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Truncated Markov chains and V -geometric ergodicity via weak perturbation theory

Loïc HERVÉ and James LEDOUX *

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Abstract

Let P be a Markov kernel on a metric space \mathbb{X} and let $V : \mathbb{X} \rightarrow [1, +\infty)$. This paper provides explicit connections between the V -geometrical ergodicity of P and that of its truncated and augmented kernels P_k . A special attention is paid to obtain an efficient way to specify the convergence rate for P from that of P_k and conversely. Furthermore, explicit bounds for the total variation distance between the invariant probability measures of P and P_k are presented. The proofs are based on the Keller-Liverani perturbation theorem which requires an accurate control of the essential spectral radius of both linear operators P and P_k on usual weighted supremum spaces. To that effect, computable bounds for the essential spectral radius of a general Markov kernel on this space are derived in terms of standard drift conditions.

AMS subject classification : 60J10; 47B07

Keywords : rate of convergence, approximations of Markov kernels, essential spectral radius, drift condition.

1 Introduction

Throughout the paper P is a Markov kernel on $(\mathbb{X}, \mathcal{X})$, where (\mathbb{X}, d) is a metric space equipped with its Borel σ -algebra \mathcal{X} . For $A \in \mathcal{X}$ and $f : \mathbb{X} \rightarrow \mathbb{C}$, f_A denotes the restriction of f to A . We introduce the following truncation approximation of P .

Definition 1.1 *Let $x_0 \in \mathbb{X}$. For every $k \in \mathbb{N}^*$, we denote by B_k the closed ball centered at x_0 with radius k and we set $B_k^c := \mathbb{X} \setminus B_k$. The σ -algebra on B_k induced by \mathcal{X} is called \mathcal{X}_k . Given $x_k \in \mathbb{X}$ such that $d(x_0, x_k) = k$, the k -th truncated (and augmented) Markov kernel P_k is defined on (B_k, \mathcal{X}_k) by*

$$\forall x \in B_k, \forall F \in \mathcal{X}_k, \quad P_k(x, F) := P(x, F) + 1_F(x_k) P(x, B_k^c). \quad (1)$$

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The associated (extended) sub-Markov kernel \widehat{P}_k on $(\mathbb{X}, \mathcal{X})$ is defined by:

$$\forall x \in \mathbb{X}, \forall A \in \mathcal{X}, \quad \widehat{P}_k(x, A) := 1_{B_k}(x) P_k(x, A \cap B_k). \quad (2)$$

For instance, when $\mathbb{X} := \mathbb{N}$, the k -th truncated (and augmented) kernel P_k of $P := (P(i, j))_{(i, j) \in \mathbb{N}^2}$ associated with $B_k := \{0, \dots, k\}$ and $x_k := k$ is the following $(k+1) \times (k+1)$ matrix:

$$P_k(i, j) := \begin{cases} P(i, j) & \text{if } (i, j) \in \{0, \dots, k\} \times \{0, \dots, k-1\} \\ \sum_{\ell \geq k} P(i, \ell) & \text{if } (i, j) \in \{0, \dots, k\} \times \{k\}. \end{cases}$$

Such a matrix is generally called a linear augmentation (in the last column here) of the $(k+1) \times (k+1)$ northwest corner truncation of P . Other kinds of augmentation, as the censored Markov chain [ZL96], could be considered. Truncation approximation of an infinite stochastic matrix has a long story (e.g. see [Sen81, Twe98, Liu10] and the references therein).

In this work, the connection between the geometrical ergodicity of P and that of its truncated kernels P_k is investigated. For $V : \mathbb{X} \rightarrow [1, +\infty)$, these properties are defined as follows.

P is said to be V -geometrically ergodic if P has an invariant probability measure π on $(\mathbb{X}, \mathcal{X})$ such that $\pi(V) := \int_{\mathbb{X}} V(x) \pi(dx) < \infty$ and there exist some rate $\rho \in (0, 1)$ and constant $C > 0$ such that for every measurable function $f : \mathbb{X} \rightarrow \mathbb{C}$ satisfying $|f| \leq V$

$$\forall n \geq 1, \quad \sup_{x \in \mathbb{X}} \frac{|(P^n f)(x) - \pi(f)|}{V(x)} \leq C \rho^n. \quad (V)$$

P_k is said to be V_{B_k} -geometrically ergodic if P_k has an invariant probability measure π_k on (B_k, \mathcal{X}_k) such that $\pi_k(V_{B_k}) < \infty$, and there exist some rate $\rho_k \in (0, 1)$ and constant $C_k > 0$ such that for every measurable function $h : B_k \rightarrow \mathbb{C}$ satisfying $|h| \leq V_{B_k}$

$$\forall n \geq 1, \quad \sup_{x \in B_k} \frac{|(P_k^n h)(x) - \pi_k(h)|}{V(x)} \leq C_k \rho_k^n. \quad (V_k)$$

Specifically, the two following issues are studied.

(Q1) When P is V -geometrically ergodic, is P_k a V_{B_k} -geometrically ergodic kernel for k large enough? If so, when (ρ, C) is known in (V), can we deduce explicit (ρ_k, C_k) in (V_k) from (ρ, C) ? Finally can we obtain an explicit bound for the total variation distance $\|\widehat{\pi}_k - \pi\|_{TV}$, where $\widehat{\pi}_k$ is the following probability measure on $(\mathbb{X}, \mathcal{X})$: $\forall A \in \mathcal{X}, \widehat{\pi}_k(1_A) := \pi_k(1_{A \cap B_k})$?

(Q2) Conversely, when P_k is V_{B_k} -geometrically ergodic for some k , is P a V -geometrically ergodic kernel? If so, when (ρ_k, C_k) is known in (V_k) , can we deduce explicit (ρ, C) in (V) from (ρ_k, C_k) ? Finally can we obtain an explicit bound for the total variation distance $\|\widehat{\pi}_k - \pi\|_{TV}$?

Although (Q1) is interesting from a theoretical point of view, the computational issues involve rather (Q2). Indeed, since the state space of P_k is a closed ball in \mathbb{X} , rate ρ_k and

constant C_k in (V_k) are expected to be tractable and to provide constant and rate (ρ, C) in (V) . For instance, when $\mathbb{X} := \mathbb{N}$, P_k is a finite stochastic matrix, while P is infinite.

Let $(\mathcal{B}_1, \|\cdot\|_1)$ be the weighted-supremum Banach space composed of measurable complex-valued functions f such that $\|f\|_1 := \sup_{x \in \mathbb{X}} |f(x)|/V(x) < \infty$. When PV/V is bounded on \mathbb{X} , both P and \widehat{P}_k are bounded linear operators on \mathcal{B}_1 . A natural way to solve (Q1) is to see \widehat{P}_k as a perturbed operator of P , and vice versa for (Q2). But the standard perturbation theory does not apply in general (even if \mathbb{X} is discrete) since $\{\widehat{P}_k\}_{k \geq 0}$ does not converge to P in operator norm on \mathcal{B}_1 (see Remark 2.2). Here we use the weak perturbation theory due to Keller and Liverani [KL99, Liv01] (see also [Bal00]) which invokes the weakened convergence property

$$\|\widehat{P}_k - P\|_{0,1} := \sup_{f \in \mathcal{B}_0, \|f\|_0 \leq 1} \|\widehat{P}_k f - P f\|_1 \xrightarrow{k \rightarrow \infty} 0, \quad (C_{0,1})$$

where \mathcal{B}_0 is the Banach space of bounded measurable \mathbb{C} -valued functions on \mathbb{X} equipped with its usual norm $\|f\|_0 := \sup_{x \in \mathbb{X}} |f(x)|$. Condition $(C_{0,1})$ holds provided that $V(x) \rightarrow +\infty$ when $d(x_0, x) \rightarrow +\infty$ (see the proof of Lemma 2.1). However the price to pay for using $(C_{0,1})$ is that two functional assumptions are needed. The first one involves the Doeblin-Fortet inequalities: such dual inequalities can be derived for P and every \widehat{P}_k from Condition **(WD)** below (see Lemma 2.5). The second one requires that the essential spectral radii of both P and \widehat{P}_k on \mathcal{B}_1 are strictly less than one in a uniform way in k . This is Condition **(ESR)** below.

Issues (Q1) and (Q2) are solved in Sections 3 and 4. The two first questions in both (Q1)-(Q2) are addressed using the first part of [Liv01, Prop. 3.1]. The question in (Q1)-(Q2) concerning $\|\widehat{\pi}_k - \pi\|_{TV}$ is solved by direct spectral arguments (see Proposition 2.6). Theorems 3.1 and 4.1 provide a positive and explicit answer to the issues (Q1) and (Q2) under the following assumptions.

Conditions (P). *The Markov kernel P and its (extended) sub-Markov kernels \widehat{P}_k satisfy the two following conditions:*

$$\exists \delta \in (0, 1), \exists L > 0, \quad PV \leq \delta V + L 1_{\mathbb{X}} \quad (\mathbf{WD})$$

$$\exists \hat{k} \in \mathbb{N}^*, \quad \hat{r} := \max(r_{ess}(P), \sup_{k \geq \hat{k}} r_{ess}(\widehat{P}_k)) < 1 \quad (\mathbf{ESR})$$

where $r_{ess}(P)$ and $r_{ess}(\widehat{P}_k)$ denote the essential spectral radius of P and \widehat{P}_k acting on \mathcal{B}_1 . We denote $\hat{\alpha} := \max(\delta, \hat{r})$.

The notion of essential spectral radius and the material on quasi-compactness used for studying Condition **(ESR)** are reported in Section 5. Some useful theoretical complements on the weak perturbation theory for truncation are postponed to Section 6.

Inequality **(WD)** is a simple and well-known drift condition introduced in [MT93] for studying the V -geometric ergodicity of P . Managing Inequality **(ESR)** is more delicate, even in discrete case. For instance, if $\mathbb{X} := \mathbb{N}$, then P_k is a finite matrix, thus $r_{ess}(\widehat{P}_k) = 0$, but we do not have $r_{ess}(P) = 0$ in general (in particular P is not compact on \mathcal{B}_1 in general) and the question of bounding $r_{ess}(P)$ must be addressed. In non-discrete case, both \widehat{P}_k and P are general operators on \mathcal{B}_1 , so that bounding their essential spectral radii is nontrivial. In

any case, the estimates of Theorems 3.1 and 4.1 are all the more precise that the bound $\hat{\alpha}$ in **(P)**, thus \hat{r} in **(ESR)**, are accurate. To that effect, it is important not to confuse $r_{ess}(P)$ with ρ in (V) (and similarly for P_k). Actually Inequality (V) gives¹ $r_{ess}(P) \leq \rho$, but this bound cannot be used in **(ESR)**. Indeed, first either ρ_k or ρ are supposed to be unknown in $(Q1)$ or $(Q2)$. Second this bound may be inaccurate since there exist Markov kernels P such that $\rho - r_{ess}(P)$ is close to one (see Definition 5.3 and (31)). Accordingly, the study of **(ESR)** requires to obtain a specific control of the essential spectral radius $r_{ess}(Q)$ of a general Markov (or sub-Markov) kernel Q acting on \mathcal{B}_1 . This study, which has its own interest, is presented in Section 5. Two bounds for $r_{ess}(Q)$ are provided. The first one (Theorem 5.4) is derived from classical drift/minorization conditions on Q . To the best of our knowledge, this general result is not known in the literature. The second bound (Proposition 5.6) is obtained if Q satisfies the weak drift condition **(WD)** and Q is compact from \mathcal{B}_0 to \mathcal{B}_1 . Proposition 5.6 is a simplified version of [Wu04, Th. 3.11]. For convenience we present a direct and short proof using [Hen93, Cor. 1]. The previous conditions are specially relevant when applied to our truncation context. Indeed, if P satisfies these conditions, then so does every \hat{P}_k . As a consequence, **(ESR)** holds with explicit bound \hat{r} under anyone of the two following assumptions:

- (i) P satisfies some classical drift/minorization conditions (see Corollary 5.1),
- (ii) P satisfies **(WD)** and $P : \mathcal{B}_0 \rightarrow \mathcal{B}_1$ is compact (see Corollary 5.2).

Under Assumption (i), \hat{r} is estimated in terms of the constants of the drift/minorization conditions. The bound $\hat{r} \leq \delta$ obtained under (ii) is more precise. The compactness property in (ii) must not be confused with that of $P : \mathcal{B}_1 \rightarrow \mathcal{B}_1$ or $P : \mathcal{B}_0 \rightarrow \mathcal{B}_0$, which are much stronger conditions. If P is an infinite matrix, then $P : \mathcal{B}_0 \rightarrow \mathcal{B}_1$ is compact when $\lim_x V(x) = +\infty$, while in general P is compact neither on \mathcal{B}_1 nor on \mathcal{B}_0 .

