From differential to difference importance measures for Markov reliability models

Phuc Do Van, Anne Barros, Christophe Bérenguer

Université de technologie de Troyes

Institut Charles Delaunay - FRE CNRS 2848

Laboratoire de Modélisation et Sûreté des Systèmes

12, rue Marie Curie - BP 2060 -10010 Troyes cedex - France

Tel : +33 3 25 71 56 33 - Fax: +33 3 25 71 56 49

Corresponding author: Christophe Bérenguer

E-mail : christophe.berenguer@utt.fr

Abstract

This paper presents the development of the differential importance measures (DIM), proposed recently for the use in risk-informed decision-making, in the context of Markov reliability models. The proposed DIM measures are essentially based on directional derivatives. They can be used to quantify the relative contribution of a component (or a group of components, a state or a group of states) of the system on the total variation of system performance provoked by the changes in system parameters values. The estimation of DIM measures at steady state using only a single sample path of a Markov process is also investigated. A numerical example of a dynamic system is finally introduced to illustrate the use of DIM measures, as well as the advantages of proposed evaluation approaches.

Keywords: reliability, sensitivity analysis, differential importance measures, Markov process

1 Introduction

Reliability importance measures providing information about the importance of components on the system performance (reliability, maintainability, safety, or any performance metrics of interest) have been widely used in reliability studies and risk analyses. They are useful tools to identify design weakness or operation bottlenecks and to suggest optimal modifications for system upgrades. Recently, a new importance measure, called Differential Importance Measure (DIM), has been introduced for use in risk-informed decision-making [1, 2, 17, 24]. DIM^I is defined as a first-order sensitivity measure that ranks the parameters of the risk model according to the fraction of the total change in the risk metric that is due to a small change in the parameters' values, taken one at a time. Since DIM^I accounts for only the first-order effects of changes of system parameters, a second-order extension of DIM^I, named DIM^{II}, is considered in [25]. However, DIM^{II} is limited to the second-order effects of changes. Hence the third and higher-order effects of changes of system parameters can not be quantified. Furthermore, several existing methods to compute DIM^I & DIM^{II} are based on the system structure function and require the assumptions of stochastic independent components. Consequently, in the realistic case of stochastic dependencies existing between some components (shared maintenance resource, cold spare, shared load, ...), and/or high-order effects requirements, the problem remains wide open.

The first objective of this paper is to develop the differential importance measures in the context of dynamic systems including inter-component, functional dependencies, or more generally, systems described by Markov models. In such systems, the (un)availability of a component does not depend only on its characteristics but also on other system parameters, and its (un)availability in the system can be different from its (un)availability out of the system, see e.g. [22]. In this context, the partial derivatives with respect to the system parameters, rather than to the components'(un)availability, appear to be more relevant and are often preferred for design purposes. Hence, for steady state, DIM^I is firstly developed based on directional derivative in the direction defined by a matrix in an appropriate space [6, 11]. The direction can be related to a given parameter, a group of parameters, or more generally, transition rates between states of the system. In its version proposed in this paper, DIM^I can be used to quantify the relative contribution of a component or a group of components, a state or a group of states, on the first-order variation of system performance. An extension of DIM^I, namely the total differential importance measure (DIM^T) taking into account all order effects of changes of system parameters, is next investigated. DIM^T can therefore provide better and more insightful results than those obtained from DIM^I, DIM^{II} and can be used with any magnitude of change in system parameters.

From a practical point of view, the system's data may not always be available. For example, the reliability behaviour (failure rate, repair rate, etc) of some components of the system may be unknown. The analytical calculation can no longer be used. The second goal of the present paper is to show how DIM^I and DIM^T at steady state can be estimated from the operating feedback data of the system, i.e. a single sample path of a Markov process, by using an estimation method mentioned in [6, 5]. Through a numerical example, it is shown that the estimated results are closely related to those obtained by the analytical method.

This paper is organized as follows. The first section is devoted to define the first-order differential importance measure (DIM^I) in the context of Markov models. The analytical calculation of DIM^I is also considered. The second section focuses on the total differential importance measure (DIM^T). A simple numerical example is introduced in this section to illustrate the use of DIM^I and DIM^T in reliability studies. The estimation of both DIM^I and DIM^T at steady state from a single sample path of a Markov process is presented in the third section. Some numerical results are in addition discussed here. Finally, the last section presents the conclusions drawn from this work.

NOTATION LIST

λ_i, μ_i	failure and repair rate of component i
$\mathbf{M}$	transition rate matrix of Markov models
$\mathbf{M}^\sharp$	group inverse of $\mathbf{M}$
$\boldsymbol{\pi}$	row vector of steady-state probabilities
X	irreducible homogenous Markov process with finite space
$\mathbf{Q}$	directional perturbation matrix
A	asymptotic performance measure of the system
$\delta^I A_\Sigma, \delta A_\Sigma$	first-order and total variation of system performance A
$\dfrac{dA}{d\mathbf{Q}}$	directional derivative of A in the direction $\mathbf{Q}$
$\text{DIM}^I, \text{DIM}^T$	first-order and total differential importance measure

2 First-order differential importance measure DIM^I

Markov processes have been widely used to analyse and assess the performances (reliability, availability, maintainability, production capacity, etc...) of many complex dynamic systems with inter-component and functional dependencies (cold spare, shared load, shared resources, ...). This section explores the development of the differential importance measure in the context of Markov models at steady state.

Consider an n-components dynamic system described by an irreducible homogenous Markov process $X = \{X_t, t \geq 0\}$ with finite state space E and the transition rate matrix $\mathbf{M}$. This Markov process is ergodic and a single stationary distribution exists [23]. The vector of steady state probabilities (stationary distribution vector) $\boldsymbol{\pi} = (\pi_1, \pi_2, ...)$ verifies the Chapman-Kolmogorov equations:

$$\boldsymbol{\pi}\mathbf{M} = 0. \tag{1}$$

The system performance (availability, production capacity, etc.) is usually considered as the expected performance function:

$$A = \mathbb{E}_{\boldsymbol{\pi}}(f) = \sum_{i \in E} \pi_i f_i = \boldsymbol{\pi}\boldsymbol{f}, \tag{2}$$

where $\mathbb{E}_{\boldsymbol{\pi}}$ denotes the expectation with respect to the steady state probabilities $\boldsymbol{\pi}$, and $\boldsymbol{f} = (f_1, f_2, ...,)^T$ is a column vector representing the performance function associated to the system states. For example, for the system availability $f_i = 1$ when the system is operational in state i and $f_i = 0$ otherwise.

2.1 Variation of system parameters

Consider now a perturbation on the transition rate matrix $\mathbf{M}$ of the Markov process and the perturbed transition rate matrix $\mathbf{M}_\delta$:

$$\mathbf{M}_\delta = \mathbf{M} + \delta\mathbf{Q} = \mathbf{M} + \mathbf{Q}^\delta, \tag{3}$$

where $\mathbf{Q}^\delta = \delta\mathbf{Q}$, δ is a real number and $\mathbf{Q}$ is a matrix representing the direction of perturbation. Within the structured perturbation framework considered in this work, the state diagram of the perturbed system remains unchanged. Hence, if an entry M_{ij} is equal to 0 (i.e. there is no link between states i and j), the corresponding entry Q_{ij} must then be equal to 0. An entry $Q_{ij} = \alpha$ different from 0 indicates that the transition rate from state i to state j is perturbed by an amount $\alpha\delta$. The only condition on the structure of $\mathbf{Q}$ to ensure that the matrix $\mathbf{M}_\delta$ remains a transition rate matrix of a Markov process is that the sum of each row of $\mathbf{Q}$ equals 0, i.e. $\mathbf{Q}\boldsymbol{e} = 0$ with $\boldsymbol{e} = (1, 1, ..., 1)^\top$.

