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► **To cite this version:**

Emeline Schmisser. Non-parametric adaptive estimation of the drift for a jump diffusion process. Stochastic Processes and their Applications, 2014, 124 (1), pp.883-914. 10.1016/j.spa.2013.09.012 . hal-00704637v3

HAL Id: hal-00704637

<https://hal.science/hal-00704637v3>

Submitted on 26 Sep 2013

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Non-parametric adaptive estimation of the drift for a jump diffusion process

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26th September 2013

Abstract

In this article, we consider a jump diffusion process $(X_t)_{t \geq 0}$ observed at discrete times $t = 0, \Delta, \dots, n\Delta$. The sampling interval Δ tends to 0 and $n\Delta$ tends to infinity. We assume that $(X_t)_{t \geq 0}$ is ergodic, strictly stationary and exponentially β -mixing. We use a penalized least-square approach to compute two adaptive estimators of the drift function b . We provide bounds for the risks of the two estimators.

1 Introduction

We consider a general diffusion with jumps:

$$dX_t = b(X_t)dt + \sigma(X_t)dW_t + \xi(X_{t-})dL_t \quad \text{and} \quad X_0 = \eta \quad (1)$$

where L_t is a centred pure jump Levy process:

$$dL_t = \int_{z \in \mathbb{R}} z (\mu(dt, dz) - dt\nu(dz))$$

with μ a random Poisson measure with intensity measure $\nu(dz)dt$ such that $\int_{z \in \mathbb{R}} z^2 \nu(dz) < \infty$. The compensated Poisson measure $\tilde{\mu}$ is defined by $\tilde{\mu}(dt, dz) = \mu(dt, dz) - \nu(dz)dt$. The random variable η is independent of $(W_t, L_t)_{t \geq 0}$. Moreover, $(W_t)_{t \geq 0}$ and $(L_t)_{t \geq 0}$ are independent.

This process is observed with high frequency (at times $t = 0, \Delta, \dots, n\Delta$ where, as n tends to infinity, the sampling interval $\Delta \rightarrow 0$ and the time of observation $n\Delta \rightarrow \infty$). It is assumed to be ergodic, stationary and exponentially β -mixing (see Masuda (2007) for sufficient conditions). Our aim is to construct a non-parametric estimator of b on a compact set A .

The non-parametric estimation of b and σ for a diffusion process observed with high-frequency is well-known (see for instance Hoffmann (1999) and Comte

et al. (2007)). Diffusion processes with jumps are used in various fields, for instance in finance, for modelling the growth of a population, in hydrology, in medical science, . . . , but there exist few results for the non-parametric estimation of b and σ . Mai (2012) and Shimizu and Yoshida (2006) construct maximum-likelihood estimators of parameters of b . Their estimators reach the standard rate of convergence, $\sqrt{n\Delta}$. Shimizu (2008) and Mancini and Renò (2011) use a kernel estimator to obtain non parametric threshold estimators of σ . Mancini and Renò (2011) also construct a non-parametric truncated estimator of b , but only when L_t is a compound Poisson process. To our knowledge, minimax rates of convergences for non-parametric estimators of b , σ or ξ for jump-diffusions processes are not available in the literature (see Hoffmann (1999) or Gobet *et al.* (2004) for rates of convergence for diffusions processes).

In this paper, we use model selection to construct two non-parametric estimators of b under the asymptotic framework $\Delta \rightarrow 0$ and $n\Delta \rightarrow \infty$. This method was introduced by Birgé and Massart (1998).

First, we introduce a sequence of linear subspaces $S_m \subseteq L^2(A)$ and, for each m , we construct an estimator \hat{b}_m of b by minimising on S_m the contrast function:

$$\gamma_n(t) = \frac{1}{n} \sum_{k=1}^n (Y_{k\Delta} - t(X_{k\Delta}))^2 \quad \text{where} \quad Y_{k\Delta} = \frac{X_{(k+1)\Delta} - X_{k\Delta}}{\Delta}.$$

We obtain a collection of estimators of the drift function b and we bound their risks (Theorem 2). Then, we introduce a penalty function to select the “best” dimension m and we deduce an adaptive estimator $\hat{b}_{\hat{m}}$. Under the assumption that ν is sub-exponential, that is if there exist two positive constants C, λ such that, for z large enough, $\nu([-z, z]^c) \leq Ce^{-\lambda z}$, the risk bound of $\hat{b}_{\hat{m}}$ is exactly the same as for a diffusion without jumps (Theorem 4) (see Comte *et al.* (2007) or Hoffmann (1999)).

In a second part, we do not assume that ν is sub-exponential and we construct a truncated estimator \tilde{b}_m of b . We minimise the contrast function

$$\tilde{\gamma}_n(t) = \frac{1}{n} \sum_{k=1}^n (Y_{k\Delta} 1_{|Y_{k\Delta}| \leq C_\Delta} - t(X_{k\Delta}))^2 \quad \text{where} \quad C_\Delta \propto \sqrt{\Delta} \ln(n)$$

in order to obtain a new estimator $\tilde{b}_{\hat{m}}$. As in the first part, we introduce a penalty function to obtain an adaptive estimator $\tilde{b}_{\tilde{m}}$. The risk bound of this adaptive estimator depends on the Blumenthal-Gettoor index of ν (Theorems 7 and 10).

In Section 2, we present the model and its assumptions. In Sections 3 and 4, we construct the estimators and bound their risks. Some simulations are presented in Section 5. Proofs are gathered in Section 6.

2 Assumptions

2.1 Assumptions on the model

We consider the following assumptions:

A 1. The functions b , σ and ξ are Lipschitz.

A 2. 1. The function σ is bounded from below and above:

$$\exists \sigma_0, \sigma_1, \forall x \in \mathbb{R}, \quad 0 < \sigma_1 \leq \sigma(x) \leq \sigma_0.$$

2. The function ξ is bounded: $\exists \xi_0, \forall x \in \mathbb{R}, \quad 0 \leq \xi(x) \leq \xi_0$.

3. The drift function b is elastic: there exists a constant M such that, for any $x \in \mathbb{R}, |x| > M: xb(x) \lesssim -|x|^2$.

4. The Lévy measure ν satisfies:

$$\nu(\{0\}) = 0, \quad \int_{-\infty}^{\infty} z^2 \nu(dz) = 1 \quad \text{and} \quad \int_{-\infty}^{\infty} z^4 \nu(dz) < \infty.$$

Under Assumption A1, the stochastic differential equation (1) admits a unique strong solution. According to Masuda (2007), under Assumptions A1 and A2, the process (X_t) admits a unique invariant probability ϖ and satisfies the ergodic theorem: for any measurable function g such that $\int |g(x)| \varpi(dx) < \infty$, when $T \rightarrow \infty$,

$$\frac{1}{T} \int_0^T g(X_s) ds \rightarrow \int g(x) \varpi(dx).$$

This distribution has moments of order 4. Moreover, Masuda (2007) also ensures that under these assumptions, the process (X_t) is exponentially β -mixing. Furthermore, if there exist two constants c and n_0 such that, for any $x \in \mathbb{R}, \xi^2(x) \geq c(1 + |x|)^{-n_0}$, then Ishikawa and Kunita (2006) ensure that a smooth transition density exists.

A 3. 1. The stationary measure ϖ admits a density π which is bounded from below and above on the compact interval A :

$$\exists \pi_0, \pi_1, \forall x \in A, \quad 0 < \pi_1 \leq \pi(x) \leq \pi_0.$$

2. The process $(X_t)_{t \geq 0}$ is stationary ($\eta \sim \varpi(dx) = \pi(x)dx$).

The first part of this assumption is automatically satisfied if $\xi = 0$ (that is if $(X_t)_{t \geq 0}$ is a diffusion process). The following proposition is very useful for the proofs. It is derived from Result 11.

Proposition 1.

Under Assumptions A1-A3, for any $p \geq 1$, there exists a constant $c(p)$ such that, if $\int_{\mathbb{R}} z^{2p} \nu(dz) < \infty$:

$$\mathbb{E} \left(\sup_{s \in [t, t+h]} (X_s - X_t)^{2p} \right) \leq c(p)h.$$

2.2 Assumptions on the approximation spaces

In order to construct an adaptive estimator of b , we use model selection: we compute a collection of estimators \hat{b}_m of b by minimising a contrast function $\gamma_n(t)$ on a vectorial subspace $S_m \subset L^2(A)$, then we choose the best possible estimator using a penalty function $pen(m)$. The collection of vectorial subspaces $(S_m)_{m \in \mathcal{M}_n}$ has to satisfy the following assumption:

A 4.

1. The subspaces S_m have finite dimension D_m .
2. The sequence of vectorial subspaces $(S_m)_{m \geq 0}$ is increasing: for any m , $S_m \subseteq S_{m+1}$.
3. Norm connexion: there exists a constant ϕ_1 such that, for any $m \geq 0$, any $t \in S_m$,

$$\|t\|_\infty^2 \leq \phi_1 D_m \|t\|_{L^2}^2$$

where $\|\cdot\|_{L^2}$ is the L^2 -norm and $\|\cdot\|_\infty$ is the sup-norm on A .

4. For any $m \in \mathbb{N}$, there exists an orthonormal basis $(\psi_\lambda)_{\lambda \in \Lambda_m}$ of S_m such that

$$\forall \lambda, \quad \text{card}(\lambda', \|\psi_\lambda \psi_{\lambda'}\|_\infty \neq 0) \leq \phi_0$$

where ϕ_0 does not depend on m .

5. For any function t belonging to the unit ball of the Besov space $\mathcal{B}_{2,\infty}^\alpha$,

$$\exists C, \forall m \quad \|t - t_m\|_{L^2}^2 \leq C D_m^{-2\alpha}$$

where t_m is the L^2 orthogonal projection of t on S_m .

The subspaces generated by piecewise polynomials, compactly supported wavelets or spline functions satisfy A4 (see DeVore and Lorentz (1993) and Meyer (1990) for instance).

3 Estimation of the drift

By analogy with Comte *et al.* (2007), we decompose $Y_{k\Delta}$ in the following way:

$$Y_{k\Delta} = \frac{X_{(k+1)\Delta} - X_{k\Delta}}{\Delta} = b(X_{k\Delta}) + I_{k\Delta} + Z_{k\Delta} + T_{k\Delta} \quad (2)$$

where

$$I_{k\Delta} = \frac{1}{\Delta} \int_{k\Delta}^{(k+1)\Delta} (b(X_s) - b(X_{k\Delta})) ds, \quad Z_{k\Delta} = \frac{1}{\Delta} \int_{k\Delta}^{(k+1)\Delta} \sigma(X_s) dW_s$$

$$T_{k\Delta} = \frac{1}{\Delta} \int_{k\Delta}^{(k+1)\Delta} \xi(X_{s-}) dL_s.$$

The terms $Z_{k\Delta}$ and $T_{k\Delta}$ are martingale increments. Let us introduce the mean square contrast function

$$\gamma_n(t) = \frac{1}{n} \sum_{k=1}^n (Y_{k\Delta} - t(X_{k\Delta}))^2. \quad (3)$$

We can always minimise $\gamma_n(t)$ on S_m , but the minimiser may be not unique. That is why we introduce the empirical risk

$$\mathcal{R}_n(t) = \mathbb{E} \left(\|t - b_A\|_n^2 \right) \quad \text{where} \quad \|t\|_n^2 = \frac{1}{n} \sum_{k=1}^n t^2(X_{k\Delta}) \quad \text{and} \quad t_A = t1_A. \quad (4)$$

We consider the asymptotic framework:

$$\Delta \rightarrow 0, \quad n\Delta \rightarrow \infty.$$

For any $m \in \mathcal{M}_n = \{m, D_m \leq \mathcal{D}_n\}$ where $\mathcal{D}_n^2 \leq n\Delta / \ln^2(n)$, we construct the regression-type estimator:

$$\hat{b}_m = \arg \min_{t \in S_m} \gamma_n(t).$$

Theorem 2.

Under Assumptions A1-A4, the risk of the estimator with fixed m satisfies:

$$\mathcal{R}_n(\hat{b}_m) \leq 3\pi_1 \|b_m - b_A\|_{L^2}^2 + 48(\sigma_0^2 + \xi_0^2) \frac{D_m}{n\Delta} + c\Delta$$

where b_m is the orthogonal (L^2) projection of b_A over the vectorial subspace S_m . The constant c is independent of m , n and Δ .

Except for the constant $(\sigma_0^2 + \xi_0^2)$ in the variance term, this is exactly the bound of the risk that Comte *et al.* (2007) found for a diffusion process without jumps.

The bias term, $\|b_m - b_A\|_{L^2}^2$, decreases when the dimension D_m increases whereas the variance term $(\sigma_0^2 + \xi_0^2)D_m/(n\Delta)$ is proportional to the dimension. Under the classical assumption $n\Delta^2 = O(1)$, the remainder term Δ is negligible. Thus we need to find a good compromise between the bias and the variance term.

Remark 3. If the regularity of the drift function is known, that is, if b belongs to a ball of a Besov space $\mathcal{B}_{2,\infty}^\alpha$, then the bias term $\|b_m - b_A\|_{L^2}^2$ is smaller than $D_m^{-2\alpha}$. The best estimator is obtained when the bias term, $\|b_m - b_A\|_{L^2}^2$, and the variance term, $cD_m(n\Delta)^{-1}$, are equal, that is for $D_{m_{opt}} = (n\Delta)^{1/(1+2\alpha)}$. In that case, the estimator risk satisfies:

$$\mathcal{R}_n(\hat{b}_{m_{opt}}) \lesssim (n\Delta)^{-2\alpha/(2\alpha+1)} + \Delta.$$

Let us introduce a penalty function *pen* such that :

$$pen(m) \geq \kappa(\sigma_0^2 + \xi_0^2) \frac{D_m}{n\Delta}$$

and set:

$$\hat{m} = \arg \min_{m \in \mathcal{M}_n} \left\{ \gamma_n(\hat{b}_m) + \text{pen}(m) \right\}.$$

We will chose κ later. We denote by $\hat{b}_{\hat{m}}$ the resulting estimator. To bound the risk of the adaptive estimator, an additional assumption is needed:

A 5. 1. *The Lévy measure ν is symmetric or the function ξ is constant.*

2. *The Lévy measure ν is sub exponential: there exist $\lambda, C > 0$ such that, for any $|z| > 1$, $\nu(\cdot - z, z)^c \leq Ce^{-\lambda|z|}$.*

Theorem 4.

Under Assumptions A1-A5, there exists a constant κ (depending only on ν) such that, if $\mathcal{D}_n^2 \leq n\Delta/\ln^2(n)$:

$$\mathbb{E} \left(\left\| \hat{b}_{\hat{m}} - b_A \right\|_n^2 \right) \lesssim \inf_{m \in \mathcal{M}_n} \left(\|b_m - b_A\|_{L^2}^2 + \text{pen}(m) \right) + \left(\Delta + \frac{1}{n\Delta} \right).$$

Remark 5. We can bound κ theoretically, however, this bound is in practice too large for the simulations. In Section 5, we calibrate κ by simulations (see Comte et al. (2007) for instance). If σ and ξ are unknown, it is possible to replace them by rough estimators (in fact, we only need upper bounds of σ_0^2 and ξ_0^2). It is also possible to performe a completely data-driven calibration of the parameters of the penalty (see Arlot and Massart (2009)).

