



**HAL**  
open science

# Non-parametric adaptive estimation of the drift for a jump diffusion process

Emeline Schmisser

► **To cite this version:**

Emeline Schmisser. Non-parametric adaptive estimation of the drift for a jump diffusion process. 2012. hal-00704637v2

**HAL Id: hal-00704637**

**<https://hal.science/hal-00704637v2>**

Preprint submitted on 12 Jun 2012 (v2), last revised 26 Sep 2013 (v3)

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

# Non-parametric adaptive estimation of the drift for a jump diffusion process

Emeline Schmisser  
Université Lille 1  
Laboratoire Paul Painlevé  
emeline.schmisser@math.univ-lille1.fr

12th June 2012

## Abstract

In this article, we consider a jump diffusion process  $(X_t)_{t \geq 0}$  observed at discrete times  $t = 0, \Delta, \dots, n\Delta$ . The sampling interval  $\Delta$  tends to 0 and  $n\Delta$  tends to infinity. We assume that  $(X_t)_{t \geq 0}$  is ergodic, strictly stationary and exponentially  $\beta$ -mixing. We use a penalized least-square approach to compute two adaptive estimators of the drift function  $b$ . We provide bounds for the risks of the two estimators.

## 1 Introduction

We consider a general diffusion with jumps:

$$dX_t = b(X_t)dt + \sigma(X_t)dW_t + \xi(X_{t-})dL_t \quad \text{and} \quad X_0 = \eta \quad (1)$$

where  $L_t$  is a centred pure jump Levy process:

$$dL_t = \int_{z \in \mathbb{R}} z (\mu(dt, dz) - dt\nu(dz))$$

with  $\mu$  a random Poisson measure with intensity measure  $\nu(dz)dt$  such that  $\int_{z \in \mathbb{R}} z^2 \nu(dz) < \infty$ . The compensated Poisson measure  $\tilde{\mu}$  is defined by  $\tilde{\mu}(dt, dz) = \mu(dt, dz) - \nu(dz)dt$ . The random variable  $\eta$  is independant of  $(W_t, L_t)_{t \geq 0}$ . Moreover,  $(W_t)_{t \geq 0}$  and  $(L_t)_{t \geq 0}$  are independant.

This process is observed with high frequency (at times  $t = 0, \Delta, \dots, n\Delta$  where  $\Delta \rightarrow 0$  and  $n\Delta \rightarrow \infty$ ). It is assumed to be ergodic, stationary and exponentially  $\beta$ -mixing (see Masuda (2007) for sufficient conditions). Our aim is to construct a non-parametric estimator of  $b$  on a compact set  $A$ .

The non-parametric estimation of  $b$  and  $\sigma$  for a diffusion process observed with high-frequency is well-known (see for instance Hoffmann (1999) and Comte *et al.* (2007)). Diffusion processes with jumps are used in various fields, for instance in finance, for modelling the growth of a population, in hydrology, in

medical science, . . . , but there exist few results for the non-parametric estimation of  $b$  and  $\sigma$ . Shimizu and Yoshida (2006) construct maximum-likelihood estimators of parameters of  $b$  and  $\sigma$ . Their estimators reach the standard rates of convergence:  $\sqrt{n\Delta}$  for the estimator of  $b$ , and  $\sqrt{n}$  for the estimator of  $\sigma$ . Shimizu (2008) and Mancini and Renò (2011) use a kernel estimator to obtain non parametric threshold estimators of  $\sigma$ . Mancini and Renò (2011) also construct a non-parametric truncated estimator of  $b$ , but only when  $L_t$  is a compound Poisson process. To our knowledge, minimax rates of convergences for non-parametric estimators of  $b$ ,  $\sigma$  or  $\xi$  are not available in the literature.

In this paper, we use model selection to construct two non-parametric estimators of  $b$  under the asymptotic framework  $\Delta \rightarrow 0$  and  $n\Delta \rightarrow \infty$ . This method was introduced by Birgé and Massart (1998).

First, we introduce a sequence of linear subspaces  $S_m \subseteq L^2(A)$  and, for each  $m$ , we construct an estimator  $\hat{b}_m$  of  $b$  by minimising on  $S_m$  the contrast function:

$$\gamma_n(t) = \frac{1}{n} \sum_{k=1}^n (Y_{k\Delta} - t(X_{k\Delta}))^2 \quad \text{where} \quad Y_{k\Delta} = \frac{X_{(k+1)\Delta} - X_{k\Delta}}{\Delta}.$$

We obtain a collection of estimators of the drift function  $b$  and we bound their risks (Theorem 1). Then, we introduce a penalty function to select the “best” dimension  $m$  and we deduce an adaptive estimator  $\hat{b}_{\hat{m}}$ . Under the assumption that  $\nu$  is sub-exponential, that is if there exist two positive constants  $C, \lambda$  such that, for  $z$  large enough,  $\nu([-z, z]^c) \leq Ce^{-\lambda z}$ , the risk bound of  $\hat{b}_{\hat{m}}$  is exactly the same as for a diffusion without jumps (Theorem 2) (see Comte *et al.* (2007) or Hoffmann (1999)).

