models (DEDS), the problem of sensitivity analysis remains widely open. In this paper we propose to use the estimation method developed by Cao in (Cao & Chen 1997) in the framework of Perturbation Analysis, to formalize several sensitivity measures in case of dynamic systems. We show with numerical examples why this method offers a promising tool for steady state sensitivity analysis of Markov Processes in reliability studies.

1 INTRODUCTION

The sensitivity analysis of the results of a system reliability study helps to identify which components contribute the most to system (un)performance (reliability, maintainability, safety, or any performance metrics of interest). Hence, the reliability sensitivity (or importance) analysis provides fruitful insight into the system behavior, helps to find design weaknesses or operation bottlenecks and suggest optimal modifications for system upgrade (improved design, better maintenance, ...). To take full advantage of reliability studies, it is thus of great importance to have at one's disposal efficient sensitivity analysis methods which can be implemented on industrial systems, without oversimplifying assumptions.

From a mathematical point of view, the sensitivity of the system reliability to a given design parameter (and, incidentally, the reliability importance of a given component) is very often defined as the partial derivative of the reliability with respect to that parameter since this derivative quantifies the effect of a small parameter change on the system reliability.

Sensitivity (or importance analysis) has been first defined for "static systems", i.e. systems described by combinatorial reliability models (fault or event trees) and several measures, both structural and probabilistic, have been proposed to assess component importance. Most of these measures are linked one to each other and the Birnbaum importance, defined as a the partial derivative of the system reliability wrt to a parameter of interest is one of the most used importance factor. A well established methodology exists to compute the sensitivity measures, the most efficient being based on binary decision diagrams (BDD), (Dutuit & Rauzy 2001).

For dynamic systems including inter-component and functional dependencies (cold spare, shared load, shared ressources, ...), and described by Markov models or, more generally, by discrete events dynamic systems models (DEDS), the problem of sensitivity analysis remains widely open. The exact solution for the sensitivity measures for a Markov model relies on the Frank's approach (Frank 1978) (the classical set of differential equations is extended to a bigger set of equations including the sensitivity factor equations), but it is computationally burdensome and almost unusable or highly inefficient on a realistic-size systems because the state-space is too big. To cope with this problem, some approximate solutions have been proposed but they are often only applicable on a limited class of systems (e.g. acyclic Markov models with no repair), (Ou & Dugan 2003). The primary objective of this contribution is thus to identify a feasible (i.e. usable on realistic size systems and realistic from a practical point of view) approach to evaluate the reliability sensitivity measures (based on the performance derivative with respect to the parameter of interest) for dynamic systems. We focus on the estimation methods developed in the framework of Perturbation Analysis (PA) which seems to be very promising especially in the context of Markov model and stationary performance measure. This presentation is organized in two parts.

In section 2, we present the main issue of *"Per-*

turbation analysis" : from a methodological point of view, the aim of this communication is to show how the perturbation analysis approach and its first variant, the Infinitesimal Perturbation Analysis (IPA) (Cao 1995), offer a promising solution to find derivative estimates, and hence reliability importance measures, for general discrete event dynamic systems via a single sample path observation.

In section 3, we focus on the application in reliability of PA approach. We show how the PA and one of its more recent variants, the perturbation realization, give a particularly suitable solution for sensitivity analysis at steady state, in the context of Markov chain modeling (Dai 1996; Cao et al. 1996; Cao & Chen 1997; Cao & Wan 1998). The estimation of derivative is made via a single sample path observation, which allows both:

- the optimization of the on-line system performance,
- the study of the system sensitivity wrt its parameters when the infinitesimal generator of the Markov process is unknown.

It provides also an efficient tool to investigate not only the importance of a given component, but also the importance of a class of components, the importance of the maintenance, and, more generally, the effect of the simultaneous change of several design parameters. Several numerical examples will illustrate the proposed approach.