The variations in the transition rate matrix affect the system performance A which becomes A_δ so that $A_\delta = \boldsymbol{\pi}_\delta \boldsymbol{f}$ where the steady state probabilities vector $\boldsymbol{\pi}_\delta$ of the perturbed system verifies:

$$\boldsymbol{\pi}_\delta \mathbf{M}_\delta = 0. \tag{4}$$

The directional derivative of the system performance A in the direction $\mathbf{Q}$ can be defined as, see [6, 7]:

$$\frac{dA}{d\mathbf{Q}} = \lim_{\delta\to 0} \frac{A_\delta - A}{\delta} = \lim_{\delta\to 0} \frac{\boldsymbol{\pi}_\delta - \boldsymbol{\pi}}{\delta}\boldsymbol{f}. \tag{5}$$

This directional derivative is used as an importance measure in reliability and productivity sensitivity analysis in [11] and [9] respectively.

The directional matrix $\mathbf{Q}$ can be used to describe the change of one parameter, a group of parameters, more generally, the change in any direction of transition rates, see [11].

If one considers the case in which the transition rate matrix is perturbed in K different directions $(\mathbf{Q}_1, \mathbf{Q}_2, \cdots, \mathbf{Q}_K)$, the perturbed transition rate matrix is then:

$$\mathbf{M}_\delta = \mathbf{M} + \delta_1\mathbf{Q}_1 + \delta_2\mathbf{Q}_2 + ... + \delta_K\mathbf{Q}_K = \mathbf{M} + \sum_{i=1}^{K} \delta_i \mathbf{Q}_i, \tag{6}$$

with $\delta_1, \delta_2, ..., \delta_K$ are the amounts of variation in directions $\mathbf{Q}_1, \mathbf{Q}_2, ..., \mathbf{Q}_K$ respectively.

2.2 Definition of DIM^I

The variations in transition rate matrix described in Equation (6) may lead to a variation of the system performance A, noted δA_Σ. If the changes of parameters are small enough, δA_Σ can be then approximated by the first-order contribution $\delta^I A_\Sigma$, see [2]:

$$\delta A_\Sigma \simeq \delta^I A_\Sigma = \sum_{i=1}^{K} \delta_i \frac{dA}{d\mathbf{Q}_i} = \sum_{i=1}^{K} \delta^I A_i,$$

where:

- $\dfrac{dA}{d\mathbf{Q}_i}$ is the directional derivative of A in the direction $\mathbf{Q}_i$ (see again Equation (5)),

- $\delta^I A_i = \delta_i \dfrac{dA}{d\mathbf{Q}_i}$ is the first-order contribution of the change in transition rate matrix with respect to the direction $\mathbf{Q}_i$ and the amount δ_i.

The first-order differential importance measure (named DIM^I [2, 18]) can be extended here by using the directional derivatives. More precisely, DIM^I in direction $\mathbf{Q}_i$ can be defined as:

$$\text{DIM}^I(\mathbf{Q}_i) = \frac{\delta^I A_i}{\delta^I A_\Sigma} = \frac{\delta_i \dfrac{dA}{d\mathbf{Q}_i}}{\sum_{j=1}^K \delta_j \dfrac{dA}{d\mathbf{Q}_j}}. \tag{7}$$

If the direction $\mathbf{Q}_i$ relates to a component (or a group of components), a state (or a group of states), DIM^I in direction $\mathbf{Q}_i$ represents then the relative contribution, on the total variation of the system performance, of a component (or a group of components), a state (or a group of states) respectively. The applications of DIM^I in reliability sensitivity analysis are described in more detail in later sections of this paper.

By construction, DIM^I owns two interesting properties:

Property 1: Additivity. If one is interested in the DIM^I for a subset of directions $\mathbf{Q}_i, \mathbf{Q}_j, ...\mathbf{Q}_s$, then:

$$\begin{aligned}\text{DIM}^I(\mathbf{Q}_i, \mathbf{Q}_j, ..., \mathbf{Q}_s) = \frac{\delta^I A_{i,j,..,s}}{\delta^I A_\Sigma} &= \frac{\delta_i \dfrac{dA}{d\mathbf{Q}_i} + \delta_j \dfrac{dA}{d\mathbf{Q}_j} + ... + \delta_s \dfrac{dA}{d\mathbf{Q}_s}}{\sum_{j=1}^K \delta_j \dfrac{dA}{d\mathbf{Q}_j}} \\ &= \text{DIM}^I(\mathbf{Q}_i) + \text{DIM}^I(\mathbf{Q}_j) + ... + \text{DIM}^I(\mathbf{Q}_s).\end{aligned}$$

This relationship shows that DIM^I is additive. This important property can be used to calculate DIM^I relating to a group components given DIM^I relating to each component in this group.

Property 2: The sum of the DIM^{I}s of all directions equals unity, that is:

$$\text{DIM}^I(\mathbf{Q}_1) + \text{DIM}^I(\mathbf{Q}_2) + ... + \text{DIM}^I(\mathbf{Q}_K) = 1.$$

This property can be used to determine the DIM^I of a direction from the others.

2.3 Calculation of DIM^I

Theorem 1. *If the transition rate matrix is perturbed in the directional matrix* $\mathbf{Q}$*, i.e. Equation (3) holds, the directional derivative of* A *in this direction* $\mathbf{Q}$ *can then be written* *as:*

$$\frac{dA}{d\mathbf{Q}} = -\boldsymbol{\pi}\mathbf{Q}\mathbf{M}^{\sharp}\boldsymbol{f}, \tag{8}$$

where $\mathbf{M}^{\sharp}$ *is the generalized inverse (or group inverse) of* $\mathbf{M}$*:* $\mathbf{M}^{\sharp} = (\mathbf{M} - \boldsymbol{e\pi})^{-1} - \boldsymbol{e\pi}$*.*

Proof: From (1) and (4), one gets

$$(\boldsymbol{\pi}_\delta - \boldsymbol{\pi})\mathbf{M}_\delta + \boldsymbol{\pi}(\mathbf{M}_\delta - \mathbf{M}) = 0,$$

so from (3)

$$-\frac{\boldsymbol{\pi}_\delta - \boldsymbol{\pi}}{\delta}\mathbf{M}_\delta = \boldsymbol{\pi}\frac{\mathbf{M}_\delta - \mathbf{M}}{\delta} = \boldsymbol{\pi}\mathbf{Q}. \tag{9}$$

Since the perturbed transition rate matrix $\mathbf{M}_\delta$ is not invertible, the generalized inverse $\mathbf{M}_\delta^{\sharp}$ of $\mathbf{M}_\delta$ is introduced, see e.g. [19]:

$$\mathbf{M}_\delta^{\sharp} = (\mathbf{M}_\delta - \boldsymbol{e\pi}_\delta)^{-1} - \boldsymbol{e\pi}_\delta,$$

and,

$$\mathbf{M}_\delta\mathbf{M}_\delta^{\sharp} = \mathbf{M}_\delta^{\sharp}\mathbf{M}_\delta = \mathbf{I} - \boldsymbol{e\pi}_\delta, \tag{10}$$

where $\mathbf{I}$ is identical matrix. Right-multiplying both sides of Equation (9) with $\mathbf{M}_\delta^{\sharp}$ and using (10), one obtains:

$$-\frac{\boldsymbol{\pi}_\delta - \boldsymbol{\pi}}{\delta}(\mathbf{I} - \boldsymbol{e\pi}_\delta) = \boldsymbol{\pi}\mathbf{Q}\mathbf{M}_\delta^{\sharp}.$$

Remember that $\boldsymbol{\pi e} = \boldsymbol{\pi}_\delta \boldsymbol{e} = 1$, hence:

$$\frac{\boldsymbol{\pi}_\delta - \boldsymbol{\pi}}{\delta} = -\boldsymbol{\pi}\mathbf{Q}\mathbf{M}_\delta^{\sharp}.$$