4 Truncated estimator of the drift

Truncated estimators are widely used for the estimation of the diffusion coefficient of a jump diffusion (see for instance Mancini and Renò (2011), Shimizu (2008) and Mai (2012)). Our aim is to construct an adaptive estimator of b even if Assumption A5 is not fulfilled. To this end, we cut off the big jumps. Let us introduce the set

$$\Omega_{X,k} = \left\{ \omega, |X_{(k+1)\Delta} - X_{k\Delta}| \leq C_\Delta \right\}$$

where $C_\Delta = (b_{max} + 3)\Delta + (\sigma_0 + 4\xi_0)\sqrt{\Delta}\ln(n)$ (with $b_{max} = \sup_{x \in A} |b(x)|$). Let us consider the random variables

$$\tilde{Y}_{k\Delta} = \frac{X_{(k+1)\Delta} - X_{k\Delta}}{\Delta} 1_{\Omega_{X,k}} 1_{X_{k\Delta} \in A}.$$

We recall here the definition of the Blumenthal-Gettoor index:

Definition 6.

The Blumenthal-Gettoor index of a Lévy measure is

$$\beta = \inf \left\{ \alpha \geq 0, \int_{|z| \leq 1} |z|^\alpha \nu(dz) < \infty \right\}.$$

A compound Poisson process has $\beta = 0$.

We assume that the following assumption is fulfilled.

- A 6. 1. For $|x|$ small, $\nu(dx)$ is absolutely continuous with respect to the Lebesgue measure ($\nu(dx) = n(x)dx$) and:

$$\exists \beta \in [0, 2[, \exists a_0, \forall x \in [-a_0, a_0], \quad n(x) \leq Cx^{-\beta-1}.$$

This implies that the Blumenthal-Gettoor index is equal to β .

2. The Lévy measure $\nu(z)$ is symmetric for z small:

$$\exists a_1 < a_0, \forall z \in [-a_1, a_1], n(z) = n(-z)$$

3. The function ξ is bounded from below: there exists $\xi_1 > 0$ such that, for any $z \in \mathbb{R}$, $0 < \xi_1 \leq \xi(z)$.

4. The functions σ and ξ are \mathcal{C}^2 , ξ' and σ' are Lipschitz.

We consider the following asymptotic framework:

$$\frac{n\Delta}{\ln^2(n)} \rightarrow \infty, \quad \Delta^{1-\beta/2} \ln^2(n) \rightarrow 0.$$

The truncated estimator \tilde{b}_m is obtained by minimising the contrast function:

$$\tilde{b}_m = \arg \min_{t \in S_m} \tilde{\gamma}_n(t) \quad \text{where} \quad \tilde{\gamma}_n(t) = \frac{1}{n} \sum_{k=1}^n \left(\tilde{Y}_{k\Delta} - t(X_{k\Delta}) \right)^2.$$

Theorem 7 : Risk of the non adaptive truncated estimator.

Under Assumptions A1-A4 and A6, for any m such that $D_m \leq \mathcal{D}_n$ where $\mathcal{D}_n^2 \leq n\Delta / \ln^2(n)$:

$$\mathbb{E} \left(\left\| \tilde{b}_m - b_A \right\|_n^2 \right) \lesssim \|b_m - b_A\|_{L^2}^2 + (\sigma_0^2 + c\Delta^{1/2-\beta/4}) \frac{D_m}{n\Delta} + \Delta^{1-\beta/2} \ln^2(n) + \frac{1}{n\Delta}.$$

The variance term is smaller than for the first estimator, but the remainder term depends on the Blumenthal-Gettoor index and is larger than for the first estimator. This remainder term is due to the fact that $\tilde{Y}_{k\Delta} = 0$ every time $|X_{(k+1)\Delta} - X_{k\Delta}| > C\Delta$: then

$$\left| \mathbb{E} \left(\tilde{Y}_{k\Delta} - b(X_{k\Delta}) \right) \right| > |\mathbb{E} (Y_{k\Delta} - b(X_{k\Delta}))|.$$

If L_t is a compound Poisson process, (which implies $\beta = 0$) or if Δ is small enough (see Remark 9), we obtain a better inequality than for the non-truncated estimator.

Remark 8. If ν is not absolutely continuous, we can prove the weaker inequality:

$$\mathbb{E} \left(\left\| \tilde{b}_m - b_A \right\|_n^2 \right) \lesssim \|b_m - b_A\|_{L^2}^2 + (\sigma_0^2 + \xi_0^2) \frac{D_m}{n\Delta} + \Delta^{1-\beta} \ln^2(n) + \frac{1}{n\Delta}.$$

In that case, \tilde{b}_m converges towards b_A only if $\beta < 1$, which implies that ν has finite variation ($\int_{\mathbb{R}} |z| \nu(dz) < \infty$). See Remark 18.

Remark 9. Assume that b_A belongs to the Besov space $\mathcal{B}_{2,\infty}^\alpha$ and that $\|b_A\|_{\mathcal{B}_{2,\infty}^\alpha} \leq 1$.

The bias-variance compromise $\|b_m - b_A\|_{L^2}^2 + D_m/n\Delta$ is minimal when $m = \log_2(n\Delta)/(1 + 2\alpha)$, and the risk satisfies:

$$\mathbb{E} \left(\left\| \tilde{b}_m - b_A \right\|_n^2 \right) \lesssim (n\Delta)^{-2\alpha/(1+2\alpha)} + \Delta^{1-\beta/2} \ln^2(n)$$

Let us set $\Delta \sim n^{-\gamma}$ with $\gamma > 0$. We have the following convergence rates:

γ	first estimator	truncated estimator
$0 < \gamma \leq \frac{2\alpha}{4\alpha+1} \leq \frac{1}{2}$	Δ	$\Delta^{1-\beta/2} \ln^2(n)$
$\frac{2\alpha}{4\alpha+1} \leq \gamma \leq \frac{2\alpha}{4\alpha+1-\beta\alpha-\beta/2} \leq \frac{1}{2(1-\beta/4)}$	$(n\Delta)^{-2\alpha/(2\alpha+1)}$	$\Delta^{1-\beta/2} \ln^2(n)$
$\frac{2\alpha+1}{4\alpha+1-\beta\alpha-\beta/2} \leq \gamma < 1$	$(n\Delta)^{-2\alpha/(2\alpha+1)}$	$(n\Delta)^{-2\alpha/(2\alpha+1)}$

If we have sufficiently high frequency data ($n\Delta^{2(1-\beta/4)} = O(1)$), then the rate of convergence is $(n\Delta)^{2\alpha/(2\alpha+1)}$ for the two estimators. The estimator of Mai (2012) converges with the corresponding parametric rate, $n\Delta$, if $n\Delta^{3/2-\gamma} = o(1)$ for $\gamma \in]0, 1/2[$.

To construct the adaptive estimator, we use the same penalty function as in the previous section:

$$\text{pen}(m) \geq \kappa (\sigma_0^2 + \xi_0^2) \frac{D_m}{n\Delta}$$

and define the adaptive estimator:

$$\tilde{m} = \arg \min_{m \in \mathcal{M}_n} \left\{ \tilde{\gamma}_n(\tilde{b}_m) + \text{pen}(m) \right\}.$$

Theorem 10 : Risk of the adaptive truncated estimator.

If Assumptions A1-A4 and A6 are satisfied, then there exists κ such that, if $\mathcal{D}_n^2 \leq n\Delta/\ln^2(n)$:

$$\mathbb{E} \left(\left\| \tilde{b}_{\tilde{m}} - b_A \right\|_n^2 \right) \lesssim \min_{m \in \mathcal{M}_n} \left(\|b_m - b_A\|_n^2 + \text{pen}(m) \right) + \Delta^{1-\beta/2} \ln^2(n) + \frac{1}{n\Delta}.$$

The adaptive estimator $\tilde{b}_{\tilde{m}}$ automatically realises the bias/variance compromise.

5 Numerical simulations and examples

5.1 Models

We consider the stochastic differential equation:

$$dX_t = b(X_t)dt + \sigma(X_t)dW_t + \xi(X_{t-})dL_t$$

where L_t is a compound Poisson process of intensity 1: $L_t = \sum_{j=1}^{N_t} \zeta_j$, with N_t a Poisson process of intensity 1 and $(\zeta_1, \dots, \zeta_n)$ are independent and identically distributed random variables independent of (N_t) . We denote by f the probability law of ζ_i .

Model 1:

$$b(x) = -2x, \quad \sigma(x) = \xi(x) = 1 \quad \text{and} \quad f(dz) = \nu(dz) = \frac{1}{2}\delta_1 + \frac{1}{2}\delta_{-1}.$$

Model 2:

$$b(x) = -(x - 1/4)^3 - (x + 1/4)^3, \quad \sigma(x) = \xi(x) = 1 \quad \text{and} \quad f(dz) = \nu(dz) = \frac{e^{-\lambda|z|}dz}{2}.$$

We can remark that the function b is not Lipschitz and therefore does not satisfy Assumption A1.

Model 3:

We consider the stochastic process of parameters

$$b(x) = -2x + \sin(3x), \quad \sigma(x) = \xi(x) = \sqrt{\frac{3+x^2}{1+x^2}}$$

and

$$f(dz) = \nu(dz) = \frac{1}{4} \sqrt{\frac{\sqrt{24}}{|z|}} e^{-\sqrt{\sqrt{24}|z|}} dz.$$

Let us remark that $\nu = f$ is not sub-exponential and does not satisfy A5. Nevertheless, this model satisfies all the assumptions of Theorem 10.

Model 4:

In this model, the Lévy process is not a compound Poisson process. We set

$$\nu(dz) = \sum_{k=0}^{\infty} 2^{k+2} (\delta_{1/2^k} + \delta_{-1/2^k}), \quad b(x) = -2x \quad \text{and} \quad \sigma(x) = \xi(x) = 1.$$

The Blumenthal-Gettoor index of this process is such that $\beta > 1$.

5.2 Simulation algorithm (Compound Poisson case)

We estimate b on the compact interval $A = [-1, 1]$.

1. Simulate random variables $(X_0, X_\Delta, \dots, X_{n\Delta})$ thanks to a Euler scheme with sampling interval $\delta = \Delta/5$. To this end, we use the same simulation scheme as Rubenthaler (2010). We simulate the times of the jumps $(\tau_1, \dots, \tau_N, \tau_{N+1})$ with $\tau_N < n\Delta \leq \tau_{N+1}$ and we fix $X_0 = 0$. If $\delta < \tau_1$, we compute

$$X_\delta = \delta b(X_0) + \sqrt{\delta} \sigma(X_0) N \quad \text{with } N \sim \mathcal{N}(0, 1).$$

If $\tau_1 < \delta$, we first compute

$$X_{\tau_1} = \tau_1 b(X_0) + \sqrt{\tau_1} \sigma(X_0) N + \xi(X_0) \zeta_1$$

with $N \sim \mathcal{N}(0, 1)$ and $\zeta_1 \sim f$ is independent of N . If $\delta < \tau_2$, we compute

$$X_\delta = (\delta - \tau_1) b(X_{\tau_1}) + \sqrt{\delta - \tau_1} \sigma(X_{\tau_1}) N'$$

else we compute

$$X_{\tau_2} = (\tau_2 - \tau_1) b(X_{\tau_1}) + \sqrt{\tau_2 - \tau_1} \sigma(X_{\tau_1}) N' + \xi(X_{\tau_1}) \zeta_2$$

where $N' \sim \mathcal{N}(0, 1)$ and ζ_2 has distribution f . N , N' , ζ_1 and ζ_2 are independent.

2. Construct the random variables

$$Y_{k\Delta} = \frac{X_{(k+1)\Delta} - X_{k\Delta}}{\Delta} \quad \text{and} \quad \tilde{Y}_{k\Delta} = \frac{X_{(k+1)\Delta} - X_{k\Delta}}{\Delta} 1_{\Omega_{X,k}} 1_{X_{k\Delta} \in A}.$$

3. We consider the vectorial subspaces $S_{m,r}$ generated by the spline functions of degree r (see for instance Schmisser (2013)). In that case $D_{m,r} = \dim(S_{m,r}) = 2^m + r$. For $r \in \{1, 2, 3\}$ and $m \in \mathcal{M}_n(r) = \{m, D_{m,r} \leq \mathcal{D}_n\}$, we compute the estimators $\hat{b}_{m,r}$ and $\tilde{b}_{m,r}$ by minimising the contrast functions γ_n and $\tilde{\gamma}_n$ on the vectorial subspaces $S_{m,r}$.
4. For the estimation algorithm, we make a selection of m and r as follows. Using the penalty function $pen(m, r) := pen(m) = \kappa(\sigma_0^2 + \xi_0^2)(2^m + r)/n\Delta$, we select the adaptive estimators $\hat{b}_{\hat{m},r}$ and $\tilde{b}_{\tilde{m},r}$, and then choose the best r by minimizing $\gamma_n(\hat{b}_{\hat{m},r}) + pen(\hat{m}, r)$ and $\tilde{\gamma}_n(\tilde{b}_{\tilde{m},r}) + pen(\tilde{m}, r)$.

To calibrate κ , we run a various number of simulations for a model with known parameters and let κ vary. When κ is too small, the value of m selected by the estimation procedure is in general very high (often maximal). When κ is too big, the estimator is always linear even if the true function is not. We used the true value of σ_0^2 and ξ_0^2 .

5.3 Results

In Figures 1-4, we simulate 5 times the process $(X_0, \dots, X_{n\Delta})$ for $\Delta = 10^{-1}$ and $n = 10^4$ and draw the obtained estimators. The two adaptive estimators are nearly superposed, moreover, they are close to the true function.

In Tables 1-4, for each value of (n, Δ) , we simulate 50 trajectories of $(X_0, X_\Delta, \dots, X_{n\Delta})$. For each path, we construct the two adaptive estimators $\hat{b}_{\hat{m}, \hat{r}}$ and $\tilde{b}_{\tilde{m}, \tilde{r}}$ and we compute the empirical errors:

$$err_1 = \left\| \hat{b}_{\hat{m}, \hat{r}} - b_A \right\|_n^2 \quad \text{and} \quad err_2 = \left\| \tilde{b}_{\tilde{m}, \tilde{r}} - b_A \right\|_n^2.$$

In order to check that our algorithm is adaptive, we also compute the minimal errors

$$emin_1 = \min_{m,r} \left\| \hat{b}_{m,r} - b_A \right\|_n^2 \quad \text{and} \quad emin_2 = \min_{m,r} \left\| \tilde{b}_{m,r} - b_A \right\|_n^2$$

and the oracles $oracle_i = err_i / emin_i$. We give the means \hat{m}_a , \hat{r}_a , \tilde{m}_a and \tilde{r}_a of the selected values \hat{m} , \hat{r} , \tilde{m} and \tilde{r} . The value $risk_i$ is the mean of err_i over the 50 simulations and or_i is the mean of $oracle_i$. The computation time for one adaptive estimator varies from 0.1 second ($\Delta = 10^{-1}$, $n = 10^3$) to 30 seconds ($\Delta = 10^{-1}$, $n = 10^4$). The empirical risk is decreasing when the product $n\Delta$ is increasing, which is coherent with the theoretical model. For Model 1, the two estimators are equivalent. When the tails of ν become larger (Models 2 and 3), the truncated estimator is better. The improvement is also more significant when the discretization distance is smaller. As on the first three models, the processes L_t are compound Poisson processes, these results were expected. The truncated estimator seems also more robust: we do not observe aberrant values (like for the first estimator in Table 2). Those aberrant values may be due to the fact that b is not Lipschitz and then $b(X_{k\Delta})$ may be quite large, and to the non-exact simulation by an Euler scheme. For Model 4, the results are slightly better for the first estimator when $\Delta = 0.1$, which is due to the fact that the remainder term is greater for the truncated estimator. When $\Delta = 10^{-2}$, the risk of the truncated estimator is lower than for the first estimator.