Let us give a brief review of previous related works. Various probabilistic methods have been developed to derive explicit rate and constant (ρ, C) in Inequality (V) from the constants of drift conditions (see [MT94, LT96, Bax05] and the references therein). To the best of our knowledge, these methods, which are not concerned with truncation approximation, provide a computable rate ρ which is often unsatisfactory except for stochastically monotone P . For truncation approximation with a discrete space \mathbb{X} , it is proved in [Twe98], that for a V -geometrically ergodic P , $\sup_{|f| \leq V} |\hat{\pi}_k(f) - \pi(f)|$ goes to 0 when $k \rightarrow +\infty$. In particular $\|\hat{\pi}_k - \pi\|_{TV}$ goes to 0. In the special case when P is stochastically monotone, the following rate of convergence is obtained [Twe98, Th. 4.2,(46)]

$$\forall m \in \mathbb{N}^*, \forall k \in \mathbb{N}^*, \quad \|\hat{\pi}_k - \pi\|_{TV} \leq c \delta^m + \frac{d m}{V(k)},$$

with explicit constants c, d only depending on constants involved in drift/minorization conditions (e.g. see (29a)-(29b)). A similar result has been obtained for polynomially ergodic Markov chains in [Liu10]. To the best of our knowledge, these results have not been extended to non-discrete \mathbb{X} . In particular, for general Markov kernels, there are no known conditions ensuring that $\lim_k \|\hat{\pi}_k - \pi\|_{TV} = 0$. The two first parts of both issues $(Q1)$ and $(Q2)$ are not discussed in [Twe98, Liu10].

¹(See e.g. [KM03] or apply Definition 5.3 with $H := \{f \in \mathcal{B}_1 : \pi(f) = 0\}$.)

A weak perturbation approach based on the refinement [Liv04] of the Keller-Liverani theorem, has been used in [FHL13] to study general perturbed Markov kernels (not necessarily defined by truncation) of a V -geometrically ergodic Markov kernel. When applied to our context, [FHL13, Th. 1] gives a positive answer to the first and third parts of (Q1). The explicit connection between (ρ_k, C_k) and (ρ, C) in (Q1) is not addressed in [FHL13]. The weak perturbation results of [KL99, Liv01] have been fully used in the framework of dynamical systems (e.g. see [Bah06, DL08, BB10, BB11, Zhe10]). There, Markov kernels and their invariant probability measure are replaced by Perron-Frobenius operators and their so-called SRB measure. We want to point out that, in contrast with the previously cited papers, the second part of [Liv01, Prop. 3.1] and the spectral rank-stability property [Liv01, Cor. 3.1] are not used to derive the bounds in Theorems 3.1 and 4.1. This new approach, in which Proposition 2.6 plays an important role, is relevant to bound $\|\widehat{\pi}_k - \pi\|_{TV}$ since it uses both smaller integers k and better constants than those which would be involved by using the weak perturbation approach of [Liv01], as argued in Sections 3 and 4.

2 Notations and preliminary results

In this section, we introduce the main notations used in the paper. Moreover we propose a collection of results which are useful to solve Issues (Q1)-(Q2) and to analyze Condition **(ESR)**. Except in Section 5, the function $V : \mathbb{X} \rightarrow [1, +\infty)$ is assumed to be of the form

$$\forall x \in \mathbb{X}, \quad V(x) := v(d(x, x_0))$$

for some unbounded increasing function $v : [0, \infty) \rightarrow [1, +\infty)$ and some $x_0 \in \mathbb{X}$. Let P be a Markov kernel on $(\mathbb{X}, \mathcal{X})$ with associated truncation approximation family $\{P_k\}_{k \geq 1}$ and extended sub-Markov kernels $\{\widehat{P}_k\}_{k \geq 1}$ on $(\mathbb{X}, \mathcal{X})$ (see Definition 1.1). For $k \geq 1$, the restriction V_{B_k} of V to $B_k := \{x \in \mathbb{X} : d(x, x_0) \leq k\}$ is denoted by V_k . We denote by $[\cdot]$ the integer part function on \mathbb{R} . Now let us introduce the functional notations.

Notations. For $\beta \in [0, 1]$, $(\mathcal{B}_\beta, \|\cdot\|_\beta)$ denotes the Banach space

$$\mathcal{B}_\beta := \left\{ f : \mathbb{X} \rightarrow \mathbb{C}, \text{ measurable} : \|f\|_\beta := \sup_{x \in \mathbb{X}} V(x)^{-\beta} |f(x)| < \infty \right\}.$$

In particular \mathcal{B}_0 is the space of bounded measurable functions on \mathbb{X} and \mathcal{B}_1 is the V -weighted supremum space defined in Introduction. We denote by $(\mathcal{L}(\mathcal{B}_\beta, \mathcal{B}_{\beta'}), \|\cdot\|_{\beta, \beta'})$ the space of all the bounded linear maps from \mathcal{B}_β to $\mathcal{B}_{\beta'}$, equipped with its usual norm:

$$\|T\|_{\beta, \beta'} = \sup \left\{ \|Tf\|_{\beta'}, f \in \mathcal{B}_\beta, \|f\|_\beta \leq 1 \right\}.$$

We write $\mathcal{L}(\mathcal{B}_\beta)$ for $\mathcal{L}(\mathcal{B}_\beta, \mathcal{B}_\beta)$ and $\|T\|_\beta$ for $\|T\|_{\beta, \beta}$ which is a slight abuse of notation. When P is V -geometrically ergodic and P_k is V_k -geometrically ergodic, we set

$$\|\widehat{\pi}_k - \pi\|_\beta := \sup_{\|f\|_\beta \leq 1} |\widehat{\pi}_k(f) - \pi(f)|. \quad (3)$$

Observe that any condition (or conclusion) concerning the action of \widehat{P}_k on \mathcal{B}_1 can be equivalently reformulated into the action of P_k on the Banach space of measurable functions

$h : B_k \rightarrow \mathbb{C}$ such that $\sup_{x \in B_k} |h(x)|/V(x) < \infty$ (which defines the norm). Actually, although \widehat{P}_k is not Markov, the V_k -geometric ergodicity of P_k shall be often expressed as the V -geometrical ergodicity of \widehat{P}_k . In fact Inequalities (V) and (V_k) read as follows

$$\begin{aligned} \text{(V)} \quad &\iff \forall n \geq 1, \quad \sup_{f \in \mathcal{B}_1, \|f\|_1 \leq 1} \|P^n f - \pi(f)1_{\mathbb{X}}\|_1 \leq C \rho^n \\ \text{(V}_k\text{)} \quad &\iff \forall n \geq 1, \quad \sup_{f \in \mathcal{B}_1, \|f\|_1 \leq 1} \|\widehat{P}_k^n f - \widehat{\pi}_k(f)1_{B_k}\|_1 \leq C_k \rho_k^n. \end{aligned} \quad (4)$$

Lemma 2.1 *Under Condition (WD) with parameters (δ, L) , we have:*

$$\forall \beta \in [0, 1), \forall k \in \mathbb{N}^*, \quad \|\widehat{P}_k - P\|_{\beta, 1} \leq \frac{\max(2(\delta + L), (\delta + L)^\beta)}{v(k)^{1-\beta}}$$

so that $\lim_k \|\widehat{P}_k - P\|_{\beta, 1} = 0$.

Proof. Let $f \in \mathcal{B}_\beta$ be such that $\|f\|_\beta \leq 1$. First, we obtain for every $x \in B_k$:

$$\begin{aligned} |(Pf)(x) - (\widehat{P}_k f)(x)| &= \left| \int_{B_k^c} f(y)P(x, dy) - \int_{B_k^c} f(x_k)P(x, dy) \right| \\ &\leq 2 \int_{B_k^c} V(y)^\beta P(x, dy) = 2 \int_{B_k^c} V(y) \frac{1}{V(y)^{1-\beta}} P(x, dy) \\ &\leq \frac{2}{v(k)^{1-\beta}} (PV)(x) \quad (\text{using } v(k) = V(x_k) \leq V(y) \text{ for } y \in B_k^c) \\ &\leq \frac{2(\delta + L)}{v(k)^{1-\beta}} V(x) \quad (\text{since } PV \leq (\delta + L)V \text{ from (WD)}). \end{aligned}$$

Second let $x \in B_k^c$. Then $(\widehat{P}_k f)(x) = 0$, so that we obtain from Jensen's inequality

$$|(Pf)(x) - (\widehat{P}_k f)(x)| \leq (PV^\beta)(x) \leq (PV(x))^\beta \leq V(x)^\beta (\delta + L)^\beta \leq \frac{(\delta + L)^\beta}{v(k)^{1-\beta}} V(x). \quad \square$$

Remark 2.2 *Assume that $\limsup_{d(x_0, x) \rightarrow +\infty} PV(x)/V(x) > 0$ and that P satisfies (WD) (so that the infimum of δ such that (WD) holds is non zero). These assumptions are satisfied in most of models involving V -geometrical ergodicity. Then the strong convergence property $\lim_k \|P - \widehat{P}_k\|_1 = 0$ required in the standard perturbation theory does not hold. Indeed, using $\widehat{P}_k V(x) = 0$ when $x \in B_k^c$, we obtain*

$$\sup_{x \in B_k^c} \frac{(PV)(x)}{V(x)} = \sup_{x \in B_k^c} \frac{|(PV)(x) - (\widehat{P}_k V)(x)|}{V(x)} \leq \|PV - \widehat{P}_k V\|_1 \leq \|P - \widehat{P}_k\|_1.$$

If $\lim_k \|P - \widehat{P}_k\|_1 = 0$, then $\lim_k \sup_{x \in B_k^c} (PV)(x)/V(x) = 0$ which can not hold from the hypothesis. Actually using the strong convergence condition leads to difficulties or restrictions in other approximation questions. For instance in [KM05], some iterate P^N of the Markov kernel P is approached by finite rank kernels in operator norm on \mathcal{B}_1 (for other purpose than truncation issue). Thus P^N is compact on \mathcal{B}_1 . Consequently this property in operator norm $\|\cdot\|_1$ leads to suppose that $r_{\text{ess}}(P) = 0$ which is restrictive. Indeed, if P is V -geometrically ergodic, then $r_{\text{ess}}(P)$ is strictly less than one but is not zero in general.

Let $(\mathcal{B}'_1, \|\cdot\|_1)$ denote the dual space of \mathcal{B}_1 . Note that we make a slight abuse of notation in writing again $\|\cdot\|_1$ for the dual norm. For any $Q \in \mathcal{L}(\mathcal{B}_1)$, we denote by Q' its adjoint operator. Define the following auxiliary semi-norm on \mathcal{B}'_1 :

$$\forall f' \in \mathcal{B}'_1, \quad \|f'\|_0 := \sup \{|f'(f)|, f \in \mathcal{B}_0, \|f\|_0 \leq 1\}. \quad (5)$$

Lemma 2.3 *Let Q be any non-negative linear operator on the space \mathcal{B}_1 . Under Condition **(WD)** the following dual Doeblin-Fortet inequality holds*

$$\forall f' \in \mathcal{B}'_1, \quad \|Q'f'\|_1 \leq \delta \|f'\|_1 + L \|f'\|_0.$$

For a Markov kernel Q , the proof of this lemma is sketched in [FHL13] and details are provided in [GHL11]. Since only the non-negativity of the operator Q plays a role in this proof, the details are omitted.

Lemma 2.4 *If P satisfies Condition **(WD)** with parameters (δ, L) then, for any $k \in \mathbb{N}^*$, \widehat{P}_k satisfies **(WD)** with the same parameters, that is: $\widehat{P}_k V \leq \delta V + L 1_{\mathbb{X}}$.*

Proof. If $x \in B_k$, then

$$\begin{aligned} (\widehat{P}_k V)(x) &= \int_{B_k} V(y)P(x, dy) + \int_{B_k^c} V(x_k)P(x, dy) \\ &\leq \int_{B_k} V(y)P(x, dy) + \int_{B_k^c} V(y)P(x, dy) = PV(x) \end{aligned}$$

so $(\widehat{P}_k V)(x) \leq \delta V(x) + L$ from **(WD)**. If $x \in B_k^c$, then: $(\widehat{P}_k V)(x) = 0$. □

Combining the two last lemmas, we obtain the following statement.