If $\delta \to 0$, one obtains:

$$\lim_{\delta\to 0}\frac{\boldsymbol{\pi}_\delta - \boldsymbol{\pi}}{\delta} = -\lim_{\delta\to 0}\boldsymbol{\pi}\mathbf{Q}\mathbf{M}_\delta^{\sharp} = -\boldsymbol{\pi}\mathbf{Q}\big[\lim_{\delta\to 0}\mathbf{M}_\delta^{\sharp}\big].$$

Since the continuity of the matrix $\mathbf{M}_\delta^{\sharp}$ is proved in [6], therefore $\lim_{\delta\to 0}\mathbf{M}_\delta^{\sharp} = \mathbf{M}^{\sharp}$. Consequently,

$$\lim_{\delta\to 0}\frac{\boldsymbol{\pi}_\delta - \boldsymbol{\pi}}{\delta} = -\boldsymbol{\pi}\mathbf{Q}\mathbf{M}^{\sharp}.$$

By using this result and the definition of the directional derivative of A in the direction $\mathbf{Q}$ (see again (5)), the final result is (8).■

Applying now Theorem 1, one obtains:

$$\delta^I A_i = \delta_i\frac{dA}{d\mathbf{Q}_i} = -\boldsymbol{\pi}\mathbf{Q}_i^\delta\mathbf{M}^{\sharp}\boldsymbol{f} \text{ with } \mathbf{Q}_i^\delta = \delta_i\mathbf{Q}_i,$$

$$\delta^I A_\Sigma = \sum_{i=1}^{K}\delta^I A_i = -\boldsymbol{\pi}\mathbf{Q}_\sum^\delta\mathbf{M}^{\sharp}\boldsymbol{f} \text{ with } \mathbf{Q}_\sum^\delta = \sum_{i=1}^{K}\mathbf{Q}_i^\delta. \tag{11}$$

Thus, equation (7) can be finally expressed as:

$$\mathrm{DIM}^I(\mathbf{Q}_i) = \frac{\boldsymbol{\pi}\mathbf{Q}_i^\delta \mathbf{M}^\sharp \boldsymbol{f}}{\boldsymbol{\pi}\mathbf{Q}_\sum^\delta \mathbf{M}^\sharp \boldsymbol{f}}. \tag{12}$$

This formula shows that the DIM^I in different directions can be easily obtained by changing only the directional matrix without additional calculations. Moreover, DIM^I for a group of directions can be directly obtained by using the following:

$$\mathrm{DIM}^I(\mathbf{Q}_i, \mathbf{Q}_j, ..., \mathbf{Q}_s) = \frac{\boldsymbol{\pi}\mathbf{Q}_{i,j,...s}^\delta \mathbf{M}^\sharp \boldsymbol{f}}{\boldsymbol{\pi}\mathbf{Q}_\sum^\delta \mathbf{M}^\sharp \boldsymbol{f}}, \text{with, } \mathbf{Q}_{i,j,...s}^\delta = \mathbf{Q}_i^\delta + \mathbf{Q}_j^\delta + ... + \mathbf{Q}_s^\delta.$$

The DIM^I can be easily calculated by using Equation (12), it does not however account for the effects of simultaneous changes of several parameters, and it can therefore be used only when the changes of parameters are small enough to neglect the interaction effects. The idea of a second-order extension of DIM^I, is considered in [25]. However, this extension is only applicable when the higher-order interaction effects are neglected. From a practical point of view, this assumption is not always true. Thus, the next section is devoted to presenting an extension of DIM^I, namely the total differential importance measure which can take into account all the higher-order interaction effects of changes in system parameters.

3 Total differential importance measure DIM^T

3.1 Exact calculation of the variation of system performance

The purpose of this subsection is to calculate precisely the variation of the system performance provoked by the change in some specific directions of the transition rate matrix.

The simplest method to calculate the variation of the system performance δA $(\delta A_\Sigma, \delta A_1, \delta A_2, ..., \delta A_K)$ relies on the use of finite differences [8]. Thus,

$$\delta A = A_\delta - A = (\boldsymbol{\pi}_\delta - \boldsymbol{\pi})\boldsymbol{f}. \tag{13}$$

This method requires however the knowledge of both a nominal model (with transition rate matrix $\mathbf{M}$) and a perturbed model (with perturbation transitions rates matrix $\mathbf{M}_\delta$). From a practical point of view, the data of the perturbed model is not always available. Hence, a simulation is needed for this case but it would be computationally burdensome. The following theorem will give an exact calculation of δA based solely on the nominal model:

Theorem 2. *If the transition rate matrix is perturbed by a perturbation matrix* $\mathbf{Q}^\delta$*, i.e. Equation (3) holds, the variation of the system performance is then:*

$$\delta A = -\boldsymbol{\pi}\mathbf{Q}^\delta\mathbf{M}^\sharp(\boldsymbol{I} + \mathbf{Q}^\delta\mathbf{M}^\sharp)^{-1}\boldsymbol{f}. \tag{14}$$

Proof: First, it is shown in [19, 6] that the relationship between the group inverse $\mathbf{M}^\sharp$ and the transition rate matrix $\mathbf{M}$ is parallel to (10):

$$\mathbf{M}\mathbf{M}^\sharp = \mathbf{I} - \boldsymbol{e}\boldsymbol{\pi}.$$

Multiplying both sides of this equation on the left with $\boldsymbol{\pi}_\delta$ and using $\mathbf{M} = \mathbf{M}_\delta - \mathbf{Q}^\delta$ and $\boldsymbol{\pi}_\delta\boldsymbol{e} = 1$, one gets:

$$\boldsymbol{\pi}_\delta\mathbf{M}_\delta\mathbf{M}^\sharp - \boldsymbol{\pi}_\delta\mathbf{Q}^\delta\mathbf{M}^\sharp = \boldsymbol{\pi}_\delta - \boldsymbol{\pi}.$$

From (4) one obtains:

$$\boldsymbol{\pi}_\delta(\mathbf{I} + \mathbf{Q}^\delta\mathbf{M}^\sharp) = \boldsymbol{\pi}.$$

It is shown in [20, 21] that $(\mathbf{I} + \mathbf{Q}^\delta\mathbf{M}^\sharp)$ is non-singular. Consequently:

$$\boldsymbol{\pi}_\delta = \boldsymbol{\pi}(\mathbf{I} + \mathbf{Q}^\delta\mathbf{M}^\sharp)^{-1},$$

or,

$$\boldsymbol{\pi}_\delta - \boldsymbol{\pi} = \boldsymbol{\pi}[(\mathbf{I} + \mathbf{Q}^\delta\mathbf{M}^\sharp)^{-1} - \mathbf{I}].$$

Taking into account that $(\mathbf{I} + \mathbf{Q}^\delta\mathbf{M}^\sharp)^{-1} - \mathbf{I} = -\mathbf{Q}^\delta\mathbf{M}^\sharp(\mathbf{I} + \mathbf{Q}^\delta\mathbf{M}^\sharp)^{-1}$, one obtains:

$$\boldsymbol{\pi}_\delta - \boldsymbol{\pi} = -\boldsymbol{\pi}\mathbf{Q}^\delta\mathbf{M}^\sharp(\mathbf{I} + \mathbf{Q}^\delta\mathbf{M}^\sharp)^{-1}. \tag{15}$$

Finally, substituting (15) for (13), Equation (14) is verified ■.

By using (14), if the transition rate matrix is perturbed by a perturbation matrix $\mathbf{Q}_i^\delta$ ($\mathbf{Q}_i^\delta = \delta\mathbf{Q}_i$), the variation of the system performance provoked by this change is then:

$$\delta A_i = -\boldsymbol{\pi}\mathbf{Q}_i^\delta \mathbf{M}^\sharp(\mathbf{I} + \mathbf{Q}_i^\delta \mathbf{M}^\sharp)^{-1}\boldsymbol{f}. \tag{16}$$

Similarly, the total variation of the system performance caused by the change in transition rate matrix $\mathbf{M}$ with the perturbation matrix $\mathbf{Q}_\sum^\delta$ ($\mathbf{Q}_\sum^\delta = \sum_{i=1}^K \mathbf{Q}_i^\delta$) is:

$$\delta A_\Sigma = -\boldsymbol{\pi}\mathbf{Q}_\sum^\delta \mathbf{M}^\sharp(\mathbf{I} + \mathbf{Q}_\sum^\delta \mathbf{M}^\sharp)^{-1}\boldsymbol{f}. \tag{17}$$

The previous development requires no assumption on δ. Hence, the total variation of the system performance can be calculated with equation (17) for any value of δ, i.e. for any magnitude of change. This interesting property relies on the invertibility of the matrix $(\mathbf{I} + \mathbf{Q}_\sum^\delta \mathbf{M}^\sharp)$ and on series convergence properties that are examined and proved in [10, 12].

