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Gradient estimates for porous medium and fast diffusion equations via FBSDE approach

Ying Hu*, Zhongmin Qian[†] and Zichen Zhang[‡]

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Abstract

In this paper, we establish several gradient estimates for the positive solution of Porous Medium Equations (PMEs) and Fast Diffusion Equations (FDEs). Our proof is probabilistic and uses martingale techniques and Forward and Backward Stochastic Differential Equations (FBSDEs).

1 Introduction

The Porous Medium Equation (PME) on \mathbb{R}^n is

$$\partial_t u = \Delta u^m, \quad \text{on } (0, \infty) \times \mathbb{R}^n, \quad (1.1)$$

where $m > 1$. It is a nonlinear version of heat equation arising in modelling fluid flow or heat transfer in physics. Moreover, this equation is degenerate on $u = 0$, hence having finite speed of propagation. On the other hand, when $m \in (0, 1)$, (1.1) is called Fast Diffusion Equation (FDE). It appears as diffusion in plasma in physics. We refer to [11] for a comprehensive study.

An important aspect in the study of (1.1) in connection with the existence / uniqueness problem and the regularity problem is to establish a priori estimates, which is an interesting question by its own. There is a large body of knowledge in this respect for the similar partial differential equations in particular those linear equations, namely for linear heat equations or in general for parabolic equations. For porous medium equations, usually only nonnegative solutions are considered for both cases where $m < 1$ or $m > 1$, as this is natural from physical point of view.

To study positive solutions, mathematically a way of utilizing the non-negativity is to do the transformation: $v = \frac{m}{m-1} u^{m-1}$. It is an analogy of the Hopf transformation for the heat equation in the sense that

$$\lim_{m \rightarrow 1} \left(v - \frac{m}{m-1} \right) = \log u .$$

It is easy to see that v evolves according to the following non-linear parabolic equation:

$$\frac{\partial v}{\partial t} = (m-1) v \Delta v + |\nabla v|^2. \quad (1.2)$$

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It is this equation which can be used to derive various a priori estimates for the non-negative solutions to (1.1).

The fact that the gradient term on the right-hand side of (1.2) is quadratic is useful for many calculations. However, the diffusion coefficient in (1.2) is degenerate at $v = 0$. A standard way to overcome this degeneracy is to consider strictly positive solutions which are bounded away from zero at first, then use an approximation argument such as on page 200 in [11] to pass to the non-negative solutions. Therefore, our result for nonnegative initial data should be understood in distribution sense. From now on, however, in order to avoid such technical steps which have been treated thoroughly in [11], by a positive solution to the porous medium equation we mean a solution which is bounded below by a positive constant, whose initial data is Lipschitz continuous. We on the other hand wish to emphasize that the a priori estimates we are going to establish in this article are independent of the positive lower bound of a positive solution.

In [1], Aronson and Benilan established that if $m > 1 - \frac{2}{n}$, then for any positive smooth solution u of (1.1),

$$\Delta v \geq -\frac{\alpha}{(m-1)t}$$

with $\alpha = \frac{n(m-1)}{(m-1)n+2}$ and $v = \frac{m}{m-1}u^{m-1}$, or equivalently

$$|\nabla v|^2 - \frac{\partial v}{\partial t} \leq \frac{\alpha v}{t}. \quad (1.3)$$

The proceeding *a priori* estimate plays a key role in the study of existence of initial value problem for PME. A lot of work have been done to improve this result. Let us recall here some results which are related to gradient estimates for positive solutions to (1.1). We apologize for any possible omissions due to authors' limited knowledge.

In [8] P. Lu, L. Ni, J. L. Vazquez and C. Villani established a local version of Aronson-Benilan's estimate for (1.1) where $m > 1 - \frac{2}{n}$ on a complete Riemannian manifold with Ricci curvature bounded from below. Assume that $\text{Ric} \geq -(n-1)K^2$ on $B(\mathcal{O}, R)$ for some $K \geq 0$, and u is a positive smooth solution u to (1.1) on $B(\mathcal{O}, R) \times [0, T]$ ($B(\mathcal{O}, R)$ denotes the ball of radius $R > 0$ with centre \mathcal{O}). If $m > 1$ and $\beta > 1$, they proved that

$$|\nabla v|^2 - \beta v_t \leq \alpha \beta^2 \left(\frac{1}{t} + C_2 K^2 v_{\max}^{R,T} \right) v + \alpha \beta^2 \frac{v_{\max}^{R,T}}{R^2} C_1 v$$

on $B(\mathcal{O}, \frac{R}{2}) \times [0, T]$, with

$$C_1 = 40(m-1)[3 + (n-1)(1 + KR)] + \frac{200\alpha\beta^2 m^2}{\beta-1}$$

and $C_2 = \frac{(m-1)(n-1)}{\beta-1}$, where $v_{\max}^{R,T} = \max_{B(\mathcal{O}, R) \times [0, T]} v$. They also established that, if $m \in (1 - \frac{2}{n}, 1)$ and $\gamma \in (0, 1)$,

$$|\nabla v|^2 - \gamma v_t \leq \frac{\alpha \gamma^2}{C_3} \left(\frac{1}{t} + C_4 \sqrt{C_3} K^2 \bar{v}_{\max}^{R,T} \right) v + \frac{\alpha \gamma^2}{C_3} \frac{\bar{v}_{\max}^{R,T}}{R^2} C_5 v$$

on $B(\mathcal{O}, \frac{R}{2}) \times [0, T]$, where

$$\begin{aligned} C_3 &= 1 - \alpha(1 - \gamma) - (1 - \gamma) \frac{(1 - \alpha + \epsilon_1)^2}{(1 - \gamma) - \alpha - (1 - \gamma)\epsilon_2^2}, \\ C_4 &= \frac{(m - 1)(n - 1)}{(\gamma - 1)\epsilon_2}, \\ C_5 &= 1600m^2 \frac{\alpha\gamma^2}{2\epsilon_1(\gamma - 1)} + 40(1 - m)[3 + (n - 1)(1 + KR)] \end{aligned}$$

and $v_{\max}^{R,T} = \max_{B(\mathcal{O}, R) \times [0, T]}(-v)$, and ϵ_1 and ϵ_2 are positive constants such that

$$\begin{aligned} 1 - \gamma - \alpha - (1 - \gamma)\epsilon_2^2 &> 0, \\ 1 - \alpha(1 - \gamma) - (1 - \gamma) \frac{(1 - \alpha + \epsilon_1)^2}{1 - \gamma - \alpha - (1 - \gamma)\epsilon_2^2} &> 0. \end{aligned}$$

