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► To cite this version:

Déborah Ferré, Loïc Hervé, James Ledoux. Regular perturbation of V -geometrically ergodic Markov chains. Journal of Applied Probability, 2013, 50 (1), pp.184-194. hal-00704685v2

HAL Id: hal-00704685

<https://hal.science/hal-00704685v2>

Submitted on 17 Jun 2012

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Regular perturbation of V -geometrically ergodic Markov chains

Déborah FERRÉ, Loïc HERVÉ, James LEDOUX *

June 17, 2012

Abstract

In this paper, new conditions for the stability of V -geometrically ergodic Markov chains are introduced. The results are based on an extension of the standard perturbation theory formulated by Keller and Liverani. The continuity and higher regularity properties are investigated. As an illustration, an asymptotic expansion of the invariant probability measure for an autoregressive model with i.i.d. noises (with a non-standard probability density function) is obtained.

AMS subject classification : 60J05, 47B07

Keywords : Stability, Spectral method

1 Introduction and statements

Let $\{P_\varepsilon\}_{|\varepsilon| < \varepsilon_0}$ be a family of transition kernels on a measurable space $(\mathbb{X}, \mathcal{X})$, where ε reads as a small perturbation parameter. Throughout the paper $V : \mathbb{X} \rightarrow [1, +\infty)$ is a fixed function. The (unperturbed) kernel P_0 is assumed to satisfy the classical V -geometrical ergodicity property, called **(VG)** in our paper, that is: P_0 admits a unique invariant probability measure π_0 , we have $\pi_0(V) < \infty$, and there exists some constants $c \in (0, +\infty)$ and $\kappa_1 \in (0, 1)$ such that:

$$\forall x \in \mathbb{X}, \quad \sup_{|f| \leq V} |\mathbb{E}[f(X_n) | X_0 = x] - \pi_0(f)| \leq c \kappa_1^n V(x). \quad (\mathbf{VG})$$

or equivalently

$$\forall x \in \mathbb{X}, \quad \sup_{|f| \leq V} |P_0^n f(x) - \pi_0(f)| \leq c \kappa_1^n V(x).$$

This means that P_0 has a spectral gap on the weighted supremum normed space $\mathcal{B}_V$ composed of the measurable functions $f : \mathbb{X} \rightarrow \mathbb{C}$ such that $\|f\|_V := \sup_{x \in \mathbb{X}} V(x)^{-1} |f(x)| < \infty$. We are interested in the two following questions. For $|\varepsilon|$ small enough,

(I) does P_ε admit an invariant probability measure, say π_ε , and is P_ε V -geometrically ergodic?

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(II) Do we have any control on $\pi_0 - \pi_\varepsilon$?

Under some classical aperiodicity and irreducibility conditions, the property **(VG)** holds true if and only if P_0 satisfies the so-called drift condition based on the notion of small set (see [MT93] for the definition of the drift condition which is not used here). Consequently a natural and efficient way to study (I) is to prove that the perturbed Markov kernel P_ε also satisfies the drift condition (e.g. see [RRS98, BRR01]). However, to the best of our knowledge, the theory of geometrical ergodic Markov chains does not provide any general answer for question (II), except in terms of weak convergence in specific cases (see [RRS98]). On the other hand the standard perturbation theory, which is a natural way to investigate (II), leads to assume the following continuity condition:

$$\|P_\varepsilon - P_0\|_{\mathcal{B}_V} := \sup_{\|f\|_V \leq 1} \|P_\varepsilon f - P_0 f\|_V \rightarrow 0 \quad \text{when } \varepsilon \rightarrow 0 \quad (1)$$

that is the operator norm of $P_\varepsilon - P_0$ on $\mathcal{B}_V$ goes to 0 as $\varepsilon \rightarrow 0$. In a series of papers, Kartashov has introduced the concept of “strongly stable Markov chain” for a Markov chain with a transition kernel P_0 such that, in some neighborhood of P_0 with respect to $\|\cdot\|_{\mathcal{B}_V}$, P_ε has a unique invariant probability measure π_ε with the property $\sup_{\|f\|_V \leq 1} |\pi_\varepsilon(f) - \pi_0(f)| \rightarrow 0$ as $\|P_\varepsilon - P_0\|_{\mathcal{B}_V} \rightarrow 0$ uniformly in this neighborhood. Strong stability is shown to be equivalent to the convergence

$$\lim_{n \rightarrow \infty} \sup_{\|f\|_V \leq 1} \left\| \frac{1}{n} \sum_{k=1}^n P_0^k f - \pi_0(f) 1_{\mathbb{X}} \right\|_V = 0.$$

Moreover, if P_0 is V -geometrically ergodic, then given $\rho \in (0, 1)$, one can consider $N \in \mathbb{N}^*$ such that $c\kappa_1^N \leq \rho$, and next if P_ε is such that $\Delta_N := \|P_0^N - P_\varepsilon^N\|_{\mathcal{B}_V} < 1 - \rho$, then P_ε is V -geometrically ergodic and $\sup_{\|f\|_V \leq 1} |\pi(f) - \pi_\varepsilon(f)| = O(\Delta_N / (1 - \rho - \Delta_N))$. We refer to [Kar96] for an overview of results in this direction and to [RA10] for a related discussion for discrete state spaces. However, as discussed in [SS00, p. 1126] and in Example 1 below, the continuity condition (1) may be restrictive.

Similar questions arise in the context of dynamical systems. To overcome the previous difficulty, Keller introduced the more general assumption

$$\lim_{\varepsilon \rightarrow 0} \sup_{\|f\|_0 \leq 1} \|P_\varepsilon f - P_0 f\|_1 = 0$$

involving two norms $\|\cdot\|_0$ and $\|\cdot\|_1$ (instead of a single one) on the space on which P_0 has a spectral gap [Kel82]. This approach has been highly enhanced by the Keller-Liverani perturbation theorem [KL99, Liv04], which has proved to be very powerful in studying the behaviour of the Sinai-Ruelle-Bowen measures of certain perturbed dynamical systems (e.g. see [Bal00, Th 2.10] and [GL06, Th. 2.8]).

The goal of this paper is to show that the Keller-Liverani theorem also provides an interesting way to investigate both the questions (I) (II) in the context of geometrical ergodic Markov chains. In this markovian context, the closest work to ours is [SS00] where Keller’s approach is used. The results of [SS00] are improved here thanks to the Keller-Liverani perturbation theorem. Furthermore in this paper, higher regularity properties than continuity are investigated in question (II). Mention that the results of [KL99, Liv04] have been already used in [Fer12] to study some stability properties of parametric autoregressive models (for different purposes from those of Proposition 1 below).