3.2 Definition of DIM^T

As in [3, 12], we propose to extend DIM to the total order and to define the total differential importance measure, denoted DIM^T, as:

$$\text{DIM}^T(\mathbf{Q}_i) = \frac{\delta A_i}{\delta A_\Sigma}. \tag{18}$$

Since the variation of system performance δA_i and δA_Σ can be exactly calculated, an "exact differential importance" is obtained which is the fraction of the total change of the system performance related to direction matrix $\mathbf{Q}_i$. Remember that this direction matrix can be associated with a component (or a group of components), as well as a state (or a group of states) of the system.

Substituting (16) and (17) for (18), DIM^T can be calculated:

$$\text{DIM}^T(\mathbf{Q}_i) = \frac{\boldsymbol{\pi}\mathbf{Q}_i^\delta \mathbf{M}^\sharp(\mathbf{I} + \mathbf{Q}_i^\delta \mathbf{M}^\sharp)^{-1}\boldsymbol{f}}{\boldsymbol{\pi}\mathbf{Q}_\sum^\delta \mathbf{M}^\sharp(\mathbf{I} + \mathbf{Q}_\sum^\delta \mathbf{M}^\sharp)^{-1}\boldsymbol{f}}. \tag{19}$$

Since DIM^T is an "exact differential importance", the results obtained from DIM^T are more precise than those provided by the first-order differential importance DIM^I. However, the calculation of DIM^I is less difficult than that of DIM^T. The comparison of both measures are discussed in more detail through the numerical example shown below.

3.3 Numerical example

This subsection shows how the differential importance measures can be used in order to analyse the system availability at steady state and to investigate the importance of a given component (or group of components) as well as the importance of a given state. The comparison between DIM^I and DIM^T is also studied.

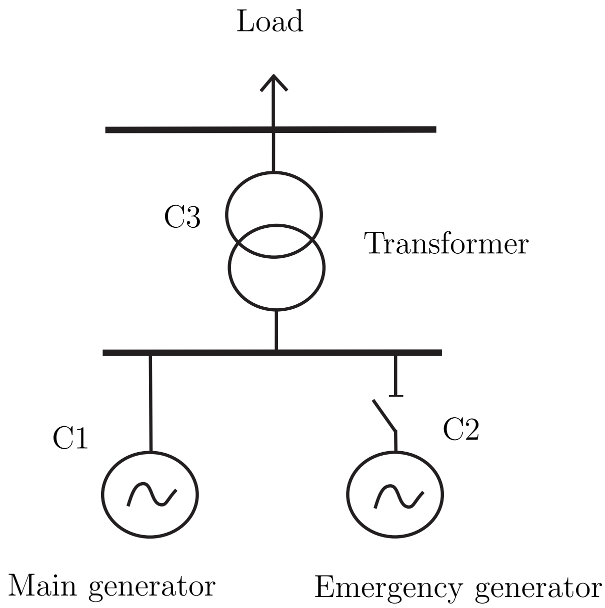

Figure 1: Power generation system.

Consider a power generation system with 3 units, whose structure is presented in Figure 1:

- units C1 and C2 are two generators supplying the power required by customers. In this system, C1 is the main generator and C2 is the emergency one. When C1 operates, C2 is on standby and it becomes an active component immediately if C1 fails. As soon as C1 is repaired, C2 stops. When both C1 and C2 fail, the maintenance priority is always given to unit C1 (see Figure 2),
- unit C3 is a power transformer that steps down generator voltages to customer voltages.

The operational modes of the system are described in Table 1 where "O" denotes an operating state, "S" denotes a standby state, and "F" denotes a failed state. The corresponding Markov process is sketched in Figure 2. One assumes that the failure rate λ_i and repair rate μ_i of unit i ($i = 1, 2, 3$) are constant and their values are reported in Table 2.

Table 1: System states

	Component			
State	C1	C2	C3	System
1	O	S	O	O
2	O	S	F	F
3	F	O	O	O
4	O	F	O	O
5	F	O	F	F
6	F	F	O	F
7	O	F	F	F

Table 2: Components' failure and repair rates.

Unit	$\lambda_i(h^{-1})$	$\mu_i(h^{-1})$
C1	0.00801	1/200
C2	0.001	1/100
C3	0.0011	1/155

According to the Markov diagram shown in Figure 2, the system availability at steady state is :

$$A = \pi_1 + \pi_3 + \pi_4 = \boldsymbol{\pi f}, \text{with } \boldsymbol{f} = (1, 0, 1, 1, 0, 0, 0)^T.$$

By resolving the Chapman-Kolmogorov equations at steady state (1), one obtains the steady state availability $A = 0.7324$.

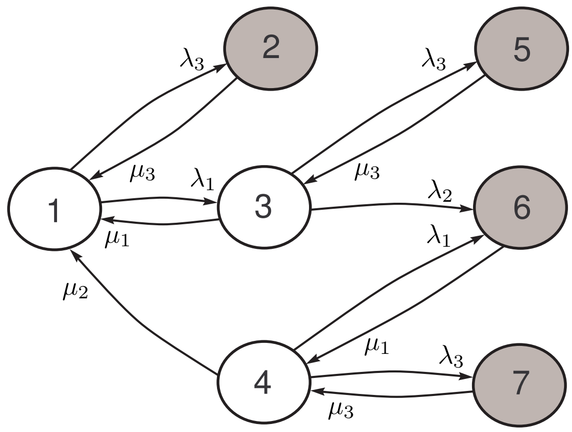

Figure 2: Markov diagram of system with priority repair of C1.

As it is usual in a DIM analysis, assume that all component failure rates are simultaneously changed. This could be due to, for example, brutal changes of environmental conditions, an overload, etc. To illustrate the application of the differential importance measures DIM^I and DIM^T, we compute the relative importance of the corresponding changes in some specific directions related to a parameter, a group of parameters or a state. The case of proportional changes in the failure rates is considered here, [2], i.e.:

$$\frac{\delta_{\lambda_1}}{\lambda_1} = \frac{\delta_{\lambda_2}}{\lambda_2} = \frac{\delta_{\lambda_3}}{\lambda_3} = \omega.$$

If this relationship is substituted in (7), the first-order differential importance measure DIM^I does not then depend on the percentage of change ω. Note however that this is not true for DIM^T, i.e. ω can not be eliminated in the calculation expression of DIM^T. In the following numerical experiments, ω varies from 1% to 100% and we compare the results given by DIM^I and DIM^T and we analyze the corresponding importance rankings for the considered change directions.

3.3.1 Variation of the system availability

These changes in failure rates lead to a variation in the system availability. Figure 3 shows the results of the approximation of this variation ($\delta^I A_\Sigma$) and its exact calculation (δA_Σ) by using the computing equations (11) and (17) respectively. Obviously, a failure rate increase results in a decrease of the system availability. The results show that the larger the ω is, the larger the difference between $\delta^I A_\Sigma$ and δA_Σ is. $\delta^I A_\Sigma$ can be used as a good approximation of δA_Σ only when ω is small.

