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Can Cosmological Perturbations Produce Early Universe Vorticity?

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Abstract. In this special issue article, based on the talk with the same title in session B5 (Theoretical and Mathematical Cosmology) at GR19, we review the case of vorticity generation in cosmology using cosmological perturbation theory. We show that, while at linear order the vorticity evolution equation has no source term in the absence of anisotropic stress, at second order vorticity is sourced by gradients in entropy and energy density perturbations. We then present some estimates for the magnitude and scale dependence of the vorticity power spectrum using simple input power spectra for the energy density and entropy perturbations. Finally, we close with possible directions for future work followed by some hints toward the observational importance of the vorticity so generated, and the possibility of primordial magnetic field generation.

1. Introduction

Vorticity is a common phenomenon in situations involving fluids in the ‘real world’ (see e.g. Refs. [1, 2]). There has also been some interest recently in studying vorticity in astrophysical scenarios, including the inter galactic medium [3, 4], but relatively little attention has been paid to the role that vorticity plays in cosmology and the early universe.

In classical fluid dynamics the evolution of an inviscid fluid in the absence of body forces is governed by the Euler, or momentum, equation

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla) \mathbf{v} = -\frac{1}{\rho} \nabla P, \quad (1.1)$$

where \mathbf{v} is the velocity vector, ρ the energy density and P the pressure of the fluid. The vorticity, $\boldsymbol{\omega}$, is a vector field and is defined as

$$\boldsymbol{\omega} \equiv \nabla \times \mathbf{v}, \quad (1.2)$$

and can be thought of as the circulation per unit area at a point in the fluid flow. An evolution equation for the vorticity can be obtained by taking the curl of Eq. (1.1), which gives

$$\frac{\partial \boldsymbol{\omega}}{\partial t} = \nabla \times (\mathbf{v} \times \boldsymbol{\omega}) + \frac{1}{\rho^2} \nabla \rho \times \nabla P. \quad (1.3)$$

The second term on the right hand side of Eq. (1.3), often called the baroclinic term in the literature, then acts as a source for the vorticity. Evidently, this term vanishes if lines of constant energy and pressure are parallel, or if the energy density or pressure are constant. A special class of fluid for which the former is true is a barotropic fluid, defined such that the equation of state is a function of the energy density only, i.e. $P \equiv P(\rho)$, and so $(1/\rho^2) \nabla \rho \times \nabla P = 0$.

For a barotropic fluid, the vorticity evolution equation, Eq. (1.3), can then be written, by using vector calculus identities, as

$$\frac{D\boldsymbol{\omega}}{Dt} \equiv \frac{\partial \boldsymbol{\omega}}{\partial t} + (\mathbf{v} \cdot \nabla) \boldsymbol{\omega} = (\boldsymbol{\omega} \cdot \nabla) \mathbf{v} - \boldsymbol{\omega} (\nabla \cdot \mathbf{v}), \quad (1.4)$$

which makes it clear that, in this case, the vorticity vector has no source, and so $\boldsymbol{\omega} = \mathbf{0}$ is a solution to Eq. (1.4).

Allowing for a more general perfect fluid with an equation of state depending not only on the energy density, but of the form $P \equiv P(S, \rho)$ will mean that, in general, the baroclinic term is no longer vanishing, and so acts as a source for the evolution of the vorticity. This is Crocco’s theorem [5] which states that vorticity generation is sourced by gradients in entropy in classical fluid dynamics.

2. Elements of Cosmological Perturbation Theory

In order to study vorticity in the early universe, we must use general relativity. Since Einstein's equations are notoriously difficult to solve, and relatively few exact, inhomogeneous solutions exist, we invoke the technique of cosmological perturbation theory. This means starting with a homogeneous and isotropic Friedmann-Robertson-Walker (FRW) spacetime as a background and adding on top layers of inhomogeneous perturbations. Of course, perturbing the geometry of the spacetime invokes perturbations in its matter content, so, for example, the energy density is perturbed as

$$\rho(x^i, \eta) = \rho_0(\eta) + \delta\rho(x^i, \eta), \quad (2.1)$$

where η is conformal time and a subscript zero denotes the homogeneous background part of the quantity. The perturbations are then expanded order by order in a series as