NOTATION LIST

D	realization matrix
$\hat{D}$	estimated realization matrix
d_{ij}	realization factor
$E\{X\}$	expected value of random variable X
$\eta_t(\theta, X)$	performance function at time t
$\overline{\eta}_t(\theta)$	performance measure at time t
$\overline{\eta}(\theta)$	limit of $\overline{\eta}_t(\theta)$
$\eta(\theta)$	performance measure on infinite horizon
$\phi_t(\theta, X)$	estimate of the performance measure derivative
λ	failure rate of one unit
m	dimension of the discrete state space
μ	repair rate of one
n	number of independent parameters
A	transition rate matrix
$\pi_i(t)$	probability of being in state S_i at time t
Q	perturbation matrix
S	discrete state space
S_i	discrete state i
θ	system parameters set
$X(x)$	random state transition instants

2 PERTURBATION ANALYSIS

2.1 *Main issue of perturbation analysis*

We consider a stochastic discrete event system defined by:

- a discrete state space noted $S = (S_1, S_2, ..., S_m)$,
- a set of parameters noted $\theta = (\theta_1, \theta_2, ..., \theta_n)$,
- a probability space (Ω, F, P) and a random vector $X = X(x), x \in \Omega$, that determines de state transition instants.

We define a performance function at time t on this system, noted $\eta_t(\theta, X)$ (it can correspond to the running state of the system, a maintenance cost, the productivity, etc...). Due to the random part of the system, the two following quantities are of interest and are considered as the performance measure of the system behavior. At transient state, we have, if it exists:

$$\overline{\eta}_t(\theta) = E\Big\{\eta_t(\theta, X)\Big\}$$

where E is the expectation with respect to the probability measure P. At steady state (whenever it exists), we have:

$$\eta(\theta) = \lim_{t \to +\infty} \eta_t(\theta, X)$$

The aim of perturbation analysis is to estimate the performance measure derivatives by analyzing a single sample path defined by $X = (X(x), x \in \Omega)$ (Cao 1995). We put by definition:

$$\frac{d\overline{\eta}_t(\theta)}{d\theta} = \lim_{\Delta\theta \to 0} \frac{\overline{\eta}_t(\theta + \Delta\theta) - \overline{\eta}_t(\theta)}{\Delta\theta} \tag{1}$$

and

$$\frac{d\eta(\theta)}{d\theta} = \lim_{\Delta\theta \to 0} \frac{\eta(\theta + \Delta\theta) - \eta(\theta)}{\Delta\theta} \tag{2}$$

Assume we could change slightly any of the system parameters, that is we could change θ into $\theta' = \theta + \Delta\theta$. We could get :

- a sample path $X = (X(x), x \in \Omega)$, of random state transition instants when the system parameters are θ,
- a sample path $X' = (X'(x), x \in \Omega)$, of random state transition instants when the system parameters are $\theta + \Delta\theta$.

A natural estimate for the performance measure derivative would be to evaluate $\overline{\eta}_t(\theta + \Delta\theta)$ with data from the sample path X' (perturbed sample path) and to estimate $\overline{\eta}_t(\theta)$ from the other sample path X (nominal sample path). However, this estimation can have many disadvantages from a practical point of view. Actually the data from perturbed sample paths are not always available because it can be impossible to change the parameters of the system and to observe the realizations of the state transition instants X'(Cao et al. 1996). In this case, simulation is required, but

it can be computationally burdensome to simulate the system behavior for each perturbation, and to evaluate the corresponding finite difference estimates. That is why in the area of Perturbation Analysis, the objective is to estimate the derivatives 1 or 2 on the basis of data observed from a nominal sample path.

2.2 *Infinitesimal Perturbation Analysis (IPA)*

The solution for this problem proposed by Infinitesimal Perturbation Analysis (IPA) is to use the following estimate $\phi_t(\theta, X)$ for the performance measure derivative:

$$\phi_t(\theta, X) = \lim_{\Delta\theta \to 0} \frac{\eta_t(\theta + \Delta\theta, X) - \eta_t(\theta, X)}{\Delta\theta}$$

where:

- $\eta_t(\theta, X)$ is the performance function of a system computed with data from the sample path X, and making the assumption that the system parameters are θ,
- $\eta_t(\theta + \Delta\theta, X)$ is the performance function of a system computed with data from the same sample path X, and making the assumption that the system parameters are $\theta + \Delta\theta$.

