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LINEAR FORMS AT A BASIS OF AN ALGEBRAIC NUMBER FIELD

Bernard de Mathan

Abstract. It was proved by Cassels and Swinnerton-Dyer that Littlewood conjecture in simultaneous Diophantine approximation holds for any pair of numbers in a cubic field. Later this result was generalized by Peck to a basis $(1, \alpha_1, \dots, \alpha_n)$ of a real algebraic number field of degree at least 3. By transference, this result provides some solutions for the dual form of Littlewood's conjecture. Here we find another solutions, and using Baker's estimates for linear forms in logarithms of algebraic numbers, we discuss whether the result is best possible.

1 Introduction

The celebrated Littlewood conjecture asserts that for any real numbers α and β , one has:

$$\inf_{q>0} q \|q\alpha\| \|q\beta\| = 0, \quad (1.1)$$

where q runs among the positive integers, and, for a real number x , $\|x\|$ is the distance between x and the nearest integer. It is not known whether this conjecture is true. If n is an integer with $n \geq 2$, one may also ask whether

$$\inf_{q>0} q \|q\alpha_1\| \cdots \|q\alpha_n\| = 0 \quad (1.2)$$

holds for any real numbers $\alpha_1, \dots, \alpha_n$. This weaker problem is neither solved.

Littlewood's conjecture has a dual form: is it true that for any real numbers $(\alpha_1, \dots, \alpha_n)$, one has

$$\inf_{x_1, \dots, x_n} \max\{|x_1|, 1\} \cdots \max\{|x_n|, 1\} \|x_1\alpha_1 + \dots + x_n\alpha_n\| = 0, \quad (1.3)$$

for (x_1, \dots, x_n) running in $\mathbf{Z}^n \setminus \{0\}$? It is well known (see [10]) that this problem is equivalent to the previous:

Theorem 1.1 ([10]) *For any real numbers $\alpha_1, \dots, \alpha_n$, conditions (1.2) and (1.3) are equivalent.*

A famous theorem of Cassels and Swinnerton-Dyer [10] asserts that, in the case $n = 2$, condition (1.2) is satisfied for any pair α_1, α_2 , of numbers in a cubic field. This result was generalized by Peck [17] who proved that:

Theorem 1.2 [17] *If $(1, \alpha_1, \dots, \alpha_n)$ is a basis of a real algebraic number field, with $n \geq 2$, there exist infinitely many integers $q > 1$ with*

$$\|q\alpha_k\| \ll q^{-1/n}(\log q)^{-1/(n-1)}, \quad 1 \leq k \leq n-1, \quad (1.4)$$

and

$$\|q\alpha_n\| \ll q^{-1/n}. \quad (1.5)$$

As usual, the Vinogradov symbol $A \ll B$, where A and B are positive quantities, means that there exists a positive real constant C such that $A \leq CB$. We shall also use the symbol $A \asymp B$ for $A \ll B$ and $B \ll A$.

The constants involved by the symbol \ll in (1.4) and (1.5) depend upon the α_k 's. We deduce from (1.4) and (1.5) that:

$$\liminf q \log q \|q\alpha_1\| \cdots \|q\alpha_n\| < +\infty, \quad (1.2')$$

which implies (1.2). In Theorem 1.2, the logarithmic factor in inequality (1.4) is close to be best possible (see [14]).

It is easy to see that Theorem 1.2 provides, via the proof of Theorem 1.1 ([10]), solutions of (1.3) satisfying the following conditions:

Corollary 1.3 *If $(1, \alpha_1, \dots, \alpha_n)$ is a basis of a real algebraic number field, with $n \geq 2$, there exist arbitrarily large real numbers M , for which there are integers x_0, \dots, x_n , not all zero, such that:*

$$|x_i| \ll M \log M^{1/(n^2(n-1))}, \quad i = 0, \dots, n-1,$$

$$|x_n| \ll M \log M^{-(n+1)/n^2},$$

and

$$|x_0 + x_1\alpha_1 + \dots + x_n\alpha_n| \ll M^{-n}.$$

We thus have

$$\liminf (\log \max_{1 \leq i \leq n} |x_i|)^{1/n} \max\{|x_1|, 1\} \cdots \max\{|x_n|, 1\} \|x_1\alpha_1 + \dots + x_n\alpha_n\| < +\infty, \quad (1.3')$$

which implies (1.3).

By the pigeon hole principle, one sees that for any real numbers $\alpha_1, \dots, \alpha_n$, and for each positive integer M , there exist integers x_0, \dots, x_n , not all zero, satisfying the conditions:

$$\max_{0 \leq i \leq n} |x_i| \leq M,$$

and

$$|x_0 + x_1\alpha_1 + \dots + x_n\alpha_n| \ll M^{-n},$$

where the constant involved is only depending upon the α_i 's (this constant can be taken equal to $1 + |\alpha_1| + \dots + |\alpha_n|$). Comparing with Theorem 1.2, one may ask whether, for a basis $1, \alpha_1, \dots, \alpha_n$, of an algebraic number field E of degree $n + 1 \geq 3$, there exist infinitely many positive integers M for which there are integers x_0, \dots, x_n , not all zero, such that

$$|x_i| \leq M, \quad 0 \leq i \leq n - 1,$$

$$|x_n| = o(M),$$

and

$$|x_0 + x_1\alpha_1 + \dots + x_n\alpha_n| \ll M^{-n}.$$

In this paper, we shall prove that there are also solutions of (1.3) satisfying the previous conditions. By using a direct method, which is an inhomogeneous version of Peck's method, we shall obtain the following result:

Theorem 1.4 *Let $(1, \alpha_1, \dots, \alpha_n)$ be a basis of a real algebraic number field E of degree $n + 1 \geq 3$ over \mathbb{Q} . Then there exist infinitely many positive integers M for which there are integers x_0, \dots, x_n , not all zero, such that*

$$\max_{0 \leq i \leq n} |x_i| = M, \quad (1.6)$$

$$|x_n| = o(M), \quad (1.7)$$

and

$$|x_0 + x_1\alpha_1 + \dots + x_n\alpha_n| \ll M^{-n}. \quad (1.8)$$

Obviously this result also provides solutions of (1.3).

One can notice that for any integers x_0, \dots, x_n , not all zero, we have

$$|x_0 + x_1\alpha_1 + \dots + x_n\alpha_n| \gg \left(\max_{0 \leq i \leq n} |x_i| \right)^{-n} \quad (1.9)$$

where the constant involved depends upon the α_i 's. Indeed, setting $x_0 + \dots + x_n\alpha_n = x$, we have for each isomorphism σ from E to \mathbb{C} ,

$$|\sigma(x)| \ll \max_{0 \leq i \leq n} |x_i|,$$

hence

$$|N_{E/\mathbb{Q}}(x)| \ll |x| \max_{0 \leq i \leq n} |x_i|^n,$$

which leads to (1.9) because, denoting by D a positive integer such that $D\alpha_i$ is an algebraic integer for each i , we see that Dx is an algebraic integer, hence we have $D^{n+1}|N_{E/\mathbb{Q}}(x)| \geq 1$. Accordingly when (1.6) and (1.8) are satisfied, we get

$$|x_0 + \dots + x_n\alpha_n| \asymp M^{-n}.$$

In the case $n = 2$, Theorem 1.4 can be improved:

Theorem 1.5 *Let $(1, \alpha_1, \alpha_2)$ be a basis of a real cubic field E over \mathbb{Q} . There exists a positive real constant κ for which there are arbitrarily large integers M , and non zero integers x_0, x_1, x_2 , with*

$$\max\{|x_0|, |x_1|, |x_2|\} = M, \quad (1.6')$$

$$|x_2| \ll M \log M^{-\kappa}, \quad (1.10)$$

and

$$|x_0 + x_1\alpha_1 + x_2\alpha_2| \ll M^{-2}. \quad (1.8')$$

We thus obtain the estimation:

$$\liminf(\log \max\{|x_1|, |x_2|\})^\kappa \max\{|x_1|, 1\} \max\{|x_2|, 1\} \|x_1\alpha_1 + x_2\alpha_2\| < +\infty,$$

which implies (1.3).

Note for comparison that Corollary 1.3 leads in this case to

$$|x_0 + x_1\alpha_1 + x_2\alpha_2| \ll M^{-2},$$

with

$$|x_1| \ll M \log M^{1/4}$$

and

$$|x_2| \ll M \log M^{-3/4}.$$

A dual form of Schmidt's Theorem [18] asserts that, if $\alpha_1, \dots, \alpha_n$, are real algebraic numbers such that $1, \alpha_1, \dots, \alpha_n$, are linearly independent over \mathbb{Q} , then for any real number $\epsilon > 0$, and for every $(x_1, \dots, x_n) \in \mathbb{Z}^n \setminus \{0\}$, one has

$$\|x_1\alpha_1 + \dots + x_n\alpha_n\| \prod_{i=1}^n \max\{1, |x_i|^{1+\epsilon}\} \gg_\epsilon 1.$$

Here, we shall prove:

Theorem 1.6 *Assume that $(1, \alpha_1, \dots, \alpha_n)$ is a basis of an algebraic number field E , with $n \geq 2$. If \mathcal{S} is an infinite set of $(n+1)$ -tuples $(x_0, \dots, x_n) \in \mathbb{Z}^{n+1}$ satisfying (1.8) and (1.6), with $M = \max_{0 \leq i \leq n} |x_i| \geq 2$, then there exists a positive real constant λ such that we have for each $(x_0, \dots, x_n) \in \mathcal{S}$,*

$$\max_{2 \leq i \leq n} |x_i| \gg M \log M^{-\lambda}. \quad (1.11)$$

Moreover, if $\mathbb{Q}(\alpha_1) \neq E$, then we get

$$\max_{2 \leq i \leq n} |x_i| \gg M. \quad (1.12)$$

The constant λ and the constants involved in (1.11) or (1.12), depend upon the α'_i 's and upon the set \mathcal{S} . Schmidt's Theorem provides in this case the lower bound

$$\max_{2 \leq i \leq n} |x_i| \gg M^{1-\epsilon}$$

for any $\epsilon > 0$. Hence, in this case, our result is more precise, but it is very particular. We are not able to obtain, in Theorem 1.6, a constant λ independent upon \mathcal{S} .