Notations. For $\beta \in [0, 1]$, we denote by $(\mathcal{B}_\beta, \|\cdot\|_\beta)$ the Banach space composed of the measurable functions $f : \mathbb{X} \rightarrow \mathbb{C}$ such that $\|f\|_\beta := \sup_{x \in \mathbb{X}} V(x)^{-\beta} |f(x)| < \infty$. Note that $\mathcal{B}_0$ corresponds to the space of bounded measurable functions on $\mathbb{X}$, with $\|f\|_0 = \sup_{x \in \mathbb{X}} |f(x)|$, and that $\mathcal{B}_1 = \mathcal{B}_V$. We denote by $(\mathcal{L}(\mathcal{B}_\beta, \mathcal{B}_{\beta'}), \|\cdot\|_{\beta, \beta'})$ the space of all the bounded linear maps from $\mathcal{B}_\beta$ to $\mathcal{B}_{\beta'}$, equipped with its usual norm: $\|T\|_{\beta, \beta'} = \sup \{\|Tf\|_{\beta'}, f \in \mathcal{B}_\beta, \|f\|_\beta \leq 1\}$. We write $\mathcal{L}(\mathcal{B}_\beta)$ for $\mathcal{L}(\mathcal{B}_\beta, \mathcal{B}_\beta)$ and $\|T\|_\beta$ for $\|T\|_{\beta, \beta}$ which is a slight abuse of notation. Let $(\mathcal{B}'_\beta, \|\cdot\|_{\mathcal{B}'_\beta})$ denote the dual space of $\mathcal{B}_\beta$. If $T \in \mathcal{L}(\mathcal{B}_\beta)$, then T^* stands for the adjoint operator of T . By definition we have $T^* \in \mathcal{L}(\mathcal{B}'_\beta)$ with the corresponding operator norm $\|T^*\|_{\mathcal{B}'_\beta} = \|T\|_\beta$. Note that T^* also defines an element of $\mathcal{L}(\mathcal{B}'_\beta, \mathcal{B}'_0)$ with corresponding operator norm $\|T^*\|_{\mathcal{B}'_\beta, \mathcal{B}'_0} \leq \|T^*\|_{\mathcal{B}'_\beta}$ from the continuous inclusion $\mathcal{B}_0 \subset \mathcal{B}_\beta$.

Each perturbed Markov kernel P_ε is assumed to continuously act on $\mathcal{B}_1$. The unperturbed kernel P_0 is assumed to satisfy **(VG)**, namely: P_0 admits a unique invariant distribution π_0 on $(\mathbb{X}, \mathcal{X})$, $\pi_0(V) < \infty$, and

$$\exists \kappa_1 \in (0, 1), \quad \|P_0^n - \pi_0(\cdot)1_{\mathbb{X}}\|_1 = O(\kappa_1^n). \quad (V_1)$$

We also assume that there exist $N \in \mathbb{N}^*$, $L \in (0, +\infty)$ and $\delta \in (0, 1)$ such that

$$\forall \varepsilon \in (-\varepsilon_0, \varepsilon_0), \quad P_\varepsilon^N V \leq \delta^N V + L 1_{\mathbb{X}}. \quad (D)$$

Under some classical aperiodicity and irreducibility assumptions, Property (V_1) is equivalent to the drift condition (see [MT93, Chapter 4, Th. 16.0.1]). Condition (D) on the family $\{P_\varepsilon\}_{|\varepsilon| < \varepsilon_0}$ is weaker than the simultaneous geometrical ergodicity condition introduced in [RRS98] since (D) involves no small set.¹

Theorem 1 *Conditions (V_1) , (D) and*

$$\lim_{\varepsilon \rightarrow 0} \|P_\varepsilon - P_0\|_{0,1} = 0 \quad (2)$$

are assumed to hold. Then, setting $\hat{\kappa} := \max(\kappa_1, \delta)$, the following statements are fulfilled:

1. *for each $\kappa \in (\hat{\kappa}, 1)$, there exists $\varepsilon_1 \in (0, \varepsilon_0]$ such that, for all $\varepsilon \in (-\varepsilon_1, \varepsilon_1)$, P_ε has a unique invariant probability measure π_ε , with $\pi_\varepsilon(V) < \infty$, such that*

$$\sup_{|\varepsilon| < \varepsilon_1} \|P_\varepsilon^n - \pi_\varepsilon(\cdot)1_{\mathbb{X}}\|_1 = O(\kappa^n). \quad (3)$$

2. *We have*

$$\lim_{\varepsilon \rightarrow 0} \sup_{\|f\|_0 \leq 1} |\pi_\varepsilon(f) - \pi_0(f)| = 0. \quad (4)$$

Note that, if each probability measure π_ε admits a density, say p_ε , with respect to a fixed positive measure ψ on $(\mathbb{X}, \mathcal{X})$, then we have $\sup_{\|f\|_0 \leq 1} |\pi_\varepsilon(f) - \pi_0(f)| = \int_{\mathbb{X}} |p_\varepsilon(x) - p_0(x)| d\psi(x)$.

Conclusion (3) means that P_ε is V -geometrically ergodic in a uniform way with respect to the perturbation parameter ε . Conclusion (4) means that the total variation norm of $\pi_\varepsilon - \pi_0$ goes to

¹In case $\varepsilon = 0$ (i.e. for a single transition kernel P_0), the connection between the “best” constants κ_1 in (V_1) and δ in (D) is discussed in details in [GHL11]. See also Remark 1.

0 when $\varepsilon \rightarrow 0$. Consequently Theorem 1 provides the same theoretical conclusions that in [Kar96, Chap. 3], but under the continuity condition (2) which is weaker than that in (1). Indeed we have: $\|P_\varepsilon - P_0\|_{0,1} \leq \|P_\varepsilon - P_0\|_1$. There are several examples (see for instance [SS00]) showing that the continuity condition (1) of [Kar96] may fail, while (2) (or (9) below) holds true. The case of an AR process is investigated in Example 1 below. Note that the uniform bound (3) can be obtained from [MT94] when the kernels P_ε are assumed to satisfy the drift condition with constants and small set which do not depend on ε . To derive (3) and (4) from the weaker assumptions (V₁) and (D), some continuity assumption must be assumed on the map $\varepsilon \mapsto P_\varepsilon$.