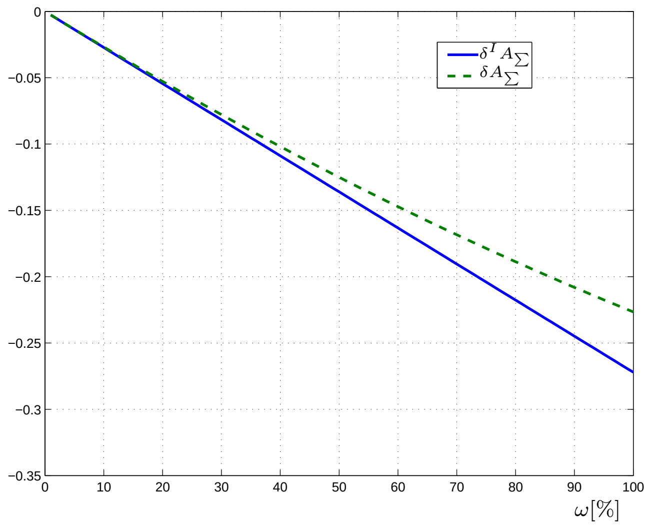

Figure 3: Variation of the system availability at steady state as a function of ω

The results show that $\delta^I A_\Sigma$ remains close to δA_Σ until around $\omega = 30\%$. In the next paragraph, it will however be shown that the differential importance measures (DIM^I and DIM^T) based on these quantities ($\delta^I A_\Sigma$ and δA_Σ respectively) can lead to different importance rankings.

3.3.2 DIMs for a component or a group of components

The application of DIMs (DIM^I and DIM^T) to quantify the relative contribution of a component, a group of components, on the variation of system availability is examined here.

A directional perturbation matrix $\mathbf{Q}_{\lambda_i}$ is first considered corresponding to changes in the direction of a single parameter of interest, e.g. the failure rate of component

i, λ_i.

Table 3: DIMs and component ranking - $\omega = 4\%$.

DIMs	λ_1	λ_2	λ_3	Order
$\text{DIM}^I(\mathbf{Q}_{\lambda_i})$	0.3264	0.3374	0.3362	C2>C3>C1
$\text{DIM}^T(\mathbf{Q}_{\lambda_i})$	0.3258	0.3360	0.3365	**C3**>**C2**>C1

DIM^I and DIM^T in the different directions $\mathbf{Q}_{\lambda_i}$ $(i = 1, 2, 3)$ are obtained by using the analytical expressions (12) and (19) respectively. Table 3 presents the results of DIM measures in the case where $\omega = 4\%$. According to these DIM^I measures, the contribution of C2 on the variation of system availability is more important than that of C3 and C1, and the components' importance ranking is C2>C3>C1. This importance ranking is not the same as the one based on DIM^T measures.

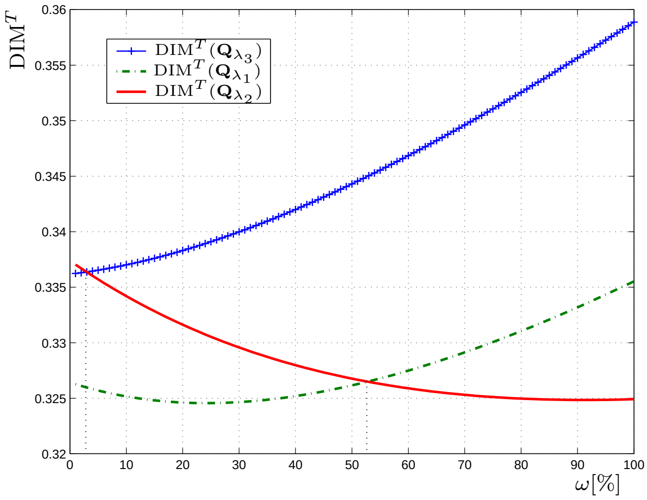

Figure 4: $\text{DIM}^T(\mathbf{Q}_{\lambda_i}), i = 1, 2, 3$, as a function of ω

Since DIM^I is independent of ω, DIM^I measures remain unchanged when ω varies and consequently, the components' importance ranking based on DIM^I does not change with ω. However, the evolution of $\text{DIM}^T(\mathbf{Q}_{\lambda_i}), i = 1, 2, 3$ represented in Figure 4 shows that DIM^T measures change significantly with ω, leading to different components' importance rankings. When $\omega \geq 4\%$, the importance rankings based on DIM^T become different from the one given by DIM^I. These results can be explained by the fact that DIM^T integrates the all-order interactions between the parameters changes, whereas DIM^I does not. They clearly show that DIM^I should be used with care when high percentages of parameters change are considered and that DIM^T should be preferred in theses cases.

From the additivity property, the results of DIM^I for the pairs of parameters are shown in Table 4. Considering these results, the groups'/components' ranking is: C3<C1<C2<(C1,C3)<(C1,C2)<(C2,C3).

Table 4: DIM^I for a group of parameters with any ω's value.

DIM^I	(λ_1, λ_2)	(λ_1, λ_3)	(λ_2, λ_3)	Order
$\mathrm{DIM}^I(\mathbf{Q}_{\lambda_i,\lambda_j})$	0.6638	0.6626	0.6736	(C1,C3)<(C1,C2)<(C2,C3)

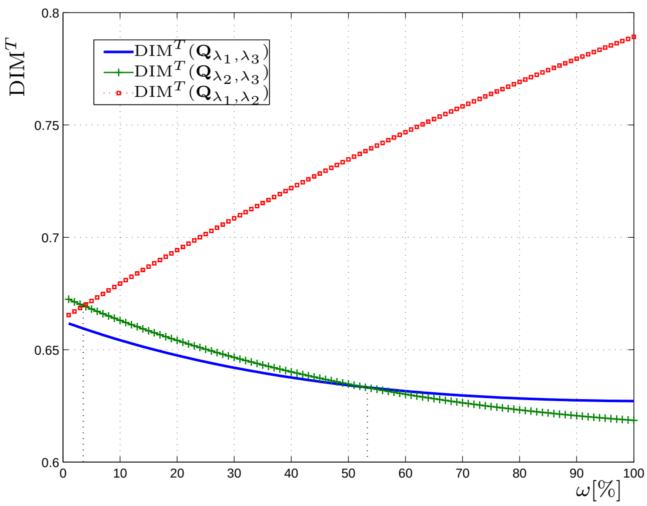

Figure 5: $\mathrm{DIM}^T(\mathbf{Q}_{\lambda_i,\lambda_j}), i, j = 1, 2, 3$, as a function of ω

From the results given in Figure 5, DIM^T measures change with ω and lead to different importance ranking when ω varies:

- when $\omega < 4\%$, the groups'/components' ranking based on DIM^T is the same as the one obtained with DIM^I ;

- when $\omega \geq 4\%$, DIM^T measures lead to a different importance ranking (and this ranking changes again when $\omega \geq 54\%$).

Note however that the components' ranking thus obtained are not absolute importance rankings, but rankings based on DIM criterion. Obviously, different rankings could be obtained if a different importance measure was used.

3.3.3 DIMs for a given state

To study the sensitivity of a given state, some specific directions of sensitivity are considered. In Table 5 and Figure 6, $\mathbf{Q}_i$ represents the direction of all failure rates corresponding to transitions exiting from the operational state i ($i = 1, 3, 4$). λ_{ij} indicates the transition rate from state i to state j. The differential importance measure of the direction $\mathbf{Q}_i$ provides the relative contribution of state i on the total variation of system availability provoked by the changes in components failure rate mentioned above.

Table 5: DIM^I for a state.

State i	Transitions	$\text{DIM}^I(\mathbf{Q}_i)$	Order
1	$\lambda_{12}, \lambda_{13}$	0.2918	2
3	$\lambda_{35}, \lambda_{36}$	0.5192	1
4	$\lambda_{46}, \lambda_{47}$	0.1890	3

The results of DIM^I measures, for any ω, are reported in Table 5. According to these measures, state 3 is the most important and state 1 is more important than state 4. This importance ranking still holds if one considers DIM^T measures for the cases where $\omega < 92\%$, see Figure 6. When $\omega \geq 92\%$ DIM^T measures lead to a different ranking: state 1 $<$ state 4 $<$ state 3.

