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► **To cite this version:**

Nadine Hilgert, André Mas, Nicolas Verzelen. Minimax adaptive tests for the Functional Linear model. 2012. hal-00704836v1

HAL Id: hal-00704836

<https://hal.science/hal-00704836v1>

Submitted on 6 Jun 2012 (v1), last revised 11 Feb 2013 (v2)

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Minimax adaptive tests for the Functional Linear model

Nadine Hilgert^{*}, André Mas[†] and Nicolas Verzelen^{*}

Abstract: We introduce two novel procedures to test the nullity of the slope function in the functional linear model with real output. The test statistics combine multiple testing ideas and random projections of the input data through functional Principal Component Analysis. Interestingly, the procedures are completely data-driven and do not require any prior knowledge on the smoothness of the slope nor on the smoothness of the covariate functions. The levels and powers against local alternatives are assessed in a nonasymptotic setting. This allows us to prove that these procedures are minimax adaptive (up to an unavoidable $\log \log n$ multiplicative term) to the unknown regularity of the slope. As a side result, the minimax separation distances of the slope are derived for a large range of regularity classes. A numerical study illustrates these theoretical results.

AMS 2000 subject classifications: Primary 62J05; secondary 62G10.

Keywords and phrases: Functional linear regression, eigenfunction, principal component analysis, adaptive testing, minimax hypothesis testing, minimax separation rate, multiple testing, ellipsoid, goodness-of-fit.

1. Introduction

Consider the following functional linear regression model where the scalar response Y is related to a square integrable random function $X(\cdot)$ through

$$Y = \omega + \int_{\mathcal{T}} X(t)\theta(t)dt + \epsilon . \quad (1)$$

Here, ω is a constant, denoting the intercept of the model, \mathcal{T} is the domain of $X(\cdot)$, $\theta(\cdot)$ is an unknown function representing the slope function, and ϵ is a centered random noise variable. In functional linear regression, much interest focuses on the nonparametric estimation of $\theta(\cdot)$ in (1), given an *i.i.d.* sample $(X_i, Y_i)_{1 \leq i \leq n}$ of (X, Y) . Testing whether θ belongs to a given finite dimensional linear subspace \mathcal{V} is a question that arises in different problems such as dimension reduction, goodness-of-fit analysis, or lack-of-effect tests of a functional variable. If the properties of estimators of θ are widely discussed in the literature, there is still a great need to have generic test procedures supported by strong theoretical properties. This is the problem addressed in the present paper.

Let us reformulate the functional model (1) as a generic linear regression model in an infinite dimensional space. The random function X is assumed to belong to some separable Hilbert space henceforth denoted \mathcal{H} endowed with the inner product $\langle \cdot, \cdot \rangle$. Examples of \mathcal{H} include $\mathcal{L}^2([0, 1])$ or Sobolev space $\mathcal{W}_2^m([0, 1])$. For the sake of clarity, we consider that $\omega = 0$ and that X and Y are centered. Thus, assuming that θ also belongs to \mathcal{H} , the statistical model (1) is rephrased as

$$Y = \langle X, \theta \rangle + \epsilon , \quad (2)$$

where ϵ is a centered random variable independent from X with unknown variance σ^2 . In the sequel, we note \mathbf{X} and \mathbf{Y} the size n vectors of i.i.d. observations X_i and Y_i ($1 \leq i \leq n$), while $\boldsymbol{\epsilon}$ stands for the size n vector of the noise.

^{*} INRA, UMR 729 MISTEA, F-34060 Montpellier, FRANCE.
e-mail: nadine.hilgert@supagro.inra.fr; nicolas.verzelen@supagro.inra.fr

[†] Université Montpellier 2, F-34000 Montpellier, FRANCE. e-mail: mas@math.univ-montp2.fr

In essence, testing a linear hypothesis of the form “ $\theta \in \mathcal{V}$ ” is as difficult as testing “ $\theta = 0$ ” when a parametric estimator of θ in \mathcal{V} is computed. Therefore we consider the problem of testing:

$$H_0 : “\theta = 0” \quad \text{against} \quad H_1 : “\theta \neq 0”$$

given an i.i.d. sample (\mathbf{X}, \mathbf{Y}) from model (2). The extension to general subspaces \mathcal{V} is developed in the discussion section.

Most testing procedures are based on ideas that have been originally developed for the estimation of θ . We briefly review the main approaches and the corresponding results in estimation.

A first class of procedures is based on the minimization of a least-square type criterion penalized by a roughness term that assesses the “plausibility” of θ . Such approaches include smoothing spline estimators [7, 14], thresholding projection estimators [9], or reproducing kernel Hilbert space methods [38]. A second class of procedures is based on the functional principal components analysis (PCA) of \mathbf{X} [10, 22]. It consists in estimating θ in a finite dimensional space spanned by the k first eigenfunctions of the empirical covariance operator of \mathbf{X} . The main difference with the previous class of estimators lies in the fact that the finite dimensional space is estimated from the observations of the process X . See the survey [11] and references therein for an overview of these two approaches.

The theoretical properties of these classes of estimators have been investigated from different viewpoints: prediction [7, 10, 14, 38] (estimation of $\langle X_{n+1}, \theta \rangle$ where X_{n+1} follows the same distribution as X), pointwise prediction [5] (estimation of $\langle x, \theta \rangle$ for a fixed $x \in \mathcal{H}$) or the inverse problem [12, 22] (estimation of θ). For these three objectives, optimal rates of convergence have been derived and some of the aforementioned procedures have been shown to asymptotically achieve this rate [5, 14, 38, 22]. Recently, some non-asymptotic results have emerged [12, 13] for estimation procedures that rely on a prescribed basis of functions (e.g. splines). Most of these estimation procedures rely on tuning parameters whose optimal value depends on quantities such as the noise variance, or the smoothness of θ . In fact, there is a longstanding gap in the literature between theory, where the variance σ^2 , the smoothness of θ and the smoothness of the covariance operator of X are generally assumed to be known, and practice where they are unknown.

The literature on tests in the functional linear model is scarce. In [6], Cardot et al. introduced a test statistic based on the k first components of the functional PCA of \mathbf{X} . Its limiting distribution is derived under H_0 and the power of the corresponding test is proved to converge to one under H_1 . The main drawback of the procedure is that the number k of components involved in the statistic has to be set. As for estimation, setting k is arguably a difficult problem. To bypass this calibration issue, one may apply a permutation approach [8] or use bootstrap methodologies [15, 21]. While the levels of the corresponding tests are asymptotically controlled, there is again no theoretical guarantee on the power.

In this paper, our objective is to introduce automatic testing procedures whose powers are optimal from a nonasymptotic viewpoint. Two such procedures are defined in Section 3. The idea underlying our approach combines Fisher-type statistics, corresponding to projections on the principal components of \mathbf{X} , with multiple testing techniques in the spirit of [3]. On one hand, these two testing procedures are completely data-driven: they do not require tuning parameters whose optimal values depend on θ , the distribution of X or on σ . On the other hand, their levels and powers are analyzed in Section 4 and 5 from a nonasymptotic viewpoint. Two types of results are provided. Upon assuming that ϵ follows a Gaussian distribution, the levels of these tests are shown to be smaller than

a prescribed number α . Under moment assumptions on ϵ and mild assumptions on the covariance of X , the level is smaller than α up to a $\log^{-1}(n)$ additional term, and a sharp control of the power is provided. Such results are comparable to state of the art results in nonparametric regression [3, 35]. However, in our case, the main difficulty in the proof is to control the randomness of the principal components of \mathbf{X} . The arguments rely on the perturbation theory of operators. While other estimation or testing procedures based on the Karhunen-Loève expansion have only been analyzed in an asymptotic setting [5, 6, 22], our nonasymptotic results rely on less restrictive assumptions on X than those commonly used in the asymptotic literature.

In Section 6, we assess the optimality of our testing procedures in the minimax adaptive sense. Let α and β be two positive numbers smaller than one. The notion of minimaxity of a level- α test T_α is related to the *separation distance* of T_α over some class of functions Θ (e.g. a Sobolev ball). Intuitively, the power of a reasonable test T_α should be large when the norm of θ is large while the power of T_α is close to α when θ is close to 0. For the problem of testing $H_0: “\theta = 0”$ against $H_{1,\Theta}: “\theta \in \Theta \setminus \{0\}”$, the separation distance corresponds to the smallest distance ρ such that T_α rejects H_0 with probability larger than $1 - \beta$ for all $\theta \in \Theta$ whose norm is larger than ρ . The smaller the separation distance, the more powerful the test T_α is. The minimax separation distance over Θ is the smallest separation distance that is achieved by a level- α test. A test achieving this minimax separation distance is said to be minimax over Θ . The notions of separation distances and minimax separation distances are formalized in Section 6.2. In the nonparametric regression setting, minimax separation distances have been derived in an asymptotic [27, 28, 29] and a nonasymptotic [2] setting.

In practice, the regularity of θ is unknown. Thus, assuming a priori that the function θ belongs to a particular smoothness class Θ and building an optimal test over Θ may lead to poor performances, for instance if $\theta \notin \Theta$. For this reason, a more ambitious issue is to build a minimax adaptive testing procedure, that is a procedure which is simultaneously minimax for a wide range of regularity classes Θ . Minimax adaptive testing procedures have already been studied in the nonparametric regression setting, from an asymptotic [35] and a nonasymptotic [3] viewpoint.

In this paper, the separation distances of our testing procedures are nonasymptotically controlled. We derive minimax separation distance in the functional model (2) for a wide class of ellipsoids. Building on the multiple testing approach of [3], we prove that our testing procedures are simultaneously minimax over this class of ellipsoids (up to an unavoidable $\log \log n$ factor). As in the estimation setting [22], the minimax separation distances involve the common regularity of θ and X .

The two testing procedures are illustrated and compared by simulations in Section 7. Extensions of the approach are discussed in Section 8. Section 9 contains the main proofs while the technical lemmas involving perturbation theory are postponed to Section 10.

2. Preliminaries

2.1. Notations

We remind that $\langle \cdot, \cdot \rangle$ and $\|\cdot\|$ respectively refer to the inner product and the corresponding norm in the Hilbert \mathcal{H} . In contrast, $\langle \cdot, \cdot \rangle_n$ and $\|\cdot\|_n$ stand for the inner product and the Euclidean norm in \mathbb{R}^n . Furthermore, \otimes refers to the tensor product. We assume henceforth that X is centered and has a second moment that is $\mathbb{E}[\|\mathbf{X}\|^2] < \infty$. The covariance operator of X is defined as the linear operator Γ defined on \mathcal{H} as follows:

$$\Gamma h = \mathbb{E}[X \otimes Xh] = \mathbb{E}[\langle h, X \rangle X] , \quad h \in \mathcal{H} .$$

It is well known that Γ is a symmetric, positive trace-class hence Hilbert-Schmidt operator, which implies that Γ is diagonalizable in an orthonormal basis. We denote $(\lambda_j)_{j \geq 1}$ the non-increasing sequence of eigenvalues of Γ , while the sequence $(V_j)_{j \geq 1}$ stands for a corresponding sequence of eigenfunctions. It follows that Γ decomposes as $\Gamma = \sum_{j=1}^{\infty} \lambda_j V_j \otimes V_j$. For any integer $k \geq 1$, we note $\Gamma_k = \sum_{j=1}^k \lambda_j V_j \otimes V_j$ the operator such that $\Gamma_k h = \Gamma h$ for $h \in \text{Vect}(V_1, \dots, V_k)$ and $\Gamma_k h = 0$ if $h \in (V_1, \dots, V_k)^\perp$.

In the sequel, C, C_1, \dots denote positive universal constants that may vary from line to line. The notation $C(\cdot)$ specifies the dependency on some quantities.

2.2. Karhunen-Loève expansion and functional PCA

We recall here a classical tool of functional data analysis : the **Karhunen-Loève expansion**, denoted KL expansion in the sequel.

Definition 2.1. *There exists an expansion of X in the basis $(V_j)_{j \geq 1}$: $X = \sum \langle X, V_j \rangle V_j$. The real random variables $\langle X, V_j \rangle$ are centered (when X is centered), uncorrelated, and with variance λ_j . As a consequence, there exists a collection $(\eta^{(j)})_{j \geq 1}$ of random variables that are centered, uncorrelated, and with unit variance such that*

$$X = \sum_{j=1}^{+\infty} \sqrt{\lambda_j} \eta^{(j)} V_j . \quad (3)$$

The decomposition is called the *KL-expansion* of X .

The eigenfunction V_j is the j -th principal direction whose amount of variance coincides with λ_j . When X is a Gaussian process, the $(\eta^{(j)})_{j \in \mathbb{N}}$ form an i.i.d sequence with $\eta^{(1)} \sim \mathcal{N}(0, 1)$. If the eigenfunctions (V_j) and the eigenvalues (λ_j) are unknown in practice, they can be estimated from the data using functional principal component analysis. In the sequel, we note $\widehat{\Gamma}_n$ the empirical covariance operator defined by

$$\widehat{\Gamma}_n h = \frac{1}{n} \sum_{i=1}^n X_i \otimes X_i h = \frac{1}{n} \sum_{i=1}^n \langle X_i, h \rangle X_i , \quad h \in \mathcal{H} .$$

Functional PCA allows to estimate (λ_j, V_j) , $j \geq 1$ by diagonalizing the empirical covariance operator $\widehat{\Gamma}_n$. These empirical counterparts of (λ_j, V_j) are denoted $(\widehat{\lambda}_j, \widehat{V}_j)$ in the sequel.

Functional PCA is usually applied as a dimension reduction technique. One of its appealing features relies on its ability to capture most of the variance of X by a k -dimensional projection on the space $\text{Vect}(\widehat{V}_1, \dots, \widehat{V}_k)$. For this reason, PCA is at the core of many procedures for functional data. After the seminal paper by Dauxois et al. [16], the convergence of the random eigenelements $(\widehat{\lambda}_j, \widehat{V}_j)$ has been assessed from an asymptotic point of view [23, 24, 25, 32]. One issue with such a dimension reduction method is the choice of the tuning parameter k , whose optimal value usually depends on unknown quantities. Besides plugging the $(\widehat{\lambda}_j, \widehat{V}_j)$ into linear estimates creates non-linearity and usually introduces stochastic dependence.

3. Testing Procedures

Our testing procedures are based on multiple testing principles. We first introduce a collection of parametric statistics. Afterwards, we merge these statistics to obtain non-parametric procedures.

3.1. Parametric statistics

In the sequel, k denotes a positive integer smaller than $n/2$. As a first step, we consider the parametric testing problem of the hypotheses:

$$H_0 : “\theta = 0” \quad \text{against} \quad H_{1,k} : “\theta \in \text{Vect}[(V_j)_{j=1,\dots,k}] \setminus \{0\}” .$$

Given a dimension k of the Karhunen-Loève expansion, we note \hat{k}^{KL} as $k \wedge \text{Rank}(\widehat{\Gamma}_n)$. In order to introduce the parametric statistic, let us restate the functional linear model into a finite dimensional linear model. We consider the response vector \mathbf{Y} of size n , the $n \times \hat{k}^{KL}$ design matrix \mathbf{W} defined by $\mathbf{W}_{i,j} = \langle X_i, \widehat{V}_j \rangle$ for $i = 1, \dots, n$, $j = 1, \dots, \hat{k}^{KL}$, the parameter vector ϑ defined by $\vartheta_j = \langle \theta, \widehat{V}_j \rangle$, $j = 1, \dots, \hat{k}^{KL}$, and the size n noise vector $\tilde{\epsilon}$ defined by $\tilde{\epsilon}_i = \epsilon_i + \langle X_i, \theta \rangle - [\mathbf{W}\vartheta]_i$. The functional linear model is equivalently written as

$$\mathbf{Y} = \mathbf{W}\vartheta + \tilde{\epsilon} .$$

Intuitively, testing “ $\vartheta = 0$ ” is a reasonable proxy for testing H_0 against $H_{1,k}$. For this reason, we propose a Fisher-type statistic.

Definition 3.1. *In the sequel, $\widehat{\Pi}_k$ stands for the orthogonal projection in \mathbb{R}^n onto the space generated by the \hat{k}^{KL} columns of \mathbf{W} . For any $k \leq n/2$, we consider the statistic $\phi_k(\mathbf{Y}, \mathbf{X})$ defined by*

$$\phi_k(\mathbf{Y}, \mathbf{X}) := \frac{\|\widehat{\Pi}_k \mathbf{Y}\|_n^2}{\|\mathbf{Y} - \widehat{\Pi}_k \mathbf{Y}\|_n^2 / (n - \hat{k}^{KL})} . \quad (4)$$

The main difference with a classical Fisher statistic comes from the fact that the projection $\widehat{\Pi}_k$ is *random*. This projector is built using the \hat{k}^{KL} first directions $(\widehat{V}_1, \widehat{V}_2, \dots, \widehat{V}_{\hat{k}^{KL}})$ of the empirical Karhunen-Loève expansion of \mathbf{X} . Let us call $\widetilde{\Pi}_k$ the orthogonal projector in \mathbb{R}^n onto the space spanned by $(\langle X_i, V_j \rangle)_{i=1,\dots,n, j=1,\dots,k}$. If we knew the basis (V_j) , $j \geq 1$ in advance, we would use this orthogonal projector instead of $\widehat{\Pi}_k$. We shall prove that, under H_0 , $\phi_k(\mathbf{Y}, \mathbf{X})/\hat{k}^{KL}$ behaves like a Fisher distribution with $(\hat{k}^{KL}, n - \hat{k}^{KL})$ degrees of freedom.

Remark 3.1 (Other interpretations of $\phi_k(\mathbf{Y}, \mathbf{X})$). *Consider $\widehat{\theta}_k$ the least-squares estimator of θ in the space generated by \widehat{V}_j , $j = 1, \dots, \hat{k}^{KL}$. It is proved in Section 9.2 that $\|\widehat{\Pi}_k \mathbf{Y}\|_n^2 = \|\widehat{\Gamma}_n^{1/2} \widehat{\theta}_k\|^2$. Thus, the numerator of (4) corresponds to some norm of $\widehat{\theta}_k$. Intuitively, the larger $\widehat{\theta}_k$, the larger the statistic $\phi_k(\mathbf{Y}, \mathbf{X})$ is. Furthermore, $\|\widehat{\Pi}_k \mathbf{Y}\|_n^2$ also expresses as the numerator of the statistic D_n considered in Cardot et al. [6] (see Section 9.2 for details).*

3.2. Multiple testing procedures

The statistic $\phi_k(\mathbf{Y}, \mathbf{X})$ depends on the choice of the size k of the expansion. However, a good choice of k depends on the unknown θ and on the unknown covariance operator. Intuitively, taking k too small does not allow to detect non-zero θ whose k -first coefficients $\langle \theta, V_i \rangle$ in the Karhunen-Loève expansion are zero. In contrast, taking k too large leads to a high-dimensional parametric test and therefore decreases the power. The trade-off between a “small” k and a “large” k is further described in Section 5. As k cannot be a priori chosen, we evaluate the statistic $\phi_k(\mathbf{Y}, \mathbf{X})$ for all k belonging to a collection \mathcal{K}_n . This multiple testing approach has already been considered in the non-parametric fixed design regression setting [3].

Consider some number $0 < \alpha < 1$. In the sequel, \mathcal{K}_n stands for a “dyadic” collection of dimensions defined by

$$\mathcal{K}_n = \{2^0, 2^1, 2^2, 2^3 \dots, \bar{k}_n\}, \quad (5)$$

where \bar{k}_n is a power of 2. In the sequel, we note $\bar{\mathcal{F}}_{D,N}(u)$ the probability that a Fisher random variables with D and N degrees of freedom is larger than u , while $\bar{\mathcal{F}}_{D,N}^{-1}(u)$ denotes the $1 - u$ quantile of a Fisher random variable.

Definition 3.2 (KL-Test). *We reject $H_0: “\theta = 0”$ when the statistic*

$$T_\alpha := \sup_{k \in \mathcal{K}_n, k \leq \text{Rank}(\hat{\Gamma}_n)} \left[\phi_k(\mathbf{Y}, \mathbf{X}) - \hat{k}^{KL} \bar{\mathcal{F}}_{\hat{k}^{KL}, n - \hat{k}^{KL}}^{-1} \{ \alpha_{\mathcal{K}_n}(\mathbf{X}) \} \right]. \quad (6)$$

is positive, where the weight $\alpha_{\mathcal{K}_n}(\mathbf{X})$ is chosen according to one of the procedures P_1 and P_2 explained below.

P₁: (Bonferroni) $\alpha_{\mathcal{K}_n}(\mathbf{X})$ is equal to $\alpha/|\mathcal{K}_n|$.

P₂: Let \mathbf{Z} be a standard Gaussian vector of size n . We take $\alpha_{\mathcal{K}_n}(\mathbf{X}) = q_{\mathbf{X}, \alpha}$, the α -quantile of the distribution of the random variable

$$\inf_{k \in \mathcal{K}_n} \bar{\mathcal{F}}_{\hat{k}^{KL}, n - \hat{k}^{KL}} \left[\phi_k(\mathbf{Z}, \mathbf{X}) / \hat{k}^{KL} \right] \quad (7)$$

conditionally to \mathbf{X} .

In the sequel, $T_\alpha^{(1)}$ (resp. $T_\alpha^{(2)}$) refers to the statistic T_α , defined with Procedure P_1 (resp. P_2).

Remark 3.2. [Computation of $q_{\mathbf{X}, \alpha}$] Let Z be a standard Gaussian random vector of size n independent of \mathbf{X} . As ϵ is independent of \mathbf{X} , the distribution of (7) conditionally to \mathbf{X} is the same as the distribution of

$$\inf_{k \in \mathcal{K}_n} \bar{\mathcal{F}}_{\hat{k}^{KL}, n - \hat{k}^{KL}} \left(\frac{\|\hat{\Pi}_k Z\|_n^2 / \hat{k}^{KL}}{\|Z - \hat{\Pi}_k Z\|_n^2 / (n - \hat{k}^{KL})} \right)$$

conditionally to \mathbf{X} . As a consequence, one can simulate a random variable that follows the same distribution as (7) conditionally to \mathbf{X} . Hence, the quantile $q_{\mathbf{X}, \alpha}$ is easily worked out applying a Monte-Carlo approach.