On the other hand, some gradient bounds of Hamilton type have been established by various authors in recent years. This is kind of estimates on the space derivatives of the solution. In L. A. Caffarelli, J. L. Vazquez and N. I. Wolanski [4], the problem of (1.1) with a compactly supported initial data has been studied thoroughly. Among of other interesting results, they proved that for $m > 1$, there exist a time $T = T(u_0) > 0$ and a constant $c = c(m, n) > 0$ such that for any $t > T$ and almost everywhere $x \in \mathbb{R}^n$

$$|\nabla v(x, t)| \leq c \left(\left(\frac{v}{t} \right)^{\frac{1}{2}} + \frac{|x|}{t} \right)$$

if $u_0 \geq 0$ is integrable with a compact support. In a recent paper [12] by X. Xu, positive solutions of (1.1) on a complete Riemannian manifold have been considered, where $m \in \left(1 - \frac{4}{n+3}, \infty\right)$ and the Ricci curvature is bounded below: $\text{Ric} \geq -k$ for some $k \geq 0$. In particular, it was proved in [12] that if $m > 1$ and if there exists a constant $\delta \in \left(0, \frac{4}{n-1}\right]$ such that

$$1 \leq \frac{v_{\max}^{R,T}}{v_{\min}^{R,T}} < \frac{1}{1 + \delta} \left(\frac{4m}{(n-1)(m-1)} + 1 \right)$$

then

$$\frac{|\nabla v|}{v_{\max}^{R,T}(1 + \delta) - v} \leq C(n, m) \left(\frac{1 + \delta}{\rho\delta R} + \frac{1}{\sqrt{\frac{m-1}{m}v_{\max}^{R,T}\delta\rho T}} + \sqrt{\frac{k}{\delta}} \right)$$

on $B(x_0, \frac{R}{2}) \times [t_0 - \frac{T}{2}, t_0]$, where

$$\rho = 2m - \frac{(n-1)(m-1)v_{\max}^{R,T}(1 + \delta) - v_{\min}^{R,T}}{2v_{\min}^{R,T}},$$

$v_{\max}^{R,T} = \sup_{B(x_0, R) \times [t_0 - T, t_0]} v$ and $v_{\min}^{R,T} = \inf_{B(x_0, R) \times [t_0 - T, t_0]} v$. In particular, if $n = 1$

$$\frac{|\nabla v|}{v_{\max}^{R,T}(1 + \delta) - v} \leq C(m) \left(\frac{1 + \delta}{\delta R} + \frac{1}{\sqrt{\frac{m-1}{m}v_{\max}^{R,T}\delta T}} \right)$$

on $B(x_0, \frac{R}{2}) \times [t_0 - \frac{T}{2}, t_0]$ for any $\delta > 0$. On the other hand, for $m \in (1 - \frac{4}{n+3}, 1)$, it was also proved in [12] that

$$\frac{|\nabla v|}{-v} \leq C(m, n) \left(\frac{1}{R} + \frac{1}{\sqrt{\frac{1-m}{m} \bar{v}_{\min}^{R,T} T}} + \sqrt{k} \right)$$

on $B(x_0, \frac{R}{2}) \times [t_0 - \frac{T}{2}, t_0]$, where $\bar{v}_{\min}^{R,T} = \inf_{B(x_0, R) \times [t_0 - T, t_0]}(-v)$. In X. Zhu [13], the author proved a similar result for positive solution of (1.1) for $m \in (1 - \frac{2}{n}, 1)$ which reads as the following

$$\frac{|\nabla v|}{\sqrt{-v}} \leq C(m, n, \bar{v}_{\max}^{R,T}) \sqrt{\bar{v}_{\max}^{R,T}} \left(\frac{1}{R} + \frac{1}{\sqrt{T}} + \sqrt{k} \right)$$

on $B(x_0, \frac{R}{2}) \times [t_0 - \frac{T}{2}, t_0]$, where $\bar{v}_{\max}^{R,T} = \sup_{B(x_0, R) \times [t_0 - T, t_0]}(-v)$.

In this paper, by using a connection of a solution and its gradient to a backward stochastic differential equation, we derive global estimates with explicit constants for the gradients of a positive solution to the porous medium equation. Our estimates, in contrast with the preceding estimates, involve only the supremum norm of the initial data rather than the norm over the solution over whole evolving time interval $[0, T]$, furthermore the constants appearing in our a priori estimate are explicit and are close to optimal. Our method uses some martingale methods and the link between PDE and BSDE borrowing some idea from the paper [6].