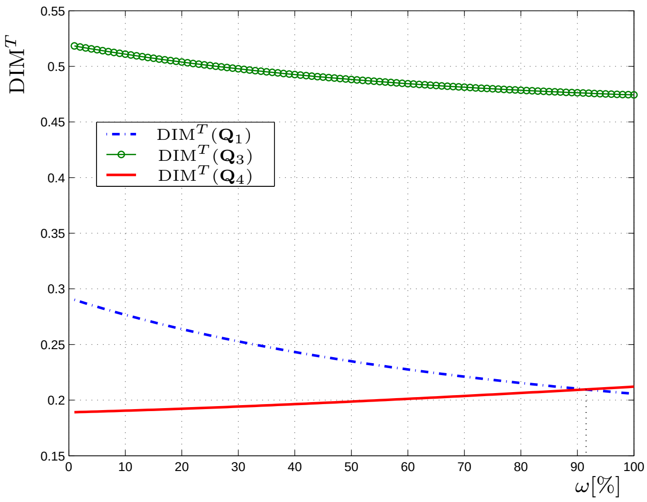

Figure 6: $\mathrm{DIM}^T(\mathbf{Q}_i), i = 1, 3, 4$, as a function of ω

4 Single sample path-based estimation

From a practical point of view, the analytical methods developed in the previous section can be used in order to compute DIM^I and DIM^T. These methods may however lead to some difficulties. The reliability behaviour (failure rate, repair rate) of some components of the system may be unknown, and the transition rate matrix of the Markov process modeling the system may even be unknown. In these cases, the analytical methods may be unusable. Moreover, these methods could be computationally burdensome or highly inefficient when the state space dimension is too high. It is therefore interesting to dispose of an effective method for estimating DIM^I and DIM^T from the observed data of the system, i.e. from a single sample path of the corresponding Markov process.

It has been shown in [6] and [11] that perturbation analysis can be a promising approach since this method can provide the estimate of the directional derivatives. As a consequence, the estimation of DIM^I can be easily obtained using perturbation analysis, see (7). In order to estimate both DIM^I and DIM^T, the approach presented here is based on a single sample path of a Markov process.

Looking at the computing equations (12) and (19), if the steady state probabilities vector $\boldsymbol{\pi}$ and the group inverse $\mathbf{M}^\sharp$ can be estimated, an evaluation of DIM^I and DIM^T can be then easily obtained. The estimation of $\boldsymbol{\pi}$ and $\mathbf{M}^\sharp$ from a single sample path of the Markov process will now be shown.

4.1 Estimation of $\boldsymbol{\pi}$ and $\mathbf{M}^\sharp$

It has been demonstrated that the transition rate matrix $\mathbf{M}$ can be estimated from a single sample path by using the maximum likelihood estimation (see [15, 16]). Note however that the matrix has to be inverted to obtain $\mathbf{M}^\sharp$ ($\mathbf{M}^\sharp = (\mathbf{M} - \boldsymbol{e\pi})^{-1} - \boldsymbol{e\pi}$). This may lead to numerical errors [13]. The interest here is in the direct estimation of the group inverse $\mathbf{M}^\sharp$ without using the transition rate matrix $\mathbf{M}$. Some numerical results will be considered in the next paragraph to show that this approach can provide a good estimation of DIM^I and DIM^T.

First, it is shown in [6] that the group inverse $\mathbf{M}^\sharp$ can be written as:

$$\begin{aligned} \mathbf{M}^\sharp &= -\int_0^\infty (\exp\{\mathbf{M}t\} - \boldsymbol{e\pi})dt, \\ &= -\lim_{T\to\infty} \Big\{ \int_0^T \exp\{\mathbf{M}t\}dt - T\boldsymbol{e\pi} \Big\}. \end{aligned} \tag{20}$$

From (20), one gets:

$$\mathbf{M}^\sharp = \lim_{T\to\infty} \mathbf{M}^\sharp(T),$$

where,

$$\mathbf{M}^\sharp(T) = -\Big\{ \int_0^T \exp\{\mathbf{M}t\}dt - T\boldsymbol{e\pi} \Big\}. \tag{21}$$

The results in [7, 5] shows that $\mathbf{M}^\sharp(T)$, for a fixed T, can be used as a good estimate of $\mathbf{M}^\sharp$.

For any constant vector $\mathcal{C} = (c_1, c_2, ...)$, let's define $\overline{\mathbf{M}}^\sharp = \mathbf{M}^\sharp + \boldsymbol{e}\mathcal{C}$. Since $\mathbf{Q}_i^\delta \boldsymbol{e} = \mathbf{0}$ and $\mathbf{Q}_\sum^\delta \boldsymbol{e} = \mathbf{0}$, one has: $\mathbf{Q}_i^\delta \overline{\mathbf{M}}^\sharp = \mathbf{Q}_i^\delta \mathbf{M}^\sharp$ and $\mathbf{Q}_\sum^\delta \overline{\mathbf{M}}^\sharp = \mathbf{Q}_\sum^\delta \mathbf{M}^\sharp$ respectively. In order to compute DIM^I and DIM^T (see (12) and (19)), $\mathbf{M}^\sharp$ can be replaced by $\overline{\mathbf{M}}^\sharp$. For example, one can simply use

$$\overline{\mathbf{M}}^\sharp(T) = -\int_0^T \exp\{\mathbf{M}t\}dt \tag{22}$$

instead of $\mathbf{M}^\sharp(T)$ as an estimate of the group inverse in (12) and (19).

Let $p_{ij}(t) = \mathbb{P}\{X_t = j | X_0 = i\}$ and $\mathbf{P}(t) = [p_{ij}(t)]_{i,j\in E}$. Then one has $\mathbf{P}(t) = \exp(\mathbf{M}t)$ (see e.g. [14]). Thus, from (22) the (i, j) entry of $\overline{\mathbf{M}}^\sharp(T)$ is

$$m_{ij}^\sharp(T) = -\int_0^T p_{ij}(t)dt. \tag{23}$$

Let us define $\epsilon^j(v)$ so that $\epsilon^k(v) = 1$ if $v = k$ and $\epsilon^k(v) = 0$ otherwise. One gets:

$$m^{\sharp}_{ij}(T) = -\mathbb{E}\Big[\int_0^T \epsilon^j(X_t)dt|X_0 = i\Big]. \tag{24}$$

Let T_k be the kth transition epoch of $\{X_t\}$ and X_k be state of $\{X_t\}$ after the kth transition. By the definition, $X_k = X_{t|t=T_k^+}$ and the steady state probability of state i is:

$$\pi_i = \lim_{N\to\infty} \frac{1}{T_N} \int_0^{T_N} \epsilon^i(X_t)dt, \text{for all } i \in E \tag{25}$$

where N represents the number of transition of the Markov process. Based on the ergodicity of the Markov process, (24) leads to:

$$m^{\sharp}_{ij}(T) = -\lim_{N\to\infty} \frac{\sum_{k=0}^{N-m} \Big\{\epsilon^i(X_k) \int_{T_k}^{T_k+T} \epsilon^j(X_t)dt\Big\}}{\sum_{k=0}^{N-m} \epsilon^i(X_k)}, \tag{26}$$

where $m = max\{x|T_{N-x} + T \leq T_N\}$. This relationships is proved in appendix A. One problem remaining here is the choice of the length T. It is shown in [7, 5] that T should be comparable to the mean of first passage time between state j and i, named $\Gamma^j(i) = \inf\{t, t > 0|X_t = i, X_0 = j\}$.

Estimation of first passage time A Markov process X_t with the transition matrix $\mathbf{M}$ is observed with the two time sequences $\{j_s\}$ and $\{i_s\}$ so that:

- $i_0 = 0$,
- j_s is the time upon which X_t is in state j for the first time after time i_{s-1},$s \geq 1$,
- i_s is the time upon which X_t is in state i for the first time after time j_s, $s \geq 1$.