6 Proofs

Let us introduce the filtration

$$\mathcal{F}_t = \sigma \left(\eta, (W_s)_{0 \leq s \leq t}, (L_s)_{0 \leq s \leq t} \right).$$

The following result is very useful. It comes from Dellacherie and Meyer (1980) (Theorem 92 Chapter VII) and Applebaum (2004), Theorem 4.4.23 p265 (Kunita's first inequality).

Result 11 (Burkholder-Davis-Gundy inequality). We have that, for any $p \geq 2$,

$$\mathbb{E} \left[\sup_{s \in [t, t+h]} \left| \int_t^s \sigma(X_u) dW_u \right|^p \middle| \mathcal{F}_t \right] \leq C_p \left(\mathbb{E} \left[\left| \int_t^{t+h} \sigma^2(X_u) du \right|^{p/2} \middle| \mathcal{F}_t \right] \right)$$

and, if $\int_{\mathbb{R}} |z|^p \nu(dz) < \infty$, as $\int_{\mathbb{R}} z^2 \nu(dz) = 1$:

$$\begin{aligned} \mathbb{E} \left[\sup_{s \in [t, t+h]} \left| \int_t^s \xi(X_{u-}) dL_u \right|^p \middle| \mathcal{F}_t \right] &\leq C_p \mathbb{E} \left[\left(\int_t^{t+h} \xi^2(X_u) du \right)^{p/2} \middle| \mathcal{F}_t \right] \\ &+ C_p \mathbb{E} \left[\left(\int_t^{t+h} |\xi(X_u)|^p du \right) \middle| \mathcal{F}_t \right] \int_{\mathbb{R}} |z|^p \nu(dz). \end{aligned}$$

6.1 Proof of Theorem 2

By (3) and (4), we get:

$$\begin{aligned} \gamma_n(t) = \frac{1}{n} \sum_{k=1}^n (Y_{k\Delta} - t(X_{k\Delta}))^2 &= \frac{1}{n} \sum_{k=1}^n (Y_{k\Delta} - b(X_{k\Delta}))^2 + \|b - t\|_n^2 \\ &+ \frac{2}{n} \sum_{k=1}^n (Y_{k\Delta} - b(X_{k\Delta})) (b(X_{k\Delta}) - t(X_{k\Delta})). \end{aligned}$$

As, by definition, $\gamma_n(\hat{b}_m) \leq \gamma_n(b_m)$, we obtain:

$$\left\| \hat{b}_m - b \right\|_n^2 \leq \|b_m - b\|_n^2 + \frac{2}{n} \sum_{k=1}^n (Y_{k\Delta} - b(X_{k\Delta})) \left(\hat{b}_m(X_{k\Delta}) - b_m(X_{k\Delta}) \right).$$

By (2), and as \hat{b}_m and b_m are supported by A ,

$$\left\| \hat{b}_m - b_A \right\|_n^2 \leq \|b_m - b_A\|_n^2 + \frac{2}{n} \sum_{k=1}^n (I_{k\Delta} + Z_{k\Delta} + T_{k\Delta}) \left(\hat{b}_m(X_{k\Delta}) - b_m(X_{k\Delta}) \right).$$

Let us introduce the unit ball

$$\mathcal{B}_m = \{t \in S_m, \|t\|_{\varpi} \leq 1\} \quad \text{where} \quad \|t\|_{\varpi}^2 = \int_A t^2(x) \varpi(dx)$$

and the englobing space $\mathcal{S}_n = \bigcup_{m \in \mathcal{M}_n} S_m$. Let us consider the set

$$\Omega_n = \left\{ \omega, \forall t \in \mathcal{S}_n, \left| \frac{\|t\|_n^2}{\|t\|_{\varpi}^2} - 1 \right| \leq \frac{1}{2} \right\}$$

where the norms $\|\cdot\|_{\varpi}$ and $\|\cdot\|_n$ are equivalent.

Step 1: bound of the risk on Ω_n Thanks to the Cauchy-Schwartz inequality, we obtain that, on Ω_n :

$$\left\| \hat{b}_m - b_A \right\|_n^2 \leq \|b_m - b_A\|_n^2 + \frac{1}{12} \left\| \hat{b}_m - b_m \right\|_n^2 + 12 \sum_{k=1}^n I_{k\Delta}^2 + \frac{1}{12} \left\| \hat{b}_m - b_m \right\|_{\mathcal{W}}^2 + 12 \sup_{t \in \mathcal{B}_m} \nu_n^2(t)$$

where

$$\nu_n(t) = \frac{1}{n} \sum_{k=1}^n (Z_{k\Delta} + T_{k\Delta}) t(X_{k\Delta}). \quad (5)$$

On Ω_n , by definition, we have:

$$\left\| \hat{b}_m - b_m \right\|_n^2 \leq 2 \left\| \hat{b}_m - b_A \right\|_n^2 + 2 \|b_m - b_A\|_n^2 \quad \text{and} \quad \left\| \hat{b}_m - b_m \right\|_{\mathcal{W}}^2 \leq 2 \left\| \hat{b}_m - b_m \right\|_n^2.$$

Thus we obtain:

$$\left\| \hat{b}_m - b_A \right\|_n^2 \leq 3 \|b_m - b_A\|_n^2 + 24 \sum_{k=1}^n I_{k\Delta}^2 + 24 \sup_{t \in \mathcal{B}_m} \nu_n^2(t).$$

The following lemma is very useful. It is derived from Proposition 1 and Result 11.

Lemma 12. 1. $\mathbb{E}(I_{k\Delta}^2) \leq c\Delta$ and $\mathbb{E}(I_{k\Delta}^4) \leq c\Delta$.

2. $\mathbb{E}(Z_{k\Delta} | \mathcal{F}_{k\Delta}) = 0$, $\mathbb{E}(Z_{k\Delta}^2 | \mathcal{F}_{k\Delta}) \leq \sigma_0^2/\Delta$ and $\mathbb{E}(Z_{k\Delta}^4 | \mathcal{F}_{k\Delta}) \leq c/\Delta^2$.

3. $\mathbb{E}(T_{k\Delta} | \mathcal{F}_{k\Delta}) = 0$, $\mathbb{E}(T_{k\Delta}^2 | \mathcal{F}_{k\Delta}) \leq \xi_0^2/\Delta$ and $\mathbb{E}(T_{k\Delta}^4 | \mathcal{F}_{k\Delta}) \leq c/\Delta^3$.

By Lemma 12, $\mathbb{E}[I_{k\Delta}^2] \leq \Delta$. It remains to bound $\mathbb{E}[\sup_{t \in \mathcal{B}_m} \nu_n^2(t)]$. We consider an orthonormal basis $(\varphi_\lambda)_{\lambda \in \Lambda_m}$ of S_m for the $L_{\mathcal{W}}^2$ -norm with $|\Lambda_m| = D_m$. Any function $t \in S_m$ can be written $t = \sum_{\lambda \in \Lambda_m} a_\lambda \varphi_\lambda$ and $\|t\|_{\mathcal{W}}^2 = \sum_{\lambda \in \Lambda_m} a_\lambda^2$. Then:

$$\begin{aligned} \sup_{t \in \mathcal{B}_m} \nu_n^2(t) &= \sup_{\sum_{\lambda} a_\lambda^2 \leq 1} \left(\sum_{\lambda \in \Lambda_m} a_\lambda \nu_n(\varphi_\lambda) \right)^2 \\ &\leq \sup_{\sum_{\lambda} a_\lambda^2 \leq 1} \left(\sum_{\lambda \in \Lambda_m} a_\lambda^2 \right) \left(\sum_{\lambda \in \Lambda_m} \nu_n^2(\varphi_\lambda) \right) \\ &= \sum_{\lambda \in \Lambda_m} \nu_n^2(\varphi_\lambda). \end{aligned}$$

It remains to bound $\mathbb{E}(\nu_n^2(\varphi_\lambda))$. By (5),

$$\begin{aligned} \mathbb{E}[\nu_n^2(\varphi_\lambda)] &= \frac{1}{n^2} \sum_{k=1}^n \mathbb{E}[\varphi_\lambda^2(X_{k\Delta}) \mathbb{E}[(Z_{k\Delta} + T_{k\Delta})^2 | \mathcal{F}_{k\Delta}]] \\ &\quad + \frac{2}{n^2} \sum_{k < l} \mathbb{E}[(Z_{k\Delta} + T_{k\Delta}) \varphi_\lambda(X_{k\Delta}) \varphi_\lambda(X_{l\Delta}) \mathbb{E}[Z_{l\Delta} + T_{l\Delta} | \mathcal{F}_{l\Delta}]] \end{aligned}$$

Thanks to Lemma 12, the second term of this inequality is null and we obtain, as $\int_{\mathbb{R}} \varphi_\lambda^2(x) \varpi(dx) = 1$:

$$\mathbb{E} [\nu_n^2(\varphi_\lambda)] \leq \frac{2(\sigma_0^2 + \xi_0^2)}{n^2 \Delta} \sum_{k=1}^n \mathbb{E} [\varphi_\lambda^2(X_{k\Delta})] = \frac{2(\sigma_0^2 + \xi_0^2)}{n\Delta}.$$

Therefore:

$$\mathbb{E} \left[\left\| \hat{b}_m - b_A \right\|_n^2 \mathbf{1}_{\Omega_n^c} \right] \leq 3 \|b_m - b_A\|_n^2 + 48(\sigma_0^2 + \xi_0^2) \frac{D_m}{n\Delta} + C\Delta.$$

Step 2: bound of the risk on Ω_n^c . The process $(X_t)_{t \geq 0}$ is exponentially β -mixing, π is bounded from below and above and $n\Delta \rightarrow \infty$. The following result is proved for $\xi = 0$ for instance in Comte *et al.* (2007) for diffusion processes, but as it relies only on the β -mixing property, we can apply it.

Result 13.

$$\mathbb{P} [\Omega_n^c] \leq \frac{1}{n^3}.$$

Let us set $e = (e_\Delta, \dots, e_{n\Delta})^*$ where $e_{k\Delta} := Y_{k\Delta} - b(X_{k\Delta}) = I_{k\Delta} + Z_{k\Delta} + T_{k\Delta}$ and $\Pi_m Y = \Pi_m (Y_\Delta, \dots, Y_{n\Delta})^* = (\hat{b}_m(X_0), \dots, \hat{b}_m(X_{n\Delta}))^*$ where Π_m is the Euclidean orthogonal projection over S_m . Then

$$\begin{aligned} \left\| \hat{b}_m - b_A \right\|_n^2 &= \left\| \Pi_m Y - b_A \right\|_n^2 = \left\| \Pi_m b_A - b_A \right\|_n^2 + \left\| \Pi_m Y - \Pi_m b_A \right\|_n^2 \\ &\leq \|b_A\|_n^2 + \|e\|_n^2. \end{aligned}$$

According to Lemma 12, Result 13 and the Cauchy-Schwarz inequality,

$$\mathbb{E} \left[\|e\|_n^2 \mathbf{1}_{\Omega_n^c} \right] \leq \left(\mathbb{E} \left[\|e\|_n^4 \right] \right)^{1/2} (\mathbb{P} (\Omega_n^c))^{1/2} \leq \frac{C}{(\Delta^3 n^3)^{1/2}} \leq \frac{C}{n\Delta}$$

and, as b is bounded on the compact set A ,

$$\mathbb{E} \left[\|b_A\|_n^2 \mathbf{1}_{\Omega_n^c} \right] \leq \left(\mathbb{E} \left[\|b_A\|_n^4 \right] \right)^{1/2} \mathbb{P} (\Omega_n^c) \lesssim \frac{1}{n^{3/2}}.$$

Collecting the results, we get:

$$\mathbb{E} \left[\left\| \hat{b}_m - b_A \right\|_n^2 \mathbf{1}_{\Omega_n^c} \right] \lesssim \frac{1}{n\Delta}$$

which ends the proof of Theorem 2.

6.2 Proof of Theorem 4

The bound of the risk on Ω_n^c is done exactly in the same way as for the non adaptive estimator. It remains thus to bound the risk on Ω_n . As in the previous proof, we get:

$$\begin{aligned} \left\| \hat{b}_{\hat{m}} - b_A \right\|_n^2 1_{\Omega_n} &\leq 3 \|b_m - b_A\|_n^2 + \frac{24}{n} \sum_{k=1}^n I_{k\Delta}^2 + 2pen(m) - 2pen(\hat{m}) \\ &+ 24 \sup_{t \in \mathcal{B}_{m, \hat{m}}} \nu_n^2(t) \end{aligned}$$

where $\mathcal{B}_{m, m'}$ is the unit ball (for the L_{ϖ}^2 -norm) of the subspace $S_m + S_{m'}$: $\mathcal{B}_{m, m'} = \{t \in S_m + S_{m'}, \|t\|_{\varpi} \leq 1\}$. Let us introduce a function $p(m, m')$ such that $12p(m, m') = pen(m) + pen(m')$. We obtain that, on Ω_n , for any $m \in \mathcal{M}_n$:

$$\begin{aligned} \left\| \hat{b}_{\hat{m}} - b_A \right\|_n^2 &\leq 3 \|b_m - b_A\|_n^2 + \frac{24}{n} \sum_{k=1}^n I_{k\Delta}^2 + 4pen(m) \\ &+ 24 \sup_{t \in \mathcal{B}_{m, \hat{m}}} (\nu_n^2(t) - p(m, \hat{m})). \end{aligned}$$

It remains to bound

$$\mathbb{E} \left[\sup_{t \in \mathcal{B}_{m, \hat{m}}} \nu_n^2(t) - p(m, \hat{m}) \right] \leq \sum_{m'} \mathbb{E} \left[\sup_{t \in \mathcal{B}_{m, m'}} \nu_n^2(t) - p(m, m') \right]_+.$$

For this purpose, we use the following proposition proved in Applebaum (2004) (Corollary 5.2.2).

Proposition 14 : exponential martingale.