$$\delta\rho = \delta\rho_1 + \frac{1}{2}\delta\rho_2 + \dots, \quad (2.2)$$

and $\rho_0 \gg \delta\rho_1 > \delta\rho_2$, where the subscripts denote the order of the perturbation. In analogy with the classical case, we want to consider perturbations of a perfect fluid with equation of state $P \equiv P(S, \rho)$. Expanding the pressure perturbation in a Taylor series gives

$$\delta P = \frac{\partial P}{\partial S} \delta S + \frac{\partial P}{\partial \rho} \delta \rho, \quad (2.3)$$

which can also be written as

$$\delta P = \delta P_{\text{nad}} + c_s^2 \delta \rho, \quad (2.4)$$

where we have introduced the non-adiabatic pressure perturbation, δP_{nad} and the adiabatic sound speed squared, c_s^2 . This can then be extended beyond linear order; see Ref. [6] for details.

The most general perturbations to the FRW line element are given by [7, 8]

$$ds^2 = a^2(\eta) \left[-(1 + 2\phi)d\eta^2 + 2B_i dx^i d\eta + (\delta_{ij} + 2C_{ij})dx^i dx^j \right], \quad (2.5)$$

where the perturbations of the spatial components of the metric can be further decomposed into scalars, vectors and tensors (characterised by their transformation behaviour on spatial hypersurfaces) as

$$B_i = B_{,i} - S_i, \quad C_{ij} = -\psi\delta_{ij} + E_{,ij} + F_{(i,j)} + \frac{1}{2}h_{ij}, \quad (2.6)$$

where S_i and F_i are divergence-free vectors, and h_{ij} is a divergence-free, traceless rank two tensor. Each metric perturbation is then expanded order by order in an analogous way to the energy density, Eq. (2.2).

Splitting the spacetime into a background and perturbation introduces spurious coordinate artefacts, or gauge modes, which have the potential to cause a problem. In order to remove these gauge modes, we need to make a gauge choice. Since there are myriad articles in the literature dedicated to this issue, we do not further comment on this here, pointing the interested reader to, for example, Refs. [7, 8, 9, 10] for more information. For the rest of this article, we will work in the uniform curvature gauge, for which $E = 0 = \psi$ and $F_i = \mathbf{0}$, and neglect tensor perturbations, since they are negligible compared to scalar perturbations. This gives the line element

$$ds^2 = a^2(\eta) \left[-(1 + 2\phi)d\eta^2 + 2B_i dx^i d\eta + \delta_{ij} dx^i dx^j \right], \quad (2.7)$$

and the fluid four-velocity,

$$u_\mu = -\frac{1}{2}a \left[2(1 + \phi_1) + \phi_2 - \phi_1^2 + v_{1k} v_1^k, -2V_{1i} - V_{2i} + 2\phi_1 B_{1i} \right], \quad (2.8)$$

where where v^i is the spatial three-velocity and $V^i \equiv v^i + B^i$.

Evolution and constraint equations are obtained, in general relativity, through the covariant conservation of energy-momentum, $\nabla_\mu T^\mu{}_\nu = 0$, and the Einstein field equations, $G^\mu{}_\nu = 8\pi G T^\mu{}_\nu$, where $T_{\mu\nu}$ is the energy-momentum tensor, and $G_{\mu\nu}$ the Einstein tensor, as usual. In cosmological perturbation theory, we obtain such equations order by order. As an example, consider the momentum conservation equation, from $\nabla_\mu T^\mu{}_i = 0$, at linear order

$$V'_{1i} + \mathcal{H}(1 - 3c_s^2)V_{1i} + \left[\frac{\delta P_1}{\rho_0 + P_0} + \phi_1 \right]_{,i} = 0, \quad (2.9)$$

where \mathcal{H} is the Hubble parameter in conformal time, defined as $\mathcal{H} = a'/a$, the prime denoting a derivative with respect to conformal time. Eq. (2.9) is the analogue of the Euler equation (1.1) in cosmological perturbation theory.