Theorem 1.1 *Let $u > 0$ be a positive solution to the porous medium equation (1.1).*

1) *If $m \in (1, 1 + \frac{2}{n})$, then*

$$|\nabla u^{\frac{3}{2}(m-1)}(t, \cdot)| \leq \frac{3 \|u_0^{m-1} - 1\|_{\infty}^2}{\sqrt{2mt}} \frac{1}{\sqrt{t}}, \quad \forall t > 0. \quad (1.4)$$

2) *If $n = 1$ and $m > 1$, then*

$$|\nabla u^{m-1}(t, \cdot)| \leq \frac{\sqrt{2(m-1) \|u_0^{m-1} - 1\|_{\infty}}}{m} \frac{1}{\sqrt{t}}, \quad \forall t > 0. \quad (1.5)$$

3) *If $1 - \frac{6}{n+8} \leq m < 1$, then*

$$|\nabla u^{1-m}(t, \cdot)| \leq \frac{\sqrt{2} \|u_0^{1-m} - 1\|_{\infty} \sqrt{\|u_0^{1-m} - 1\|_{\infty} + 1}}{\sqrt{m}} \frac{1}{\sqrt{t}}, \quad \forall t > 0. \quad (1.6)$$

4) *If $m \in (\frac{n-1}{n+3}, 1)$, then*

$$|\nabla \log u(t, \cdot)| \leq \frac{2 \sqrt{\frac{m}{1-m} \|u_0^{1-m} - 1\|_{\infty}}}{m^2 \sqrt{|2m - 4 - \frac{\sqrt{2}}{m} \beta_2|}} \frac{1}{\sqrt{t}}, \quad \forall t > 0 \quad (1.7)$$

where β_2 is given in equation (4.21).

This paper is organized as follows: in the next section, we introduce the PME, SDE and BSDE and their link. In section 3, we use martingale method to state the main result on gradient estimate when $m > 1$. The last section is devoted to the gradient estimate when $0 < m < 1$.

2 Fundamental equations and FBSDE

Consider a positive solution u to the porous medium equation (1.1). We will use the following transformation:

$$f = \frac{m}{m-1}(u^{m-1} - 1)$$

in the case that $m > 1$. Since

$$f = \frac{m}{m-1}(u^{m-1} - u^0) \rightarrow \log u \quad \text{as } m \downarrow 1,$$

the transformation coincides with the Hopf transformation for the linear heat equation, though in the case that $m \in (0, 1)$ we will use a slightly different transformation, see (2.5) below. Then

$$(m-1)f + m = mu^{m-1} > 0,$$

a fact will be used throughout the paper.

It is easy to verify that f satisfies equations:

$$\nabla f = mu^{m-2}\nabla u, \quad |\nabla f|^2 = m^2u^{2m-4}|\nabla u|^2, \quad (2.1)$$

$$\begin{aligned} \Delta f &= \frac{m}{m-1}\Delta(u^m)^{\frac{m-1}{m}} \\ &= (u^m)^{\frac{m-1}{m}-1}\Delta u^m - \frac{1}{m}m^2u^{-m-1}u^{2m-2}|\nabla u|^2 \\ &= u^{-1}\Delta u^m - \frac{1}{m}u^{-m+1}|\nabla f|^2 \end{aligned} \quad (2.2)$$

and

$$\frac{\partial}{\partial t}f = mu^{m-2}\frac{\partial}{\partial t}u. \quad (2.3)$$

Hence

$$\Delta f = u^{-m+1}\frac{1}{m}\partial_t f - \frac{1}{m}u^{-m+1}|\nabla f|^2$$

that is

$$\frac{\partial}{\partial t}f = ((m-1)f + m)\Delta f + |\nabla f|^2. \quad (2.4)$$

If $m \in (0, 1)$, then we use a slight different transformation

$$f = \frac{m}{1-m}(u^{1-m} - 1). \quad (2.5)$$

Then f is a solution to

$$\frac{\partial}{\partial t}f = \frac{m^2}{(1-m)f + m}\Delta f + \frac{m^2(2m-1)}{((1-m)f + m)^2}|\nabla f|^2. \quad (2.6)$$

The link between Partial Differential Equation (PDE) and Backward Stochastic Differential Equation (BSDE) is by now well known, see, e.g. [10], [9], [2]. In this section, we introduce the

SDE and BSDE associated with PME (1.1) and recall this link in our framework. Throughout this paper, we suppose that u is a bounded smooth solution to (1.1).

Given $T > 0$. We start with the PDE (2.4), and write down a Forward-Backward Stochastic Differential Equation (FBSDE) for this PDE. Let (Ω, \mathcal{F}, P) be a complete probability space, and W be an n -dimensional Brownian motion on it. And let $\{\mathcal{F}_t\}$ be the natural augmented filtration of W .

Let $X = (X_t)$ be the diffusion process defined to be the unique (strong) solution flow of the following stochastic differential equation (SDE)

$$dX = \sqrt{2((m-1)f(T-t, X) + m)} \circ dW, \quad X_0 = x \quad (2.7)$$

where $\circ dW$ denotes the Stratonovich differentiation, which may be written in terms of Itô integral:

$$dX = \sqrt{2((m-1)f(T-t, X) + m)} dW + \frac{m-1}{2} \nabla f(T-t, X) dt. \quad (2.8)$$

Let $Y = f(T-\cdot, X)$, $U = 2((m-1)Y + m)$, $Z^i = \sqrt{U} \frac{\partial f}{\partial x^i}(T-\cdot, X)$ and $Z = (Z^i) = \sqrt{U} \nabla f(T-\cdot, X)$. Then

$$dX = \sqrt{U} dW + \frac{m-1}{2\sqrt{U}} Z dt, \quad X_0 = x. \quad (2.9)$$

Itô's formula applying to $f(T-t, X_t)$ yields

$$Y_T - Y_t = \int_t^T Z dW + \int_t^T \frac{m-3}{2} \frac{|Z|^2}{U} ds,$$

that is

$$dY = Z dW + \frac{m-3}{2} \frac{|Z|^2}{U} dt, \quad Y_T = f(0, X_T). \quad (2.10)$$

The coupled system (2.9) and (2.10) form a system of FBSDEs.

Next, let us recall the representation of Z by the stochastic flow of Y .