Hence $\phi_t(\theta, X)$ is defined with one single sample path $(X(x), x \in \Omega)$. Moreover, looking at the mean and the limit of this estimate we get :

$$E\Big\{\phi_t(\theta, X)\Big\} =$$

$$E\Big\{\lim_{\Delta\theta \to 0} \frac{\eta_t(\theta + \Delta\theta, X) - \eta_t(\theta, X)}{\Delta\theta}\Big\}$$

and

$$\lim_{t \to +\infty} \phi_t(\theta, X) =$$

$$\lim_{t \to +\infty} (\lim_{\Delta\theta \to 0} \frac{\eta_t(\theta + \Delta\theta, X) - \eta_t(\theta, X)}{\Delta\theta})$$

Hence the estimate $\Phi_t(\theta, X)$ can be used if it can be proved at least for the studied application case that it is unbiased and/or consistent, i.e.:

$$E\Big\{\phi_t(\theta, X)\Big\} = \frac{d}{d\theta} E\Big\{\eta_t(\theta, X)\Big\} = \frac{d\overline{\eta}_t(\theta)}{d\theta}$$

$$\lim_{t \to +\infty} \phi_t(\theta, X) = \frac{d\eta(\theta)}{d\theta}$$

The main issue of IPA are first to find an algorithm that estimates the function $\phi_t(\theta, X)$ and then to prove that the estimate $\Phi_t(\theta, X)$ verifies one or both of these properties. It is not always the case, especially when the performance measure is discontinuous with respect to θ. That is why, smoothed perturbation analysis, finite perturbation analysis and perturbation realization are developed for the cases when IPA fails (Cao 1995; Cao & Chen 1997). The two first methods work well for some class of problems but imply a higher analytical difficulties and higher computational difficulties (Cao 1995). The last one is more promising especially in the context of Markov model and stationary performance measures. We think it can be of great interest for application in reliability and we explain the main concepts in section 3.

3 APPLICATIONS IN RELIABILITY

Markov processes are widely used in reliability to study the performance measure (reliability studies, Availability, Maintainability, etc...) of many complex dynamic systems with inter-component and functional dependencies (cold spare, shared load, shared resources, ...). For such systems, the performance measure derivatives calculation is of great interest but the exact solutions are often computationally burdensome. The methods proposed directly by Perturbation Analysis and Infinitesimal Perturbation Analysis can be an alternative solution to exact methods, but in the specific context of Markov modeling, a more recent development of PA, *"the perturbation realization"* presented by Cao in (Cao & Chen 1997), is of great interest from a practical point of view. By explaining in this section the main concepts of this method we aim at showing its advantages in realistic working condition.

3.1 *Perturbation analysis & Markov process*

The aim of this subsection is to transpose the PA concept into the Markov model formulation. We consider an irreducible Markov process with a finite state space. Hence, this process is ergodic and there exists a single stationary distribution, (Ross 1996). This Markov process is characterized by:

- a finite state space $S = \Big\{S_1, S_2, ..., S_m\Big\}$,
- a set of parameters noted θ which represents the transition rates, and determines the transition rates matrix noted A.
- a random vector $X = (X_n, n \geq 0)$, that determines the state transition instants.
- a probability vector $\pi = (\pi_1, \pi_2, ..., \pi_m)$ that indicates the probability that the system is in each state S_i in steady state.

We define a performance function $\eta_t(\theta, X)$ that associates a state or a group of states to a real number. For the sake of clarity, we note it $\eta_t(A, X)$ in the following. The stationary performance measure we often

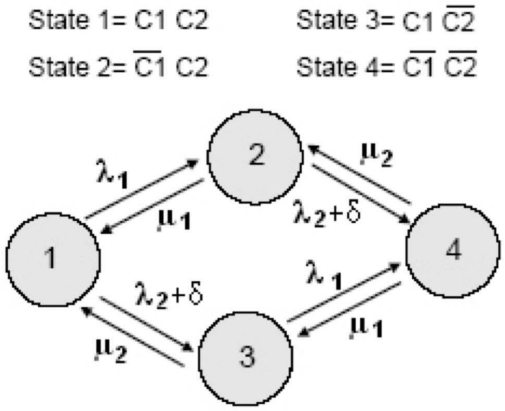

Figure 1: Perturbation on λ_2

need in the context of reliability studies is the limit of the expected performance function defined as:

$$\overline{\eta}(A) = \lim_{t \to +\infty} E_\pi \Big\{ \eta_t(A, X) \Big\}$$

For example, $\eta_t(A, X)$ can be the function that equals 1 when the system is running and that equals 0 when the system is failed. In this case, the expected performance measure in steady state will be:

$$\overline{\eta}(A) = \sum_{i \in \mathbf{W}} \pi_i$$

with W is the set of running states of the system. $\overline{\eta}(A)$ corresponds here to the asymptotic availability of the system.