Remark 3.3. The statistic T_α corresponds to a multiple testing procedures against the hypothesis $H_{1,k}: “\theta \in \text{Vect}[(V_j)_{j=1, \dots, k}] \setminus \{0\}”$ for $k \in \mathcal{K}_n$. Not restricting ourselves to a single parametric test but to a collection of parametric tests will allow our test to be adaptive to the the regularity of Γ and θ as explained in Sections 5 and 6.

Remark 3.4. [Choice of \bar{k}_n] In practice, we advise to take $\bar{k}_n = 2^{\lfloor \log_2 n \rfloor - 1}$ which lies between $n/4$ and $n/2$. This choice is supported by practical experiences in connection with results obtained in sections 4 and 5 with a Gaussian noise ϵ . Nevertheless, some of the theoretical results will require to take a slightly smaller value for \bar{k}_n .

4. Size and comparison of the procedures P_1 and P_2

In the mathematical analysis, we consider two settings. On one hand, we control explicitly the size of the tests when the noise ϵ is normally distributed. On the other hand, we bound the size of the tests when the noise is only constrained to admit a fourth moment. Nevertheless, this second control requires additional assumptions on the process X and its proof is more involved.

4.1. Gaussian noise

We consider the following assumption:

$$\mathbf{A.1} \quad \epsilon \text{ follows a Gaussian distribution } \mathcal{N}(0, \sigma^2) . \quad (8)$$

First, we prove that the size of $T_\alpha^{(1)}$ and $T_\alpha^{(2)}$ is explicitly controlled under this Gaussian assumption.

Proposition 4.1 (Size of $T_\alpha^{(1)}$ and $T_\alpha^{(2)}$ under Gaussian errors). *Under Assumption **A.1** and if $\bar{k}_n \leq n/2$, we have for any $n \geq 2$,*

$$\begin{aligned} \mathbb{P}_0(T_\alpha^{(1)} > 0) &\leq \alpha , \\ \mathbb{P}_0(T_\alpha^{(2)} > 0) &= \alpha . \end{aligned}$$

4.2. Non-Gaussian noise

In this part, the noise ϵ is only assumed to admit a fourth order moment, but we perform additional assumptions on X and \bar{k}_n .

$$\mathbf{B.1} \quad \sup_{j \geq 1} \mathbb{E}[(\eta^{(j)})^4] \leq C_1 \text{ and } \frac{\mathbb{E}[\epsilon^4]}{\sigma^4} \leq C_2 , \quad (9)$$

where C_1 and C_2 are two positive constants.

$$\mathbf{B.2} \quad \text{For some } \gamma > 0, (j\lambda_j((\log^{1+\gamma} j) \vee 1))_{j \geq 1} \text{ is decreasing and } \text{Rank}(\Gamma) = \infty . \quad (10)$$

$$\mathbf{B.3} \quad \bar{k}_n \leq n^{1/4} / \log^4(n) . \quad (11)$$

Assumption **B.1** is classical, since we need to control second order moments for the empirical covariance operator $\widehat{\Gamma}_n$. This comes down to inspecting the behavior of the fourth order moments of the $\eta^{(j)}$'s. The second part of **B.2** ensures that the framework is truly functional. The first part of **B.2** is mild and holds for an X that may have very irregular paths (it holds for the Brownian motion for which $\lambda_j \propto j^{-2}$) and for classical examples of eigenvalue sequences: with polynomial decay, exponential decay, or even Laurent sequences such as $\lambda_j = j^{-\delta} \cdot \log^{-\nu}(j)$ for $\delta > 1$ and $\nu \geq 0$. In fact, **B.2** is less restrictive than assumptions commonly used in the literature [5, 6, 22] since it does not require any spacing control between the eigenvalues.

The restriction **B.3** on the dimension of the projection is classical for the analysis of statistical procedures based on the Karhunen-Loève expansion. If we knew the eigenfunctions V_k of Γ in advance, we could consider larger dimensions \bar{k}_n . The estimation of the eigenfunctions V_k becomes more difficult when k increases. By considering dimensions \bar{k}_n that satisfy Assumption **B.3**, we prove in the next theorem that the *random* projector $\widehat{\Pi}_k$ concentrates well around its mean. It may be noticed that this assumption links \bar{k}_n and n independently from the eigenvalues hence from any prior knowledge on the data.

Theorem 4.2 (Size of $T_\alpha^{(1)}$). *Under Assumptions **B.1**, **B.2**, and **B.3**, there exist a positive constants $C(\alpha, \gamma)$ and C_2 such that the following holds. For any $n \geq C_2$, we have*

$$\mathbb{P}_0 \left[T_\alpha^{(1)} > 0 \right] \leq \alpha + \frac{C(\alpha, \gamma)}{\log(n)} .$$

The proof of Theorem 4.2 relies on perturbation theory for random operators (see Section 10).

Remark 4.1. *In the proof of Theorem 4.2, we show that, under H_0 , the statistic $\phi_k(\mathbf{Y}, \mathbf{X})$ behaves like a χ^2 distribution with k degrees of freedom. In contrast, we show in the proof of Proposition 4.1 that, under H_0 , the statistic $\phi_k(\mathbf{Y}, \mathbf{X})/\hat{k}^{KL}$ exactly follows a Fisher distribution with $(\hat{k}^{KL}, n - \hat{k}^{KL})$ degrees of freedom.*

4.3. Comparison of $T_\alpha^{(1)}$ and $T_\alpha^{(2)}$

The test $T_\alpha^{(2)}$ is always more powerful than the corresponding test $T_\alpha^{(1)}$ as shown in the next proposition.

Proposition 4.3. *For any parameter $\theta \neq 0$, the tests $T_\alpha^{(1)}$ and $T_\alpha^{(2)}$ satisfy*

$$\mathbb{P}_\theta \left(T_\alpha^{(2)} > 0 \mid \mathbf{X} \right) \geq \mathbb{P}_\theta \left(T_\alpha^{(1)} > 0 \mid \mathbf{X} \right) \quad \mathbf{X} \text{ a.s. .} \quad (12)$$

On one hand, the choice of Procedure P_1 is valid even for a non-Gaussian noise and avoids the computation of the quantile $q_{\mathbf{X}, \alpha}$. On the other hand, the test $T_\alpha^{(2)}$ has a size α when the error is Gaussian and is more powerful than the corresponding test with Procedure P_1 . This comparison is numerically illustrated in Section 7.

5. Power

As $T_\alpha^{(2)}$ is more powerful than $T_\alpha^{(1)}$, we only consider the power of $T_\alpha^{(1)}$. Intuitively, the larger the signal-to-noise ratio $\mathbb{E}[\langle X, \theta \rangle^2] / \sigma^2 = \|\Gamma^{1/2}\theta\|^2 / \sigma^2$ is, the easier we can reject H_0 . For this reason, we study how large $\|\Gamma^{1/2}\theta\|^2$ has to be, so that the test $T_\alpha^{(1)}$ rejects H_0 with probability larger than $1 - \beta$ for a prescribed positive number β .

As for the type I error, the power of the test is evaluated under two kinds of assumption. When the noise is normally distributed, we control the type II error under very weak assumptions on X . When the noise only admits a fourth moment, a sharper bound on the power is provided under additional assumptions on X . As in the previous section, the proof of this second result is more involved.

5.1. Gaussian noise

Proposition 5.1 (Power under Gaussian errors). *Let α and β be fixed. There exists positive constants C , $C_1(\beta)$, and C_2 such that the following holds. Suppose that $\alpha \geq \exp(-n/20)$, $\beta \geq C/n$ and that Assumptions B.1 and A.1 are true. Then, $\mathbb{P}_\theta(T_\alpha^{(1)} > 0) \geq 1 - \beta$ for any θ belonging to the set*

$$\Theta(\beta) := \left\{ \theta \in \mathcal{H}, \|\Gamma^{1/2}\theta\|^2 \geq \inf_{k \in \mathcal{K}_n} \left[C_1(\beta) \left(\lambda_{k+1} + \sum_{j \geq k+1} \frac{\lambda_j}{\sqrt{n}} \right) \|\theta\|^2 + \Delta(k) \right] \right\}. \quad (13)$$

where

$$\Delta(k) := C_2 \frac{\sigma^2}{n} \left(\sqrt{k \log \left(\frac{\log n}{\alpha \beta} \right)} + \log \left(\frac{\log n}{\alpha \beta} \right) \right). \quad (14)$$

Remark 5.1. Observe that this proposition does not only evaluate the power of $T_\alpha^{(1)}$ for a particular alternative but also provides a uniform bound on the power. In order to interpret the result, let us fix some $k \in \mathcal{K}_n$ larger than $\log \log n$. Then, the test $T_\alpha^{(1)}$ rejects H_0 with probability larger than $1 - \beta$ if

$$\|\Gamma^{1/2}\theta\|^2 \geq C_1(\beta) \left(\lambda_{k+1} + \sum_{j \geq k+1} \frac{\lambda_j}{\sqrt{n}} \right) \|\theta\|^2 + C_2 \frac{\sigma^2}{n} \sqrt{k \log \left(\frac{\log n}{\alpha\beta} \right)}. \quad (15)$$

Remark 5.2. If we knew that θ belongs to the space spanned by the k first eigenvectors (V_1, \dots, V_k) and if we knew these k eigenvectors in advance, then we could consider the statistic defined by

$$\tilde{\phi}_k(\mathbf{X}, \mathbf{Y}) := \frac{\|\tilde{\Pi}_k \mathbf{Y}\|_n^2}{\|\mathbf{Y} - \tilde{\Pi}_k \mathbf{Y}\|_n^2} - \bar{\mathcal{F}}_{k, n-k}^{-1}(\alpha),$$

where $\tilde{\Pi}_k$ is the projection in \mathbb{R}^n onto the space spanned by $(\langle X_i, V_j \rangle)_{i=1, \dots, n, j=1, \dots, k}$. The corresponding test is optimal in the minimax sense and rejects H_0 with probability larger than $1 - \beta$ when

$$\|\Gamma^{1/2}\theta\|^2 \geq C(\alpha, \beta) \sqrt{k} \sigma^2 / n. \quad (16)$$

See [37] for a proof when X is a Gaussian process, the extension to non Gaussian processes being straightforward. In (15), we do not exactly recover the term $\sqrt{k} \sigma^2 / n$ because we face three difficulties:

1. First, we do not assume that θ belongs to the space spanned by (V_1, \dots, V_k) . With the statistic $\phi_k(\mathbf{Y}, \mathbf{X})$, we only capture the projection of θ onto $\text{span}(V_1, \dots, V_k)$. As a consequence, there is a remaining bias term in (15)

$$\mathbb{E} \left[\left\{ \sum_{j=k+1}^{\infty} \langle \theta, V_j \rangle \langle X, V_j \rangle \right\}^2 \right] = \sum_{j=k+1}^{\infty} \lambda_j \langle \theta, V_j \rangle^2 \leq \lambda_{k+1} \|\theta\|^2.$$

2. Second, we do not know the eigenvectors (V_1, \dots, V_k) , and we have to estimate them. We pay a price $\sum_{j \geq k+1} \lambda_j / \sqrt{n}$ for this. In fact, the term $\sum_{j \geq k+1} \lambda_j / \sqrt{n}$ is negligible in front of the term λ_{k+1} if the eigenvalues have an exponential decrease or polynomial decrease $\lambda_j = j^{-1-\tau}$ with $\tau > 1/2$.
3. In the test $T_\alpha^{(1)}$, we do not fix a value for k but we rather consider a collection \mathcal{K}_n . As a consequence, the rejection region $\Theta(\beta)$ is a union of $|\mathcal{K}_n|$ rejection regions. When k increases, the bias term $[\lambda_{k+1} + \sum_{j \geq k+1} \lambda_j / \sqrt{n}] \|\theta\|^2$ decreases but the quantity $\Delta(k)$ increases. Comparing (15) and (16), we observe the rejection region of the procedure $T_\alpha^{(1)}$ almost contains the rejection region of the best statistic $\tilde{\phi}_k(\mathbf{X}, \mathbf{Y})$ with $k \in \mathcal{K}_n$ without knowing Γ , nor θ in advance. The price to pay for this feature is of order $\sqrt{\log \log n}$ in the term $\Delta(k)$ in (14). We shall see in Section 6.3 that this $\sqrt{\log \log n}$ term is in fact unavoidable.

5.2. Non-Gaussian noise

In order to lower bound the power of $T_\alpha^{(1)}$ for non Gaussian errors, we need a slightly stronger assumption:

B.4

$$\sup_{j \geq 1} \mathbb{E}[(\eta^{(j)})^8] \leq C. \quad (17)$$

Theorem 5.2 (Power under non-Gaussian errors). *Let α and β be fixed. Under **B.1 – 4**, there exist positive constants $C(\gamma)$, C_1 , C_2 , and C_3 such that the following holds. Assume that $\alpha \geq e^{-\sqrt{n}}$, $\beta \geq C(\gamma)/\log(n)$, and that $n \geq C_3$. Then, $\mathbb{P}_\theta(T_\alpha^{(1)} > 0) \geq 1 - \beta$ for any θ belonging to the set $\Theta'(\beta)$ made of all $\theta \in \mathcal{H}$ satisfying*

$$\|\Gamma^{1/2}\theta\|^2 \geq \inf_{k \in \mathcal{K}_n} C_1 \|(\Gamma^{1/2} - \Gamma_k^{1/2})\theta\|^2 + C_2 \frac{\sigma^2}{n} \left(\sqrt{k \log \left(\frac{\log n}{\alpha\beta} \right)} + \log \left(\frac{\log n}{\alpha\beta} \right) \right). \quad (18)$$

Remark 5.3. *Since $\|(\Gamma^{1/2} - \Gamma_k^{1/2})\theta\|^2 \leq \lambda_{k+1}\|\theta\|^2$, observe that $\Theta(\beta)$ contains $\Theta'(\beta)$ (up to the choice of the constants in (18) and (13)). Thus, we get a stronger result than the corresponding bound obtained in Proposition 5.1. As a consequence, the remarks made after Proposition 5.1 still hold for Theorem 5.2.*

Remark 5.4 (Joint regularity of Γ and θ). *Looking more precisely at the bias term, we obtain*

$$\|(\Gamma^{1/2} - \Gamma_k^{1/2})\theta\|^2 = \sum_{j=k+1}^{\infty} \lambda_j \langle \theta, V_j \rangle^2.$$

Consequently, the bias term does not only depend on the rate of convergence of the eigenvalues of Γ , it also depends on the behavior of the sequence $\lambda_j \langle \theta, V_j \rangle_H^2$. In other words, the joint regularity of the covariance operator Γ and of θ (in the expansion of (V_j) , $j \geq 1$) plays a role in studying the bias term. Thus, the faster the term $\sum_{j=k+1}^{\infty} \lambda_j \langle \theta, V_j \rangle^2$ converges to zero (with $\|\Gamma^{1/2}\theta\|$ fixed), the more powerful the test is.

6. Minimax properties

6.1. Definitions

In this section, we assess the optimality of the procedure $T_\alpha^{(1)}$. To this end, we study the optimal power of a level- α test, when θ is assumed to have a known regularity.

Definition 6.1 (Ellipsoids). *Given a non increasing sequence $(a_i)_{i \geq 1}$ and a positive number $R > 0$, we define the ellipsoid $\mathcal{E}_a(R)$ by*

$$\mathcal{E}_a(R) := \left\{ \theta \in \mathcal{H} : \sum_{k=1}^{+\infty} \frac{\langle \theta, V_k \rangle^2}{a_k^2} \leq R^2 \sigma^2 \right\}.$$

The ellipsoid $\mathcal{E}_a(R)$ contains all the elements $\theta \in \mathcal{H}$ that have a given regularity in the basis (V_k) , $k \geq 1$. In other words, it prescribes the rate of convergence of $\langle \theta, V_k \rangle$ towards 0. The faster a_k goes to zero, the more regular θ is assumed to be.

We take some positive numbers α and β such that $\alpha + \beta < 1$. Let us consider a test T taking its values in $\{0, 1\}$. For any subset $\mathcal{C} \subset \mathcal{H} \times \mathbb{R}^+$, $\beta[T; \mathcal{C}]$ denotes the supremum of type II errors of the test T for all parameters $(\theta, \sigma) \in \mathcal{C}$:

$$\beta[T; \mathcal{C}] = \sup_{(\theta, \sigma) \in \mathcal{C}} \mathbb{P}_\theta[T = 0].$$

The (α, β) -**separation distance** of the test T over the ellipsoid $\mathcal{E}_a(R)$, noted $\rho[T; \mathcal{E}_a(R)]$ is the minimal number $\rho > 0$ such that T rejects H_0 with probability larger than $1 - \beta$ for all $\theta \in \mathcal{E}_a(R)$ and $\sigma > 0$ such that $\|\Gamma^{1/2}\theta\|^2/\sigma^2 \geq \rho^2$. Hence, $\rho[T; \mathcal{E}_a(R)]$

corresponds to the minimal distance such that the hypotheses $\{\theta = 0, \sigma > 0\}$ and $\{\theta \in \mathcal{E}_a(R), \sigma > 0, \|\Gamma^{1/2}\theta\|^2/\sigma^2 \geq \rho^2\}$ are well separated by T .

$$\rho[T; \mathcal{E}_a(R)] := \inf \left\{ \rho > 0, \quad \beta \left[T; \left\{ \theta \in \mathcal{E}_a(R), \sigma > 0, \frac{\|\Gamma^{1/2}\theta\|^2}{\sigma^2} \geq \rho^2 \right\} \right] \leq \beta \right\} .$$

By definition, T has a power larger than $1 - \beta$ for all $\theta \in \mathcal{E}_a(R)$ and $\sigma > 0$ such that $\|\Gamma^{1/2}\theta\|^2/\sigma^2 \geq \rho^2[T, \mathcal{E}_a(R)]$.

Definition 6.2 (Minimax Separation distance). *We consider*

$$\rho^*[\alpha; \mathcal{E}_a(R)] := \inf_{T_\alpha} \rho[T_\alpha; \mathcal{E}_a(R)] , \quad (19)$$

where the infimum run over all level- α tests. This quantity is called the (α, β) -**minimax separation distance** over the ellipsoid $\mathcal{E}_a(R)$.

Remark 6.1. *The notion of (α, β) -minimax separation distance is a non asymptotic counterpart of the detection boundaries studied in the Gaussian sequence model [17]. Furthermore, as the variance σ^2 is unknown, this definition of the minimax separation distance considers the power of the testing procedures for all possible values of σ^2 .*

6.2. Minimax separation distance over an ellipsoid

Proposition 6.3 (Minimax lower bound over an ellipsoid). *There exists a constant $C(\alpha, \beta)$ such that the following holds. Let us assume that X is a Gaussian process and that ϵ follows a Gaussian distribution. For any ellipsoid $\mathcal{E}_a(R)$, we have*

$$\rho^*[\alpha; \mathcal{E}_a(R)] \geq \rho_{a,R,n}^2 := \sup_{k \geq 1} \left[C(\alpha, \beta) \left(\frac{\sqrt{k}}{n} \right) \wedge (R^2 a_k^2 \lambda_k) \right] .$$

In other words, for any test T_α of level α , we have

$$\beta \left[T_\alpha; \left\{ \theta \in \mathcal{E}_a(R), \sigma > 0, \frac{\|\Gamma^{1/2}\theta\|^2}{\sigma^2} \geq \rho_{a,R,n}^2 \right\} \right] \geq \beta .$$

Consequently, the (α, β) minimax-separation distance over $\mathcal{E}_a(R)$ is lower bounded by $\rho_{a,R,n}^2$. The next proposition states the corresponding upper bound.

Proposition 6.4 (Minimax upper bound). *Under B.1 – 4, there exist positive constants $C(\gamma)$, C_2 , $C_3(\alpha, \gamma)$, and $C_4(\alpha, \beta)$ such that the following holds. Given an ellipsoid $\mathcal{E}_a(R)$, we define the quantity k_n^* as*

$$k_n^* := \inf \left\{ k \geq 1, \quad a_k^2 \lambda_k R^2 \leq \frac{\sqrt{k}}{n} \right\} . \quad (20)$$

Assume that $\alpha \geq e^{-\sqrt{n}}$, $\beta \geq C(\gamma)/\log(n)$, $n \geq C_2$, and $k_n^* \leq n^{1/4}/\log^4(n)$. Consider the statistic

$$T_{\alpha, k_n^*} := \phi_{k_n^*}^{KL}(\mathbf{Y}, \mathbf{X}) - k_n^* \bar{\mathcal{F}}_{k_n^*, n-k_n^*}^{-1}(\alpha) . \quad (21)$$

The corresponding test has a size smaller than $\alpha + C_3(\alpha, \gamma)/\log(n)$ and is minimax over $\mathcal{E}_a(R)$:

$$\beta \left[T_{\alpha, k_n^*}; \left\{ \theta \in \mathcal{E}_a(R), \sigma > 0, \frac{\|\Gamma^{1/2}\theta\|^2}{\sigma^2} \geq C_4(\alpha, \beta) \rho_{a,R,n}^2 \right\} \right] \leq \beta . \quad (22)$$

As a consequence, the test T_{α, k_n^*} is minimax over $\mathcal{E}_a(R)$, that is its (α, β) -separation distance equals (up to a multiplicative constant) the (α, β) minimax separation distance. Interestingly, the upper bound (22) does not require the error ϵ to be normally distributed.

Remark 6.2. *As a consequence, the (α, β) -minimax separation distance over $\mathcal{E}_a(R)$ is of order*

$$\rho_{a,R,n}^2 := \sup_{k \geq 1} \left[C(\alpha, \beta) \left(\frac{\sqrt{k}}{n} \right) \wedge (R^2 a_k^2 \lambda_k) \right].$$

It depends on the behavior of the non-increasing sequence $(\lambda_k a_k^2)$, where the sequence of eigenvalues (λ_k) prescribes the “regularity” of the process X and the sequence (a_k) prescribes the regularity of θ . In order to grasp the quantity $\rho_{a,R,n}^2$, let us specify some examples of sequences $\lambda_k a_k^2$:

- *Polynomial decay.* If $\lambda_k a_k^2 = k^{-s}$ with $s > 7/2$, then the (α, β) -minimax separation is of order

$$R^{2/(1+2s)} n^{-2s/(1+2s)}.$$

- *Exponential decay.* If $\lambda_k a_k^2 = e^{-sk}$ with $s > 0$, then the (α, β) -separation distance of $T_\alpha^{(1)}$ over $\mathcal{E}_a(R)$ is of order

$$\frac{\sqrt{\log(n)}}{\sqrt{sn}}.$$

Let us now study the power of $T_\alpha^{(1)}$ over the ellipsoid $\mathcal{E}_a(R)$. In the sequel, $\lfloor \cdot \rfloor$ stands for the integer part, while $\log_2(\cdot)$ corresponds to the binary logarithm.

