



# Optimal starting times, stopping times and risk measures for algorithmic trading

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# Optimal starting times, stopping times and risk measures for algorithmic trading

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## Abstract

We derive explicit recursive formulas for Target Close (TC) and Implementation Shortfall (IS) in the Almgren-Chriss framework. We explain how to compute the optimal starting and stopping times for IS and TC, respectively, given a minimum trading size. We also show how to add a minimum participation rate constraint (Percentage of Volume, PVol) for both TC and IS.

We also study an alternative set of risk measures for the optimisation of algorithmic trading curves. We assume a self-similar process (e.g. Lévy process, fractional Brownian motion or fractal process) and define a new risk measure, the  $p$ -variation, which reduces to the variance if the process is a Brownian motion. We deduce the explicit formula for the TC and IS algorithms under a self-similar process.

We show that there is an equivalence between self-similar models and a family of risk measures called  $p$ -variations: assuming a self-similar process and calibrating empirically the parameter  $p$  for the  $p$ -variation yields the same result as assuming a Brownian motion and using the  $p$ -variation as risk measure instead of the variance. We also show that  $p$  can be seen as a measure of the aggressiveness:  $p$  increases if and only if the TC algorithm starts later and executes faster.

From the explicit expression of the TC algorithm one can compute the sensitivities of the curve with respect to the parameters up to any order. As an example, we compute the first order sensitivity with respect to both a local and a global surge of volatility.

Finally, we show how the parameter  $p$  of the  $p$ -variation can be implied from the optimal starting time of TC, and that under this framework  $p$  can be viewed as a measure of the joint impact of market impact (i.e. liquidity) and volatility.

**Keywords:** Quantitative Finance, High-Frequency Trading, Algorithmic Trading, Optimal Execution, Market Impact, Risk Measures, Self-similar Processes, Fractal Processes.

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# 1 Introduction

**Purpose of the paper.** Trading algorithms are used by asset managers to control the trading rate of a large order, performing a balance between trading fast to minimise exposure to market risk and trading slow to minimise market impact (for an overview of quantitative trading methods, see [Lehalle, 2012] and [Abergel et al., 2012]). This balance is usually captured via a cost function which takes into account two joint effects, namely the market impact and the market risk. The first frameworks to be proposed were [Bertsimas and Lo, 1998] and [Almgren and Chriss, 2000], the latter using a mean-variance criteria. More sophisticated cost functions have been already proposed in the academic literature, leading to the use of different optimization approaches like stochastic control (see [Bouchard et al., 2011] or [Guéant et al., 2011]) or stochastic algorithms (see [Pagès et al., 2012]).

From the practitioners' viewpoint, the cost function to choose is far from obvious. An easiest way to proceed is to replace the choice of the cost function by observable features of the market. This is the approach chosen in this paper, where the cost function belongs to the family of mean plus  $p$ -variation, extending the frameworks of both Almgren-Chriss and Gatheral-Schied (see [Gatheral et al., 2010]). Instead of complex cost functions and compute-intensive parameter calibration, this paper proposes a simpler approach that covers a large class of parametrized cost function already in use by practitioners. We calibrate the parameters from observable variables like stopping times and maximum participation rates.

This approach is very flexible and customisable. Indeed, since it depends on a fine-tuning parameter, a practitioner can either calibrate the parameters or modify it by hand to fit her risk budget. A good example is the maximum participation rate: actually most practitioners are using a mean-variance criteria with an arbitrary risk aversion parameter, but add a “control layer” to their algorithms in order to ensure that the participation on real time will never be more than a pre-determined threshold  $q$  (i.e. the trading algorithm will never buy or sell more than  $q$  times the volume traded by the whole market). Here we propose a way to include this constraint into the full optimisation process, at the very first step of the process. Moreover, some traders know that they would like to see a given algorithm finish a buy of  $v^*$  orders on a given stock in no more than  $h$  hours; again we propose a way to implicit the parameters of the cost function to achieve this.

Another important parameter of the cost function is the power of the  $p$ -variation. The question that arises is: would it better to choose  $p = 2$  and implement a mean variance criterion, or to choose  $p = 1$  and be consequently less risk averse? This question is linked to an implicit belief in the mean-reversion and scaling properties of the price formation process [Mandelbrot, 1997]. This paper explores the impact of  $p$  on the properties of the obtained optimal trading curve.

**An optimal trading framework for the target close and implementation shortfall benchmarks with percentage of volume constraints.** A TC (Target Close) algorithm is a trading strategy that aims to execute a certain amount of shares as near as possible to the closing auction price. Since the benchmark with respect to which the TC algorithm is measured is the closing price, the trader has interest in executing most of her order at the close auction. However, if the number of shares to trade is too large, the order cannot be totally executed at the close auction without moving the price too much due to its market impact [Gatheral and Schied, 2012]. Therefore, the trader has to trade some shares during the continuous auction phase (i.e. before the close) following one of the now well-known optimal trading algorithms available, e.g. mean-variance optimisation (following [Almgren and Chriss, 2000]) or stochastic control (like in [Bouchard et al., 2011]).

As we have mentioned above, this paper will stay close to the original Almgren-Chriss framework,

extending the risk measure from the variance to a general  $p$ -variation criterion. The goal of this paper is to explain the practical interpretation of the  $p$ -variation parameter used in the optimisation scheme and show how to choose them optimally in practice.

**Inverting the optimal liquidation problem putting the emphasis on observables of the obtained trading process.** Since the TC (Target Close) benchmark can be compared to a “reverse IS” (Implementation Shortfall) (see equation (8) and following for details), the starting time for a TC is as important as the ending time for an IS. For practitioners this distinction is even more critical since shortening the trading duration of an IS because of an interesting price opportunity can always be justified, but beginning sooner or later than an “*expected optimal start time*” for a TC is more difficult to explain.

The paper also shows that the results obtained for the TC criterion can be applied to the IS criterion because TC and IS are both sides of the same coin. Indeed, on the one hand, the TC has a pre-determined end time, its benchmark is the price at the end of the execution and the starting time is unknown. On the other hand, IS has a pre-determined starting time, its benchmark is the price at the beginning of the execution and the stopping time is unknown. Therefore, there is no surprise that the recursive formula for IS turns out to be exactly the same as that for TC but with the time running backwards.

It is customary for practitioners to put constraints on the maximum participation rate of a trading algorithm (say 20% of the volume traded by the market). Therefore, it is of paramount importance to find a systematic way of computing the starting time of a TC under a percentage of volume (PVol) constraint. Such an “*optimal trading policy under PVol constraint*” is properly defined and solved in this paper. A numerical example with real data is provided, where the optimal trading curves and their corresponding optimal starting times are computed.

Solving the TC problem under constraints allows us to analyze the impact of the parameters of the optimisation criterion on observable variables of the trading process. It should be straightforward for quantitative traders the task to implement our results numerically, i.e. to choose the characteristics of the trading process they would like to target and then infer the proper value of the parameters of the criterion they need.

**Link between a mean  $p$ -variation criterion and self similar price formation processes.** Almgren and Chriss [Almgren and Chriss, 2000] developed a mean-variance framework to trade IS (Implementation Shortfall) portfolios driven by a Brownian motion. More recently, Lehalle [Lehalle, 2009] extended the model to Gaussian portfolios whilst Gatheral and Schied [Gatheral and Schied, 2012] addressed the same problem for the geometric Brownian motion. In this article we extend the analysis to a broad class of non-Brownian models, the so-called self-similar models, which include Lévy processes and fractional Brownian motion (for empirical studies about the self-similarity of intraday data, see [Xu and Gençay, 2003], [Müller et al., 1990] or [Cont et al., 1997]). We study in detail the relationship between the exponent of self-similarity, the choice of the risk measure and the level of aggressiveness of the algorithm. We show that there are two opposite approaches that nevertheless give the same recursive trading formula: one assumes a self-similar process, estimates the exponent of self-similarity  $H$  and chooses the  $p$ -variation via  $p = 1/H$ ; the other assumes a classical Brownian motion and chooses the  $p$ -variation as the risk measure instead of the variance.