Let $J_j^i = \frac{\partial}{\partial x^j} X^i$ be the Jacobi matrix, its inverse K_j^i . Let $Y^i = \frac{\partial}{\partial x^i} Y$ and $Z_j^i = \frac{\partial}{\partial x^j} Z^i$. We will use the following relation

$$Y^i = \frac{\partial}{\partial x^i} Y = \frac{\partial}{\partial X^k} f(T-t, X) J_i^k = \frac{1}{\sqrt{U}} Z^k J_i^k$$

so that $Z^i = \sqrt{U} Y^l K_l^i$, here and thereafter the Einstein convention is enforced, that is, repeated indices are summed up within their range.

Finally, we are in a position to write down the stochastic differential for Z from this representation. The following computation involving the tangent process is standard, see for detail K. Elworthy [5], N. Ikeda and S. Watanabe [7].

Differentiating the SDE for X (forward equation) to obtain the SDE for J :

$$\begin{aligned} dJ_j^i &= \frac{(m-1)Y^j}{\sqrt{U}} dW^i - \frac{(m-1)^2}{2U^{3/2}} Y^j Z^i dt \\ &\quad + \frac{m-1}{2\sqrt{U}} Z_j^i dt. \end{aligned}$$

We then derive the equation for K by using $KJ = JK = I$, which is given by

$$\begin{aligned} dK_j^i &= -\frac{m-1}{U} Z^j K_k^i dW^k - \frac{m-1}{2\sqrt{U}} K_k^i K_j^l Z_l^k dt \\ &\quad + \frac{3(m-1)^2}{2U^2} K_k^i Z^k Z^j dt. \end{aligned} \quad (2.11)$$

Differentiating the backward equation we obtain a SDE for Y^i :

$$dY^i = Z_i^k dW^k + \frac{(m-3)Z^k Z_i^k}{U} dt - \frac{(m-3)(m-1)}{U^2} |Z|^2 Y^i dt. \quad (2.12)$$

Thus, using (2.11) and (2.12), together with integration by parts finally in a position to write down the equation for Z , which takes the following form:

$$\begin{aligned} dZ^i &= \sqrt{U} K_i^l Z_l^k \left(dW^k + \frac{(3m-7)Z^k}{2} \frac{Z^k}{U} dt \right) \\ &\quad - \frac{(m-3)(m-1)}{2} \frac{Z^i |Z|^2}{U^2} dt \\ &\quad - \frac{m-1}{\sqrt{U}} Z^i K_k^l Z_l^k dt. \end{aligned} \quad (2.13)$$

3 Porous Medium Equations

In this section we develop explicit estimates for the derivative of a positive solution u to the porous equation (1.1), where $m > 1$, and $f = \frac{m}{m-1}(u^{m-1} - 1)$.

3.1 Gradient Estimates

The first ingredient of our approach is a Girsanov transformation of probability. In order to have some flexibility, we introduce a family of change of probability depending on a parameter ε .

Lemma 3.1 $\frac{Z}{U} \cdot W$ is a BMO martingale.

Making change of probability and define

$$d\tilde{W}^k = dW^k - \varepsilon \frac{Z^k}{U} dt. \quad (3.1)$$

Then (2.13) and (2.10) can be rewritten as

$$\begin{aligned} dZ^i &= \sqrt{U} K_i^l Z_l^k d\tilde{W}^k - \frac{(m-3)(m-1)}{2} \frac{Z^i |Z|^2}{U^2} dt \\ &\quad + \frac{(3m-7+2\varepsilon)}{2} K_i^l Z_l^k \frac{Z^k}{\sqrt{U}} dt - \frac{m-1}{\sqrt{U}} Z^i K_k^l Z_l^k dt \end{aligned} \quad (3.2)$$

and

$$dY = Z^k d\tilde{W}^k + \frac{m-3+2\varepsilon}{2} \frac{|Z|^2}{U} dt. \quad (3.3)$$

According to [6], we work out Doob-Meyer's decomposition for $|Z|^2$ under the probability Q , and identify conditions under which $|Z|^2$ is a Q -submartingale.

Lemma 3.2 Let $\theta_k = \sum_l K_i^l Z_l^k$, and $\delta = 3m - 7 + 2\varepsilon$. Then

$$\begin{aligned} d|Z|^2 &= 2\sqrt{U} Z^i K_i^l Z_l^k d\tilde{W}^k \\ &+ \left[(n+3 - (n+1)m)(m-1) + (m-1)\delta - \frac{\delta^2}{4} \right] \frac{|Z|^4}{U^2} dt + A, \end{aligned} \quad (3.4)$$

where

$$\begin{aligned} A &= U \sum_{i \neq k} \left| \sum_l K_i^l Z_l^k + \frac{\delta}{2U^{3/2}} Z^i Z^k \right|^2 \\ &+ U \left| \theta_k - \frac{1}{2U^{3/2}} (\delta |Z^k|^2 - 2(m-1)|Z|^2) \right|^2. \end{aligned} \quad (3.5)$$

Proof. By (3.2) and integration by parts we obtain

$$\begin{aligned} d|Z|^2 &= 2\sqrt{U} Z^i K_i^l Z_l^k d\tilde{W}^k + (3-m)(m-1) \frac{|Z|^4}{U^2} dt \\ &+ U \sum_{k=1}^n \left[\frac{\theta_k}{U^{3/2}} (\delta |Z^k|^2 - 2(m-1)|Z|^2) + |\theta_k|^2 \right] dt \\ &+ U \sum_{i \neq k} \left(\left| \sum_l K_i^l Z_l^k \right|^2 + \frac{\delta}{U^{3/2}} Z^i Z^k K_i^l Z_l^k \right) dt. \end{aligned} \quad (3.6)$$