3. Vorticity in Cosmology

In general relativity, the vorticity tensor is defined as the projected anti-symmetrised covariant derivative of the fluid four velocity, that is [8]

$$\omega_{\mu\nu} = \mathcal{P}_\mu{}^\alpha \mathcal{P}_\nu{}^\beta u_{[\alpha;\beta]}, \quad (3.1)$$

where $\mathcal{P}_{\mu\nu}$ is the projection tensor into the instantaneous fluid rest space and is given by

$$\mathcal{P}_{\mu\nu} = g_{\mu\nu} + u_\mu u_\nu. \quad (3.2)$$

Note that, in analogy with the classical case, it is possible to define a vorticity vector as $\omega_\mu = \frac{1}{2}\varepsilon_{\mu\nu\gamma}\omega^{\nu\gamma}$, where $\varepsilon_{\mu\nu\gamma}$ is the covariant permutation tensor in the fluid rest space (see Ref. [11, 12]).

The vorticity tensor can then be decomposed in the usual way, up to second order in perturbation theory, as $\omega_{ij} \equiv \omega_{1ij} + \frac{1}{2}\omega_{2ij}$. Working in the uniform curvature gauge, and considering only scalar and vector perturbations, we can obtain the components of the vorticity tensor by substituting the expressions for the fluid four velocity, Eq. (2.8), along with the metric tensor into Eq. (3.1). At first order this gives us

$$\omega_{1ij} = aV_{1[i,j]}, \quad (3.3)$$

and at second order

$$\omega_{2ij} = aV_{2[i,j]} + 2a \left[V'_{1[i}V_{1j]} + \phi_{1,[i}(V_1 + B_1)_{j]} - \phi_1 B_{1[i,j]} \right]. \quad (3.4)$$

3.1. Vorticity Evolution

In order to obtain an evolution equation for the vorticity at first order, we take the time derivative of Eq. (3.3). Simplifying, and using the first order equations, gives

$$V'_{1[i,j]} = -\mathcal{H}(1 - 3c_s^2)V_{1[i,j]} = -\frac{1}{a}\mathcal{H}(1 - 3c_s^2)\omega_{1ij}, \quad (3.5)$$

and so, from Eq. (3.3),

$$\omega'_{1ij} - 3\mathcal{H}c_s^2\omega_{1ij} = 0. \quad (3.6)$$

This reproduces the well known result that, during radiation domination, $|\omega_{1ij}\omega_1^{ij}| \propto a^{-2}$ in the absence of an anisotropic stress term [8].

At second order things are somewhat more complicated. We now need to take the time derivative of Eq. (3.4), and use the first order evolution and constraint equations, as well as the second order conservation equations in order to eliminate all the metric perturbation variables. We omit the derivation and the governing equations in this review article (see Ref. [13] for details of the derivation and the governing equations), but calculations give

$$\begin{aligned} \omega'_{2ij} - 3\mathcal{H}c_s^2\omega_{2ij} + 2 \left[\left(\frac{\delta P_1 + \delta \rho_1}{\rho_0 + P_0} \right)' + V_{1,k}^k - X_{1,k}^k \right] \omega_{1ij} \\ + 2(V_1^k - X_1^k)\omega_{1ij,k} - 2(X_{1,j}^k - V_{1,j}^k)\omega_{1ik} + 2(X_{1,i}^k - V_{1,i}^k)\omega_{1jk} \\ = \frac{a}{\rho_0 + P_0} \left\{ 3\mathcal{H}(V_{1i}\delta P_{\text{nad}1,j} - V_{1j}\delta P_{\text{nad}1,i}) \right. \\ \left. + \frac{1}{\rho_0 + P_0} (\delta\rho_{1,j}\delta P_{\text{nad}1,i} - \delta\rho_{1,i}\delta P_{\text{nad}1,j}) \right\}, \end{aligned} \quad (3.7)$$

where X_{1i} is given entirely in terms of matter perturbations as

$$X_{1i} = \nabla^{-2} \left[\frac{4\pi G a^2}{\mathcal{H}} (\delta\rho_{1,i} - \mathcal{H}(\rho_0 + P_0)V_{1i}) \right]. \quad (3.8)$$