Typically Condition (V₁) is the assumption AI in [SS00] (see the remark following their assumptions). Condition (2) is weaker than the assumption AII in [SS00] involving the sequence $(P_\varepsilon^n)_{n \geq 1}$ (here Assumption AII is only required for $n = 1$). Mention that, given any $r \in (\hat{\kappa}, 1)$ and setting $\eta := 1 - \ln r / \ln \delta$ ($\eta \in (0, 1)$), the proof of Theorem 1 and further results in [KL99] ensure the following:

$$\exists D_r \in (0, +\infty), \quad \sup_{\|f\|_0 \leq 1} |\pi_\varepsilon(f) - \pi_0(f)| \leq D_r (\|P_\varepsilon - P_0\|_{0,1})^\eta. \quad (5)$$

This provides an alternative statement to [SS00, Th. 3.1], which states an inequality of type (5) in assuming the existence of π_ε . The V -geometrical ergodicity of the perturbed kernel P_ε was an open question in [SS00]. Actually, under their assumptions, Condition (D) is a quite natural hypothesis for P_ε to inherit the V -geometrical ergodicity of the unperturbed kernel P_0 (note that Condition (D) is not so far from the drift condition). In the proof of Theorem 1, Condition (D) is viewed as a Doeblin-Fortet inequality on the dual space of $\mathcal{B}_1$ in order to use the Keller-Liverani theorem.

Example 1 (AR process) Assume that $\mathbb{X} := \mathbb{R}$ and $(X_n)_{n \in \mathbb{N}}$ is the autoregressive model defined by

$$n \in \mathbb{N}^*, \quad X_n = \alpha X_{n-1} + \vartheta_n \quad (6)$$

where X_0 is a real-valued random variable, $\alpha \in (-1, 1)$, and $(\vartheta_n)_{n \geq 1}$ is a sequence of i.i.d. real-valued random variables, independent of X_0 . Assume that ϑ_1 has a Lebesgue probability density function on $\mathbb{X}$, say $\nu(\cdot)$, and admits a first moment, $\int |x| \nu(x) dx < \infty$. We know that $(X_n)_{n \in \mathbb{N}}$ is a Markov chain with transition kernel

$$P_\alpha(x, A) = \int_{\mathbb{R}} 1_A(\alpha x + y) \nu(y) dy = \int_{\mathbb{R}} 1_A(y) \nu(y - \alpha x) dy. \quad (7)$$

Set $V(x) := 1 + |x|$, $x \in \mathbb{R}$. It is known (e.g. [Wu04, Sec. 8] or [GHL11, Sec. 5.5]) that, for each $\alpha \in (-1, 1)$, $(X_n)_{n \in \mathbb{N}}$ is V -geometrically ergodic with an invariant distribution π_α . Next, given any $\alpha_0 \in (0, 1)$, it can be easily checked that the family $\{P_\alpha, \alpha \in (-\alpha_0, \alpha_0)\}$ satisfies Condition (D) with $N = 1$ and with any $\delta \in (\alpha_0, 1)$. Finally we prove below that, for every $\alpha_0 \in (-1, 1)$, the kernels $P_{\alpha_0 + \varepsilon}$ satisfy the weak continuity condition (2). The previous facts ensure that Theorem 1 applies to the family $(P_{\alpha_0 + \varepsilon})_\varepsilon$, so that the total variation norm of $\pi_\alpha - \pi_{\alpha_0}$ goes to 0 when $\alpha \rightarrow \alpha_0$. Note that no specific assumptions are required on the density ν , excepted that it admits a first moment. To the best of our knowledge this last result is new. Anyway, it cannot be deduced from a known method as in [Kar96] since Condition (1) does not hold, as proved afterwards.

First we show that, for any $\alpha_0 \in (-1, 1)$, we have $\lim_{\alpha \rightarrow \alpha_0} \|P_\alpha - P_{\alpha_0}\|_{0,1} = 0$, that is

$$\forall \varepsilon > 0, \quad \exists \eta > 0, \quad |\alpha - \alpha_0| < \eta \implies \|P_\alpha - P_{\alpha_0}\|_{0,1} = \sup_{\|f\|_0 \leq 1} \sup_{x \in \mathbb{R}} \frac{|P_\alpha f(x) - P_{\alpha_0} f(x)|}{V(x)} < \varepsilon.$$

From (7), we have $P_\alpha f(x) = \int_{\mathbb{R}} f(y) \nu(y - \alpha x) dy$ so that

$$\|P_\alpha - P_{\alpha_0}\|_{0,1} \leq \sup_{x \in \mathbb{R}} \frac{\|\nu(\cdot - \alpha x) - \nu(\cdot - \alpha_0 x)\|_{\mathbb{L}^1(\mathbb{R})}}{V(x)}.$$

Let $\varepsilon > 0$. Since $V(x) \rightarrow +\infty$ when $x \rightarrow \infty$, we can choose a constant $A > 0$ such that $4/V(x) < \varepsilon$ for any $|x| > A$. Therefore, using that ν is a probability density function, we obtain

$$\sup_{|x| > A} \frac{\|\nu(\cdot - \alpha x) - \nu(\cdot - \alpha_0 x)\|_{\mathbb{L}^1(\mathbb{R})}}{V(x)} \leq \sup_{|x| > A} \frac{2}{V(x)} < \varepsilon.$$

Next, the map from $\mathbb{R}$ into $\mathbb{L}^1(\mathbb{R})$ defined by $t \mapsto \nu(\cdot - t)$ is continuous, so that we can pick $\eta' > 0$ such that

$$\forall |x| \leq A, \quad |\alpha x - \alpha_0 x| \leq |\alpha - \alpha_0| A < \eta' \implies \|\nu(\cdot - \alpha x) - \nu(\cdot - \alpha_0 x)\|_{\mathbb{L}^1(\mathbb{R})} < \varepsilon.$$

Therefore, given that $V \geq 1$, there exists $\eta := \eta'/A$ such that

$$|\alpha - \alpha_0| < \eta \implies \sup_{|x| \leq A} \frac{\|\nu(\cdot - \alpha x) - \nu(\cdot - \alpha_0 x)\|_{\mathbb{L}^1(\mathbb{R})}}{V(x)} \leq \sup_{|x| \leq A} \|\nu(\cdot - \alpha x) - \nu(\cdot - \alpha_0 x)\|_{\mathbb{L}^1(\mathbb{R})} < \varepsilon.$$

Second, let us check that, whatever the density $\nu(\cdot)$ is, the strong continuity condition (1) is never fulfilled. Let us consider a positive real number a such that

$$\int_{-a}^a \nu(y) dy \neq \int_{-2a}^{-a} \nu(y) dy + \int_a^{2a} \nu(y) dy.$$