While

$$\begin{aligned} &\frac{\theta_k}{U^{3/2}} (\delta |Z^k|^2 - 2(m-1)|Z|^2) + |\theta_k|^2 \\ &= \left| \theta_k - \frac{1}{2U^{3/2}} (\delta |Z^k|^2 - 2(m-1)|Z|^2) \right|^2 \\ &\quad - \frac{\delta^2}{4U^3} |Z^k|^4 + (m-1)\delta |Z^k|^2 \frac{|Z|^2}{U^3} - \frac{|Z|^4}{U^3} (m-1)^2 \end{aligned}$$

and the last term on the right-hand side of (3.6) can be handled as the following

$$\begin{aligned} &\sum_{i \neq k} \left[\left| \sum_l K_i^l Z_l^k \right|^2 + \frac{\delta}{U^{3/2}} Z^i Z^k \sum_l K_i^l Z_l^k \right] \\ &= \sum_{i \neq k} \left| \sum_l K_i^l Z_l^k + \frac{\delta}{2U^{3/2}} Z^i Z^k \right|^2 - \frac{\delta^2}{4U^3} \sum_{i \neq k} |Z^i|^2 |Z^k|^2 \end{aligned}$$

so that (3.6) yields (3.4). ■

Lemma 3.3 Let us suppose $m \leq 1 + \frac{2}{n}$ and choose $\delta = 2(m-1)$. Then $|Z|^2$ becomes a Q -submartingale.

Proof. Choosing $\delta = 2(m-1)$ in (3.4), we deduce that

$$d|Z|^2 \geq 2\sqrt{U}Z^i K_i^l Z_l^k d\tilde{W}^k + (n+2-nm)(m-1)\frac{|Z|^4}{U^2}dt, \quad (3.7)$$

from which we deduce the result. ■

Finally from the BSDE (3.3) satisfied by (Y, Z) under Q and Lemma 2.2 in [3],

$$E^Q\left[\int_0^T |Z|^2 dt\right] \leq 4\|f_0\|_\infty^2.$$

This together with the fact that $|Z|^2$ is a Q -submartingale gives us the main result of this section.

Theorem 3.4 *Suppose $m \leq 1 + \frac{2}{n}$. Then*

$$((m-1)f(T, x) + m)|\nabla f(T, x)|^2 \leq \frac{2\|f_0\|_\infty^2}{T} \quad (3.8)$$

which is equivalent to (1.4).

Proof. Since

$$E^Q\left[\int_0^T |Z|^2 dt\right] \leq 4\|f_0\|_\infty^2$$

and $|Z|^2$ is a Q -submartingale, we deduce that

$$T|Z_0|^2 \leq 4\|f_0\|_\infty^2$$

which yields (3.8). ■

3.2 One-dimensional case

In Theorem 3.4, we have given an estimate on $((m-1)f+m)|\nabla f|^2$. The objective of this subsection is to try to obtain an estimate on $|\nabla f|^2$. The idea comes from the observation that from BSDE (3.3),

$$\left|\frac{m-3+2\varepsilon}{2}\right| E^Q\left[\int_0^T \frac{|Z|^2}{U} dt\right] \leq 2\|f_0\|_\infty.$$

Hence it is sufficient to give some condition under which $M = \frac{|Z|^2}{U}$ is a Q -submartingale.

Lemma 3.5 *Let $M = \frac{|Z|^2}{U}$ and $\beta = 2\varepsilon - 3 - m$. Then*

$$\begin{aligned} dM &= \left(\frac{2}{\sqrt{U}}Z^i K_i^l Z_l^k - 2(m-1)U^{-2}|Z|^2 Z^k\right) d\tilde{W}^k \\ &\quad + ((m-1)^2(1-n) - \beta^2)\frac{|Z|^4}{U^3}dt + B, \end{aligned} \quad (3.9)$$

where

$$\begin{aligned} B &= \sum_{k=1}^n \left| \theta_k + \frac{\beta|Z^k|^2 - 2(m-1)|Z|^2}{2U^{3/2}} \right|^2 \\ &\quad + \sum_{i \neq k} \left| \sum_l K_i^l Z_l^k + \frac{\beta}{2U^{3/2}} Z^i Z^k \right|^2. \end{aligned} \quad (3.10)$$

Proof. Recall that, by (3.6),

$$\begin{aligned}
d|Z|^2 &= 2\sqrt{U}Z^i K_i^l Z_l^k d\tilde{W}^k + (3-m)(m-1)\frac{|Z|^4}{U^2}dt \\
&+ U \sum_{k=1}^n \left[\frac{\theta_k}{U^{3/2}} \left((3m-7+2\varepsilon)|Z^k|^2 - 2(m-1)|Z|^2 \right) + |\theta_k|^2 \right] dt \\
&+ U \sum_{i \neq k} \left(\left| \sum_l K_i^l Z_l^k \right|^2 + \frac{(3m-7+2\varepsilon)}{U^{3/2}} Z^i Z^k K_i^l Z_l^k \right) dt, \tag{3.11}
\end{aligned}$$

where $\theta_k = \sum_l K_k^l Z_l^k$, $U = 2(m-1)Y + 2m$, and

$$dY = Z^k d\tilde{W}^k + \frac{m-3+2\varepsilon}{2} \frac{|Z|^2}{U} dt. \tag{3.12}$$

According to Itô's formula

$$\begin{aligned}
d\frac{1}{U} &= d[2(m-1)Y + 2m]^{-1} \\
&= -2(m-1)U^{-2} Z^k d\tilde{W}^k - (1+2\varepsilon-3m)(m-1)\frac{|Z|^2}{U^3} dt,
\end{aligned}$$

so that, integrating by parts we obtain

$$\begin{aligned}
dM &= \frac{2}{\sqrt{U}} Z^i K_i^l Z_l^k d\tilde{W}^k - 2(m-1)U^{-2} |Z|^2 Z^k d\tilde{W}^k \\
&+ 2(m+1-\varepsilon)(m-1)\frac{|Z|^4}{U^3} dt \\
&+ \sum_{k=1}^n \left[\left(\beta |Z^k|^2 - 2(m-1)|Z|^2 \right) \frac{\theta_k}{U^{3/2}} + |\theta_k|^2 \right] dt \\
&+ \sum_{i \neq k} \left(\left| \sum_l K_i^l Z_l^k \right|^2 + \frac{\beta}{U^{3/2}} Z^i Z^k \sum_l K_i^l Z_l^k \right) dt. \tag{3.13}
\end{aligned}$$