Note that, if we take for instance $E = \mathbb{Q}(\sqrt{2}, \sqrt{3})$, and $\alpha_1 = \sqrt{2}$, $\alpha_2 = \sqrt{3}$, $\alpha_3 = \sqrt{6}$, then Theorem 1.6 implies that if we consider any infinite set \mathcal{S} of $(x_0, x_1, x_2, x_3) \in \mathbb{Z}^4 \setminus \{0\}$ for which

$$|x_0 + x_1\alpha_1 + x_2\alpha_2 + x_3\alpha_3| \ll \left(\max_{0 \leq i \leq 3} |x_i|\right)^{-3},$$

then there is at most one index $0 \leq i \leq 3$ such that

$$|x_i| = o\left(\max_{0 \leq j \leq 3} |x_j|\right)$$

for every $(x_0, x_1, x_2, x_3) \in \mathcal{S}$. Thus Theorem 1.4 cannot be improved by setting condition (1.7) for two indices.

2 A metrical point of view.

Theorems 1.4, 1.5 and 1.6 can also be examined from a metrical point of view. For any real numbers $\omega_1, \dots, \omega_n$, and every positive real number ν , we can consider the following diophantine problems:

(P₁): given positive real constants C and C' , does there exist infinitely many $(n+1)$ -tuples $(x_0, x_1, \dots, x_n) \in \mathbb{Z}^{n+1}$, with $\max_{1 \leq i \leq n-1} |x_i| > 1$, such that

$$|x_0 + x_1\omega_1 + \dots + x_n\omega_n| \leq C \left(\max_{1 \leq i \leq n-1} |x_i|\right)^{-n} \quad (2.1)$$

and

$$|x_n| \leq C' \left(\max_{1 \leq i \leq n-1} |x_i|\right) (\log \left(\max_{1 \leq i \leq n-1} |x_i|\right))^{-\nu}; \quad (2.2)$$

(P₂): does there exist infinitely many $(n+1)$ -tuples $(x_0, x_1, \dots, x_n) \in \mathbb{Z}^{n+1}$, with $|x_1| > 1$, such that

$$|x_0 + x_1\omega_1 + \dots + x_n\omega_n| \leq C |x_1|^{-n} \quad (2.1')$$

and

$$|x_i| \leq C' |x_1| (\log |x_1|)^{-\nu}, \quad 2 \leq i \leq n. \quad (2.3)$$

By comparison with (1.3) or with Schmidt's Theorem, one can also consider the multiplicative problem:

(P₃): does there exist infinitely many n -tuples $(x_1, \dots, x_n) \in \mathbb{Z}^n$ such that

$$\|x_1\omega_1 + \dots + x_n\omega_n\| \prod_{i=1}^n \max\{1, |x_i|\} (\log \prod_{i=1}^n \max\{1, |x_i|\})^\nu \leq C. \quad (2.4)$$

Given integers x_1, \dots, x_n , not all zero, and a real number r with $0 < r \leq 1/2$, the set $A(x_1, \dots, x_n, r)$ of $(\omega_1, \dots, \omega_n) \in \mathbb{R}/\mathbb{Z}$ such that $\|x_1\omega_1 + \dots + x_n\omega_n\| \leq r$, has measure (for the Haar measure):

$$\mu(A(x_1, \dots, x_n, r)) = 2r.$$

Now it is easy to see that the series $\sum_{(x_1, \dots, x_n)} (\max_{1 \leq i \leq n-1} |x_i|)^{-n}$, where (x_1, \dots, x_n) runs among the n -tuples in \mathbb{Z}^n with $\max_{1 \leq i \leq n-1} |x_i| > 1$, satisfying (2.2), is convergent for $\nu > 1$, since we are led to the convergent series $\sum_{m \geq 2} m^{-1} (\log m)^{-\nu}$. Hence it follows from the usual Borel-Cantelli Lemma that for almost all $(\omega_1, \dots, \omega_n) \in \mathbb{R}^n$ (in the sense of the Lebesgue measure), there are only finitely many $(x_0, x_1, \dots, x_n) \in \mathbb{Z}^{n+1}$ satisfying (2.1) and (2.2), with $\nu > 1$.

In [20] (page 162, Theorem 1), we find the following result:

Theorem 2.1 ([20]) *Let n be an integer with $n \geq 2$. For every primitive vector $(x_1, \dots, x_n) \in \mathbf{Z}^n$, let $A(x_1, \dots, x_n)$ be a measurable subset of \mathbf{R}/\mathbf{Z} . Suppose that*

$$\sum_{(x_1, \dots, x_n)=1} \mu(A(x_1, \dots, x_n)) = +\infty.$$

Then for almost all $(\omega_1, \dots, \omega_n) \in \mathbf{R}^n$, in the sense of Lebesgue measure, there are infinitely many primitive vectors $(x_1, \dots, x_n) \in \mathbf{Z}^n$ such that

$$x_1\omega_1 + \dots + x_n\omega_n \in A(x_1, \dots, x_n) \pmod{1}.$$

It is easy to see that the series $\sum x_1^{-n}$ where (x_1, \dots, x_n) runs among the primitive vectors of \mathbf{Z}^n with $\max_{1 \leq i \leq n-1} |x_i| = x_1 > 1$, satisfying (2.2), is divergent for $\nu = 1$. Indeed, in order to ensure that (x_1, \dots, x_n) is primitive, it is enough to take x_1 and x_n coprime. Thus, using the Euler function φ , and recalling that

$$\sum_{1 \leq k \leq K} \frac{\varphi(k)}{k} \sim \frac{6}{\pi^2} K, \quad (2.5)$$

we are led to the sum

$$\sum_{x_1 \geq 2} x_1^{-2} \sum_{1 \leq x_n \leq x_1 / \log x_1} \varphi(x_n) / x_n \gg \sum_{x_1 \geq 2} x_1^{-1} (\log x_1)^{-1} = +\infty.$$

Accordingly, for almost all $(\omega_1, \dots, \omega_n) \in \mathbf{R}^n$, there are infinitely many vectors $(x_1, \dots, x_n) \in \mathbf{Z}^n$ satisfying (2.1) and (2.2), with $\nu = 1$. One can thus expect the value $\kappa = 1$ in Theorem 1.5, and Theorem 1.4 presumably holds true when replacing condition (1.7) by

$$|x_n| \ll M(\log M)^{-1}.$$

Similarly the series $\sum |x_1|^{-n}$ where (x_1, \dots, x_n) runs among the vectors of \mathbf{Z}^n with $|x_1| > 1$, satisfying (2.3), is convergent if $\nu > 1/(n-1)$, and divergent for $\nu = 1/(n-1)$ (even when we restrict ourselves to primitive vectors). Hence, for $\nu > 1/(n-1)$ (respectively $\nu = 1/(n-1)$), for almost all $(\omega_1, \dots, \omega_n) \in \mathbf{R}^n$, there are finitely (respectively infinitely) many $(x_1, \dots, x_n) \in \mathbf{Z}^n$ satisfying (2.1') and (2.3). The expected value of the constant λ in Theorem 1.6 is thus $\lambda = 1/(n-1)$.

One can also study problem (P₃) by the same method. Although the convergence case follows from a much more general result in [4], let us give a short proof of the following assertion: for $\nu > n$, one has

$$S = \sum_{\substack{\max_{1 \leq i \leq n} |x_i| \geq 2 \\ 1 \leq i \leq n}} \left(\prod_{1 \leq i \leq n} \max\{1, |x_i|\} \right)^{-1} (\log \prod_{1 \leq i \leq n} \max\{1, |x_i|\})^{-\nu} < +\infty.$$

Indeed, considering for non-negative integers (m_1, \dots, m_n) with $m_1 + \dots + m_n \geq 1$, the n -tuples $(x_1, \dots, x_n) \in \mathbf{Z}^n$ with

$$4^{m_i} \leq \max\{1, |x_i|\} < 4^{m_i+1},$$

we see that

$$S \ll \sum_{m_1 + \dots + m_n \geq 1} (m_1 + \dots + m_n)^{-\nu} < \infty.$$

Thus, if $\nu > n$, then for almost all $(\omega_1, \dots, \omega_n) \in \mathbb{R}^n$, there are only finitely many $(x_1, \dots, x_n) \in \mathbb{Z}^n$ satisfying (2.4).