Corollary 6.5 (Power of $T_\alpha^{(1)}$ over ellipsoids). *Under B.1, B.2, and B.4, there exist positive constants $C(\gamma)$, C_1 , C_2 , $C_3(\alpha, \beta)$, and $C_4(\alpha, \beta)$ such that the following holds. Assume that $\alpha \geq e^{-\sqrt{n}}$, that $\beta \geq C(\gamma)/\log(n)$, and $n \geq C_2$. Consider the test $T_\alpha^{(1)}$ with $\bar{k}_n = 2^{\lfloor \log_2[n^{1/4}/\log^4(n)] \rfloor}$.*

1. *For any ellipsoid $\mathcal{E}_a(R)$, we define $\Theta_\mathcal{E}[\beta, a, R]$ as the set of function $\theta \in \mathcal{H}$ that satisfy*

$$\frac{\|\Gamma^{1/2}\theta\|^2}{\sigma^2} \geq C_3(\alpha, \beta) \inf_{k=1,2,4,\dots,\bar{k}_n} \left[\lambda_{k+1} a_{k+1}^2 R^2 + \frac{\sigma^2}{n} \left(\sqrt{k \log \log n} + \log \log n \right) \right]. \quad (23)$$

We have $\inf_{\theta \in \Theta_\mathcal{E}[\beta, a, R] \cap \mathcal{E}_a(R)} \mathbb{P}_\theta(T_\alpha^{(1)} > 0) \geq 1 - \beta$.

2. *For any ellipsoid $\mathcal{E}_a(R)$, we consider k_n^* as in (20). If $\log \log(n) \leq k_n^* \leq \bar{k}_n$, then*

$$\beta \left[T_\alpha^{(1)}; \left\{ \theta \in \mathcal{E}_a(R), \sigma > 0, \frac{\|\Gamma^{1/2}\theta\|^2}{\sigma^2} \geq C_4(\alpha, \beta) \sqrt{\log \log n} \rho_{a,R,n}^2 \right\} \right] \leq \beta.$$

Remark 6.3. *If we compare Corollary 6.5 with the minimax lower bound of Proposition 6.3, we observe that the separation distance only match up to a factor of order $\sqrt{\log \log n}$. As a consequence, $T_\alpha^{(1)}$ is almost minimax over $\mathcal{E}_a(R)$.*

Let us compare the performance of the tests T_{α, k_n^*} and $T_\alpha^{(1)}$. On one hand, the test T_{α, k_n^*} defined in Proposition 6.4 is minimax over the ellipsoid $\mathcal{E}_a(R)$ but it requires the prior knowledge of the ellipsoid $\mathcal{E}_a(R)$ and of the sequence of eigenvalues (λ_k) , which is unlikely in practice. On the other hand, the test $T_\alpha^{(1)}$ is only minimax up to a possible $\sqrt{\log \log n}$ multiplicative term. Nevertheless, $T_\alpha^{(1)}$ does not require the knowledge of the ellipsoid $\mathcal{E}_a(R)$ or of the eigenvalues (λ_k) . In the next subsection, we prove that this $\sqrt{\log \log n}$ loss is unavoidable when the ellipsoid $\mathcal{E}_a(R)$ is unknown.

6.3. Adaptation to the regularity

Let us derive the *simultaneous* minimax separation distance over a collection of ellipsoids.

Proposition 6.6 (Minimax lower bounds over a collection of nested ellipsoids). *There exists a positive constant $C(\alpha, \beta)$ such that the following holds. Let us assume that X is a Gaussian process, that the noise ϵ follows a Gaussian distribution, and that the rank of Γ is infinite. For any ellipsoid $\mathcal{E}_a(R)$, we set*

$$\tilde{\rho}_{a,R,n}^2 := \sup_{k \geq 1} \left[C(\alpha, \beta) \left(\frac{\sqrt{\log \log(k \vee 3)} \sqrt{k}}{n} \right) \wedge (R^2 a_k^2 \lambda_k) \right].$$

For any non increasing sequence $(a_k)_{k \geq 1}$ and any test T of level α , we have

$$\beta \left[T; \bigcup_{R>0} \left\{ \theta \in \mathcal{E}_a(R), \sigma > 0, \frac{\|\Gamma^{1/2}\theta\|^2}{\sigma^2} \geq \tilde{\rho}_{a,R,n}^2 \right\} \right] \geq \beta.$$

As a consequence, there is a $\sqrt{\log \log n}$ price to pay if we simultaneously consider a nested collection of ellipsoids. Such impossibility for perfect adaptation has already been observed for the testing problem in the classical nonparametric regression framework [35].

Remark 6.4. *In order to compare the lower and upper bound of Proposition 6.6 and Corollary 6.5, we specify the sequence $\lambda_k a_k^2$:*

- *Polynomial decay.* If $\lambda_k a_k^2 = k^{-s}$, then the (α, β) -separation distance of $T_\alpha^{(1)}$ over $\mathcal{E}_a(R)$ is of order

$$R^{2/(1+2s)} \left(\frac{\sqrt{\log \log(n)}}{n} \right)^{2s/(1+2s)},$$

for $s > 7/2$. By Proposition 6.6, this rate is optimal for adaptation.

- *Exponential decay.* If $\lambda_k a_k^2 = e^{-sk}$, then the (α, β) -separation distance of $T_\alpha^{(1)}$ over $\mathcal{E}_a(R)$ is of order

$$\frac{\sqrt{\log(n) \log \log(n)}}{\sqrt{sn}},$$

for any $s > 0$. By Proposition 6.6, this rate is almost optimal for adaptation (up to a $\sqrt{\log \log(n)}/\log \log \log n$ term).

In conclusion, the procedure $T_\alpha^{(1)}$ is adaptive to the unknown regularity of θ , to the unknown regularity of the eigenvalues $(\lambda_k)_{k \geq 1}$ and to the unknown noise variance σ^2 . Interestingly, the minimax rate of testing depends on the decay of the non-increasing sequence $(\lambda_k a_k^2)_{k \geq 1}$.

7. Simulations

7.1. Experiments

Setting. The performances of the procedures $T_\alpha^{(1)}$ and $T_\alpha^{(2)}$ are illustrated for various choices of the function θ . In all experiments, the noise ϵ follows a standard Gaussian distribution with unit variance, while the process X is a Brownian motion defined on

$[0, 1]$. The eigenfunctions and eigenvalues of the covariance operator of the Brownian motion have been computed in Ash & Gardner [1]:

$$\lambda_j = \frac{1}{(j - 0.5)^2 \pi^2} \quad \text{and} \quad V_j(t) = \sqrt{2} \sin((j - 0.5)\pi t), \quad t \in [0, 1], \quad j = 1, 2, \dots$$

In practice $X(t)$ has been simulated using a truncated version of the Karhunen Loève expansion $\sum_{j=1}^{100} \sqrt{\lambda_j} \eta^{(j)} V_j(t)$, where the $(\eta^{(j)})_{j \in \mathbb{N}}$ form an i.i.d. sequence of standard normal variables. The function $X(t)$ is observed on 1000 evenly spaced points in $[0, 1]$.

Testing procedure. For each experiment, we perform the tests $T_\alpha^{(1)}$ (procedure P_1) and $T_\alpha^{(2)}$ (procedure P_2) with $\bar{k}_n = 2^{\lceil \log_2 n - 1 \rceil}$. The quantile $q_{\mathbf{X}, \alpha}$ involved in P_2 is computed by Monte Carlo simulations. For each experiment, we use 1000 random simulations to estimate this quantile.

Choice of θ .

1. In the first experiment, we fix $\theta = 0$ as a way to evaluate the sizes of the testing procedures.
2. In the second experiment, we build directly the function θ in the KL basis of X . The set Θ_{KL} is made of all the functions $\theta_{B, \xi}$ with $B > 0$, $\xi > 0$, and

$$\theta_{B, \xi}(t) := \frac{B}{\sqrt{\sum_{k=1}^{+\infty} k^{-2\xi-1}}} \sum_{j=1}^{100} j^{-\xi-0.5} V_j(t), \quad (24)$$

where ξ is a smoothness parameter. Observe that B stands for the l_2 norm of the function $\theta_{B, \xi}$. As shown on Figure 1, the smoothness of $\theta_{B, \xi} \in \Theta_{KL}$ increases with ξ . This particular choice of θ aims to illustrating the rates of testing shown in Section 5 since we have here an expression of the joint regularity of θ and Γ :

$$\|(\Gamma^{1/2} - \Gamma_k^{1/2})\theta\|^2 = \frac{B^2}{\pi^2 \left(\sum_{l=1}^{+\infty} l^{-2\xi-1} \right)} \sum_{j=k+1}^{100} (j - 0.5)^{-2} j^{-2\xi-1}.$$

In practice, we fix $\xi = 0.1, 0.5, 1$ and $B = 0.1, 0.5, 1$.

3. In the third experiment, we consider the set Θ_G of functions

$$\theta_{B, \tau}(t) = B \exp \left[-\frac{(t - 0.5)^2}{2\tau^2} \right] \left[\int_0^1 \exp \left[-\frac{(x - 0.5)^2}{\tau^2} \right] dx \right]^{-1/2},$$

with $B > 0$ and $\tau > 0$. Here, B stands for the l_2 norm of $\theta_{B, \tau}$ and τ is a smoothness parameter. In fact, $\theta_{B, \tau}(t)$ corresponds (up to a constant) to the density of a normal variable with mean 0.5 and variance τ^2 . As τ decreases to 0, $\theta_{B, \tau}$ converges to a Dirac function centered on 0.5. In practice, we fix $\tau = 0.01, 0.02, 0.05$ and $B = 0.5, 1, 2$.

Number of experiments. We have set $n = 100$ and $n = 500$. For each set of parameters (n, B, ξ) or (n, B, τ) , 10 000 trials were run to estimate the percentages of rejection of H_0 (ie. the percentages of positive values of $T_\alpha^{(1)}$ and $T_\alpha^{(2)}$ with $\alpha = 5\%$), along with their 95% confidence intervals.

7.2. Results

The two procedures P_1 and P_2 have been implemented in R [33] on a 3 GHz Intel Xeon processor, with a 4000KB cache size and 8GB total physical memory.

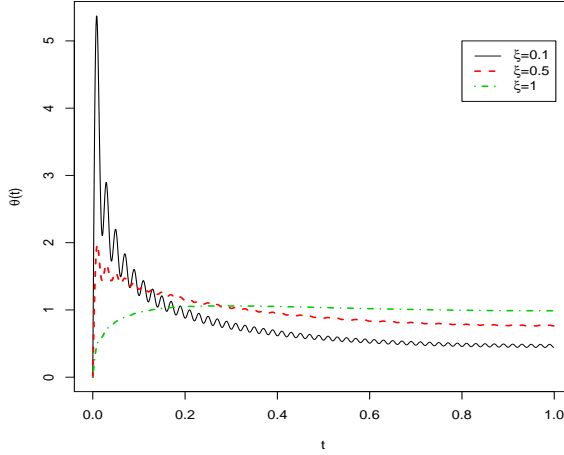


FIGURE 1. Three functions θ in Θ_{KL} when $B = 1$.

TABLE 1

First simulation study: Null hypothesis is true. Percentages of rejection of H_0 and 95% confidence intervals

	$n = 100$		$n = 500$	
$T_\alpha^{(1)}$	3.47		2.61	
CI_α^1	3.11	3.83	2.3	2.92
$T_\alpha^{(2)}$	4.97		5.26	
CI_α^2	4.54	5.4	4.82	5.7

First setting. The percentages of rejection of $T_\alpha^{(1)}$ and $T_\alpha^{(2)}$ under H_0 with $n = 100$ and $n = 500$ are provided in Table 1. As expected, the size of $T_\alpha^{(1)}$ decreases when n increases because we pay a price for the Bonferroni correction. The size of $T_\alpha^{(2)}$ remains close to the nominal level $\alpha = 5\%$.

Second setting. Tables 2 and 3 depict the results for $\theta \in \Theta_{KL}$ with $n = 100$ and $n = 500$ respectively. As expected, the power of the procedures is increasing with B as $\|\theta\|$ becomes larger. Furthermore, the power also increases with ξ . This corroborates the rates stated in Section 5, since the function $\theta_{B,\xi}$ becomes smoother when ξ increases. In every setting the test $T_\alpha^{(2)}$ with the second procedure performs better than $T_\alpha^{(1)}$.

Third setting. The results of the last experiment are provided in Tables 4 for $n = 100$ and 5 for $n = 500$. Again, the power is increasing with B , n and τ . Here, τ does not directly correspond to the rate of convergence of the sequence $(\int_0^1 \theta_{B,\tau} V_j(t) dt)$, $j \geq 1$ as ξ does in the last example. Nevertheless, it is difficult to detect a function $\theta_{B,\tau}$ when τ decreases, that is when $\theta_{B,\tau}$ becomes close to a Dirac function.

In each setting, the test under P_2 is more powerful than the test under P_1 . Nevertheless, the procedure (P_2) is slightly slower as it requires to evaluate the quantile $q_{\mathbf{X},\alpha}$ by a Monte-Carlo method. Under P_1 , the mean computation time is 9 seconds for $n = 100$ and 12 seconds for $n = 500$. In contrast, it respectively equals 11 and 18 seconds under P_2 .

TABLE 2
 Second simulation study: $\theta \in \Theta_{KL}$, $n = 100$. Percentages of rejection of H_0 and 95% confidence intervals

		$B = 0.1$		$B = 0.5$		$B = 1$	
$\xi = 0.1$	$T_\alpha^{(1)}$	3.88		21.41		77.24	
	CI_α^1	3.5	4.26	20.61	22.21	76.42	78.06
	$T_\alpha^{(2)}$	5.8		26.38		81.78	
$\xi = 0.5$	CI_α^2	5.34	6.26	25.52	27.24	81.02	82.54
	$T_\alpha^{(1)}$	4.74		46.47		98.68	
	CI_α^1	4.32	5.16	45.49	47.45	98.46	98.9
$\xi = 1$	$T_\alpha^{(2)}$	6.65	7.14	51.81	53.77	98.87	99.25
	$T_\alpha^{(1)}$	4.8		62.67		99.75	
	CI_α^1	4.38	5.22	61.72	63.62	99.65	99.85
	$T_\alpha^{(2)}$	7.07		68.3		99.84	
	CI_α^2	6.57	7.57	67.39	69.21	99.76	99.92

TABLE 3
 Second simulation study: $\theta \in \Theta_{KL}$, $n = 500$. Percentages of rejection of H_0 and 95% confidence intervals

		$B = 0.1$		$B = 0.5$		$B = 1$	
$\xi = 0.1$	$T_\alpha^{(1)}$	5.17		86.98		100	
	CI_α^1	4.74	5.6	86.32	87.64	100	100
	$T_\alpha^{(2)}$	8.48		90.89		100	
$\xi = 0.5$	CI_α^2	7.93	9.03	90.33	91.45	100	100
	$T_\alpha^{(1)}$	8.81		99.85		100	
	CI_α^1	8.25	9.37	99.77	99.93	100	100
$\xi = 1$	$T_\alpha^{(2)}$	13.07		99.88		100	
	CI_α^2	12.41	13.73	99.81	99.95	100	100
	$T_\alpha^{(1)}$	11.38		99.99		100	
	CI_α^1	10.76	12	99.97	100	100	100
	$T_\alpha^{(2)}$	16.13		100		100	
	CI_α^2	15.41	16.85	100	100	100	100

8. Discussion

Two testing procedures of the nullity of the slope function θ have been proposed in this paper. They are completely data-driven and benefit from optimal properties assessed in a nonasymptotic setting. We address here some extensions of our results.

Although we focused on the null-hypothesis “ $H_0: \theta = 0$ ”, our approach easily extends to linear hypotheses $H_{0,\mathcal{V}}$: “ $\theta \in \mathcal{V}$ ”, where \mathcal{V} is a given finite dimensional subspace of \mathcal{H} of dimension $p < n/2$. As previously, the procedure relies on parametric statistics for testing $H_{0,\mathcal{V}}$ against $H_{1,k,\mathcal{V}}$: “ $\theta \in (\text{Vect}(V_1, \dots, V_k) + \mathcal{V}) \setminus \mathcal{V}$ ”, where k is a positive integer. We consider the $n \times \hat{k}^{KL}$ design matrix \mathbf{W} defined by $\mathbf{W}_{i,j} = \langle X_i, \hat{V}_j \rangle$ for $i = 1, \dots, n$, $j = 1, \dots, \hat{k}^{KL}$. The space generated by the \hat{k}^{KL} columns of the matrix W is denoted $\underline{\mathcal{W}}_{\hat{k}^{KL}}$. Considering a basis (ξ_1, \dots, ξ_p) of \mathcal{V} , we define $\underline{\mathcal{V}}_p$ as the space generated by the p columns of the matrix whose $(ij)^{th}$ element is $\langle X_i, \xi_j \rangle$. In the sequel, $\hat{\Pi}_{k,\mathcal{V}}$ stands for the orthogonal projection in \mathbb{R}^n onto $\underline{\mathcal{V}}_p^\perp \cap \underline{\mathcal{W}}_{\hat{k}^{KL}}$ of dimension less or equal to \hat{k}^{KL} ,

TABLE 4

Third simulation study: $\theta \in \Theta_G$, $n = 100$. Percentage of rejection of H_0 and 95% confidence interval

		$B = 0.5$		$B = 1$		$B = 2$	
$\tau = 0.01$	$T_\alpha^{(1)}$	4.94		11.85		46.69	
	CI_α^1	4.52	5.36	11.22	12.48	45.71	47.67
	$T_\alpha^{(2)}$	7.25		15.49		53.56	
$\tau = 0.02$	CI_α^2	6.74	7.76	14.78	16.2	52.58	54.54
	$T_\alpha^{(1)}$	7.33		23.09		80.26	
	CI_α^1	6.82	7.84	22.26	23.92	79.48	81.04
$\tau = 0.05$	$T_\alpha^{(2)}$	10		28.54		84.04	
	CI_α^2	9.41	10.59	27.65	29.43	83.32	84.76
	$T_\alpha^{(1)}$	13.85		56.51		99.48	
$\tau = 0.05$	CI_α^1	13.17	14.53	55.54	57.48	99.34	99.62
	$T_\alpha^{(2)}$	18.13		63.09		99.65	
	CI_α^2	17.37	18.89	62.14	64.04	99.53	99.77

TABLE 5

Third simulation study: $\theta \in \Theta_G$, $n = 500$. Percentage of rejection of H_0 and 95% confidence interval

		$B = 0.5$		$B = 1$		$B = 2$	
$\tau = 0.01$	$T_\alpha^{(1)}$	12.41		54.6		99.75	
	CI_α^1	11.76	13.06	53.62	55.58	99.65	99.85
	$T_\alpha^{(2)}$	17.99		63.16		99.98	
$\tau = 0.02$	CI_α^2	17.24	18.74	62.21	64.11	99.81	99.95
	$T_\alpha^{(1)}$	26.11		88.91		100	
	CI_α^1	25.25	26.97	88.29	89.53	100	100
$\tau = 0.05$	$T_\alpha^{(2)}$	33.95		92.62		100	
	CI_α^2	33.02	34.88	92.11	93.13	100	100
	$T_\alpha^{(1)}$	65.38		99.95		100	
$\tau = 0.05$	CI_α^1	64.45	66.31	99.91	99.99	100	100
	$T_\alpha^{(2)}$	72.74		99.99		100	
	CI_α^2	71.87	73.61	99.97	100	100	100

while $\widehat{\Pi}_\mathcal{V}$ stands for the orthogonal projection onto \mathcal{V}_p . Then, we consider the following parametric statistic:

$$\phi_{k,\mathcal{V}}(\mathbf{Y}, \mathbf{X}) := \frac{\|\widehat{\Pi}_{k,\mathcal{V}} \mathbf{Y}\|_n^2}{\|\mathbf{Y} - \widehat{\Pi}_{k,\mathcal{V}} \mathbf{Y} - \widehat{\Pi}_\mathcal{V} \mathbf{Y}\|_n^2 / [n - \dim(\mathcal{V}_p + \mathcal{W}_{\hat{k}^{KL}})]}. \quad (25)$$

Under $H_{0,\mathcal{V}}$, $\phi_{k,\mathcal{V}}(\mathbf{Y}, \mathbf{X})/\hat{k}^{KL}$ behaves like a Fisher distribution with $(\dim(\mathcal{V}_p^\perp \cap \mathcal{W}_{\hat{k}^{KL}}), n - \dim(\mathcal{V}_p + \mathcal{W}_{\hat{k}^{KL}}))$ degrees of freedom. The proof is the same as that for $\phi_k(\mathbf{Y}, \mathbf{X})$. In typical situations, we have $\dim(\mathcal{V}_p^\perp \cap \mathcal{W}_{\hat{k}^{KL}}) = k$ and $\dim(\mathcal{V}_p + \mathcal{W}_{\hat{k}^{KL}}) = k + p$. We reject $H_{0,\mathcal{V}}$ when the statistic

$$T_{\alpha,\mathcal{V}} := \sup_{k \in \mathcal{K}_n, k \leq \text{Rank}(\widehat{\Gamma}_n)} \left[\phi_{k,\mathcal{V}}(\mathbf{Y}, \mathbf{X}) - \hat{k}^{KL} \bar{\mathcal{F}}_{\dim(\mathcal{V}_p^\perp \cap \mathcal{W}_{\hat{k}^{KL}}), n - \dim(\mathcal{V}_p + \mathcal{W}_{\hat{k}^{KL}})}^{-1} \{ \alpha_{\mathcal{K}_n}(\mathbf{X}) \} \right] \quad (26)$$

is positive, where the weight $\alpha_{\mathcal{K}_n}(\mathbf{X})$ is chosen according to procedure P_1 (Bonferroni) or a slight variation of P_2 (Monte-Carlo). All the results stated for $T_\alpha^{(1)}$ and $T_\alpha^{(2)}$ are still valid with $T_{\alpha,\mathcal{V}}$. The extension to affine subspaces \mathcal{V} is also possible.