In a second part, we do not assume that  $\nu$  is subexponential and we construct a truncated estimator  $\tilde{b}_m$  of  $b$ . We minimise the contrast function

$$\tilde{\gamma}_n(t) = \frac{1}{n} \sum_{k=1}^n (Y_{k\Delta} 1_{|Y_{k\Delta}| \leq C_\Delta} - t(X_{k\Delta}))^2 \quad \text{where} \quad C_\Delta \propto \sqrt{\Delta} \ln(n)$$

in order to obtain a new estimator  $\tilde{b}_m$ . As in the first part, we introduce a penalty function to obtain an adaptive estimator  $\tilde{b}_{\hat{m}}$ . The risk bound of this adaptive estimator depends on the Blumenthal-Gettoor index of  $\nu$  (Theorems 3 and 4).

In Section 2, we present the model and its assumptions. In Sections 3 and 4, we construct the estimators and bound their risks. Some simulations are presented in Section 5. Proofs are gathered in Sections 6 and 7.

## 2 Assumptions

### 2.1 Assumptions on the model

We consider the following assumptions:

A 1. *The functions  $b, \sigma$  and  $\xi$  are Lipschitz.*

A 2. 1. The function  $\sigma$  is bounded from below and above:

$$\exists \sigma_0, \sigma_1, \forall x \in \mathbb{R}, \quad 0 < \sigma_1 \leq \sigma(x) \leq \sigma_0.$$

2. The function  $\xi$  is bounded:  $\exists \xi_0, \forall x \in \mathbb{R}, \quad 0 \leq \xi(x) \leq \xi_0$ .

3. The drift function  $b$  is elastic: there exists a constant  $M$  such that, for any  $x \in \mathbb{R}, |x| > M$ :  $xb(x) \lesssim -|x|^2$ .

4. The Lévy measure  $\nu$  satisfies:

$$\nu(\{0\}) = 0, \quad \int_{-\infty}^{\infty} z^2 \nu(dz) = 1 \quad \text{and} \quad \int_{-\infty}^{\infty} z^4 \nu(dz) < \infty.$$

Under Assumption A1, the stochastic differential equation (1) admits a unique strong solution. According to Masuda (2007), under Assumptions A1 and A2, the process  $(X_t)$  admits a unique invariant probability  $\varpi$  and satisfies the ergodic theorem: for any measurable function  $g$  such that  $\int |g(x)| \varpi(dx) < \infty$ , when  $T \rightarrow \infty$ ,

$$\frac{1}{T} \int_0^T g(X_s) ds \rightarrow \int g(x) \varpi(dx).$$

This distribution has moments of order 4. Moreover, Masuda (2007) also ensures that under these assumptions, the process  $(X_t)$  is exponentially  $\beta$ -mixing. Furthermore, if there exist two constants  $c$  and  $n_0$  such that, for any  $x \in \mathbb{R}$ ,  $\xi^2(x) \geq c(1 + |x|)^{-n_0}$ , then Ishikawa and Kunita (2006) ensure that a smooth transition density exists.

A 3. 1. The stationary measure  $\varpi$  admits a density  $\pi$  which is bounded from below and above on the compact interval  $A$ :

$$\exists \pi_0, \pi_1, \forall x \in A, \quad 0 < \pi_1 \leq \pi(x) \leq \pi_0.$$

2. The process  $(X_t)_{t \geq 0}$  is stationary ( $\eta \sim \varpi(dx) = \pi(x)dx$ ).

The following proposition very useful for the proofs is proved later.

**Proposition 1.**

Under Assumptions A1-A3, for any  $p \geq 1$ , there exists a constant  $c(p)$  such that, if  $\int_{\mathbb{R}} z^{2p} \nu(dz) < \infty$ :

$$\mathbb{E} \left( \sup_{s \in [t, t+h]} (X_s - X_t)^{2p} \right) \leq c(p)h.$$

## 2.2 Assumptions on the approximation spaces

In order to construct an adaptive estimator of  $b$ , we use model selection: we compute a collection of estimators  $\hat{b}_m$  of  $b$  by minimising a contrast function  $\gamma_n(t)$  on a vectorial subspace  $S_m \subset L^2(A)$ , then we choose the best possible estimator using a penalty function  $pen(m)$ . The collection of vectorial subspaces  $(S_m)_{m \in \mathcal{M}_n}$  has to satisfy the following assumption:

A 4.

1. The subspaces  $S_m$  have finite dimension  $D_m$ .
2. The sequence of vectorial subspaces  $(S_m)_{m \geq 0}$  is increasing: for any  $m$ ,  $S_m \subseteq S_{m+1}$ .
3. Norm connexion: there exists a constant  $\phi_1$  such that, for any  $m \geq 0$ , any  $t \in S_m$ ,

$$\|t\|_\infty^2 \leq \phi_1 D_m \|t\|_{L^2}^2$$

where  $\|\cdot\|_{L^2}$  is the  $L^2$ -norm and  $\|\cdot\|_\infty$  is the sup-norm on  $A$ .

4. For any  $m \in \mathbb{N}$ , there exists an orthonormal basis  $(\psi_\lambda)_{\lambda \in \Lambda_m}$  of  $S_m$  such that

$$\forall \lambda, \quad \text{card}(\lambda', \|\psi_\lambda \psi_{\lambda'}\|_\infty \neq 0) \leq \phi_0$$

where  $\phi_0$  does not depend on  $m$ .

5. For any function  $t$  belonging to the unit ball of the Besov space  $\mathcal{B}_{2,\infty}^\alpha$ ,

$$\exists C, \forall m \quad \|t - t_m\|_{L^2}^2 \leq C 2^{-2m\alpha}$$

where  $t_m$  is the  $L^2$  orthogonal projection of  $t$  on  $S_m$ .