Now, consider the sum

$$S' = \sum_{(x_1, \dots, x_n)=1} \prod_{1 \leq i \leq n} x_i^{-1} (\log \prod_{1 \leq i \leq n} x_i)^{-n},$$

where $x_i \geq 1$ and $\prod_{1 \leq i \leq n} x_i > 1$. We deduce from (2.5) that

$$\sum_{4^{m_1} \leq x_1 < 4^{m_1+1}} \sum_{4^{m_2} \leq x_2 < 4^{m_2+1}, (x_1, x_2)=1} 1 \gg 4^{m_1+m_2}.$$

We thus have

$$\sum_{4^{m_i} \leq x_i < 4^{m_i+1}, (x_1, \dots, x_n)=1} 1 \gg 4^{m_1+\dots+m_n},$$

hence we get

$$S' \gg \sum_{m_1+\dots+m_n \geq 1} (m_1 + \dots + m_n)^{-n} = +\infty.$$

Accordingly, Sprindzük's Theorem applies, and we conclude that, for almost all $(\omega_1, \dots, \omega_n) \in (\mathbb{R}/\mathbb{Z})^n$, there are infinitely many n -tuples $(x_1, \dots, x_n) \in \mathbb{Z}^n$ satisfying (2.4) with $\nu = n$. Hence, one can expect that for algebraic real numbers $\alpha_1, \dots, \alpha_n$, such that $1, \alpha_1, \dots, \alpha_n$, are linearly independent over \mathbb{Z} , the inequality (2.4) is satisfied by infinitely many $(x_1, \dots, x_n) \in \mathbb{Z}^n$ for $\nu = n$, and only by finitely many if $\nu > n$. Of course, proving such statements is out of reach, however, in Theorem 1.5, the value $\kappa=1$, or maybe any value less than 1, is consistent with the proof that we give (although we are unable to obtain such values). On the other hand, the comparison between metrical results, and results concerning particular numbers, may obviously be wrong. For instance, considering a diophantine inequality in the form

$$\|x_1\omega_1 + \dots + x_n\omega_n\| \leq \psi\left(\max_{1 \leq i \leq n} |x_i|\right), \quad (2.6)$$

where ψ is a non negative monotonic function over \mathbb{N} , Khintchine has proved that for almost all $(\omega_1, \dots, \omega_n) \in \mathbb{R}^n$, there are infinitely many $(x_1, \dots, x_n) \in \mathbb{Z}^n \setminus \{0\}$ satisfying (2.6) whenever $\sum_{m \geq 1} m^{n-1} \psi(m) = +\infty$ (actually, the monotonicity assumption is unnecessary if $n \geq 2$, see for instance [3]; more general results may also be found in [2]). Nevertheless, if $(1, \alpha_1, \dots, \alpha_n)$ is a basis of a real algebraic number field, this conclusion is false when $\psi(m) = o(m^{-n})$, since in this case (1.9) holds.

The logarithmic exponent n for the dual Littlewood conjecture is the same as for the direct conjecture. It was proved by D.C Spencer [19] that for each positive real number $\nu > n$, and for almost all $(\omega_1, \dots, \omega_n) \in \mathbb{R}^n$, one has

$$\lim q(\log q)^\nu \|q\omega_1\| \cdots \|q\omega_n\| = +\infty,$$

and P. Gallagher [12] proved that for almost all $(\omega_1, \dots, \omega_n)$, one has

$$\liminf q(\log q)^n \|q\omega_1\| \cdots \|q\omega_n\| = 0.$$

We refer to [8] for a more precise result. A mixed Littlewood-type problem was also studied, and it was proved in [6] that, given n distinct prime numbers p_1, \dots, p_n , where $n \geq 1$, for almost all $\omega \in \mathbb{R}$, one has

$$\lim q(\log q)^\nu |q|_{p_1} \cdots |q|_{p_n} \|q\omega\| = +\infty$$

when $\nu > n + 1$, and

$$\liminf q(\log q)^{n+1} |q|_{p_1} \cdots |q|_{p_n} \|q\omega\| = 0.$$

About the mixed Littlewood-type problem, analogues of Theorems 1.5 and 1.6 have already been established for quadratic numbers ([15], [13]).

3 Proof of Theorems 1.4 and 1.5.

3.1 Some notations.

Let σ_k ($0 \leq k \leq n$) be the isomorphisms from E into \mathbb{C} , where $\sigma_0 = \text{id}$. Denoting by $r_1 \geq 1$ the number of isomorphisms from E into \mathbb{R} , we can suppose that σ_k is real for $0 \leq k < r_1$. The number of non real isomorphisms from E into \mathbb{C} is an even number $2r_2 \geq 0$ such that $r_1 + 2r_2 = n + 1$, and we may also suppose that for $r_1 \leq k < r_1 + r_2$ we have

$$\overline{\sigma_k(x)} = \sigma_{k+r_2}(x)$$

for every $x \in E$.

Recall that we put $\alpha_0 = 1$. Using the \mathbb{Q} -linear form $\text{Tr} = \sigma_0 + \dots + \sigma_n$ on E , we consider the dual basis $(\beta_0, \dots, \beta_n)$ of $(\alpha_0, \dots, \alpha_n)$ for the bilinear form on $E \times E$, $(x, y) \mapsto \text{Tr}(xy)$. That means that $\text{Tr}(\alpha_j \beta_k) = \delta_{j,k}$, where $\delta_{j,j} = 1$ and $\delta_{j,k} = 0$ if $k \neq j$. If we have an element $x \in E$,

$$x = \sum_{j=0}^n x_j \alpha_j,$$

we can calculate the coordinates $x_j \in \mathbb{Q}$ by

$$x_j = \text{Tr}(x\beta_j) = \sum_{k=0}^n \sigma_k(x\beta_j). \quad (3.1)$$

Let \mathcal{O}_E be the ring of algebraic integers in E . Let D be a positive integer such that $D\alpha_j$ and $D\beta_j$ are in \mathcal{O}_E for each $j = 0, \dots, n$. If $x \in \mathcal{O}_E$, then the number $Dx\beta_j$ is an algebraic integer for each j , hence it follows from (3.1) that $Dx_j \in \mathbb{Z}$. We thus have

$$D\mathcal{O}_E \subset \mathbb{Z} + \dots + \mathbb{Z}\alpha_n \subset \frac{1}{D}\mathcal{O}_E,$$

and similarly,

$$D\mathcal{O}_E \subset \mathbb{Z}\beta_0 + \dots + \mathbb{Z}\beta_n \subset \frac{1}{D}\mathcal{O}_E.$$

It is well known (see [22] for instance) that the units group of E is the product of $\{\pm 1\}$ by a free multiplicative group of rank $r = r_1 + r_2 - 1$. Accordingly there exist units $\epsilon_1, \dots, \epsilon_r$, which are multiplicatively independent, and replacing if necessary ϵ_i by ϵ_i^2 , we can suppose that for each $i = 1, \dots, r$, we have $\sigma_k(\epsilon_i) > 0$ for $0 \leq k < r_1$. Then $N_{E/\mathbb{Q}}(\epsilon_i) = 1$.

3.2 Peck's units.

We shall call a *Peck's system* a set \mathcal{U} of units η in E satisfying the condition

$$|\sigma_j(\eta)| \asymp_{\mathcal{U}} |\sigma_k(\eta)|, \quad 1 \leq j < k \leq n, \quad (3.2)$$

the constants involved in inequalities (3.2) depending upon the set \mathcal{U} . Note that condition (3.2) is also equivalent to

$$|\sigma_k(\eta)| \asymp_{\mathcal{U}} |\eta|^{-1/n}, \quad k = 1, \dots, n. \quad (3.3)$$

First we prove the following Lemma:

Lemma 3.1 *There exist a positive real constant C and a Peck's system \mathcal{U} of units in E such that for every positive real number K , one can find $\eta \in \mathcal{U}$ with*

$$\frac{1}{C}K \leq \eta \leq CK. \quad (3.4)$$

Proof. We shall find η in the form

$$\eta = \epsilon_1^{\mu_1} \dots \epsilon_r^{\mu_r},$$

where $\mu_i \in \mathbb{Z}$. Note that the set of linear equations:

$$\lambda_1 \log \epsilon_1 + \dots + \lambda_r \log \epsilon_r = \log K \quad (3.5)$$

and

$$\lambda_1 \log |\sigma_k(\epsilon_1)| + \dots + \lambda_r \log |\sigma_k(\epsilon_r)| = -\frac{1}{n} \log K, \quad 1 \leq k \leq n, \quad (3.6)$$

has a *real* solution $(\lambda_1, \dots, \lambda_r)$. Indeed, considering the equations (3.6), with $1 \leq k \leq r$, we get a Cramer system, since the determinant $\det(\log |\sigma_k(\epsilon_i)|)_{\substack{1 \leq i \leq r \\ 1 \leq k \leq r}}$ is not zero. Accordingly, these equations have a real solution $(\lambda_1, \dots, \lambda_r)$. Now for $r+1 = r_1 + r_2 \leq k \leq n$, we have $|\sigma_k(\epsilon_i)| = |\sigma_{k-r_2}(\epsilon_i)|$, hence $(\lambda_1, \dots, \lambda_r)$ satisfies the equations (3.6) for each $1 \leq k \leq n$. Since $N_{E/\mathbb{Q}}(\epsilon_k) = 1$, we have

$$\log \epsilon_i = -\sum_{k=1}^n \log |\sigma_k(\epsilon_i)|,$$

and thus we obtain (3.5) by adding the equations (3.6), for $1 \leq k \leq n$. Then choosing integers μ_k such that $\lambda_k - 1/2 \leq \mu_k < \lambda_k + 1/2$, we get

$$\mu_1 \log \epsilon_1 + \dots + \mu_r \log \epsilon_r = \log K + O(1)$$

and

$$\mu_1 \log |\sigma_k(\epsilon_1)| + \dots + \mu_r \log |\sigma_k(\epsilon_r)| = -\frac{1}{n} \log K + O(1)$$

for $1 \leq k \leq n$, which lead to (3.3) and (3.4). □

We shall then describe the solutions of

$$|x_0 + x_1 \alpha_1 + \dots + x_n \alpha_n| \ll \left(\max_{0 \leq j \leq n} |x_j| \right)^{-n}, \quad (3.7)$$

where the x_j 's are integers, not all zero.