In the same way the starting time of a TC or the ending time of an IS can be used as an observable to infer values of parameters of the optimization program, the maximum participation rate is expressed as a function of  $p$  for a mean  $p$ -variance criterion. In the light of this, a quantitative trader who has chosen

to trade no more than 30% of the market volume during a given time interval, can modify the value of  $p$  to fine-tune her execution and respect her constraints.

This paper formalizes an innovative approach of optimal trading based on observable variables, risk budget and participation constraints. By doing so, it opens the door to a framework close to risk neutral valuation of derivative products in optimal trading: instead of choosing the measure under which to compute the expectation of the payoff (because optimal trading is always considered under historical measure), we propose to infer the value of some parameters of the cost function so that the trading process will satisfy some observable characteristics (start time, end time, maximum participation rate, etc). In this framework, instead of being hedged with respect to market prices, the trader will be hedged with respect to the risk-performance profile of an ideal trading process, i.e. a proxy defined a priori.

Notice that we have chosen to extend the usual mean-variance criterion rather than going to more non-parametric approaches like stochastic control. The main reason for this approach is because our framework allows more explicit recurrent formulas, not to mention that our method can be easily extended to any other optimal trading framework.

**Organisation of the article.** In Section 2 we derive a nonlinear, explicit recursive formula for both the TC and IS algorithms with a nonlinear market impact. We explain how to build a TC algorithm under a maximum participation rate constraint (percentage of volume, PVol). We provide a numerical example using real data, in which we computed the trading curves and their optimal starting time. All our computations can be also applied to IS.

In Section 3 we extend the analysis for a class of non-Brownian models called self-similar processes, which include Lévy Processes, fractional Brownian motion and fractal processes. We define an *ad hoc* risk measure, denoted  $p$ -variation, which renders the process linear in time. We show numerically that the exponent of self-similarity  $H$  can be viewed as a fine-tuning parameter for the level of aggressiveness of the TC algorithm under PVol constraint.

In Section 4 we assess the effect of the parameter  $p$  in terms of risk management. We show the existence of an equivalence between risk measures of  $p$ -variation type and self-similar models of exponent  $H$ : choosing a self-similar model, estimating  $H$  and defining  $p = 1/H$  for the risk measure yields the same trading curve as if instead we assume a Brownian motion but change the risk measure from variance to  $p$ -variation. We also study the effect of  $p$  on the starting times for TC and the slopes of the corresponding trading curves. We finish with a discussion on how  $p$  could be implied and used. As a conclusive remark, we show how the parameter  $p$  of the  $p$ -variation can be implied from the optimal starting time of TC, and in that framework  $p$  can be viewed measure of the joint impact of market impact (i.e. liquidity) and volatility.

## 2 Optimal starting and stopping times

### 2.1 A review of the mean-variance optimisation of Almgren-Chriss

This section recalls the framework, notation and results in Almgren and Chriss [Almgren and Chriss, 2000] and Lehalle [Lehalle, 2009]. Suppose we want to trade an asset  $S$  throughout a time horizon  $T > 0$ . As-

sume that we have already set the trading schedule, i.e. we will do  $N$  trades at evenly distributed times

$$0 = t_0 < t_1 < t_2 < \dots < t_N = T.$$

The goal is, given a volume to execute  $v^*$ , find the optimal quantity of shares  $v_n$  to execute at time  $t_n$  that minimise the joint effect of market impact and market risk under the constraint

$$\sum_{i=1}^N v_i = v^* > 0. \quad (1)$$

Define  $\tau = t_n - t_{n-1}$  and assume that the price dynamics follows a Brownian motion, i.e.

$$S_{n+1} = S_n + \sigma_{n+1} \tau^{1/2} \varepsilon_{n+1}, \quad (2)$$

where  $(\varepsilon_n)_{1 \leq n \leq N}$  are i.i.d. normal random variables of mean zero and variance 1, and  $(\sigma_n)_{1 \leq n \leq N}$  are the historical volatilities at the trading times  $(t_n)_{1 \leq n \leq N}$ . Following Almgren and Chriss [Almgren and Chriss, 2000] and Lehalle [Lehalle, 2009], we will model the temporary market impact as a function  $h(v_n)$ , i.e. depending solely on what happens at each trading time. Under this framework, the wealth process (i.e. the full trading revenue upon completion of all trades) is

$$\begin{aligned} W &= \sum_{n=1}^N q_n v_n (S_n + q_n h(v_n)) \\ &= \sum_{n=1}^N q_n v_n S_n + \sum_{n=1}^N v_n h(v_n), \end{aligned} \quad (3)$$

where  $q_n = 1$  if we buy at time  $t_n$  and  $q_n = -1$  if we sell. Under the new variables

$$x_n := \sum_{i=n}^N v_i \quad \Longleftrightarrow \quad v_n = x_n - x_{n+1},$$

and for long-only portfolios (i.e.  $q_n = +1$ ), the wealth process becomes

$$W(x_1, \dots, x_N) = S_0 v^* + \sum_{n=1}^N \sigma_n \tau^{1/2} \varepsilon_n x_n + \sum_{n=1}^N (x_n - x_{n+1}) h(x_n - x_{n+1}). \quad (4)$$

The expectation and variance of the wealth process (3) are, respectively,

$$\mathbb{E}(W) = S_0 v^* + \sum_{n=1}^N (x_n - x_{n+1}) h(x_n - x_{n+1}), \quad \mathbb{V}(W) = \sum_{n=1}^N \sigma_n^2 \tau x_n^2.$$

Therefore, the corresponding mean-variance cost functional for a level of risk aversion  $\lambda > 0$  is

$$\begin{aligned} J_\lambda(x_1, \dots, x_N) &= \mathbb{E}(W) + \lambda \mathbb{V}(W) \\ &= S_0 v^* + \sum_{n=1}^N (x_n - x_{n+1}) h(x_n - x_{n+1}) + \lambda \sum_{n=1}^N \sigma_n^2 \tau x_n^2. \end{aligned} \quad (5)$$

In order to find the optimal trading curve we have to find the points  $(x_1, \dots, x_N)$  that solve the system

$$\frac{\partial J_\lambda}{\partial x_n} = 0, \quad n = 1, \dots, N.$$

If the market impact function  $v_n \mapsto h(v_n)$  is strictly monotone and differentiable for positive values, e.g.  $h(s) = s^\gamma$  with  $\gamma > 0$ , we obtain an explicit recursive algorithm of the form

$$x_{n+1} = f(x_n, x_{n-1})$$

with the constraints  $x_0 = v^*$  and  $x_{N+1} = 0$ .

## 2.2 Derivation of the Target Close (TC) algorithm

As in Almgren [Almgren, 2003], Almgren *et al* [Almgren, 2003] and Bouchaud [Bouchaud, 2010], we will consider a power market impact function, i.e.

$$h(v_n) = \kappa \sigma_n \tau^{1/2} \left( \frac{v_n}{V_n} \right)^\gamma, \quad (6)$$

where  $v_n$  is the amount of shares executed at the  $n$ -th pillar (i.e. at time  $t_n$ ),  $V_n$  is the (historical) volume at the  $n$ -th pillar,  $\sigma_n$  is the normalised volatility at the  $n$ -th pillar, and  $\kappa$  and  $\gamma$  are positive constants. Under this framework, the wealth process (3) takes the form

$$W = \sum_{n=1}^N v_n S_n + \sum_{n=1}^N \kappa \sigma_n \tau^{1/2} v_n \left( \frac{v_n}{V_n} \right)^\gamma, \quad (7)$$

where  $N$  is the number of slices in the trading algorithm. The first term in the right-hand side of (7) is the cost of executing  $v^* := \sum_{i=1}^N v_i$  shares; the second term models the market impact of the execution as a power law of the percentage of volume executed at each pillar  $n = 1, \dots, N$ .