Lemma 2.5 *If P satisfies Condition **(WD)** with parameters (δ, L) , then the kernels $Q := P$ and $Q := \widehat{P}_k$ for $k \geq 1$ satisfy the following uniform Doeblin-Fortet inequality on \mathcal{B}'_1 :*

$$\forall f' \in \mathcal{B}'_1, \quad \|Q'f'\|_1 \leq \delta \|f'\|_1 + L \|f'\|_0. \quad (6)$$

Under the weak drift condition **(WD)**, we set for every $\beta \in [0, 1]$

$$M_\beta \equiv M_\beta(\delta, L) := \begin{cases} \frac{1 - \delta^\beta + L^\beta}{1 - \delta^\beta} & \text{if } \beta \in (0, 1] \\ 1 & \text{if } \beta = 0. \end{cases}$$

The next proposition is relevant to solve the last question in issues (Q1)-(Q2).

Proposition 2.6 *Assume that P is V -geometrically ergodic and satisfies Condition **(WD)**, and that P_k is V_k -geometrically ergodic.*

(a) For any $\beta \in [0, 1)$ we have:

$$\forall R > 1, \forall n \geq 1, \quad \|\widehat{\pi}_k - \pi\|_\beta \leq \frac{C_k \rho_k^n + C \rho^n}{\|1_{\mathbb{X}}\|_1} + \frac{d_\beta}{(R-1)^2} \frac{R^{n+1}}{v(k)^{1-\beta}} \quad (7a)$$

$$\text{with } d_\beta \equiv d_\beta(\delta, L) := \|1_{\mathbb{X}}\|_1^{-1} M_\beta M_1 \max(2(\delta + L), (\delta + L)^\beta). \quad (7b)$$

In particular, if (V_k) holds for every $k \geq 1$ with $\sup_k C_k < \infty$ and $\sup_k \rho_k < 1$, then we have $\lim_k \|\widehat{\pi}_k - \pi\|_\beta = 0$ for any $\beta \in [0, 1)$.

(b) Setting $\widehat{\rho}_k := \max(\rho, \rho_k)$, we have:

$$\forall \sigma \in (0, 1], \quad \|\widehat{\pi}_k - \pi\|_{TV} \leq \left(\frac{C_k + C}{\widehat{\rho}_k \|1_{\mathbb{X}}\|_1} + \frac{d_0 \widehat{\rho}_k^\sigma}{(1 - \widehat{\rho}_k^\sigma)^2} \right) \frac{1}{v(k)^{1/(1+\sigma)}}. \quad (8)$$

Proof. Note that $\widehat{P}_k V^\beta \leq P V^\beta \leq \delta^\beta V^\beta + L^\beta$ for any $\beta \in (0, 1]$ (use **(WD)** and Jensen's inequality when $\beta \in (0, 1)$). Iterating this inequality gives

$$\forall \beta \in (0, 1], \quad \max\left(\sup_{n \geq 0} \|P^n\|_\beta, \sup_{n \geq 0} \|\widehat{P}_k^n\|_\beta\right) \leq M_\beta. \quad (9)$$

Moreover we have $\widehat{P}_k 1_{B_k} = 1_{B_k}$ and $P 1_{\mathbb{X}} = 1_{\mathbb{X}}$, so that the spectral radius of P and \widehat{P}_k on both \mathcal{B}_β and \mathcal{B}_1 is equal to 1. If $\beta := 0$, then (9) is fulfilled with $M_0 = 1$ since $\widehat{P}_k 1_{\mathbb{X}} \leq P 1_{\mathbb{X}} = 1_{\mathbb{X}}$. Now let $R > 1$ and let $\beta \in [0, 1]$. From the equality $(zI - T)^{-1} = \sum_{n \geq 0} z^{-(n+1)} T^n$ applied to $T := P$ and $T := \widehat{P}_k$ with $|z| = R$, we obtain

$$\max\left(\sup_{z \in \mathbb{C}: |z|=R} \|(zI - \widehat{P}_k)^{-1}\|_\beta, \sup_{z \in \mathbb{C}: |z|=R} \|(zI - P)^{-1}\|_\beta\right) \leq \frac{M_\beta}{R-1}.$$

Moreover the spectral theory gives

$$\widehat{P}_k^n - P^n = \frac{1}{2i\pi} \oint_{|z|=R} z^n [(zI - \widehat{P}_k)^{-1} - (zI - P)^{-1}] dz \quad (10)$$

and the following equality is obvious:

$$(zI - \widehat{P}_k)^{-1} - (zI - P)^{-1} = (zI - \widehat{P}_k)^{-1} (\widehat{P}_k - P) (zI - P)^{-1}.$$

Thus we obtain for every $f \in \mathcal{B}_\beta$

$$\|(\widehat{P}_k^n - P^n)f\|_1 \leq \frac{M_\beta M_1}{(R-1)^2} R^{n+1} \|\widehat{P}_k - P\|_{\beta,1} \|f\|_\beta.$$

Using the triangle inequality, it follows that

$$\begin{aligned} |\widehat{\pi}_k(f) - \pi(f)| \|1_{\mathbb{X}}\|_1 &= \|\widehat{\pi}_k(f) 1_{\mathbb{X}} - \pi(f) 1_{\mathbb{X}}\|_1 \\ &\leq \|\widehat{\pi}_k(f) 1_{\mathbb{X}} - \widehat{P}_k^n f\|_1 + \|\widehat{P}_k^n f - P^n f\|_1 + \|P^n f - \pi(f) 1_{\mathbb{X}}\|_1 \\ &\leq C_k \rho_k^n \|f\|_1 + \frac{M_\beta M_1}{(R-1)^2} R^{n+1} \|\widehat{P}_k - P\|_{\beta,1} \|f\|_\beta + C \rho^n \|f\|_1. \end{aligned}$$

Inequality (7a) then follows from $\|f\|_1 \leq \|f\|_\beta$ and Lemma 2.1. The last result of Assertion (a) is easily deduced from a usual "two-epsilon" argument. Inequality (8) follows from (7a) with $\beta := 0$, $R := \widehat{\rho}_k^{-\sigma}$ and $n := \lfloor -\ln v(k) / ((1 + \sigma) \ln \widehat{\rho}_k) \rfloor$. \square

3 From V -geometric ergodicity to V_k -geometric ergodicity

For any $(a, \theta) \in \mathbb{C} \times (0, +\infty)$, let us define $D(a, \theta) := \{z \in \mathbb{C} : d(a, z) < \theta\}$ and $\overline{D}(a, \theta) := \{z \in \mathbb{C} : d(a, z) \leq \theta\}$. For any $T \in \mathcal{L}(\mathcal{B}_1)$, the spectrum of T is denoted by $\sigma(T)$, and for any $(r, \vartheta) \in (0, 1)^2$, we introduce the following subsets of the complex plane

$$\mathcal{V}(r, \vartheta, T) := \{z \in \mathbb{C} : |z| \leq r \text{ or } d(z, \sigma(T)) \leq \vartheta\} \quad \text{and} \quad \mathcal{V}(r, \vartheta, T)^c := \mathbb{C} \setminus \mathcal{V}(r, \vartheta, T), \quad (11)$$

where $d(z, \sigma(T)) := \inf\{d(z, \lambda), \lambda \in \sigma(T)\}$. Recall that $\hat{\alpha}$ is defined in Conditions **(P)**. For any $r > \hat{\alpha}$ and $\vartheta > 0$, we set:

$$H \equiv H(r, \vartheta, P) := \sup_{z \in \mathcal{V}(r, \vartheta, P)^c} \|(zI - P)^{-1}\|_1 \quad (12a)$$

$$n_1 \equiv n_1(r) := \left\lfloor \frac{\ln 2}{\ln(r/\hat{\alpha})} \right\rfloor \quad n_2 \equiv n_2(r, \vartheta, P) := \left\lfloor \frac{3HL(L+3)\ln 2}{r^{n_1} \ln(r/\hat{\alpha})} \right\rfloor \quad (12b)$$

$$\varepsilon_1 \equiv \varepsilon_1(r, \vartheta, P) := \frac{r^{n_1+n_2}}{8L(HL + (1-r)^{-1})}. \quad (12c)$$

The following theorem provides a positive answer to Issue (Q1).

Theorem 3.1 *Let P be a V -geometrically ergodic Markov kernel with rate $\rho \in (0, 1)$ in (V). Assume that Conditions **(P)** hold. Let $(r, \vartheta) \in (0, 1)^2$ be such that*

$$\max(\hat{\alpha}, \rho) + \vartheta < r < 1 - \vartheta. \quad (13)$$

Then, for any $k \in \mathbb{N}^$ such that the convergence condition $(C_{0,1})$ holds at rate ε_1 , that is*

$$\|\widehat{P}_k - P\|_{0,1} := \sup_{\|f\|_0 \leq 1} \|\widehat{P}_k f - P f\|_1 \leq \varepsilon_1, \quad (\mathcal{E}_{0,1})$$

and such that (V_k) holds with some rate ρ_k satisfying

$$\rho_k < 1 - \vartheta,$$

the following inequalities hold:

1. *For any measurable function $h : B_k \rightarrow \mathbb{C}$ such that $|h| \leq V_k$, we have*

$$\forall n \geq 1, \quad \sup_{x \in B_k} \frac{|(P_k^n h)(x) - \pi_k(h)|}{V(x)} \leq c r^{n+1} \quad (14a)$$

$$\text{with } c \equiv c(r, \vartheta, P) := \frac{4(L+1)}{r^{n_1}(1-r)} + \frac{1}{2\varepsilon_1}. \quad (14b)$$

2. *For any $\beta \in [0, 1)$, we have the following estimate of $\|\widehat{\pi}_k - \pi\|_\beta$*

$$\forall R > 1, \quad \forall n \geq 1, \quad \|\widehat{\pi}_k - \pi\|_\beta \leq \frac{C\rho^n + c r^{n+1}}{\|1_{\mathbb{X}}\|_1} + \frac{d_\beta}{(R-1)^2} \frac{R^{n+1}}{v(k)^{1-\beta}}, \quad (15)$$

where the constants C and $d_\beta \equiv d_\beta(\delta, L)$ are from (V) and (7b) respectively. In particular we have with $\xi := \max(\rho, r)$:

$$\forall \sigma \in (0, 1], \quad \|\widehat{\pi}_k - \pi\|_{TV} \leq \left(\frac{C + c r}{\xi \|1_{\mathbb{X}}\|_1} + \frac{d_0 \xi^\sigma}{(1 - \xi^\sigma)^2} \right) \frac{1}{v(k)^{1/(1+\sigma)}}. \quad (16)$$

Actually, under Conditions **(P)** and (V) , the conclusions (14a)-(16) hold true for k large enough. Indeed, the fact that Condition $(\mathcal{E}_{0,1})$ is fulfilled for k large enough follows from Lemma 2.1: more specifically, a sufficient condition for $(\mathcal{E}_{0,1})$ to hold is that $k \geq k_1$, with

$$k_1 \equiv k_1(r, \vartheta, P) := \inf \left\{ k \in \mathbb{N}, \quad v(k) \geq \frac{\max(2(\delta + L), 1)}{\varepsilon_1} \right\}. \quad (17)$$

That (V_k) holds with some rate $\rho_k < 1 - \vartheta$ for k large enough is more difficult to establish. This follows from Proposition 6.1 which ensures that a sufficient condition for such a property to hold is that $k \geq k_0$, with k_0 defined in (37). However this k_0 is very large (typically $k_0 \approx k_1^3$), while (V_k) with $\rho_k < 1 - \vartheta$ may be fulfilled for $k \ll k_0$. This explains why k_0 is not introduced in Theorem 3.1. In fact Theorem 3.1 proceeds as follows. Under Conditions **(P)** and (V) , let ϑ be such that $0 < \vartheta < (1 - \max(\hat{\alpha}, \rho))/2$. One can choose r close to ρ satisfying (13). Pick the first k such that Conditions $(\mathcal{E}_{0,1})$ and (V_k) with some rate $\rho_k < 1 - \vartheta$ hold true. Such a k exists from the previous discussion. Then Inequality (14a) reads as follows: P_k is V_k -geometrically ergodic with the new rate r in (V_k) and explicit constant c . Of course, Theorem 3.1 is only relevant when the rate ρ and the bound C in (V) are known. That ρ must be known is necessary to choose (r, ϑ) in (13). Constant C must be known for an effective computation of an upper bound of $H \equiv H(r, \vartheta, P)$ in (12a) (see Remark 3.2). Note that H is involved in the definition of $\varepsilon_1(r, \vartheta, P)$, thus in the computation of the bound $c(r, \vartheta, P)$ in (14b), thus finally in the bounds in (15) and (16).