A perturbation on one or more parameters of the system is equivalent to a perturbation in the transition rates matrix A. It leads to a perturbed transition matrix $A_\delta = A + \delta Q$ where δ is a small real number and Q is a matrix in which a 0 indicates that the parameter at the same place in A is not perturbed and a number α different from 0 indicates that the parameter at the same place in A is perturbed of an amount $\alpha\delta$. The only condition on the structure of Q is that the matrix A_δ is also a transition matrix, that is the sum of its lines equals 0. As an example, we consider two units C1 and C2 in a parallel structure with constant failure rates λ_1 and λ_2 and constant repair rates μ_1 and μ_2. Each component can be running or failed (failed states are noted $\overline{C}_1$ and $\overline{C}_2$). The Markov graph of this system is sketched in Figures 1 and 2. We consider two types of perturbation: a perturbation on one specific parameter in Figure 1 (on λ_2), and the perturbation on the probability of being in one specific state in Figure 2 (state number 3). These perturbations correspond to two different matrixes Q:

$$Q_1 = \begin{pmatrix} -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \; Q_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 3 & 0 & -1 & -2 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

The stationary performance measure of the perturbed Markov process (that is the Markov process with transition matrix A_δ) is noted $\overline{\eta}(A_\delta)$. Hence, we

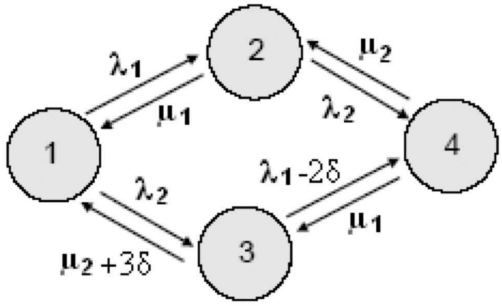

Figure 2: Perturbation on state 3

can define the derivative of $\overline{\eta}(A)$ in the direction of Q as:

$$\frac{d\overline{\eta}(A)}{dQ} = \lim_{\delta \to 0} \frac{\overline{\eta}(A_\delta) - \overline{\eta}(A)}{\delta}$$

Cao showed in (Cao & Chen 1997) that this derivative can be estimated with one single sample path and without using IPA estimate. We note $X^i = (X_t^i, t \geq 0)$ and $X^j = (X_t^j, t \geq 0)$ the Markov process with the same transition rate matrix A and with different initial states $X_0^i = i$ and $X_0^j = j$. Let us define d_{ij} the realization factor as ($i, j = 1, ..., m$):

$$d_{ij} = E\Big\{ \int_0^{+\infty} \big[\eta(A, X_t^i) - \eta(A, X_t^j)\big] dt \Big\}$$

d_{ij} is the expected long term effect of a change of the initial state on the measure performance. It is called in (Cao & Chen 1997) " *realization factor*". If the Markov processs are irreductible, then there exists a random time L such that $X_L^i = X_L^j$. Hence:

$$d_{ij} = E\Big\{ \int_0^{L} \big[\eta(A, X_t^i) - \eta(A, X_t^j\big] dt \Big\}$$

So the effect on an infinite horizon can be estimated on a finite horizon.

We also define $S^j(i)$:

$$S^j(i) = \inf \Big\{ t : t \geq 0, X_t^j = i \Big\}$$

$S^j(i)$ corresponds to the minimal time elapsed between a transition in state j and a transition in state i. It is proved in (Cao & Chen 1997) that:

$$d_{ij} = E\Big\{ \int_0^{S^j(i)} \big[(\eta(A, X_t^i) - \overline{\eta}(A)\big] dt \Big\} \qquad (3)$$

and that:

$$\frac{d\overline{\eta}}{dQ} = \pi Q D^T \pi^T \qquad (4)$$

D is called the "*pertubation realization matrix*" whose components are $d_{ij}, i, j = 1, ..., m$. Equations 3 and 4 allow the estimation of the quantities d_{ij} and of the derivative $\frac{d\overline{\eta}}{dQ}$ with a single sample path of the

Markov process $X^i = (X_t^i, t \geq 0)$: we can estimate $S^j(i)$, and π (and consequently $\overline{\eta}(A)$) only with the observation of the transition instants of X^i.