For a TC algorithm, the benchmark is the closing price. Therefore, the wealth process relative to this benchmark is

$$W^\# = W - S_N \sum_{n=1}^N v_n. \quad (8)$$

Assume a Brownian motion model for the asset, i.e.

$$S_{n+1} = S_n + \sigma_{n+1} \tau^{1/2} \varepsilon_{n+1}.$$

Under the change of variables

$$x_n := \sum_{i=1}^n v_i \quad \Longleftrightarrow \quad v_n = x_n - x_{n-1}$$

the relative wealth process takes the form

$$\begin{aligned} W^\sharp &= -\sum_{n=1}^N x_n \sigma_n \tau^{1/2} \varepsilon_n + \sum_{n=1}^N \kappa \sigma_n \tau^{1/2} \frac{(x_n - x_{n-1})^{\gamma+1}}{V_n^\gamma} \\ &= \left( -\sum_{n=1}^N x_n \sigma_n \varepsilon_n + \sum_{n=1}^N \kappa \sigma_n \frac{(x_n - x_{n-1})^{\gamma+1}}{V_n^\gamma} \right) \tau^{1/2}. \end{aligned}$$

Since the time-step  $\tau^{1/2}$  is a constant multiplicative factor, we can consider a normalised relative wealth

$$\tilde{W} := \frac{W^\sharp}{\tau^{1/2}}.$$

We are not losing any generality with the normalisation because it is equivalent to use a normalised volatility  $\tilde{\sigma}_n := \sigma_n \tau^{1/2}$ . Under this new framework, the average and variance of  $\tilde{W}$  are, respectively,

$$\mathbb{E}(\tilde{W}) = \sum_{n=1}^N \kappa \sigma_n \frac{(x_n - x_{n-1})^{\gamma+1}}{V_n^\gamma}, \quad \mathbb{V}(\tilde{W}) = \sum_{n=1}^N x_n^2 \sigma_n^2.$$

The corresponding mean-variance functional is thus

$$\begin{aligned} J_\lambda(x_1, \dots, x_N) &= \mathbb{E}(\tilde{W}) + \lambda \mathbb{V}(\tilde{W}) \\ &= \sum_{n=1}^N \kappa \sigma_n \frac{(x_n - x_{n-1})^{\gamma+1}}{V_n^\gamma} + \lambda \sum_{n=1}^N x_n^2 \sigma_n^2. \end{aligned}$$

The optimal trading curve is determined by solving

$$\frac{\partial J_\lambda}{\partial x_n} = 0, \quad n = 1, \dots, N,$$

i.e.

$$\kappa \sigma_n (\gamma + 1) \frac{(x_n - x_{n-1})^\gamma}{V_n^\gamma} - \kappa \sigma_{n+1} (\gamma + 1) \frac{(x_{n+1} - x_n)^\gamma}{V_{n+1}^\gamma} + 2\lambda \sigma_n^2 x_n = 0.$$

Returning to the variables  $v_n$  we get

$$\kappa \sigma_n (\gamma + 1) \left( \frac{v_n}{V_n} \right)^\gamma - \kappa \sigma_{n+1} (\gamma + 1) \left( \frac{v_{n+1}}{V_{n+1}} \right)^\gamma + 2\lambda \sigma_n^2 \left( \sum_{i=1}^n v_i \right) = 0.$$

Finally, we obtain an explicit, nonlinear recursive formula of the optimal trading curve for a TC algorithm:

$$v_{n+1} = V_{n+1} \left[ \frac{\sigma_n}{\sigma_{n+1}} \left( \frac{v_n}{V_n} \right)^\gamma + \frac{2\lambda}{\kappa(\gamma + 1)} \frac{\sigma_n^2}{\sigma_{n+1}} \left( \sum_{i=1}^n v_i \right) \right]^{1/\gamma}. \quad (9)$$

### 2.3 Derivation of the Implementation Shortfall (IS) algorithm

For an IS algorithm, the starting time is given and we have to find the optimal stopping time for our execution. Since the benchmark is the price at the moment when the execution starts, the relative wealth of an IS algorithm is

$$W^\sharp = W - S_1 \sum_{n=1}^N v_n.$$

Using the change of variables

$$x_n := \sum_{i=n}^N v_i \quad \Longleftrightarrow \quad v_n = x_n - x_{n+1},$$

and equations (4) and (12) it can be shown that the relative wealth process is

$$\begin{aligned} W^\sharp &= \sum_{n=1}^N x_n \sigma_n \tau^{1/2} \varepsilon_n + \sum_{n=1}^N \kappa \sigma_n \tau^{1/2} \frac{(x_n - x_{n-1})^{\gamma+1}}{V_n^\gamma} \\ &= \left( \sum_{n=1}^N x_n \sigma_n \varepsilon_n + \sum_{n=1}^N \kappa \sigma_n \frac{(x_n - x_{n-1})^{\gamma+1}}{V_n^\gamma} \right) \tau^{1/2}. \end{aligned}$$

As in the TC case, we can consider a normalised relative wealth

$$\tilde{W} := \frac{W^\sharp}{\tau^{1/2}},$$

whose average and variance are, respectively,

$$\mathbb{E}(\tilde{W}) = \sum_{n=1}^N \kappa \sigma_n \frac{(x_n - x_{n+1})^{\gamma+1}}{V_n^\gamma}, \quad \mathbb{V}(\tilde{W}) = \sum_{n=1}^N x_n^2 \sigma_n^2.$$

In consequence, the corresponding mean-variance functional is

$$\begin{aligned} J_\lambda(x_1, \dots, x_N) &= \mathbb{E}(\tilde{W}) + \lambda \mathbb{V}_p(\tilde{W}) \\ &= \sum_{n=1}^N \kappa \sigma_n \frac{(x_n - x_{n+1})^{\gamma+1}}{V_n^\gamma} + \lambda \sum_{n=1}^N x_n^2 \sigma_n^2. \end{aligned}$$

The optimal trading curve is determined by solving

$$\frac{\partial J_\lambda}{\partial x_n} = 0, \quad n = 1, \dots, N,$$

i.e.

$$\kappa \sigma_n (\gamma + 1) \frac{(x_n - x_{n+1})^\gamma}{V_n^\gamma} - \kappa \sigma_{n-1} (\gamma + 1) \frac{(x_{n-1} - x_n)^\gamma}{V_{n+1}^\gamma} + 2\lambda \sigma_n^2 x_n = 0.$$

Returning to the variables  $v_n$  we get

$$\kappa \sigma_n (\gamma + 1) \left( \frac{v_n}{V_n} \right)^\gamma - \kappa \sigma_{n-1} (\gamma + 1) \left( \frac{v_{n-1}}{V_{n+1}} \right)^\gamma + 2\lambda \sigma_n^2 \left( \sum_{i=n}^N v_i \right) = 0.$$

We thus obtain the recursive nonlinear formula for the optimal IS trading curve:

$$v_{n-1} = V_{n-1} \left[ \frac{\sigma_n}{\sigma_{n-1}} \left( \frac{v_n}{V_n} \right)^\gamma + \frac{2\lambda}{\kappa(\gamma+1)} \frac{\sigma_n^2}{\sigma_{n-1}} \left( \sum_{i=n}^N v_i \right) \right]^{1/\gamma}. \quad (10)$$

Observe that (10) is essentially the same recursive formula than the TC formula (14), the only difference being that TC is forward in time whilst IS is backwards in time. Therefore, all the analysis we will perform for the TC algorithm can be applied as well for the IS algorithm: it suffices to run the TC algorithm backwards in time to compute IS. Moreover, following this backwards time idea, it is easy to replicate for IS everything that we will say on TC, especially adding a maximum participation rate constraint and computing the starting time.

## 2.4 Adding constraints: Percentage of Volume (PVol)

The TC algorithm can have a constraint of , meaning that the size of each slice cannot exceed a fixed percentage of the current available volume. This algorithm is called Percentage of Volume (PVol). Under a PVol constraint, the trading slices  $v_n$  of the TC algorithm satisfy the constraint

$$v_n \leq qV_n, \quad q \in (0, 1).$$

It is worth to notice that the PVol algorithm is not a solution of the Almgren-Chriss optimisation. Indeed, if it were then

$$\frac{v_n}{V_n} = p \quad \forall n = 1, \dots, N,$$

and from (9) we would have that

$$\sum_{i=1}^n v_i = 0.$$

In consequence, since  $v_n \geq 0$  it follows that  $v_n = 0$  for all  $n$ , which contradicts (1).

In general, if two adjacent pillars  $n$  and  $n+1$  satisfy the PVol constraint then the previous argument shows that  $v_i = 0$  for all  $i = 1, \dots, N$ . Therefore, the two algorithms TC and PVol are mutually exclusive. This implies that a classical optimisation scheme of TC with the PVol constraint via Lagrange multipliers is not straightforward, to say the least. We thus have to find another way to obtain a solution of the TC algorithm under the PVol constraint.