The subspaces generated by piecewise polynomials, compactly supported wavelets or spline functions satisfy A4 (see DeVore and Lorentz (1993) and Meyer (1990) for instance).

## 3 Estimation of the drift

By analogy with Comte *et al.* (2007), we decompose  $Y_{k\Delta}$  in the following way:

$$Y_{k\Delta} = \frac{X_{(k+1)\Delta} - X_{k\Delta}}{\Delta} = b(X_{k\Delta}) + I_{k\Delta} + Z_{k\Delta} + T_{k\Delta} \quad (2)$$

where

$$I_{k\Delta} = \frac{1}{\Delta} \int_{k\Delta}^{(k+1)\Delta} (b(X_s) - b(X_{k\Delta})) ds, \quad Z_{k\Delta} = \frac{1}{\Delta} \int_{k\Delta}^{(k+1)\Delta} \sigma(X_s) dW_s$$

$$T_{k\Delta} = \frac{1}{\Delta} \int_{k\Delta}^{(k+1)\Delta} \xi(X_{s-}) dL_s.$$

The terms  $Z_{k\Delta}$  and  $T_{k\Delta}$  are martingale increments. Let us introduce the mean square contrast function

$$\gamma_n(t) = \frac{1}{n} \sum_{k=1}^n (Y_{k\Delta} - t(X_{k\Delta}))^2 \quad (3)$$

and the empirical risk

$$\mathcal{R}_n(t) = \|t - b_A\|_n^2 \quad \text{where} \quad \|t\|_n^2 = \frac{1}{n} \sum_{k=1}^n t^2(X_{k\Delta}) \quad \text{and} \quad t_A = t1_A. \quad (4)$$

We consider the asymptotic framework:

$$\Delta \rightarrow 0, \quad n\Delta \rightarrow \infty.$$

For any  $m \in \mathcal{M}_n = \{m, D_m \leq \mathcal{D}_n\}$  where  $\mathcal{D}_n^2 \leq n\Delta / \ln^2(n)$ , we construct the regression-type estimator:

$$\hat{b}_m = \arg \min_{t \in S_m} \gamma_n(t).$$

**Theorem 1.**

*Under Assumptions A1-A4, the risk of the estimator with fixed  $m$  satisfies:*

$$\mathcal{R}_n(\hat{b}_m) \leq 3\pi_1 \|b_m - b_A\|_{L^2}^2 + 48(\sigma_0^2 + \xi_0^2) \frac{D_m}{n\Delta} + c\Delta$$

where  $b_m$  is the orthogonal ( $L^2$ ) projection of  $b_A$  over the vectorial subspace  $S_m$ . The constant  $c$  is independent of  $m$ ,  $n$  and  $\Delta$ .

Except for the constant  $(\sigma_0^2 + \xi_0^2)$  in the variance term, this is exactly the bound of the risk that Comte *et al.* (2007) found for a diffusion process without jumps.

The bias term,  $\|b_m - b_A\|_{L^2}^2$ , decreases when the dimension  $D_m$  increases whereas the variance term  $(\sigma_0^2 + \xi_0^2)D_m/(n\Delta)$  is proportional to the dimension. Under the classical assumption  $n\Delta^2 = O(1)$ , the remainder term  $\Delta$  is negligible. Thus we need to find a good compromise between the bias and the variance term.

*Remark 1. If the regularity of the drift function is known, that is, if  $b$  belongs to a ball of a Besov space  $\mathcal{B}_{2,\infty}^\alpha$ , then the bias term  $\|b_m - b_A\|_{L^2}^2$  is smaller than  $D_m^{-2\alpha}$ . The best estimator is obtained for  $D_{m_{opt}} = (n\Delta)^{1/(1+2\alpha)}$  and the estimator risk satisfies:*

$$\mathcal{R}_n(\hat{b}_{m_{opt}}) \lesssim (n\Delta)^{-2\alpha/(2\alpha+1)} + \Delta.$$

Let us introduce a penalty function  $pen$  such that :

$$pen(m) \geq \kappa(\sigma_0^2 + \xi_0^2) \frac{D_m}{n\Delta}$$

and set:

$$\hat{m} = \arg \min_{m \in \mathcal{M}_n} \left\{ \gamma_n(\hat{b}_m) + \text{pen}(m) \right\}.$$

We will chose  $\kappa$  later. We denote by  $\hat{b}_{\hat{m}}$  the resulting estimator. To bound the risk of the adaptive estimator, an additional assumption is needed:

- A 5. 1. *The Lévy measure  $\nu$  is symmetric or the function  $\xi$  is constant.*  
 2. *The Lévy measure  $\nu$  is sub exponential: there exist  $\lambda, C > 0$  such that, for any  $|z| > 1$ ,  $\nu(\cdot - z, z)^c \leq Ce^{-\lambda|z|}$ .*

**Theorem 2.**

*Under Assumptions A1-A5, there exists a constant  $\kappa$  (depending only on  $\nu$ ) such that, if  $\mathcal{D}_n^2 \leq n\Delta / \ln^2(n)$ :*

$$\mathbb{E} \left( \left\| \hat{b}_{\hat{m}} - b_A \right\|_n^2 \right) \lesssim \inf_{m \in \mathcal{M}_n} \left( \|b_m - b_A\|_{L^2}^2 + \text{pen}(m) \right) + \left( \Delta + \frac{1}{n\Delta} \right).$$

We can bound  $\kappa$  theoretically, however, this bound is in practice too large for the simulations. In Section 5, we calibrate  $\kappa$  by simulations (see Comte *et al.* (2007) for instance). The adaptive estimator automatically realises the bias-variance compromise. Moreover, this is the same oracle inequality as for a diffusion process without jumps.