One can adopt a similar approach in the context of a prescribed basis (as wavelet, spline or Fourier basis) instead of the eigenfunctions basis $(\widehat{V}_1, \dots, \widehat{V}_k)$. The size and the power of the corresponding procedures are in fact easier to derive than for a Karhunen-Loève approach as we do not have to control the randomness of the basis. We refer for instance to [3] for such results in a fixed design regression problems.

Finally, our approach can also extend to functional linear regression models with functional outputs building on the parametric statistic of [30].

9. Main proofs

In this section, we describe the core of the proofs. Some arguments are based on perturbation theory and are postponed to the next section.

9.1. Additional notations

Given any integer $k < \text{Rank}(\Gamma)$, we recall that $\Gamma_k = \sum_{j=1}^k \lambda_j V_j \otimes V_j$, where \otimes stands for the tensor product. Similarly, $\widehat{\Gamma}_{n,k} := \sum_{j=1}^k \widehat{\lambda}_j \widehat{V}_j \otimes \widehat{V}_j$ denotes its empirical counterpart. For any $k < \text{Rank}(\Gamma)$, we note Π_k the orthogonal projection in \mathcal{H} onto the space spanned by V_j , $j = 1, \dots, k$, while $\widehat{\Pi}_k$ stands for the orthogonal projection onto the space spanned by \widehat{V}_j , $j = 1, \dots, k \wedge \text{Rank}(\widehat{\Gamma}_n)$.

In order to translate the definition of the testing procedure into functional data analysis framework, we shall use $\Delta = \mathbb{E}(\langle X, \cdot \rangle Y)$. We note $\Delta_n = \sum_{i=1}^n \langle X_i, \cdot \rangle \mathbf{Y}_i / n$ its empirical counterpart. For any $k \leq \text{Rank}(\Gamma)$, we note $A_k = \sum_{j=1}^k \lambda_j^{-1/2} \langle V_j, \cdot \rangle V_j$ and $\widehat{A}_k = \sum_{j=1}^{k \wedge \text{Rank}(\widehat{\Gamma}_n)} \widehat{\lambda}_j^{-1/2} \langle \widehat{V}_j, \cdot \rangle \widehat{V}_j$ its empirical counterpart.

Let S be a bounded linear operator on the Hilbert space \mathcal{H} . The corresponding operator norm will be denoted $\|\cdot\|_\infty$ where $\|S\|_\infty = \sup_{x \in \mathbf{B}(\mathbf{0}, \mathbf{1})} \|S(x)\|$ and $\mathbf{B}(\mathbf{0}, \mathbf{1})$ stands for the unit ball of \mathcal{H} . Let T be a Hilbert-Schmidt operator. $\|\cdot\|_{HS}$ denotes the Hilbert-Schmidt norm and tr stands for the classical trace (defined for trace-class operators). We recall that $\|T\|_{HS}^2 = \text{tr}(T^*T)$.

In the sequel, we note $\bar{\chi}_k(u)$ the probability that a χ^2 variables with k degrees of freedom is larger than u , while $\bar{\chi}_k^{-1}(u)$ denotes the $1 - u$ quantile of a χ^2 random variable.

9.2. Connection between $\phi_k(\mathbf{Y}, \mathbf{X})$ and the procedure of Cardot et al. [6]

In fact, the numerator of the statistic ϕ_k is exactly the same as the test statistic $\|\sqrt{n}\widehat{A}_k\Delta_n\|^2$ introduced by Cardot et al. [6], that is:

$$\phi_k(\mathbf{Y}, \mathbf{X}) = \frac{\|\widehat{\Pi}_k \mathbf{Y}\|_n^2}{\|\mathbf{Y} - \widehat{\Pi}_k \mathbf{Y}\|_n^2 / (n - \hat{k}^{KL})} = \frac{\|\sqrt{n}\widehat{A}_k\Delta_n\|^2}{\|\mathbf{Y} - \widehat{\Pi}_k \mathbf{Y}\|_n^2 / (n - \hat{k}^{KL})}. \quad (27)$$

Proof of Equation (27). Consider the least-squares $\widehat{\theta}_k$ estimator of θ in the space generated by \widehat{V}_j , $j = 1, \dots, \hat{k}^{KL}$. It follows that $\|\widehat{\Pi}_k \mathbf{Y}\|_n^2 = n \langle \widehat{\theta}_k, \widehat{\Gamma}_n \widehat{\theta}_k \rangle$. Since $\widehat{\theta}_k = \widehat{\Gamma}_{n,k}^- \Delta_n$ where $\widehat{\Gamma}_{n,k}^-$ is the Moore-Penrose pseudo-inverse of $\widehat{\Gamma}_{n,k}$, we obtain

$$\|\widehat{\Pi}_k \mathbf{Y}\|_n^2 = n \langle \widehat{\Gamma}_{n,k}^- \Delta_n, \widehat{\Gamma}_n \widehat{\Gamma}_{n,k}^- \Delta_n \rangle = n \langle \widehat{A}_k \Delta_n, \widehat{A}_k \widehat{\Gamma}_n \widehat{\Gamma}_{n,k}^- \Delta_n \rangle = n \|\widehat{A}_k \Delta_n\|^2.$$

□

9.3. Proof of the type I error bounds

We prove first Theorem 4.2 then Propositions 4.1 and 4.3.

Proof of Theorem 4.2. First, we prove that $\hat{k}^{KL} = k$ with large probability.

Lemma 9.1. *Consider the event \mathcal{A}_n defined by*

$$\mathcal{A}_n = \left\{ \sup_{1 \leq j \leq \bar{k}_n} \frac{|\hat{\lambda}_j - \lambda_j|}{\min\{\lambda_j - \lambda_{j+1}, \lambda_{j-1} - \lambda_j\}} \geq 1/2 \right\}. \quad (28)$$

Under Assumption B.2, we have

$$\mathbb{P}(\mathcal{A}_n) \leq C(\gamma) \left[\frac{\bar{k}_n^3 \log^2(\bar{k}_n \vee e)}{n} \right] \leq C(\gamma) \frac{1}{\log^2(n)}, \quad (29)$$

where γ is a positive constant involved in Assumption B.2.

This result is proved Section 10. Observe that under the event $\bar{\mathcal{A}}_n$, we have $\hat{k}^{KL} = k$ for all $k \leq \bar{k}_n$. Consequently, we can replace \hat{k}^{KL} by k in the definition of the test statistic up to an event of probability less than $C(\gamma)/\log(n)$.

In the sequel, we use the alternative expression (27) of ϕ_k and we replace \hat{k}^{KL} by k . The proof is split into the three main lemmas 9.2 - 9.4 first announced then derived in sequence. In the first lemma, we state that $\|\sqrt{n}A_k\Delta_n\|^2/\sigma^2$ behaves like a χ^2 distribution. The second lemma tells us that $\|\sqrt{n}A_k\Delta_n\|^2/\sigma^2$ is close to $\|\sqrt{n}\hat{A}_k\Delta_n\|^2/\sigma^2$ and contains the encapsulated Lemma 9.6. Finally, we prove in the last lemma that $\|\mathbf{Y} - \hat{\Pi}_k\mathbf{Y}\|_n^2/n$ concentrates well around σ^2 . An additional result regarding the chi-square distribution is needed in Lemma 9.5.

Lemma 9.2. *For any $k \geq 1$ and any $x > 0$, we have*

$$|\mathbb{P}(\|\sqrt{n}A_k\Delta_n\|^2 \geq x) - \bar{\chi}_k(x/\sigma^2)| \leq C \frac{k^{3/2} \mathbb{E}[\epsilon^4]^{3/4}}{\sqrt{n} \sigma^3} \sup_{1 \leq j \leq k} \mathbb{E}[(\eta^{(j)})^4]^{3/4} \leq \frac{C}{\log^2(n)},$$

uniformly over all $k \leq \bar{k}_n$.

Lemma 9.3. *Assume that B.1–B.3 hold. For all $M > 0$, $x > 0$, $k \leq \bar{k}_n$, and all $n \geq 1$, we have*

$$\begin{aligned} \mathbb{P} \left[\frac{|\|\sqrt{n}\hat{A}_k\Delta_n\|^2 - \|\sqrt{n}A_k\Delta_n\|^2|}{\|\sqrt{n}\hat{A}_k\Delta_n\|^2} \geq x \right] &\leq \mathbb{P} \left(\|\sqrt{n}A_k\Delta_n\| \leq \frac{\sqrt{k}}{M} \right) + \mathbb{P}[\mathcal{A}_n] \\ &+ C(\gamma)\sigma^2 \frac{M^2}{x^2 \wedge x} \left[\frac{k^2 \log^2(k \vee e)}{n} \vee \frac{1}{\sqrt{n}} \vee \frac{\bar{k}_n^{5/2} \log(\bar{k}_n \vee e)}{kn} \right], \end{aligned} \quad (30)$$

uniformly over all $k \leq \bar{k}_n$.

By Lemma 11.1 in [36], we know that for any $0 < x < 1$ and any integer $d \geq 1$,

$$\mathbb{P} \left[\chi^2(d) \leq de^{-1}x^{2/d} \right] \leq x.$$

Taking $M = \log(n)/\sigma$, we get from Lemma 9.2 and the last deviation inequality that

$$\mathbb{P} \left(\|\sqrt{n}A_k\Delta_n\| \leq \frac{\sqrt{k}}{M} \right) \leq \frac{C}{\log^2(n)} + \left(\frac{\sqrt{e}}{\log(n)} \right)^k,$$

uniformly over all $k \leq \bar{k}_n$. Taking $x_{n,k} = 1/(k \log^2(n))$ in (30) and applying Lemma 9.1, we get

$$\begin{aligned} & \mathbb{P} \left[\frac{|\|\sqrt{n}\widehat{A}_k\Delta_n\|^2 - \|\sqrt{n}A_k\Delta_n\|^2|}{\|\sqrt{n}\widehat{A}_k\Delta_n\|^2} \geq \frac{1}{k \log^2(n)} \right] \\ & \leq C(\gamma) \left[\frac{k^4 \log^8(n)}{n} \vee \frac{k^2 \log^6(n)}{\sqrt{n}} \vee \frac{\bar{k}_n^{7/2} \log^7(n)}{n^{3/4}} \right] + \frac{C'}{\log^2(n)} + \left(\frac{\sqrt{e}}{\log(n)} \right)^k \\ & \leq \frac{C(\gamma)}{\log^2(n)} + \left(\frac{\sqrt{e}}{\log(n)} \right)^k, \end{aligned} \quad (31)$$

uniformly over all $k \leq \bar{k}_n$ by Assumption B.3.

Lemma 9.4. *We have*

$$\mathbb{P} \left[\left| \frac{\|\mathbf{Y} - \widehat{\Pi}_k \mathbf{Y}\|_n^2}{n\sigma^2} - 1 \right| \geq \frac{k \log^2(n)}{n} + C \sqrt{\frac{\log \log n}{n}} \right] \leq \frac{C}{\log^2(n)} + \frac{C'}{\sqrt{n}},$$

uniformly over all $k \leq \bar{k}_n$.

Gathering these three lemmas, we get

$$\begin{aligned} & \mathbb{P} \left[\frac{\|\sqrt{n}\widehat{A}_k\Delta_n\|^2}{\|\mathbf{Y} - \widehat{\Pi}_k \mathbf{Y}\|_n^2/n} \geq k \bar{\mathcal{F}}_{k,n-k}^{-1}(\alpha/|\mathcal{K}_n|) \right] \\ & \leq \mathbb{P} \left[\frac{\|\sqrt{n}A_k\Delta_n\|^2}{(1-x_{n,k})\sigma^2} \geq k \left(1 - e^{-\sqrt{n}} - \frac{k \log^2(n)}{n} \right) \bar{\mathcal{F}}_{k,n-k}^{-1}(\alpha/|\mathcal{K}_n|) \right] \\ & \quad + \mathbb{P} \left[\left| \frac{\|\mathbf{Y} - \widehat{\Pi}_k \mathbf{Y}\|_n^2}{n\sigma^2} - 1 \right| \geq \frac{k \log^2(n)}{n} + e^{-\sqrt{n}} \right] \\ & \quad + \mathbb{P} \left[\frac{|\|\sqrt{n}\widehat{A}_k\Delta_n\|^2 - \|\sqrt{n}A_k\Delta_n\|^2|}{\|\sqrt{n}\widehat{A}_k\Delta_n\|^2} \geq x_{n,k} \right] \\ & \leq \bar{\chi}_k \left[k \left(1 - e^{-\sqrt{n}} - \frac{k \log^2(n)}{n} - x_{n,k} \right) \bar{\mathcal{F}}_{k,n-k}^{-1}(\alpha/|\mathcal{K}_n|) \right] \\ & \quad + \frac{C(\gamma)}{\log^2(n)} + \left(\frac{\sqrt{e}}{\log(n)} \right)^k, \end{aligned} \quad (32)$$

uniformly over all $k \leq \bar{k}_n$.

Lemma 9.5. *For n larger than some numerical constant, we have*

$$\bar{\chi}_k \left[k \left(1 - e^{-\sqrt{n}} - \frac{k \log^2(n)}{n} - \frac{1}{k \log^2(n)} \right) \bar{\mathcal{F}}_{k,n-k}^{-1}(\alpha/|\mathcal{K}_n|) \right] \leq \frac{\alpha}{|\mathcal{K}_n|} \left(1 + \frac{C(\alpha)}{\log(n)} \right). \quad (33)$$

The proof of this lemma is postponed to Appendix A. Coming back to (32), we derive that

$$\mathbb{P} \left[\frac{\|\sqrt{n}\widehat{A}_k\Delta_n\|^2}{\|\mathbf{Y} - \widehat{\Pi}_k \mathbf{Y}\|_n^2/n} \geq k \bar{\mathcal{F}}_{k,n-k}^{-1}(\alpha/|\mathcal{K}_n|) \right] \leq \frac{\alpha}{|\mathcal{K}_n|} \left(1 + \frac{C(\alpha)}{\log(n)} \right) + \frac{C'(\gamma)}{\log^2(n)} + \left(\frac{\sqrt{e}}{\log(n)} \right)^k.$$

Applying an union bound over all $k \in \mathcal{K}_n$, we conclude that

$$\mathbb{P} \left[T_\alpha^{(1)} > 0 \right] \leq \alpha + \frac{C(\alpha, \gamma)}{\log(n)}.$$

□

Proof of Lemma 9.2. Let us fix some $k > 0$. For any $1 \leq j \leq k$, we have

$$\sqrt{n}\langle \Delta_n, A_k V_j \rangle = \frac{1}{\sqrt{n}} \sum_{i=1}^n \langle X_i, \frac{V_j}{\sqrt{\lambda_j}} \rangle \epsilon_i = \frac{1}{\sqrt{n}} \sum_{i=1}^n \boldsymbol{\eta}_i^{(j)} \epsilon_i$$

For any $1 \leq j_1 < j_2 \leq k$ and $1 \leq i \leq j$, the random variables $\boldsymbol{\eta}_i^{(j_1)} \epsilon_i$ and $\boldsymbol{\eta}_i^{(j_2)} \epsilon_i$ are uncorrelated. By the central limit theorem, we conclude that $\|\sqrt{n} A_k \Delta_n\|^2 / \sigma^2$ converges in distribution towards a $\chi^2(k)$ random variable, at least when k is fixed.

In order to precisely control the tails of $\|\sqrt{n} A_k \Delta_n\|^2$, the central limit theorem is not sufficient. We need a Berry-Esseen type inequality. Let us call W_i the vector of size k who j -th component is $\boldsymbol{\eta}_i^{(j)} \epsilon_i$. We note $\|W_i\|_k$ its Euclidean norm. By Assumption **B.1**, we have

$$\mathbb{E} [\|W_i\|_k^3] \leq k^{3/2} \mathbb{E} [\epsilon^4]^{3/4} \sup_{1 \leq j \leq k} \mathbb{E} [(\eta^{(j)})^4]^{3/4}.$$

Applying the second part of Theorem 1.1 in Bentkus [4], we obtain

$$\sup_{x>0} |\mathbb{P}(\|\sqrt{n} A_k \Delta_n\|^2 \geq x) - \bar{\chi}_k(x/\sigma^2)| \leq C \frac{k^{3/2} \mathbb{E} [\epsilon^4]^{3/4}}{\sqrt{n} \sigma^3} \sup_{1 \leq j \leq k} \mathbb{E} [(\eta^{(j)})^4]^{3/4}.$$

We conclude by applying Assumption **B.3**. □

Proof of Lemma 9.3. From $\|b\|^2 - \|a\|^2 = 2 \langle a, b - a \rangle + \|b - a\|^2$, we get

$$\frac{|\|b\|^2 - \|a\|^2|}{\|a\|^2} \leq \frac{\|b - a\|}{\|a\|} \left(2 + \frac{\|b - a\|}{\|a\|} \right).$$

Consequently, we have for any $M > 0$ and any $x > 0$

$$\begin{aligned} & \mathbb{P} \left(\frac{\left| \|\sqrt{n} \widehat{A}_k \Delta_n\|^2 - \|\sqrt{n} A_k \Delta_n\|^2 \right|}{\|\sqrt{n} A_k \Delta_n\|^2} \geq x \right) \\ & \leq \mathbb{P} \left(\frac{\|\sqrt{n} (\widehat{A}_k - A_k) \Delta_n\|}{\|\sqrt{n} A_k \Delta_n\|} \geq \frac{x}{4} \wedge \sqrt{\frac{x}{2}} \right) \\ & \leq \mathbb{P} \left(\|\sqrt{n} (\widehat{A}_k - A_k) \Delta_n\| \mathbf{1}_{\overline{\mathcal{A}}_n} \geq \frac{\sqrt{k}(x \wedge \sqrt{x})}{4M} \right) + \mathbb{P} \left(\|\sqrt{n} A_k \Delta_n\| \leq \frac{\sqrt{k}}{M} \right) + \mathbb{P}[\mathcal{A}_n] \\ & \leq \frac{4M^2}{k(x^2 \wedge x)} \mathbb{E} \left[\|\sqrt{n} (\widehat{A}_k - A_k) \Delta_n\|^2 \mathbf{1}_{\overline{\mathcal{A}}_n} \right] + \mathbb{P} \left(\|\sqrt{n} A_k \Delta_n\| \leq \frac{\sqrt{k}}{M} \right) + \mathbb{P}[\mathcal{A}_n]. \quad (34) \end{aligned}$$

In order to conclude, we only need to bound $\mathbb{E}[\|\sqrt{n} (\widehat{A}_k - A_k) \Delta_n\|^2 \mathbf{1}_{\overline{\mathcal{A}}_n}]$. Noticing that $\widehat{A}_k - A_k$ only depends on the X_i 's, we derive that

$$\begin{aligned} \mathbb{E} \left[\|\sqrt{n} (\widehat{A}_k - A_k) \Delta_n\|^2 \mathbf{1}_{\overline{\mathcal{A}}_n} \right] &= \frac{1}{n} \mathbb{E} \left[\|\sqrt{n} (\widehat{A}_k - A_k) X_1 \boldsymbol{\varepsilon}_1\|^2 \mathbf{1}_{\overline{\mathcal{A}}_n} \right] \\ &= \frac{\sigma^2}{n} \mathbb{E} \left[\|\sqrt{n} (\widehat{A}_k - A_k) X_1\|^2 \mathbf{1}_{\overline{\mathcal{A}}_n} \right] \\ &= \frac{\sigma^2}{n} \mathbb{E} \left[\left(\|\widehat{A}_k X_1\|^2 + \|A_k X_1\|^2 - 2 \langle \widehat{A}_k X_1, A_k X_1 \rangle \right) \mathbf{1}_{\overline{\mathcal{A}}_n} \right]. \end{aligned}$$

We deal with each term separately:

$$\begin{aligned}
\mathbb{E} \left[\|\widehat{A}_k X_1\|^2 \mathbf{1}_{\overline{\mathcal{A}}_n} \right] &= \mathbb{E} \left[\text{tr} \left(\widehat{A}_k (X_1 \otimes X_1) \widehat{A}_k \right) \mathbf{1}_{\overline{\mathcal{A}}_n} \right] = \mathbb{E} \left[\text{tr} \left(\widehat{A}_k \widehat{\Gamma}_n \widehat{A}_k \right) \mathbf{1}_{\overline{\mathcal{A}}_n} \right] \\
&= \mathbb{E} \left[\text{tr} \widehat{\Pi}_k \mathbf{1}_{\overline{\mathcal{A}}_n} \right] \leq k \mathbb{P} [\overline{\mathcal{A}}_n] , \\
\mathbb{E} \left[\|A_k X_1\|^2 \mathbf{1}_{\overline{\mathcal{A}}_n} \right] &= \mathbb{E} \left[\text{tr} \left(A_k \widehat{\Gamma}_n A_k \right) \mathbf{1}_{\overline{\mathcal{A}}_n} \right] \\
&= \mathbb{E} \left[\text{tr} (A_k \Gamma A_k) \mathbf{1}_{\overline{\mathcal{A}}_n} \right] + \mathbb{E} \left[\text{tr} \left(A_k (\widehat{\Gamma}_n - \Gamma) A_k \right) \mathbf{1}_{\overline{\mathcal{A}}_n} \right] \\
&= \mathbb{E} \left[\text{tr} \Pi_k \mathbf{1}_{\overline{\mathcal{A}}_n} \right] - \mathbb{E} \left[\text{tr} \left(A_k (\widehat{\Gamma}_n - \Gamma) A_k \right) \mathbf{1}_{\overline{\mathcal{A}}_n} \right] \\
&\leq k \mathbb{P} [\overline{\mathcal{A}}_n] + \sqrt{\mathbb{E} \left[\text{tr}^2 \left(A_k (\widehat{\Gamma}_n - \Gamma) A_k \right) \right]} \sqrt{\mathbb{P} [\overline{\mathcal{A}}_n]} , \\
\mathbb{E} \left[\left\langle \widehat{A}_k X_1, A_k X_1 \right\rangle \mathbf{1}_{\overline{\mathcal{A}}_n} \right] &= \mathbb{E} \left[\text{tr} \left(\widehat{A}_k \widehat{\Gamma}_n A_k \right) \mathbf{1}_{\overline{\mathcal{A}}_n} \right] = \mathbb{E} \left[\text{tr} \left(\Gamma_k^{-1/2} \widehat{\Gamma}_{n,k}^{1/2} \right) \mathbf{1}_{\overline{\mathcal{A}}_n} \right] \\
&= k \mathbb{P} (\overline{\mathcal{A}}_n) + \mathbb{E} \left[\text{tr} \left\{ \Gamma_k^{-1/2} \left(\widehat{\Gamma}_{n,k}^{1/2} - \Gamma_k^{1/2} \right) \mathbf{1}_{\overline{\mathcal{A}}_n} \right\} \right] .
\end{aligned}$$

It follows that

$$\begin{aligned}
\mathbb{E} \left[\left\| \sqrt{n} \left(\widehat{A}_k - A_k \right) \Delta_n \right\|^2 \mathbf{1}_{\overline{\mathcal{A}}_n} \right] &\leq 2\sigma^2 \mathbb{E} \left[\text{tr} \left\{ \Gamma_k^{-1/2} \left(\Gamma_k^{1/2} - \widehat{\Gamma}_{n,k}^{1/2} \right) \right\} \mathbf{1}_{\overline{\mathcal{A}}_n} \right] \\
&\quad + \sigma^2 \sqrt{\mathbb{E} \left[\text{tr}^2 \left(A_k (\widehat{\Gamma}_n - \Gamma) A_k \right) \right]} \sqrt{\mathbb{P} [\overline{\mathcal{A}}_n]} . \quad (35)
\end{aligned}$$

Lemma 9.6. *Under Assumptions B.1 and B.2, we have for all $n \geq 1$,*

$$\mathbb{E} \left[\text{tr} \left\{ \Gamma_k^{-1/2} \left(\Gamma_k^{1/2} - \widehat{\Gamma}_{n,k}^{1/2} \right) \right\} \mathbf{1}_{\overline{\mathcal{A}}_n} \right] \leq C(\gamma) \frac{k^3 [\log^2(k) \vee 1]}{n} + C' \frac{k}{\sqrt{n}} \quad (36)$$

uniformly over all $k \leq \bar{k}_n$.