Let $(Y_t)_{t \geq 0}$ satisfy:

$$Y_t = \int_0^t F_s dW_s + \int_0^t K_s dL_s - \int_0^t \left[\frac{F_s^2}{2} + \int_{\mathbb{R}} (e^{K_s z} - 1 - K_s z) \nu(dz) \right] ds$$

where F_s and K_s are locally integrable and predictable processes. If for any $t > 0$,

$$\mathbb{E} \left[\int_0^t \int_{|z| > 1} |e^{K_s z} - 1| \nu(dz) ds \right] < \infty,$$

then e^{Y_t} is a \mathcal{G}_t -local martingale where $\mathcal{G}_t = \sigma(W_s, L_s, 0 \leq s \leq t)$.

For any $\varepsilon \leq \varepsilon_1 := (\lambda \wedge 1)/(2 \|t\|_{\infty} \xi_0)$ where λ is defined in Assumption A5, for any $t \geq 0$

$$\int_0^t \int_{|z| \geq 1} (\exp(\varepsilon t (X_{k\Delta}) \xi(X_s) z) - 1) \nu(dz) 1_{s \in]k\Delta, (k+1)\Delta]} ds < \infty.$$

Let us introduce the two Markov processes

$$A_{\varepsilon,t} := \varepsilon^2 \sum_{k=0}^n t^2(X_{k\Delta}) \int_0^t \sigma^2(X_s) 1_{s \in]k\Delta, (k+1)\Delta]} ds$$

and

$$B_{\varepsilon,t} := \sum_{k=0}^n \int_0^t \int_{\mathbb{R}} (\exp(\varepsilon t(X_{k\Delta})\xi(X_s)z) - \varepsilon t(X_{k\Delta})\xi(X_s)z - 1) 1_{s \in]k\Delta, (k+1)\Delta]} \nu(dz) ds$$

and the following martingale:

$$M_t = \int_0^t \sum_{k=0}^n 1_{s \in]k\Delta, (k+1)\Delta]} t(X_{k\Delta-}) (\sigma(X_s) dW_s + \xi(X_{s-}) dL_s).$$

By Proposition 14,

$$Y_{\varepsilon,s} := \varepsilon M_s - A_{\varepsilon,s} - B_{\varepsilon,s}$$

is such that $e^{Y_{\varepsilon,s}}$ is a local martingale.

Bound of $A_{\varepsilon,s}$ and $B_{\varepsilon,s}$. We obtain easily that $A_{\varepsilon,s} \leq A_{\varepsilon,(n+1)\Delta} \leq \varepsilon^2 n\Delta \|t\|_n^2 \sigma_0^2$. Under Assumption A5, ξ is constant or ν is symmetric, and therefore

$$B_{\varepsilon,s} \leq B_{\varepsilon,(n+1)\Delta} \leq \Delta \sum_{k=0}^n \int_{\mathbb{R}} (\exp(\varepsilon t(X_{k\Delta})\xi_0 z) - \varepsilon t(X_{k\Delta-})\xi_0 z - 1) \nu(dz).$$

As $\int_{\mathbb{R}} z^2 \nu(dz) = 1$, for any $\alpha \leq 1$,

$$\int_{-1}^1 (\exp(\alpha z) - \alpha z - 1) \nu(dz) \leq \alpha^2 \int_{-1}^1 z^2 \nu(dz) \leq \alpha^2.$$

Moreover, by integration by parts, for any $\alpha \leq (1 \wedge \lambda)/2$,

$$\begin{aligned} \int_{[-1,1]^c} (\exp(\alpha z) - \alpha z - 1) \nu(dz) &\leq (e^\alpha - \alpha - 1) \nu([1, +\infty[) + (e^{-\alpha} + \alpha - 1) \nu(]-\infty, -1]) \\ &\quad + \int_1^{+\infty} \alpha (e^{\alpha z} - 1) \nu([-z, z]^c) dz \end{aligned}$$

By assumption A5, $\nu([-z, z]^c) \leq C e^{-\lambda z}$ and then

$$\int_{[-1,1]^c} (\exp(\alpha z) - \alpha z - 1) \nu(dz) \leq 2\alpha^2 \nu([-1, 1]^c) + C e^{-\lambda} \frac{\alpha}{\lambda} \left(\frac{e^\alpha}{1 - \alpha/\lambda} - 1 \right) \leq C' \alpha^2.$$

Then $B_{\varepsilon,s} \lesssim n\Delta \varepsilon^2 \xi_0^2 \|t\|_n^2$. There exists a constant c such that, for any $\varepsilon < \varepsilon_1$,

$$A_{\varepsilon,s} + B_{\varepsilon,s} \leq c \frac{n\Delta \varepsilon^2 (\sigma_0^2 + \xi_0^2) \|t\|_n^2}{(1 - \varepsilon/\varepsilon_1)}.$$

Bound of $\mathbb{P}(\nu_n(t) \geq \eta, \|t\|_n^2 \leq \zeta^2)$. The process $\exp(Y_{\varepsilon,t})$ is a local martingale, then there exists an increasing sequence (τ_N) of stopping times such that $\lim_{N \rightarrow \infty} \tau_N = \infty$ and $\exp(Y_{\varepsilon,t \wedge \tau_N})$ is a \mathcal{F}_t -martingale. For any $\varepsilon < \varepsilon_1$, and all N ,

$$\begin{aligned} E &:= \mathbb{P}\left(M_{(n+1)\Delta \wedge \tau_N} \geq n\Delta\eta, \|t\|_n^2 \leq \zeta^2\right) \\ &\leq \mathbb{P}\left(M_{(n+1)\Delta \wedge \tau_N} \geq n\Delta\eta, A_{(n+1)\Delta \wedge \tau_N} + B_{(n+1)\Delta \wedge \tau_N} \leq \frac{cn\Delta\varepsilon^2(\sigma_0^2 + \xi_0^2)\zeta^2}{(1 - \varepsilon/\varepsilon_1)}\right) \\ &\leq \mathbb{E}\left(\exp(Y_{\varepsilon,(n+1)\Delta \wedge \tau_N}) \exp\left(-n\Delta\eta\varepsilon + \frac{cn\Delta\varepsilon^2(\xi_0^2 + \sigma_0^2)\zeta^2}{(1 - \varepsilon/\varepsilon_1)}\right)\right). \end{aligned}$$

As $\exp(Y_{\varepsilon,t \wedge \tau_N})$ is a martingale, $\mathbb{E}(\exp(Y_{\varepsilon,t \wedge \tau_N})) = 1$ and

$$E \leq \exp\left(-n\Delta\eta\varepsilon + \frac{cn\Delta\varepsilon^2(\xi_0^2 + \sigma_0^2)\zeta^2}{(1 - \varepsilon/\varepsilon_1)}\right).$$

Letting N tend to infinity, by dominated convergence, and as $\nu_n(t) = n\Delta M_{(n+1)\Delta}$, we obtain that

$$\mathbb{P}\left(\nu_n(t) \geq \eta, \|t\|_n^2 \leq \zeta^2\right) \leq \exp\left(-n\Delta\eta\varepsilon + \frac{cn\Delta\varepsilon^2(\xi_0^2 + \sigma_0^2)\zeta^2}{(1 - \varepsilon/\varepsilon_1)}\right).$$

It remains to minimise this inequality in ε . Let us set

$$\varepsilon = \frac{\eta}{2c(\sigma_0^2 + \xi_0^2)\zeta^2/\Delta + \eta/\varepsilon_1} < \varepsilon_1.$$

We get:

$$\mathbb{P}\left(\nu_n(t) \geq \eta, \|t\|_n^2 \leq \zeta^2\right) \leq \exp\left(-\frac{\eta^2 n \Delta}{4c((\sigma_0^2 + \xi_0^2)\zeta^2 + c'\eta\xi_0\|t\|_\infty)}\right).$$

The following lemma concludes the proof. It is proved thanks to a $L^2_\infty - L^\infty$ chaining technique. See Comte (2001), proof of Proposition 4, and Schmisser (2010), Appendix D.3.

Lemma 15.

There exists a constant κ such that:

$$\mathbb{E}\left[\sup_{t \in \mathcal{B}_{m,m'}} \nu_n^2(t) - p(m, m')\right] \lesssim \kappa(\xi_0^2 + \sigma_0^2) \frac{D^{3/2}}{n\Delta} e^{-\sqrt{D}}$$

where $D = \dim(S_m + S_{m'})$.

As $\sum_D D^{3/2} e^{-\sqrt{D}} \leq \sum_{k=0}^{+\infty} k^{3/2} e^{-\sqrt{k}} < \infty$, we obtain that

$$\mathbb{E}\left[\sup_{t \in \mathcal{B}_{m,\hat{m}}} \nu_n^2(t) - p(m, \hat{m})\right] \leq \sum_{m' \in \mathcal{M}_n} \mathbb{E}\left[\sup_{t \in \mathcal{B}_{m,m'}} \nu_n^2(t) - p(m, m')\right] \lesssim \kappa \frac{\xi_0^2 + \sigma_0^2}{n\Delta}.$$

6.3 Proof of Theorem 7

We recall that

$$\Omega_{X,k} = \left\{ \omega, |X_{(k+1)\Delta} - X_{k\Delta}| \leq C_\Delta = (b_{max} + 3)\Delta + (\sigma_0 + 4\xi_0)\sqrt{\Delta \ln(n)} \right\}.$$

Let us introduce the set

$$\Omega_{N,k} = \left\{ \omega, N'_{k\Delta} = 0 \right\}$$

where $N'_{k\Delta}$ is the number of jumps of size larger than $\Delta^{1/4}$ occurring in the time interval $]k\Delta, (k+1)\Delta]$:

$$N'_{k\Delta} = \mu \left(]k\Delta, (k+1)\Delta], \left[-\Delta^{1/4}, \Delta^{1/4} \right]^c \right).$$

We have that

$$\begin{aligned} \tilde{Y}_{k\Delta} &= Y_{k\Delta} 1_{\Omega_{X,k}} 1_{X_{k\Delta} \in A} \\ &= b_A(X_{k\Delta}) - b_A(X_{k\Delta}) 1_{\Omega_{X,k}^c \cap (X_{k\Delta} \in A)} + I_{k\Delta} 1_{\Omega_{X,k} \cap (X_{k\Delta} \in A)} + \tilde{Z}_{k\Delta} + \tilde{T}_{k\Delta} \\ &\quad + (Z_{k\Delta} + T_{k\Delta}) 1_{\Omega_{X,k} \cap \Omega_{N,k}^c \cap (X_{k\Delta} \in A)} + \mathbb{E} \left((Z_{k\Delta} + T_{k\Delta}) 1_{\Omega_{X,k} \cap \Omega_{N,k} \cap (X_{k\Delta} \in A)} \mid \mathcal{F}_{k\Delta} \right). \end{aligned}$$

where

$$\tilde{Z}_{k\Delta} = Z_{k\Delta} 1_{\Omega_{X,k} \cap \Omega_{N,k} \cap (X_{k\Delta} \in A)} - \mathbb{E} \left(Z_{k\Delta} 1_{\Omega_{X,k} \cap \Omega_{N,k} \cap (X_{k\Delta} \in A)} \mid \mathcal{F}_{k\Delta} \right)$$

and

$$\tilde{T}_{k\Delta} = T_{k\Delta} 1_{\Omega_{X,k} \cap \Omega_{N,k} \cap (X_{k\Delta} \in A)} - \mathbb{E} \left(T_{k\Delta} 1_{\Omega_{X,k} \cap \Omega_{N,k} \cap (X_{k\Delta} \in A)} \mid \mathcal{F}_{k\Delta} \right).$$

As previously, we only bound the risk on Ω_n . Let us set

$$\tilde{v}_n(t) := \frac{1}{n} \sum_{k=1}^n t(X_{k\Delta}) \left(\tilde{Z}_{k\Delta} + \tilde{T}_{k\Delta} \right).$$

We have that

$$\begin{aligned} \left\| \tilde{b}_m - b_A \right\|_n^2 1_{\Omega_n} &\leq 3 \|b_m - b_A\|_n^2 + 24 \sup_{t \in \mathcal{B}_m} \tilde{v}_n^2(t) + \frac{224}{n} \sum_{k=1}^n \left(I_{k\Delta}^2 + b_A^2(X_{k\Delta}) 1_{\Omega_{X,k}^c} \right) \\ &\quad + \frac{224}{n} \sum_{k=1}^n (Z_{k\Delta}^2 + T_{k\Delta}^2) 1_{\Omega_{X,k} \cap \Omega_{N,k}^c \cap (X_{k\Delta} \in A)} \\ &\quad + \frac{224}{n} \sum_{k=1}^n \left(\mathbb{E} \left[(Z_{k\Delta} + T_{k\Delta}) 1_{\Omega_{X,k} \cap \Omega_{N,k} \cap (X_{k\Delta} \in A)} \mid \mathcal{F}_{k\Delta} \right] \right)^2. \end{aligned}$$

The following lemma is proved later.

Lemma 16. 1. $\mathbb{P}(\Omega_{X,k}^c \cap (X_{k\Delta} \in A)) \lesssim \Delta^{1-\beta/2}$.

2. $\mathbb{P}(\Omega_{X,k} \cap \Omega_{N,k}^c \cap (X_{k\Delta} \in A)) \lesssim \Delta^{2-\beta/2}$.

3. $(\mathbb{E} [(Z_{k\Delta} + T_{k\Delta}) 1_{\Omega_{N,k} \cap \Omega_{X,k} \cap (X_{k\Delta} \in A)} | \mathcal{F}_{k\Delta}])^2 \lesssim \ln^2(n) \Delta^{1-\beta/2}$.