From (9) we see that given  $v_n$ , the corresponding  $v_{n+1}$  depends on  $\sum_{i=1}^n v_i$ , i.e. the cumulative execution up to  $n$ , which implies that curve is in general increasing. Therefore, in order to satisfy the constraint of maximum percentage of volume (PVol), if the total volume to execute is large then the algorithm has to be divided into two patterns:

1. As long as the constraint of maximum participation rate (PVol) is not reached, we execute the slices according to the Almgren-Chriss recursive formula. This corresponds to the TC pattern.
2. As soon as the PVol constraint is attained, the algorithm executes the minimum between the TC curve and PVol curve.

Loosely speaking, we start with a TC algorithm, but once the slices are *saturated* we switch to a PVol algorithm until the end of the execution. However, it can happen that the algo switches back to TC if the PVol curve is bigger at a further pillar; this situation is exceptional though, save for cases where the volume curve presents sharp peaks or gaps.

It is worth to mention that adding a PVol constraint to IS is the same as adding the constraint for TC and running the TC algorithm backwards.

## 2.5 Computing the optimal stopping time for TC

Let us describe in detail all the steps of our TC algorithm under PVol constraint. Let  $n_0$  be the starting time,  $n_1$  the switching time (i.e. when we change from TC to PVol) and  $\alpha$  the number of shares we trade at pillar  $n_0$ .

1. According to the historical estimates of the available volume at the close auction, plus the desired participation rate, we define the execution at the pillar  $n = 103$  (the close auction), denoted  $v^\sharp$ .
2. We compute the Almgren-Chriss algorithm for the residual shares  $v^b = v^* - v^\sharp$  i.e. the shares to execute in continuous, outside of the close auction. We start with  $n_0 = 1$  and  $n_1 = 102$  and launch the TC recursive argument (9). Since the algorithm is completely determined by  $v_1 = \alpha$ , it suffices to find the right  $\alpha$  such that the cumulative shares at  $n_1 = 102$  are equal to  $v^b$ .
3. We compare the trading curve of the previous step with the PVol curve. If the PVol constraint is satisfied then we are done. If not, we *saturate* pillar  $n = 102$  with the PVol constraint and redefine the parameters:  $v^b$  is now the shares to execute outside both pillars 103 and 102, i.e.  $v^b = v^b - v_{102}$ , whilst  $n_1$  is set to 101, i.e.  $n_1 = n_1 - 1$ .
4. Eventually, we will obtain a TC curve starting at  $n_0 = 1$  that switches to PVol at  $n_1 \leq 102$ , satisfying the constraint. Moreover, the algorithm finds the right  $\alpha$  at  $n_0 = 1$  such that the total execution from  $n = 1$  to  $n = 103$  is equal to  $v^*$ . Remark that the whole algorithm executes TC between  $n_0 = 1$  to  $n_1$ , PVol between  $n_1$  and  $n = 102$ , and the desired participation at the close auction at  $n = 103$ .
5. In order to find the right starting time  $n_0$ , we define a minimal trading size for each slice, that we denote  $\alpha_{min}$ . Let  $\alpha_0$  be the minimum of the trading curve we have already found in the previous step. If  $\alpha_0 < \alpha_{min}$  then we advance one pillar, i.e.  $n_0$  passes from 1 to 2, and we recompute the trading curve. In order to continue hitting the target we change  $\alpha$  for the cumulative trades of the previous step up to pillar  $n = 2$ . We continue until we find the first pillar  $n_0$  such that  $\alpha_0 \geq \alpha_{min}$ ; notice that in this case  $\alpha$  will be the cumulative trades of the previous step up to the pillar  $n_0$ .

Therefore,  $n_1$  is determined by the PVol constraint whilst  $n_0$  is determined by the minimal trading size constraint  $\alpha_{min}$ . Notice however that the optimal starting pillar  $n_0$  is determined after  $n_1$ , which implies that  $n_0$  depends not only on  $\alpha_0$  but also on the rest of the parameters, in particular the PVol curve, the participation rate at the close auction and the market impact parameters.

Observe that there is a systematic way of computing the stopping pillar for an IS algorithm: it corresponds to the *backwards or symmetrical image* of the starting time we computed for the TC algorithm.

## 2.6 Numerical results

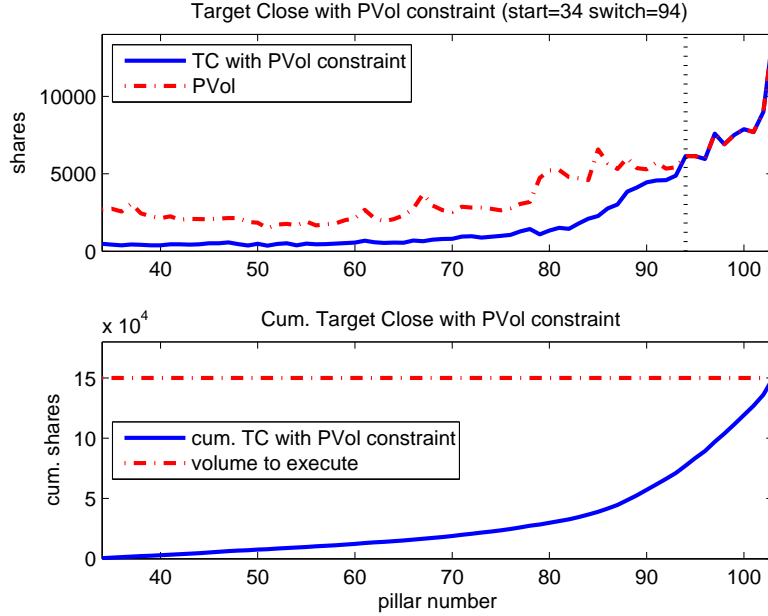


Figure 1: Cumulative TC curves under PVol constraint.

In the first plot of Figure 1 we have the TC curve (solid line) under PVol constraint vs the PVol curve (broken line) of stock AIRP.PA (Air Liquide). In the second plot we have the cumulative execution of the TC curve (solid line) under PVol constraint vs the volume to execute  $v^*$  (broken line). The parameters we used are  $v^* = 150,000$  shares,  $\alpha_{min} = 500$  shares and a maximum participation of 20% in both the continuous trading period and the close auction. The historical volatility and volume curves, the market impact parameters  $\kappa$  and  $\gamma$  and the risk-aversion coefficient  $\lambda$  were provided by the Quantitative Research at Cheuvreux - Cr dit Agricole.

In Figure 1 we can also observe that the algorithm finds the optimal starting time at pillar  $n = 34$  (beginning of the horizontal axis), at which it starts to execute the order following the TC algorithm based on the Almgren-Chriss optimisation. At pillar  $n = 94$  (vertical line) the algorithm switches to PVol in order to satisfy the constraint. Moreover, during the whole execution, the PVol constraint has been satisfied. In the second plot we can see that the TC algorithm under PVol constraint successfully executed the whole order.

### 3 Non-Brownian models: self-similar processes

#### 3.1 The $p$ -variation model

Let  $p > 1$  and  $y = (y_1, \dots, y_N) \in \mathbb{R}^N$  be a random vector of mean zero. We define the  $p$ -variation of  $y$  as

$$\mathbb{V}_p(y) := \sum_{n=1}^N \mathbb{E}[|y_n|^p].$$

The  $p$ -variation  $\mathbb{V}_p(y)$  and the  $l_p$ -norm in  $\mathbb{R}^N$  are related via

$$\|y\|_p = \mathbb{V}_p(y)^{1/p}.$$

Notice that if  $y = (y_1, \dots, y_N)$  is a time series of i.i.d. random variables then the 2-variation reduces to the variance, i.e.