Such a real number exists for any density $\nu(\cdot)$ (if the previous terms coincide for every $a > 0$, then we obtain a contradiction when $a \rightarrow +\infty$). Let $\alpha_0 \in (0, 1)$ be fixed. For each $\alpha \in (\alpha_0, 1)$, define

$$x_\alpha := \frac{a}{\alpha - \alpha_0} > 0.$$

Note that $-a + \alpha_0 x_\alpha < -a + \alpha x_\alpha < a + \alpha_0 x_\alpha < a + \alpha x_\alpha$. Next let us introduce the following function

$$\forall y \in \mathbb{R}, \quad f_\alpha(y) := y \mathbf{1}_{[a + \alpha_0 x_\alpha, a + \alpha x_\alpha]}(y) - y \mathbf{1}_{[-a + \alpha_0 x_\alpha, -a + \alpha x_\alpha]}(y).$$

From (7), we have

$$\begin{aligned} (P_\alpha f_\alpha)(x_\alpha) &= \int_0^a (y + \alpha x_\alpha) \nu(y) dy - \int_{-2a}^{-a} (y + \alpha x_\alpha) \nu(y) dy \\ (P_{\alpha_0} f_\alpha)(x_\alpha) &= \int_a^{2a} (y + \alpha_0 x_\alpha) \nu(y) dy - \int_{-a}^0 (y + \alpha_0 x_\alpha) \nu(y) dy. \end{aligned}$$

Set $J := \int_{-a}^a y \nu(y) dy - \int_{-2a}^{-a} y \nu(y) dy - \int_a^{2a} y \nu(y) dy$. We obtain

$$(P_\alpha f_\alpha)(x_\alpha) - (P_{\alpha_0} f_\alpha)(x_\alpha) = J + \alpha x_\alpha \left(\int_0^a \nu(y) dy - \int_{-2a}^{-a} \nu(y) dy \right) + \alpha_0 x_\alpha \left(\int_{-a}^0 \nu(y) dy - \int_a^{2a} \nu(y) dy \right).$$

Since $V(x_\alpha) = 1 + x_\alpha$ and $x_\alpha \rightarrow +\infty$ as $\alpha \downarrow \alpha_0$, it follows that

$$\lim_{\alpha \downarrow \alpha_0} \frac{(P_\alpha f_\alpha)(x_\alpha) - (P_{\alpha_0} f_\alpha)(x_\alpha)}{V(x_\alpha)} = \alpha_0 \left(\int_{-a}^a \nu(y) dy - \int_{-2a}^{-a} \nu(y) dy - \int_a^{2a} \nu(y) dy \right) \neq 0.$$

Finally, from

$$\|P_\alpha - P_{\alpha_0}\|_1 = \|P_\alpha - P_{\alpha_0}\|_{\mathcal{B}_V} = \sup_{|f| \leq V} \sup_{x \in \mathbb{R}} \frac{|(P_\alpha f)(x) - (P_{\alpha_0} f)(x)|}{V(x)} \geq \frac{|(P_\alpha f_\alpha)(x_\alpha) - (P_{\alpha_0} f_\alpha)(x_\alpha)|}{V(x_\alpha)},$$

we deduce that $\|P_\alpha - P_{\alpha_0}\|_{\mathcal{B}_V}$ does not go to 0 when $\alpha \downarrow \alpha_0$.

Reinforcing the continuity condition (2) of Theorem 1 allows us to obtain the following refinement.

Corollary 1 Assume that Conditions (V_1) and (D) hold and that there exists $\beta \in (0, 1]$ such that²

$$\exists \kappa_\beta \in (0, 1), \quad \|P_0^n - \pi_0(\cdot)1_{\mathbb{X}}\|_\beta = O(\kappa_\beta^n) \quad (V_\beta)$$

$$\text{and } \lim_{\varepsilon \rightarrow 0} \|P_\varepsilon - P_0\|_{\beta, 1} = 0. \quad (9)$$

Then there exist $\varepsilon_1 \in (0, \varepsilon_0]$ (from Theorem 1) and $C \in (0, +\infty)$ such that:

$$\forall \varepsilon \in (-\varepsilon_1, \varepsilon_1), \quad \sup_{\|f\|_\beta \leq 1} |\pi_\varepsilon(f) - \pi_0(f)| \leq C \|P_\varepsilon - P_0\|_{\beta, 1}.$$

Remark 1 Let P be any Markov kernel on $\mathbb{X}$. If we have $P^N V \leq \delta^N V + L 1_{\mathbb{X}}$ for some $N \in \mathbb{N}^*$, $L \in (0, +\infty)$ and $\delta \in (0, 1)$ and if P^ℓ is compact from $\mathcal{B}_0$ to $\mathcal{B}_V$ (for some $\ell \geq 1$), then P is a power-bounded quasi-compact operator on $\mathcal{B}_V$. Moreover its essential spectral radius is such that $r_{\text{ess}}(P) \leq \delta$. Actually, denoting by $\delta_V(P)$ the infimum of the real numbers δ for which the above inequality holds for some $N \in \mathbb{N}^*$ and $L \in (0, +\infty)$, we have $r_{\text{ess}}(P) \leq \delta_V(P)$, with equality in many cases. This follows from the Doeblin-Fortet inequalities (11) and [Hen93], since $(P^*)^\ell$ is compact from $\mathcal{B}'_V$ to $\mathcal{B}'_0$. If in addition P satisfies aperiodicity and irreducibility conditions, then P is V -geometrically ergodic. Such results are fully detailed and applied in [GHL11] (without any perturbation issues). Now, if $\rho_V(P)$ denotes the infimum bound of the real numbers κ_1 such that (V_1) holds, then we have $\rho_V(P) \geq r_{\text{ess}}(P)$. If there are eigenvalues λ of P on $\mathcal{B}_V$ such that $r_{\text{ess}}(P) < |\lambda| < 1$, then $\rho_V(P)$ is the maximal modulus of such eigenvalues (which are in finite number from the definition of $r_{\text{ess}}(P)$); if not, we have $\rho_V(P) = r_{\text{ess}}(P)$. For instance, if $(X_n)_{n \in \mathbb{N}}$ is defined by (6) and if the probability density function $\nu(\cdot)$ of the noise has a moment of order $r \in [1, +\infty)$, then $(X_n)_{n \in \mathbb{N}}$ is $(1 + |\cdot|)^r$ -geometrically ergodic with $r_{\text{ess}}(P) \leq |\alpha|^r$ and $\rho_V(P) = |\alpha|$. See [Wu04, Sec. 8] or [GHL11, Sec. 5] for details.