$\{j_s\}$ and $\{i_s\}$ are well defined on a sample path. Now define: $L^j_s(i) = i_s - j_s$ for $s \geq 1$ (see Figure 7). The mean of a first passage time between state j and i is :

$$\Gamma^j(i) = \lim_{n\to\infty} \frac{1}{n} \sum_{s=1}^{n} L^j_s(i).$$

4.2 Numerical results

Reconsidering the example presented in subsection 3.3, the DIMs' measures mentioned above are obtained by using the analytical calculation under the hypothesis

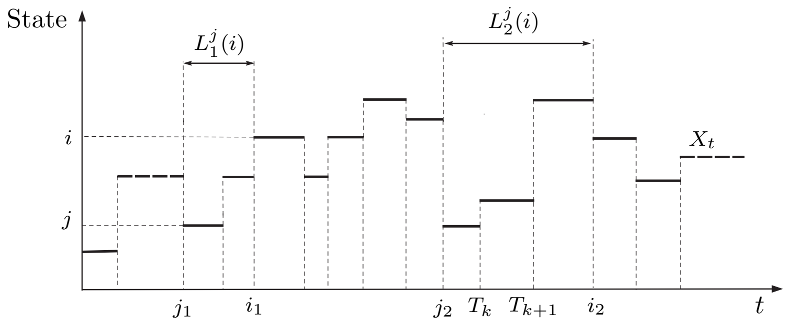

Figure 7: Illustration of $L_s^j(i)$ on observed single sample path X_t.

that the transition rate matrix $\mathbf{M}$ of the Markov process is available. Assume now that $\mathbf{M}$ is unknown and that the only available data is a single sample path of the Markov process. Since the realistic data set of this trajectory is not available here, it need to be simulated with the parameter values given in Table 2. The goal is to estimate these DIM measures from this data set using the estimation approach. In fact, the simulation is made for 7.10^4 transitions.

Table 6: Estimation of DIMs for a component - $\omega = 20\%$

DIMs	λ_1	λ_2	λ_3
$\hat{\text{DIM}}^I(\mathbf{Q}_{\lambda_i})$	0.3260	0.3376	0.3361
$\hat{\text{DIM}}^T(\mathbf{Q}_{\lambda_i})$	0.3248	0.3316	0.3378

Based on this simulated data set, the steady state vector $\boldsymbol{\pi}$ and the group inverse $\mathbf{M}^\sharp$ have been estimated by using (25) and (26) respectively. All the results presented in Tables 6 and 7 are obtained by changing only the directional matrix in (12) and (19). The numerical results show that the estimated values are very close to those given by the analytical method presented in the previous section.

To illustrate the convergence of the proposed estimation approach, Figures 8, 9 and 10 sketch the evolution of the estimators $\hat{\text{DIM}}^T(\mathbf{Q}_{\lambda_i})$ $(i = 1, 2, 3)$ as a function of the sample size for the case where $\omega = 20\%$.

Table 7: Estimation of DIMs for a state - $\omega = 20\%$.

State i	Transitions	$\widehat{\mathrm{DIM}}^I(\mathbf{Q}_i)$	$\widehat{\mathrm{DIM}}^T(\mathbf{Q}_i)$
1	$\lambda_{12}, \lambda_{13}$	0.2912	0.2650
3	$\lambda_{35}, \lambda_{36}$	0.5189	0.5015
4	$\lambda_{46}, \lambda_{47}$	0.1896	0.1938

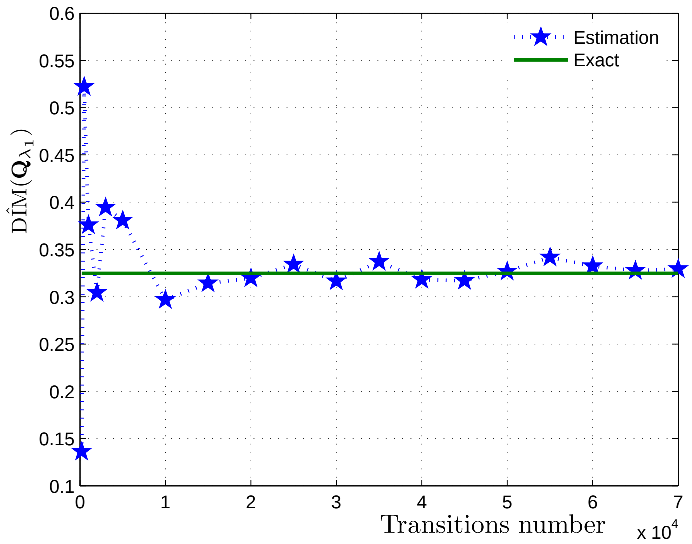

Figure 8: Estimator $\widehat{\mathrm{DIM}}^T(\mathbf{Q}_{\lambda_1})$ as a function of the sample size, case $\omega = 20\%$

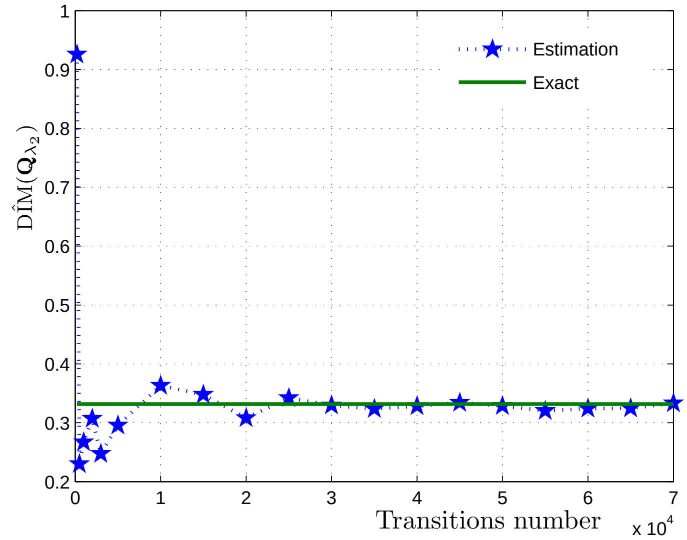

Figure 9: Estimator $\hat{\mathrm{DIM}}^T(\mathbf{Q}_{\lambda_2})$ as a function of the sample size, case $\omega = 20\%$

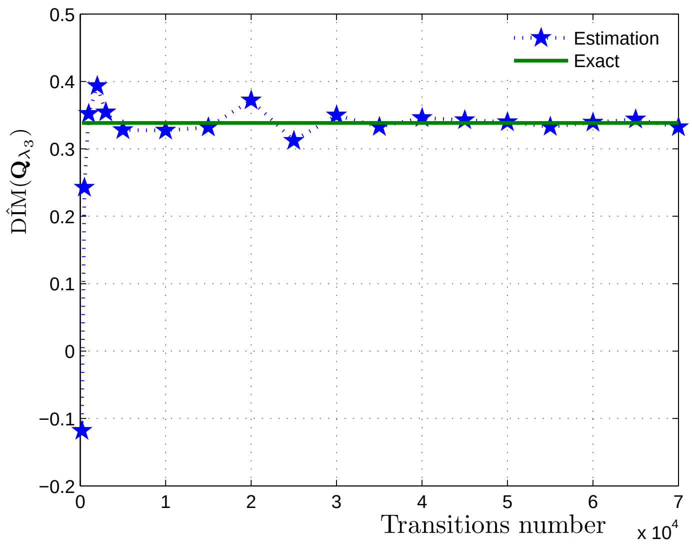

Figure 10: Estimators $\hat{\mathrm{DIM}}^T(\mathbf{Q}_{\lambda_3})$ as functions of the sample size, case $\omega = 20\%$

5 Conclusions

In this work, the differential importance measures at different orders (including the total order) are extended to dynamic systems including e.g. inter-components or functional dependencies, described by Markov models. DIM measures permit quantifying the relative contribution of a component (or a group of components, a state or a group of states) on the total variation of system performance provoked by the changes in system parameters values. When compared to the DIM^I measure, the proposed total order measure DIM^T provides more insightful results to analyse the system performance variation in response to parameters changes. Moreover, DIM^T can be used with any magnitude of change in system parameters.