When \mathbb{X} is discrete, it is proved in [Twe98] that $\sup_{|f| \leq V} |\hat{\pi}_k(f) - \pi(f)| = \|\pi_k - \pi\|_1$ goes to 0 when $k \rightarrow +\infty$. The case $\beta = 1$ is not covered by (15). Under Conditions **(P)** and (V) , we deduce from (15) and the previous discussion that

$$\forall \beta \in [0, 1), \quad \lim_k \|\hat{\pi}_k - \pi\|_\beta = 0.$$

To the best of our knowledge, this property (even in case $\beta = 0$) was unknown for non-discrete state spaces. Moreover the rate of convergence for $\|\hat{\pi}_k - \pi\|_{TV}$ obtained in discrete case in [Twe98] (see Introduction) has been derived under the stochastic monotonicity assumption. The bounds (15) and (16), which are also valid for general set \mathbb{X} , do not require this assumption. Note that the expected rate of convergence $O(1/v(k))$ for the total variation distance $\|\hat{\pi}_k - \pi\|_{TV}$ is approached in (16) when $\sigma \rightarrow 0$, but with constant increasing to $+\infty$.

The weak perturbation theorem [KL99] has been used in many papers to study the total variation distance between the perturbed and unperturbed SRB measures associated with certain dynamical systems (see [Liv01] and the related works cited at the end of Introduction). The method developed in these works involves the whole results [Liv01, Prop. 3.1, Cor. 3.1]. When applied to our truncation approximation context, this method would require: first to test the convergence condition $(\mathcal{E}_{0,1})$ at rate ε_0 defined in (37) (in substance $\varepsilon_0 \approx \varepsilon_1^3$) in order to apply the spectral rank-stability property [Liv01, Cor. 3.1]; second to use the second part of [Liv01, Prop. 3.1] also based on ε_0 in order to bound $\|\hat{\pi}_k - \pi\|_{TV}$. This would lead to bounds of the form $\|\hat{\pi}_k - \pi\|_{TV} \leq D v(k)^{-\eta}$ with integers $k \approx O(v^{-1}(H^3))$ and $D = O(H^2)$ (typically $\eta \approx 1/3$). The method proposed in this paper is different and original, so that it provides a new weak perturbation approach. The key idea is that Proposition 2.6 allows us to derive the bounds in (15)-(16) from the property (14a) which is only based on the first part of [Liv01, Prop. 3.1]. This alternative approach requires to test the convergence condition

$(\mathcal{E}_{0,1})$ at rate ε_1 and the condition (V_k) with rate $\rho_k < 1 - \vartheta$. The constant c in (14b), thus the constants in (15) (16), do not involve the second part of [Liv01, Prop. 3.1]. Accordingly, using (14b) and the definition of ε_1 in (12c), the constant in (16) is only $O(H)$ in place of $O(H^2)$. Since H may be large, this improvement is relevant. See Appendix B for details.

Remark 3.2 *When P is V -geometrically ergodic, we have for any $(r, \vartheta) \in (0, 1)^2$ such that $\rho + \vartheta < r < 1 - \vartheta$:*

$$H(r, \vartheta, P) = \sup_{z \in \overline{D}(0, r)^c \cap \overline{D}(1, \vartheta)^c} \|(zI - P)^{-1}\|_1 \leq \frac{\pi(V)\|1_{\mathbb{X}}\|_1}{\vartheta} + \frac{C}{r - \rho} \leq \frac{\pi(V)\|1_{\mathbb{X}}\|_1 + C}{\vartheta}.$$

Since π may be unknown, note that $\pi(V) \leq L/(1 - \delta)$ under **(WD)**. See details in Appendix A.

Using the convergence condition $(\mathcal{E}_{0,1})$ and Lemma 2.5, Theorem 3.1 is derived from the first part of [Liv01, Prop. 3.1] as follows.

Proof of Theorem 3.1. We use the functional notations introduced at the beginning of Section 2, as well as the duality notations before Lemma 2.3. The operator norm on the space $\mathcal{L}(\mathcal{B}'_1)$ of bounded linear operators on \mathcal{B}'_1 is still denoted by $\|\cdot\|_1$. For any $T \in \mathcal{L}(\mathcal{B}_1)$, recall that $\|T'\|_1 = \|T\|_1$ and $\sigma(T) = \sigma(T')$. Given any real numbers $r > \hat{\alpha}$ and $\vartheta > 0$, we use the notations introduced in (12a)-(12c). Note that H in (12a) may be equivalently defined with P' in place of P since the resolvents of P and P' have the same norm in $\mathcal{L}(\mathcal{B}_1)$ and $\mathcal{L}(\mathcal{B}'_1)$ respectively and $\mathcal{V}(r, \vartheta, P) = \mathcal{V}(r, \vartheta, P')$.

The first assertion of Theorem 3.1 is proved in two steps. In a first step, we show that, under Conditions **(P)** and $(\mathcal{E}_{0,1})$ for some $k \in \mathbb{N}^*$, we have with $c(r, \vartheta, P)$ given in (14b)

$$\sigma(\widehat{P}_k) \subset \mathcal{V}(r, \vartheta, P) \quad \text{and} \quad \sup_{z \in \mathcal{V}(r, \vartheta, P)^c} \|(zI - \widehat{P}_k)^{-1}\|_1 \leq c(r, \vartheta, P). \quad (18)$$

Clearly (18) will hold true if we establish that, under Conditions **(P)** and $(\mathcal{E}_{0,1})$ for some $k \in \mathbb{N}^*$, the following properties are valid:

$$\sigma(\widehat{P}_k') \subset \mathcal{V}(r, \vartheta, P) \quad \text{and} \quad \sup_{z \in \mathcal{V}(r, \vartheta, P)^c} \|(zI - \widehat{P}_k')^{-1}\|_1 \leq c(r, \vartheta, P). \quad (19)$$

In fact we prove that (19), so (18), hold for any $\vartheta > 0$ and $r > \hat{\alpha}$ (this fact will be used in the sequel of the paper as in Lemma 6.7). To that effect, the Keller-Liverani perturbation theorem [KL99] is applied to the adjoint operators of P and \widehat{P}_k acting on \mathcal{B}'_1 . Moreover we use the explicit constants given in [Liv01]. Let us use the auxiliary semi-norm $\|\cdot\|_0$ on \mathcal{B}'_1 introduced in (5). We have that $\forall f' \in \mathcal{B}'_1$, $\|f'\|_0 \leq \|f'\|_1$. Moreover we have for any $f' \in \mathcal{B}'_1$

$$\|P'f'\|_0 \leq \|f'\|_0 \quad \text{and} \quad \forall k \geq 1, \|\widehat{P}_k'f'\|_0 \leq \|f'\|_0.$$

Indeed, for $K := P$ and $K := \widehat{P}_k$, we have $\|Kf\|_0 \leq \|f\|_0$ from the positivity of K and $K1_{\mathbb{X}} \leq 1_{\mathbb{X}}$, so that we obtain for any $f' \in \mathcal{B}'_1$ and $f \in \mathcal{B}_0$, $\|f\|_0 \leq 1$:

$$|(K'f')(f)| = |f'(Kf)| \leq \|f'\|_0 \|Kf\|_0 \leq \|f'\|_0.$$

Next, we know from Lemma 2.5 that $K' := P'$ and $K' := \widehat{P}_k'$ (for every $k \geq 1$) satisfy the uniform Doeblin-Fortet inequality (6) on \mathcal{B}'_1 . Moreover we obtain by duality and from Inequality ($\mathcal{E}_{0,1}$)

$$\|\widehat{P}_k' - P'\|_{1,0} := \sup_{f' \in \mathcal{B}'_1, \|f'\|_{\mathcal{B}'_1} \leq 1} \|\widehat{P}_k' f' - P' f'\|_0 = \|\widehat{P}_k - P\|_{0,1} \leq \varepsilon_1.$$

Finally, from Inequality (**ESR**), P' and \widehat{P}_k' are quasi-compact on \mathcal{B}'_1 with essential spectral radius less than $\hat{r} \leq \hat{\alpha}$. The previous facts and the first part of [Liv01, Prop. 3.1] give (19).

In a second step, we prove that Inequality (14a) holds under the assumptions of Theorem 3.1. Since P is V -geometrically ergodic, we have $\sigma(P) \subset D(0, \rho) \cup \{1\}$. Thus, it follows from (13) and the first inclusion in (18) that $\sigma(\widehat{P}_k) \subset \overline{D}(0, r) \cup \overline{D}(1, \vartheta)$. Moreover, since \widehat{P}_k is assumed to be V -geometrically ergodic with rate $\rho_k < 1 - \vartheta$, we obtain that

$$\sigma(\widehat{P}_k) \subset \overline{D}(0, r) \cup \{1\} \quad (20)$$

and that $(1/2i\pi) \oint_{|z-1|=\vartheta} (zI - \widehat{P}_k)^{-1} dz = \widehat{\pi}_k(\cdot)1_{B_k}$ from standard spectral theory. Thus we have for any $\kappa \in (r, 1 - \vartheta)$

$$\begin{aligned} \forall n \geq 1, \quad \widehat{P}_k^n &= \frac{1}{2i\pi} \oint_{|z-1|=\vartheta} (zI - \widehat{P}_k)^{-1} dz + \frac{1}{2i\pi} \oint_{|z|=\kappa} z^n (zI - \widehat{P}_k)^{-1} dz \\ &= \widehat{\pi}_k(\cdot)1_{B_k} + \frac{1}{2i\pi} \oint_{|z|=\kappa} z^n (zI - \widehat{P}_k)^{-1} dz. \end{aligned}$$

Finally it follows from (18) that $\|\widehat{P}_k^n - \widehat{\pi}_k(\cdot)1_{B_k}\|_1 \leq c(r, \vartheta, P) \kappa^{n+1}$. Since $\kappa \in (r, 1 - \vartheta)$ is arbitrary, this inequality holds with r in place of κ , and it obviously gives (14a).

The second assertion of Theorem 3.1 follows from Proposition 2.6 and the proof of Theorem 3.1 is complete. \square

4 From V_k -geometric ergodicity to V -geometric ergodicity

Let $\rho_V(P)$ denote the spectral gap of P , that is the infimum bound of the rates ρ in Inequality (V). Recall that $\rho_V(P)$ is unknown in general and that, even when it is known, finding an explicit constant C associated with $\rho > \rho_V(P)$ in (V) is a difficult question. As mentioned in Introduction, there exist various methods [MT94, LT96, Bax05] providing rates and constants (ρ, C) in Inequality (V), but they yield a rate ρ which is much far from $\rho_V(P)$ (except in specific cases). The first purpose of Theorem 4.1 below is to provide a rate r_k in (V) from ρ_k in (V_k) , which is all the more close to $\rho_V(P)$ that k is large and ρ_k in (V_k) is close to the spectral gap of P_k . Such a ρ_k is expected to be computable since the state space associated with P_k is a closed ball of \mathbb{X} (a finite set when \mathbb{X} is discrete). The second purpose of Theorem 4.1 is to provide an explicit constant $c_k \equiv c(r_k)$ associated with rate r_k in (V). Then Proposition 2.6 gives a bound for $\|\widehat{\pi}_k - \pi\|_{TV}$.

Let us briefly explain why the passage from the V_k -geometric ergodicity of P_k to the V -geometric ergodicity of P is theoretically more difficult than the converse one studied in

Section 3. Assume that Conditions **(P)** hold. Let $r > \hat{\alpha}$ and $\vartheta > 0$. With $\varepsilon_1(r, \vartheta, P)$ defined in (12c), Property (18) reads as follows

$$\|\widehat{P}_k - P\|_{0,1} \leq \varepsilon_1(r, \vartheta, P) \implies \sigma(\widehat{P}_k) \subset \mathcal{V}(r, \vartheta, P), \quad \sup_{z \in \mathcal{V}(r, \vartheta, P)^c} \|(zI - \widehat{P}_k)^{-1}\|_1 < \infty. \quad (21)$$

Next, for any fixed $k \in \mathbb{N}^*$, exchanging the role of P and \widehat{P}_k in (18) gives the following implication:

$$\|P - \widehat{P}_k\|_{0,1} \leq \varepsilon_1(r, \vartheta, \widehat{P}_k) \implies \sigma(P) \subset \mathcal{V}(r, \vartheta, \widehat{P}_k), \quad \sup_{z \in \mathcal{V}(r, \vartheta, \widehat{P}_k)^c} \|(zI - P)^{-1}\|_1 < \infty. \quad (22)$$

There is a significant difference between the implications (21) and (22) since the inequality $\|\widehat{P}_k - P\|_{0,1} \leq \varepsilon_1(r, \vartheta, P)$ in (21) is satisfied for k large enough from Lemma 2.1 (see (17)), while the inequality $\|P - \widehat{P}_k\|_{0,1} \leq \varepsilon_1(r, \vartheta, \widehat{P}_k)$ in (22) could fail for every k . Fortunately, the previous failure cannot occur when Conditions **(P)** hold (see Proposition 6.4).