By using the elementary equalities

$$\begin{aligned}
&\sum_{i \neq k} \left(\left| \sum_l K_i^l Z_l^k \right|^2 + \frac{\beta}{U^{3/2}} Z^i Z^k \sum_l K_i^l Z_l^k \right) \\
&= \sum_{i \neq k} \left| \sum_l K_i^l Z_l^k + \frac{\beta}{2U^{3/2}} Z^i Z^k \right|^2 - \frac{\beta^2}{4} \frac{1}{U^3} \sum_{i \neq k} |Z^i|^2 |Z^k|^2
\end{aligned}$$

and

$$\begin{aligned}
& \sum_{k=1}^n \left[\left(\beta |Z^k|^2 - 2(m-1)|Z|^2 \right) \frac{\theta_k}{U^{3/2}} + |\theta_k|^2 \right] \\
= & \sum_{k=1}^n \left| \theta_k + \frac{\beta |Z^k|^2 - 2(m-1)|Z|^2}{2U^{3/2}} \right|^2 \\
& - \frac{\beta^2}{4U^3} \sum_{k=1}^n |Z^k|^4 + (m-1)(\beta - nm + n) \frac{|Z|^4}{U^3}.
\end{aligned}$$

Thus (3.9) follows from (3.13). ■

It is trivial to see that if $n = 1$ and $\beta = 0$, then M is a Q -submartingale.

Lemma 3.6 *If $n = 1$, and $\varepsilon = \frac{3+m}{2}$, then M is a Q -submartingale.*

Theorem 3.7 *If $n = 1$ and $m > 1$, then*

$$|\nabla f(T, x)|^2 \leq \frac{2\|f_0\|_\infty}{mT}$$

which is equivalent to (1.5).

Proof. From BSDE (3.3) for (Y, Z) , we know that

$$mE^Q\left[\int_0^T M_t dt\right] \leq 2\|f_0\|_\infty,$$

from which we deduce

$$|\nabla f(T, x)|^2 \leq \frac{2\|f_0\|_\infty}{mT}.$$

■

4 Fast diffusion Equations

4.1 Forward-Backward Systems

If $0 < m < 1$ and u is a positive solution to $\partial_t u = \Delta u^m$ we use a different change of variable $f = \frac{m}{1-m}(u^{1-m} - 1)$. Note that

$$f = \frac{m}{1-m}(u^{1-m} - u^0) \rightarrow \log u \quad \text{as } m \uparrow 1.$$

Then

$$\partial_t f = \frac{m^2}{(1-m)f+m} \Delta f + \frac{m^2(2m-1)}{((1-m)f+m)^2} |\nabla f|^2. \quad (4.1)$$

As in Section 2, we first derive a coupled system of FBSDEs for this PDE. Run a diffusion X with generator $L = \frac{m^2}{(1-m)f+m} \Delta$. Consider the following diffusion

$$dX = m\sqrt{2}((1-m)f(T-\cdot, X) + m)^{-\frac{1}{2}} \circ dW, \quad X_0 = x. \quad (4.2)$$

Let $Y = f(T - \cdot, X)$,

$$U = (1 - m)f(T - \cdot, X) + m \text{ and } Z = \frac{m\sqrt{2}}{\sqrt{U}}\nabla f(T - \cdot, X). \quad (4.3)$$

Then, we may rewrite the SDE (4.2) as the following (in Itô's integral)

$$dX = \frac{m\sqrt{2}}{\sqrt{U}}dW - \frac{\sqrt{2}m(1-m)}{4} \frac{Z}{U^{\frac{3}{2}}}dt. \quad (4.4)$$

For simplicity, let $J_j^i = \frac{\partial}{\partial x^j}X^i$, K_j^i is the inverse of J . Let $Y_j = \frac{\partial}{\partial x^j}Y$, $U_j = \frac{\partial}{\partial x^j}U$, $Z_j^\alpha = \frac{\partial}{\partial x^j}Z^\alpha$. Then

$$U_j = (1 - m) \sum_{i=1}^n J_j^i \frac{\partial f}{\partial x^i}(T - \cdot, X) = (1 - m)Y_j \quad (4.5)$$

and

$$Y_j = \sum_{i=1}^n \frac{\partial}{\partial X^i} f(T - t, X) J_j^i = \frac{\sqrt{U}}{\sqrt{2m}} Z^i J_j^i. \quad (4.6)$$

Therefore

$$Z^i = \sqrt{2m}U^{-\frac{1}{2}}Y_j K_i^j. \quad (4.7)$$

As $Z = \frac{m\sqrt{2}}{\sqrt{U}}\nabla f(T - \cdot, X)$, we have

$$|\nabla f|^2(T - \cdot, X) = \frac{1}{2m^2}U|Z|^2.$$

By Itô's formula to $f(T - \cdot, X)$,

$$dY = Z \cdot dW - \frac{(3m-1)|Z|^2}{4U}dt. \quad (4.8)$$

(4.4) and (4.8) form a coupled system of FBSDEs. Now differentiating in the initial data we deduce that

$$dY_j = Z_j^i dW^i - \frac{(3m-1)}{2} Z_j^i \frac{Z^i}{U} dt + \frac{(3m-1)(1-m)}{4} \frac{|Z|^2 Y_j}{U^2} dt. \quad (4.9)$$