$$\mathbb{V}_2(y) = \text{Var}(y).$$

Moreover, it is easy to show that the  $p$ -variation defines a metric on  $\mathbb{R}^N$ , and since all norms in  $\mathbb{R}^N$  are equivalent there exist  $0 < \beta_1 < \beta_2$  such that

$$\beta_1 \|y\|_p \leq \|y\|_2 \leq \beta_2 \|y\|_p.$$

Therefore, the variance (i.e. the 2-variation) and the  $p$ -variation are two equivalent metrics on  $\mathbb{R}^N$ , and in particular

$$\mathbb{V}_p(y) \sim \mathbb{V}_2(y)^{p/2}. \quad (11)$$

Now let us define the  $p$ -variation for a special family of functions of random variables. Let  $y = (y_1, \dots, y_N)$  a random vector of mean zero and consider the function  $F : \mathbb{R}^N \rightarrow \mathbb{R}$  defined as

$$F(y) = \sum_{n=1}^N y_n.$$

We define the  $p$ -variation of  $F$  as

$$\mathbb{V}_p(F) = \sum_{n=1}^N \mathbb{E}[|y_n|^p].$$

Observe that if  $y = (y_1, \dots, y_N)$  is a time series of mean zero then  $\mathbb{V}_p(F)$  is the sample  $p$ -th moment of the time series  $y$  multiplied by  $N$ . Finally, for general functions  $F$  such that

$$F - \mathbb{E}(F) = \sum_{n=1}^N y_n$$

we define their  $p$ -variation as

$$\mathbb{V}_p(F) := \mathbb{V}_p(F - \mathbb{E}(F)) = \mathbb{V}_p(y).$$

It is worth o remark that if the random variables  $y = (y_1, \dots, y_N)$  are i.i.d. of mean zero and variance 1 then the 2-variation and the variance of  $y$  coincide:

$$\mathbb{V}_2(F) = \text{Var}(y).$$

### 3.2 Optimal trading algorithms using $p$ -variance as risk measure

Let  $H \in (0, 1)$  and assume that the price dynamics is self-similar, i.e.

$$S_{n+1} = S_n + \sigma_{n+1} \tau^H \varepsilon_{n+1}, \quad (12)$$

where  $H \in (0, 1)$  and  $(\varepsilon_n)_{1 \leq n \leq N}$  are i.i.d. random variables such that  $\mathbb{E}[\varepsilon_n] = 0$ . We will assume a power market impact of the form

$$h(v_n) = \kappa \sigma_n \tau^H \left( \frac{v_n}{V_n} \right)^\gamma. \quad (13)$$

In order to use the  $p$ -variation as a risk measure, we have to choose the right  $p$ . From (12) we see that if  $H = 1/2$  we recover the classical Brownian motion, for which the variance is the most common choice for a risk measure. In this case we have  $H = 1/2$  and  $p = 2$ , which implies that the risk measure is linear in time. This suggests that the correct choice of  $p$  is  $p = 1/H$ , since it is the only  $p$  that renders the risk  $p$ -variation as a risk measure linear in time.

We would like to remark that the idea of a risk measure that is linear in time was also introduced by Gatheral and Schied [Gatheral and Schied, 2012], where the risk measure was the expectation of the time-average. The advantage of our approach is that we do not fix a priori the dynamics of the price process. Indeed, we first find empirically the right exponent of self-similarity  $H$  and then we choose the correct risk measure via  $p = 1/H$ .

In order to derive the recursive formula for a process following (12), we normalise the relative wealth as in the previous case of Brownian motion. Under this framework, the normalised relative wealth of a TC algorithm is

$$\tilde{W} = - \sum_{n=1}^N x_n \sigma_n \varepsilon_n + \sum_{n=1}^N \kappa \sigma_n \frac{(x_n - x_{n-1})^{\gamma+1}}{V_n^\gamma}.$$

We will assume that the process (12) is normalised, i.e.  $\mathbb{E}[|\varepsilon_n|^p] = 1$  for all  $n$ . In the case of Brownian motion ( $H = 1/2$  and  $p = 2$ ) this is equivalent to suppose that the increments  $(\varepsilon_n)_{1 \leq n \leq N}$  have variance 1. Under this framework, the average and  $p$ -variation of  $\tilde{W}$  are

$$\mathbb{E}(\tilde{W}) = \sum_{n=1}^N \kappa \sigma_n \frac{(x_n - x_{n-1})^{\gamma+1}}{V_n^\gamma}, \quad \mathbb{V}_p(\tilde{W}) = \sum_{n=1}^N x_n^p \sigma_n^p.$$

Therefore, the corresponding  $p$ -functional is

$$\begin{aligned} J_p(x_1, \dots, x_N) &= \mathbb{E}(\tilde{W}) + \lambda \mathbb{V}_p(\tilde{W}) \\ &= \sum_{n=1}^N \kappa \sigma_n \frac{(x_n - x_{n-1})^{\gamma+1}}{V_n^\gamma} + \lambda \sum_{n=1}^N x_n^p \sigma_n^p. \end{aligned}$$

The optimal trading curve is determined by solving

$$\frac{\partial J_p}{\partial x_n} = 0, \quad n = 1, \dots, N,$$

i.e.

$$\kappa\sigma_n(\gamma+1)\frac{(x_n-x_{n-1})^\gamma}{V_n^\gamma} - \kappa\sigma_{n+1}(\gamma+1)\frac{(x_{n+1}-x_n)^\gamma}{V_{n+1}^\gamma} + p\lambda\sigma_n^p x_n^{p-1} = 0.$$

Returning to the variables  $v_n$  we get

$$\kappa\sigma_n(\gamma+1)\left(\frac{v_n}{V_n}\right)^\gamma - \kappa\sigma_{n+1}(\gamma+1)\left(\frac{v_{n+1}}{V_{n+1}}\right)^\gamma + p\lambda\sigma_n^p\left(\sum_{i=1}^n v_i\right)^{p-1} = 0.$$

We thus obtain the recursive nonlinear formula for the optimal TC trading curve for a self-similar process:

$$v_{n+1} = V_{n+1} \left[ \frac{\sigma_n}{\sigma_{n+1}} \left(\frac{v_n}{V_n}\right)^\gamma + \frac{p\lambda}{\kappa(\gamma+1)} \frac{\sigma_n^p}{\sigma_{n+1}} \left(\sum_{i=1}^n v_i\right)^{p-1} \right]^{1/\gamma}. \quad (14)$$

As in the Brownian case, if we were interested in the IS algorithm then an argument similar to the previous one would show that the optimal trading curve for IS satisfies

$$v_{n-1} = V_{n-1} \left[ \frac{\sigma_n}{\sigma_{n-1}} \left(\frac{v_n}{V_n}\right)^\gamma + \frac{p\lambda}{\kappa(\gamma+1)} \frac{\sigma_n^p}{\sigma_{n-1}} \left(\sum_{i=n}^N v_i\right)^{p-1} \right]^{1/\gamma}. \quad (15)$$

### 3.3 Examples of self-similar processes

Amongst the class of continuous stochastic processes that admit a discretisation of the form (12), we have three processes in mind: Lévy processes, fractional Brownian motion and fractal processes (for more details we suggest Bacry *et al* [Bacry et al., 2001], Bouchaud and Potters [Bouchaud and Potters, 2004], Embrechts [Embrechts, 2002], Mandelbrot [Mandelbrot and Hudson, 2004] and Mantegna and Stanley [Mantegna and Stanley, 1994]):

- **Truncated Lévy processes.**  $p$ -stable Lévy processes are the only self-similar processes satisfying (12) with  $H = 1/p$  and with independent, stationary increments. If  $p = 2$  we recover the classical Brownian motion. However, for such processes the  $p$ -th moment is infinite, and as such they cannot be used in our framework. Nevertheless, one can consider the so-called *truncated* Lévy distributions, which are Lévy within a bounded interval and exponential on the tails. This allows moments of any order, in particular the  $p$ -th moment, whilst within the bounded interval we keep the self-similarity given by (12).
- **Fractional Brownian motion.** The fractional Brownian motion is the only self-similar process with stationary, Gaussian increments. Its exponent of self-similarity  $H$  is called *Hurst exponent*). If  $H = 1/2$  we recover the classical Brownian motion. The fractional Brownian motion has moments of all orders, hence the  $p$ -variation is well-defined and we can apply our model. However, for  $H \neq 1/2$  the increments are auto-correlated (positively if  $H > 1/2$  and negatively if  $H < 1/2$ ) and our model does not take into account the auto correlations. Therefore, we can consider our model as an approximation when auto correlations are weak with respect to the market impact and the  $p$ -variance.