Now we introduce the material used in Theorem 4.1 which solves Issue (Q2). Under Conditions **(P)**, for any $\vartheta > 0$, $k \in \mathbb{N}^*$ and $r_k \in (\hat{\alpha}, 1)$, we introduce the following constants:

$$\varepsilon_1(k) \equiv \varepsilon_1(r_k, \vartheta, \widehat{P}_k) := \frac{r_k^{n_1(k) + n_2(k)}}{8L(H_k L + (1 - r_k)^{-1})} \quad (23a)$$

with

$$H_k \equiv H(r_k, \vartheta, \widehat{P}_k) := \sup_{z \in \mathcal{V}(r_k, \vartheta, \widehat{P}_k)^c} \|(zI - \widehat{P}_k)^{-1}\|_1 \quad (23b)$$

$$n_1(k) \equiv n_1(r_k) := \left\lfloor \frac{\ln 2}{\ln(r_k/\hat{\alpha})} \right\rfloor \quad n_2(k) \equiv n_2(r_k, \vartheta, P_k) := \left\lfloor \frac{3H_k L(L+3)\ln 2}{r_k^{n_1(k)} \ln(r_k/\hat{\alpha})} \right\rfloor. \quad (23c)$$

Moreover we denote by \mathcal{I}_ϑ the following subset of integers :

$$\mathcal{I}_\vartheta := \{ k \in \mathbb{N}^* : P_k \text{ satisfies } (V_k) \text{ with some rate } \rho_k < 1 - 2\vartheta \}.$$

Theorem 4.1 *Assume that P is V -geometrically ergodic with some rate $\rho \in (0, 1)$ and that Conditions **(P)** hold. Let ϑ be such that*

$$0 < \vartheta < \frac{1 - \max(\hat{\alpha}, \rho)}{3}. \quad (24)$$

Then, for any $k \in \mathcal{I}_\vartheta$ and for any r_k such that

$$\max(\hat{\alpha}, \rho_k) + \vartheta < r_k < 1 - \vartheta, \quad (25)$$

the assertions (a) and (b) below hold true provided that the convergence condition $(C_{0,1})$ holds at rate $\varepsilon_1(k)$, that is

$$\|P - \widehat{P}_k\|_{0,1} \leq \varepsilon_1(k). \quad (\mathcal{E}_{0,1}(k))$$

(a) *The iterates of P converge to $\pi(\cdot)1_{\mathbb{X}}$ with the following explicit rate of convergence:*

$$\forall n \geq 1, \quad \|P^n - \pi(\cdot)1_{\mathbb{X}}\|_1 \leq c_k r_k^{n+1} \quad (26a)$$

$$\text{with } c_k \equiv c_k(r_k, \vartheta) := \frac{4(L+1)}{r_k^{n_1(k)}(1-r_k)} + \frac{1}{2\varepsilon_1(k)}. \quad (26b)$$

(b) The following inequality is valid for any $\beta \in [0, 1)$:

$$\forall R > 1, \forall n \geq 1, \quad \|\widehat{\pi}_k - \pi\|_\beta \leq \frac{C_k \rho_k^n + c_k r_k^{n+1}}{\|1_{\mathbb{X}}\|_1} + \frac{d_\beta}{(R-1)^2} \frac{R^{n+1}}{v(k)^{1-\beta}}, \quad (27)$$

where the constants C_k , and $d_\beta \equiv d_\beta(\delta, L)$ are from (V_k) and (7b) respectively. In particular we have for any $\sigma \in (0, 1]$ with $\widehat{\rho}_k := \max(r_k, \rho_k)$

$$\|\widehat{\pi}_k - \pi\|_{TV} \leq \left(\frac{C_k + c_k r_k}{\widehat{\rho}_k} + \frac{d_0 \widehat{\rho}_k^\sigma}{(1 - \widehat{\rho}_k^\sigma)^2} \right) \frac{1}{v(k)^{1/(1+\sigma)}}. \quad (28)$$

Theorem 4.1 can be established as Theorem 3.1 by exchanging the role of P and \widehat{P}_k . Moreover, under the assumptions of Theorem 4.1, the conclusions (26a)-(28) are valid for k large enough. Indeed, first Proposition 6.1 ensures that a sufficient condition for k to belong to I_ϑ is that $k \geq k_0$ with k_0 given in (37) (use $\max(\widehat{\alpha}, \rho) + \vartheta < 1 - 2\vartheta$ from (24)). Second Proposition 6.4 ensures that a sufficient condition for the convergence property $(\mathcal{E}_{0,1}(k))$ to hold is that $k \in [\tilde{k}, +\infty) \cap \mathcal{I}_\vartheta$ for some integer \tilde{k} . This \tilde{k} is very large (greater than k_0). Since integer $k \in \mathcal{I}_\vartheta$ satisfying $(\mathcal{E}_{0,1}(k))$ may exist for $k \ll \max(\tilde{k}, k_0)$, an effective application of Theorem 4.1 needs to find such a k with a value as small as possible. This can be done in testing the validity of both properties $k \in \mathcal{I}_\vartheta$ and $(\mathcal{E}_{0,1}(k))$ for increasing values of k . Note that ρ_k will be all the more tractable that the previous k is small (when \mathbb{X} is discrete ρ_k is linked to the second eigenvalue of the finite matrix P_k).

In Theorem 4.1, some preliminary rate ρ in (V) is assumed to be available, but it is worth noticing that no constant associated with ρ in (V) is required to apply Theorem 4.1. Inequality (26a) provides a new rate r_k in (V) , but also an explicit associated constant c_k . Note that the preliminary rate ρ in (V) affects the bound c_k in (26b). In particular, if ϑ is small, then $c_k \equiv c_k(r_k, \vartheta)$ may be large since $c_k \approx O(1/\varepsilon_1(k)) = O(H_k)$, with $H_k \geq \sup_{|z-1|=\vartheta} \|(zI - \widehat{P}_k)^{-1}\|_1 \rightarrow +\infty$ when $\vartheta \rightarrow 0$. When the new rate r_k can be chosen significantly smaller than ρ (see (25)), then Theorem 4.1 can be applied once again with a new initial rate $\rho' < \rho$, so with a new value $\vartheta' > \vartheta$. The resulting rate r'_k in (26a) will be larger than r_k since $\vartheta' > \vartheta$ (see (25)), but the associated constant c'_k will be smaller than c_k because $H'_k \equiv H(r'_k, \vartheta', \widehat{P}_k) < H_k$. More generally, an efficient use of Theorem 4.1 needs to make a trade-off between the rate and the associated constant in (26a). Finally note that H_k can be bounded under Condition (V_k) following the lines of Remark 3.2 with \widehat{P}_k in place of P .

The comments on the improvements on truncation approximations and on the new weak perturbation approach providing Theorem 3.1, naturally extend to Theorem 4.1. Here they are even more relevant since the computational issues involve (Q2) (rather than (Q1)). In particular, using a preliminary rate ρ in (V) allows us to derive the bounds of Theorem 4.1 from the first part of [Liv01, Prop. 3.1]. Again Proposition 2.6 plays an important role. The benefits in terms of constants in (28) are similar to those obtained for (16) in Theorem 3.1.

5 Quasi-compactness on \mathcal{B}_1 and application to truncation issues

The new rates provided in (V_k) by Theorem 3.1, or in (V) by Theorem 4.1, greatly depend on the upper bound \widehat{r} of $r_{ess}(P)$ and all the $r_{ess}(\widehat{P}_k)$ in Inequality **(ESR)** of Conditions **(P)**

(e.g. see (25) and (26a)). The next corollaries provide estimates of \hat{r} under two different kind of assumptions on the Markov kernel P .

Corollary 5.1 *Assume that P satisfies the following drift/minorization conditions: there exist a bounded set $S \in \mathcal{X}$ and a positive measure ν on $(\mathbb{X}, \mathcal{X})$ such that*

$$\exists \delta \in (0, 1), \exists L > 0, \quad PV \leq \delta V + L 1_S, \quad (29a)$$

$$\forall x \in \mathbb{X}, \forall A \in \mathcal{X}, \quad P(x, A) \geq \nu(1_A) 1_S(x). \quad (29b)$$

Let $k_S := \inf\{k \geq 1 : S \subset B_k \text{ and } \nu(1_{B_k}) > 0\}$. Then Condition **(ESR)**, so Conditions **(P)**, hold with $\hat{k} := k_S$ and

$$\hat{r} \leq \frac{\delta \nu(1_{B_{k_S}}) + \tau_{k_S}}{\nu(1_{B_{k_S}}) + \tau_{k_S}} \quad \text{where } \tau_{k_S} := \max(0, L - \nu(V 1_{B_{k_S}})).$$

In particular, if the set S is an atom for P , then $\hat{r} \leq \delta$.

Corollary 5.2 *Assume that P is compact from \mathcal{B}_0 into \mathcal{B}_1 and satisfies Condition **(WD)**. Then Condition **(ESR)**, so Conditions **(P)**, hold with $\hat{k} := 1$ and $\hat{r} \leq \delta$.*

These corollaries follow from Theorem 5.4 and Proposition 5.6 on the essential spectral radius of a general Markov (or sub-Markov) kernel Q acting on \mathcal{B}_1 .

Let us recall some basic facts on quasi-compactness and on the essential spectral radius of linear bounded operators on a complex Banach space $(\mathcal{B}, \|\cdot\|)$ (e.g. see [HH01]). Let T be a bounded linear operator on \mathcal{B} with positive spectral radius $r(T) := \lim_n \|T^n\|^{1/n}$, where $\|\cdot\|$ also stands for the operator norm on \mathcal{B} . For the sake of simplicity, we assume that $r(T) := 1$ (if not, replace T with $r(T)^{-1}T$). We denote by I the identity operator on \mathcal{B} . The simplest definition of quasi-compactness is the following one (to be compared with the reduction of matrices or compact operators).

Definition 5.3 *T is quasi-compact on \mathcal{B} if there exist $r_0 \in (0, 1)$, $m \in \mathbb{N}^*$ and $(\lambda_i, p_i) \in \mathbb{C} \times \mathbb{N}^*$ for $i = 1, \dots, m$ such that:*

$$\mathcal{B} = \bigoplus_{i=1}^m \text{Ker}(T - \lambda_i I)^{p_i} \oplus H, \quad (30a)$$

where the λ_i 's are such that

$$|\lambda_i| \geq r_0 \quad \text{and} \quad 1 \leq \dim \text{Ker}(T - \lambda_i I)^{p_i} < \infty, \quad (30b)$$

and H is a closed T -invariant subspace such that

$$\sup_{h \in H, \|h\| \leq 1} \|T^n h\| = O(r_0^n). \quad (30c)$$

Concerning the essential spectral radius of T , denoted by $r_{ess}(T)$, here it is enough to have in mind that, if T is quasi-compact on \mathcal{B} , then we have (e.g. see [Hen93])

$$r_{ess}(T) := \inf \{ r_0 \in (0, 1) \text{ s.t. we have (30a), (30b), (30c) } \}. \quad (31)$$

It is also well-known that $r_{ess}(T) := \lim_n(\inf \|T^n - K\|)^{1/n}$, where the infimum is taken over the ideal of compact operators K on \mathcal{B} (e.g. see [Nev64]). Consequently, T is quasi-compact if and only if there exist some $n_0 \in \mathbb{N}^*$ and some compact operator K_0 on \mathcal{B} such that $r(T^{n_0} - K_0) < 1$. Under the previous condition we have

$$r_{ess}(T) \leq (r(T^{n_0} - K_0))^{1/n_0}. \quad (32)$$

Finally recall that $r_{ess}(T) = r_{ess}(T^p)^{1/p}$ for every $p \geq 1$ since $\lim_n(\inf \|T^n - K\|)^{1/n} = \lim_k(\inf \|T^{pk} - K\|)^{1/(pk)}$.

Throughout the two next subsections, we consider a general measurable space $(\mathbb{X}, \mathcal{X})$, a function $V : \mathbb{X} \rightarrow [1, +\infty)$ and a nonnegative kernel Q on $(\mathbb{X}, \mathcal{X})$ such that $\|QV\|_1 < \infty$. So Q continuously acts on \mathcal{B}_1 . The first bound for $r_{ess}(Q)$ is derived from a result on positive operators [Sch71]. The second one is obtained by duality from the quasi-compactness criteria of [Hen93].

5.1 Bounds on $r_{ess}(Q)$ under drift/minorization conditions

It is well-known (see [MT93]) that, under usual irreducibility and aperiodicity assumptions, if the Markov kernel Q satisfies the drift/minorization conditions (29a)-(29b) then Q is V -geometrical ergodic. In the next theorem, no irreducibility/aperiodicity condition is assumed, and a simple bound of $r_{ess}(Q)$ is given in terms of the parameters involved in (29a)-(29b).