Now we are interested in asymptotic expansions of the perturbed invariant distributions. Under Condition **(VG)** and the continuity condition (1), expansions related to the generalized potential $R = (I - P_0 + \Pi_0)^{-1}$, with $\Pi_0(\cdot) = \pi_0(\cdot)1_{\mathbb{X}}$, are given in [Kar81, Kar96, HH03], namely

$$\pi_\varepsilon = \pi_0(I - D_\varepsilon R)^{-1} = \sum_{k \geq 0} \pi_0(D_\varepsilon R)^k = \pi_0 + \sum_{k=1}^n \pi_0(D_\varepsilon R)^k + O((D_\varepsilon R)^{n+1}),$$

when $D_\varepsilon = P_\varepsilon - P_0$ is such that $\|D_\varepsilon R\|_1 < 1$. In general this is not a Taylor expansion, except for instance when D_ε has the form $D_\varepsilon = \varepsilon D$ with $D \in \mathcal{L}(\mathcal{B}_1)$. This special case is discussed in [AAN04]. In fact, for general perturbations, even in the case when (1) is fulfilled, obtaining Taylor expansions for π_ε

²Under the drift condition, (V_1) implies (V_β) by Jensen's inequality for $\beta \in (0, 1]$.

causes difficulties when the derivatives of the perturbed kernels $P_\varepsilon(x, \cdot)$ w.r.t. to ε yield some weights (see the term x^k in (15) for order of regularity $k \geq 1$). For this question, the derivation procedure (see [GL06, HP10]) based on the Keller-Liverani theorem is of great interest.

To illustrate this approach, we only consider the special instance of autoregressive model introduced in Example 1. More specifically, we assume that $(X_n)_{n \in \mathbb{N}}$ is given by (6) and that the probability density function $\nu(\cdot)$ of the noise has a moment of order r , $\int |x|^r \nu(x) dx < \infty$, for some $r \in [1, +\infty)$. Set $V(x) := (1 + |x|)^r$, $x \in \mathbb{R}$. Then, for each $\alpha \in (-1, 1)$, $(X_n)_{n \in \mathbb{N}}$ is V -geometrically ergodic with an invariant distribution π_α . Moreover we assume that r is not an integer (ie. $r > \lfloor r \rfloor$ where $\lfloor \cdot \rfloor$ stands for the integer part function on $\mathbb{R}$), that $\nu(\cdot)$ is positive and $\lfloor r \rfloor + 1$ -times continuously differentiable on $\mathbb{R}$, with

$$j = 1, \dots, \lfloor r \rfloor + 1, \quad \sup_{t \in \mathbb{R}} \frac{|\nu^{(j)}(t)|}{\nu(t)} < \infty. \quad (10)$$

Finally suppose that for all $x_0 \in \mathbb{R}$, there exist a neighborhood V_{x_0} of x_0 and a non-negative measurable function $q_{x_0}(\cdot)$ such that $\int_{\mathbb{R}} (1 + |y|)^r q_{x_0}(y) dy < \infty$ and: $\forall y \in \mathbb{R}, \forall v \in V_{x_0}, \nu(y + v) \leq q_{x_0}(y)$.

Proposition 1 *Under the previous assumptions, there exists $\beta \equiv \beta(r) \in (0, 1)$ such that the map $\alpha \mapsto \pi_\alpha$ is $\lfloor r \rfloor$ -times continuously differentiable from $(-1, 1)$ to the dual space $\mathcal{B}'_\beta$ of $\mathcal{B}_\beta$. In particular, for all $\alpha \in (-1, 1)$, there exist $\lfloor r \rfloor$ signed measures $\mu_{\alpha,1}, \dots, \mu_{\alpha,\lfloor r \rfloor}$ on $\mathbb{R}$ such that:*

$$\forall A \in \mathcal{X}, \quad \pi_{\alpha+\varepsilon}(A) = \pi_\alpha(A) + \sum_{j=1}^{\lfloor r \rfloor} \frac{\varepsilon^j}{j!} \mu_{\alpha,j}(A) + \varepsilon^{\lfloor r \rfloor} R_\varepsilon(A) \quad \text{with} \quad \lim_{\varepsilon \rightarrow 0} \sup_{A \in \mathcal{X}} |R_\varepsilon(A)| = 0.$$

The previous conditions (10) are well suited to densities of the form $\nu(x) = c(1 + |x|)^{-\gamma}$. For other forms of densities, they can be easily adapted in order that Proposition 1 works.

2 Proofs

Proof of Theorem 1. We shall repeatedly use the fact that a bounded linear operator (between two normed vector spaces) and its adjoint have the same norm. The adjoint of the rank-one projection $\Pi_0(f) = \pi_0(f)1_{\mathbb{X}}$ is defined by: $\forall f' \in \mathcal{B}'_1, \Pi_0^*(f') := f'(1_{\mathbb{X}})\pi_0$.

First, Assumption (V₁) gives

$$\|(P_0^*)^n - \Pi_0^*\|_{\mathcal{B}'_1} = O(\kappa_1^n).$$

Second, it follows from the continuity assumption (2) of Theorem 1 that

$$\lim_{\varepsilon \rightarrow 0} \|P_\varepsilon^* - P_0^*\|_{\mathcal{B}'_1, \mathcal{B}'_0} = 0,$$

Third, Assumption (D) gives:

$$\forall \varepsilon \in (-\varepsilon_0, \varepsilon_0), \forall f' \in \mathcal{B}'_1, \quad \|P_\varepsilon^{*N} f'\|_{\mathcal{B}'_1} \leq \delta^N \|f'\|_{\mathcal{B}'_1} + L \|f'\|_{\mathcal{B}'_0}. \quad (11)$$

Indeed, recall that $\mathcal{B}'_1$ and $\mathcal{B}'_0$ are Banach lattices and that, for each $g' \in \mathcal{B}'_1$, we have $\|g'\|_{\mathcal{B}'_0} = \| |g'| \|_{\mathcal{B}'_0} = \langle |g'|, 1_{\mathbb{X}} \rangle$ and $\|g'\|_{\mathcal{B}'_1} = \| |g'| \|_{\mathcal{B}'_1} = \langle |g'|, V \rangle$, where $\langle \cdot, \cdot \rangle$ stands for the duality brackets in

both $\mathcal{B}'_0 \times \mathcal{B}_0$ and $\mathcal{B}'_1 \times \mathcal{B}_1$. Next, observe that, for every $f' \in \mathcal{B}'_1$ and every $f \in \mathcal{B}_1$ such that $|f| \leq V$, we have $|\langle (P^*)^N f', f \rangle| \leq \langle |f'|, P^N V \rangle$. Hence (D) gives (11) as claimed.