- **Multifractal processes.** Multifractal processes are defined as follows. Given a stochastic process  $X(t)$  its fluctuation is defined as

$$\delta_l X(t) := X(t+l) - X(t).$$

For any  $q > 0$  define

$$m(q, l) := \mathbb{E}[|\delta_l X(t)|^q].$$

We say that  $X(t)$  is multifractal of exponents  $\zeta(q)$  if for any  $q > 0$  there exists  $K(q) > 0$  such that

$$m(q, l) = K(q)l^{\zeta(q)}.$$

In the case where  $\zeta(q)$  is linear, i.e.  $\zeta(q) = qH$  the process  $X(t)$  is called monofractal. If  $\zeta(q)$  is not linear then  $X(t)$  is called multifractal. Notice that all self-similar processes are monofractal, in particular the fractional Brownian motion and Lévy processes. However, we will continue to use the term *self-similar*, even for monofractal process, since it is more common in the literature.

### 3.4 Numerical results

In Figure 2 we plotted three TC curves under the PVol constraint for three different self-similarity exponents  $H$ , which gives three different  $p$ 's for the  $p$ -variation (recall  $p = 1/H$ ).

| $H$  | $p$ | start pillar | switch pillar |
|------|-----|--------------|---------------|
| 0.55 | 1.8 | 17           | 102           |
| 0.50 | 2.0 | 34           | 94            |
| 0.45 | 2.2 | 50           | 89            |

Our numerical example renders the following evidence, which has been found in all runs we have performed:

- If  $H$  increases then the starting pillar of the execution decreases, i.e. the execution starts earlier.
- If  $H$  increases then the pillar at which we switch from TC to PVol increases, i.e. the PVol constraint is saturated later.

Since starting the execution later and saturating the PVol constraint earlier is related to higher levels of aggressiveness, we can infer from Figure 2 that the level of aggressiveness of TC under the PVol constraint decreases as  $H$  increases. This finding is quite natural if we assume that the model is a fractional Brownian motion and  $H$  is the Hurst exponent:

- For  $H < 1/2$  the process has negative auto correlations, i.e. it behaves as a mean-reverting process. Therefore, the market impact is reduced because prices go back to their level after an execution. In consequence, we can execute the order faster than in the case of a classical Brownian motion: we start the execution later and we go as fast as possible, and as such we saturate the constraint earlier.
- For  $H > 1/2$  the process has positive auto correlations, i.e. it has a trend. Therefore, the market impact is of paramount importance because if we execute too fast then prices will move in the wrong direction. In consequence, we start the execution earlier and we go as slow as possible, and as such we saturate the constraint later.

In the next section we will study in detail, in the TC algorithm without PVol constraint, the relation between the risk measures of  $p$ -variation type and both the starting time and the slope at the last pillar.

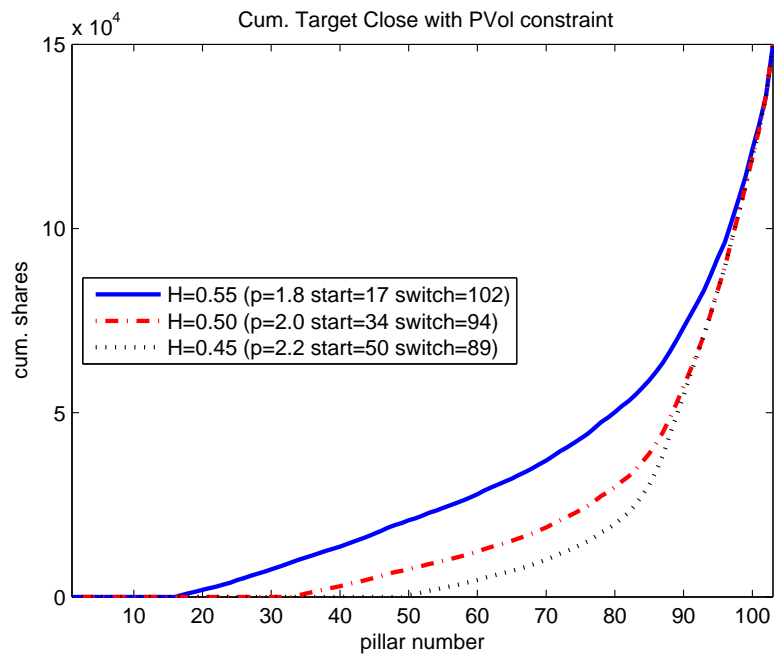


Figure 2: Cumulative TC curves under PVol constraint for different values of  $H$ .

## 4 Assessing the effects of the risk measure

In this section we consider the TC algorithm without PVol constraint and without participation at the close auction (see Figure 4). We show that the choice of the parameter  $p$  for the  $p$ -variation plays a crucial role in both the model of the asset we are trading and the weight we give to the market risk with respect to the market impact.

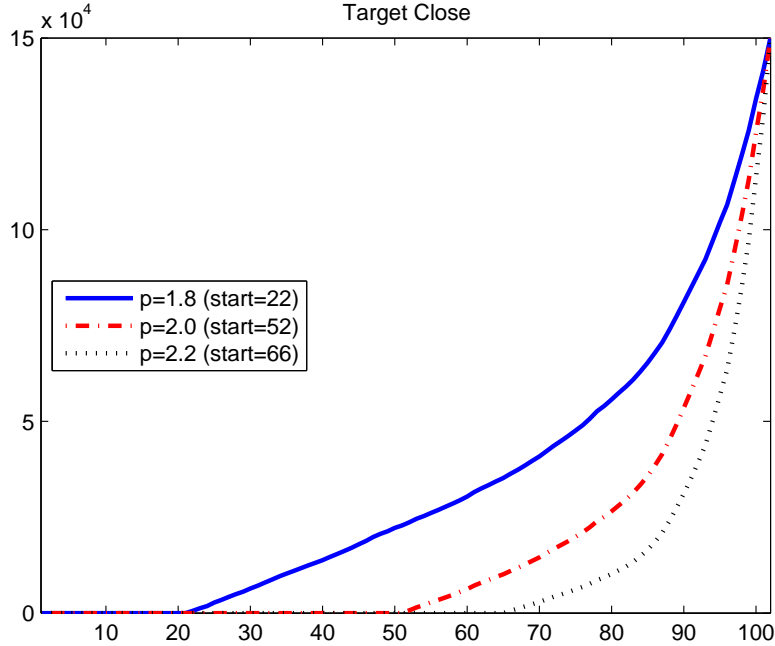


Figure 3: TC curves. Notice that the number of shares traded at every pillar is above the minimum  $\alpha_{min} = 500$ .

### 4.1 Equivalence between risk measures and models

If we compare the recursive formula (9) for a Brownian motion with the recursive formula (14) for a self-similar process, we see that the former can be recovered from the latter by setting  $p = 2$ . Therefore, formula (14) can be derived for a Brownian model when the risk measure is not the variance but the  $p$ -variation.

In consequence, (14) is independent of the model we choose for the asset: assuming a self-similar process, estimating empirically its exponent of self-similarity  $H$  and defining  $p = 1/H$  is equivalent to assuming a Brownian model, choosing  $p$  and using the  $p$ -variation as the risk measure instead of the variance.

A direct consequence of this analysis is that the risk measure is of paramount importance. Indeed, it not only determines the weight we impose to the market risk but it is implicitly related to a model choice. Indeed, choosing  $p > 2$  is equivalent to choosing a self-similar process with  $H < 1/2$ , which for the

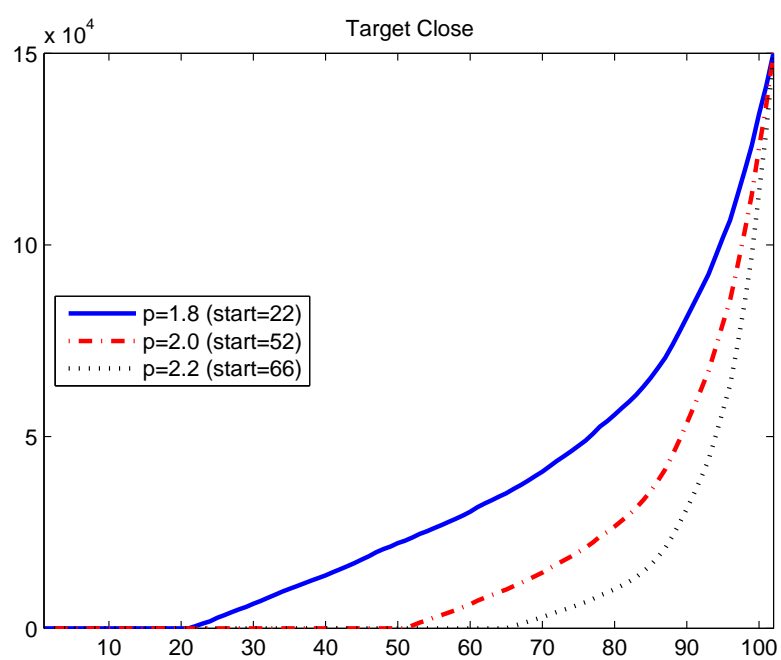


Figure 4: Cumulative TC curves.

fractional Brownian motion would mean a process with negative auto correlations, i.e. a mean-reverting process. On the other hand, choosing  $p < 2$  implies  $H > 1/2$ , hence the corresponding fractional Brownian motion has positive auto correlations, i.e. it has a trend.