Hence, thanks to the ergodicity of the considered process, the realization matrix gives an estimate of the stationary performance measures that:

- can be evaluated from one single sample path. This is very interesting from a practical point of view for on-line performance optimization, when the parameters are impossible to change intentionally, or when the simulation of each perturbed path is computationally burdensome.
- can be evaluated without knowing the infinitesimal generator A of the Markov process.
- is valid even if the performance measure is not continuous wrt each parameter. This situation can exist in reliability studies with some failure rates and/or repair rates closed to zero.
- can be evaluated in any direction, only by changing the matrix Q.

Let us note that the estimation of the matrix D can also be computationally burdensome because it must be led for each couple (i, j) (complexity of order $o(m^2)$). That is why an approximated estimate (with potential vector) is proposed in (Cao & Wan 1998) which reduce the complexity of the calculation to the order $o(m)$.

The convergence of these estimators based on the realization matrix has been studied and proved by X.R. Cao in (Cao & Chen 1997).

3.2 *Numerical examples*

The numerical results presented in this section are obtained with simulated operating feedback data. The aim is to show how the estimation of realization factors can help for the sensitivity study of stationary measures in reliability studies. A first simple case is studied to make a connection between the Birnbaum factor and a derivative in the direction of Q. Then, a more complex case is presented to enhance the advantages of the realization matrix from a practical point of view. In both cases, the transition rates matrix is supposed to be unknown, the performance measure is the asymptotic availability and the simulations are led for 100000 transitions time.

Consider first the two-unit system sketched in Figure 1. In this case, the asymptotic availability can be obviously calculated with the Kolmogorov equations in steady state:

$$\begin{aligned}
-(\lambda_1+\lambda_2)\pi_1 + \lambda_1\pi_2 + \lambda_2\pi_3 &= 0 \\
\mu_1\pi_1 - (\mu_1+\lambda_2)\pi_2 + \lambda_2\pi_4 &= 0 \\
\mu_2\pi_1 - (\lambda_1+\mu_2)\pi_3 + \lambda_1\pi_4 &= 0 \\
\mu_2\pi_2 + \mu_1\pi_3 - (\mu_1+\mu_2)\pi_4 &= 0 \\
\pi_1+\pi_2+\pi_3+\pi_4 &= 1
\end{aligned}$$

The availability of the system is:

$$\overline{\eta}(A) = \pi_1+\pi_2+\pi_3 = \frac{\mu_1\mu_2+\mu_1\lambda_2+\mu_2\lambda_1}{(\lambda_1+\mu_1)(\lambda_2+\mu_2)}$$

We first generate a set of data with transition instants in case $\lambda_1 = 0.01, \lambda_2 = 0.01, \mu_1 = 0.05, \mu_2 = 0.05$. Then we estimate $S^j(i)$ and π and we obtain the following estimation for the matrix D.

$$\hat{D} = \begin{pmatrix} 0 & -1.3648 & -1.4528 & -11.2053 \\ 1.3648 & 0 & -0.0850 & -9.8446 \\ 1.4528 & 0.0850 & 0 & -9.8063 \\ 11.2053 & 9.8446 & 9.8063 & 0 \end{pmatrix}$$

and for the steady-state probability vector: $\hat{\Pi} = (0.6898, 0.1406, 0.1413, 0.0283)$.

Now let us consider the perturbation on one specific parameter (for example on λ_2, with derivative η in the direction Q_1). Then, the perturbation estimation with $\hat{D}$, $\hat{\pi}$, Q_1 and Equation 4 gives: $\frac{\hat{d\eta}}{dQ_1} = -2.3854$. In comparison, the analytical calculation of the partial derivative gives $\frac{\partial\eta}{\partial\lambda_1} = -2.3148$. In this case, the perturbation realization leads to the estimation of partial derivatives.

Now let us consider the perturbation matrix Q_2. In this case, the perturbation corresponds to the perturbation on one specific state (state 3) and no more on one parameter. We obtain with the same estimation $\hat{D}$, and $\hat{\pi}$: $\frac{\hat{\partial\eta}}{\partial Q_2} = 3.3741$. This derivative means that if we increase the repair rate of an amount 3δ, and if we decrease the failure rate of an amount 2δ, then the availability of the whole system will increase of an amount 3.374δ. This value quantifies the gain for the system availability if we change the probability of being in state 3. Since state 3 is a running state and the derivative is calculated in a direction Q_2 that increases the probability of in staying state 3. The availability increases and we can quantify at which speed.