Lemma 3.2 *Let γ be a non-zero number in \mathcal{O}_E and let \mathcal{U} be a Peck's system of units in E with*

$$|\eta| \ll 1 \quad (3.8)$$

for each $\eta \in \mathcal{U}$. Then the $(n+1)$ -tuples of integers (x_0, \dots, x_n) such that

$$D\gamma\eta = x_0 + x_1\alpha_1 + \dots + x_n\alpha_n,$$

where $\eta \in \mathcal{U}$, satisfy (3.7). Conversely, let \mathcal{S} be a set of $(n+1)$ -tuples of integers $(x_0, \dots, x_n) \neq (0, \dots, 0)$ satisfying (3.7), and let \mathcal{U} be a Peck's system of units in E satisfying Lemma 3.1. Then there exists a finite set Γ of non zero elements $\gamma \in E$ such that for each element $(x_0, \dots, x_n) \in \mathcal{S}$, we can write

$$x_0 + \dots + x_n\alpha_n = \gamma\eta,$$

with $\gamma \in \Gamma$ and $\eta \in \mathcal{U}$. Further $D\gamma$ lies in \mathcal{O}_E .

Proof. If $\gamma \in \mathcal{O}_E$, then $\gamma\eta \in \mathcal{O}_E$ for every unit η of E , hence we can write

$$D\gamma\eta = x_0 + x_1\alpha_1 + \dots + x_n\alpha_n,$$

where $x_j \in \mathbb{Z}$. If we have a set of units η satisfying (3.3) and (3.8), we get

$$|\sigma_k(D\gamma\eta\beta_j)| \ll |\eta|^{-1/n}, \quad j = 0, \dots, n, \quad k = 1, \dots, n,$$

hence by (3.1) and (3.8),

$$|x_j| \ll |\eta|^{-1/n},$$

which leads immediately to (3.7).

Conversely, if a set \mathcal{S} of $(n+1)$ -tuples $(x_0, \dots, x_n) \in \mathbb{Z}^{n+1} \setminus \{0\}$ satisfies (3.7), let us consider

$$x = x_0 + x_1\alpha_1 + \dots + x_n\alpha_n.$$

As \mathcal{U} satisfies (3.4), we can find a unit $\eta \in \mathcal{U}$ such that

$$|x| \asymp \eta,$$

and therefore,

$$|x|^{-1/n} \asymp |\sigma_k(\eta)|, \quad 1 \leq k \leq n.$$

Set then $\gamma = x\eta^{-1}$. We have

$$|\gamma| \ll 1.$$

Moreover, by (3.7), we have for each $k = 1, \dots, n$,

$$|\sigma_k(x)| \ll \max_{0 \leq j \leq n} |x_j| \ll_{\mathcal{S}} |x|^{-1/n} \ll_{\mathcal{S}} |\sigma_k(\eta)|,$$

hence

$$|\sigma_k(\gamma)| \ll_{\mathcal{S}} 1.$$

Further $Dx \in \mathcal{O}_E$, hence $D\gamma$ is an algebraic integer. Now, there exist only a finite number of algebraic integers $\gamma' = D\gamma$ in E such that $|\sigma_k(\gamma')| \ll_{\mathcal{S}} 1$ for each $k = 0, \dots, n$, thus Lemma 3.2 is proved. □

3.3 Some lemmas.

Lemma 3.3 *Let η_m be a sequence of units in E such that*

$$\lim \eta_m = 0. \quad (3.9)$$

Suppose that for each $k = 1, 2, \dots, n$, the sequence $\sigma_k(\eta_m)/|\sigma_1(\eta_m)|$ has a limit ℓ_k in \mathbb{C} , with $\ell_k \neq 0$. Let $\gamma \neq 0$ be a number in E , and suppose that

$$\sum_{k=1}^n \sigma_k(\gamma\beta_n)\ell_k = 0. \quad (3.10)$$

Then, setting

$$\gamma\eta_m = x_{0,m} + \dots + x_{n,m}\alpha_n,$$

we have

$$|x_{k,m}| \ll |\eta_m|^{-1/n}, \quad 0 \leq k < n,$$

and

$$|x_{n,m}| = o(|\eta_m|^{-1/n}).$$

Proof. The set of units η_m satisfying condition (3.2), it is a Peck's system, and by (3.9), condition (3.8) is also satisfied. Then Lemma 3.2 ensures that $|x_{k,m}| \ll |\eta_m|^{-1/n}$ for each $0 \leq k \leq n$. Moreover, as we have, by (3.1),

$$x_{n,m} = \text{Tr}\gamma\eta_m\beta_n = \gamma\eta_m\beta_n + \sum_{k=1}^n \sigma_k(\gamma\beta_n)\sigma_k(\eta_m),$$

we get

$$|x_{n,m}| \ll |\eta_m|^{-1/n} \left| \sum_{k=1}^n \sigma_k(\gamma\beta_n) \frac{\sigma_k(\eta_m)}{|\sigma_1(\eta_m)|} \right| + |\eta_m|. \quad (3.11)$$

□

Note that in Lemma 3.3 it is enough that $\sigma_k(\eta_m)/|\sigma_1(\eta_m)|$ has a non-zero limit ℓ_k for $1 \leq k \leq r_1 + r_2 - 1$. Indeed, for $r_1 + r_2 \leq k \leq n$, $\sigma_k(\eta_m)$ and $\sigma_{k-r_2}(\eta_m)$ being conjugate complex numbers, we have

$$\ell_k = \overline{\ell_{k-r_2}}.$$

Accordingly condition (3.10) may be written:

$$\sum_{1 \leq k < r_1} \sigma_k(\gamma\beta_n)\ell_k + 2 \sum_{r_1 \leq k \leq r_1+r_2-1} \Re(\sigma_k(\gamma\beta_n)\ell_k) = 0. \quad (3.12)$$

Then, using the determination of $\arg z$ such that $\arg z \in [0, 2\pi[$ for any non zero complex number z , we define the vectors $\underline{v}_j = (v_{j,1}, v_{j,2}, \dots, v_{j,n}) \in \{0\} \times \mathbb{R}^{n-1}$, for $1 \leq j \leq n$, as follows. Recall that $r = r_1 + r_2 - 1$, and set

$$v_{j,k} = \log |\sigma_k(\epsilon_j)| - \log |\sigma_1(\epsilon_j)|, \quad 1 \leq j \leq r, 1 \leq k \leq r, \quad (3.13)$$

$$v_{j,k} = \arg \sigma_k(\epsilon_j), \quad 1 \leq j \leq r, r < k \leq n, \quad (3.14)$$

$$v_{j,k} = 2\delta_{j,k}\pi, \quad r < j \leq n, \quad 1 \leq k \leq n \quad (3.15)$$

($\delta_{j,j} = 1$ and $\delta_{j,k} = 0$ if $j \neq k$). Note that we have

$$\log |\sigma_k(\epsilon_j)| - \log |\sigma_1(\epsilon_j)| = v_{j,k-r_2}, \quad 1 \leq j \leq r, \quad r_1 + r_2 \leq k \leq n,$$

and

$$\arg \sigma_k(\epsilon_j) \equiv -v_{j,k+r_2} \pmod{2\pi}, \quad 1 \leq j \leq r, \quad r_1 \leq k < r_1 + r_2.$$

It will be useful to note that if we have integers $\lambda_1, \dots, \lambda_n$, and if we put

$$\epsilon = \prod_{j=1}^r \epsilon_j^{\lambda_j},$$

then we deduce from formulae (3.13), (3.14) and (3.15), that

$$\log \frac{|\sigma_k(\epsilon)|}{|\sigma_1(\epsilon)|} = \sum_{0 \leq j \leq r} \lambda_j v_{j,k} = \sum_{0 \leq j \leq n} \lambda_j v_{j,k}, \quad 1 \leq k \leq r,$$

and

$$\arg \sigma_k(\epsilon) \equiv - \sum_{0 \leq j \leq r} \lambda_j v_{j,k+r_2} \equiv - \sum_{0 \leq j \leq n} \lambda_{j,k} v_{j,k+r_2} \pmod{2\pi}, \quad r_1 \leq k \leq r.$$

Hence we have

$$\frac{\sigma_k(\epsilon)}{|\sigma_1(\epsilon)|} = e^{\sum_{0 \leq j \leq n} \lambda_j v_{j,k}}, \quad 1 \leq k < r_1, \quad (3.16)$$

$$\frac{\sigma_k(\epsilon)}{|\sigma_1(\epsilon)|} = e^{\sum_{0 \leq j \leq n} \lambda_j (v_{j,k} - i v_{j,k+r_2})}, \quad r_1 \leq k < r_1 + r_2, \quad (3.17)$$

and

$$\frac{\sigma_k(\epsilon)}{|\sigma_1(\epsilon)|} = e^{\sum_{0 \leq j \leq n} \lambda_j (v_{j,k-r_2} + i v_{j,k})}, \quad r_1 + r_2 \leq k \leq n, \quad (3.18)$$