In summary:

**Proposition 1** *In order to obtain the recursive formula (14) for a TC algorithm via an Almgren-Chriss optimisation, the following two paths are equivalent:*

1. *Assuming a self-similar model for the asset, calibrating empirically its exponent of self-similarity  $H$  and choosing the  $p$ -variation as the risk measure with  $p = 1/H$ .*
2. *Assuming a Brownian motion model for the asset and choosing the  $p$ -variation as the risk measure.*

## 4.2 First-order effects

Using the explicit expression (14) we can find explicitly the sensitivity of the curve with respect to the risk parameter  $p$ .

**Proposition 2**

$$\frac{\partial v_{n+1}}{\partial p} = A_{n+1} \frac{\partial v_n}{\partial p} + B_{n+1} \left( \sum_{i=1}^n \frac{\partial v_i}{\partial p} \right) + C_{n+1}, \quad (16)$$

where

$$\begin{aligned} A_{n+1} &= \left( \frac{\sigma_n}{\sigma_{n+1}} \right) \left( \frac{V_{n+1}}{V_n} \right)^\gamma \left( \frac{v_n}{v_{n+1}} \right)^{\gamma-1} > 0, \\ B_{n+1} &= \frac{\lambda p(p-1) v_{n+1}^{1-\gamma} V_{n+1}^\gamma \sigma_n^p}{k\gamma(\gamma+1)\sigma_{n+1}} \left( \sum_{i=1}^n v_i \right)^{p-2} > 0, \\ C_{n+1} &= \frac{\lambda p(p-1) v_{n+1}^{1-\gamma} V_{n+1}^\gamma \sigma_n^p}{k\gamma(\gamma+1)\sigma_{n+1}} \left\{ 1 + p \log \left[ \sigma_n \left( \sum_{i=1}^n v_i \right) \right] \right\}. \end{aligned}$$

Moreover, if

$$\sigma_n \left( \sum_{i=1}^n v_i \right) > e^{-1/p} \quad (17)$$

then  $C_{n+1} > 0$  and there exist  $D_{n+1} > 0$  and  $E_{n+1} > 0$  such that

$$\frac{\partial v_{n+1}}{\partial p} = D_{n+1} \frac{\partial v_1}{\partial p} + E_{n+1}. \quad (18)$$

**Proof:** Equation (16) is a straightforward computation using (14) and recalling that  $v_i = v_i(p)$  for all  $i = 1, \dots, n$ . Equation (18) can be obtained from (16) and (17) via a recursive argument.  $\square$

From Proposition 2 it follows that studying the slope of the cumulative TC curve at the first pillar  $n = 1$  is equivalent to study the slope at the last pillar  $n = N$ . Moreover, if (17) holds then using (18) we can assess the effect of  $p$  on the curves when we fix the first or last pillar:

- If we fix the starting pillar  $v_1$  then  $\frac{\partial v_1}{\partial p} = 0$ , which implies that  $\frac{\partial v_N}{\partial p} > 0$ . Therefore, if  $p$  increases then the final execution  $v_N$  increases.
- If we fix the final pillar  $v_N$  then  $\frac{\partial v_N}{\partial p} = 0$ , which implies that  $\frac{\partial v_1}{\partial p} < 0$ . Therefore, if  $p$  increases then the first pillar  $v_1$  decreases.

Finally, from the explicit expression (14) we can obtain any desired sensitivity up to any order. For example, the first-order sensitivity to the volatility is as follows.

**Proposition 3**

$$\frac{\partial v_{n+1}}{\partial \sigma_{n+1}} = -\frac{1}{\gamma} \left( \frac{1}{\sigma_{n+1}} \right)^{1+1/\gamma} \left[ \sigma_n \left( \frac{V_{n+1}}{V_n} \right)^\gamma v_n^\gamma + \frac{\lambda p V_{n+1}^\gamma \sigma_n^p}{k(\gamma+1)} \left( \sum_{i=1}^n v_i \right)^{p-1} \right]^{1/\gamma} < 0.$$

If the volatility is constant, i.e.  $\sigma_i = \sigma$  for all  $i = 1, \dots, n$  then

$$\frac{\partial v_{n+1}}{\partial \sigma} = \frac{\lambda p(p-1) v_{n+1}^{1-\gamma} V_{n+1}^\gamma \sigma^{p-2}}{k\gamma(\gamma+1)} \left( \sum_{i=1}^n v_i \right)^{p-1} > 0.$$

We can interpret the difference in sign in Proposition 3 as follows.

- If we have already executed the orders up to pillar  $n$  and at pillar  $n+1$  there is a sudden surge of volatility then the algorithm will execute less shares  $v_{n+1}$  than what it would have done if the surge did not take place. Indeed, if  $\sigma_{n+1}$  rises then both the market impact and market risk at pillar  $n+1$  rise, hence in order to counter this *local rise of market impact and market risk*, or equivalently to *keep the local risk budget constant*, the algorithm trades less.
- If the volatility is constant then according to (14) the volatility only appears on the market risk. Therefore, in order to minimise the *global rise on market risk* the algorithm will trade faster, i.e. will execute more shares per pillar.

### 4.3 Risk measures, starting times and slopes for the TC algorithm

The aggressiveness of a TC algorithm can be measured in terms of both the starting time and the slope of the trading curve: an algorithm is more aggressive if it starts later and it executes *faster* the trades i.e. it puts more shares per pillar and the rate of change between consecutive pillars is bigger.

Our numerical simulations confirm the analytical results of the previous section: the optimal starting time and the slope of the trading curve are both monotone increasing in  $p$  (see Figures 5 and 6, respectively). Under this framework,  $p$  can be viewed as a tuning parameter for aggressiveness:

**Proposition 4** *The parameter  $p$  in the recursive formula (14) measures the level of aggressiveness of the TC algorithm. More precisely:*

1.  $p$  increases  $\iff$  the optimal starting time increases.
2.  $p$  increases  $\iff$  the slope of the cumulative trading curve at the last pillar increases.

*In consequence,  $p$  increases if and only if the TC algorithm is more aggressive, i.e. it starts the execution later and executes more at each pillar.*

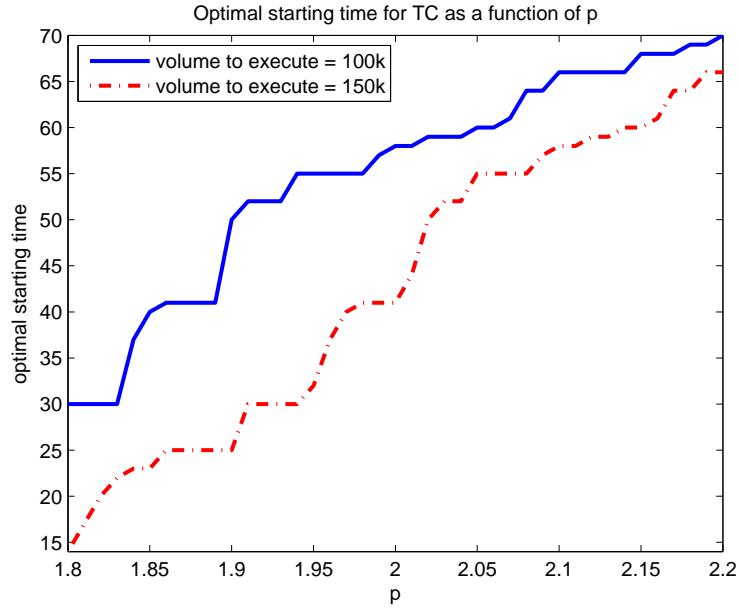


Figure 5: The optimal start time is increasing in  $p$ .