Theorem 5.4 *Let Q be a Markov kernel satisfying the drift/minorization conditions (29a)-(29b). Then Q is a power-bounded quasi-compact operator on \mathcal{B}_1 with*

$$r_{ess}(Q) \leq \frac{\delta \nu(1_{\mathbb{X}}) + \tau}{\nu(1_{\mathbb{X}}) + \tau} \quad \text{where } \tau := \max(0, L - \nu(V)). \quad (33)$$

In the unpublished paper [Hen06, Th. IV.2], the quasi-compactness of a Markov kernel Q on \mathcal{B}_1 is proved under the assumptions of Theorem 5.4, and the V -geometrical ergodicity of Q is then obtained as a corollary under the additional irreducibility and aperiodicity assumptions (see [Hen06, Cor. IV.3] and the related classical references therein). The bound obtained for $r_{ess}(Q)$ in [Hen06, Th. IV.2] is less tractable than (33) since it is expressed in terms of the hitting time for S . Also mention that, in the unpublished paper [Del97], the quasi-compactness of Markov kernels is obtained on the subspace of continuous functions of \mathcal{B}_1 under some drift/minorization conditions. No bound on the essential spectral radius is presented in [Del97].

The short proof of Theorem 5.4 illuminates the role of the drift and minorization conditions to obtain good spectral properties of Q on \mathcal{B}_1 . In particular, using [Hen06, Cor. IV.3], this provides a simple proof of the fact that under the assumptions of Theorem 5.4, together with irreducibility and aperiodicity assumptions, Q is V -geometrically ergodic.

Proof of Theorem 5.4. Condition (29a) for Q implies that $QV \leq \delta V + L1_{\mathbb{X}}$. Thus Q is power-bounded on \mathcal{B}_1 . Then, from $Q1_{\mathbb{X}} = 1_{\mathbb{X}}$ and $1_{\mathbb{X}} \in \mathcal{B}_1$, we have $r(Q) = 1$. Moreover, since $\|QV\|_1 < \infty$, we deduce from (29b) that $\nu(V) < \infty$. Thus we can define the following

rank-one operator on \mathcal{B}_1 : $Tf := \nu(f)1_S$. Set $R := Q - T$. From $T \geq 0$ and from (29b), it follows that $0 \leq R \leq Q$, so $r(R) \leq 1$. Set $r := r(R)$. If $r = 0$, then Q is quasi-compact with $r_{ess}(Q) = 0$ from (32). Now assume that $r \in (0, 1]$. Then, we know from [Sch71, App., Cor.2.6] that there exists a nontrivial nonnegative continuous linear form η on \mathcal{B}_1 such that $\eta \circ R = r\eta$. From $Q = T + R$, we have $\eta \circ Q = \eta \circ T + r\eta$, thus $\eta(Q1_{\mathbb{X}}) = \eta(1_{\mathbb{X}}) = \eta(T1_{\mathbb{X}}) + r\eta(1_{\mathbb{X}})$. Hence $\eta(T1_{\mathbb{X}}) = (1 - r)\eta(1_{\mathbb{X}})$, from which we deduce that

$$\eta(1_S) = \frac{(1 - r)\eta(1_{\mathbb{X}})}{\nu(1_{\mathbb{X}})} \leq \frac{(1 - r)\eta(V)}{\nu(1_{\mathbb{X}})}.$$

Next, we have $RV = QV - TV = QV - \nu(V)1_S \leq \delta V + L1_S - \nu(V)1_S = \delta V + (L - \nu(V))1_S$. Hence, setting $\tau := \max(0, L - \nu(V)) \geq 0$,

$$r\eta(V) = \eta(RV) \leq \delta\eta(V) + \tau\eta(1_S) \leq \delta\eta(V) + \tau\frac{(1 - r)\eta(V)}{\nu(1_{\mathbb{X}})}. \quad (34)$$

Since $\eta \neq 0$, we have $\eta(V) > 0$, and since $\delta \in (0, 1)$, we cannot have $r = 1$. Thus $r \in (0, 1)$, and Q is quasi-compact from (32) with $r_{ess}(Q) \leq r(Q - T) = r$. Then (34) gives (33). \square

Remark 5.5 Recall that $A \in \mathcal{X}$ is said to be an atom with respect to the Markov kernel Q if for any $(a, a') \in A$, $Q(a, \cdot) = Q(a', \cdot)$. Any Markov model having a regenerative structure is concerned with such a property (e.g. see [Num84, Asm03]). If Q satisfies (29a) with $S := A$, then

$$r_{ess}(Q) \leq \delta.$$

Indeed, note that Q satisfies the minorization condition (29b) with A and $\nu(1_{\cdot}) := Q(a_0, \cdot)$ for any $a_0 \in A$. Choose $L := \sup_{x \in A} (QV)(x)$ in (29a). Since A is an atom, we have $L = (QV)(a_0) = \nu(V)$ so that $\tau = 0$ in (33).

Proof of Corollary 5.1. We must prove that Condition **(ESR)** holds with $\hat{k} := k_S$. To that effect, Theorem 5.4 is applied to P_k and P . Note that, for any $k \geq k_S$, the Markov kernel P_k satisfies the drift/minorization conditions (29a)-(29b) w.r.t. V_k and set S . Indeed, let $x \in B_k$. First we have for any $F \in \mathcal{X}_k$: $P_k(x, F) \geq P(x, F) \geq \nu(1_F)1_S(x)$. Thus P_k satisfies the minorization condition with set S and restriction of ν to (B_k, \mathcal{X}_k) . Second, we obtain $P(V1_{B_k^c})(x) \geq V(x_k)P(x, B_k^c)$. Next, from

$$(P_k V_k)(x) = (PV)(x) - (P(V1_{B_k^c}))(x) + V(x_k)P(x, B_k^c),$$

it follows that $(P_k V_k)(x) \leq (PV)(x) \leq \delta V_k(x) + L1_S(x)$. Now set $\tau_k := \max(0, L - \nu(V1_{B_k}))$ for any $k \geq 1$. Then applying Theorem 5.4 to P_k gives $r_{ess}(P_k) \leq (\delta\nu(1_{B_k}) + \tau_k) / (\nu(1_{B_k}) + \tau_k)$, where $r_{ess}(P_k)$ is the essential spectral radius of P_k acting on the Banach space composed of measurable functions $f : B_k \rightarrow \mathbb{C}$ such that $\sup_{x \in B_k} |f(x)|/V(x) < \infty$ (which defines the norm on this space). Finally, it is easily checked that $r_{ess}(\hat{P}_k) \leq r_{ess}(P_k)$, so that we obtain

$$\forall k \geq k_S, \quad r_{ess}(\hat{P}_k) \leq \frac{\delta\nu(1_{B_k}) + \tau_k}{\nu(1_{B_k}) + \tau_k} \leq \frac{\delta\nu(1_{B_{k_S}}) + \tau_{k_S}}{\nu(1_{B_{k_S}}) + \tau_{k_S}}. \quad (35)$$

The last inequality follows from the monotonicity of maps $t \mapsto (\delta t + \tau_k)/(t + \tau_k)$ and $t \mapsto (\delta \nu(1_{B_{k_S}}) + t)/(\nu(1_{B_{k_S}}) + t)$. Finally, with $\tau := \max(0, L - \nu(V))$, Theorem 5.4 gives

$$r_{ess}(P) \leq \frac{\delta \nu(1_{\mathbb{X}}) + \tau}{\nu(1_{\mathbb{X}}) + \tau} \leq \frac{\delta \nu(1_{B_{k_S}}) + \tau_{k_S}}{\nu(1_{B_{k_S}}) + \tau_{k_S}}.$$

When S is an atom for P , then the property $\hat{r} \leq \delta$ follows from Remark 5.5 since S is also an atom for P_k . \square

5.2 Bound on $r_{ess}(Q)$ under a weak drift condition

Let Q be a nonnegative kernel on $(\mathbb{X}, \mathcal{X})$ acting on \mathcal{B}_1 and such that $r(Q) = 1$. The key idea to obtain quasi-compactness in Proposition 5.6 below is to use the dual Doeblin-Fortet inequality obtained under Condition (WD) in Lemma 2.3. Despite its great simplicity, this duality approach seems to be unknown in the literature. In particular it allows us to greatly simplify the arguments used in [Wu04] since the well known statement [Hen93, Cor. 1] gives the bound $r_{ess}(Q) \leq \delta$ provided that Q^ℓ is compact from \mathcal{B}_0 to \mathcal{B}_1 for some $\ell \geq 1$. Simple sufficient conditions for this compactness property are presented in [GHL11]. For instance, this holds for any discrete Markov chains, for some CHARN models or for functional autoregressive models on $\mathbb{X} := \mathbb{R}^q$ with absolutely continuous noise with respect to the Lebesgue measure. Our compactness condition is much simpler than the assumptions of [Wu04] based on sophisticated parameters $\beta_w(Q)$ and $\beta_\tau(Q)$ as measure of non-compactness of Q . Precise comparisons with [Wu04] and complements are presented in [GHL11, Sect. 2.3].

Proposition 5.6 *Let Q be a nonnegative kernel on $(\mathbb{X}, \mathcal{X})$ acting on \mathcal{B}_1 such that $r(Q) = 1$. If Q satisfies Condition (WD) and if $Q^\ell : \mathcal{B}_0 \rightarrow \mathcal{B}_1$ is compact for some $\ell \geq 1$, then Q is a power-bounded quasi-compact operator on \mathcal{B}_1 , with $r_{ess}(Q) \leq \delta$.*

Proof. Iterating (WD) easily ensures that $\sup_k \|Q^k V\|_1 < \infty$, so that Q is power-bounded on \mathcal{B}_1 . Since $r_{ess}(Q) = (r_{ess}(Q^\ell))^{1/\ell}$, we only consider the case $\ell := 1$, that is $Q : \mathcal{B}_0 \rightarrow \mathcal{B}_1$ is compact. Let \mathcal{B}'_1 and \mathcal{B}'_0 denote the dual spaces of \mathcal{B}_1 and \mathcal{B}_0 respectively. Let Q' denote the adjoint operator of Q on \mathcal{B}'_1 . In fact, we prove that Q' is a quasi-compact operator on \mathcal{B}'_1 with $r_{ess}(Q') \leq \delta$, so that Q satisfies the same properties on \mathcal{B}_1 . Since the operator $Q : \mathcal{B}_0 \rightarrow \mathcal{B}_1$ is assumed to be compact, so is $Q' : \mathcal{B}'_1 \rightarrow \mathcal{B}'_0$. Then we deduce from the Doeblin-Fortet inequality of Lemma 2.3 and from [Hen93, Cor. 1] that Q' is a quasi-compact operator on \mathcal{B}'_1 with $r_{ess}(Q') \leq \delta$. \square

Recall that, if P satisfies Condition (WD), then so does each \hat{P}_k (Lemma 2.4). Thus \hat{P}_k is power-bounded. Moreover we have $\hat{P}_k 1_{B_k} = 1_{B_k}$. It follows that the sub-Markov kernel \hat{P}_k is such that $r(\hat{P}_k) = 1$. Then Corollary 5.2 clearly follows from Proposition 5.6, Lemma 2.4 and the next lemma.

Lemma 5.7 *If P is compact from \mathcal{B}_0 to \mathcal{B}_1 , then so is each sub-Markov operator \hat{P}_k .*

Proof. For the sake of simplicity, the index k in \hat{P}_k and B_k is omitted. Let $\{f_n\}_{n \in \mathbb{N}}$ be a sequence in the unit ball of \mathcal{B}_0 . Since P is compact from \mathcal{B}_0 into \mathcal{B}_1 and $\{f_n(x_k)\}_{n \in \mathbb{N}}$ is

bounded in \mathbb{C} , there exists a subsequence $n_j \uparrow +\infty$ such that $\lim_j P(1_B f_{n_j}) = g$ in \mathcal{B}_1 and $\lim_j f_{n_j}(x_k) = z_0$ for some $g \in \mathcal{B}_1$ and $z_0 \in \mathbb{C}$. From $\widehat{P}f_{n_j} = 1_B[P(1_B f_{n_j}) + f_{n_j}(x_k)P(\cdot, B^c)]$, it follows that $\lim_j \widehat{P}f_{n_j} = 1_B(g + z_0 P(\cdot, B^c))$ in \mathcal{B}_1 . Thus \widehat{P} is compact from \mathcal{B}_0 into \mathcal{B}_1 . \square

Remark 5.8 *If Conditions (29a)-(29b) are fulfilled for some iterate Q^N in place of Q (with parameters $\delta_N < 1$, $L_N > 0$ and positive measure $\nu_N(\cdot)$), then the conclusions of Theorem 5.4 are valid with (33) replaced by*

$$r_{\text{ess}}(Q) = r_{\text{ess}}(Q^N)^{1/N} \leq \left(\frac{\delta_N \nu_N(1_{\mathbb{X}}) + \tau_N}{\nu_N(1_{\mathbb{X}}) + \tau_N} \right)^{\frac{1}{N}} \quad \text{where } \tau_N := \max(0, L_N - \nu_N(V)).$$

Similarly Proposition 5.6 still holds when **(WD)** is replaced by the following condition

$$\exists \delta \in (0, 1), \exists N \in \mathbb{N}^*, \exists L > 0, \quad Q^N V \leq \delta^N V + L 1_{\mathbb{X}}. \quad (\text{WDN})$$

Indeed the dual Doeblin-Fortet of Lemma 2.3 extends under **(WDN)** by replacing Q' and δ by Q'^N and δ^N respectively. In some cases, $r_{\text{ess}}(Q)$ can be computed by using **(WDN)**, see [GHL11].