The three previous facts show that $\{P_\varepsilon^*\}_{|\varepsilon| < \varepsilon_0}$ satisfies the assumptions of [Liv04] on $\mathcal{B}'_1$ (see also [HP10] for the use of the perturbation Keller-Liverani theorem in a Markov context). Therefore, for all $\kappa \in (\widehat{\kappa}, 1)$, there exists $\varepsilon_1 \in (0, \varepsilon_0]$ such that, for all $\varepsilon \in (-\varepsilon_1, \varepsilon_1)$, the following properties hold: there exist $\lambda_\varepsilon \in \mathbb{C}$ satisfying $\lim_{\varepsilon \rightarrow 0} \lambda_\varepsilon = 1$ and a rank-one projection Π'_ε on $\mathcal{B}'_1$ such that

$$\sup_{|\varepsilon| < \varepsilon_1} \|(P_\varepsilon^*)^n - \lambda_\varepsilon^n \Pi'_\varepsilon\|_{\mathcal{B}'_1} = O(\kappa^n) \quad \text{and} \quad \lim_{\varepsilon \rightarrow 0} \|\Pi'_\varepsilon - \Pi_0^*\|_{\mathcal{B}'_1, \mathcal{B}'_0} = 0. \quad (12)$$

Up to reduce ε_1 , we have $\lambda_\varepsilon = 1$. Indeed one may assume that $|\lambda_\varepsilon| > \kappa$. Then we deduce from $1_{\mathbb{X}} = P_\varepsilon^n 1_{\mathbb{X}}$ and (12) that, for any $f' \in \mathcal{B}'_1$, we have $\lim_n \lambda_\varepsilon^{-n} \langle f', 1_{\mathbb{X}} \rangle = \lim_n \lambda_\varepsilon^{-n} \langle (P_\varepsilon^*)^n f', 1_{\mathbb{X}} \rangle = \langle \Pi'_\varepsilon f', 1_{\mathbb{X}} \rangle$. Thus $(\lambda_\varepsilon^{-n})_{n \in \mathbb{N}}$ converges in $\mathbb{C}$, so that we have, either $\lambda_\varepsilon = 1$, or $|\lambda_\varepsilon| > 1$. Moreover the sequence $(P_\varepsilon^n)_{n \in \mathbb{N}}$ is bounded in $\mathcal{L}(\mathcal{B}_V)$ from (D). Thus $((P_\varepsilon^*)^n)_{n \in \mathbb{N}}$ is bounded in $\mathcal{L}(\mathcal{B}'_V)$, and (12) then implies that $(\lambda_\varepsilon^n)_{n \in \mathbb{N}}$ is bounded in $\mathbb{C}$. Thus $\lambda_\varepsilon = 1$.

Hence $((P_\varepsilon^*)^n)_{n \in \mathbb{N}}$ is Cauchy, and so is $(P_\varepsilon^n)_{n \in \mathbb{N}}$ in $\mathcal{L}(\mathcal{B}_1)$. Therefore

$$\exists \Pi_\varepsilon \in \mathcal{L}(\mathcal{B}_1), \quad \lim_n P_\varepsilon^n = \Pi_\varepsilon \text{ in } \mathcal{L}(\mathcal{B}_1). \quad (13)$$

Thus $(P_\varepsilon^n)^* \rightarrow \Pi_\varepsilon^*$ in $\mathcal{L}(\mathcal{B}'_1)$, and so $\Pi'_\varepsilon = \Pi_\varepsilon^*$. We have obtained that

$$\sup_{|\varepsilon| < \varepsilon_1} \|P_\varepsilon^n - \Pi_\varepsilon\|_1 = O(\kappa^n) \quad \text{and} \quad \lim_{\varepsilon \rightarrow 0} \|\Pi_\varepsilon - \Pi_0\|_{0,1} = 0.$$

From (13) it follows that Π_ε is a positive projection on $\mathcal{B}_1$ satisfying $\Pi_\varepsilon P_\varepsilon = P_\varepsilon \Pi_\varepsilon = \Pi_\varepsilon$. Let us prove that Π_ε is rank-one. We know that Π'_ε is rank-one, namely $\Pi'_\varepsilon(\cdot) := \langle \phi_\varepsilon, \cdot \rangle a'_\varepsilon$ for some $\phi_\varepsilon \neq 0$ in the dual space of $\mathcal{B}'_1$ and for some $a'_\varepsilon \neq 0$ in $\mathcal{B}'_1$. From $\Pi_\varepsilon^* = \Pi'_\varepsilon$, we obtain $\text{Ker } \Pi_\varepsilon = \text{Ker } a'_\varepsilon$. Thus $\text{Ker } \Pi_\varepsilon$ is of codimension one, so that we have $\dim \text{Im } \Pi_\varepsilon = 1$ (use $\mathcal{B}_1 = \text{Ker } \Pi_\varepsilon \oplus \text{Im } \Pi_\varepsilon$).

The last fact shows that $\Pi_\varepsilon(\cdot) := e'_\varepsilon(\cdot) 1_{\mathbb{X}}$ for some $e'_\varepsilon \geq 0$ in $\mathcal{B}'_1$. Since $\forall A \in \mathcal{X}$, $\lim_n P_\varepsilon^n(x_0, A) = e'_\varepsilon(1_A)$ (for some fixed $x_0 \in \mathbb{X}$) and $P_\varepsilon^n(x_0, \cdot)$ is a probability measure on $(\mathbb{X}, \mathcal{X})$ for each $n \geq 1$, we deduce from the Vitali-Hahn-Saks theorem that $A \mapsto \pi_\varepsilon(A) := e'_\varepsilon(1_A)$ is a probability measure on $(\mathbb{X}, \mathcal{X})$. Clearly π_ε is P_ε -invariant. Next π_ε and e'_ε coincide on $\mathcal{B}_0$: indeed every $f \in \mathcal{B}_0$ can be approached uniformly on $\mathbb{X}$ by a sequence $(f_n)_n$ of simple functions, so that $\pi_\varepsilon(f) = \lim_n \pi_\varepsilon(f_n) = \lim_n e'_\varepsilon(f_n) = e'_\varepsilon(f)$ (the last convergence holds since $e'_\varepsilon \in \mathcal{B}'_1$). We obtain $\pi_\varepsilon(V 1_{[V \leq n]}) = e'_\varepsilon(V 1_{[V \leq n]}) \leq e'_\varepsilon(V) < \infty$ for every $n \geq 1$ since $e'_\varepsilon \geq 0$. The monotone convergence theorem gives $\pi_\varepsilon(V) < \infty$. Thus $f \mapsto \pi_\varepsilon(f)$ is in $\mathcal{B}'_1$. Since $\forall f \in \mathcal{B}_1$, $\lim_n P_\varepsilon^n f = e'_\varepsilon(f) 1_{\mathbb{X}}$ in $\mathcal{B}_1$, we deduce that $e'_\varepsilon = \pi_\varepsilon$ on $\mathcal{B}_1$ from the invariance of π_ε . Finally the last assertion of Theorem 1 follows from $\lim_{\varepsilon \rightarrow 0} \|\Pi_\varepsilon - \Pi_0\|_{0,1} = 0$. $\square$