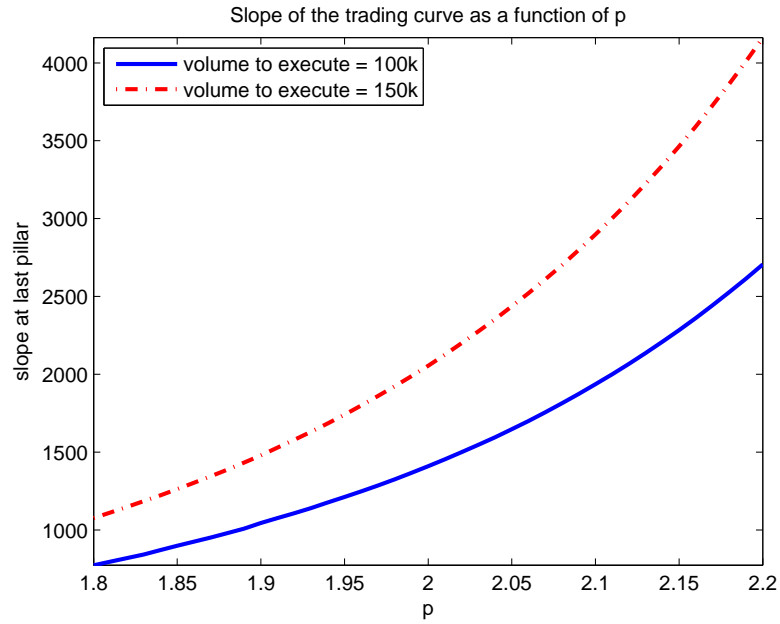


Figure 6: The slope at the last pillar is increasing in  $p$ .

#### 4.4 Implied $p$ -variation for CAC40 and link with liquidity

In Figure 7 we plotted the starting times as a function of  $p$  for 39 out of 40 stocks in the CAC40 index.

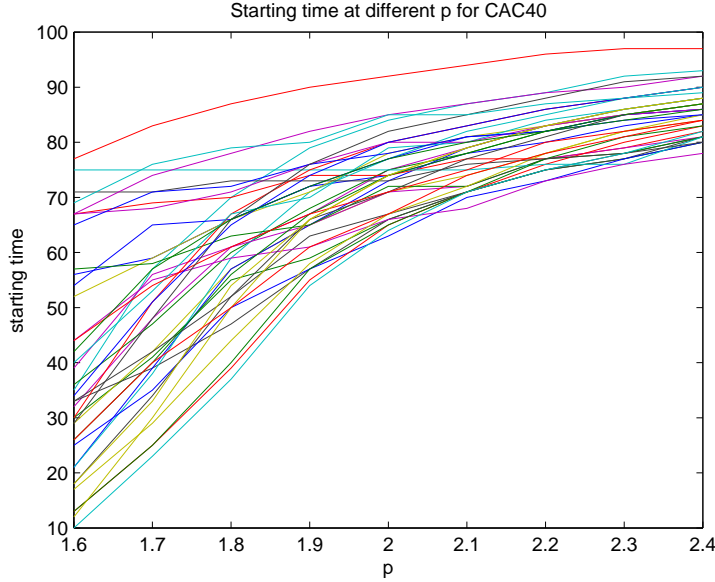


Figure 7: Cumulative TC curves.

The  $p$  parameter has been considered so far as an input whilst the starting time  $n_0$  was an output. However, since  $p \mapsto n_0(p)$  is increasing, we can consider the inverse problem: given a starting time  $n^\sharp$  there is a  $p$  such that the trading curve for the TC algorithm executes the total number of shares between  $n_0$  and the last trade before the closing auction. The  $p$  is not unique because  $n_0$  can only take discrete values, which implies that  $p \mapsto n_0(p)$  is piecewise constant. However, it can be rendered unique if we define the implied  $p$  as

$$p := \sup\{p' : n_0(p') = n^\sharp\}.$$

In consequence, we have an implied  $p$  for the CAC40 index: given a common starting time  $n^\sharp$ , for each stock in the CAC40 index we find their  $p$  such that  $n_0(p) = n^\sharp$ . For the numerical simulations in Figure 7 we chose  $n^\sharp = 77$ , which corresponds to the opening of the NYSE in the US, a very important time for European traders. We supposed that, for each name on the CAC40 index, we have to execute 6% of the total daily volume. For volume curves, volatility curves and market impact parameters we used those provided by the Quantitative Research at Cheuvreux.

The statistics of the implied  $p$  are summarised in the next table:

|                      |      |                     |      |
|----------------------|------|---------------------|------|
| <b>minimum</b>       | 1.60 | <b>quantile 25%</b> | 2.00 |
| <b>maximum</b>       | 2.40 | <b>median</b>       | 2.10 |
| <b>mean</b>          | 2.11 | <b>quantile 75%</b> | 2.20 |
| <b>std deviation</b> | 0.17 |                     |      |

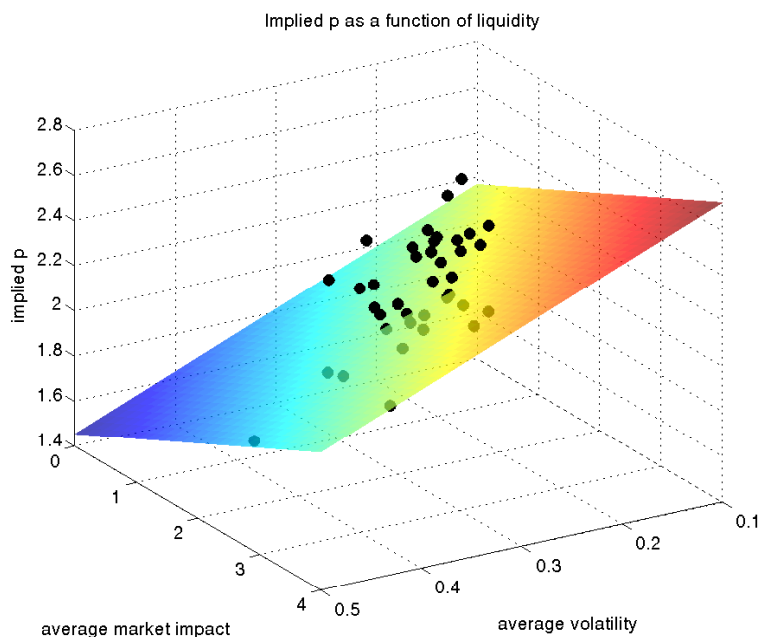
The implied  $p$  can be very useful for executing portfolios: we can synchronise all assets in our basket, so that all executions start at the same time  $n^\#$ . In this way we can assess and compare the market impact of the individual executions on the same ground, just as options traders use the implied volatility for that purpose. Of course, for real portfolio execution this is far from optimal, but at least it is a first step towards a systematic, quantitative measure of portfolio execution.

Finally, the implied  $p$  can be viewed as a measure of the joint impact of the volatility and the liquidity, the latter modelled as the market impact. In order to illustrate that fact, we performed a linear regression on the implied  $p$  with respect to the average volatility per year and the average market impact per pillar given by (13). The coefficients of the linear regression are:

$$\text{implied } p \cong 2.35 + 0.14 \times \text{market impact} - 1.79 \times \text{volatility} \quad (19)$$

with  $R^2 = 0.27$ . The results of (19) and Figure 8 can be interpreted as follows:

- If the market impact decreases then the TC algorithm will start the execution later because the execution will have a smaller effect on moving the price in the wrong direction. However, since the starting pillar has been fixed, the TC algorithm compensates this fact by playing less aggressively, i.e. by decreasing  $p$ .
- If the volatility increases then the TC algorithm would like to start later in order to avoid paying the market risk. However, since the starting time is fixed, the algorithm plays less aggressive and thus it decreases  $p$ .

Figure 8: Implied  $p$  and liquidity

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