The proof of Lemma 9.6 is postponed to Section 10. Let us compute the last term

$$\begin{aligned}
\mathbb{E} \left[\text{tr}^2 \left(A_k (\widehat{\Gamma}_n - \Gamma) A_k \right) \right] &= \mathbb{E} \left[\left(\sum_{j=1}^k \left(\sum_{i=1}^n \frac{[\boldsymbol{\eta}_i^{(j)}]^2}{n} - 1 \right) \right)^2 \right] \\
&\leq \frac{k^2}{n} \sup_{j \geq 1} \text{Var} \left[(\eta^{(j)})^2 \right] \leq C \frac{k^2}{n} ,
\end{aligned}$$

by Assumption B.1. Combining this bound with (29), we get

$$\sqrt{\mathbb{E} \left[\text{tr}^2 \left(A_k (\widehat{\Gamma}_n - \Gamma) A_k \right) \right]} \sqrt{\mathbb{P} [\overline{\mathcal{A}}_n]} \leq C(\gamma) \frac{\bar{k}_n^{5/2} \log(\bar{k}_n \vee e)}{n} . \quad (37)$$

Gathering Lemma 9.6 with (34), (35), and (37) allows us to conclude. \square

Proof of Lemma 9.4. We have $\|\mathbf{Y} - \widehat{\Pi}_k \mathbf{Y}\|_n^2 = \|\mathbf{Y}\|_n^2 - \|\widehat{\Pi}_k \mathbf{Y}\|_n^2$. By the Central limit Theorem, the classical Berry-Esseen inequality, and a classical deviation inequality of χ^2 random variables (e.g. Lemma 1 in [31]), we get

$$\mathbb{P} \left[\left| \frac{\|\mathbf{Y}\|_n^2}{n\sigma^2} - 1 \right| \geq 2\sqrt{\frac{\log(1/x)}{n}} + 2\frac{\log(1/x)}{n} \right] \leq 2x + C \frac{\mathbb{E}(|\epsilon|^3)}{\sigma^3 \sqrt{n}} , \quad (38)$$

for any $x > 0$. Let us compute the expectation of $\|\widehat{\Pi}_k \mathbf{Y}\|_n^2$.

$$\begin{aligned} \mathbb{E} \left[\|\widehat{\Pi}_k \mathbf{Y}\|_n^2 \right] &= \mathbb{E} \left[\mathbb{E} \left\{ \|\widehat{\Pi}_k \mathbf{Y}\|_n^2 \mid \mathbf{X} \right\} \right] = \mathbb{E} \left[\text{tr}[\mathbf{Y}^* \widehat{\Pi}_k \mathbf{Y}] \mid \mathbf{X} \right] = \sigma^2 \mathbb{E} \left[\text{tr}[\widehat{\Pi}_k] \mid \mathbf{X} \right] \\ &= \sigma^2 \mathbb{E} \left[\text{tr} \left[\widehat{\Pi}_k \right] \right] \leq \sigma^2 k \end{aligned}$$

Applying Markov's inequality to $\|\widehat{\Pi}_k \mathbf{Y}\|_n^2$ and gathering this deviation inequality with (38), we conclude that

$$\mathbb{P} \left[\left| \frac{\|\mathbf{Y} - \widehat{\Pi}_k \mathbf{Y}\|_n^2}{n\sigma^2} - 1 \right| \geq \frac{k \log^2(n)}{n} + 8\sqrt{\frac{\log \log n}{n}} \right] \leq \frac{3}{\log^2(n)} + \frac{C}{\sqrt{n}},$$

uniformly over all $k \leq \bar{k}_n$. □

Proof of Proposition 4.1. Let us assume that ϵ follows a Gaussian distribution and that $\theta = 0$. Conditionally on \mathbf{X} , the statistic $\phi_k(\mathbf{Y}, \mathbf{X})/\hat{k}$ defined in (3.2) follows a Fisher distribution with $(\hat{k}, n - \hat{k})$ degrees of freedom. Hence, conditionally on \mathbf{X} , the multiple testing procedure $T_\alpha^{(1)}$ is a Bonferroni procedure of Fisher statistics and its size is smaller than α . Reintegrating with respect to \mathbf{X} , we derive that the size of $T_\alpha^{(1)}$ is smaller than α .

Let us turn to the second result. We have

$$\mathbb{P}_0(T_\alpha^{(2)} \mid \mathbf{X}) = \alpha \quad \mathbf{X} \text{ p.s. .}$$

The result follows from the fact that $q_{\mathbf{X}, \alpha}$ satisfies

$$\mathbb{P}_0 \left(\sup_{k \in \mathcal{K}_n} \left\{ \frac{(n - \hat{k}^*) \|\widehat{\Pi}_k \epsilon\|_n^2}{\hat{k}^* \|\epsilon - \widehat{\Pi}_k \epsilon\|_n^2} - \bar{\mathcal{F}}_{\hat{k}^*, n - \hat{k}^*}^{-1}(q_{\mathbf{X}, \alpha}^*) \right\} > 0 \mid \mathbf{X} \right) = \alpha .$$

□

Proof of Proposition 4.3. This proof follows the same steps as the proof of Proposition 3.2 in [37]. □

9.4. Proofs of the type II error bounds

Proof of Proposition 5.1. Let us first work conditionally to \mathbf{X} . In this case, the design \mathbf{X} and the projection $\widehat{\Pi}_k$ are considered as fixed. Thus, the statistic $T_\alpha^{(1)}$ is analogous to the procedure of Baraud et al. [3]. By Theorem 3 in [3], we have $\mathbb{P}_\theta(T_\alpha^{(1)} > 0) \geq 1 - \beta/2$ if θ belongs to the set $\Theta_n(\beta/2, \mathbf{X}) := \left\{ \theta \in \mathcal{H}, n\langle \theta, \widehat{\Gamma}_n \theta \rangle \geq \inf_{k \in \mathcal{K}_n} \Delta(\theta, k, \mathbf{X}) \right\}$, where

$$\Delta(\theta, k, \mathbf{X}) := C_1 \langle \theta, \widehat{\Pi}_{k, KL}^\perp \widehat{\Gamma}_n \theta \rangle + C_2 \sqrt{\hat{k}^{KL} \log \left(\frac{2 \log n}{\alpha \beta} \right)} \sigma^2 + C_3 \log \left(\frac{2 \log n}{\alpha \beta} \right) \sigma^2 . \quad (39)$$

since $\alpha \geq \exp(-n/20)$, $\beta \geq C/n$. We have $n\langle \theta, \widehat{\Gamma}_n \theta \rangle = \sum_{i=1}^n \langle X_i, \theta \rangle^2$. By Assumption B.1, we get

$$\mathbb{E}[\langle X, \theta \rangle^4] = \mathbb{E} \left[\left(\sum_{j=1}^{\infty} \sqrt{\lambda_j} \theta_j \eta^{(j)} \right)^4 \right] \leq C \langle \theta, \Gamma \theta \rangle^4 .$$

Applying Chebychev's inequality, we have

$$\langle \theta, \widehat{\Gamma}_n \theta \rangle \geq \mathbb{E}[\langle X, \theta \rangle^2] / 2, \text{ with probability larger than } 1 - \beta/4 \text{ as long as } \beta \geq C/n. \quad (40)$$

Let us fix some $k \in \mathcal{K}_n$. We have $\langle \theta, \widehat{\Pi}_{\widehat{k}^{KL}}^\perp \widehat{\Gamma}_n \theta \rangle^2 \leq \|\theta\|^2 \widehat{\lambda}_{\widehat{k}^{KL}+1}$. Observe that $\widehat{k}^{KL} + 1 < k + 1$ only if $\widehat{\lambda}_{\widehat{k}^{KL}+1} = 0$. Consequently, we always have $\langle \theta, \widehat{\Pi}_{\widehat{k}^{KL}}^\perp \widehat{\Gamma}_n \theta \rangle \leq \|\theta\|^2 \widehat{\lambda}_{k+1}$.

To conclude it is sufficient to provide an upper bound $\widehat{\lambda}_k$ with high probability. By definition of $\widehat{\lambda}_{k+1}$, we have

$$\widehat{\lambda}_{k+1} = \inf_{W, \text{Codim}(W)=k} \sup_{z \in W, \|z\|=1} \langle z, \widehat{\Gamma}_n z \rangle \leq \sup_{z \in \text{Vect}(V_{k+1}, \dots), \|z\|=1} \langle z, \widehat{\Gamma}_n z \rangle,$$

implying that

$$\widehat{\lambda}_k \leq \|\Pi_k^\perp \widehat{\Gamma}_n \Pi_k^\perp\|_\infty \leq \lambda_{k+1} + \|\Pi_k^\perp (\Gamma - \widehat{\Gamma}_n) \Pi_k^\perp\|_\infty \leq \lambda_{k+1} + \|\Pi_k^\perp (\Gamma - \widehat{\Gamma}_n) \Pi_k^\perp\|_{HS}.$$

Hence, it is sufficient to bound the Hilbert Schmidt norm $\|\Pi_k^\perp (\Gamma - \widehat{\Gamma}_n) \Pi_k^\perp\|_{HS}$ in probability. By Jensen's inequality, we have $\mathbb{E}[\|\Pi_k^\perp (\Gamma - \widehat{\Gamma}_n) \Pi_k^\perp\|_{HS}] \leq \mathbb{E}[\|\Pi_k^\perp (\Gamma - \widehat{\Gamma}_n) \Pi_k^\perp\|_{HS}^2]^{1/2}$ and simple calculations lead to

$$\mathbb{E}[\|\Pi_k^\perp (\Gamma - \widehat{\Gamma}_n) \Pi_k^\perp\|_{HS}^2] = \frac{1}{n} \mathbb{E}[\|\Pi_k^\perp \Gamma \Pi_k^\perp - (\Pi_k^\perp X) \otimes (\Pi_k^\perp X)\|_{HS}^2].$$

By Assumption **B.1**, we conclude that

$$\mathbb{E}[\|\Pi_k^\perp \Gamma \Pi_k^\perp - (\Pi_k^\perp X) \otimes (\Pi_k^\perp X)\|_{HS}^2] \leq \mathbb{E}[\|\Pi_k^\perp X\|^4] \leq C \left(\sum_{j \geq k+1} \lambda_j \right)^2.$$

By Markov's inequality, we conclude that $\widehat{\lambda}_{k+1} \leq \lambda_k + C(\beta) \sum_{j \geq k+1} \frac{\lambda_j}{\sqrt{n}}$ with probability larger than $1 - \beta/4$. Gathering this probability bound with (39) and (40), we derive that $P_\theta(T_\alpha^{(1)} > 0) \geq 1 - \beta$ if θ satisfies for some $k \in \mathcal{K}_n$,

$$\|\Gamma^{1/2} \theta\|^2 \geq \frac{C_1}{n} \|\theta\|^2 \left[\lambda_k + C(\beta) \sum_{j \geq k+1} \lambda_j / \sqrt{n} \right] + C_2 \frac{\sigma^2}{n} \left[\sqrt{k \log \left(\frac{2 \log n}{\alpha \beta} \right)} + \log \left(\frac{2 \log n}{\alpha \beta} \right) \right].$$

□

Proof of Theorem 5.2. Arguing as in the beginning of the proof of Theorem 4.2, we can replace \widehat{k}^{KL} by k in the definition of the statistic (6). Consider some $k \in \mathcal{K}_n$, the numerator of $\phi_k(\mathbf{Y}, \mathbf{X})$ is broken down as follows

$$\left\| \sqrt{n} \widehat{A}_k \Delta_n \right\|^2 \leq \left\| \sqrt{n} A_k \Delta_n \right\|^2 + \left[\left\| \sqrt{n} \widehat{A}_k \Delta_n \right\|^2 - \left\| \sqrt{n} A_k \Delta_n \right\|^2 \right].$$

Let us call A_1 and A_2 the two terms of this expression. Using the decomposition $Y = \langle X, \theta \rangle + \epsilon$, observe that $\Delta_n = \widehat{\Gamma}_n \theta + \Delta_{n,1}$, where:

$$\Delta_{n,1} = \sum_{i=1}^n \langle X_i, \cdot \rangle \epsilon_i / n.$$

Lemma 9.7 (Control of A_1). *For any $\beta \in (0, 1)$, we have*

$$\mathbb{P} \left(\left\| \sqrt{n} A_k \Delta_n \right\|^2 \geq k \sigma^2 + \frac{n}{5} \|\Gamma_k^{1/2} \theta\|^2 - 2 \sigma^2 \sqrt{k \log \left(\frac{2}{\beta} \right)} - 10 \sigma^2 \log \left(\frac{2}{\beta} \right) \right) \geq 1 - \frac{\beta}{2} - \frac{C}{\log(n)},$$

uniformly over all $k \leq \bar{k}_n$.

Lemma 9.8. *Assume that B.1–B.3 hold. For any $x > 0$, $k \leq \bar{k}_n$, and $n \geq 1$ we have*

$$\begin{aligned} \mathbb{P} \left[\|\sqrt{n}(\widehat{A}_k - A_k)\widehat{\Gamma}_n\theta\| \geq x \right] &\leq \mathbb{P}[\mathcal{A}_n] + \frac{C'}{\log(n)} \\ &+ C(\gamma) \frac{\log(n)}{x^2} \|\Gamma^{1/2}\theta\|^2 \left[\frac{k^3 \log^2(k \vee e)}{n} \vee \frac{k}{\sqrt{n}} \vee \frac{\bar{k}_n^{5/2} \log(\bar{k}_n \vee e)}{n} \right], \end{aligned} \quad (41)$$

$$\mathbb{P} \left[\|\sqrt{n}(\widehat{A}_k - A_k)\Delta_{n,1}\| \geq x \right] \leq C(\gamma) \frac{\sigma^2}{x^2} \left[\frac{k^3 \log^2(k \vee e)}{n} \vee \frac{k}{\sqrt{n}} \vee \frac{\bar{k}_n^{5/2} \log(\bar{k}_n \vee e)}{n} \right] + \mathbb{P}[\mathcal{A}_n].$$

uniformly over all $k \leq \bar{k}_n$.

Fixing $x = \|\Gamma^{1/2}\theta\|/\log(n)$ in the first bound and $x = \sigma/\log(n)$ in the second bound and applying Assumption B.3, we derive that

$$\|\sqrt{n}(A_k - \widehat{A}_k)\Delta_n\| \leq \frac{\|\Gamma^{1/2}\theta\| + \sigma}{\log(n)},$$

with probability larger than $1 - C(\gamma)/[n^{1/4} \log(n)]$.

For any positive numbers a and b , we have $a^2 \geq b^2 - 2b|a - b| \geq b^2(1 - 1/(\sqrt{k} \log n)) - |a - b|^2 \sqrt{k} \log(n)$. Hence,

$$\|\sqrt{n}\widehat{A}_k\Delta_n\|^2 \geq \|\sqrt{n}A_k\Delta_n\|^2 \left[1 - (\sqrt{k} \log(n))^{-1} \right] - \sqrt{k} \log(n) \|\sqrt{n}(A_k - \widehat{A}_k)\Delta_n\|^2.$$

Gathering the two last bounds with Lemma 9.7, we derive that

$$\begin{aligned} \|\sqrt{n}\widehat{A}_k\Delta_n\|^2 &\geq k\sigma^2 - 2(\|\Gamma^{1/2}\theta\|^2 + \sigma^2) \frac{\sqrt{k}}{\log(n)} \\ &+ \left[C_1 n \|\Gamma_k^{1/2}\theta\|^2 - C_2 \sigma^2 \left(\sqrt{k \log\left(\frac{2}{\beta}\right)} + \log\left(\frac{2}{\beta}\right) \right) \right] \left[1 - (\sqrt{k} \log(n))^{-1} \right], \end{aligned}$$

with probability larger than $1 - \beta/2 - C(\gamma)/\log(n)$.

Lemma 9.9 (Control of the denominator). *We have*

$$\frac{\|\mathbf{Y} - \widehat{\Pi}_k \mathbf{Y}\|_n^2}{n - k} \leq \left(\sigma^2 + \|\Gamma^{1/2}\theta\|^2 \right) \left[1 + C \left(\frac{k}{n} + \frac{\sqrt{\log(n)}}{\sqrt{n}} \right) \right],$$

with probability larger than $1 - \log^{-1}(n)$.

We derive from the previous results that with probability larger than $1 - \beta/2 - C(\gamma)/\log(n)$, the statistic $\phi_k(\mathbf{Y}, \mathbf{X})$ is lower bounded by

$$\frac{k\sigma^2 + \|\Gamma_k^{1/2}\theta\|^2 \left[C_1 n - 2\sqrt{k}/\log(n) \right] - C_2 \sigma^2 \left(\sqrt{k \log(1/\beta)} + \log(1/\beta) \right)}{\left(\sigma^2 + \|\Gamma^{1/2}\theta\|^2 \right) \left[1 + C_3 \left(\frac{k}{n} + \frac{\sqrt{\log(n)}}{\sqrt{n}} \right) \right]}.$$

By Lemma 1 in [3], we can upper bound the quantile of Fisher distribution

$$k\bar{\mathcal{F}}_{k, n-k}^{-1}(\alpha/|\mathcal{K}_n|) \leq k + C \left[\sqrt{k \log\left(\frac{|\mathcal{K}_n|}{\alpha}\right)} + \log\left(\frac{|\mathcal{K}_n|}{\alpha}\right) \right],$$

since we assume that $\log(|\mathcal{K}_n|/\alpha) \leq \log(n) + \log(1/\alpha) \leq 2\sqrt{n}$. By Assumption **B.3**, we derive that $\phi_k(\mathbf{Y}, \mathbf{X}) - k\bar{\mathcal{F}}_{k,n-k}^{-1}(\alpha/|\mathcal{K}_n|)$ is positive with probability larger than $1 - \beta/2 - C(\gamma) \log^{-1}(n)$ if

$$\begin{aligned} & \|\Gamma_k^{1/2}\theta\|^2 \left[C_1 n - 2\sqrt{k}/\log(n) \right] - C_2 \sigma^2 \left(\sqrt{k \log(1/\beta)} + \log(1/\beta) \right) \\ & \geq C_3 \sigma^2 \left[\sqrt{k \log\left(\frac{|\mathcal{K}_n|}{\alpha}\right)} + \log\left(\frac{|\mathcal{K}_n|}{\alpha}\right) \right] + \sigma^2 C_4 \left(\frac{k^2}{n} + \frac{k\sqrt{\log(n)}}{\sqrt{n}} \right) \\ & \quad + \|\Gamma^{1/2}\theta\|^2 C_5 \left[k \vee \log\left(\frac{|\mathcal{K}_n|}{\alpha}\right) \right]. \end{aligned}$$

Since $\log(|\mathcal{K}_n|/\alpha) \leq 2\sqrt{n}$ and since $k \leq n^{1/4}$, we derive that for n larger than a numerical quantity, $\phi_k(\mathbf{Y}, \mathbf{X}) - k\bar{\mathcal{F}}_{k,n-k}^{-1}(\alpha/|\mathcal{K}_n|)$ is positive with probability larger than $1 - \beta/2 - C(\gamma) \log^{-1}(n)$ if

$$\|\Gamma^{1/2}\theta\|^2 \geq C_1 \|\Gamma^{1/2} - \Gamma_k^{1/2}\theta\|^2 + \sigma^2 \frac{C_2}{n} \left[\sqrt{k \log\left(\frac{|\mathcal{K}_n|}{\alpha\beta}\right)} + \log\left(\frac{|\mathcal{K}_n|}{\beta\alpha}\right) \right].$$