We consider now a more complex system sketched in Figure 3. The corresponding Markov process is drawn in Figure 4. This structure includes a cold spare (component 4) and a shared load (when components 2,3, 4 are failed, if component 1 is running, its failure rate increases from λ_1 to $\overline{\lambda}_1$)

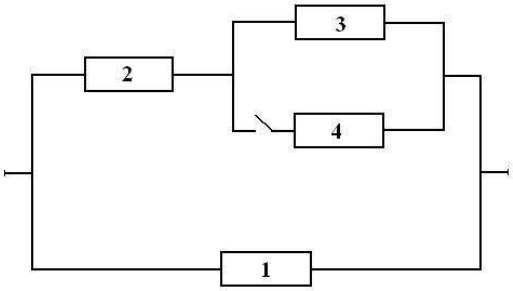

Figure 3: System structure

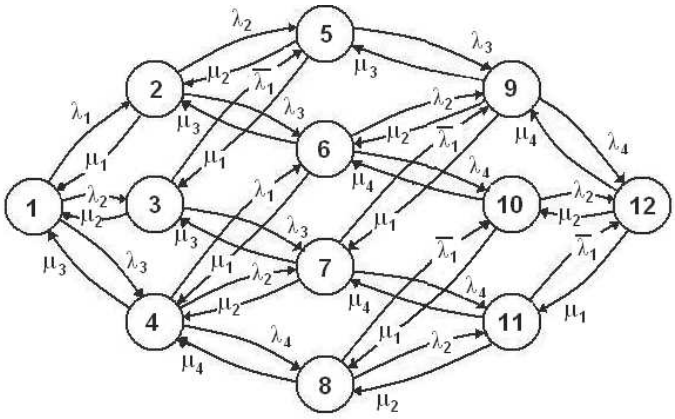

Figure 4: Markov process

In this case, the analytical calculation is burdensome and we present only the results obtained with the estimation of the realization matrix D. The data are simulated with parameters $\lambda_1 = 0.01, \overline{\lambda}_1 = 0.015, \lambda_2 = 0.01, \lambda_3 = 0.01, \lambda_4 = 0.01, \mu_1 = 0.05, \mu_2 = 0.05, \mu_3 = 0.05, \mu_4 = 0.05$. Hence the repair rates and the failure rates are of the same order of magnitude. The matrix D and the steady state vector π have been estimated one time and all the results presented in Tables 1,2, 3, 4, are obtained by changing only the matrix Q in Equation 4.

Tables 1 and 2 give the partial derivative estimates. We can observe that an increase of a failure rate leads to a decrease of the availability, and on the contrary, an increase of the repair rate μ_i leads to an increase of the availability. If we look at the repair rates, we can classify the components according to their importance factor: $C3 < C4 < C2 < C1$. Component 1 is the most "critical" since the whole system recovers when it recovers and the repair rates are of the same order of magnitude. If we look at the failure rates, we get a different ranking because of the shared load: $C3 < C4 < C1(\lambda_1) < C1(\overline{\lambda}_1) < C2$. The system is more sensitive to a perturbation of $\overline{\lambda}_1$ than a perturbation of λ_1 since when the failure rate of component 1 equals $\overline{\lambda}_1$, components 2, 3, and 4 are failed. So a failure of component 1 implies a failure of the whole system. What is more, the system sensitivity to component 1 is shared between the sensitivity to $\overline{\lambda}_1$ and to λ_1. Hence the above classification does not mean that component 2 is more critical than component 1. At last, in case of repair rate and failure rate, we can

$\frac{\partial \eta}{\partial \lambda_1}$	$\frac{\partial \eta}{\partial \overline{\lambda}_1}$	$\frac{\partial \eta}{\partial \lambda_2}$	$\frac{\partial \eta}{\partial \lambda_3}$	$\frac{\partial \eta}{\partial \lambda_4}$
-0.9876	-1.4779	-2.8559	-0.4075	-0.5156

Table 1: Sensitivity analysis to failure rates

$\frac{\partial \eta}{\partial \mu_1}$	$\frac{\partial \eta}{\partial \mu_2}$	$\frac{\partial \eta}{\partial \mu_3}$	$\frac{\partial \eta}{\partial \mu_4}$
0.6591	0.5183	0.0822	0.1045

Table 2: Sensitivity analysis to repair rates

see that $C3$ is less critical than $C4$. This is due to the cold spare: when component 4 is running, component 3 is already failed so the impact of component 4 on the system state is more important.