On the other hand, differentiating X in the initial data in the stochastic differential equation (4.4) we obtain

$$dJ_j^i = -\frac{(1-m)}{2} \frac{Z^a}{U} J_j^a dW^i + \frac{3}{2} \frac{(1-m)^2}{4} \frac{Z^i Z^a}{U^2} J_j^a dt - \frac{\sqrt{2}m(1-m)}{4} \frac{Z_j^i}{U^{\frac{3}{2}}} dt. \quad (4.10)$$

It follows that, by using $KJ = I$, to obtain

$$dK_j^i = \frac{(1-m)}{2} \frac{K_k^i Z^j}{U} dW^k + \frac{\sqrt{2}m(1-m)}{4} \frac{K_c^i K_j^k Z_k^c}{U^{\frac{3}{2}}} dt - \frac{1}{2} \frac{(1-m)^2}{4} \frac{K_k^i Z^k Z^j}{U^2} dt. \quad (4.11)$$

We then can derive the stochastic differential equation for Z^i . Indeed by integrating by parts the following relation $Z^i = \sqrt{2m}U^{-\frac{1}{2}}Y_j K_i^j$, we deduce

$$\begin{aligned} \frac{1}{\sqrt{2m}} dZ^i &= \frac{\sum_l K_i^l Z_l^k}{\sqrt{U}} dW^k - \frac{3(3m-1)(m-1)}{8} \frac{Y_j K_i^j |Z^k|^2}{\sqrt{U}U} dt \\ &\quad - \frac{5m-1}{4} \frac{K_i^c Z_c^k Z^k}{U\sqrt{U}} dt - \frac{m-1}{2} \frac{Z_j^k K_k^j Z^i}{U\sqrt{U}} dt. \end{aligned} \quad (4.12)$$

4.2 Gradient Estimates

We are now in a position to establish the following

Theorem 4.1 *If $1 - \frac{6}{n+8} \leq m \leq 1$, then*

$$|\nabla f(T, x)| \leq \frac{\sqrt{2}\|f_0\|_\infty\sqrt{(1-m)\|f_0\|_\infty + m}}{m\sqrt{T}} \quad (4.13)$$

which yields (1.6).

From (4.12), we have

$$\begin{aligned} d|Z|^2 &= 2\sqrt{2}m \frac{K_i^j Z_j^k Z^i}{\sqrt{U}} dW^k + \frac{3}{4}(3m-1)(1-m) \frac{|Z|^4}{U^2} dt \\ &\quad - \frac{\sqrt{2}(5m-1)m}{2} \frac{K_i^j Z_j^k Z^i Z^k}{U\sqrt{U}} dt + \sqrt{2}(1-m)m \frac{Z_j^k K_k^j |Z|^2}{U\sqrt{U}} dt \\ &\quad + 2m^2 \sum_k \left| \frac{K_i^j Z_j^k}{\sqrt{U}} \right|^2 dt. \end{aligned} \quad (4.14)$$

Let us return to the equation for Y , which may be written as

$$dY = \sum_k Z^k \left(dW^k - \varepsilon \frac{Z^k}{U} dt \right) - \left(\frac{3m-1}{4} - \varepsilon \right) \frac{|Z|^2}{U} dt. \quad (4.15)$$

Hence we first define a new probability such that

$$d\tilde{W}^k = dW^k - \varepsilon \frac{Z^k}{U} dt$$

is a Brownian motion. Then

$$\begin{aligned} d|Z|^2 &= 2\sqrt{2}m \frac{K_i^j Z_j^k Z^i}{\sqrt{U}} d\tilde{W}^k + \frac{3}{4}(3m-1)(1-m) \frac{|Z|^4}{U^2} dt \\ &\quad + \sum_k \left[\left(\delta |Z^k|^2 + \sqrt{2}(1-m)m |Z|^2 \right) \frac{1}{U} \frac{\theta_k}{\sqrt{U}} + 2m^2 \left| \frac{\theta_k}{\sqrt{U}} \right|^2 \right] dt \\ &\quad + \sum_{k \neq i} \left[2m^2 \left| \frac{\sum_j K_i^j Z_j^k}{\sqrt{U}} \right|^2 dt + \delta \frac{Z^i Z^k}{U} \frac{\sum_j K_i^j Z_j^k}{\sqrt{U}} \right] dt \end{aligned} \quad (4.16)$$

where

$$\delta = 2\sqrt{2}m\varepsilon - \frac{\sqrt{2}(5m-1)m}{2}, \quad \theta_k = \sum_j K_k^j Z_j^k.$$

Since

$$\begin{aligned}
& \sum_k \left[\left(\delta |Z^k|^2 + \sqrt{2}(1-m)m|Z|^2 \right) \frac{1}{U} \frac{\theta_k}{\sqrt{U}} + 2m^2 \left| \frac{\theta_k}{\sqrt{U}} \right|^2 \right] \\
&= 2m^2 \sum_k \left| \frac{\theta_k}{\sqrt{U}} \right|^2 + \left(\delta |Z^k|^2 + \sqrt{2}(1-m)m|Z|^2 \right) \frac{1}{4m^2 U} \\
&\quad - \left(\frac{n(1-m)^2}{4} + \frac{\sqrt{2}\delta(1-m)}{4m} \right) \frac{|Z|^4}{U^2} - \frac{\delta^2}{8m^2 U^2} \sum_k |Z^k|^4
\end{aligned}$$

and

$$\begin{aligned}
& \sum_{k \neq i} \left[2m^2 \left| \frac{\sum_j K_i^j Z_j^k}{\sqrt{U}} \right|^2 dt + \delta \frac{Z^i Z^k}{U} \frac{\sum_j K_i^j Z_j^k}{\sqrt{U}} \right] \\
&= 2m^2 \sum_{k \neq i} \left| \frac{\sum_j K_i^j Z_j^k}{\sqrt{U}} + \frac{\delta}{4m^2} \frac{Z^i Z^k}{U} \right|^2 - \frac{\delta^2}{8m^2 U^2} \sum_{k \neq i} |Z^i|^2 |Z^k|^2.
\end{aligned}$$