□

Proof of Lemma 9.7. As explained in the proof of Lemma 9.2, $\sqrt{n}/\sigma A_k \Delta_{n,1}$ converges to a Gaussian process whose covariance operator Σ_k is defined by $\Sigma_k = \sum_{j=1}^k \langle V_j, \cdot \rangle V_j$. For $j = 1, \dots, k$, we define $\xi_j = (\lambda_j^{1/2} \sqrt{\text{Var}([\eta^{(j)}]^2)} \langle \theta, V_j \rangle)^{-1}$ if $\langle \theta, V_j \rangle^2 \neq 0$ and $\xi_j = 0$ else. Consider the operator $D_k = \sum_{j=1}^k \xi_j \langle V_j, \cdot \rangle V_j$. For any $j = 1, \dots, k$ such that $\xi_j \neq 0$, we have

$$\sqrt{n} \langle D_k A_k \hat{\Gamma}_n \theta, V_j \rangle - \sqrt{\frac{n}{\text{Var}([\eta^{(j)}]^2)}} = \frac{\sum_{i=1}^n [\boldsymbol{\eta}_i^{(j)}]^2 - 1}{\sqrt{n \text{Var}([\eta^{(j)}]^2)}}.$$

As a consequence, $\sqrt{n}(D_k A_k \hat{\Gamma}_n \theta - D_k \Gamma_k^{1/2} \theta)$ converges in distribution towards a Gaussian process whose covariance operator Σ'_k is defined by $\Sigma'_k = \sum_{j=1}^k \langle V_j, \cdot \rangle V_j \mathbf{1}_{\xi_j \neq 0}$. Furthermore, the processes $\sqrt{n}/\sigma A_k \Delta_{n,1}$ and $\sqrt{n}(D_k A_k \hat{\Gamma}_n \theta - D_k \Gamma_k^{1/2} \theta)$ are asymptotically independent. Let us consider the random vector Z of size $k' := k + \#\{j \in \{1, \dots, k\} : \xi_j \neq 0\}$ such that $Z_j = \epsilon/\sigma \eta^{(j)}$ if $j = 1, \dots, k$ and $Z_j = ([\eta^{(j)}]^2 - 1)/\sqrt{\text{Var}([\eta^{(j)}]^2)}$ if $j > k$. Let us upper bound $\mathbb{E}[\|Z\|_{k'}^3]$

$$\mathbb{E}[\|Z\|_{k'}^3] \leq C k^{3/2} \left(\frac{\mathbb{E}[\epsilon^4]^{3/4}}{\sigma^3} \max_{1 \leq j \leq k} \mathbb{E}[(\eta^{(j)})^4]^{3/4} \vee \max_{1 \leq j \leq k} \mathbb{E}[(\eta^{(j)})^8]^{3/4} \right).$$

We note $\mathbf{Z}_1, \dots, \mathbf{Z}_n$ the n observations of the vector Z , based on $\boldsymbol{\eta}_i^{(j)}$ and ϵ_i for $i = 1, \dots, n$. By Assumptions **B.1** and **B.4**, we can apply the Berry-Esseen type inequality of Bentkus (Theorem 1.1 in [4]) in dimension k' . For any convex set \mathcal{A} , we obtain

$$\begin{aligned} & \left| \mathbb{P} \left(\sum_{i=1}^n \frac{\mathbf{Z}_i}{\sqrt{n}} \in \mathcal{A} \right) - \mathbb{P} [\mathcal{N}_{k'}(0, I_{k'}) \in \mathcal{A}] \right| \leq C \frac{k^{7/4}}{\sqrt{n}} \\ & \quad \times \left(\frac{\mathbb{E}[\epsilon^4]^{3/4}}{\sigma^3} \max_{1 \leq j \leq k} \mathbb{E}[(\eta^{(j)})^4]^{3/4} \vee \max_{1 \leq j \leq k} \mathbb{E}[(\eta^{(j)})^8]^{3/4} \right). \end{aligned}$$

Moreover, this last quantity is smaller than $C n^{-1/16} \log^{-7}(n)$ uniformly over all $k \leq \bar{k}_n$ by Assumption **B.3**. Consider a standard Gaussian vector (u_1, \dots, u_{2k}) . We define the

random vector W by

$$W = \sum_{j=1}^k \left(\sqrt{n\lambda_j} \langle \theta, V_j \rangle + \sqrt{\lambda_j} \langle \theta, V_j \rangle \sqrt{\text{Var}([\eta^{(j)}]^2)} u_j + \sigma u_{j+k} \right)^2 .$$

We derive from the definition of W and the previous Berry-Esseen inequality that

$$\sup_{x>0} \left| \mathbb{P} \left(\left\| \sqrt{n} A_{k,n} \left(\widehat{\Gamma}_n \theta + \Delta_{n,1} \right) \right\|^2 \geq x \right) - \mathbb{P}(W \geq x) \right| \leq \frac{C}{\log^7(n)} .$$

Conditionally to (u_1, \dots, u_k) , W/σ^2 follows a non-central χ^2 distribution with k degrees of freedom and non-centrality parameter

$$V := \sum_{j=1}^k \left(\sqrt{n\lambda_j} \langle \theta, V_j \rangle + \sqrt{\lambda_j} \langle \theta, V_j \rangle \sqrt{\text{Var}([\eta^{(j)}]^2)} u_j \right)^2 / \sigma^2 .$$

By a deviation inequality on non-central χ^2 distributions (e.g. Eq.18 in [3]), we derive that, conditionally to (u_1, \dots, u_k) ,

$$W \geq k\sigma^2 + \frac{4}{5}V\sigma^2 - 2\sigma^2 \sqrt{k \log(2/\beta)} - 10\sigma^2 \log(2/\beta) ,$$

with probability larger than $1 - \beta/2$. The non-centrality parameter V is polynomial function of independent normal variables. Applying a deviation inequality for normal variables, we derive that $V \geq n/4 \|\Gamma_k^{1/2} \theta\|^2 / \sigma^2$ with probability larger than $1 - \sum_{j=1}^k \exp[-n \text{Var}([\eta^{(j)}]^2)/8]$. All in all, we conclude that

$$\left\| \sqrt{n} A_{k,n} \left(\widehat{\Gamma}_n \theta + \Delta_{n,1} \right) \right\|^2 \geq k\sigma^2 + \frac{n}{5} \|\Gamma_k^{1/2} \theta\|^2 - 2\sigma^2 \sqrt{k \log(2/\beta)} - 10\sigma^2 \log(2/\beta) ,$$

with probability larger than $1 - \beta/2 - C/\log^7(n) - n \exp[-C'n]$. □

Proof of Lemma 9.8. The second lower bound has already been studied in the proof of Theorem 4.2. For any $x > 0$, we have by Markov's Inequality

$$\mathbb{P} \left(\sqrt{n} \|(\widehat{A}_k - A_k) \Delta_{n,1}\| \geq x \right) \leq \frac{1}{x^2} \mathbb{E} \left[\|(\widehat{A}_k - A_k) \Delta_{n,1}\|^2 \mathbf{1}_{\overline{\mathcal{A}}_n} \right] + \mathbb{P}[\mathcal{A}_n] .$$

We have shown in the proof of Lemma 9.3 that

$$\mathbb{E} \left[\|(\widehat{A}_k - A_k) \Delta_{n,1}\|^2 \mathbf{1}_{\overline{\mathcal{A}}_n} \right] \leq C(\gamma) \sigma^2 \left[\frac{k^3 \log^2(k \vee e)}{n} \vee \frac{k}{\sqrt{n}} \vee \frac{\bar{k}_n^{5/2} \log(\bar{k}_n \vee e)}{n} \right] .$$

Let us turn to the other term $\mathbb{P}[\|(\widehat{A}_k - A_k) \widehat{\Gamma}_n \theta\| \geq x]$. First, we control $\sum_{i=1}^n \langle X_i, \theta \rangle^2 / n$ with larger probability. Note that $\mathbb{E}[\sum_{i=1}^n \langle X_i, \theta \rangle^2 / n] = \|\Gamma^{1/2} \theta\|^2$. Let us define the event $\mathcal{U}_n := \{\frac{1}{n} \sum_{i=1}^n \langle X_i, \theta \rangle^2 > \log(n) \|\Gamma^{1/2} \theta\|^2\}$. Applying Markov's inequality, we get

$$\mathbb{P}[\mathcal{U}_n] \leq \frac{1}{\log(n)} . \quad (42)$$

We bound $\mathbb{P}[\|(\widehat{A}_k - A_k)\widehat{\Gamma}_n\theta\| \geq x]$ as follows

$$\begin{aligned}
\mathbb{P}\left[\|(\widehat{A}_k - A_k)\widehat{\Gamma}_n\theta\| \geq x\right] &\leq \mathbb{P}\left[\left\{\|(\widehat{A}_k - A_k)\widehat{\Gamma}_n\theta\| \geq x\right\} \cap \overline{\mathcal{U}_n} \cap \overline{\mathcal{A}_n}\right] + \mathbb{P}[\mathcal{U}_n \cup \mathcal{A}_n] \\
&\leq \frac{1}{x^2} \mathbb{E}\left[\|(\widehat{A}_k - A_k)\widehat{\Gamma}_n\theta\|^2 \mathbf{1}_{\overline{\mathcal{U}_n} \cap \overline{\mathcal{A}_n}}\right] + \mathbb{P}[\mathcal{U}_n \cup \mathcal{A}_n] \\
&\leq \frac{1}{x^2} \mathbb{E}\left[\|(\widehat{A}_k - A_k)\widehat{\Gamma}_n^{1/2}\|_{HS}^2 \|\widehat{\Gamma}_n^{1/2}\theta\|^2 \mathbf{1}_{\overline{\mathcal{U}_n} \cap \overline{\mathcal{A}_n}}\right] + \mathbb{P}[\mathcal{U}_n \cup \mathcal{A}_n] \\
&\leq \frac{\log(n)}{x^2} \|\Gamma^{1/2}\theta\|^2 \mathbb{E}\left[\|(\widehat{A}_k - A_k)\widehat{\Gamma}_n^{1/2}\|_{HS}^2 \mathbf{1}_{\overline{\mathcal{A}_n}}\right] + \mathbb{P}[\mathcal{U}_n \cup \mathcal{A}_n],
\end{aligned} \tag{43}$$

As a consequence, we have to investigate

$$\begin{aligned}
\|(\widehat{A}_k - A_k)\widehat{\Gamma}_n^{1/2}\|_{HS}^2 &= \text{tr}\left((\widehat{A}_k - A_k)\widehat{\Gamma}_n(\widehat{A}_k - A_k)\right) \\
&= \text{tr}\widehat{A}_k\widehat{\Gamma}_n\widehat{A}_k - \text{tr}\widehat{A}_k\widehat{\Gamma}_n A_k - \text{tr}A_k\widehat{\Gamma}_n\widehat{A}_k + \text{tr}A_k\widehat{\Gamma}_n A_k.
\end{aligned}$$

Arguing as in the proof of Lemma 9.3, we take the expectation

$$\begin{aligned}
&\mathbb{E}\left[\|(\widehat{A}_k - A_k)\widehat{\Gamma}_n^{1/2}\|_{HS}^2 \mathbf{1}_{\overline{\mathcal{A}_n}}\right] \\
&= \mathbb{E}\left[\text{tr}\widehat{\Pi}_k \mathbf{1}_{\overline{\mathcal{A}_n}}\right] + \mathbb{E}\left[\text{tr}\Pi_k \mathbf{1}_{\overline{\mathcal{A}_n}}\right] + \mathbb{E}\left[\text{tr}\left(A_k(\widehat{\Gamma}_n - \Gamma)A_k\right) \mathbf{1}_{\overline{\mathcal{A}_n}}\right] - 2\mathbb{E}\left[\text{tr}\widehat{\Gamma}_{n,k}^{1/2}\Gamma_k^{-1/2} \mathbf{1}_{\overline{\mathcal{A}_n}}\right] \\
&\leq 2\mathbb{E}\left[\left\{k - \text{tr}\widehat{\Gamma}_{n,k}^{1/2}\Gamma_k^{-1/2}\right\} \mathbf{1}_{\overline{\mathcal{A}_n}}\right] + \sqrt{\mathbb{E}\left[\text{tr}^2\left(A_k(\widehat{\Gamma}_n - \Gamma)A_k\right)\right]} \sqrt{\mathbb{P}[\overline{\mathcal{A}_n}]} \\
&\leq 2\mathbb{E}\left[\text{tr}\left\{\Gamma_k^{-1/2}\left(\Gamma_k^{1/2} - \widehat{\Gamma}_{n,k}^{1/2}\right)\right\} \mathbf{1}_{\overline{\mathcal{A}_n}}\right] + \sqrt{\mathbb{E}\left[\text{tr}^2\left(A_k(\widehat{\Gamma}_n - \Gamma)A_k\right)\right]} \sqrt{\mathbb{P}[\overline{\mathcal{A}_n}]}.
\end{aligned}$$

These expectations have already been upper bounded in the proof of Lemma 9.6. By (36) and (37), we know that

$$\mathbb{E}\left[\|(\widehat{A}_k - A_k)\widehat{\Gamma}_n^{1/2}\|_{HS}^2 \mathbf{1}_{\overline{\mathcal{A}_n}}\right] \leq C(\gamma) \left[\frac{k^3 \log^2(k \vee e)}{n} \vee \frac{k}{\sqrt{n}} \vee \frac{\bar{k}_n^{5/2} \log(\bar{k}_n \vee e)}{n} \right].$$

Gathering this last bound with (42) and (43), we conclude that

$$\begin{aligned}
\mathbb{P}\left(\sqrt{n} \|(\widehat{A}_k - A_k)\widehat{\Gamma}_n\theta\| \geq x\right) &\leq \frac{1}{\log(n)} + \mathbb{P}[\mathcal{A}_n] \\
&\quad + C(\gamma) \frac{\log(n)}{x^2} \|\Gamma^{1/2}\theta\|^2 \left[\frac{k^3 \log^2(k \vee e)}{n} \vee \frac{k}{\sqrt{n}} \vee \frac{\bar{k}_n^{5/2} \log(\bar{k}_n \vee e)}{n} \right].
\end{aligned}$$

□

Proof of Lemma 9.9. Observe that $\|\mathbf{Y} - \widehat{\Pi}_k \mathbf{Y}\|_n^2 \leq \|\mathbf{Y}\|_n^2$. The random variable Y is centered and with variance $\sigma^2 + \|\Gamma^{1/2}\theta\|^2$. By Assumption B.1 and Tchebychev's inequality we have

$$\mathbb{P}\left[\frac{\|\mathbf{Y}\|_n^2}{n} \geq \left(1 + C\sqrt{\frac{\log n}{n}}\right) (\sigma^2 + \|\Gamma^{1/2}\theta\|^2)\right] \leq \frac{1}{\log(n)}.$$

Hence, we conclude that

$$\|\mathbf{Y} - \widehat{\Pi}_k \mathbf{Y}\|_n^2 / (n - k) \leq (\sigma^2 + \|\Gamma^{1/2}\theta\|^2) \left[1 + C\left(\frac{k}{n} + \frac{\sqrt{\log(n)}}{\sqrt{n}}\right)\right],$$

with probability larger than $1 - 1/\log(n)$.

□

9.5. Proofs of the minimax bounds

Here again we privilege a grouping of the proofs of Propositions 6.3 and 6.6 for the sake of clarity since they are connected.

Proof of Proposition 6.3. For any dimension $k \geq 1$, we define $r_k^2 = C(\alpha, \beta) \frac{\sqrt{k}}{n} \wedge \lambda_k a_k^2 R^2$, where the constant $C(\alpha, \beta)$ will be fixed later. For any $\theta \in \text{Vect}(V_1, \dots, V_k)$ such that $\|\Gamma^{1/2}\theta\|^2/\sigma^2 \leq r_k^2$, we have

$$\sum_{j=1}^k \frac{\langle \theta, V_j \rangle^2}{a_j^2} \leq \frac{1}{\lambda_k a_k^2} \sum_{j=1}^k \langle \theta, V_j \rangle^2 \lambda_j \leq \frac{r_k^2 \sigma^2}{a_k^2 \lambda_k} \leq R^2 \sigma^2$$

since $r_k^2 \leq \lambda_k a_k^2 R^2$ and since the $\lambda_j a_j^2$'s are non increasing. As a consequence,

$$\left\{ \theta \in \text{Vect}(V_1, \dots, V_k), \|\Gamma^{1/2}\theta\|^2/\sigma^2 = r_k^2 \right\} \subset \left\{ \theta \in \mathcal{E}_a(R), \|\Gamma^{1/2}\theta\|^2/\sigma^2 \geq r_k^2 \right\} .$$

Since X is a centered Gaussian process, $(\langle X, V_1 \rangle, \dots, \langle X, V_k \rangle)$ is a centered Gaussian vector. Assuming that θ belongs to $\text{Vect}(V_1, \dots, V_k)$ and that (V_1, \dots, V_k) is known, the functional linear model translates as a linear Gaussian model with Gaussian design as studied in [37]:

$$Y = \sum_{j=1}^k \langle X, V_j \rangle \langle \theta, V_j \rangle + \epsilon .$$

By Proposition 4.2 in [37], there exists a constant $C(\alpha, \beta)$, such that for any test T of level α , we have

$$\beta \left[T; \left\{ \theta \in \text{Vect}(V_1, \dots, V_k), \sigma > 0, \|\Gamma^{1/2}\theta\|^2 \geq C(\alpha, \beta) \frac{\sqrt{k}}{n} \sigma^2 \right\} \right] \geq \beta .$$

Gathering this last bound for all $k \geq 1$ allows us to conclude. \square

Proof of Proposition 6.6. As in the last proof, we shall adapt results for the Gaussian linear regression model with Gaussian design. Let $k_n^*(R) \in \mathbb{N}^*$ be an integer that achieves the supremum of $\tilde{r}_k^2 = C(\alpha, \beta) \sqrt{k} \log \log(k \vee 3)/n \wedge R^2 a_k^2 \lambda_k$. We note as in the last proof that for any $R > 0$ and $k_n^*(R)$ in \mathbb{N}^* ,

$$\left\{ \theta \in \text{Vect}(V_1, \dots, V_{k_n^*(R)}), \frac{\|\Gamma^{1/2}\theta\|^2}{\sigma^2} = \tilde{r}_{k_n^*(R)}^2 \right\} \subset \left\{ \theta \in \mathcal{E}_a(R), \frac{\|\Gamma^{1/2}\theta\|^2}{\sigma^2} \geq \tilde{r}_{k_n^*(R)}^2 \right\} .$$

Thus, we obtain

$$\begin{aligned} & \bigcup_{k \geq 1} \left\{ \theta \in \text{Vect}(V_1, \dots, V_k), \frac{\|\theta\|^2}{\text{Var}(Y) - \|\theta\|^2} = C(\alpha, \beta) \sqrt{k \log \log(k \vee 3)/n} \right\} \\ & \subset \bigcup_{R > 0} \left\{ \theta \in \text{Vect}(V_1, \dots, V_{k_n^*(R)}), \frac{\|\Gamma^{1/2}\theta\|^2}{\sigma^2} = r_{k_n^*(R)}^2 \right\} \\ & \subset \bigcup_{R > 0} \left\{ \theta \in \mathcal{E}_a(R), \frac{\|\Gamma^{1/2}\theta\|^2}{\sigma^2} \geq r_{D^*(R)}^2 \right\} . \end{aligned}$$

Hence, we only have to provide a minimax lower bound for simultaneously testing over a family of nesting linear spaces. Letting p go to infinity in Proposition 5.5 in [37], we

obtain that

$$\beta \left[\bigcup_{k \geq 1} \left\{ \theta \in \text{Vect}(V_1, \dots, V_k), \frac{\|\Gamma^{1/2}\theta\|^2}{\text{Var}(Y) - \|\Gamma^{1/2}\theta\|^2} = C(\alpha, \beta) \sqrt{k \log \log(k \vee 3)/n} \right\} \right] \geq \beta,$$

which allows us to conclude. \square

Proof of Proposition 6.4. This proposition is proved following the same arguments as Theorems 4.2 and 5.2. The proof only differs by the fact that we only consider here one parametric test, while we consider a multiple testing procedure based on $\log(n)$ parametric tests in the Theorems 4.2 and 5.2. This explains why there is no $\log \log n$ term in Proposition 6.4. \square

Proof of Corollary 6.5. This is direct consequence of Theorem 5.2. \square

10. Proofs based on perturbation theory

10.1. Preliminary facts about perturbation theory

This section is devoted to the arguments relying on perturbation theory. It may be useful to have basic notions about spectral representation of bounded operators and perturbation theory. We refer to Dunford and Schwartz [18, Chapter VII.3] or to Gohberg et al. [19, 20] for an introduction to functional calculus for operators related with Riesz integrals. Roughly speaking, several results mentioned below are based on an extension of the classical residue formula on the complex plane (see Rudin [34]) to analytic functions still defined on the complex plane but with values in the space of operators. The introduction of Gohberg et al. [19, pp. 4-16] illustrates well this extension. Let us denote \mathcal{B}_j the oriented circle of the complex plane with center λ_j and radius $\delta_j/2$ where δ_j is defined by

$$\delta_j = \min \{ \lambda_j - \lambda_{j+1}, \lambda_{j-1} - \lambda_j \}. \quad (44)$$

The open domain whose boundary is $\mathcal{C}_k := \bigcup_{j=1}^k \mathcal{B}_j$ is not connected but we can apply the functional calculus for bounded operators (see Dunford and Schwartz [18, Section VII.3, Definitions 8 and 9]). Using this formalism it is easy to prove the following formulas :

$$\Pi_k = \frac{1}{2\pi i} \int_{\mathcal{C}_k} (zI - \Gamma)^{-1} dz \quad \text{and} \quad \Gamma_k^{1/2} = \frac{1}{2\pi i} \int_{\mathcal{C}_k} z^{1/2} (zI - \Gamma)^{-1} dz.$$

The same is true with the random operator $\widehat{\Gamma}_n$, but the contour \mathcal{C}_k must be replaced by its random counterpart $\widehat{\mathcal{C}}_k = \bigcup_{j=1}^{k \wedge \text{Rank}(\widehat{\Gamma}_n)} \widehat{\mathcal{B}}_j$ where each $\widehat{\mathcal{B}}_j$ is a random ball of the complex plane with center $\widehat{\lambda}_j$ and a radius $\widehat{\delta}_j/2 = \min\{\widehat{\lambda}_j - \widehat{\lambda}_{j+1}, \widehat{\lambda}_{j-1} - \widehat{\lambda}_j\}$. We start with some lemmas.