Let us now consider other directions, that is other matrixes Q which lead to other directional derivatives. We do not consider here perturbation on one specific state, that is when a parameter is perturbed, it is perturbed in the whole transition rates matrix. In Table 3, we put directions (λ_i, λ_j) to indicate that both parameters are perturbed of the same amount δ. These perturbations correspond to the derivative in the direction of the line of equation: $y = x$ ($\lambda_i = x, \lambda_j$=y). Hence, they give an indicator of the system sensibility to a group of components. We find that the group $(C3, C4)$ is less critical than component 2. This is due to the cold spare: when C3 fails, component C4 can be switched on immediately to replace C3, whereas when C2 fails, there is no replacement. Hence, the group $(C3, C4)$ can be considered as a component which is less critical than C2. What is more, the group $(C2, C4)$ is more critical than the group $(C2, C3)$ because when $C4$ is running, the component C3 is already failed. At last, we verify that a group (C_2, C_i) is more critical than C_2 alone and that a group C_2, C_3, C_4 with three components is more critical than any groups with one or two of these components. Of course, this analysis is valid if the failure and repair rates are of the same order of magnitude. If it was not the case, we could easily identify in the same way some groups of lower size that are more critical because of the high failure rate of one of their components.

The last example we present is given in Table 4. In this case, we tuned the perturbation on parameter μ_2, such that a change of an amount δ on the failure rate λ_2 is totally compensated. From a practical point of view, we optimized the maintenance parameter, such that a perturbation on the failure rate has no impact on the system availability. The approximate optimal solution we found corresponds to a deriva-

tive in the direction of the line of equation: $y = 5.5x$ ($x = \lambda_2, y = \mu_2$). A study of the estimate of the derivative $\frac{d\overline{\eta}}{dQ}$ can be possible in many cases because we need only to change the matrix Q in Equation 4. Hence in few iterations we can approximate the value of the repair rate that compensates an increase of the failure rate for one component. Some isolines can be easily drawn to indicate which joint parameters variations do not affect the system availability.

Direction	(λ_3, λ_4)	(λ_2, λ_3)	(λ_2, λ_4)	$(\lambda_2, \lambda_3, \lambda_4)$
$\frac{d\eta}{dQ}$	-0.9231	-3.2634	-3.3715	-3.7790

Table 3: Sensitivity analysis to a group of components

Direction	$(\lambda_2, 5.5 * \mu_2)$
$\frac{d\eta}{dQ}$	-0.0053

Table 4: Sensitivity analysis to a component

4 CONCLUSION

The results presented in this paper are a natural extension of the classical sensitivity analysis developed for "static systems". The main idea is to obtain the derivatives of a performance measure without using exact or approximates methods which are burdensome, and without using finite difference estimates which require data from both the nominal and the perturbed system behavior. Actually, the data of the perturbed system can be unavailable in many realistic cases when the parameters can not be intentionally modified (for economic or safety reasons for example).

With Perturbation Analysis and Infinitesimal Perturbation Analysis, methods have been developed to estimate the sensitivity measure of discrete events dynamic systems models on the basis of the nominal system behavior only. In the framework of Markov process modeling, the estimation method presented by Cao in (Cao & Chen 1997) is particularly well formalized. From a practical point of view, it allows the estimation of sensitivity measures on the basis of operating feedback data in nominal conditions, without knowing the generator of the underlying Markov process. We have shown in this paper that many different sensitivity measures (sensitivity to one or more parameters with any directional derivative, sensitivity to the probability of being in a state) can be led with no additional calculations and can be used in many reliability studies: identification of a group of critical components, adaptation of the maintenance parameters to keep a constant availability level in case of components degradation, etc...

This paper is a first step towards the formalization of practical tools and sensitivity measures for importance analysis of dynamic systems. Our further research focuses on more detailed applications of the perturbation realization to the sensitivity studies of dynamic systems and the development of methods to analyse the transient state of a Markov processes.

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