According to Lemma 12, $\mathbb{E}(I_{k\Delta}^2) \leq \Delta$. As b is bounded on the compact set A , $\mathbb{E} [b_A^2(X_{k\Delta}) 1_{\Omega_{X,k}^c}] \lesssim \mathbb{P}(\Omega_{X,k}^c) \lesssim \Delta^{1-\beta/2}$. Moreover, on $\Omega_{X,k}$,

$$\begin{aligned} (Z_{k\Delta} + T_{k\Delta})^2 1_{\Omega_{X,k} \cap (X_{k\Delta} \in A)} &= \left(\frac{X_{(k+1)\Delta} - X_{k\Delta}}{\Delta} - b_A(X_{k\Delta}) - I_{k\Delta} \right)^2 1_{\Omega_{X,k}} 1_{X_{k\Delta} \in A} \\ &\lesssim \frac{\ln^2(n)}{\Delta} + b_A^2(X_{k\Delta}) + I_{k\Delta}^2 \end{aligned}$$

and then

$$\begin{aligned} E &:= \mathbb{E} \left[(Z_{k\Delta} + T_{k\Delta})^2 1_{\Omega_{X,k} \cap \Omega_{N,k}^c \cap (X_{k\Delta} \in A)} \right] \\ &\lesssim \left(\frac{\ln^2(n)}{\Delta} + b_{max}^2 \right) \mathbb{P}(\Omega_{X,k} \cap \Omega_{N,k}^c \cap (X_{k\Delta} \in A)) + \mathbb{E}(I_{k\Delta}^2) \\ &\lesssim \ln^2(n) \Delta^{1-\beta/2}. \end{aligned}$$

It remains to bound $\mathbb{E}(\sup_{t \in \mathcal{B}_m} \tilde{\nu}_n^2(t))$. In the same way as in Subsection 6.1, we get:

$$\begin{aligned} \mathbb{E} \left(\sup_{t \in \mathcal{B}_m} \tilde{\nu}_n^2(t) \right) &\leq \sum_{\lambda \in \Lambda_m} \mathbb{E}(\tilde{\nu}_n^2(\varphi_\lambda)) \leq \frac{2D_m}{n} \mathbb{E}(\tilde{Z}_\Delta^2 + \tilde{T}_\Delta^2) \\ &\leq \frac{2D_m}{n} \mathbb{E}(Z_\Delta^2 + T_\Delta^2) \leq 2(\sigma_0^2 + \xi_0^2) \frac{D_m}{n\Delta}. \end{aligned}$$

We have that $\mathbb{E}(\tilde{Z}_\Delta^2) \leq \mathbb{E}(Z_\Delta^2) \leq \frac{\sigma_0^2}{\Delta}$. Moreover,

$$\begin{aligned} \mathbb{E}(\tilde{T}_{k\Delta}^2) &\lesssim \mathbb{E}(T_{k\Delta}^2 1_{\Omega_{X,k} \cap \Omega_{N,k}}) - (\mathbb{E}(T_{k\Delta} 1_{\Omega_{X,k} \cap \Omega_{N,k}}))^2 \\ &\lesssim \mathbb{E}(T_{k\Delta}^2 1_{\Omega_{N,k}}) + \ln^2(n) \Delta^{1-\beta/2} \\ &\lesssim \Delta^{1/2-\beta/4}. \end{aligned}$$

Then $\mathbb{E}(\sup_{t \in \mathcal{B}_m} \tilde{\nu}_n^2(t)) \leq (n\Delta)^{-1} D_m(\sigma_0^2 + o(1))$.

6.3.1 Proof of Lemma 16

Result 17. Let β be the Blumenthal-Gettoor index of L_t . Then:

$$\nu([-z, z]^c) \lesssim z^{-\beta}, \quad \int_{|x| \leq z \wedge a_0} x^2 \nu(dx) \lesssim z^{2-\beta} \quad \text{and} \quad \int_{|x| \leq z \wedge a_0} x^4 \nu(dx) \lesssim z^{4-\beta}.$$

The constant a_0 is defined in A6.

Bound of $\mathbb{P}(\Omega_{X,k}^c \cap (X_{k\Delta} \in A))$. We have:

$$\mathbb{P}(\Omega_{X,k}^c \cap (X_{k\Delta} \in A)) = \mathbb{P}(\{|X_{(k+1)\Delta} - X_{k\Delta}| > C_\Delta\} \cap (X_{k\Delta} \in A)).$$

We know that $X_{(k+1)\Delta} - X_{k\Delta} = b(X_{k\Delta}) + I_{k\Delta} + Z_{k\Delta} + T_{k\Delta}$. Then

$$\begin{aligned} \mathbb{P}(\Omega_{X,k}^c \cap (X_{k\Delta} \in A)) &\leq \mathbb{P}(|\Delta I_{k\Delta}| \geq \Delta) \\ &\quad + \mathbb{P}(|\Delta Z_{k\Delta}| \geq \sigma_0 \sqrt{\Delta} \ln(n)) + \mathbb{P}(|\Delta T_{k\Delta}| \geq \xi_0 \sqrt{\Delta} \ln(n)). \end{aligned}$$

By a Markov inequality and Lemma 12, we obtain:

$$\mathbb{P}(|\Delta I_{k\Delta}| \geq \Delta) \leq \frac{\mathbb{E}(\Delta^2 I_{k\Delta}^2)}{\Delta^2} \lesssim \Delta. \quad (6)$$

By Proposition 14, the process $\exp\left(c \int_0^t \sigma(X_{s-}) dW_s - c^2 \int_0^t \sigma^2(X_s) ds\right)$ is a local martingale (as σ is bounded, it is in fact a martingale, see Liptser and Shiryaev (2001), pp 229-232). Then, by a Markov inequality:

$$\mathbb{P}(|\Delta Z_{k\Delta}| \geq \sigma_0 \sqrt{\Delta} \ln(n)) \leq \frac{2}{n} \mathbb{E} \left[\exp \left(\frac{\sqrt{\Delta} Z_{k\Delta}}{\sigma_0} \right) \right] \lesssim \frac{1}{n}. \quad (7)$$

To bound inequality (6), it remains to bound $\mathbb{P}(|\Delta T_{k\Delta}| \geq \xi_0 \sqrt{\Delta} \ln(n))$. Let us set

$$T_{k\Delta} = T_{k\Delta}^{(1)} + T_{k\Delta}^{(2)} + T_{k\Delta}^{(3)} \quad \text{where} \quad T_{k\Delta}^{(i)} = \frac{1}{\Delta} \int_{k\Delta}^{(k+1)\Delta} \xi(X_{s-}) dL_s^{(i)}$$

with

$$\begin{aligned} L_t^{(1)} &= \int_0^t \int_{[-\sqrt{\Delta}, \sqrt{\Delta}]} z \tilde{\mu}(ds, dz), \quad L_t^{(2)} = \int_0^t \int_{[-\Delta^{1/4}, -\sqrt{\Delta}] \cup [\sqrt{\Delta}, \Delta^{1/4}]} z \tilde{\mu}(ds, dz) \\ L_t^{(3)} &= \int_0^t \int_{[-\Delta^{1/4}, \Delta^{1/4}]^c} z \tilde{\mu}(ds, dz). \end{aligned}$$

Let us set $N_{k\Delta}'' = \mu\left(]k\Delta, (k+1)\Delta], [-\sqrt{\Delta}, \sqrt{\Delta}]^c\right)$. By Result 17, we have:

$$\mathbb{P}\left(|T_{k\Delta}^{(2)} + T_{k\Delta}^{(3)}| > 0\right) = \mathbb{P}\left(N_{k\Delta}'' \geq 1\right) \lesssim \Delta \nu\left([- \sqrt{\Delta}, \sqrt{\Delta}]^c\right) \lesssim \Delta^{1-\beta/2}.$$

It remains to bound $\mathbb{P}\left[|\Delta T_{k\Delta}^{(1)}| \geq 2\xi_0 \sqrt{\Delta} \ln(n)\right]$. We have that:

$$\mathbb{P}\left[|\Delta T_{k\Delta}^{(1)}| \geq 2\xi_0 \sqrt{\Delta} \ln(n)\right] \leq 2\mathbb{P}\left[\exp\left(\varepsilon \int_{k\Delta}^{(k+1)\Delta} \xi(X_{s-}) dL_s^{(1)}\right) \geq n^{2\varepsilon\xi_0 \sqrt{\Delta}}\right].$$

By Proposition 14, for any ε ,

$$D_t := \exp\left(\varepsilon \int_{k\Delta}^t \xi(X_{s-}) dL_s^{(1)} - \int_{k\Delta}^t \int_{|z| \leq \sqrt{\Delta}} (\exp(\varepsilon z \xi(X_{s-})) - 1 - \varepsilon z \xi(X_{s-})) \nu(dz)\right)$$

is a local martingale. Let us set $\varepsilon = 1/(2\xi_0\Delta^{1/2})$. There exists an increasing sequence of stopping times τ_N such that, for any N ,

$$\begin{aligned} F &:= \mathbb{P} \left[\exp \left(\frac{1}{2\xi_0\Delta^{1/2}} \int_{k\Delta}^{(k+1)\Delta \wedge \tau_N} \xi(X_{s-}) dL_s^{(1)} \right) \geq n \right] \\ &\leq n^{-1} \mathbb{E} \left(\exp \left(\int_{k\Delta}^{(k+1)\Delta \wedge \tau_N} \int_{|z| \leq \sqrt{\Delta}} \left(\exp \left(\frac{z\xi(X_{s-})}{2\xi_0\Delta^{1/2}} \right) - 1 - \frac{z\xi(X_{s-})}{2\xi_0\Delta^{1/2}} \right) \nu(dz) \right) \right) \\ &\leq n^{-1} \exp \left(2\Delta \int_{|z| \leq \sqrt{\Delta}} \frac{\xi_0^2 z^2}{4\xi_0^2 \Delta} \nu(dz) \right) \leq n^{-1} \exp \left(\int_{\mathbb{R}} z^2 \nu(dz) \right) \leq n^{-1}. \end{aligned}$$

When $N \rightarrow \infty$, by dominated convergence, we obtain:

$$\mathbb{P} \left(\left| \Delta T_{k\Delta}^{(1)} \right| \geq \xi_0 \sqrt{\Delta} \ln(n) \right) \lesssim n^{-1}. \quad (8)$$

Bound of $\mathbb{P} \left(\Omega_{X,k} \cap \Omega_{N,k}^c \cap (X_{k\Delta} \in A) \right)$. We recall that $N'_{k\Delta} = \mu \left(]k\Delta, (k+1)\Delta], [-\Delta^{1/4}, \Delta^{1/4}]^c \right)$. We have:

$$\Omega_{N,k}^c = \left\{ N'_{k\Delta} = 1 \right\} \cup \left\{ N'_{k\Delta} \geq 2 \right\}$$

with

$$\mathbb{P} \left(N'_{k\Delta} = 1 \right) \lesssim \Delta^{1-\beta/4} \quad \text{and} \quad \mathbb{P} \left(N'_{k\Delta} \geq 2 \right) \lesssim \Delta^{2-\beta/2}.$$

Then $\mathbb{P} \left(\Omega_{N,k}^c \cap \left\{ N'_{k\Delta} \geq 2 \right\} \right) \lesssim \Delta^{2-\beta/2}$. We can write:

$$\begin{aligned} G &:= \mathbb{P} \left(\Omega_{X,k} \cap (X_{k\Delta} \in A) \cap (N'_{k\Delta} = 1) \right) \\ &\leq \mathbb{P} \left(N'_{k\Delta} = 1 \right) \mathbb{P} \left(\left| \Delta T_{k\Delta}^{(2)} + \Delta T_{k\Delta}^{(3)} \right| \leq 2C_\Delta \mid N'_{k\Delta} = 1 \right) \\ &+ \mathbb{P} \left(N'_{k\Delta} = 1 \right) \mathbb{P} \left(\left\{ \left| \Delta T_{k\Delta}^{(2)} + \Delta T_{k\Delta}^{(3)} \right| \geq 2C_\Delta \mid N'_{k\Delta} = 1 \right\} \cap \Omega_{X,k} \cap (X_{k\Delta} \in A) \right). \end{aligned}$$

By (6), (7) and (8), we obtain:

$$\begin{aligned} H &:= \mathbb{P} \left(\left\{ \left| \Delta T_{k\Delta}^{(2)} + \Delta T_{k\Delta}^{(3)} \right| \geq 2C_\Delta \mid N'_{k\Delta} = 1 \right\} \cap \Omega_{X,k} \cap (X_{k\Delta} \in A) \right) \\ &\leq \mathbb{P} \left(\Delta \left| b_A(X_{k\Delta}) + I_{k\Delta} + Z_{k\Delta} + T_{k\Delta}^{(1)} \right| > C_\Delta \right) \\ &\lesssim \Delta + n^{-1}. \end{aligned}$$

It remains to bound $J := \mathbb{P} \left(\left| \Delta T_{k\Delta}^{(2)} + \Delta T_{k\Delta}^{(3)} \right| \leq 2C_\Delta \mid N'_{k\Delta} = 1 \right)$. If $N'_{k\Delta} = 1$, then $\left| \Delta T_{k\Delta}^{(3)} \right| = \left| \int_{k\Delta}^{(k+1)\Delta} \xi(X_{s-}) dL_s^{(3)} \right| \geq \xi_1 \Delta^{1/4}$. Then $J \leq \mathbb{P} \left(\Delta \left| T_{k\Delta}^{(2)} \right| \geq \xi_1 \Delta^{1/4} - 2C_\Delta \right)$.

Let us set $n_0 = \left\lceil \frac{1}{1-\beta/2} \right\rceil$ and $a = (\xi_0 n_0)^{-1} (\xi_1 \Delta^{1/4} - 2C_\Delta)$. We have:

$$\begin{aligned} J &\leq \mathbb{P}[\mu(\cdot)k\Delta, (k+1)\Delta, [-a, a]^c] \geq 1 \\ &+ \mathbb{P}\left[\mu(\cdot)k\Delta, (k+1)\Delta, [-a, -\Delta^{1/2}] \cup [\Delta^{1/2}, a] \geq n_0\right] \\ &\leq \Delta\nu([-a, a]^c) + \Delta^{n_0}\nu([-\Delta^{1/2}, \Delta^{1/2}]^c)^{n_0} \\ &\lesssim \Delta^{1-\beta/4} + \Delta. \end{aligned}$$

Then $\mathbb{P}(\Omega_{X,k} \cap \Omega_{N,k}^c) \leq \mathbb{P}(N'_{k\Delta} = 1)\Delta^{1-\beta/4} + \mathbb{P}(N'_{k\Delta} = 2) \lesssim \Delta^{2-\beta/2}$.

Bound of $(\mathbb{E}[(Z_{k\Delta} + T_{k\Delta}) 1_{\Omega_{X,k} \cap \Omega_{N,k} \cap (X_{k\Delta} \in A)} | \mathcal{F}_{k\Delta}])^2$.