## 4 Truncated estimator of the drift

Truncated estimators are widely used for the estimation of the diffusion coefficient of a jump diffusion (see for instance Mancini and Renò (2011) and Shimizu (2008)). Our aim is to construct an adaptive estimator of  $b$  even if Assumption A5 is not fulfilled. To this end, we cut off the big jumps. Let us introduce the set

$$\Omega_{X,k} = \left\{ \omega, |X_{(k+1)\Delta} - X_{k\Delta}| \leq C_\Delta \right\}$$

where  $C_\Delta = (b_{max} + 3)\Delta + (\sigma_0 + 4\xi_0)\sqrt{\Delta} \ln(n)$  (with  $b_{max} = \sup_{x \in A} |b(x)|$ ). Let us consider the random variables

$$\tilde{Y}_{k\Delta} = \frac{X_{(k+1)\Delta} - X_{k\Delta}}{\Delta} 1_{\Omega_{X,k}} 1_{X_{k\Delta} \in A}.$$

We recall here the definition of the Blumenthal-Gettoor index:

**Definition 1.**

*The Blumenthal-Gettoor index of a Lévy measure is*

$$\beta = \inf \left\{ \alpha \geq 0, \int_{|z| \leq 1} |z|^\alpha \nu(dz) < \infty \right\}.$$

A compound Poisson process has  $\beta = 0$ .

We assume that the following assumption is fulfilled.

A 6. 1. The Lévy measure  $\nu$  is symmetric.

2. For  $|x|$  small,  $\nu(dx)$  is absolutely continuous with respect to the Lebesgue measure ( $\nu(x) = n(x)dx$ ) and:

$$\exists \beta \in [0, 2], \exists a_0, \forall x \in [-a_0, a_0], \quad n(x) \leq Cx^{-\beta-1}.$$

This implies that the Blumenthal-Gettoor index is equal to  $\beta$ .

3. The function  $\xi$  is bounded from below: there exists  $\xi_1 > 0$  such that, for any  $z \in \mathbb{R}$ ,  $0 < \xi_1 \leq \xi(z)$ .

4. The functions  $\sigma$  and  $\xi$  are  $\mathcal{C}^2$ , and  $\xi'$  and  $\sigma'$  are Lipschitz.

We consider the following asymptotic framework:

$$\frac{n\Delta}{\ln^2(n)} \rightarrow 0, \quad \Delta^{1-\beta/2} \ln^2(n) \rightarrow 0.$$

The truncated estimator  $\tilde{b}_m$  is obtained by minimising the contrast function:

$$\tilde{b}_m = \arg \min_{t \in S_m} \tilde{\gamma}_n(t) \quad \text{where} \quad \tilde{\gamma}_n(t) = \frac{1}{n} \sum_{k=1}^n \left( \tilde{Y}_{k\Delta} - t(X_{k\Delta}) \right)^2.$$

**Theorem 3 : Risk of the non adaptive truncated estimator.**

Under Assumptions A1-A4 and A6, for any  $m$  such that  $D_m \leq \mathcal{D}_n$  where  $\mathcal{D}_n^2 \leq n\Delta / \ln^2(n)$ :

$$\mathbb{E} \left( \left\| \tilde{b}_m - b_A \right\|_n^2 \right) \lesssim \|b_m - b_A\|_{L^2}^2 + (\sigma_0^2 + \xi_0^2) \frac{D_m}{n\Delta} + \Delta^{1-\beta/2} \ln^2(n) + \frac{1}{n\Delta}.$$

The terms of the rest depend on the Blumenthal-Gettoor index and are larger than for the first estimator. Nevertheless, if  $L_t$  is a compound Poisson process, then  $\beta = 0$  and we obtain (up to a logarithm factor) the same inequality as for the non-truncated estimator.

*Remark 2.* If  $\nu$  is not absolutely continuous, we can prove the weaker inequality:

$$\mathbb{E} \left( \left\| \tilde{b}_m - b_A \right\|_n^2 \right) \lesssim \|b_m - b_A\|_{L^2}^2 + (\sigma_0^2 + \xi_0^2) \frac{D_m}{n\Delta} + \Delta^{1-\beta} \ln^2(n) + \frac{1}{n\Delta}.$$

In that case,  $\tilde{b}_m$  converges towards  $b_A$  only if  $\beta < 1$ , which implies that  $\nu$  has finite intensity ( $\int_{\mathbb{R}} |z| \nu(dz) < \infty$ ).

*Remark 3.* Assume that  $b_A$  belongs to the Besov space  $\mathcal{B}_{2,\infty}^\alpha$  and that  $\|b_A\|_{\mathcal{B}_{2,\infty}^\alpha} \leq 1$ .