In fact, even assuming zero first order vorticity, that is $\omega_{1ij} = 0$, the second order vorticity evolves as

$$\omega'_{2ij} - 3\mathcal{H}c_s^2\omega_{2ij} = \frac{2a}{\rho_0 + P_0} \left\{ 3\mathcal{H}V_{1[i}\delta P_{\text{nad1},j]} + \frac{\delta\rho_{1,[j}\delta P_{\text{nad1},i]}}{\rho_0 + P_0} \right\}, \quad (3.9)$$

and so we see that there is a non zero source term for the vorticity at second order in perturbation theory which is, in analogy with classical fluid dynamics, made up of gradients in entropy and density perturbations. Note that, in the absence of a non-adiabatic pressure perturbation, we recover the result of Ref. [11] that there is no vorticity generation.

3.2. First Estimates: Magnitude and Scale Dependence

In this section we present an estimate of the magnitude and scale dependence of the power spectrum for the vorticity, assuming simple power-law input power spectra. The results presented here are calculated in detail in Ref. [12].

Working now in the radiation era, and neglecting the vector perturbations (the first term on the right hand side of Eq. (3.9)), we obtain

$$\omega'_{2ij} - \mathcal{H}\omega_{2ij} = \frac{9a}{8\rho_0^2}\delta\rho_{1,[j}\delta P_{\text{nad1},i]} \equiv S_{ij}, \quad (3.10)$$

and we can define the power spectrum of the vorticity in the usual way, in Fourier space, as

$$\langle \omega_2^*(\mathbf{k}_1, \eta)\omega_2(\mathbf{k}_2, \eta) \rangle = \frac{2\pi}{k^3}\delta(\mathbf{k}_1 - \mathbf{k}_2)\mathcal{P}_\omega(k, \eta). \quad (3.11)$$

We take the following input power spectra

$$\delta\rho_1(k, \eta) = A \left(\frac{k}{k_0} \right) \left(\frac{\eta}{\eta_0} \right)^{-4}, \quad \delta P_{\text{nad1}}(k, \eta) = D \left(\frac{k}{k_0} \right)^2 \left(\frac{\eta}{\eta_0} \right)^{-5}, \quad (3.12)$$

where the time evolution of the energy density is obtained by solving the first order equations, and the wavenumber dependence from relating an initial ansatz to WMAP data. The non-adiabatic pressure input spectrum comes from demanding that it decays faster than the energy density, and that its spectrum is bluer than that of the energy density. The amplitudes can also be related to WMAP parameters. These input power spectra allow us to compute the power spectrum for the vorticity analytically, giving

$$\begin{aligned} \mathcal{P}_\omega(k, \eta) &= \frac{81}{256}k_0^5 \frac{\eta^2}{4\pi^4} \ln^2 \left(\frac{\eta}{\eta_0} \right) \left(\frac{\alpha(k_0)}{1 - \alpha(k_0)} \right)^2 k_0^{-12} \Delta_{\mathcal{R}}^8 k_c^5 \\ &\times \left[\frac{16}{135} \frac{k_c^4}{k_0^4} \left(\frac{k}{k_0} \right)^7 + \frac{12}{245} \frac{k_c^2}{k_0^2} \left(\frac{k}{k_0} \right)^9 - \frac{4}{1575} \left(\frac{k}{k_0} \right)^{11} \right], \quad (3.13) \end{aligned}$$

where k_c denotes the large wavenumber or small scale integration cutoff, and the other parameters are given in Ref. [14]. Substituting in an illustrative value for the cutoff,

$k_c = 10\text{Mpc}^{-1}$, and using the values of the WMAP parameters from Ref. [14], taking a conservative value of 10% of the maximum value for α , we obtain the following vorticity power spectrum

$$\mathcal{P}_\omega(k, \eta) = \eta^2 \ln^2 \left(\frac{\eta}{\eta_0} \right) \left[0.87 \times 10^{-2} \left(\frac{k}{k_0} \right)^7 + 3.73 \times 10^{-11} \left(\frac{k}{k_0} \right)^9 - 7.71 \times 10^{-20} \left(\frac{k}{k_0} \right)^{11} \right] \text{Mpc}^2. \quad (3.14)$$

This spectrum has a non-negligible magnitude which depends upon the small scale cut-off k_c and the chosen parameters. As this is a second order effect the magnitude is somewhat surprising. As can be seen, the result has a dependence on the wavenumber to the power of at least seven for our choice of non-adiabatic pressure input spectrum.