If σ and ξ are constants. Let us set $E := (\mathbb{E}[(Z_{k\Delta} + T_{k\Delta}) 1_{\Omega_{X,k} \cap \Omega_{N,k} \cap (X_{k\Delta} \in A)} | \mathcal{F}_{k\Delta}])^2$ and

$$\Omega_{I,k} = \left\{ \omega, |I_{k\Delta}| \leq 1, \cap |\Delta Z_{k\Delta}| \leq \sigma_0 \sqrt{\Delta} \ln(n), \cap \left| \Delta T_{k\Delta}^{(1)} \right| \leq 2\xi_0 \sqrt{\Delta} \ln(n) \right\}.$$

By (6), (7) and (8), $\mathbb{P}(\Omega_{I,k}^c) \leq \Delta + n^{-1}$. Then, by a Markov inequality:

$$E \lesssim \Delta \ln^2(n) + (\mathbb{E}[(Z_{k\Delta} + T_{k\Delta}) 1_{\Omega_{X,k} \cap \Omega_{N,k} \cap \Omega_{I,k} \cap (X_{k\Delta} \in A)} | \mathcal{F}_{k\Delta}])^2.$$

Let us introduce the set $\Omega_{ZT,k} := \{\omega, |Z_{k\Delta} + T_{k\Delta}| \leq C_\Delta \Delta^{-1} - b_{max} - 1\}$. On $\Omega_{I,k}$, $|I_{k\Delta}| \leq 1$ and therefore:

$$\Omega_{ZT,k} \cap \Omega_{I,k} \subseteq \Omega_{X,k} \cap \Omega_{I,k} \subseteq \{\omega, |Z_{k\Delta} + T_{k\Delta}| \leq C_\Delta \Delta^{-1} + b_{max} + 1\} \cap \Omega_{I,k}.$$

Then

$$E \lesssim \Delta \ln^2(n) + F^2 + G^2$$

where $F = \mathbb{E}[(Z_{k\Delta} + T_{k\Delta}) 1_{\Omega_{ZT,k} \cap \Omega_{N,k} \cap \Omega_{I,k} \cap (X_{k\Delta} \in A)} | \mathcal{F}_{k\Delta}]$ and

$G = \mathbb{E}[(Z_{k\Delta} + T_{k\Delta}) 1_{\Omega_{ZT,k}^c \cap \Omega_{X,k} \cap \Omega_{N,k} \cap \Omega_{I,k} \cap (X_{k\Delta} \in A)} | \mathcal{F}_{k\Delta}]$. As σ and ξ are constants, the terms

$$Z_{k\Delta} = \frac{\sigma_0}{\Delta} \int_{k\Delta}^{(k+1)\Delta} dW_s \quad \text{and} \quad T_{k\Delta} = \frac{\xi_0}{\Delta} \int_{k\Delta}^{(k+1)\Delta} dL_s$$

are centred and independent. Then $F = 0$. Moreover, on $\Omega_{N,k}$, $T_{k\Delta}^{(3)} = 0$. Then

$$|G| \lesssim \left| \mathbb{E} \left[(Z_{k\Delta} + T_{k\Delta}^{(1)} + T_{k\Delta}^{(2)}) 1_{\Omega_{X,k} \cap \Omega_{ZT,k}^c \cap \Omega_{N,k} \cap \Omega_{I,k} \cap (X_{k\Delta} \in A)} \middle| \mathcal{F}_{k\Delta} \right] \right|.$$

Let us set $c_b = b_{max} + 1$. On $\Omega_{I,k} \cap \Omega_{X,k}$, $|Z_{k\Delta} + T_{k\Delta}^{(1)} + T_{k\Delta}^{(2)}| \lesssim \ln(n)\Delta^{-1/2}$, and

$$\begin{aligned} |G| &\lesssim \frac{\ln(n)}{\sqrt{\Delta}} \left(\mathbb{P} \left(\left| Z_{k\Delta} + T_{k\Delta}^{(1)} + T_{k\Delta}^{(2)} \right| \in [C_\Delta \Delta^{-1} - c_b, C_\Delta \Delta^{-1} + c_b] 1_{\Omega_{I,k}} \right) \right) \\ &= 2 \frac{\ln(n)}{\sqrt{\Delta}} \int_{\mathbb{R}} \mathbb{P} \left(T_{k\Delta}^{(2)} \in [C_\Delta \Delta^{-1} - c_b - x, C_\Delta \Delta^{-1} + c_b - x] 1_{\Omega_{I,k}} \right) \\ &\times \mathbb{P} \left(Z_{k\Delta} + T_{k\Delta}^{(1)} \in dx \middle| T_{k\Delta}^{(2)} \in [C_\Delta \Delta^{-1} - c_b - x, C_\Delta \Delta^{-1} + c_b - x] 1_{\Omega_{I,k}} \right). \end{aligned}$$

On $\Omega_{I,k}$, $\left|Z_{k\Delta} + T_{k\Delta}^{(1)}\right| \leq (\sigma_0 + 2\xi_0) \ln(n) \Delta^{-1/2}$. Then

$$|G| \lesssim \frac{\ln(n)}{\sqrt{\Delta}} \left[\sup_{C \geq \xi_0 \ln(n) \Delta^{-1/2}} \mathbb{P} \left(T_{k\Delta}^{(2)} \in [C, C + 2c_b] \right) \right]. \quad (9)$$

We recall that $L_t^{(2)}$ is a compound Poisson process in which all the jumps are greater than $\sqrt{\Delta}$ and smaller than $\Delta^{1/4}$. Let us denote by τ_i the times of the jumps of size in $[\sqrt{\Delta}, \Delta^{1/4}]$ and by ζ_i the size of the jumps. We set $a_j = \xi_0^{-1} C \Delta - \sum_{i=1}^{j-1} \zeta_i$ and $c := \xi_0^{-1} (2b_{max} + 2)$. Then, as ξ is constant equal to ξ_0 :

$$\begin{aligned} H &:= \mathbb{P} \left(T_{k\Delta}^{(2)} \in [C, C + 2b_{max} + 2] \right) \\ &\leq \sum_{j=1}^{\infty} \mathbb{P} \left(j \text{ jumps} \geq \sqrt{\Delta}, \text{ last jump} \in [a_j, a_j + c\Delta] \right) \\ &\lesssim 2 \sup_{a \geq \sqrt{\Delta}} \mathbb{P} (1 \text{ jump} \in [a, a + c\Delta]) = 2\Delta \sup_{a \geq \sqrt{\Delta}} \nu([a, a + c\Delta]). \end{aligned}$$

By A6,

$$H \lesssim \Delta \sup_{a \geq \sqrt{\Delta}} \left[\frac{1}{a^\beta} - \frac{1}{(a + c\Delta)^\beta} \right] \lesssim \sqrt{\Delta} \Delta^{1-\beta/2} \quad (10)$$

and, by (9) and (10),

$$E \lesssim \Delta \ln^2(n) + \frac{\ln^2(n)}{\Delta} \Delta \Delta^{2-\beta} \lesssim \Delta \ln^2(n) + \Delta^{2-\beta} \ln^2(n).$$

Remark 18. If ν is not absolutely continuous, inequality 10 is not valid. We obtain:

$$H \lesssim 2\Delta \sup_{a \geq \sqrt{\Delta}} \nu([a, a + c\Delta]) \lesssim \Delta^{1-\beta/2}$$

Therefore

$$E \leq \Delta \ln^2(n) + G^2 \lesssim \Delta \ln^2(n) + \Delta^{1-\beta} \ln^2(n).$$

If σ or ξ are not constants. The problem is that $Z_{k\Delta}$ and $T_{k\Delta}$ are not symmetric and we can't apply directly the previous method. We replace them by two centred terms. The following lemma is very useful.

Lemma 19.

Let f be a \mathcal{C}^2 function such that f and f' are Lipschitz. Let us set, for any $t \in]k\Delta, (k+1)\Delta[$:

$$\psi_f(X_{k\Delta}, t) = f'(X_{k\Delta}) \left(\sigma(X_{k\Delta}) \int_{k\Delta}^t dW_s + \xi(X_{k\Delta}) \int_{k\Delta}^t z \tilde{\mu}(ds, dz) \right).$$

We have:

$$\mathbb{E} \left[(f(X_t) - f(X_{k\Delta}) - \psi_f(X_{k\Delta}, t))^2 1_{\Omega_{N,k}} 1_{X_{k\Delta} \in A} \right] \lesssim \Delta^{2-\beta/4}.$$

Lemma 4 is proved below. Let us set

$$\bar{Z}_{k\Delta} = \frac{1}{\Delta} \int_{k\Delta}^{(k+1)\Delta} (\sigma(X_{k\Delta}) + \psi_\sigma(X_{k\Delta,s})) dW_s,$$

$$\bar{T}_{k\Delta}^{(i)} = \frac{1}{\Delta} \int_{k\Delta}^{(k+1)\Delta} (\xi(X_{k\Delta}) + \psi_\xi(X_{k\Delta,s})) dL_s^{(i)} \quad \text{and} \quad \bar{T}_{k\Delta} = \bar{T}_{k\Delta}^{(1)} + \bar{T}_{k\Delta}^{(2)} + \bar{T}_{k\Delta}^{(3)}.$$

The terms $\bar{Z}_{k\Delta}$ and $\bar{T}_{k\Delta}$ are symmetric. By lemma 19,

$$\begin{aligned} \mathbb{E} \left[(\bar{Z}_{k\Delta} - Z_{k\Delta})^2 1_{\Omega_{N,k}} 1_{X_{k\Delta} \in A} \right] &= \frac{1}{\Delta^2} \mathbb{E} \left[\int_{k\Delta}^{(k+1)\Delta} (\sigma(X_s) - \sigma(X_{k\Delta}) - \psi_\sigma(X_{k\Delta,s}))^2 ds \right] \\ &\lesssim \Delta^{1-\beta/4}. \end{aligned} \quad (11)$$

We prove in the same way that

$$\mathbb{E} \left[(\bar{T}_{k\Delta} - T_{k\Delta})^2 1_{\Omega_{N,k}} 1_{X_{k\Delta} \in A} \right] \leq \Delta^{1-\beta/4}. \quad (12)$$

Let us set $U_{k\Delta} = \Delta^{-1} \xi(X_{k\Delta-}) \int_{k\Delta}^{(k+1)\Delta} dL_s^{(2)}$. By Result 11 and Proposition 1,

$$\mathbb{E} \left[\Delta^2 (\bar{T}_{k\Delta}^{(2)} - U_{k\Delta})^2 \right] = \mathbb{E} \left[\int_{k\Delta}^{(k+1)\Delta} \int_{\mathbb{R}} (\psi_\xi(X_{k\Delta,s}))^2 z^2 \nu(dz) ds \right] \leq \Delta^{2-\beta/4}. \quad (13)$$

Let us introduce the set

$$\begin{aligned} \bar{\Omega}_{I,k} &= \{ \omega, |I_{k\Delta}| + |Z_{k\Delta} - \bar{Z}_{k\Delta}| + |T_{k\Delta} - \bar{T}_{k\Delta}| \leq 3 \} \\ &\cap \left\{ |\Delta \bar{Z}_{k\Delta}| \leq \sigma_0 \sqrt{\Delta} \ln(n) + \Delta, \left| \Delta \bar{T}_{k\Delta}^{(1)} \right| \leq 2\xi_0 \sqrt{\Delta} \ln(n) + \Delta \right\} \\ &\cap \left\{ \left| \Delta (\bar{T}_{k\Delta}^{(2)} - U_{k\Delta}) \right| \leq \xi_0 \sqrt{\Delta} \right\}. \end{aligned}$$

By (6), (7), (8), (11), (12), (13) and Markov inequalities, we obtain:

$$\mathbb{P}(\bar{\Omega}_{I,k}^c) \lesssim \Delta^{1-\beta/4} + \frac{1}{n}. \quad (14)$$

Then

$$\begin{aligned} E &:= \left(\mathbb{E} \left[(Z_{k\Delta} + T_{k\Delta}) 1_{\Omega_{X,k} \cap \Omega_{N,k} \cap (X_{k\Delta} \in A)} \middle| \mathcal{F}_{k\Delta} \right] \right)^2 \\ &\lesssim \Delta^{1-\beta/2} \ln^2(n) + \left(\mathbb{E} \left[(\bar{Z}_{k\Delta} + \bar{T}_{k\Delta}) 1_{\Omega_{X,k} \cap \Omega_{N,k} \cap (X_{k\Delta} \in A) \cap \bar{\Omega}_{I,k}} \middle| \mathcal{F}_{k\Delta} \right] \right)^2. \end{aligned} \quad (15)$$

Let us introduce the set:

$$\bar{\Omega}_{ZT,k} := \{ \omega, |\bar{Z}_{k\Delta} + \bar{T}_{k\Delta}| \leq C_\Delta \Delta^{-1} - b_{max} - 3 \}.$$

We have that

$$\bar{\Omega}_{ZT,k} \cap \bar{\Omega}_{I,k} \subseteq \Omega_{X,k} \cap \bar{\Omega}_{I,k} \subseteq \{ \omega, |\bar{Z}_{k\Delta} + \bar{T}_{k\Delta}| \leq C_\Delta \Delta^{-1} + b_{max} + 3 \} \cap \bar{\Omega}_{I,k}.$$

Given the filtration $\mathcal{F}_{k\Delta}$, the sum $\bar{Z}_{k\Delta} + \bar{T}_{k\Delta}$ is symmetric. Then

$$\mathbb{E} \left[(\bar{Z}_{k\Delta} + \bar{T}_{k\Delta}) 1_{\bar{\Omega}_{ZT,k} \cap \Omega_{N,k} \cap (X_{k\Delta} \in A)} \middle| \mathcal{F}_{k\Delta} \right] = 0.$$

Moreover, on $\Omega_{N,k}$, $\bar{T}_{k\Delta}^{(3)} = 0$. Then, by (15),

$$E \lesssim \Delta^{1-\beta/2} \ln^2(n) + G^2 + H^2$$

where $G := \mathbb{E} \left[(\bar{Z}_{k\Delta} + \bar{T}_{k\Delta}^{(1)} + \bar{T}_{k\Delta}^{(2)}) 1_{\Omega_{X,k} \cap \Omega_{ZT,k}^c \cap \Omega_{N,k} \cap \Omega_{I,k} \cap (X_{k\Delta} \in A)} \middle| \mathcal{F}_{k\Delta} \right]$ and $H := \mathbb{E} \left[(\bar{Z}_{k\Delta} + \bar{T}_{k\Delta}^{(1)} + \bar{T}_{k\Delta}^{(2)}) 1_{\Omega_{X,k} \cap \Omega_{ZT,k} \cap \Omega_{N,k} \cap \Omega_{I,k}^c \cap (X_{k\Delta} \in A)} \middle| \mathcal{F}_{k\Delta} \right]$. We have that $H^2 \lesssim \Delta^{-1} \ln^2(n) \mathbb{P}^2(\Omega_{I,k}^c) \lesssim \Delta^{1-\beta/2} \ln^2(n)$. The end of the proof is the same as in the case of σ and ξ constants. We obtain that

$$|G| \lesssim \frac{\ln(n)}{\sqrt{\Delta}} \sup_{C \geq \kappa_0 \ln(n) \Delta^{-1/2}} \mathbb{P}(U_{k\Delta} \in [C, C + 2b_{max} + 6]) \lesssim \sqrt{\Delta} \Delta^{1-\beta/2}.$$

6.3.2 Proof of Lemma 19

According to the Itô formula (see for instance Applebaum (2004), Theorem 4.4.7 p251), we have that

$$f(X_t) - f(X_{k\Delta}) = I_1 + I_2 + I_3 + I_4$$

where

$$\begin{aligned} I_1 &= \int_{k\Delta}^t f'(X_s) \sigma(X_s) dW_s, & I_2 &= \int_{k\Delta}^t \int_{\mathbb{R}} (f(X_{s^-} + z\xi(X_{s^-})) - f(X_{s^-})) \tilde{\mu}(ds, dz) \\ I_3 &= \int_{k\Delta}^t \int_{z \in \mathbb{R}} [f(X_s + z\xi(X_s)) - f(X_s) - z\xi(X_s) f'(X_s)] \nu(dz) ds \\ I_4 &= \int_{k\Delta}^t [f'(X_s) b(X_s) + f''(X_s) \sigma^2(X_s) / 2] ds. \end{aligned}$$

By Proposition 1, for any $t \leq (k+1)\Delta$, we have:

$$\begin{aligned} Q &:= \mathbb{E} \left[\left(I_1 - f'(X_{k\Delta}) \sigma(X_{k\Delta}) \int_{k\Delta}^t dW_s \right)^2 \right] \\ &= \mathbb{E} \left[\left(\int_{k\Delta}^t (\sigma(X_s) f'(X_s) - \sigma(X_{k\Delta}) f'(X_{k\Delta})) dW_s \right)^2 \right] \\ &= \int_{k\Delta}^t (\sigma(X_s) f'(X_s) - \sigma(X_{k\Delta}) f'(X_{k\Delta}))^2 ds \lesssim \Delta^2. \end{aligned}$$