Lemma 10.1. *Assume that for some $\gamma > 0$, the sequence $(j\lambda_j \log^{1+\gamma}(j \vee 2))_{j \in \mathbb{N}^*}$ decreases. Then, we have*

$$\sum_{j \geq 1, j \neq k} \frac{\lambda_j}{|\lambda_k - \lambda_j|} \leq C(\gamma)k [\log k \vee 1].$$

The proof is postponed to Appendix A For any positive integer j , let us define the event

$$\mathcal{E}_{j,n} := \left\{ \sup_{z \in \widehat{\mathcal{B}}_j} \left\| (zI - \Gamma)^{-1/2} \left(\widehat{\Gamma}_n - \Gamma \right) (zI - \Gamma)^{-1/2} \right\|_{\infty} \geq 1/2 \right\}.$$

Lemma 10.2. *Suppose that Assumption B.2 holds. For any $j \geq 1$, We have the two following bounds*

$$\begin{aligned} \mathbb{E} \sup_{z \in \mathcal{B}_j} \left\| (zI - \Gamma)^{-1/2} (\widehat{\Gamma}_n - \Gamma) (zI - \Gamma)^{-1/2} \right\|_{HS}^2 &\leq \frac{C(\gamma)}{n} [j(\log j \vee 1)]^2, \\ \mathbb{P}(\mathcal{E}_{j,n}) &\leq \frac{C(\gamma)}{n} [j(\log j \vee 1)]^2. \end{aligned}$$

Proof of Lemma 10.2. The second bound straightforwardly follows from the first bound by Markov inequality. Fix $z \in \mathcal{B}_j$. We have

$$\begin{aligned} &\left\| (zI - \Gamma)^{-1/2} (\widehat{\Gamma}_n - \Gamma) (zI - \Gamma)^{-1/2} \right\|_{HS}^2 \\ &= \sum_{l=1}^{+\infty} \sum_{k=1}^{+\infty} \left\langle (zI - \Gamma)^{-1/2} (\widehat{\Gamma}_n - \Gamma) (zI - \Gamma)^{-1/2} V_l, V_k \right\rangle^2 = \sum_{l,k=1}^{+\infty} \frac{\left\langle (\widehat{\Gamma}_n - \Gamma) V_l, V_k \right\rangle^2}{|z - \lambda_l| |z - \lambda_k|}. \end{aligned}$$

Since for $z = \lambda_j + \frac{\delta_j}{2} e^{i\theta} \in \mathcal{B}_j$ and $i \neq j$

$$|z - \lambda_i| = \left| \lambda_j - \lambda_i + \frac{\delta_j}{2} e^{i\theta} \right| \geq |\lambda_j - \lambda_i| - \frac{\delta_j}{2} \geq |\lambda_j - \lambda_i| / 2,$$

we have

$$\begin{aligned} \sum_{l,k=1}^{+\infty} \frac{\left\langle (\widehat{\Gamma}_n - \Gamma) V_l, V_k \right\rangle^2}{|z - \lambda_l| |z - \lambda_k|} &\leq 4 \sum_{\substack{l,k=1, \\ l,k \neq j}}^{+\infty} \frac{\left\langle (\widehat{\Gamma}_n - \Gamma) V_l, V_k \right\rangle^2}{|\lambda_j - \lambda_l| |\lambda_j - \lambda_k|} + 2 \sum_{\substack{k=1, \\ k \neq j}}^{+\infty} \frac{\left\langle (\widehat{\Gamma}_n - \Gamma) V_j, V_k \right\rangle^2}{\delta_j |\lambda_j - \lambda_k|} \\ &\quad + \frac{\left\langle (\widehat{\Gamma}_n - \Gamma) V_j, V_j \right\rangle^2}{\delta_j^2}. \end{aligned}$$

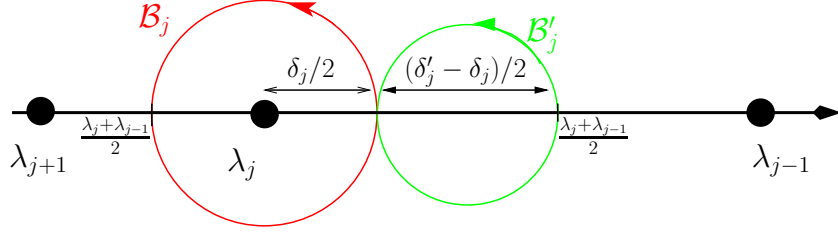
$$\begin{aligned} \mathbb{E} \left[\sum_{l,k=1}^{+\infty} \frac{\left\langle (\widehat{\Gamma}_n - \Gamma) V_l, V_k \right\rangle^2}{|z - \lambda_l| |z - \lambda_k|} \right] &\leq \frac{C}{n} \left[\sum_{\substack{l,k=1, \\ l,k \neq j}}^{+\infty} \frac{\lambda_k \lambda_l}{|\lambda_j - \lambda_k| (|\lambda_j - \lambda_l|)} + \sum_{\substack{k=1, \\ k \neq j}}^{+\infty} \frac{\lambda_k \lambda_j}{\delta_j |\lambda_j - \lambda_k|} + \frac{\lambda_j^2}{\delta_j^2} \right] \\ &\leq \frac{C'}{n} \left[\left(\sum_{k \geq 1, k \neq j}^{\infty} \frac{\lambda_k}{|\lambda_k - \lambda_j|} \right)^2 + \frac{\lambda_j^2}{|\lambda_j - \lambda_{j+1}|^2} + \frac{\lambda_j^2}{|\lambda_{j-1} - \lambda_j|^2} \right]. \end{aligned}$$

Applying Lemma 10.1 and Assumption B.2 allows us to conclude. \square

10.2. Proof of Lemma 9.1

For any $2 \leq j \leq \bar{k}_n$, we define $\delta'_j := \max(\lambda_j - \lambda_{j+1}, \lambda_{j-1} - \lambda_j)$. Then, we build an oriented circle \mathcal{B}'_j on the complex plane of radius $(\delta'_j - \delta_j)/4$ in such a way that any real number between $(\lambda_j + \lambda_{j+1})/2$ and $(\lambda_j + \lambda_{j-1})/2$ is either inside \mathcal{B}_j or \mathcal{B}'_j . See Figure 2 for an example of \mathcal{B}_j and \mathcal{B}'_j .

Lemma 9.1 is a straightforward consequence of the two following lemmas. Let us define $T_n(z) = (zI - \Gamma)^{-1/2} (\widehat{\Gamma}_n - \Gamma) (zI - \Gamma)^{-1/2}$ and $S_n(z) = (zI - \Gamma)^{1/2} (zI - \widehat{\Gamma}_n)^{-1} (zI - \Gamma)^{1/2}$.

FIGURE 2. Contours \mathcal{B}_j and \mathcal{B}'_j

Lemma 10.3. We have $\mathcal{A}_n \subset \mathcal{E}_n \cup \mathcal{E}'_n \cup \left\{ \widehat{\lambda}_1 \geq \frac{3\lambda_1 - \lambda_2}{2} \right\}$, where

$$\mathcal{E}_n := \left\{ \sup_{1 \leq j \leq \bar{k}_n} \sup_{z \in \mathcal{B}_j} \|T_n(z)\|_\infty \geq 0.5 \right\}, \quad \mathcal{E}'_n := \left\{ \sup_{2 \leq j \leq \bar{k}_n} \sup_{z \in \mathcal{B}'_j} \|T_n(z)\|_\infty \geq 0.5 \right\}.$$

Lemma 10.4. Under Assumptions **B.1** and **B.2**, we have

$$\mathbb{P}(\mathcal{E}_n) \leq C_1(\gamma) \frac{\bar{k}_n^3 \log^2(\bar{k}_n)}{n}, \quad \mathbb{P}(\mathcal{E}'_n) \leq C_2(\gamma) \frac{\bar{k}_n^3 \log^2(\bar{k}_n)}{n}, \quad \mathbb{P}\left[\widehat{\lambda}_1 \geq \frac{3\lambda_1 - \lambda_2}{2}\right] \leq \frac{C_3(\gamma)}{n}.$$

Proof of Lemma 10.3. Suppose that four following events hold: 1) $\widehat{\Gamma}_n$ has no eigenvalue on all the contours \mathcal{B}_j and \mathcal{B}'_j . 2) For each $1 \leq j \leq \bar{k}_n$, $\widehat{\Gamma}_n$ has exactly one eigenvalue inside the circle \mathcal{B}_j . 3) For each $2 \leq j \leq \bar{k}_n$, $\widehat{\Gamma}_n$ has no eigenvalue inside the circle \mathcal{B}'_j . 4) $\widehat{\lambda}_1 < (3\lambda_1 - \lambda_2)/2$. In such a case, the event $\overline{\mathcal{A}}_n$ is true. As a consequence, \mathcal{A}_n is included in the union of the four following events denoted \mathcal{D}_1 , \mathcal{D}_2 , \mathcal{D}_3 and \mathcal{D}_4 .

- For some $1 \leq j \leq \bar{k}_n$, $\widehat{\Gamma}_n$ has an eigenvalue that lies on the contours \mathcal{B}_j and \mathcal{B}'_j .
- For some $1 \leq j \leq \bar{k}_n$, $\widehat{\Gamma}_n$ has either 0 or more than 2 eigenvalues inside the circle \mathcal{B}_j .
- For some $2 \leq j \leq \bar{k}_n$, $\widehat{\Gamma}_n$ has at least 1 eigenvalue inside the circle \mathcal{B}'_j .
- $\widehat{\lambda}_1 \geq (3\lambda_1 - \lambda_2)/2$.

We shall prove that $\mathcal{D}_1 \subset \mathcal{E}_n \cup \mathcal{E}'_n$, that $\mathcal{D}_2 \setminus \mathcal{D}_1 \subset \mathcal{E}_n$ and that $\mathcal{D}_3 \setminus \mathcal{D}_1 \subset \mathcal{E}'_n$.

Event \mathcal{D}_1 . Assume that an eigenvalue of $\widehat{\Gamma}_n$ lies exactly on some contour $\mathcal{B}_j \cup \mathcal{B}'_j$. Let us call $\widehat{\lambda}$ such an eigenvalue and \widehat{V} a corresponding eigenvector. We have

$$\begin{aligned} T_n(\widehat{\lambda})(\widehat{\lambda}I - \Gamma)^{1/2}\widehat{V} &= (\widehat{\lambda}I - \Gamma)^{-1/2}(\widehat{\Gamma}_n - \Gamma)\widehat{V} \\ &= (\widehat{\lambda}I - \Gamma)^{-1/2}(\widehat{\lambda}I - \Gamma)\widehat{V} = (\widehat{\lambda}I - \Gamma)^{1/2}\widehat{V}. \end{aligned}$$

Since $\widehat{\lambda}$ is not an eigenvalue of Γ , we have $(\widehat{\lambda}I - \Gamma)^{1/2}\widehat{V} \neq 0$ so that $\sup_{z \in \mathcal{B}_j \cup \mathcal{B}'_j} \|T_n(z)\|_\infty \geq 1$. Hence, $\mathcal{D}_1 \subset \mathcal{E}_n \cup \mathcal{E}'_n$.

Event $\mathcal{D}_2 \setminus \mathcal{D}_1$. Assume that $\mathcal{D}_2 \setminus \mathcal{D}_1$ is true. It follows that for some $1 \leq j^* \leq \bar{k}_n$ the operator $(2\pi\iota)^{-1} \int_{\mathcal{B}_{j^*}} (zI - \widehat{\Gamma}_n)^{-1} dz$ is an orthogonal projector $\pi_{\widehat{W}_{j^*}}$ on a space \widehat{W}_{j^*} of dimension different from one. In contrast, $(2\pi\iota)^{-1} \int_{\mathcal{B}_{j^*}} (zI - \Gamma)^{-1} dz$ is the orthogonal projector $\pi_{V_{j^*}}$ on V_{j^*} . Consider

$$\frac{1}{2\pi\iota} \int_{\mathcal{B}_{j^*}} \left[(zI - \widehat{\Gamma}_n)^{-1} - (zI - \Gamma)^{-1} \right] dz = \pi_{\widehat{W}_{j^*}} - \pi_{V_{j^*}}.$$

If $\dim(\widehat{W}_{j^*}) = 0$, then $\|\pi_{\widehat{W}_{j^*}} - \pi_{j^*}\|_\infty = 1$. If $\dim(\widehat{W}_{j^*}) \geq 2$, then there exists a vector \widehat{V} in \widehat{W}_{j^*} such that $\pi_{j^*}\widehat{V} = 0$. As a consequence, we have $\|\pi_{\widehat{W}_{j^*}} - \pi_{j^*}\|_\infty \geq 1$. For any $z \in \mathcal{B}_{j^*}$, $S_n(z)$ is well defined since no eigenvalue of $\widehat{\Gamma}_n$ lies on \mathcal{B}_{j^*} .

It follows that

$$\begin{aligned}
1 &\leq \frac{1}{2\pi} \int_{\mathcal{B}_{j^*}} \left\| (zI - \widehat{\Gamma}_n)^{-1} (\widehat{\Gamma}_n - \Gamma) (zI - \Gamma)^{-1} \right\|_\infty dz \\
&\leq \frac{1}{2\pi} \int_{\mathcal{B}_{j^*}} \left\| (zI - \widehat{\Gamma}_n)^{-1} (zI - \Gamma)^{1/2} T_n(z) (zI - \Gamma)^{-1/2} \right\|_\infty dz \\
&\leq \frac{1}{2\pi} \int_{\mathcal{B}_{j^*}} \left\| (zI - \Gamma)^{1/2} (zI - \widehat{\Gamma}_n)^{-1} (zI - \Gamma)^{1/2} \right\|_\infty \|T_n(z)\|_\infty \left\| (zI - \Gamma)^{-1/2} \right\|_\infty^2 dz \\
&\leq \sup_{z \in \mathcal{B}_{j^*}} \|S_n(z)\|_\infty \|T_n(z)\|_\infty, \tag{45}
\end{aligned}$$

since $\|(zI - \Gamma)^{-1}\|_\infty \geq 2/\delta_j$. Moreover, we have $S_n(z)(I - T_n(z)) = I$. We can assume that $\sup_{z \in \mathcal{B}_{j^*}} \|T_n(z)\|_\infty < 0.9$, otherwise \mathcal{E}_n is true. Then, we have $\|S_n(z)\|_\infty \leq (1 - \|T_n(z)\|_\infty)^{-1}$. Gathering this bound with (45) leads to $\sup_{z \in \mathcal{B}_{j^*}} \|T_n(z)\|_\infty \geq 0.5$, which allows us to conclude that $\mathcal{D}_2 \setminus \mathcal{D}_1 \subset \mathcal{E}_n$.

Event $\mathcal{D}_3 \setminus \mathcal{D}_1$. Assume that $\mathcal{D}_3 \setminus \mathcal{D}_1$ is true. Arguing as for \mathcal{D}_2 , we derive that for some $2 \leq j^* \leq \bar{k}_n$, we have:

$$\frac{1}{2\pi} \left\| \int_{\mathcal{B}'_{j^*}} (zI - \widehat{\Gamma}_n)^{-1} (\widehat{\Gamma}_n - \Gamma) (zI - \Gamma)^{-1} dz \right\|_\infty \geq 1. \tag{46}$$

We have proved above that

$$(zI - \widehat{\Gamma}_n)^{-1} (\widehat{\Gamma}_n - \Gamma) (zI - \Gamma)^{-1} = (zI - \Gamma)^{-1/2} S_n(z) T_n(z) (zI - \Gamma)^{-1/2} dz,$$

where $S_n(z) = (I - T_n(z))^{-1}$ is well defined for any $z \in \mathcal{B}'_{j^*}$. By a straightforward induction, we get for any positive integer p

$$\begin{aligned}
\int_{\mathcal{B}'_{j^*}} (zI - \widehat{\Gamma}_n)^{-1} (\widehat{\Gamma}_n - \Gamma) (zI - \Gamma)^{-1} dz &= \sum_{k=1}^p \int_{\mathcal{B}'_{j^*}} (zI - \Gamma)^{-1/2} T_n^k(z) (zI - \Gamma)^{-1/2} dz \\
&\quad + \int_{\mathcal{B}'_{j^*}} (zI - \Gamma)^{-1/2} S_n(z) T_n^p(z) (zI - \Gamma)^{-1/2} dz.
\end{aligned}$$

Observe that each integral $\int_{\mathcal{B}'_{j^*}} (zI - \Gamma)^{-1/2} T_n^k(z) (zI - \Gamma)^{-1/2} dz$ is zero since the operator $(zI - \Gamma)^{-1/2}$ has no pole inside \mathcal{B}'_{j^*} . Assume that the event \mathcal{E}'_n does not hold. Then, we can bound $\|S_n(z)\|_\infty$ by $(1 - \|T_n(z)\|_\infty)^{-1}$ as above. As a consequence, we obtain that for any positive integer p ,

$$\begin{aligned}
&\frac{1}{2\pi} \left\| \int_{\mathcal{B}'_{j^*}} (zI - \widehat{\Gamma}_n)^{-1} (\widehat{\Gamma}_n - \Gamma) (zI - \Gamma)^{-1} dz \right\|_\infty \\
&\leq \frac{1}{2\pi} \int_{\mathcal{B}'_{j^*}} \left\| (zI - \Gamma)^{-1/2} \right\|_\infty^2 \frac{\|T_n(z)\|_\infty^p}{1 - \|T_n(z)\|_\infty} dz \leq \frac{\delta'_j - \delta_j}{2^p \delta_j}.
\end{aligned}$$

Taking p large enough in this last upper bound contradicts (46). Thus, $(\mathcal{D}_3 \setminus \mathcal{D}_1) \cap \overline{\mathcal{E}'_n} = \emptyset$, which allows us to conclude. \square

Proof of Lemma 10.4. The first bound is a straightforward consequence of Lemma 10.2 since $\mathcal{E}_n = \cup_{j \in \mathcal{K}_n} \mathcal{E}_{j,n}$. The second bound proceeds from the same approach as Lemma 10.2.

Let us turn to the third bound. By Weyl's theorem, (e.g. Theorem 4.3.1 in [26]), we have $|\widehat{\lambda}_1 - \lambda_1| \leq \|\widehat{\Gamma}_n - \Gamma\|_\infty$ so that

$$\begin{aligned} \mathbb{P} \left[\widehat{\lambda}_1 \geq \frac{3\lambda_1 - \lambda_2}{2} \right] &\leq \mathbb{P} \left[|\widehat{\lambda}_1 - \lambda_1| \geq \frac{\lambda_1 - \lambda_2}{2} \right] \leq \mathbb{P} \left[\|\widehat{\Gamma}_n - \Gamma\|_\infty \geq \frac{\lambda_1 - \lambda_2}{2} \right] \\ &\leq \mathbb{P} \left[\|\widehat{\Gamma}_n - \Gamma\|_{HS} \geq \frac{\lambda_1 - \lambda_2}{2} \right] \leq \frac{4}{(\lambda_1 - \lambda_2)^2} \mathbb{E} \left[\|\widehat{\Gamma}_n - \Gamma\|_{HS}^2 \right]. \end{aligned}$$

We have

$$\mathbb{E} \left[\|\widehat{\Gamma}_n - \Gamma\|_{HS}^2 \right] = \sum_{k,l=1}^{\infty} \mathbb{E} \left[\langle (\widehat{\Gamma}_n - \Gamma)V_k, V_l \rangle^2 \right] \leq \frac{C}{n} \left(\sum_{k=1}^{\infty} \lambda_k \right)^2,$$

by Assumption **B.1**. By Assumption **B.2**, $2\lambda_2 \leq \lambda_1$. Applying Lemma 10.1, we get

$$\mathbb{P} \left[\widehat{\lambda}_1 \geq \frac{3\lambda_1 - \lambda_2}{2} \right] \leq \frac{C}{n} \left(\frac{\sum_{k=1}^{\infty} \lambda_k}{\lambda_1} \right)^2 \leq \frac{C(\gamma)}{n}.$$

□

10.3. Proof of Lemma 9.6

In order to upper bound this expectation, we set $\widehat{\lambda}_j = 0$ for any $j > \text{Rank}(\widehat{\Gamma}_n)$. We have

$$\begin{aligned} \text{tr} \left[\Gamma_k^{-1/2} \left(\Gamma_k^{1/2} - \widehat{\Gamma}_{n,k}^{1/2} \right) 1_{\overline{\mathcal{A}}_n} \right] &= k 1_{\overline{\mathcal{A}}_n} - \sum_{j=1}^k \sum_{l=1}^k \sqrt{\frac{\widehat{\lambda}_j}{\lambda_l}} \langle V_l, \widehat{V}_j \rangle^2 1_{\overline{\mathcal{A}}_n} \\ &\leq k 1_{\overline{\mathcal{A}}_n} - \sum_{j=1}^k \sqrt{\frac{\widehat{\lambda}_j}{\lambda_j}} \langle V_j, \widehat{V}_j \rangle^2 1_{\overline{\mathcal{A}}_n} \\ &\leq \sum_{j=1}^k \left(1 - \langle V_j, \widehat{V}_j \rangle^2 \right) 1_{\overline{\mathcal{A}}_n} + \sum_{j=1}^k \left| \sqrt{\frac{\widehat{\lambda}_j}{\lambda_j}} - 1 \right| 1_{\overline{\mathcal{A}}_n} \\ &\leq \sum_{j=1}^k \left(1 - \langle V_j, \widehat{V}_j \rangle^2 \right) 1_{\overline{\mathcal{A}}_n} + \sum_{j=1}^k \frac{|\widehat{\lambda}_j - \lambda_j|}{\lambda_j} 1_{\overline{\mathcal{A}}_n}, \end{aligned}$$

where the last equation follows from the upper bound $|\sqrt{1+x} - 1| \leq |x|$ for any $x \geq -1$. Observe that under the event $\overline{\mathcal{A}}_n$, $(\widehat{\lambda}_j - \lambda_j)/\lambda_j \leq 1/2$. Applying Lemma 10.2, we obtain the following bound

$$\begin{aligned} &\mathbb{E} \left[\text{tr} \left[\Gamma_k^{-1/2} \left(\Gamma_k^{1/2} - \widehat{\Gamma}_{n,k}^{1/2} \right) 1_{\overline{\mathcal{A}}_n} \right] \right] \\ &\leq \sum_{j=1}^k \mathbb{E} \left[\left(1 - \langle V_j, \widehat{V}_j \rangle^2 \right) 1_{\overline{\mathcal{A}}_n \cap \overline{\mathcal{E}}_{j,n}} \right] + \sum_{j=1}^k \mathbb{E} \left[\frac{|\widehat{\lambda}_j - \lambda_j|}{\lambda_j} 1_{\overline{\mathcal{A}}_n \cap \overline{\mathcal{E}}_{j,n}} \right] + \sum_{j=1}^k \frac{3}{2} \mathbb{P}[\mathcal{E}_{j,n}] \\ &\leq \sum_{j=1}^k \mathbb{E} \left[\left(1 - \langle V_j, \widehat{V}_j \rangle^2 \right) 1_{\overline{\mathcal{A}}_n \cap \overline{\mathcal{E}}_{j,n}} \right] + \sum_{j=1}^k \mathbb{E} \left[\frac{|\widehat{\lambda}_j - \lambda_j|}{\lambda_j} 1_{\overline{\mathcal{A}}_n \cap \overline{\mathcal{E}}_{j,n}} \right] + C(\gamma) \frac{k^3 (\log^2(k) \vee 1)}{n}. \end{aligned} \tag{47}$$