Hence

$$d|Z|^2 = 2\sqrt{2}m \frac{Z^i K_i^j Z_j^k}{\sqrt{U}} d\tilde{W}^k + (A+B)dt + \frac{G(\delta)}{4} \frac{|Z|^4}{U^2} dt \quad (4.17)$$

where

$$G(\delta) = 3(3m-1)(1-m) - n(1-m)^2 - \frac{\sqrt{2}(1-m)}{m} \delta - \frac{1}{2m^2} \delta^2, \quad (4.18)$$

$$A = 2m^2 \sum_{k \neq i} \left| \frac{\sum_j K_i^j Z_j^k}{\sqrt{U}} + \frac{\delta}{4m^2} \frac{Z^i Z^k}{U} \right|^2$$

and

$$B = 2m^2 \sum_k \left| \frac{\theta_k}{\sqrt{U}} + \left(\delta |Z^k|^2 + \sqrt{2}(1-m)m|Z|^2 \right) \frac{1}{4m^2 U} \right|^2.$$

Setting

$$\delta = -\sqrt{2}m(1-m)$$

then

$$G(\delta) = [(n+8)m - 2 - n](1-m) \geq 0$$

if and only if

$$1 - \frac{6}{n+8} \leq m \leq 1.$$

Using the same argument as that in Theorem 3.4, we obtain (4.13).

4.3 Estimates on Logarithm of Gradient

Let us prove the following

Theorem 4.2 *If $1 > m \geq \frac{n-1}{n+3}$, then*

$$|\nabla \log u(T, x)| \leq \frac{2\sqrt{\|f_0\|_\infty}}{m^2 \sqrt{T} \left| 2m - 4 - \frac{\sqrt{2}}{m} \beta_2 \right|}, \quad (4.19)$$

where β_2 is given in (4.21), which yields (1.7)

The idea of the proof for this theorem is to identify conditions under which $M = \frac{|Z|^2}{U}$ is a Q -submartingale.

Since

$$d\frac{1}{U} = -(1-m)U^{-2} \sum_k Z^k d\tilde{W}^k + (1-m)\left(\frac{3}{4} - \varepsilon - \frac{m}{4}\right)U^{-2} \frac{|Z|^2}{U} dt,$$

together with (4.16) we get

$$\begin{aligned} UdM &= 2\sqrt{2}m \frac{K_i^j Z_j^k Z^i}{\sqrt{U}} d\tilde{W}^k - (1-m) \frac{|Z|^2}{U} Z^k d\tilde{W}^k \\ &+ \sum_k \left[\left(\beta |Z^k|^2 + \sqrt{2}(1-m)m|Z|^2 \right) \frac{1}{U} \frac{\theta_k}{\sqrt{U}} + 2m^2 \left| \frac{\theta_k}{\sqrt{U}} \right|^2 \right] dt \\ &+ \sum_{k \neq i} \left[2m^2 \left| \frac{\sum_j K_i^j Z_j^k}{\sqrt{U}} \right|^2 dt + \beta \frac{Z^i Z^k}{U} \frac{\sum_j K_i^j Z_j^k}{\sqrt{U}} dt \right] \\ &+ (1-m)(2m - \varepsilon) \frac{|Z|^4}{U^2} dt \end{aligned}$$

where

$$\beta = \delta - 2\sqrt{2}m(1-m), \quad \delta = 2\sqrt{2}m\varepsilon - \frac{\sqrt{2}(5m-1)m}{2}$$

so that

$$\varepsilon = \frac{\sqrt{2}\beta}{4m} + \frac{m+3}{4}.$$

Finally,

$$\begin{aligned} UdM &= 2\sqrt{2}m \frac{K_i^j Z_j^k Z^i}{\sqrt{U}} d\tilde{W}^k - (1-m) \frac{|Z|^2}{U} Z^k d\tilde{W}^k \\ &+ H(\beta) \frac{|Z|^4}{4U^2} + Ldt, \end{aligned}$$

where

$$H(\beta) = (1-m) \left((7+n)m - 3 - n \right) - \frac{2\sqrt{2}\beta(1-m)}{m} - \frac{\beta^2}{2m^2},$$

and

$$L = 2m^2 \sum_k \left| \frac{\theta_k}{\sqrt{U}} + \left(\beta |Z^k|^2 + \sqrt{2}(1-m)m|Z|^2 \right) \frac{1}{4m^2U} \right|^2 + 2m^2 \sum_{k \neq i} \left| \frac{\sum_j K_i^j Z_j^k}{\sqrt{U}} + \frac{\beta}{4m^2} \frac{Z^i Z^k}{U} \right|^2.$$

Hence, if $m \geq \frac{n-1}{n+3}$, then if $\beta \in [\beta_1, \beta_2]$, with

$$\beta_1 = -2\sqrt{2}m(1-m) - m\sqrt{2(1-m)((3+n)m+1-n)} \quad (4.20)$$

and

$$\beta_2 = -2\sqrt{2}m(1-m) + m\sqrt{2(1-m)((3+n)m+1-n)}. \quad (4.21)$$

Recall that

$$dY = \sum_k Z^k d\tilde{W}^k - \left(2m - 4 - \frac{\sqrt{2}}{m}\beta \right) \frac{|Z|^2}{4U} dt$$

and

$$d\tilde{W}^k = dW^k - \left(\frac{\sqrt{2}}{m}\beta + 3 + m \right) \frac{Z^k}{4U} dt$$

Hence,

$$\left| 2m - 4 - \frac{\sqrt{2}}{m}\beta \right| \int_0^T \frac{|Z|^2}{4U} dt \leq 2\|f_0\|_\infty.$$

The same argument as that in Section 3.2 leads (4.19).

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