Let us first prove:

Lemma 3.4 *The vectors \underline{V}_j , for $1 \leq j \leq n$, are \mathbb{Z} -linearly independent.*

Proof. Suppose that

$$\lambda_1 \underline{V}_1 + \dots + \lambda_n \underline{V}_n = 0,$$

where the λ_j 's are rational integers. Set

$$\epsilon = \prod_{j=1}^r \epsilon_j^{\lambda_j}.$$

It follows from (3.16), (3.17) and (3.18), that we have for each $k = 1, \dots, n$,

$$\sigma_k(\epsilon) = \sigma_1(\epsilon).$$

Indeed, if $r_1 > 1$, then $\sigma_1(\epsilon) > 0$, and if $r_1 = 1$, then (3.17) implies that $|\sigma_1(\epsilon)| = \sigma_1(\epsilon)$. We conclude that ϵ is a rational number, and since it is a positive unit, we have $\epsilon = 1$. As $\epsilon_1, \dots, \epsilon_r$, are multiplicatively independent, that proves that

$$\lambda_1 = \dots = \lambda_r = 0.$$

Then we have

$$\lambda_{r+1}\underline{V}_{r+1} + \dots + \lambda_n\underline{V}_n = 0,$$

and in particular,

$$\lambda_{r+1}v_{r+1,k} + \dots + \lambda_nv_{n,k} = 0,$$

for $k = r + 1, \dots, n$. Hence we get

$$\lambda_{r+1} = \dots = \lambda_n = 0.$$

□

It is well known that:

Lemma 3.5 *If n vectors $\underline{V}_j \in \{0\} \times \mathbf{R}^{n-1}$ are linearly independent over \mathbf{Z} , then the subgroup $\mathbf{Z}\underline{V}_1 + \dots + \mathbf{Z}\underline{V}_n$ of $\{0\} \times \mathbf{R}^{n-1}$ is not discrete.*

Lemma 3.6 *There exists a sequence u_m of units in E satisfying the conditions*

$$\lim_m u_m = 0 \tag{3.9}$$

and

$$\lim_m \frac{\sigma_k(u_m)}{|\sigma_1(u_m)|} = 1, \quad k = 1, \dots, n. \tag{3.19}$$

Proof. It follows from Lemmas 3.4 and 3.5 that there exists a sequence $\mu_{1,m}\underline{V}_1 + \dots + \mu_{n,m}\underline{V}_n$ of all distinct vectors in $\mathbf{Z}\underline{V}_1 + \dots + \mathbf{Z}\underline{V}_n$ which tends to 0. Set:

$$u_m = \epsilon_1^{\mu_{1,m}} \dots \epsilon_r^{\mu_{r,m}}.$$

By formulae (3.16) and (3.17), we have

$$\frac{\sigma_k(u_m)}{|\sigma_1(u_m)|} = e^{\sum_{1 \leq j \leq n} \mu_{j,m} v_{j,k}}, \quad 1 \leq k < r_1,$$

and

$$\frac{\sigma_k(u_m)}{|\sigma_1(u_m)|} = e^{\sum_{1 \leq j \leq n} \mu_{j,m} (v_{j,k} - i v_{j,k+r_2})}, \quad r_1 \leq k \leq r,$$

hence condition (3.19) is satisfied. Moreover, observe that the group $\mathbf{Z}\underline{V}_{r+1} + \dots + \mathbf{Z}\underline{V}_n$ is discrete, since it follows from (3.15) that

$$\|\nu_{r+1}\underline{V}_{r+1} + \dots + \nu_n\underline{V}_n\|_\infty = 2\pi \max_{r+1 \leq k \leq n} |\nu_j|.$$

Hence the r -tuples $(\mu_{1,m}, \dots, \mu_{r,m})$ are all distinct for large m , since if we have $\mu_{j,m} = \mu_{j,p}$ for each $j = 1, \dots, r$, and $\mu_{j,m} \neq \mu_{j,p}$ for at least one index j with $r < j \leq n$, then we get

$$\|(\mu_{1,m} - \mu_{1,p})\underline{V}_1 + \dots + (\mu_{n,m} - \mu_{n,p})\underline{V}_n\| \geq 2\pi.$$

Thus $\max_{1 \leq j \leq r} |\mu_{j,m}|$ tends toward infinity. As the matrix $(\log |\sigma_k(\epsilon_j)|)_{\substack{1 \leq k \leq r \\ 1 \leq j \leq r}}$ is invertible, we have

$$\lim_m \max_{1 \leq k \leq r} |\log |\sigma_k(u_m)|| = +\infty,$$

and in view of (3.19), we get

$$\lim_m |\log |\sigma_1(u_m)|| = +\infty.$$

Replacing, if necessary, for some m , $(\mu_{1,m}, \dots, \mu_{n,m})$ by $-(\mu_{1,m}, \dots, \mu_{n,m})$, that is to say u_m by u_m^{-1} , we may suppose that

$$\lim_m |\sigma_1(u_m)| = +\infty.$$

Now condition (3.19) implies that

$$\lim_m |\sigma_k(u_m)| = +\infty, \quad (3.20)$$

for each $k = 1, \dots, n$. As

$$u_m \prod_{1 \leq k \leq n} \sigma_k(u_m) = 1,$$

condition (3.9) follows then from (3.20). □

We shall also use the well-known fact:

Lemma 3.7 *For each $z \in E$, set*

$$\begin{aligned} \tilde{\sigma}_k(z) &= \sigma_k(z), & 0 \leq k < r_1, \\ \tilde{\sigma}_k(z) &= \Re(\sigma_k(z)), & r_1 \leq k < r_1 + r_2, \end{aligned}$$

and

$$\tilde{\sigma}_k(z) = \Im(\sigma_k(z)), \quad r_1 + r_2 \leq k \leq n.$$

Define a map σ from E to \mathbb{R}^{n+1} by

$$\sigma(z) = (\tilde{\sigma}_0(z), \dots, \tilde{\sigma}_n(z)).$$

Then $\sigma(E)$ is everywhere dense in \mathbb{R}^{n+1} .

Proof. Consider the linear forms f_k on \mathbb{R}^{n+1} such that

$$f_k(y_0, \dots, y_n) = y_0 \tilde{\sigma}_k(\alpha_0) + \dots + y_n \tilde{\sigma}_k(\alpha_n), \quad 0 \leq k \leq n,$$

and define the linear application of \mathbb{R}^{n+1} into itself, $f = (f_0, \dots, f_n)$. This is an automorphism of \mathbb{R}^{n+1} since the matrix $(\sigma_k(\alpha_j))_{\substack{0 \leq k \leq n \\ 0 \leq j \leq n}}$ is invertible, its inverse matrix being the matrix $(\sigma_k(\beta_j))_{\substack{0 \leq j \leq n \\ 0 \leq k \leq n}}$.

Hence $\sigma(E) = f(\mathbb{Q}^{n+1})$ is everywhere dense in \mathbb{R}^{n+1} . □

Then we prove that:

Lemma 3.8 *There exist a number $\delta \neq 0$ in E , and a point $w = (w_1, w_2, \dots, w_n)$, with $w_1 = 0$, in the closure of $\mathbb{Z}V_1 + \dots + \mathbb{Z}V_n$ in \mathbb{R}^n such that*

$$\sum_{1 \leq k < r_1} \sigma_k(\delta) e^{w_k} + 2 \sum_{r_1 \leq k \leq r_1 + r_2 - 1} \Re(\sigma_k(\delta) e^{w_k - i w_{k+r_2}}) = 0. \quad (3.21)$$

Proof. By Lemmas 3.4 and 3.5, the closure of $\mathbb{Z}V_1 + \dots + \mathbb{Z}V_n$ in \mathbb{R}^n is a non-discrete closed subgroup, included in the closed subspace $\{0\} \times \mathbb{R}^{n-1}$. Accordingly this subgroup contains a subspace $\mathbb{R}V$, where $V = (v_1, v_2, \dots, v_n) = (0, v_2, \dots, v_n)$ is a non zero vector of $\{0\} \times \mathbb{R}^{n-1}$ ([11], [5]). For each real number t , consider the \mathbb{R} -linear form over \mathbb{R}^n

$$\Phi_t(X_1, \dots, X_n) = \sum_{1 \leq k < r_1} X_k e^{tv_k} + 2 \sum_{r_1 \leq k \leq r_1 + r_2 - 1} \Re((X_k + iX_{k+r_2})e^{t(v_k - iv_{k+r_2})}).$$

It is easy to see that there are real values t_0 and t_1 such that the linear forms Φ_{t_0} and Φ_{t_1} are not proportional. Indeed, as $v_1 = 0$, if these linear forms were proportional each to the others, they would be all equal, which is impossible since one at least of the v_k 's is not zero. We may thus find real numbers t_0 and t_1 , and $(\delta_1, \dots, \delta_n) \in \mathbb{R}^n$ such that $\Phi_{t_0}(\delta_1, \dots, \delta_n) < 0$ and $\Phi_{t_1}(\delta_1, \dots, \delta_n) > 0$. Now, by Lemma 3.7, there exists $\delta \in E$ such that $\Phi_{t_0}(\tilde{\sigma}_1(\delta), \dots, \tilde{\sigma}_n(\delta)) < 0$ and $\Phi_{t_1}(\tilde{\sigma}_1(\delta), \dots, \tilde{\sigma}_n(\delta)) > 0$. Accordingly, we have $\delta \neq 0$, and there exists $t \in \mathbb{R}$ such that $\Re h_t(\tilde{\sigma}_1(\delta), \dots, \tilde{\sigma}_n(\delta)) = 0$. That means that

$$\sum_{1 \leq k < r_1} \sigma_k(\delta) e^{tv_k} + 2 \sum_{r_1 \leq k \leq r_1 + r_2 - 1} \Re(\sigma_k(\delta) e^{t(v_k - iv_{k+r_2})}) = 0.$$