Proof of Corollary 1. Let $\kappa \in (\max(\kappa_\beta, \kappa_1), 1)$, and let Γ be the oriented circle in $\mathbb{C}$, centered at $z = 1$ and with radius less than $(1 - \kappa)/2$. Note that the assumptions of Corollary 1 imply those of Theorem 1. We know from [Liv04] and the proof of Theorem 1 that $M := \sup_{z \in \Gamma, |\varepsilon| < \varepsilon_1} \|(zI - P_\varepsilon)^{-1}\|_1 < \infty$. Moreover, the rank-one eigenprojection Π_ε is from the standard spectral theory:

$$\Pi_\varepsilon = \frac{1}{2i\pi} \oint_\Gamma (zI - P_\varepsilon)^{-1} dz. \quad (14)$$

Next (V_β) gives $M_0 := \sup_{z \in \Gamma} \|(zI - P_0)^{-1}\|_\beta < \infty$. Assume that $\|1_{\mathbb{X}}\|_1 = 1$ (to simplify). Then, from

$$(zI - P_\varepsilon)^{-1} - (zI - P_0)^{-1} = (zI - P_\varepsilon)^{-1} (P_\varepsilon - P_0) (zI - P_0)^{-1},$$

we obtain that for every $f \in \mathcal{B}_\beta$

$$|\pi_\varepsilon(f) - \pi_0(f)| = \|\Pi_\varepsilon(f) - \Pi_0(f)\|_1 \leq \frac{(1 - \kappa)}{2} M \|P_\varepsilon - P_0\|_{\beta,1} M_0 \|f\|_\beta.$$

□

Proof of Proposition 1. Recall that $V(x) := (1 + |x|)^r$, $x \in \mathbb{R}$. Let $a_0 \in (0, 1)$. We apply the derivation procedure of [HP10, App. A] to the family $\{P_\alpha, \alpha \in (-a_0, a_0)\}$ with respect to the Banach spaces $\mathcal{B}_\beta$. Let $\beta \in (0, 1]$. For any $\alpha \in (-a_0, a_0)$, the Markov kernel P_α of the autoregressive model $(X_n)_{n \in \mathbb{N}}$ satisfies the drift condition and the aperiodicity/irreducibility assumptions of [MT93]. Hence each P_α satisfies (V_β) . In the same way, $\{P_\alpha\}_{|\alpha| < a_0}$ satisfies (D), so that it also satisfies (D) with respect to the function V^β thanks to Jensen's inequality. Let $\alpha \in (-a_0, a_0)$. Then $\lim_{\varepsilon \rightarrow 0} \|P_{\alpha+\varepsilon} - P_\alpha\|_{0,\beta} = 0$ from Lemma 1 below. Hence the perturbation Keller-Liverani theorem can be applied in $\mathcal{L}(\mathcal{B}_\beta)$ to the family $(P_{\alpha+\varepsilon})_\varepsilon$ (as seen with $\beta = 1$ in the proof of Theorem 1). This gives the following spectral properties: there exists $\kappa_{\alpha,\beta} \in (0, 1)$ such that, for all $\kappa \in (\kappa_{\alpha,\beta}, 1)$, there exists $\hat{\varepsilon} > 0$ such that the resolvents $(zI - P_{\alpha+\varepsilon})^{-1}$ are well-defined and uniformly bounded in $\mathcal{L}(\mathcal{B}_\beta)$ provided that $|\varepsilon| \leq \hat{\varepsilon}$ and $z \in \mathbb{C}$ satisfies $|z| \geq \kappa$ and $|z - 1| \geq (1 - \kappa)/2$.

Now let us introduce the formal derivative operators of $\alpha \mapsto P_\alpha$:

$$\forall x \in \mathbb{R}, \quad (P_{k,\alpha}f)(x) = (-1)^k x^k \int_{\mathbb{R}} f(y) \nu^{(k)}(y - \alpha x) dy. \quad (15)$$

Lemma 1 *Let $(\beta, \beta') \in [0, 1]^2$.*

- (i) *If $\beta + k/r < \beta' \leq 1$ for $k \in \{0, \dots, \lfloor r \rfloor\}$, then $\alpha \mapsto P_{k,\alpha}$ is continuous from $(-a_0, a_0)$ to $\mathcal{L}(\mathcal{B}_\beta, \mathcal{B}_{\beta'})$.*
- (ii) *If $\beta + (k+1)/r < \beta' \leq 1$ for $k \in \{0, \dots, \lfloor r \rfloor - 1\}$, then $\alpha \mapsto P_{k,\alpha}$ is continuously differentiable from $(-a_0, a_0)$ to $\mathcal{L}(\mathcal{B}_\beta, \mathcal{B}_{\beta'})$, with $\frac{dP_{k,\alpha}}{d\alpha}(\alpha) = P_{k+1,\alpha}$.*

In a first stage, let us admit that Lemma 1 holds and apply [HP10, App. A]. For $\beta \in (0, 1)$ and $\sigma > 0$, we set: $T_0(\beta) = \beta + \sigma/r$ and $T_1(\beta) = \beta + (1 + \sigma)/r$. Lemma 1 gives for $j = 1, \dots, \lfloor r \rfloor$:

- if $T_0(\beta) \in (0, 1]$, then the map $\alpha \mapsto P_\alpha$ is continuous from $(-a_0, a_0)$ to $\mathcal{L}(\mathcal{B}_\beta, \mathcal{B}_{T_0(\beta)})$,
- if $T_1(T_0 T_1)^{j-1}(\beta) \in (0, 1]$, then the map $\alpha \mapsto P_\alpha$ is j -times continuously differentiable from $(-a_0, a_0)$ to $\mathcal{L}(\mathcal{B}_\beta, \mathcal{B}_{T_1(T_0 T_1)^{j-1}(\beta)})$.