The bias-variance compromise  $\|b_m - b_A\|_{L^2}^2 + D_m/n\Delta$  is minimum when  $m = \log_2(n\Delta)/(1 + 2\alpha)$ , and the risk satisfies:

$$\mathbb{E} \left( \left\| \tilde{b}_m - b_A \right\|_n^2 \right) \lesssim (n\Delta)^{-2\alpha/(1+2\alpha)} + \Delta^{1-\beta/2} \ln^2(n)$$

Let us set  $\Delta \sim n^{-\gamma}$  with  $\gamma > 0$ . We have the following convergence rates:



$\gamma$	first estimator	truncated estimator
$0 < \gamma \leq \frac{2\alpha}{4\alpha+1} \leq \frac{1}{2}$	$\Delta$	$\Delta^{1-\beta/2} \ln^2(n)$
$\frac{2\alpha}{4\alpha+1} \leq \gamma \leq \frac{2\alpha}{4\alpha+1-\beta\alpha-\beta/2} \leq \frac{1}{2(1-\beta/4)}$	$(n\Delta)^{-2\alpha/(2\alpha+1)}$	$\Delta^{1-\beta/2} \ln^2(n)$
$\frac{2\alpha+1}{4\alpha+1-\beta\alpha-\beta/2} \leq \gamma < 1$	$(n\Delta)^{-2\alpha/(2\alpha+1)}$	$(n\Delta)^{-2\alpha/(2\alpha+1)}$

If we have sufficiently high frequency data ( $n\Delta^{2(1-\beta/4)} = O(1)$ ), then the rate of convergence is the same for the two estimators.

To construct the adaptive estimator, we use the same penalty function as in the previous section:

$$\text{pen}(m) \geq \kappa (\sigma_0^2 + \xi_0^2) \frac{D_m}{n\Delta}$$

and define the adaptive estimator:

$$\tilde{m} = \arg \min_{m \in \mathcal{M}_n} \left\{ \tilde{\gamma}_n(\tilde{b}_m) + \text{pen}(m) \right\}.$$

**Theorem 4 : Risk of the adaptive truncated estimator.**

If Assumptions A1-A4 and A6 are satisfied, then there exists  $\kappa$  such that, if  $\mathcal{D}_n^2 \leq n\Delta / \ln^2(n)$ :

$$\mathbb{E} \left( \left\| \tilde{b}_{\tilde{m}} - b_A \right\|_n^2 \right) \lesssim \min_{m \in \mathcal{M}_n} \left( \|b_m - b_A\|_n^2 + \text{pen}(m) \right) + \Delta^{1-\beta/2} \ln^2(n) + \frac{1}{n\Delta}.$$

The adaptive estimator  $\tilde{b}_{\tilde{m}}$  automatically realises the bias/variance compromise if the frequency of data is sufficiently high.

## 5 Numerical simulations and examples

### 5.1 Compound Poisson models

We consider the stochastic differential equation:

$$dX_t = b(X_t)dt + \sigma(X_t)dW_t + \xi(X_{t-})dL_t$$

where  $L_t$  is a compound Poisson process of intensity 1:  $L_t = \sum_{j=1}^{N_t} \zeta_j$ , with  $N_t$  a Poisson process of intensity 1 and  $(\zeta_1, \dots, \zeta_n)$  are independent and identically distributed random variables independent of  $(N_t)$ . We denote by  $f$  the probability law of  $\zeta_i$ .

**Model 1:**

$$b(x) = -2x, \quad \sigma(x) = \xi(x) = 1 \quad \text{and} \quad f(dz) = \nu(dz) = \frac{1}{2}\delta_1 + \frac{1}{2}\delta_{-1}.$$

**Model 2:**

$$b(x) = -(x - 1/4)^3 - (x + 1/4)^3, \quad \sigma(x) = \xi(x) = 1 \quad \text{and} \quad f(dz) = \nu(dz) = \frac{e^{-\lambda|z|} dz}{2}.$$

We can remark that the function  $b$  is not Lipschitz and therefore does not satisfy Assumption A1.

**Model 3:**

We consider the stochastic process of parameters

$$b(x) = -2x + \sin(3x), \quad \sigma(x) = \xi(x) = \sqrt{\frac{3+x^2}{1+x^2}}$$

and

$$f(dz) = \nu(dz) = \frac{1}{4} \sqrt{\frac{\sqrt{24}}{|z|}} e^{-\sqrt{\sqrt{24}|z|}} dz.$$

Let us remark that  $\nu = f$  is not sub-exponential and does not satisfy A5.

**5.2 Simulation algorithm (Compound Poisson case)**

We estimate  $b$  on the compact interval  $A = [-1, 1]$ .

1. Simulate random variables  $(X_0, X_\Delta, \dots, X_{n\Delta})$  thanks to a Euler scheme with sampling interval  $\delta = \Delta/5$ . To this end, we use the same simulation scheme as Rubenthaler (2010). We simulate the times of the jumps  $(\tau_1, \dots, \tau_N, \tau_{N+1})$  with  $\tau_N < n\Delta \leq \tau_{N+1}$  and we fix  $X_0 = 0$ . If  $\delta < \tau_1$ , we compute

$$X_\delta = \delta b(X_0) + \sqrt{\delta} \sigma(X_0) N \quad \text{with} \quad N \sim \mathcal{N}(0, 1)..$$

If  $\tau_1 < \delta$ , we first compute

$$X_{\tau_1} = \tau_1 b(X_0) + \sqrt{\tau_1} \sigma(X_0) N + \xi(X_0) \zeta_1$$

with  $N \sim \mathcal{N}(0, 1)$  and  $\zeta_1 \sim f$  is independent of  $N$ . If  $\delta < \tau_2$ , we compute

$$X_\delta = (\delta - \tau_1) b(X_{\tau_1}) + \sqrt{\delta - \tau_1} \sigma(X_{\tau_1}) N'$$

else we compute

$$X_{\tau_2} = (\tau_2 - \tau_1) b(X_{\tau_1}) + \sqrt{\tau_2 - \tau_1} \sigma(X_{\tau_1}) N' + \xi(X_{\tau_1}) \zeta_2$$

where  $N' \sim \mathcal{N}(0, 1)$  and  $\zeta_2$  has distribution  $f$ .  $N, N', \zeta_1$  and  $\zeta_2$  are independent.