We can write:

$$\begin{aligned}
E &:= \mathbb{E} \left[\left(I_2 - f'(X_{k\Delta})\xi(X_{k\Delta-}) \int_{k\Delta}^t dL_s^{(1)} + dL_s^{(2)} \right)^2 1_{\Omega_{N,k}} \right] \\
&\leq 2 \int_{k\Delta}^t \int_{|z| \leq \Delta^{1/4}} \mathbb{E} \left[(f(X_s + z\xi(X_s)) - f(X_s) - z\xi(X_s)f'(X_s))^2 \right] \nu(dz) ds \\
&+ 2 \int_{k\Delta}^t \int_{|z| \leq \Delta^{1/4}} \mathbb{E} \left[z^2 (\xi(X_s)f'(X_s) - \xi(X_{k\Delta})f'(X_{k\Delta}))^2 \right] \nu(dz) ds.
\end{aligned}$$

The function f is \mathcal{C}^2 , then, by the Taylor formula, for any $s \in [k\Delta, t]$, $z \in \mathbb{R}$, there exists $\zeta_{s,z}$ in $[X_s, X_s + z\xi(X_s)]$ such that:

$$f(X_s + z\xi(X_s)) - f(X_s) - z\xi(X_s)f'(X_s) = \frac{z^2 \xi^2(X_s)}{2} f''(\zeta_{s,z}).$$

Then, as ξ and f'' are bounded:

$$\mathbb{E} \left[(f(X_s + z\xi(X_s)) - f(X_s) - z\xi(X_s)f'(X_s))^2 \right] = \frac{z^4}{4} \mathbb{E} \left[(\xi(X_s)f''(\zeta_{s,z}))^2 \right] \lesssim z^4$$

and, by Result 17, for any $t \leq (k+1)\Delta$,

$$\begin{aligned}
F &:= \int_{k\Delta}^t \int_{|z| \leq \Delta^{1/4}} \mathbb{E} \left[(f(X_s + z\xi(X_s)) - f(X_s) - z\xi(X_s)f'(X_s))^2 \right] \nu(dz) ds \\
&\lesssim \Delta \int_{|z| \leq \Delta^{1/4}} z^4 \nu(dz) \lesssim \Delta^{2-\beta/4}.
\end{aligned}$$

The functions ξ and f' are Lipschitz, then by Proposition 1,

$$\mathbb{E} \left[z^2 (\xi(X_s)f'(X_s) - \xi(X_{k\Delta})f'(X_{k\Delta}))^2 \right] \lesssim z^2 \mathbb{E} \left[(X_s - X_{k\Delta})^2 \right] \lesssim \Delta z^2$$

and consequently, for any $t \leq (k+1)\Delta$:

$$\int_{k\Delta}^t \int_{|z| \leq \Delta^{1/4}} \mathbb{E} \left[z^2 (\xi(X_s)f'(X_s) - \xi(X_{k\Delta})f'(X_{k\Delta}))^2 \right] \nu(dz) ds \lesssim \Delta^{2-\beta/4}$$

then $E \lesssim \Delta^{2-\beta/4}$. By the same way, we obtain that

$$\mathbb{E} [I_3^2] \leq \mathbb{E} \left[\int_{k\Delta}^t \int_{|z| \leq \Delta^{1/4}} \left(\frac{z^2 \xi^2(X_s)}{2} f''(\zeta_{s,z}) \right)^2 \nu(dz) ds \right] \lesssim \Delta^{2-\beta/4}.$$

The functions b and f' are Lipschitz and f'' and σ are bounded, then, for any $t \leq (k+1)\Delta$:

$$\mathbb{E} [I_4^2] \lesssim \Delta \int_{k\Delta}^t (1 + \mathbb{E} [X_s^4]) ds \lesssim \Delta^2.$$

Then, for any $t \leq (k+1)\Delta$:

$$\mathbb{E} [(f(X_t) - f(X_{k\Delta}) - \psi_f(X_{k\Delta}, t))] \leq \Delta^{2-\beta/4}.$$

6.4 Proof of Theorem 10

As previously, we only bound the risk on Ω_n . As in Subsection 6.2, we introduce the function $p(m, m')$ such that $p(m, m') = 12(\text{pen}(m) + \text{pen}(m'))$. On Ω_n , for any $m \in \mathcal{M}_n$, we have:

$$\begin{aligned} \left\| \tilde{b}_{\tilde{m}} - b_A \right\|_n^2 &\leq 3 \|b_m - b_A\|_n^2 + \frac{224}{n} \sum_{k=1}^n b_A^2(X_{k\Delta}) 1_{\Omega_{X,k}^c} + I_{k\Delta}^2 + 2(Z_{k\Delta}^2 + T_{k\Delta}^2) 1_{\Omega_{X,k} \cap \Omega_{Z,k}^c} \\ &+ \frac{224}{n} \sum_{k=1}^n (\mathbb{E} [(Z_{k\Delta} + T_{k\Delta}) 1_{\Omega_{X,k} \cap \Omega_{Z,k}} | \mathcal{F}_{k\Delta}])^2 \\ &+ 24 \sup_{t \in \mathcal{B}_{m, \tilde{m}}} (\tilde{\nu}_n^2(t) - p(m, \tilde{m})) + 4\text{pen}(m). \end{aligned}$$

It remains only to bound

$$\mathbb{E} \left[\sup_{t \in \mathcal{B}_{m, \tilde{m}}} (\tilde{\nu}_n^2(t) - p(m, \tilde{m})) \right] \leq \sum_{m'} \mathbb{E} \left[\sup_{t \in \mathcal{B}_{m, m'}} (\tilde{\nu}_n^2(t) - p(m, \tilde{m})) \right].$$

As in the proof of Theorem 4, we bound the quantity

$$\mathbb{E} \left[\exp \left(\varepsilon t(X_{k\Delta}) (\tilde{Z}_{k\Delta} + \tilde{T}_{k\Delta}) \right) \middle| \mathcal{F}_{k\Delta} \right].$$

We have that

$$\mathbb{E} \left[\exp(\varepsilon t(X_{k\Delta}) Z_{k\Delta}) 1_{\Omega_{N,k}} \middle| \mathcal{F}_{k\Delta} \right] \leq \exp \left(\frac{\varepsilon^2 \sigma_0^2 t^2(X_{k\Delta})}{2\Delta} \right).$$

The truncated Lévy process $\tilde{L}_t = \int_0^t \int_{|z| \leq \Delta^{1/4}} z \tilde{\mu}(ds, dz)$ satisfies Assumption A5 and then there exists a constant c such that:

$$\mathbb{E} \left[\exp(\varepsilon t(X_{k\Delta}) T_{k\Delta}) 1_{\Omega_{N,k}} \middle| \mathcal{F}_{k\Delta} \right] \leq \exp \left(\frac{c\varepsilon^2 \xi_0^2 t^2(X_{k\Delta})}{\Delta(1 - \varepsilon/\varepsilon_1)} \right).$$

As $Z_{k\Delta} 1_{\Omega_{N,k}}$ and $T_{k\Delta} 1_{\Omega_{N,k}}$ are centred, we obtain:

$$\mathbb{E} \left[\exp(\varepsilon |t(X_{k\Delta}) (Z_{k\Delta} + T_{k\Delta})|) 1_{\Omega_{N,k}} \middle| \mathcal{F}_{k\Delta} \right] \leq 2 \exp \left(\frac{c\varepsilon^2 (\sigma_0^2 + \xi_0^2) t^2(X_{k\Delta})}{\Delta(1 - \varepsilon/\varepsilon_1)} \right)$$

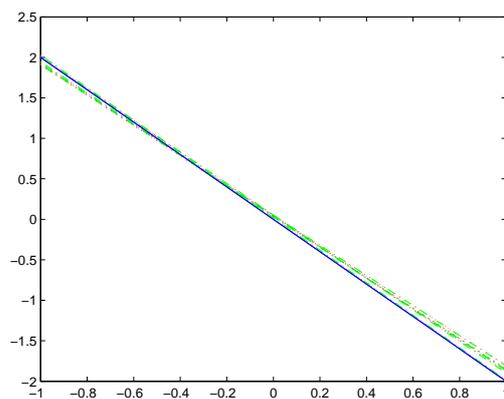
and then

$$\mathbb{E} \left[\exp \left(\varepsilon \left| t(X_{k\Delta}) (\tilde{Z}_{k\Delta} + \tilde{T}_{k\Delta}) \right| \right) 1_{\Omega_{N,k} \cap \Omega_{X,k}} \middle| \mathcal{F}_{k\Delta} \right] \leq 2 \exp \left(\frac{c\varepsilon^2 (\sigma_0^2 + \xi_0^2) t^2(X_{k\Delta})}{\Delta(1 - \varepsilon/\varepsilon_1)} \right).$$

We conclude as in the proof of Theorem 4.

Figure 1: Model 1: Ornstein-Uhlenbeck and binomial law

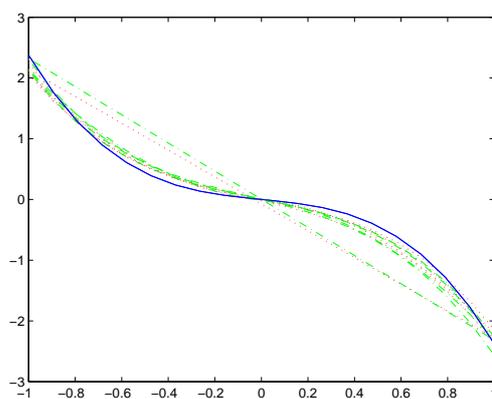
$$b(x) = -2x, \sigma(x) = \xi(x) = 1 \text{ and binomial law}$$



- : true function -.-: first estimator ...: truncated estimator
 $n = 10^4$ et $\Delta = 10^{-1}$

Figure 2: Model 2: Double well and Laplace law

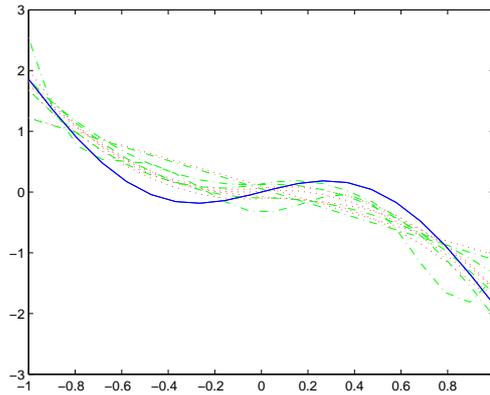
$$b(x) = -(x - 1/4)^3 - (x + 1/4)^3, \quad \sigma = \xi = 1 \text{ and Laplace law}$$



- : true function -.-: first estimator ...: truncated estimator
 $n = 10^4$ et $\Delta = 10^{-1}$

Figure 3: Model 3: Sine function

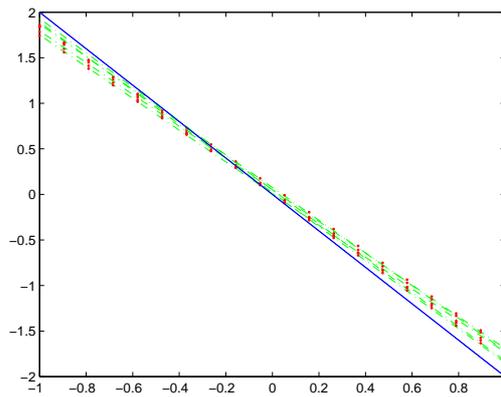
$$b(x) = -2x + \sin(3x), \sigma(x) = \xi(x) = \sqrt{(3+x^2)/(1+x^2)} \text{ jumps not sub-exponential}$$



— : true function -.-: first estimator ...: truncated estimator
 $n = 10^4$ et $\Delta = 10^{-1}$

Figure 4: Model 4: Lévy process

$$b(x) = -2x, \sigma(x) = \xi(x) = 1 \text{ jumps Lévy}$$



— : true function -.-: first estimator ...: truncated estimator
 $n = 10^4$ et $\Delta = 10^{-1}$

Table 1: Model 1: Ornstein-Uhlenbeck and binomial law

$b(x) = -2x$, $\sigma(x) = \xi(x) = 1$ and compound Poisson process (binomial law)

n	Δ	first estimator				truncated estimator			
		\hat{m}_a	\hat{r}_a	$risk_1$	or_1	\tilde{m}_a	\tilde{r}_a	$risk_2$	or_2
10^3	10^{-1}	0	1.02	0.044	1.3	0	1.02	0.044	1.3
10^4	10^{-1}	0	1.02	0.011	1.3	0	1.02	0.011	1.3
10^3	10^{-2}	0	1.02	0.55	1.04	0	1.02	0.55	1.04
10^4	10^{-2}	0	1	0.047	1	0	1	0.047	1
5.10^4	10^{-2}	0.04	1	0.010	1.4	0	1	0.0053	1

\hat{m}_a, \hat{r}_a and \tilde{m}_a, \tilde{r}_a : average values of \hat{m}, \hat{r} and \tilde{m}, \tilde{r} on the 50 simulations.

$risk_1$ and $risk_2$: means of the empirical errors of the adaptive estimators.

or_1 and or_2 : means of *oracle* =empirical error of the adaptive estimator / empirical error of the best possible estimator.

Table 2: Model 2: Double well and Laplace law

$b(x) = -(x - 1/4)^3 - (x + 1/4)^3$, $\sigma(x) = \xi(x) = 1$ and Laplace law.

n	Δ	first estimator				truncated estimator			
		\hat{m}_a	\hat{r}_a	$risk_1$	or_1	\tilde{m}_a	\tilde{r}_a	$risk_2$	or_2
10^3	10^{-1}	0.02	1.0	0.12	3.1	0.02	1.0	0.12	3.1
10^4	10^{-1}	1.7	2.1	2e96	51	0.4	2.1	0.04	1.5
10^3	10^{-2}	0.26	1.2	1.8	3.1	0.06	1	0.51	1.4
10^4	10^{-2}	0.12	1.5	0.16	1.8	0.08	1.2	0.13	2.4
5.10^4	10^{-2}	0.30	2.5	0.035	1.6	0.26	2.5	0.019	1.8

\hat{m}_a, \hat{r}_a and \tilde{m}_a, \tilde{r}_a : average values of \hat{m}, \hat{r} and \tilde{m}, \tilde{r} on the 50 simulations.

$risk_1$ and $risk_2$: means of the empirical errors of the adaptive estimators.

or_1 and or_2 : means of *oracle* =empirical error of the adaptive estimator / empirical error of the best possible estimator.