6 Complements on the weak perturbation for truncation

Assume that Conditions **(P)** hold. In this section, we use the second part of [Liv01, Prop. 3.1] and the spectral rank-stability property [Liv01, Cor. 3.1] to prove the two following properties:

- if P is V -geometrically ergodic, then P_k is V_k -geometrically ergodic for k large enough. See Proposition 6.1 (which gives a more precise conclusion);
- there exists an integer \tilde{k} such that, for every $k \geq \tilde{k}$, the condition $\|P - \widehat{P}_k\|_{0,1} \leq \varepsilon_1(k)$ holds, where $\varepsilon_1(k)$ is defined in (23a). See Proposition 6.4.

As discussed in Sections 3 and 4, the previous properties may be valid for k much smaller than the integers k_0 and \tilde{k} provided in this section. It is the reason why these two integers are not introduced in Theorems 3.1 and 4.1. However, since the V_k -geometrical ergodicity of P_k is required for some k in these two theorems, as well as the condition $\|P - \widehat{P}_k\|_{0,1} \leq \varepsilon_1(k)$ in the second theorem, obtaining the theoretical existence of k_0 and \tilde{k} is important.

6.1 Complements on Theorem 3.1

For any $r \in (0, 1)$, set $\eta \equiv \eta(r) := 1 - \ln r / \ln \hat{\alpha} \in (0, 1)$, and with $\varepsilon_1 \equiv \varepsilon_1(r, \vartheta, P)$ defined in (12c)

$$\varepsilon_2 \equiv \varepsilon_2(r, \vartheta, P) := \left\{ \frac{r^{n_1}}{4L(H(2L+3) + 2(1+L) + (1-r)^{-1})} \right\}^{\frac{1}{\eta}}, \quad (36a)$$

$$\text{and } \varepsilon_0 \equiv \varepsilon_0(r, \vartheta, P) := \min(\varepsilon_1, \varepsilon_2). \quad (36b)$$

We know from Lemma 2.1 that a sufficient condition for inequality $\|\widehat{P}_k - P\|_{0,1} \leq \varepsilon_0$ to hold is that $k \geq k_0$, with $k_0 \equiv k_0(\vartheta, r, P)$ defined as

$$k_0(r, \vartheta, P) := \inf \left\{ k \in \mathbb{N}, \quad v(k) \geq \frac{\max(2(\delta + L), 1)}{\varepsilon_0} \right\}. \quad (37)$$

Proposition 6.1 *Assume that P is V -geometrically ergodic with rate ρ and that Conditions **(P)** hold. Let $\vartheta > 0$ be such that $\max(\hat{\alpha}, \rho) < 1 - 2\vartheta$ and r be such that $\max(\hat{\alpha}, \rho) + \vartheta < r < 1 - \vartheta$. If $\|\widehat{P}_k - P\|_{0,1} \leq \varepsilon_0$ for some $k \in \mathbb{N}^*$, then Inequality (V_k) holds with $\rho_k := r$. In particular this conclusion is valid for any $k \geq k_0$.*

Note that, since (typically) $\eta \approx 1/3$ in (36a), we have $\varepsilon_0 = O(1/H^3)$, while ε_1 in (12c) is only $O(1/H)$. Consequently $k_0 \approx v^{-1}(H^3)$ may be very large.

Using duality as in the proof of Theorems 3.1, Proposition 6.1 is based on the next lemma which follows from the rank-stability property of spectral projections [Liv01, cor. 3.1].

Lemma 6.2 *Assume that Condition **(P)** holds and that, for $r > \hat{\alpha}$ and $\vartheta > 0$, we have $\|\widehat{P}_k - P\|_{0,1} \leq \varepsilon_0(r, \vartheta, P)$. Then the spectral projections of P' and \widehat{P}'_k associated with any connected component of $\mathcal{V}(r, \vartheta, P)$ (not containing 0) have the same rank.*

Proof of Proposition 6.1. From the V -geometrical ergodicity of P , we have $\sigma(P') = \sigma(P) \subset D(0, \rho) \cup \{1\}$. Thus it follows from (19) that $\sigma(\widehat{P}'_k) \subset \overline{D}(0, r) \cup \overline{D}(1, \vartheta)$. Next, from $\|\widehat{P}_k - P\|_{0,1} \leq \varepsilon_0$ and Lemma 6.2, we obtain that $\sigma(\widehat{P}'_k) \cap \overline{D}(1, \vartheta) = \{\lambda\}$ for some eigenvalue $\lambda \in \mathbb{C}$ of \widehat{P}'_k and that there exists an associated rank-one projection $\widehat{\Pi}'_{k,\lambda}$ on \mathcal{B}'_1 such that we have for any $\kappa \in (r, 1 - \vartheta)$:

$$\forall n \geq 1, \quad \widehat{P}'_k{}^n - \lambda^n \widehat{\Pi}'_{k,\lambda} = \frac{1}{2i\pi} \oint_{|z|=\kappa} z^n (zI - \widehat{P}'_k)^{-1} dz.$$

It follows from (19) that

$$\forall n \geq 1, \quad \|\widehat{P}'_k{}^n - \lambda^n \widehat{\Pi}'_{k,\lambda}\|_1 \leq c(r, \vartheta, P) \kappa^{n+1}. \quad (38)$$

Since κ is arbitrarily close to r , this gives the expected conclusion in Proposition 6.1 using duality and the next lemma. \square

Lemma 6.3 *The eigenvalue λ in (38) is equal to 1. Moreover there exists a P_k -invariant probability measure π_k on (B_k, \mathcal{X}_k) such that $\pi_k(V_k) < \infty$, and the rank-one projection $\widehat{\Pi}'_{k,\lambda}$ is the adjoint of the following rank-one projection $\widehat{\Pi}_k$ on \mathcal{B}_1 :*

$$\forall f \in \mathcal{B}_1, \quad \widehat{\Pi}_k f := \pi_k(f_{B_k}) 1_{B_k}.$$

Proof. Since $|\lambda| > \kappa$, we deduce from $\widehat{P}_k 1_{B_k} = 1_{B_k}$ and (38) that, for any $f' \in \mathcal{B}'_1$, we have $\lim_n \lambda^{-n} f'(1_{B_k}) = \lim_n \lambda^{-n} (\widehat{P}'_k{}^n f')(1_{B_k}) = (\widehat{\Pi}'_{k,\lambda} f')(1_{B_k})$. Since there exists $f' \in \mathcal{B}'_1$ such that $f'(1_{B_k}) \neq 0$, $\{\lambda^{-n}\}_{n \in \mathbb{N}}$ converges in \mathbb{C} . Thus either $\lambda = 1$, or $|\lambda| > 1$. Moreover the

sequence $\{\widehat{P}_k^n\}_{n \in \mathbb{N}}$ is bounded in $\mathcal{L}(\mathcal{B}_1)$ from Lemma 2.4. Thus $\{\widehat{P}_k'^n\}_{n \in \mathbb{N}}$ is bounded in $\mathcal{L}(\mathcal{B}'_1)$, so that Inequality (38) implies that $\{\lambda^n\}_{n \in \mathbb{N}}$ is bounded in \mathbb{C} . Therefore $\lambda = 1$.

Now we omit λ in $\widehat{\Pi}'_{k,\lambda}$. We can prove as in [FHL13, proof of Th. 1] that $\widehat{\Pi}'_k$ is the adjoint of the rank-one projection $\widehat{\Pi}_k$ defined on \mathcal{B}_1 by:

$$\forall f \in \mathcal{B}_1, \quad \widehat{\Pi}_k f := \lim_n \widehat{P}_k^n f \text{ in } \mathcal{B}_1.$$

Moreover, since $\widehat{\Pi}_k$ is rank-one, it has the form: $\forall f \in \mathcal{B}_1, \widehat{\Pi}_k f = \widehat{e}_k'(f) \phi_k$ for some non-negative element $\widehat{e}_k' \in \mathcal{B}'_1$ and some nonnegative function $\phi_k \in \mathcal{B}_1$. From $\widehat{P}_k 1_{B_k} = 1_{B_k}$, it is easily checked that $\phi_k = 1_{B_k}$ (up to a multiplicative constant). Thus $\widehat{\Pi}_k$ may be assumed of the form $\widehat{\Pi}_k = \widehat{e}_k'(\cdot) 1_{B_k}$. More precisely, from $f = f 1_{B_k} + f 1_{B_k^c}$ and $\widehat{P}_k(f 1_{B_k^c}) = 0$, we obtain $\widehat{\Pi}_k f = \widehat{\Pi}_k(f 1_{B_k})$, hence: $\forall f \in \mathcal{B}_1, \widehat{e}_k'(f) = \widehat{e}_k'(f 1_{B_k})$. It follows from the Vitaly-Hahn-Sacks theorem that $\widehat{e}_k'(f 1_{B_k}) = \pi_k(f 1_{B_k})$ for every $f \in \mathcal{B}_1$, where π_k is a probability measure on B_k such that $\pi_k(V_k) < \infty$ (see [FHL13]). \square

6.2 Complements on Theorem 4.1

Proposition 6.4 *Assume that Conditions (P) hold. Let $\vartheta \in (0, (1-\hat{\alpha})/2)$. Then there exists $\tilde{k} \equiv \tilde{k}(\vartheta) \in \mathbb{N}^*$ such that, for every $k \in [\tilde{k}, +\infty) \cap \mathcal{I}_\vartheta$, the property $(\mathcal{E}_{0,1}(k))$ holds.*

This proposition is inspired from [Liv01, Lem. 4.2]. Since the proof in [Liv01] is only sketched and the choice of r_k (involved in (23a)) must be carefully examined, the derivation of Proposition 6.4 is detailed in this subsection. First note that, from Definition 5.3 and Condition (ESR), all the spectral values of P and \widehat{P}_k strictly larger than $\hat{\alpha}$ are eigenvalues since $\hat{\alpha} \geq \hat{r}$. More precisely, for any $R > \hat{\alpha}$, the operators P and \widehat{P}_k have a finite number of eigenvalues of modulus larger than R .

Lemma 6.5 *Assume that Conditions (P) hold. Let $R > \hat{\alpha}$ and $\theta > 0$. Then there exists $\tilde{k}_0 \equiv \tilde{k}_0(R, \theta) \in \mathbb{N}^*$ such that, for any eigenvalue of P satisfying $|\lambda| > R$ and for every $k \geq \tilde{k}_0$, the open disk $D(\lambda, \theta)$ contains at least an eigenvalue of \widehat{P}_k .*

Proof. Let \mathcal{D}_R denote the set of the eigenvalues z of P such that $|z| > R$. Define $a := R - \hat{\alpha}$, $b := \min\{|z - z'|, z, z' \in \mathcal{D}_R, z \neq z'\}$ and $c := \min\{|z| - R, z \in \mathcal{D}_R\}$. Without loss of generality we can suppose that $\theta < \min(a, b/2, c/2)$. Let $\vartheta \in (0, \theta)$ and $r := \hat{\alpha} + \theta$. Let \tilde{k}_0 be the smallest integer such that $v(\tilde{k}_0) \geq \max(2(\delta + L), 1)/\varepsilon_0$ with $\varepsilon_0 \equiv \varepsilon_0(r, \vartheta, P)$ given in (36b). From Lemma 2.1: $k \geq \tilde{k}_0 \Rightarrow \|\widehat{P}_k - P\|_{0,1} \leq \varepsilon_0$. Note that assumptions of Lemma 6.2 are satisfied.