In the sequel, π_j stands for the orthogonal projector associated to the single j -th eigenvector V_j while $\widehat{\pi}_j$ refers to its empirical counterpart. Applying functional calculus tools for linear operators, we get for any $1 \leq j \leq k$

$$\begin{aligned} 1 - \langle V_j, \widehat{V}_j \rangle^2 &= \langle V_j, V_j \rangle^2 - \langle V_j, \widehat{V}_j \rangle^2 = \langle (\pi_j - \widehat{\pi}_j) V_j, V_j \rangle \\ &= \frac{1}{2\pi i} \left[\int_{\mathcal{B}_j} \langle (zI - \Gamma)^{-1} V_j, V_j \rangle dz - \int_{\widehat{\mathcal{B}}_j} \langle (zI - \widehat{\Gamma}_n)^{-1} V_j, V_j \rangle dz \right], \end{aligned}$$

which looks like the definition of Π_k given in the first paragraph of Section 10.1 (note that only the contour changed). Under the event $\overline{\mathcal{A}}_n$, $\widehat{\lambda}_j$ lies inside the circle \mathcal{B}_j . In fact, $(zI - \widehat{\Gamma}_n)^{-1}$ has only one pole inside the circle \mathcal{B}_j at $z = \widehat{\lambda}_j$. As a consequence, we have almost surely

$$\int_{\widehat{\mathcal{B}}_j} \langle (zI - \widehat{\Gamma}_n)^{-1} V_j, V_j \rangle dz \mathbf{1}_{\overline{\mathcal{A}}_n} = \int_{\mathcal{B}_j} \langle (zI - \widehat{\Gamma}_n)^{-1} V_j, V_j \rangle dz \mathbf{1}_{\overline{\mathcal{A}}_n},$$

so that

$$\left(1 - \langle V_j, \widehat{V}_j \rangle^2 \right) \mathbf{1}_{\overline{\mathcal{E}}_{j,n} \cap \overline{\mathcal{A}}_n} = \frac{1}{2\pi i} \int_{\mathcal{B}_j} \left\langle \left\{ (zI - \Gamma)^{-1} - (zI - \widehat{\Gamma}_n)^{-1} \right\} V_j, V_j \right\rangle dz \mathbf{1}_{\overline{\mathcal{E}}_{j,n} \cap \overline{\mathcal{A}}_n}. \quad (48)$$

Working out this integral, we get

$$\begin{aligned} &\int_{\mathcal{B}_j} \left\langle \left\{ (zI - \Gamma)^{-1} - (zI - \widehat{\Gamma}_n)^{-1} \right\} V_j, V_j \right\rangle dz \\ &= - \int_{\mathcal{B}_j} \langle (zI - \Gamma)^{-1} (\widehat{\Gamma}_n - \Gamma) (zI - \Gamma)^{-1} V_j, V_j \rangle \\ &\quad - \int_{\mathcal{B}_j} \left\langle (zI - \widehat{\Gamma}_n)^{-1} (\widehat{\Gamma}_n - \Gamma) (zI - \Gamma)^{-1} (\widehat{\Gamma}_n - \Gamma) (zI - \Gamma)^{-1} V_j, V_j \right\rangle dz. \end{aligned}$$

The first term is $\int_{\mathcal{B}_j} (z - \lambda_j)^{-2} \langle (\widehat{\Gamma}_n - \Gamma) V_j, V_j \rangle dz$. Thus, it is null almost surely by the Cauchy integration theorem. Define $S_n(z) = (zI - \Gamma)^{1/2} (zI - \widehat{\Gamma}_n)^{-1} (zI - \Gamma)^{1/2}$ and $T_n(z) = (zI - \Gamma)^{-1/2} (\widehat{\Gamma}_n - \Gamma) (zI - \Gamma)^{-1/2}$. For any fixed z , we have $S_n(z) = [I - T_n(z)]^{-1}$. Thus, it comes from (48) that

$$\begin{aligned} &\mathbb{E} \left[\left(1 - \langle V_j, \widehat{V}_j \rangle^2 \right) \mathbf{1}_{\overline{\mathcal{E}}_{j,n} \cap \overline{\mathcal{A}}_n} \right] \\ &\leq \mathbb{E} \left[\left| \frac{1}{2\pi i} \int_{\mathcal{B}_j} \langle (zI - \Gamma)^{-1/2} S_n(z) T_n^2(z) (zI - \Gamma)^{-1/2} V_j, V_j \rangle \mathbf{1}_{\overline{\mathcal{E}}_{j,n} \cap \overline{\mathcal{A}}_n} dz \right| \right] \\ &\leq C \delta_j \mathbb{E} \left[\sup_{z \in \mathcal{B}_j} \left\{ \|S_n(z)\|_\infty \mathbf{1}_{\overline{\mathcal{E}}_{j,n}} \|T_n(z)\|_\infty^2 \left\| (zI - \Gamma)^{-1} \right\|_\infty \right\} \right] \\ &\leq C \delta_j \sup_{z \in \mathcal{B}_j} \left\| (zI - \Gamma)^{-1} \right\|_\infty \mathbb{E} \left[\sup_{z \in \mathcal{B}_j} \|T_n(z)\|_\infty^2 \right] \leq C(\gamma) \frac{j^2 (\log^2(j) \vee 1)}{n}, \quad (49) \end{aligned}$$

since $\sup_{z \in \mathcal{B}_j} \left\| (zI - \Gamma)^{-1} \right\|_\infty \leq 2\delta_i^{-1}$, $\sup_{z \in \mathcal{B}_j} \|S_n(z)\|_\infty \mathbf{1}_{\overline{\mathcal{E}}_{j,n}} \leq 2$ and $\mathbb{E}[\sup_{z \in \mathcal{B}_j} \|T_n(z)\|_\infty^2] \leq \frac{C(\gamma)}{n} j^2 (\log^2(j) \vee 1)$ by Lemma 10.2. Hence, we obtain an upper bound for the first term in (47)

$$\mathbb{E} \left[\left(k - \mathbb{E} \sum_{j=1}^k \langle V_j, \widehat{V}_j \rangle^2 \right) \mathbf{1}_{\overline{\mathcal{A}}_n} \right] \leq C(\gamma) \frac{k^3 (\log^2(k) \vee 1)}{n}. \quad (50)$$

We turn to the second term in (47). We only provide a sketch of the proof since the approach is the same as the first term in (47). We have

$$\widehat{\lambda}_j - \lambda_j = \text{tr} \left(\widehat{\Gamma}_n \widehat{\pi}_j - \Gamma \pi_j \right) = \text{tr} \left(\widehat{\Gamma}_n (\widehat{\pi}_j - \pi_j) \right) + \text{tr} \left((\widehat{\Gamma}_n - \Gamma) \pi_j \right),$$

so that

$$\frac{|\widehat{\lambda}_j - \lambda_j|}{\lambda_j} \mathbf{1}_{\overline{\mathcal{A}}_n \cap \overline{\mathcal{E}}_{j,n}} \leq \frac{|\text{tr} \left(\widehat{\Gamma}_n (\widehat{\pi}_j - \pi_j) \right)|}{\lambda_j} \mathbf{1}_{\overline{\mathcal{A}}_n \cap \overline{\mathcal{E}}_{j,n}} + \frac{|\text{tr} \left((\widehat{\Gamma}_n - \Gamma) \pi_j \right)|}{\lambda_j} \mathbf{1}_{\overline{\mathcal{A}}_n \cap \overline{\mathcal{E}}_{j,n}}. \quad (51)$$

The second term in this decomposition is bounded as follows

$$\mathbb{E} \left[\frac{|\text{tr} \left((\widehat{\Gamma}_n - \Gamma) \pi_j \right)|}{\lambda_j} \mathbf{1}_{\overline{\mathcal{A}}_n \cap \overline{\mathcal{E}}_{j,n}} \right] \leq \mathbb{E} \left[\frac{|\langle (\widehat{\Gamma}_n - \Gamma) V_j, V_j \rangle|}{\lambda_j} \right] \leq \frac{1}{\sqrt{n}}. \quad (52)$$

We turn to $\mathbb{E} \left[\left| \text{tr} \left(\widehat{\Gamma}_n (\widehat{\pi}_j - \pi_j) \right) \right| \mathbf{1}_{\overline{\mathcal{A}}_n \cap \overline{\mathcal{E}}_{j,n}} \right]$ and we use the same method as above for bounding $(1 - \langle V_j, \widehat{V}_j \rangle)^2$.

$$\begin{aligned} & \frac{1}{\lambda_j} \text{tr} \left[\widehat{\Gamma}_n (\widehat{\pi}_j - \pi_j) \right] \mathbf{1}_{\overline{\mathcal{A}}_n \cap \overline{\mathcal{E}}_{j,n}} \\ &= \frac{1}{\lambda_j 2\pi\iota} \text{tr} \left[\int_{\mathcal{B}_j} \widehat{\Gamma}_n (zI - \widehat{\Gamma}_n)^{-1} (\widehat{\Gamma}_n - \Gamma) (zI - \Gamma)^{-1} dz \right] \mathbf{1}_{\overline{\mathcal{A}}_n \cap \overline{\mathcal{E}}_{j,n}} \\ &= \frac{1}{\lambda_j 2\pi\iota} \text{tr} \left[\int_{\mathcal{B}_j} \left(\widehat{\Gamma}_n (zI - \widehat{\Gamma}_n)^{-1} - \Gamma (zI - \Gamma)^{-1} \right) (\widehat{\Gamma}_n - \Gamma) (zI - \Gamma)^{-1} dz \right] \mathbf{1}_{\overline{\mathcal{A}}_n \cap \overline{\mathcal{E}}_{j,n}} \\ &= \frac{1}{\lambda_j 2\pi\iota} \text{tr} \left[\int_{\mathcal{B}_j} z (zI - \widehat{\Gamma}_n)^{-1} (\widehat{\Gamma}_n - \Gamma) (zI - \Gamma)^{-1} (\widehat{\Gamma}_n - \Gamma) (zI - \Gamma)^{-1} dz \right] \mathbf{1}_{\overline{\mathcal{A}}_n \cap \overline{\mathcal{E}}_{j,n}} \\ &= \frac{1}{\lambda_j 2\pi\iota} \text{tr} \left[\int_{\mathcal{B}_j} z (zI - \Gamma)^{-1/2} S_n(z) T_n^2(z) (zI - \Gamma)^{-1/2} dz \right] \mathbf{1}_{\overline{\mathcal{A}}_n \cap \overline{\mathcal{E}}_{j,n}}. \end{aligned}$$

From the upper bound

$$\begin{aligned} & \left| \text{tr} \left[(zI - \Gamma)^{-1/2} S_n(z) T_n^2(z) (zI - \Gamma)^{-1/2} \right] \right| \\ & \leq \| (zI - \Gamma)^{-1/2} S_n T_n \|_{HS} \| T_n (zI - \Gamma)^{-1/2} \|_{HS} \leq \| (zI - \Gamma)^{-1} \|_{\infty} \| S_n \|_{\infty} \| T_n \|_{HS}^2, \end{aligned}$$

we derive as in the proof of (50)

$$\begin{aligned} & \frac{1}{\lambda_j} \mathbb{E} \left[\left| \text{tr} \left(\widehat{\Gamma}_n (\widehat{\pi}_j - \pi_j) \right) \right| \mathbf{1}_{\overline{\mathcal{A}}_n \cap \overline{\mathcal{E}}_{j,n}} \right] \\ & \leq \frac{C}{\lambda_j} \mathbb{E} \left[\int_{\mathcal{B}_j} |z| \| (zI - \Gamma)^{-1} \|_{\infty} \| S_n(z) \|_{\infty} \| T_n(z) \|_{HS}^2 dz \mathbf{1}_{\overline{\mathcal{E}}_{j,n}} \right] \\ & \leq C \mathbb{E} \left[\sup_{z \in \mathcal{B}_j} \| T_n(z) \|_{HS}^2 \mathbf{1}_{\overline{\mathcal{E}}_{j,n}} \right] \leq C(\gamma) \frac{j^2 (\log^2 j \vee 1)}{n}. \end{aligned}$$

Gathering (51) and (52) with this last bound, we get

$$\sum_{j=1}^k \mathbb{E} \left[\left| \frac{\widehat{\lambda}_j - \lambda_j}{\lambda_j} \right| \mathbf{1}_{\overline{\mathcal{A}}_n \cap \overline{\mathcal{E}}_{j,n}} \right] \leq C(\gamma) \frac{k^3 (\log^2(k) \vee 1)}{n} + C' \frac{k}{\sqrt{n}},$$

Combining this last bound with (47) and (50) allows us to conclude.

Appendix A: Technical details

Proof of Lemma 9.5. First, we use the following bound that will be proved at the end of the proof.

$$k\bar{\mathcal{F}}_{k,n-k}^{-1}(\alpha/|\mathcal{K}_n|) \geq \bar{\chi}_k^{-1}\left(\frac{\alpha}{|\mathcal{K}_n|} + \frac{1}{n}\right) / \left(1 + 4\sqrt{\frac{\log(n)}{n}}\right). \quad (53)$$

As a consequence, we have

$$\begin{aligned} & \bar{\chi}_k \left[k \left(1 - e^{-\sqrt{n}} - \frac{k \log^2(n)}{n} - \frac{1}{k \log^2(n)} \right) \bar{\mathcal{F}}_{k,n-k}^{-1} \left(\frac{\alpha}{|\mathcal{K}_n|} \right) \right] \\ & \leq \bar{\chi}_k \left[\frac{\left(1 - e^{-\sqrt{n}} - \frac{k \log^2(n)}{n} - \frac{1}{k \log^2(n)} \right) \bar{\chi}_k^{-1} \left(\frac{\alpha}{|\mathcal{K}_n|} + \frac{1}{n} \right)}{1 + 4\sqrt{\frac{\log(n)}{n}}} \right]. \end{aligned}$$

Since for any $0 < u < 1$ and any integer $k \geq 1$, we have $\bar{\chi}_k^{-1}(u) \leq k + 2\sqrt{\log(1/u)k} + 2\log(1/u)$ (e.g. Lemma 1 in [31]), it follows from Assumption **B.3** that

$$\begin{aligned} & \frac{\left(1 - e^{-\sqrt{n}} - \frac{k \log^2(n)}{n} - \frac{1}{k \log^2(n)} \right) \bar{\chi}_k^{-1} \left(\frac{\alpha}{|\mathcal{K}_n|} + \frac{1}{n} \right)}{\left(1 + 4\sqrt{\frac{\log(n)}{n}} \right)} \\ & \geq \bar{\chi}_k^{-1} \left(\frac{\alpha}{|\mathcal{K}_n|} + \frac{1}{n} \right) - C(\alpha) [k \vee \log(n)] \left[\sqrt{\frac{\log(n)}{n}} \vee \frac{1}{n^{3/4} \log^2(n)} \vee \frac{1}{k \log^2(n)} \right] \\ & \geq \bar{\chi}_k^{-1} \left(\frac{\alpha}{|\mathcal{K}_n|} + \frac{1}{n} \right) - \frac{C(\alpha)}{\log(n)}. \end{aligned}$$

Let us note $f_{\chi_k}(x)$ the density at x of a χ^2 random variable with k degrees of freedom. Consider some positive numbers x and u such that $x \geq u$.

$$\frac{\bar{\chi}_k(x-u)}{\bar{\chi}_k(x)} = 1 + \frac{\mathbb{P}(x-u \leq \chi^2(k) \leq x)}{\bar{\chi}_k(x)} \leq 1 + u \frac{f_{\chi_k}(x-u)}{\bar{\chi}_k(x)} \leq 1 + ue^{u/2} \frac{f_{\chi_k}(x)}{\bar{\chi}_k(x)},$$

since $f_{\chi_k}(x) = x^{k/2-1}e^{-x/2}/[2^{k/2}\Gamma(k/2)]$. By integration by part, one observes that $f_{\chi_k}(x)/\bar{\chi}_k(x) \leq 1/2$. As a consequence, we have $\bar{\chi}_k(x-u) \leq \bar{\chi}_k(x)[1 + u/2e^{u/2}]$ for any $u \leq x$. This upper bound also holds when $u > x$.

$$\begin{aligned} \bar{\chi}_k \left[\bar{\chi}_k^{-1} \left(\frac{\alpha}{|\mathcal{K}_n|} + \frac{1}{n} \right) - \frac{C(\alpha)}{\log(n)} \right] & \leq \bar{\chi}_k \left[\bar{\chi}_k^{-1} \left(\frac{\alpha}{|\mathcal{K}_n|} + \frac{1}{n} \right) \right] \left[1 + \frac{C_2(\alpha)}{\log(n)} \right] \\ & \leq \frac{\alpha}{|\mathcal{K}_n|} \left(1 + \frac{C_3(\alpha)}{\log(n)} \right), \end{aligned}$$

which allows us to derive the desired result.

To finish the proof, we need to prove (53). Let X_k and X_{n-k} respectively denote two independent random variables that follow a χ^2 distribution with k and $n-k$ degrees of freedom. Moreover we define $F_{k,n-k}$ as $X_k(n-k)/(X_{n-k}k)$. Since $\bar{\chi}_k^{-1}(u) \leq k +$

$2\sqrt{\log(1/u)k} + 2\log(1/u)$, we have

$$\begin{aligned} \frac{\alpha}{|\mathcal{K}_n|} &= \mathbb{P} \left[X_k \geq \bar{\chi}_k^{-1} \left(\frac{\alpha}{|\mathcal{K}_n|} + \frac{1}{n} \right) \right] - \frac{1}{n} \\ &\leq \mathbb{P} \left[kF_{k,n-k} \geq \frac{\bar{\chi}_k^{-1} \left(\frac{\alpha}{|\mathcal{K}_n|} \right)}{1 + 4\sqrt{\frac{\log(n)}{n}}} \right] + \mathbb{P} \left[\frac{X_{n-k}}{n-k} \geq 1 + 4\sqrt{\frac{\log(n)}{n}} \right] - \frac{1}{n} \\ &\leq \mathbb{P} \left[kF_{k,n-k} \geq \frac{\bar{\chi}_k^{-1} \left(\frac{\alpha}{|\mathcal{K}_n|} \right)}{1 + 4\sqrt{\frac{\log(n)}{n}}} \right], \end{aligned}$$

so that $k\bar{\mathcal{F}}_{k,n-k}^{-1}(\alpha/|\mathcal{K}_n|) \geq \bar{\chi}_k^{-1} \left(\frac{\alpha}{|\mathcal{K}_n|} + \frac{1}{n} \right) / (1 + 4\sqrt{\frac{\log(n)}{n}})$ for $k \leq n/2$ and n large enough. \square

Proof of Lemma 10.1. Since $(j\lambda_j)_{j \in \mathbb{N}}$ is a decreasing sequence $j\lambda_j \geq k\lambda_k$ for $k > j$. Hence, we get

$$\sum_{j=1}^{k-1} \frac{\lambda_j}{\lambda_j - \lambda_k} \leq k \sum_{j=1}^{k-1} (k-j)^{-1} = k \sum_{j=1}^{k-1} j^{-1}.$$

Similarly $\sum_{j=k+1}^{2k} \lambda_j / (\lambda_k - \lambda_j) \leq k \sum_{j=k+1}^{2k} (j-k)^{-1} = k \sum_{j=1}^k j^{-1}$. Now we focus on $\sum_{j \geq 2k+1} \lambda_j / (\lambda_k - \lambda_j)$. The assumption on the eigenvalues implies that for $j \geq k$,

$$k \log^{1+\gamma}(k \vee 2) (\lambda_k - \lambda_j) \geq (j \log^{1+\gamma} j - k \log^{1+\gamma}(k \vee 2)) \lambda_j.$$

Thus, we get

$$\lambda_j / (\lambda_k - \lambda_j) \leq [(j \log^{1+\gamma} j / k \log^{1+\gamma}(k \vee 2)) - 1]^{-1}$$

and

$$\sum_{j \geq 2k+1} \lambda_j / (\lambda_k - \lambda_j) \leq \int_{2k}^{+\infty} \left[\frac{x \log^{1+\gamma} x}{k \log^{1+\gamma}(k \vee 2)} - 1 \right]^{-1} dx.$$

Then for $x \geq 2k$, we have

$$\frac{x \log^{1+\gamma} x}{k \log^{1+\gamma} k} \geq \frac{2k \log^{1+\gamma} 2k}{k \log^{1+\gamma}(k \vee 2)} \geq 2,$$

so that

$$\frac{x \log^{1+\gamma} x}{k \log^{1+\gamma} k} - 1 \geq \frac{1}{2} \frac{x \log^{1+\gamma} x}{k \log^{1+\gamma} k}.$$

It follows that

$$\int_{2k}^{+\infty} \left[\frac{x \log^{1+\gamma} x}{k \log^{1+\gamma} k} - 1 \right]^{-1} dx \leq 2k \log^{1+\gamma} k \int_{2k}^{+\infty} \frac{dx}{x \log^{1+\gamma} x} = \frac{2k \log^{1+\gamma} k}{\gamma \log^\gamma 2k} \leq \frac{2k \log k}{\gamma}$$

\square

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