Table 3: Model 3: Sine function and jumps not sub-exponential

$$b(x) = -2x + \sin(3x), \quad \sigma(x) = \xi(x) = \sqrt{(3+x^2)/(1+x^2)} \quad \text{and} \quad \nu(dz) \propto e^{-\sqrt{az}}/\sqrt{z} dz$$

n	Δ	first estimator				truncated estimator			
		\hat{m}_a	\hat{r}_a	$risk_1$	or_1	\tilde{m}_a	\tilde{r}_a	$risk_2$	or_2
10^3	10^{-1}	0.34	1.2	0.76	3.6	0.04	1.2	0.28	1.9
10^4	10^{-1}	0.8	2.2	0.082	1.3	0.68	2.2	0.073	1.2
10^3	10^{-2}	0.96	1.2	18	6.3	0.02	1.2	1.3	1.2
10^4	10^{-2}	0.78	1.4	1.5	4.3	0.12	1.4	0.24	3.3
$5 \cdot 10^4$	10^{-2}	0.92	2.3	0.24	4.3	0.70	2.3	0.039	1.3

\hat{m}_a, \hat{r}_a and \tilde{m}_a, \tilde{r}_a : average values of \hat{m}, \hat{r} and \tilde{m}, \tilde{r} on the 50 simulations.
 $risk_1$ and $risk_2$: means of the empirical errors of the adaptive estimators.
 or_1 and or_2 : means of *oracle* = empirical error of the adaptive estimator / empirical error of the best possible estimator.

Table 4: Model 4: Lévy process

$$b(x) = -2x, \quad \sigma(x) = \xi(x) = 1 \quad \text{and} \quad \nu(dz) = \sum_{k=0}^{\infty} 2^{k+2} (\delta_{2^{-k}} + \delta_{-2^{-k}})$$

n	Δ	first estimator				truncated estimator			
		\hat{m}_a	\hat{r}_a	$risk_1$	or_1	\tilde{m}_a	\tilde{r}_a	$risk_2$	or_2
10^3	10^{-1}	0.04	1.06	0.110	1.86	0.02	1.06	0.111	1.95
10^4	10^{-1}	0.06	1.06	0.0172	1.26	0.06	1.06	0.0176	1.22
10^3	10^{-2}	0.1	1.04	1.17	1.88	0	1.04	0.61	1.12
10^4	10^{-2}	0.04	1.08	0.11	1.25	0.02	1.08	0.068	1.25
$5 \cdot 10^4$	10^{-2}	0.08	1.16	0.023	1.71	0	1.16	0.011	1.09

\hat{m}_a, \hat{r}_a and \tilde{m}_a, \tilde{r}_a : average values of \hat{m}, \hat{r} and \tilde{m}, \tilde{r} on the 50 simulations.
 $risk_1$ and $risk_2$: means of the empirical errors of the adaptive estimators.
 or_1 and or_2 : means of *oracle* = empirical error of the adaptive estimator / empirical error of the best possible estimator.

7 Auxiliary proofs

7.1 Decomposition on a lattice

Proposition 20.

If there exist some constants c_1, c_2 and K independent of D, n, Δ, b and σ and two constants α and β independent of n and D such that, for any function $t \in S_m + S'_m$:

$$\forall \eta, \zeta > 0, \forall t \in S_m + S_{m'} \quad \|t\|_\infty \leq C\zeta, \mathbb{P} \left(f_n(t) \geq \eta, \|t\|_n^2 \leq \zeta^2 \right) \leq K \exp \left(-\frac{\eta^2 n \beta}{(c_1 \alpha^2 \zeta^2 + 2C c_2 \alpha \eta \zeta)} \right),$$

then there exist some constants C and κ depending only of ν such that, if $D \leq n\beta$:

$$\mathbb{E} \left[\sup_{t \in \mathcal{B}_{m,m'}} f_n^2(t) - \frac{\kappa \alpha^2 D}{n\beta} \right]_+ \leq CK \frac{\kappa \alpha^2 D^{3/2} e^{-\sqrt{D}}}{n\beta}.$$

Let us consider an orthonormal (for the L^2_σ -norm) basis $(\psi_\lambda)_{\lambda \in \Lambda_{m,m'}}$ of $S_{m,m'} = S_m + S_{m'}$ such that

$$\forall \lambda, \quad \text{card}(\{\lambda', \|\psi_\lambda \psi_{\lambda'}\| \neq 0\}) \leq \phi_2.$$

Let us set

$$\bar{r}_{m,m'} = \frac{1}{\sqrt{D}} \sup_{\beta \neq 0} \frac{\|\sum_\lambda \beta_\lambda \psi_\lambda\|_\infty}{|\beta|_\infty}.$$

We obtain that

$$\left\| \sum_\lambda \beta_\lambda \psi_\lambda \right\|_\infty \leq \phi_2 |\beta|_\infty \sup_\lambda \|\psi_\lambda\|_\infty \quad \text{et} \quad \|\psi_\lambda\|_\infty \leq \sqrt{D} \|\psi_\lambda\|_{L^2} \leq \pi_1 \sqrt{D} \|\psi_\lambda\|_\sigma$$

then

$$\bar{r}_{m,m'} \leq \bar{r} := \phi_2 \pi_1.$$

We need a lattice of which the infinite norm is bounded. We use Lemma 9 of Barron *et al.* (1999):

Result 21. There exists a δ_k -lattice T_k of $L^2_\sigma \cap (S_m + S_{m'})$ such that

$$|T_k \cap \mathcal{B}_{m,m'}| \leq (5/\delta^k)^D$$

where $\delta_k = 2^{-k}/5$. Let us denote by $p_k(u)$ the orthogonal projection of u on T_k . For any $u \in S_{m,m'}$, $\|u - p_k(u)\|_\pi \leq \delta_k$ and

$$\sup_{u \in p_k^{-1}(t)} \|u - t\|_\infty \leq \bar{r}_{m,m'} \delta_k \leq \bar{r} \delta_k.$$

Let us set $H_k = \ln(|T_k \cap \mathcal{B}_{m,m'}|)$. We have that:

$$H_k \leq D \ln(5/\delta_k) = D(k \ln(2) + \ln(5/\delta_0)) \leq C(k+1)D.$$

The decomposition of u_k on the δ_k -lattice must be done very carefully: the norms $\|u_k - u_{k-1}\|_{\varpi}$ and $\|u_k - u_{k-1}\|_{\infty}$ must be controlled. Let us set

$$\mathcal{E}_k = \{u_k \in T_k \cap \mathcal{B}_{m,m'}, \quad \|u - u_k\|_{\varpi} \leq \delta_k \quad \text{et} \quad \|u - u_k\|_{\infty} \leq \bar{r}\delta_k\}.$$

We have that $\ln(|\mathcal{E}_k|) \leq H_k$. For any function $u \in \mathcal{B}_{m,m'}$, there exist a series $(u_k)_{k \geq 0} \in \prod_k \mathcal{E}_k$ such that

$$u = u_0 + \sum_{k=1}^{\infty} (u_k - u_{k-1}).$$

Let us consider $(\eta_k)_{k \geq 0}$ and $\eta \in \mathbb{R}$ such that $\eta_0 + \sum_{k=1}^{\infty} \eta_k \leq \eta$. We obtain:

$$\begin{aligned} \mathbb{P}\left(\sup_{u \in \mathcal{B}_{m,m'}} |f_n(u)| > \eta\right) &\leq \mathbb{P}\left(\exists (u_k) \in \prod \mathcal{E}_k, \left|f_n(u_0) + \sum_{k=1}^{\infty} f_n(u_k - u_{k-1})\right| > \eta_0 + \sum_{k=1}^{\infty} \eta_k\right) \\ &\leq P_1 + \sum_{k=1}^{\infty} P_{2,k} \end{aligned} \quad (16)$$

where

$$P_1 = \sum_{u_0 \in \mathcal{E}_0} \mathbb{P}(|f_n(u_0)| > \eta_0) \quad \text{and} \quad P_{2,k} = \sum_{u_k \in \mathcal{E}_k} \mathbb{P}(|f_n(u_k - u_{k-1})| > \eta_k).$$

As $u_0 \in T_0$, $\|u_0\|_{\varpi} \leq 1$ and $\|u_0\|_{\infty} \leq \bar{r}\sqrt{D}$. Moreover, $\|u_0\|_n^2 \leq 3/2 \|u_0\|_{\varpi}^2 \leq 3\delta_0/2$. Then

$$\mathbb{P}(|f_n(u_0)| > \eta_0) = \mathbb{P}\left(|f_n(u_0)| > \eta_0, \|u_0\|_n^2 \leq 3\delta_0/2\right).$$

There exist two constants c'_1 and c'_2 depending only on δ_0 and \bar{r} such that

$$\mathbb{P}(|f_n(u_0)| > \eta_0) \leq K \exp\left(-\frac{n\beta\eta_0^2}{c'_1\alpha^2 + 2c'_2\sqrt{D}\alpha\eta_0}\right).$$

Let us set x_0 such that $\eta_0 = \alpha\left(\sqrt{c'_1(x_0/\beta)} + c'_2\sqrt{D}(x_0/\beta)\right)$. Then:

$$x_0 \leq \frac{\beta\eta_0^2}{c'_1\alpha^2 + 2c'_2\sqrt{D}\alpha\eta_0}$$

and

$$\mathbb{P}(f_n(u_0) > \eta_0) \leq K \exp(-nx_0).$$

Then

$$P_1 \leq K \sum_{u_0 \in \mathcal{E}_0} \exp(-nx_0) \leq K \exp(H_0 - nx_0). \quad (17)$$

We have that

$$\|u_k - u_{k-1}\|_\pi^2 \leq 2 \left(\|u - u_{k-1}\|_\pi^2 + \|u - u_k\|_\pi^2 \right) \leq 5\delta_{k-1}^2/2$$

then $\|u_k - u_{k-1}\|_n^2 \leq 15\delta_{k-1}^2/4$. As $u_{k-1}, u_k \in \mathcal{E}_{k-1} \times \mathcal{E}_k$, it follows that $\|u_k - u_{k-1}\|_\infty^2 \leq 5\delta_{k-1}^2 \bar{r}^2/2$. There exists two constants c_3 and c_4 such that:

$$\begin{aligned} \mathbb{P}_n(|f_n(u_k - u_{k-1})| > \eta_k) &= \mathbb{P}_n\left(|f_n(u_k - u_{k-1})| > \eta_k, \|u_k - u_{k-1}\|_n^2 \leq 15\delta_{k-1}^2/4\right) \\ &\leq K \exp\left(-\frac{n\beta\eta_k^2}{c_3\alpha^2\delta_{k-1}^2 + 2c_4\alpha\delta_{k-1}}\right). \end{aligned}$$

Let us fix x_k such that $\eta_k = \delta_{k-1}a \left(\sqrt{c_3(x_k/\beta)} + c_4(x_k/\beta) \right)$. We obtain:

$$x_k \leq \frac{\beta\eta_k^2}{c_3\alpha^2\delta_{k-1}^2 + 2c_4\alpha\delta_{k-1}}$$

and

$$\mathbb{P}(|f_n(u_k - u_{k-1})| > \eta_k) \leq K \exp(-nx_k).$$

Then, $P_{2,k} \leq K \exp(H_{k-1} + H_k - nx_k)$ and

$$P_2 = \sum_{k=1}^{\infty} P_{2,k} \leq K \sum_{k=1}^{\infty} \exp(H_{k-1} + H_k - nx_k). \quad (18)$$

Let us set $\tau > 0$ and choose (x_k) (and then (η_k)) such that

$$\begin{cases} \sqrt{D}nx_0 = H_0 + D + \tau \\ nx_k = H_{k-1} + H_k + (k+1)D + \tau. \end{cases}$$

Collecting the results, we obtain, by (16), (17) and (18):

$$\mathbb{P}\left(\sup_{u \in \mathcal{B}_{m,m'}} |f_n(u)| > \eta\right) \leq C \left(e^{-D}e^{-\tau} + e^{-\sqrt{D}}e^{-\tau/\sqrt{D}} \right). \quad (19)$$

It remains to compute η^2 . We denote by C a constant depending only on δ_0 and \bar{r} . This constant may vary from one line to another. We have that:

$$\eta = \sum_{k=0}^{\infty} \eta_k \leq C\alpha \left(\sum_{k=1}^{\infty} \delta_{k-1} \left(\sqrt{\frac{x_k}{\beta}} + \frac{x_k}{\beta} \right) \right) + \alpha \left(\sqrt{\frac{x_0}{\beta}} + \sqrt{D}\frac{x_0}{\beta} \right).$$

Let us recall that $H_k = C(k+1)D$. Then, $nx_k = C(3k+2)D + \tau$, $\sqrt{D}nx_0 = CD + \tau$ and

$$\sum_{k=0}^{\infty} \frac{\delta_{k-1}x_k}{\beta} \leq \frac{1}{n\beta} \sum_{k=0}^{\infty} 2^{-(k-1)} (C(3k+2)D + \tau) \leq C \frac{D + \tau}{n\beta}.$$

Moreover,

$$\sum_{k=0}^{\infty} \delta_{k-1} \sqrt{\frac{x_k}{\beta}} \leq C \frac{\sqrt{D} + \sqrt{\tau}}{\sqrt{n\beta}}.$$

As $D/n\beta \leq 1$, there exists a constant κ such that

$$\eta^2 \leq \kappa \alpha^2 \left(\frac{D}{n\beta} + 2 \frac{\tau}{n\beta} + \frac{\tau^2}{n^2 \beta^2} \right).$$

Then, according to (19):

$$\mathbb{P} \left(\sup_{u \in \mathcal{B}_{m,m'}} f_n^2(u) > \kappa \alpha^2 \left(\frac{D}{n\beta} + 2 \frac{\tau}{n\beta} + \frac{\tau^2}{n^2 \beta^2} \right) \right) \leq C \left(e^{-D-\tau} + e^{-\sqrt{D}-\tau/\sqrt{D}} \right). \quad (20)$$

Furthermore

$$\begin{aligned} E &:= \mathbb{E} \left(\left[\sup_{u \in \mathcal{B}_{m,m'}} f_n^2(u) - \kappa \alpha^2 \frac{D}{n\beta} \right]_+ \right) \\ &= \int_0^\infty \mathbb{P} \left(\sup_{u \in \mathcal{B}_{m,m'}} f_n^2(u) > \kappa \alpha^2 \frac{D}{n\beta} + \tau \right) d\tau \end{aligned}$$

Setting $\tau = \kappa \alpha^2 (2y/n\beta + y^2/n^2\beta^2)$, it follows:

$$E = C \gamma^2 \int_0^\infty \mathbb{P} \left(\sup_{u \in \mathcal{B}_{m,m'}} f_n^2(u) > \kappa \alpha^2 \left(\frac{D}{n\beta} + 2 \frac{y}{n\beta} + \frac{y^2}{n^2 \beta^2} \right) \right) \left(\frac{2}{n\beta} + \frac{2y}{n^2 \beta^2} \right) dy.$$

By (20),

$$\begin{aligned} E &= C \kappa \alpha^2 \left(e^{-D} + e^{-\sqrt{D}} \right) \left(\frac{1}{n\beta} \int_0^\infty y e^{-y/\sqrt{D}} dy \right) \\ &\leq C \frac{\kappa \alpha^2}{n\beta} D^{3/2} e^{-\sqrt{D}}. \end{aligned}$$

Acknowledgement: the author wishes to thank M. Reiss and V. Genon-Catalot for helpful discussions.

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