4. Future Directions

In the above section we have presented estimates for the vorticity power spectrum based on simple, power-law input power spectra. While this is a good first approximation, in order to obtain more realistic estimates of the magnitude of the early universe vorticity, we need to go beyond the simple ansatz for the non-adiabatic pressure perturbation input spectrum.

One way in which a non-adiabatic pressure perturbation can be generated is through the relative entropy perturbation between two or more fluids or scalar fields. For example, the relative entropy or isocurvature perturbation, at first order, between two fluids denoted with subscripts A and B is [15]

$$\mathcal{S}_{AB} = 3\mathcal{H} \left(\frac{\delta\rho_B}{\rho'_{0B}} - \frac{\delta\rho_A}{\rho'_{0A}} \right). \quad (4.1)$$

In a system consisting of multiple fluids, the non-adiabatic pressure perturbation is split as [8, 10]

$$\delta P_{\text{nad}} = \delta P_{\text{intr}} + \delta P_{\text{rel}}, \quad (4.2)$$

where the first term is the contribution from the intrinsic entropy perturbation of each fluid, and the second term is due to the relative entropy perturbation between each fluid, \mathcal{S}_{AB} , and is defined as

$$\delta P_{\text{rel}} \equiv \frac{1}{6\mathcal{H}\rho'_0} \sum_{A,B} \rho'_{0A}\rho'_{0B} (c_B^2 - c_A^2) \mathcal{S}_{AB}, \quad (4.3)$$

where c_A^2 and c_B^2 are the adiabatic sound speed of each fluid. Thus, for a multiple fluid system, even when the intrinsic entropy perturbation is zero for each fluid, there is a non-vanishing overall non-adiabatic pressure perturbation. This can be extended

to the case of scalar fields by using standard techniques of treating the fields as fluids (see e.g. Ref. [6]). Therefore a possible next step will involve using the relative entropy spectrum calculated from multi-field inflation as an input for the non-adiabatic pressure perturbation. This will enable us to obtain a more realistic description of induced vorticity in the early universe.

5. Discussion and Conclusions

In this brief article, we have reviewed current progress in the generation of vorticity in the early universe through non-linear cosmological perturbations. We started out by considering the familiar case of classical fluid dynamics and showed that the evolution equation for the classical vorticity contains a source term which is only non-zero if the equation of state is a function of two variables: the energy density and the entropy. We then briefly introduced some elements of cosmological perturbation theory and, working in the uniform curvature gauge, derived evolution equations for the vorticity. At linear order we reproduce the well known result that vorticity decays with the expansion of the universe in the absence of anisotropic stress. However we showed that vorticity is sourced is at second order by gradients in energy density and non-adiabatic pressure perturbations. This is in analogy with the classical case.

We then presented some first estimates of the power spectrum of the induced vorticity by using simple power laws as the input power spectra for the energy density and the entropy perturbations. The results show that the magnitude of the vorticity power spectrum under this approximation is non-negligible and the amplification due to the large power of k is huge. Finally, we briefly touched on the non-adiabatic pressure perturbation created by relative entropy perturbations in multi-fluid systems which will allow us to go beyond the simple approximation in the future.

A non-zero vorticity at second order in perturbation theory has important consequences for the generation of magnetic fields, as it has been long known that vorticity and magnetic fields are closely related (see Refs. [16, 17]). Previous works either used momentum exchange between multiple fluids to generate vorticity, as in Refs. [18, 19, 20, 21, 22, 23, 24], or used intermediate steps to first generate vorticity for example by using shock fronts as in Ref. [25]. However, we do not require such additional steps. Therefore, an important extension to the work presented in this review is to consider the magnetic fields generated by our mechanism which could be an important step in answering the unknown question regarding the origin of the primordial magnetic field.

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