2. Construct the random variables

$$Y_{k\Delta} = \frac{X_{(k+1)\Delta} - X_{k\Delta}}{\Delta} \quad \text{and} \quad \tilde{Y}_{k\Delta} = \frac{X_{(k+1)\Delta} - X_{k\Delta}}{\Delta} 1_{\Omega_{X,k}} 1_{X_{k\Delta} \in A}.$$

3. We consider the vectorial subspaces  $S_{m,r}$  generated by the spline functions of degree  $r$  (see for instance Schmisser (2009a)). In that case  $D_{m,r} = \dim(S_{m,r}) = 2^m + r$ . For  $r \in \{1, 2, 3\}$  and  $m \in \mathcal{M}_n(r) = \{m, D_{m,r} \leq \mathcal{D}_n\}$ , we compute the estimators  $\hat{b}_{m,r}$  and  $\tilde{b}_{m,r}$  by minimising the contrast functions  $\gamma_n$  and  $\tilde{\gamma}_n$  on the vectorial subspaces  $S_{m,r}$ .
4. For the estimation algorithm, we make a selection of  $m$  and  $r$  as follows. Using the penalty function  $pen(m, r) := pen(m) = \kappa(\sigma_0^2 + \xi_0^2)(2^m + r)/n\Delta$ , we select the adaptive estimators  $\hat{b}_{\hat{m},r}$  and  $\tilde{b}_{\tilde{m},r}$ , and then choose the best  $r$  by minimizing  $\gamma_n(\hat{b}_{\hat{m},r}) + pen(\hat{m}, r)$  and  $\tilde{\gamma}_n(\tilde{b}_{\tilde{m},r}) + pen(\tilde{m}, r)$ .

### 5.3 Results

In Figures 1-3, we simulate 5 times the process  $(X_0, \dots, X_{n\Delta})$  for  $\Delta = 10^{-1}$  and  $n = 10^4$  and draw the obtained estimators. The two adaptive estimators are nearly superposed, moreover, they are close to the true function.

In Tables 1-3, for each value of  $(n, \Delta)$ , we simulate 50 trajectories of  $(X_0, X_\Delta, \dots, X_{n\Delta})$ . For each path, we construct the two adaptive estimators  $\hat{b}_{\hat{m},\hat{r}}$  and  $\tilde{b}_{\tilde{m},\tilde{r}}$  and we compute the empirical errors:

$$err_1 = \left\| \hat{b}_{\hat{m},\hat{r}} - b_A \right\|_n^2 \quad \text{and} \quad err_2 = \left\| \tilde{b}_{\tilde{m},\tilde{r}} - b_A \right\|_n^2.$$

In order to check that our algorithm is adaptive, we also compute the minimal errors

$$emin_1 = \min_{m,r} \left\| \hat{b}_{m,r} - b_A \right\|_n^2 \quad \text{and} \quad emin_2 = \min_{m,r} \left\| \tilde{b}_{m,r} - b_A \right\|_n^2$$

and the oracles  $oracle_i = err_i / emin_i$ . We give the means  $\hat{m}_a, \hat{r}_a, \tilde{m}_a$  and  $\tilde{r}_a$  of the selected values  $\hat{m}, \hat{r}, \tilde{m}$  and  $\tilde{r}$ . The value  $risk_i$  is the mean of  $err_i$  over the 50 simulations and  $or_i$  is the mean of  $oracle_i$ .

The empirical risk is decreasing when the product  $n\Delta$  is increasing, which is coherent with the theoretical model. For Model 1, the two estimators are equivalent. When the tails of  $\nu$  become larger (Models 2 and 3), the truncated estimator is better. The improvement is also more significant when the discretisation path is smaller. As on the three models, the processes  $L_t$  are compound Poisson processes, these results were expected. The truncated estimator seems also more robust: we don't observe aberrant values (like for the first estimator in Table 2). This aberrant value may be due to the fact that  $b$  is not Lipschitz and then  $b(X_{k\Delta})$  may be quite large.

## 6 Proofs

Let us introduce the filtration

$$\mathcal{F}_t = \sigma \left( \eta, (W_s)_{0 \leq s \leq t}, (L_s)_{0 \leq s \leq t} \right).$$

The following result is very useful. It comes from Dellacherie and Meyer (1980) (Theorem 92 Chapter VII) and Applebaum (2004), Theorem 4.4.23 p265 (Kunita's first inequality).