As tv lies in the closure of $\mathbb{Z}V_1 + \dots + \mathbb{Z}V_n$, Lemma 3.8 is proved. \square

3.4 Proof of Theorem 1.4.

By Lemma 3.8, we can find a point $\underline{W} = (w_1, \dots, w_n)$ in the closure of $\mathbb{Z}V_1 + \dots + \mathbb{Z}V_n$ in \mathbb{R}^n , and $\delta \in E$ with $\delta \neq 0$, such that (3.21) is satisfied. Setting $\gamma = \delta/\beta_n$, we thus have

$$\sum_{1 \leq k < r_1} \sigma_k(\gamma\beta_n) e^{w_k} + 2 \sum_{r_1 \leq k \leq r_1 + r_2 - 1} \Re(\sigma_k(\gamma\beta_n) e^{w_k - iw_{k+r_2}}) = 0. \quad (3.22)$$

Moreover, as we can replace γ by $\Delta\gamma$, where Δ is a non-zero integer, we can suppose that γ lies in $D\mathcal{O}_E$. There exists a sequence of integer n -tuples $(\lambda_{1,m}, \dots, \lambda_{n,m})_{m \in \mathbb{N}}$ such that

$$\underline{W} = \lim_{m \rightarrow +\infty} \lambda_{1,m} \underline{V}_1 + \dots + \lambda_{n,m} \underline{V}_n.$$

If we set

$$\eta_m = \prod_{1 \leq j \leq r} \epsilon_j^{\lambda_{j,m}},$$

it follows from formulae (3.16) and (3.17) that

$$\lim_{m \rightarrow +\infty} \frac{\sigma_k(\eta_m)}{|\sigma_1(\eta_m)|} = e^{w_k}, \quad 1 \leq k < r_1, \quad (3.23)$$

and

$$\lim_{m \rightarrow +\infty} \frac{\sigma_k(\eta_m)}{|\sigma_1(\eta_m)|} = e^{w_k - iw_{k+r_2}}, \quad r_1 \leq k \leq r_1 + r_2 - 1. \quad (3.24)$$

These conditions are unchanged if we replace η_m by $\eta_m u_{h(m)}$, where u_m is a sequence of units satisfying conditions (3.9) and (3.19) of Lemma 3.6, and h is a strictly increasing map of the set \mathbb{N} of natural integers into itself. Choosing $h(m)$ sufficiently large, we can suppose that $\lim_m \eta_m u_{h(m)} = 0$,

and thus, replacing if necessary η_m by $\eta_m u_{h(m)}$, we have constructed a sequence of units η_m satisfying conditions (3.23), (3.24) and (3.9). Then, by (3.22), Lemma 3.3 applies. Setting

$$\gamma\eta_m = x_{0,m} + x_{1,m}\alpha_1 + \dots + x_{n,m}\alpha_n,$$

with integers x_0, \dots, x_n , we get

$$|x_{k,m}| \ll |\eta_m|^{-1/n}, \quad 0 \leq k < n,$$

and

$$|x_{n,m}| = o(|\eta_m|^{-1/n}).$$

Thus Theorem 1.4 is proved.

3.5 Proof of Theorem 1.5

First consider the case of a totally real cubic field E . We keep the notations as above, with $n = 2$. Let γ be a non zero number in $D\mathcal{O}_E$. Consider a sequence of Peck's units $\eta_m > 0$ satisfying (3.9). If we write

$$\gamma\eta_m = x_{0,m} + x_{1,m}\alpha_1 + x_{2,m}\alpha_2$$

then, by Lemma 3.2, we have

$$|x_{1,m}| \ll \eta_m^{-1/2}$$

and (3.11) can then be rewritten:

$$|x_{2,m}| \ll \eta_m^{-1/2} \left| \frac{\sigma_2(\eta_m)}{\sigma_1(\eta_m)} + \frac{\sigma_1(\gamma)\sigma_1(\beta_2)}{\sigma_2(\gamma)\sigma_2(\beta_2)} \right| + \eta_m. \quad (3.11)'$$

First, we choose $\gamma > 0$ in such a way that

$$\frac{\sigma_1(\gamma)\sigma_1(\beta_2)}{\sigma_2(\gamma)\sigma_2(\beta_2)} < 0,$$

which is possible by Lemma 3.7. We shall find a sequence of units $\eta_m = \epsilon_1^{\mu_{1,m}} \epsilon_2^{\mu_{2,m}}$ satisfying the conditions

$$\lim \eta_m = 0 \quad (3.9)$$

and

$$\left| \frac{\sigma_2(\eta_m)}{\sigma_1(\eta_m)} + \frac{\sigma_1(\gamma)\sigma_1(\beta_2)}{\sigma_2(\gamma)\sigma_2(\beta_2)} \right| \ll |\log \eta_m|^{-\kappa}. \quad (3.25)$$

Condition (3.25) will be satisfied if

$$\left| \log \frac{\sigma_2(\eta_m)}{\sigma_1(\eta_m)} - \log \frac{-\sigma_1(\gamma)\sigma_1(\beta_2)}{\sigma_2(\gamma)\sigma_2(\beta_2)} \right| \ll |\log \eta_m|^{-\kappa}.$$

Conditions (3.9) and (3.25) may thus be written

$$\left| \mu_{1,m} \log \frac{\sigma_2(\epsilon_1)}{\sigma_1(\epsilon_1)} + \mu_{2,m} \log \frac{\sigma_2(\epsilon_2)}{\sigma_1(\epsilon_2)} - \log \frac{-\sigma_1(\gamma)\sigma_1(\beta_2)}{\sigma_2(\gamma)\sigma_2(\beta_2)} \right| \ll |\log \eta_m|^{-\kappa}, \quad (3.26)$$

with

$$\log \eta_m = \mu_{1,m} \log \epsilon_1 + \mu_{2,m} \log \epsilon_2 \rightarrow -\infty . \quad (3.27)$$

In order to find such sequences of integers $\mu_{1,m}, \mu_{2,m}$, we shall use the following ‘‘transference’’ result:

Lemma 3.9 *Let θ be a real number. Suppose that there exists a real constant $\nu \geq 1$ such that*

$$\|q\theta\| \geq Cq^{-\nu} \quad (3.28)$$

holds for every positive integer q . Let ν' be a real number with $0 < \nu' < 1/\nu$. Then for every real number t and for each large real number X , there exists a positive integer q with $q \asymp X$ and

$$\|q\theta - t\| \ll X^{-\nu'} .$$

This lemma is actually true with $\nu' = 1/\nu$. It is classical when $\nu = 1$ (see [9] for instance). Also a proof of this lemma is given in the book of Y. Meyer [16] in the case $\nu = 1$ (then $\nu' = 1$, Theorem V, page 10), and it is easy to adapt this proof for any $\nu \geq 1$ (with $\nu' = 1/\nu$). Nevertheless Lemma 3.9 follows directly from a more general result of [7] (take $m = n = 1$ in [7], Theorem page 2). This result implies that for $0 < \nu' < 1/\nu$, for every real number t and each large real number X , there exists $x \in \mathbf{Z}$, with $|x| \leq X$ and $\|x\theta - t\| \leq X^{-\nu'}$. Since in Lemma 3.9, we need $q > 0$, and $q \asymp X$, let us consider the largest denominator of convergent of θ , Q , such that $Q \leq X$. If Q' is the denominator of the next convergent, then we have $CQ^{-\nu} \leq \|Q\theta\| < 1/Q'$, hence $Q \leq X < Q' \leq C^{-1}Q^\nu$. Let k be the smallest positive integer such that $kQ > X$, hence $k \geq 2$ and $(k-1)Q \leq X < kQ \leq 2X$. If we consider $q = x + 2kQ$, then we have

$$X \leq q \leq 5X .$$

As

$$\|2kQ\theta\| \leq 2k/Q' \leq 4X/(QQ') \leq 4/Q \leq 4C^{-1/\nu} X^{-1/\nu} ,$$

we get

$$\|q\theta - t\| \leq (4C^{-1/\nu} + 1)X^{-\nu'} .$$

Obviously, replacing t by $-t$, we may also obtain $-5X \leq q \leq -X$ in Lemma 3.9. □