Next, consider any $k \geq \tilde{k}_0$ and any $\lambda \in \mathcal{D}_R$. Let $\theta' \in (\vartheta, \theta)$ be such that $(zI - \widehat{P}_k)^{-1}$ is well-defined in $\mathcal{L}(\mathcal{B}_1)$ for every $z \in C(\lambda, \theta') := \{z \in \mathbb{C} : |z - \lambda| = \theta'\}$. Such θ' exists from the quasi-compactness of \widehat{P}_k . Note that $(zI - P)^{-1}$ is also well-defined in $\mathcal{L}(\mathcal{B}_1)$ for every $z \in C(\lambda, \theta')$, more precisely: $C(\lambda, \theta') \subset \mathcal{V}(r, \vartheta, P)^c$. Indeed let $z \in C(\lambda, \theta')$. Then

$$|z| \geq |\lambda| - \theta' \geq R + c - \theta' > R + 2\theta - \theta' > R + \vartheta,$$

thus $d(z, \overline{D}(0, R)) > \vartheta$. Next, let $z' \in \sigma(P)$ be such that $|z'| > R$. Then $z' \in \mathcal{D}_R$. If $z' = \lambda$ then $|z' - z| = |\lambda - z| = \theta' > \vartheta$. If $z' \neq \lambda$ then, using the triangle inequality $|z' - \lambda| \leq |z' - z| + |z - \lambda|$, we obtain $|z - z'| \geq b - \theta' > b - \theta > \theta > \vartheta$. We have proved that $d(z, \sigma(P)) > \vartheta$. Finally we have $|z| > r$ since $R > r$ (use $r = \hat{\alpha} + \theta < \hat{\alpha} + a = \hat{\alpha} + R - \hat{\alpha} = R$). Thus $z \in \mathcal{V}(r, \vartheta, P)^c$.

Now, the spectral projections

$$\Pi'_\lambda := \frac{1}{2i\pi} \oint_{C(\lambda, \theta')} (zI - P')^{-1} dz \quad \text{and} \quad \Pi'_{k, \lambda} := \frac{1}{2i\pi} \oint_{C(\lambda, \theta')} (zI - \widehat{P}'_k)^{-1} dz$$

have the same rank from $k \geq \widetilde{k}_0$ and Lemma 6.2. Since Π'_λ has a nonzero rank from $\lambda \in \sigma(P')$, so is $\Pi'_{k, \lambda}$. Thus we have $D(\lambda, \theta') \cap \sigma(\widehat{P}_k) \neq \emptyset$. \square

Now, to prove Proposition 6.4, we consider any $\vartheta \in (0, (1 - \hat{\alpha})/2)$ and we set $\tilde{r} := \hat{\alpha} + \vartheta/2$.

Lemma 6.6 *There exists $\widetilde{k}_0 \equiv \widetilde{k}_0(\tilde{r}, \vartheta) \in \mathbb{N}^*$ such that $\forall k \geq \widetilde{k}_0$, $\mathcal{V}(\tilde{r}, \vartheta/4, P) \subset \mathcal{V}(\tilde{r}, \vartheta, \widehat{P}_k)$.*

Proof. Let $u \in \mathcal{V}(\tilde{r}, \vartheta/4, P)$. Thus $|u| \leq \tilde{r}$ or $d(u, \sigma(P)) \leq \vartheta/4$. If $|u| \leq \tilde{r}$, then $u \in \mathcal{V}(\tilde{r}, \vartheta, \widehat{P}_k)$. Now assume that $|u| > \tilde{r}$ and $d(u, \sigma(P)) \leq \vartheta/4$. Since $\sigma(P)$ is compact, there exists $\lambda \in \sigma(P)$ such that $|u - \lambda| \leq \vartheta/4$. We have $|\lambda| > \hat{\alpha} + \vartheta/4$ from

$$|\lambda| \geq |u| - \frac{\vartheta}{4} > \tilde{r} - \frac{\vartheta}{4} = \hat{\alpha} + \frac{\vartheta}{4}.$$

Then it follows from Lemma 6.5 with $R := \hat{\alpha} + \vartheta/4$ and $\theta := \vartheta/4$ that there exists $\widetilde{k}_0 \equiv \widetilde{k}_0(R, \theta) \in \mathbb{N}^*$ such that, for every $k \geq \widetilde{k}_0$, the disk $D(\lambda, \vartheta/4)$ contains an eigenvalue of \widehat{P}_k , say λ_k . We obtain $d(u, \sigma(\widehat{P}_k)) \leq \vartheta$ since $|u - \lambda_k| \leq |u - \lambda| + |\lambda - \lambda_k| \leq \vartheta/2$. Thus $u \in \mathcal{V}(\tilde{r}, \vartheta, \widehat{P}_k)$. \square

From the definition of $\mathcal{V}(\tilde{r}, \vartheta, \widehat{P}_k)$, we have: $z \in \mathcal{V}(\tilde{r}, \vartheta, \widehat{P}_k)^c \Rightarrow d(z, \sigma(\widehat{P}_k)) > \vartheta$. Thus, the following constant is well-defined for every $k \geq 1$:

$$\widetilde{H}_k := \sup_{z \in \mathcal{V}(\tilde{r}, \vartheta, \widehat{P}_k)^c} \|(zI - \widehat{P}_k)^{-1}\|_1. \quad (39)$$

Lemma 6.7 *The sequence $\{\widetilde{H}_k\}_{k \geq 1}$ is bounded.*

Proof. Let $\varepsilon_1(\tilde{r}, \vartheta/4, P)$ be defined as in (12c). From Lemma 2.1, there exists $k_1 \equiv k_1(\tilde{r}, \vartheta) \in \mathbb{N}^*$ such that

$$\forall k \geq k_1, \quad \|\widehat{P}_k - P\|_{0,1} \leq \varepsilon_1(\tilde{r}, \vartheta/4, P).$$

It follows from Lemma 6.6 that

$$\forall k \geq \widetilde{k}_0, \quad \mathcal{V}(\tilde{r}, \vartheta, \widehat{P}_k)^c \subset \mathcal{V}(\tilde{r}, \vartheta/4, P)^c$$

so that, for every $k \geq \max(\widetilde{k}_0, k_1)$ we have from (18) with $r := \tilde{r}$ and $\vartheta/4$ in place of ϑ :

$$\sup_{z \in \mathcal{V}(\tilde{r}, \vartheta, \widehat{P}_k)^c} \|(zI - \widehat{P}_k)^{-1}\|_1 \leq \sup_{z \in \mathcal{V}(\tilde{r}, \vartheta/4, P)^c} \|(zI - \widehat{P}_k)^{-1}\|_1 \leq c(\tilde{r}, \vartheta/4, P) < \infty.$$

This gives the expected assertion. \square

Proof of Proposition 6.4. Let $H_k \equiv H(r_k, \vartheta, \widehat{P}_k)$ be defined by (23b). Then $H_k \leq \widetilde{H}_k$. Indeed we have $\mathcal{V}(r_k, \vartheta, \widehat{P}_k)^c \subset \mathcal{V}(\tilde{r}, \vartheta, \widehat{P}_k)^c$ since (use (25))

$$\tilde{r} = \hat{\alpha} + \frac{\vartheta}{2} \leq \max(\hat{\alpha}, \rho_k) + \vartheta < r_k.$$

It follows from Lemma 6.7 that $\{H_k\}_{k \geq 1}$ is bounded. Then, the sequences $\{n_1(k)\}_{k \geq 1}$ and $\{n_2(k)\}_{k \geq 1}$ given in (23c) are bounded since $\hat{\alpha} + \vartheta < r_k < 1 - \vartheta$. Therefore, the sequence $\{\varepsilon_1(k)\}_{k \geq 1}$ in (23a) is uniformly bounded away from zero, that is $\alpha_1 := \inf_{k \geq 1} \varepsilon_1(k) > 0$. Finally, from Lemma 2.1, there exists $\tilde{k} \in \mathbb{N}$ such that: $\forall k \geq \tilde{k}$, $\|\widehat{P}_k - P\|_{0,1} \leq \alpha_1$. Thus, for every $k \geq \tilde{k}$, $\|\widehat{P}_k - P\|_{0,1} \leq \varepsilon_1(k)$. \square

7 Conclusion

In all the cases when the probabilistic works cited in Introduction do not provide a satisfactory rate ρ in (V) , Theorem 4.1 can be applied to obtain a new rate r_k and constant c_k in (V) from (ρ_k, C_k) in (V_k) (see (26a)). This new pair (r_k, c_k) will be all the more interesting that

1. the pair (ρ_k, C_k) in (V_k) is precise;
2. the bound \hat{r} on essential spectral radii in Condition **(ESR)** is accurate.

The corresponding bound (28) for $\|\widehat{\pi}_k - \pi\|_{TV}$ then involves a constant depending on both (ρ_k, C_k) and (r_k, c_k) . The point 1. is of computational nature. The point 2. is addressed in Corollaries 5.1-5.2. In several cases, \hat{r} is less than the contractive coefficient δ in **(WD)** (see Corollary 5.2 and the atomic case in Corollary 5.1). It would be of interest to know whether inequality $\hat{r} \leq \delta$ extends to other situations, and more generally whether the general bound in Corollary 5.1 can be improved.

Our approach to solve Issues (Q1)-(Q2) could be used in other contexts where the weak perturbation theorem is applied. In particular, for issue (Q2) in the context of dynamical systems, the whole results of [Liv01] could be used in a first stage to obtain a preliminary (possibly poor) rate of convergence ρ for the unperturbed Perron-Frobenius operator. Then, applying Theorem 4.1 in a second stage will provide new (better in general) rates of convergence and associated constants. Finally Proposition 2.6 will give an explicit bound for the difference between perturbed and unperturbed SRB measures, which should be better than the one derived from the spectral rank-stability property of [Liv01, Cor. 3.1] and the second part of [Liv01, Prop. 3.1].

A Complements on Remark 3.2

Let us prove the inequality of Remark 3.2 under (V) . For any $g \in \mathcal{B}_1$, write $g = (g - \pi(g)1_{\mathbb{X}}) + \pi(g)1_{\mathbb{X}}$. We have

$$(zI - P)^{-1}(\pi(g)1_{\mathbb{X}}) = \frac{\pi(g)}{z - 1}1_{\mathbb{X}},$$

while we obtain from (V) applied to the function $g - \pi(g)1_{\mathbb{X}}$ and from $|z| > \rho$:

$$(zI - P)^{-1}(g - \pi(g)1_{\mathbb{X}}) = \sum_{n \geq 0} z^{-(n+1)} P^n (g - \pi(g)1_{\mathbb{X}}).$$

It follows from $|\pi(g)| \leq \pi(V)\|g\|_1$ that

$$\begin{aligned} \|(zI - P)^{-1}g\|_1 &= \left\| \sum_{n \geq 0} z^{-(n+1)} P^n (g - \pi(g)1_{\mathbb{X}}) + \frac{\pi(g)}{z-1} 1_{\mathbb{X}} \right\|_1 \\ &\leq \left(C \sum_{n \geq 0} |z|^{-(n+1)} \rho^n + \frac{\pi(V)}{|z-1|} \|1_{\mathbb{X}}\|_1 \right) \|g\|_1 \\ &\leq \left(\frac{C}{|z| - \rho} + \frac{\pi(V)\|1_{\mathbb{X}}\|_1}{|z-1|} \right) \|g\|_1. \end{aligned}$$

The expected inequalities in Remark 3.2 are obtained using $|z| - \rho > r - \rho > \vartheta$ and $|z-1| > \vartheta$ in the definition of $H(r, \vartheta, P)$.

B Further comments on the results of [Liv01]

The use of the second part of [Liv01, Prop. 3.1] provides larger bounds for $\|\widehat{\pi}_k - \pi\|_{TV}$ than in (16). In fact, the following inequality can be derived from [KL99, Liv01] (by using duality as in the proof of Theorem 3.1 and Proposition 6.1):

$$\sup_{z \in \mathcal{V}(r, \vartheta, T)^c} \|(zI - \widehat{P}'_k)^{-1} - (zI - P')^{-1}\|_{01} \leq d \|\widehat{P}_k - P\|_{0,1}^{\eta}, \quad (40)$$

with $d := aH + bH^2$, where $a \equiv a(r)$ and $b \equiv b(r)$ are defined by

$$a := \frac{1 + 8(1+L)^2(2(1+L) + (1-r)^{-1})}{(1-r)r^{n_1}} \quad b := 2 \left\{ L + \frac{4(1+L)^2(2L+3) + L}{(1-r)r^{n_1}} \right\}.$$

Using standard spectral calculus, [Liv01, Cor. 3.1] (see Lemma 6.2) and finally Lemma 2.1, we obtain

$$\forall k \geq k_0, \quad \|\widehat{\pi}_k - \pi\|_{TV} \leq \frac{d \vartheta \max(2^\eta(\delta + L)^\eta, 1)}{v(k)^\eta}.$$

The previous bound is worst than in (16) since $d = O(H^2)$ while c in bound (16) is $c \equiv c(r, \vartheta, P) = O(H)$ (see (14b)).

The previous comments are also valid for Theorem 4.1 by replacing P by \widehat{P}_k , so H by H_k .

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