*Result 1 (Burkholder-Davis-Gundy inequality).* We have that, for any  $p \geq 2$ ,

$$\mathbb{E} \left[ \sup_{s \in [t, t+h]} \left| \int_t^s \sigma(X_u) dW_u \right|^p \middle| \mathcal{F}_t \right] \leq C_p \left( \mathbb{E} \left[ \left| \int_t^{t+h} \sigma^2(X_u) du \right|^{p/2} \middle| \mathcal{F}_t \right] \right)$$

and, if  $\int_{\mathbb{R}} |z|^p \nu(dz) < \infty$ , as  $\int_{\mathbb{R}} z^2 \nu(dz) = 1$ :

$$\begin{aligned} \mathbb{E} \left[ \sup_{s \in [t, t+h]} \left| \int_t^s \xi(X_{u-}) dL_u \right|^p \middle| \mathcal{F}_t \right] &\leq C_p \mathbb{E} \left[ \left( \int_t^{t+h} \xi^2(X_u) du \right)^{p/2} \middle| \mathcal{F}_t \right] \\ &+ C_p \mathbb{E} \left[ \left( \int_t^{t+h} |\xi(X_u)|^p du \right) \middle| \mathcal{F}_t \right] \int_{\mathbb{R}} |z|^p \nu(dz). \end{aligned}$$

## 6.1 Proof of Proposition 1

By Result 1, there exists a constant  $c_p$  such that:

$$\begin{aligned} \mathbb{E} \left[ \sup_{s \in [t, t+h]} (X_s - X_t)^{2p} \middle| \mathcal{F}_t \right] &\leq c(p) \left( \left[ \left( \int_t^{t+h} |b(X_s)| ds \right)^{2p} \middle| \mathcal{F}_t \right] \right) \\ &+ c(p) \mathbb{E} \left[ \left( \int_t^{t+h} \sigma^2(X_s) ds \right)^p \middle| \mathcal{F}_t \right] \\ &+ c(p) \left( \mathbb{E} \left[ \left( \int_t^{t+h} \xi^2(X_s) ds \right)^p + \int_t^{t+h} \xi^{2p}(X_s) ds \middle| \mathcal{F}_t \right] \right). \end{aligned}$$

Then, as  $\xi$  and  $\sigma$  are bounded and  $b$  Lipschitz (and thus sub-linear), there exists a constant  $C_b$  such that:

$$\mathbb{E} \left[ \sup_{s \in [t, t+h]} (X_s - X_t)^{2p} \middle| \mathcal{F}_t \right] \leq c(p) \left( \sigma_0^{2p} h^p + \xi_0^{2p} (h + h^p) \right) + c(p) h^{2p-1} C_b \int_t^{t+h} \mathbb{E} [X_s^{2p} | \mathcal{F}_t] ds.$$

As  $(X_t)$  is stationary, we obtain the expected result.

## 6.2 Proof of Theorem 1

By (3) and (4), we get:

$$\begin{aligned} \gamma_n(t) = \frac{1}{n} \sum_{k=1}^n (Y_{k\Delta} - t(X_{k\Delta}))^2 &= \frac{1}{n} \sum_{k=1}^n (Y_{k\Delta} - b(X_{k\Delta}))^2 + \|b - t\|_n^2 \\ &+ \frac{2}{n} \sum_{k=1}^n (Y_{k\Delta} - b(X_{k\Delta})) (b(X_{k\Delta}) - t(X_{k\Delta})). \end{aligned}$$

As, by definition,  $\gamma_n(\hat{b}_m) \leq \gamma_n(b_m)$ , we obtain:

$$\left\| \hat{b}_m - b \right\|_n^2 \leq \|b_m - b\|_n^2 + \frac{2}{n} \sum_{k=1}^n (Y_{k\Delta} - b(X_{k\Delta})) \left( \hat{b}_m(X_{k\Delta}) - b_m(X_{k\Delta}) \right).$$

By (2), and as  $\hat{b}_m$  and  $b_m$  are supported by  $A$ ,

$$\left\| \hat{b}_m - b_A \right\|_n^2 \leq \|b_m - b_A\|_n^2 + \frac{2}{n} \sum_{k=1}^n (I_{k\Delta} + Z_{k\Delta} + T_{k\Delta}) \left( \hat{b}_m(X_{k\Delta}) - b_m(X_{k\Delta}) \right).$$

Let us set introduce the unit ball

$$\mathcal{B}_m = \{t \in S_m, \|t\|_{\varpi} \leq 1\} \quad \text{where} \quad \|t\|_{\varpi}^2 = \int_A t^2(x) \varpi(dx)$$

and the englobing space  $\mathcal{S}_n = \bigcup_{m \in \mathcal{M}_n} S_m$ . Let us consider the set

$$\Omega_n = \left\{ \omega, \forall t \in \mathcal{S}_n, \left| \frac{\|t\|_n^2}{\|t\|_{\varpi}^2} - 1 \right| \leq \frac{1}{2} \right\}$$

where the norms  $\|\cdot\|_{\varpi}$  and  $\|\cdot\|_n$  are equivalent.