Now it follows from Lemma 3.4 that $\log(\sigma_2(\epsilon_1)/\sigma_1(\epsilon_1))$ and $\log(\sigma_2(\epsilon_2)/\sigma_1(\epsilon_2))$ are linearly independent over \mathbf{Z} . Then by classical results about linear forms in logarithms of algebraic numbers [1], [21], there exists a constant $\nu \geq 1$ such that for each pair (μ_1, μ_2) of integers not both zero, we have

$$\left| \mu_1 \log \frac{\sigma_2(\epsilon_1)}{\sigma_1(\epsilon_1)} + \mu_2 \log \frac{\sigma_2(\epsilon_2)}{\sigma_1(\epsilon_2)} \right| \gg \max\{|\mu_1|, |\mu_2|\}^{-\nu} ,$$

hence

$$\left\| \begin{array}{c} \log \frac{\sigma_2(\epsilon_1)}{\sigma_1(\epsilon_1)} \\ q \frac{\log \frac{\sigma_2(\epsilon_1)}{\sigma_1(\epsilon_1)}}{\log \frac{\sigma_2(\epsilon_2)}{\sigma_1(\epsilon_2)}} \\ \log \frac{\sigma_2(\epsilon_2)}{\sigma_1(\epsilon_2)} \end{array} \right\| \gg q^{-\nu} ,$$

for every positive integer q . Accordingly Lemma 3.9 applies. Let κ be real constant with $0 < \kappa < 1/\nu$. If m is a sufficiently large positive integer, then we see that there exists a positive (respectively negative) integer $\mu_{1,m}$ with $|\mu_{1,m}| \asymp m$ such that

$$\left| \mu_{1,m} \log \frac{\sigma_2(\epsilon_1)}{\sigma_1(\epsilon_1)} + \mu_{2,m} \log \frac{\sigma_2(\epsilon_2)}{\sigma_1(\epsilon_2)} - \log \frac{-\sigma_1(\gamma)\sigma_1(\beta_k)}{\sigma_2(\gamma)\sigma_2(\beta_k)} \right| \ll m^{-\kappa}. \quad (3.29)$$

Recall that

$$\begin{vmatrix} \log \sigma_1(\epsilon_1) & \log \sigma_2(\epsilon_1) \\ \log \sigma_1(\epsilon_2) & \log \sigma_2(\epsilon_2) \end{vmatrix} \neq 0.$$

As

$$\log \epsilon_i = -\log \sigma_1(\epsilon_i) - \log \sigma_2(\epsilon_i), \quad i = 1, 2,$$

we get thus

$$\log \epsilon_1 \log \frac{\sigma_2(\epsilon_2)}{\sigma_1(\epsilon_2)} - \log \epsilon_2 \log \frac{\sigma_2(\epsilon_1)}{\sigma_1(\epsilon_1)} \neq 0.$$

As (3.29) implies that

$$\left| \mu_{1,m} \log \frac{\sigma_2(\epsilon_1)}{\sigma_1(\epsilon_1)} + \mu_{2,m} \log \frac{\sigma_2(\epsilon_2)}{\sigma_1(\epsilon_2)} \right| \ll 1,$$

writing

$$\begin{aligned} & (\mu_{1,m} \log \epsilon_1 + \mu_{2,m} \log \epsilon_2) \log \frac{\sigma_2(\epsilon_2)}{\sigma_1(\epsilon_2)} - \left(\mu_{1,m} \log \frac{\sigma_2(\epsilon_1)}{\sigma_1(\epsilon_1)} + \mu_{2,m} \log \frac{\sigma_2(\epsilon_2)}{\sigma_1(\epsilon_2)} \right) \log \epsilon_2 = \\ & \mu_{1,m} \left(\log \epsilon_1 \log \frac{\sigma_2(\epsilon_2)}{\sigma_1(\epsilon_2)} - \log \epsilon_2 \log \frac{\sigma_2(\epsilon_1)}{\sigma_1(\epsilon_1)} \right), \end{aligned}$$

we see that for m large, we can choose the sign of $\mu_{1,m}$ in (3.29) in a such way that

$$\mu_{1,m} \log \epsilon_1 + \mu_{2,m} \log \epsilon_2 < 0$$

and

$$-(\mu_{1,m} \log \epsilon_1 + \mu_{2,m} \log \epsilon_2) \asymp m.$$

Hence, setting $\eta_m = \epsilon_{1,m}^{\mu_{1,m}} \epsilon_{2,m}^{\mu_{2,m}}$, we obtain a sequence of units η_m satisfying (3.9) and (3.25). Thus the integers $x_{0,m}$, $x_{1,m}$, and $x_{2,m}$, such that

$$\gamma \eta_m = x_{0,m} + x_{1,m} \alpha_1 + x_{2,m} \alpha_2,$$

satisfy

$$|x_{0,m} + x_{1,m} \alpha_1 + x_{2,m} \alpha_2| \asymp \eta_m,$$

with

$$\begin{aligned} |x_{1,m}| & \ll \eta_m^{-1/2}, \\ |x_{2,m}| & \ll \eta_m^{-1/2} |\log \eta_m|^{-\kappa}, \end{aligned}$$

and

$$-\log \eta_m \asymp m.$$

Theorem 1.5 is thus proved in this case.

The case where the cubic field E can be embedded in a unique way in the real numbers field can be treated in a similar way. As the units group of E has rank 1, we set then $\eta_m = \epsilon_1^{\mu_{1,m}}$. We assume that $0 < \epsilon_1 < 1$, and we take an integer $\mu_{1,m} > 0$. The set of units e^μ , with $\mu \in \mathbb{Z}$, being a Peck's system, considering as above $\gamma\eta_m = x_{0,m} + x_{1,m}\alpha_1 + x_{2,m}\alpha_2$, where γ is a non zero number in $D\mathcal{O}_E$, we have

$$|x_{1,m}| \ll \eta_m^{-1/2},$$

and

$$|x_2| \ll \eta_m^{-1/2} \left| \frac{\overline{\sigma_1(\eta_m)}}{\sigma_1(\eta_m)} + \frac{\sigma_1(\gamma)\sigma_1(\beta_2)}{\sigma_1(\gamma)\sigma_1(\beta_2)} \right| + \eta_m \ll \eta_m^{-1/2} \left\| \frac{1}{\pi} (\arg \sigma_1(\eta_m) + \arg \sigma_1(\gamma\beta_2)) + \frac{1}{2} \right\| + \eta_m,$$

i.e.,

$$|x_2| \ll \eta_m^{-1/2} \left\| \mu_1 \frac{\arg \sigma_1(\epsilon_1)}{\pi} + \frac{\arg \sigma_1(\gamma\beta_2)}{\pi} + \frac{1}{2} \right\| + \eta_m.$$

We then proceed as above. □

4 Proof of Theorem 1.6

Consider an infinite sequence $(x_{0,m}, \dots, x_{n,m})$ of distinct integer $(n+1)$ -tuples such that

$$|x_{0,m} + \dots + x_{n,m}\alpha_n| \max_{0 \leq i \leq n} |x_{i,m}|^n \asymp 1$$

and

$$|x_{j,m}| = o(\max_{0 \leq i \leq n} |x_{i,m}|), \quad 2 \leq j \leq n. \quad (4.1)$$

By Lemma 3.2, we can suppose that there exist $\gamma \in \mathcal{O}_E$, with $\gamma \neq 0$, and a sequence η_m of Peck's units in E , in the form

$$\eta_m = \epsilon_1^{\mu_{1,m}} \dots \epsilon_r^{\mu_{r,m}},$$

where $\mu_{j,m} \in \mathbb{Z}$, with

$$x_{0,m} + x_{1,m}\alpha_1 + \dots + x_{n,m}\alpha_n = \gamma\eta_m.$$

By (3.1), we have

$$x_{j,m} = \text{Tr } \gamma\eta_m\beta_j = \gamma\eta_m\beta_j + \sum_{k=1}^n \sigma_k(\gamma\eta_m\beta_j).$$

As

$$|\sigma_k(\eta_m)| \asymp |\sigma_1(\eta_m)| \asymp \eta_m^{-1/n} \asymp \max_{0 \leq i \leq n} |x_{i,m}|,$$

it follows from (4.1) that

$$\lim_{m \rightarrow \infty} \sum_{k=1}^n \frac{\sigma_k(\gamma\eta_m\beta_j)}{\sigma_1(\gamma\eta_m\beta_j)} = 0, \quad j = 2, \dots, n. \quad (4.2)$$

We can choose the indices m in an infinite subset of \mathbb{N} , in order to make convergent, for each $k = 1, \dots, n$, the bounded sequence $\sigma_k(\eta_m)/\sigma_1(\eta_m)$, with a limit $\tau_k \in \mathbb{C}$ ($\tau_1 = 1$). As

$$\left| \frac{\sigma_k(\eta_m)}{\sigma_1(\eta_m)} \right| \gg 1,$$

we have $\tau_k \neq 0$ for each k . Moreover, by (4.2), these limits satisfy

$$\sum_{k=1}^n \sigma_k(\gamma\beta_j)\tau_k = 0, \quad j = 2, \dots, n. \quad (4.3)$$

Now observe that the matrix $(\sigma_k(\beta_j))_{\substack{1 \leq j \leq n \\ 1 \leq k \leq n}}$ is invertible. Indeed, the matrix $(\sigma_k(\beta_j))_{\substack{0 \leq j \leq n \\ 0 \leq k \leq n}}$ being invertible, there is at least one index ℓ , with $0 \leq \ell \leq n$, such that the we have a minor

$$\det(\sigma_k(\beta_j))_{\substack{1 \leq j \leq n \\ k \neq \ell}} \neq 0.$$

The $\sigma_\ell \circ \sigma_k$'s, where $1 \leq k \leq n$, being the σ_k 's with $0 \leq k \leq n$ and $k \neq \ell$, we have

$$\det(\sigma_k(\beta_j))_{\substack{1 \leq j \leq n \\ k \neq \ell}} = \pm \sigma_\ell(\det(\sigma_k(\beta_j))_{\substack{1 \leq j \leq n \\ 1 \leq k \leq n}}),$$

hence we get

$$\det(\sigma_k(\beta_j))_{\substack{1 \leq j \leq n \\ 1 \leq k \leq n}} \neq 0.$$

Thus the system of linear equations

$$\sum_{k=1}^n \sigma_k(\beta_j)X_k = 0, \quad 2 \leq j \leq n, \quad (4.4)$$

has a space of solutions $(X_k)_{1 \leq k \leq n} \in \mathbb{C}^n$ of dimension 1. By (4.3), $(\sigma_k(\gamma)\tau_k)_{1 \leq k \leq n}$ is a solution of (4.4). Therefore the system (4.4) has a unique solution $(t_k)_{1 \leq k \leq n}$ with $t_1 = \sigma_1(\gamma)$. Thus all the convergent subsequences of the initial sequence $((\sigma_k(\eta_m)/\sigma_1(\eta_m))_{1 \leq k \leq n})$ in \mathbb{C}^n have the same limit $(t_k/\sigma_k(\gamma))_{1 \leq k \leq n}$. As this sequence is bounded, finally the initial sequence is convergent for $m \in \mathbb{N}$, toward the limit $(\tau_k)_{1 \leq k \leq n}$, where $\tau_k = t_k/\sigma_k(\gamma)$.