In order to compute DIMs (DIM^I and DIM^T) at steady state, both analytical and estimation methods are investigated. Particularly, the second method based on a single sample path of a Markov process can provide a very good estimation result. From a practical point of view, this approach can be therefore a powerful tool to estimate DIMs' measures from the operating feedback data of the system without knowing components reliability behaviour (failure and/or repair rate,...), and consequently, the transition rate matrix of a Markov model.

This paper is the development of our research in the framework of the importance analysis of dynamic systems presented in part in [10]. Our future research work will focus on the development DIM^T measure in the transient state, as well as more detailed applications of these measures to decision-making in reliability engineering, e.g. to the optimization of maintenance policies.

Appendix

A Proof of (26)

The proof of (26) is based on a fundamental theorem on ergodicity, see e.g. [4]. Let $X = \{X_k | k \geq 0\}$, be an ergodic process with finite space. The process $Z = \{Z_k | k \geq 0\}$ with $Z_k = \epsilon^i(X_k) \int_{T_k}^{T_k+T} \epsilon^j(X_t)dt$ is then ergodic. Thus,

$$\lim_{N\to\infty} \frac{\sum_{k=0}^{N-m} \Big\{\epsilon^i(X_k) \int_{T_k}^{T_k+T} \epsilon^j(X_t)dt\Big\}}{\sum_{k=0}^{N-m} \epsilon^i(X_k)}$$
$$= \lim_{N\to\infty} \Big\{\frac{N-m+1}{\sum_{k=0}^{N-m} \epsilon^i(X_k)}\Big\}\Big\{\frac{\sum_{k=0}^{N-m} \Big\{\epsilon^i(X_k) \int_{T_k}^{T_k+T} \epsilon^j(X_t)dt\Big\}}{N-m+1}\Big\}. \qquad (27)$$

Note that:

$$\mathbb{E}\Big[X_k = i\Big] = \lim_{N\to\infty} \frac{\sum_{k=0}^{N-m} \epsilon^i(X_k)}{N-m+1},$$

and,

$$\mathbb{E}[Z_k] = \mathbb{E}\Big[\epsilon^i(X_k) \int_{T_k}^{T_k+T} \epsilon^j(X_t)dt\Big]$$
$$= \lim_{N\to\infty} \frac{\sum_{k=0}^{N-m} \Big\{\epsilon^i(X_k) \int_{T_k}^{T_k+T} \epsilon^j(X_t)dt\Big\}}{N-m+1}.$$

Consequently, (27) can be expressed as:

$$\lim_{N\to\infty} \frac{\sum_{k=0}^{N-m} \Big\{\epsilon^i(X_k) \int_{T_k}^{T_k+T} \epsilon^j(X_t)dt\Big\}}{\sum_{k=0}^{N-m} \epsilon^i(X_k)} = \frac{\mathbb{E}\Big[\epsilon^i(X_k) \int_{T_k}^{T_k+T} \epsilon^j(X_t)dt\Big]}{\mathbb{E}\Big[X_k = i\Big]}$$
$$= \mathbb{E}\Big[\int_0^T \epsilon^j(X_t)dt | X_0 = i\Big] \quad \blacksquare$$

References

[1] E. Borgonovo, G. Apostolakis, S. Tarantola, and A. Saltelli. Comparison of global sensitivity analysis techniques and importance measures in PSA. *Reliability Engineering and System Safety*, 79:175–185, 2003.

[2] E. Borgonovo and G.-E. Apostolakis. A new importance measure for risk-informed decision making. *Reliability Engineering and System Safety*, 72(2):193–212, 2001.

[3] E. Borgonovo. The Reliability Importance of Components and Prime Implicants in Coherent and Non-Coherent Systems Including Total-Order Interactions. *Submitted to European Journal of Operational Research*, 2009.

[4] L. Breiman. *Probability*. Reading, MA:Addison-Wesley, 1968.

[5] X.-R. Cao. The MacLaurin series for performance functions of Markov chains. *Advances in Applied Probability*, 30:676–692, 1998.

[6] X.-R. Cao and H.-F. Chen. Perturbation realization, potentials, and sensitivity analysis of Markov processes. *IEEE Transactions on Automatic Control*, 42(10):1382–1393, 1997.

[7] X.-R. Cao and Y.-W. Wan. Algorithms for sensitivity analysis of Markov systems through potential and perturbation realization. *IEEE Transactions on Automatic Control*, 6(4):482–494, 1998.

[8] L. Dai. Rate of convergence for derivative estimation of discrete-time Markov chains via finite-difference approximation with common random numbers. *SIAM Journal on Applied Mathematics*, 57(3):731–751, 1997.

[9] P. Do Van, A. Barros, and C. Bérenguer. Importance measure on finite time horizon and application to Markovian multi-state production systems. *Journal of Risk and Reliability*, 222:449–461, 2008.

[10] P. Do Van, A. Barros, and C. Bérenguer. A new result on the differential importance measures of Markov systems. In *Ninth International Probabilistic Safety Assessment and Management Conference - Proc.PSAM9, 18-23 may 2008, Hong Kong, China.*

[11] P. Do Van, A. Barros, and C. Bérenguer. Reliability importance analysis of Markovian systems at steady state using perturbation analysis. *Reliability Engineering and System Safety*, 93(11):1605–1615, 2008.

[12] P. Do Van. Contribution to the development and to the study of reliability importance measures for Markovian systems. *PhD thesis, Troyes University of Technology (UTT), France, October 2008.*

[13] P. S. Dwyer and F. V. Waugh. On errors in matrix inversion. *Journal of the American Statistical Association*, 48(262):289–319, 1953.

[14] G. George Yin and Q. Zhang. *Continuous-Time Markov Chains and Applications.* Application of Mathematics. Springer, 1998.

[15] M. Jacobsen. *Statistical Analysis of Counting Process.* Lecture Notes in Statistics 12. Spinger-Verlag, New York, 1982.

[16] U. Kuchler and M. Sorensen. *Exponential Families of Stochastic Processes*. Springer Series in Statistics. Spinger, 1997.

[17] M. Marseguerra and E. Zio. Monte Carlo estimation of the differential importance measure: application to the protection system of a nuclear reactor. *Reliability Engineering and System Safety*, 86(1):11–24, 2004.

[18] M. Marseguerra, E. Zio, and L. Podofillini. First-order differential sensitivity analysis of a nuclear safety system by Monte Carlo simulation. *Reliability Engineering and System Safety*, 90(2):162–168, 2005.

[19] J. Meyer, C.D. The role of the group generalized inverse in the theory of finite Markov chains. *SIAM Rev.*, 17:443–464, 1975.

[20] J. Meyer, C.D. The condition of a finite Markov chain and perturbation bounds for the limiting probabilities. *SIAM J. Algebraic Discrete Math.*, 1:273–283, 1980.

[21] J. Meyer, C.D. Sensitivity of the stationary distribution of Markov chain. *SIAM J. Matrix Anl. Appl.*, 15(3):715–728, 1994.

[22] Y. Ou and J. Bechta-Dugan. Approximate sensitivity analysis for acyclic Markov reliability models. *IEEE Transactions on Reliability*, 52(2):220–231, 2003.

[23] S. Ross. *Stochastic Processes*. Wiley Series in Probability and Statistics. John Wiley & Sons, Inc., 1996.

[24] G. Vinod, H. Kushwaha, A. Verma, and A. Srividya. Importance measures in ranking piping components for risk informed in-service inspection. *Reliability Engineering and System Safety*, 80(2):107–113, 2003.

[25] E. Zio and L. Podofillini. Accounting for components interactions in the differential importance measure. *Reliability Engineering and System Safety*, 91:1163–1174, 2006.