Let $\beta_r \in (0, 1 - \lfloor r \rfloor/r)$, and let $\sigma > 0$ such that $\beta_r + [(2\lfloor r \rfloor + 1)\sigma + \lfloor r \rfloor]/r = 1$. In other words we have $(T_0 T_1)^{\lfloor r \rfloor} T_0(\beta_r) = 1$. Let $\alpha_0 \in (-a_0, a_0)$. Then it follows from [HP10, Prop. A.1] that there exists $\tilde{\kappa} \in (0, 1)$ such that, for all $z \in \mathcal{D} := \{z \in \mathbb{C} : |z| \geq \tilde{\kappa}, |z - 1| \geq (1 - \tilde{\kappa})/2\}$, the map $\alpha \mapsto (zI - P_\alpha)^{-1}$ is $\lfloor r \rfloor$ -times continuously differentiable from some open interval I_{α_0} centered at α_0 into $\mathcal{L}(\mathcal{B}_{\beta_r}, \mathcal{B}_1)$. Moreover the derivatives (up to the order $\lfloor r \rfloor$) of the last map are uniformly bounded in $(\alpha, z) \in I_{\alpha_0} \times \mathcal{D}$. Next, let us define $\Pi_\alpha(f) = \pi_\alpha(\cdot)1_{\mathbb{R}}$ for any $f \in \mathcal{B}_{\beta_r}$. From the previous facts and standard spectral calculus (see (14)), the map $\alpha \mapsto \Pi_\alpha(f)$ is $\lfloor r \rfloor$ -times continuously differentiable

from I_{α_0} to $\mathcal{L}(\mathcal{B}_{\beta_r}, \mathcal{B}_1)$, so is $\alpha \mapsto \pi_\alpha(\cdot)$ from I_{α_0} to $\mathcal{B}'_{\beta_r}$. Since α_0 is any element in $(-a_0, a_0)$, with arbitrary $a_0 \in (-1, 1)$, this proves the first assertion of Proposition 1.

Finally let $\alpha \in (-1, 1)$. For all borel set A of $\mathbb{R}$, we have $\frac{d\pi_\alpha}{d\alpha}(A) = \lim_{h \rightarrow 0} (\pi_{\alpha+h}(A) - \pi_\alpha(A))/h$. Since $\pi_{\alpha+h}(\cdot) - \pi_\alpha(\cdot)$ is a (signed) measure on $\mathbb{R}$, it follows from the Vitali-Hahn-Saks theorem that there exists a (signed) measure $\mu_{\alpha,1}$ on $\mathbb{R}$ such that we have: $\frac{d\pi_\alpha}{d\alpha}(A) = \mu_{\alpha,1}(A)$. An obvious induction gives the same conclusion for the derivatives of order $j = 2, \dots, [r]$. $\square$

Proof of Lemmas 1. Set $A_k := \sup_{t \in \mathbb{R}} |\nu^{(k)}(t)|/\nu(t)$ and $A := \max_{0 \leq k \leq [r]+1} A_k$. Note that there exists $B \equiv B(\beta)$ such that, for all $\alpha \in [-a_0, a_0]$, we have $P_\alpha V^\beta \leq BV^\beta$.

For $\alpha \in (-a_0, a_0)$, $\beta \in (0, 1]$, $f \in \mathcal{B}_\beta$ and $x \in \mathbb{R}$, we denote $F_k(\alpha) := (P_{k,\alpha}f)(x)$. From the assumptions on ν and Lebesgue's theorem, for every $k = 0, \dots, [r]$, F_k is differentiable on $[-a_0, a_0]$, with $\frac{\partial F_k}{\partial \alpha}(\alpha) = F_{k+1}(\alpha)$.

Let $k \in \{0, \dots, [r]\}$, $(\beta, \beta') \in [0, 1]^2$ such that $\beta + k/r < \beta' \leq 1$, and let $0 < \sigma \leq 1$ be such that $\beta + (k + \sigma)/r = \beta'$. For any $(\alpha, \alpha') \in (-a_0, a_0)^2$, we obtain (use Taylor expansion of F_k for (16b)):

$$|F_k(\alpha) - F_k(\alpha')| \leq 2A_k B \|f\|_\beta V(x)^{\beta + \frac{k}{r}}. \quad (16a)$$

$$|F_k(\alpha) - F_k(\alpha')| \leq |\alpha - \alpha'| A_{k+1} B \|f\|_\beta V(x)^{\beta + \frac{k+1}{r}}. \quad (16b)$$

Multiplying (16a) (to the power $1 - \sigma$) by (16b) (to the power σ) gives:

$$|F_k(\alpha) - F_k(\alpha')| \leq 2|\alpha - \alpha'|^\sigma AB \|f\|_\beta V(x)^{\beta + \frac{k+\sigma}{r}},$$

which rewrites as: $\|P_{k,\alpha}f - P_{k,\alpha'}f\|_{\beta'} \leq 2AB |\alpha - \alpha'|^\sigma \|f\|_\beta$. The first assertion of Lemma 1 is proved.

Next let $k \in \{0, \dots, [r] - 1\}$, $(\beta, \beta') \in [0, 1]^2$ such that $\beta + (k + 1)/r < \beta' \leq 1$, and let $0 < \sigma \leq 1$ be such that $\beta + (k + 1 + \sigma)/r = \beta'$. For any $(\alpha, \alpha') \in (-a_0, a_0)^2$, we obtain:

$$|F_k(\alpha) - F_k(\alpha') - (\alpha - \alpha')F_{k+1}(\alpha')| \leq 2|\alpha - \alpha'| A_{k+1} B \|f\|_\beta V(x)^{\beta + \frac{k+1}{r}} \quad (17a)$$

$$|F_k(\alpha) - F_k(\alpha') - (\alpha - \alpha')F_{k+1}(\alpha')| \leq \frac{|\alpha - \alpha'|^2}{2} A_{k+2} B \|f\|_\beta V(x)^{\beta + \frac{k+2}{r}}. \quad (17b)$$

Then, by multiplying (17a) (with the power $1 - \sigma$) and (17b) (with the power σ), we obtain:

$$|F_k(\alpha) - F_k(\alpha') - (\alpha - \alpha')F_{k+1}(\alpha')| \leq 2|\alpha - \alpha'|^{1+\sigma} AB \|f\|_\beta V(x)^{\beta + \frac{k+1+\sigma}{r}},$$

which rewrites as: $\|P_{k,\alpha}f - P_{k,\alpha'}f - (\alpha - \alpha')P_{k+1,\alpha'}f\|_{\beta'} \leq 2AB |\alpha - \alpha'|^{1+\sigma} \|f\|_\beta$. $\square$

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