It is easy to make explicit the solutions of (4.4). We have

$$\sum_{k=0}^n \sigma_k(\beta_j)\sigma_k(\alpha_i) = \text{Tr}\alpha_i\beta_j = 0, \quad i = 0, 1, \quad 2 \leq j \leq n,$$

hence

$$\sum_{k=1}^n \sigma_k(\beta_j)(\sigma_k(\alpha_1) - \alpha_1) = 0, \quad 2 \leq j \leq n.$$

Since $\alpha_1 \notin \mathbb{Q}$, there is at least one index k , with $1 \leq k \leq n$, such that $\sigma_k(\alpha_1) \neq \alpha_1$, hence the solutions of (4.4) are

$$t_k = c(\sigma_k(\alpha_1) - \alpha_1),$$

where c is any complex constant. We thus get

$$\tau_k = c \frac{\sigma_k(\alpha_1) - \alpha_1}{\sigma_k(\gamma)}.$$

That is impossible if there exists $1 \leq k \leq n$ such that $\sigma_k(\alpha_1) = \alpha_1$, that is to say, if $\mathbb{Q}(\alpha_1) \neq E$, since we must have $\tau_k \neq 0$ for each k . The second part of Theorem 1.6. is thus proved. If $\mathbb{Q}(\alpha_1) = E$, then $\sigma_k(\alpha_1) \neq \alpha_1$ for each $k = 1, \dots, n$, and we get

$$\tau_k = \frac{(\sigma_k(\alpha_1) - \alpha_1)\sigma_1(\gamma)}{(\sigma_1(\alpha_1) - \alpha_1)\sigma_k(\gamma)}. \quad (4.5)$$

Moreover, the unique solution (X_1, \dots, X_n) of the system (4.4) with $X_1 = 0$ being $(0, \dots, 0)$, the matrix $(\sigma_k(\beta_j))_{\substack{2 \leq j \leq n \\ 2 \leq k \leq n}}$ is invertible. As, by (4.3), we can write

$$\sum_{k=1}^n \frac{\sigma_k(\gamma\beta_j)\sigma_k(\eta_m)}{\sigma_1(\gamma\beta_j)\sigma_1(\eta_m)} = \frac{1}{\sigma_1(\gamma\beta_j)} \sum_{k=2}^n \sigma_k(\gamma\beta_j) \left(\frac{\sigma_k(\eta_m)}{\sigma_1(\eta_m)} - \tau_k \right),$$

then we conclude that

$$\max_{2 \leq j \leq n} \left| \sum_{k=1}^n \frac{\sigma_k(\gamma\beta_j)\sigma_k(\eta_m)}{\sigma_1(\gamma\beta_j)\sigma_1(\eta_m)} \right| \asymp \max_{2 \leq k \leq n} \left| \frac{\sigma_k(\eta_m)}{\sigma_1(\eta_m)} - \tau_k \right|. \quad (4.6)$$

Further, it is impossible that there exist large m such that

$$\frac{\sigma_k(\eta_m)}{\sigma_1(\eta_m)} - \tau_k = 0, \quad k = 2, \dots, n.$$

Indeed for such m , we should have, by (3.1), $x_{j,m} = \gamma\eta_m\beta_j$ for each $j = 2, \dots, k$, hence $0 < |x_{j,m}| < 1$ when m is large, which is impossible since $x_{j,m}$ must be an integer. Choosing any determination of the complex logarithm, we can write

$$\max_{2 \leq k \leq n} \left| \frac{\sigma_k(\eta_m)}{\sigma_1(\eta_m)} - \tau_k \right| \asymp \max_{2 \leq k \leq n} \left\| \text{Log} \frac{\sigma_k(\eta_m)}{\sigma_1(\eta_m)} - \text{Log} \tau_k \right\|_{2i\pi\mathbb{Z}},$$

where we denote, for $Z \in \mathbb{C}$,

$$\|Z\|_{2i\pi\mathbb{Z}} = \min_{q \in \mathbb{Z}} |Z - 2i\pi q|.$$

Now, recalling that

$$\eta_m = \epsilon_1^{\mu_{1,m}} \dots \epsilon_r^{\mu_{r,m}},$$

where the μ_j 's are integers, we see that

$$\max_{2 \leq k \leq n} \left| \frac{\sigma_k(\eta_m)}{\sigma_1(\eta_m)} - \tau_k \right| \asymp \max_{2 \leq k \leq n} \left\| \mu_{1,m} \text{Log} \frac{\sigma_k(\epsilon_1)}{\sigma_1(\epsilon_1)} + \dots + \mu_{r,m} \text{Log} \frac{\sigma_k(\epsilon_r)}{\sigma_1(\epsilon_r)} - \text{Log} \tau_k \right\|_{2i\pi\mathbb{Z}}.$$

The τ_k 's being, by (4.5), algebraic numbers, and one at least among the numbers $\|\mu_{1,m} \text{Log} \frac{\sigma_k(\epsilon_1)}{\sigma_1(\epsilon_1)} + \dots + \mu_{r,m} \text{Log} \frac{\sigma_k(\epsilon_r)}{\sigma_1(\epsilon_r)} - \text{Log} \tau_k\|_{2i\pi\mathbb{Z}}$ ($2 \leq k \leq n$) non vanishing, the usual estimations of linear forms in logarithms of algebraic numbers [1], [21], prove that there exists a positive constant λ such that

$$\max_{2 \leq k \leq n} \left| \frac{\sigma_k(\eta_m)}{\sigma_1(\eta_m)} - \tau_k \right| \gg \left(\max_{1 \leq j \leq r} |\mu_{j,m}| \right)^{-\lambda}.$$

As the matrix $(\log |\sigma_k(\epsilon_j)|)_{\substack{1 \leq k \leq r \\ 1 \leq j \leq r}}$ is invertible, we have

$$\max_{1 \leq j \leq r} |\mu_{j,m}| \asymp \max_{1 \leq k \leq r} |\log |\sigma_k(\eta_m)|| \asymp |\log \eta_m|,$$

hence we get

$$\max_{2 \leq k \leq n} \left| \frac{\sigma_k(\eta_m)}{\sigma_1(\eta_m)} - \tau_k \right| \gg |\log \eta_m|^{-\lambda},$$

and by (4.6),

$$\max_{2 \leq j \leq n} \left| \sum_{k=1}^n \frac{\sigma_k(\gamma\beta_j)\sigma_k(\eta_m)}{\sigma_1(\gamma\beta_j)\sigma_1(\eta_m)} \right| \gg |\log \eta_m|^{-\lambda}. \quad (4.7)$$

Then we deduce from (4.7) and (3.1) that

$$\max_{2 \leq j \leq n} |x_{j,m}| \gg \eta_m^{-1/n} |\log \eta_m|^{-\lambda}.$$

As

$$\eta_m^{-1/n} \asymp \max_{0 \leq j \leq n} |x_{j,m}|,$$

we thus get

$$\max_{2 \leq j \leq n} |x_{j,m}| \gg \left(\max_{0 \leq j \leq n} |x_{j,m}| \right) \left(\log \max_{0 \leq j \leq n} |x_{j,m}| \right)^{-\lambda},$$

which is the first part of Theorem 1.6. □

5 Open problems.

It would be interesting to obtain other solutions of (1.2) or (1.3). It seems difficult to find solutions (x_1, \dots, x_n) of (1.2) or (1.3) under the hypothesis of Theorem 1.2, with a large difference between $\log |x_i|$ and $\log |x_j|$ for some indices $i \neq j$. In the case where $\mathbb{Q}(\alpha_1) = E$ and $n \geq 3$, we do not know whether it is possible to improve Theorem 1.4 by setting the condition (1.7) for more than one index $2 \leq i \leq n$. Except for the case $n = 2$, we do not either know whether condition (1.7) can be replaced by a more precise condition, for instance, $|x_n| \ll M \log M^{-\kappa}$ with a positive constant κ . Also it would be interesting to examine whether it is possible to obtain an analogue of Theorem 1.6 for the solutions of (1.3) given by Corollary 1.3.

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