**Step 1: bound of the risk on  $\Omega_n$**  Thanks to the Cauchy-Schwartz inequality, we obtain that, on  $\Omega_n$ :

$$\left\| \hat{b}_m - b_A \right\|_n^2 \leq \|b_m - b_A\|_n^2 + \frac{1}{12} \left\| \hat{b}_m - b_m \right\|_n^2 + 12 \sum_{k=1}^n I_{k\Delta}^2 + \frac{1}{12} \left\| \hat{b}_m - b_m \right\|_{\varpi}^2 + 12 \sup_{t \in \mathcal{B}_m} \nu_n^2(t)$$

where

$$\nu_n(t) = \frac{1}{n} \sum_{k=1}^n (Z_{k\Delta} + T_{k\Delta}) t(X_{k\Delta}). \quad (5)$$

On  $\Omega_n$ , by definition, we have:

$$\left\| \hat{b}_m - b_m \right\|_n^2 \leq 2 \left\| \hat{b}_m - b_A \right\|_n^2 + 2 \|b_m - b_A\|_n^2 \quad \text{and} \quad \left\| \hat{b}_m - b_m \right\|_{\varpi}^2 \leq 2 \left\| \hat{b}_m - b_m \right\|_n^2.$$

Thus we obtain:

$$\left\| \hat{b}_m - b_A \right\|_n^2 \leq 3 \|b_m - b_A\|_n^2 + 24 \sum_{k=1}^n I_{k\Delta}^2 + 24 \sup_{t \in \mathcal{B}_m} \nu_n^2(t).$$

The following lemma is very useful. It is proved later.

**Lemma 1.** 1.  $\mathbb{E}(I_{k\Delta}^2) \leq c\Delta$  and  $\mathbb{E}(I_{k\Delta}^4) \leq c\Delta$ .

2.  $\mathbb{E}(Z_{k\Delta} | \mathcal{F}_{k\Delta}) = 0$ ,  $\mathbb{E}(Z_{k\Delta}^2 | \mathcal{F}_{k\Delta}) \leq \sigma_0^2/\Delta$  and  $\mathbb{E}(Z_{k\Delta}^4 | \mathcal{F}_{k\Delta}) \leq c/\Delta^2$ .

3.  $\mathbb{E}(T_{k\Delta} | \mathcal{F}_{k\Delta}) = 0$ ,  $\mathbb{E}(T_{k\Delta}^2 | \mathcal{F}_{k\Delta}) \leq \xi_0^2/\Delta$  and  $\mathbb{E}(T_{k\Delta}^4 | \mathcal{F}_{k\Delta}) \leq c/\Delta^3$ .

By Lemma 1,  $\mathbb{E}[I_{k\Delta}^2] \leq \Delta$ . It remains to bound  $\mathbb{E}[\sup_{t \in \mathcal{B}_m} \nu_n^2(t)]$ . We consider an orthonormal basis  $(\varphi_\lambda)_{\lambda \in \Lambda_m}$  of  $S_m$  for the  $L^2_{\varpi}$ -norm with  $|\Lambda_m| = D_m$ . Any function  $t \in S_m$  can be written  $t = \sum_{\lambda \in \Lambda_m} a_\lambda \varphi_\lambda$  and  $\|t\|_{\varpi}^2 = \sum_{\lambda \in \Lambda_m} a_\lambda^2$ . Then:

$$\begin{aligned} \sup_{t \in \mathcal{B}_m} \nu_n^2(t) &= \sup_{\sum_{\lambda} a_\lambda^2 \leq 1} \left( \sum_{\lambda \in \Lambda_m} a_\lambda \nu_n(\varphi_\lambda) \right)^2 \\ &\leq \sup_{\sum_{\lambda} a_\lambda^2 \leq 1} \left( \sum_{\lambda \in \Lambda_m} a_\lambda^2 \right) \left( \sum_{\lambda \in \Lambda_m} \nu_n^2(\varphi_\lambda) \right) \\ &= \sum_{\lambda \in \Lambda_m} \nu_n^2(\varphi_\lambda). \end{aligned}$$

It remains to bound  $\mathbb{E}(\nu_n^2(\varphi_\lambda))$ . By (5),

$$\begin{aligned} \mathbb{E}[\nu_n^2(\varphi_\lambda)] &= \frac{1}{n^2} \sum_{k=1}^n \mathbb{E}[\varphi_\lambda^2(X_{k\Delta}) \mathbb{E}[(Z_{k\Delta} + T_{k\Delta})^2 | \mathcal{F}_{k\Delta}]] \\ &\quad + \frac{2}{n^2} \sum_{k < l}^n \mathbb{E}[(Z_{k\Delta} + T_{k\Delta}) \varphi_\lambda(X_{k\Delta}) \varphi_\lambda(X_{l\Delta}) \mathbb{E}[Z_{l\Delta} + T_{l\Delta} | \mathcal{F}_{l\Delta}]] \end{aligned}$$

Thanks to Lemma 1, the second term of this inequality is null and we obtain, as  $\int_{\mathbb{R}} \varphi_\lambda^2(x) \varpi(dx) = 1$ :

$$\mathbb{E}[\nu_n^2(\varphi_\lambda)] \leq \frac{2(\sigma_0^2 + \xi_0^2)}{n^2 \Delta} \sum_{k=1}^n \mathbb{E}[\varphi_\lambda^2(X_{k\Delta})] = \frac{2(\sigma_0^2 + \xi_0^2)}{n \Delta}.$$

Therefore:

$$\mathbb{E} \left[ \left\| \hat{b}_m - b_A \right\|_n^2 1_{\Omega_n^c} \right] \leq 3 \|b_m - b_A\|_n^2 + 48(\sigma_0^2 + \xi_0^2) \frac{D_m}{n \Delta} + C \Delta.$$

**Step 2: bound of the risk on  $\Omega_n^c$ .** The process  $(X_t)_{t \geq 0}$  is exponentially  $\beta$ -mixing,  $\pi$  is bounded from below and above and  $n \Delta \rightarrow \infty$ . The following result is proved for  $\xi = 0$  for instance in Comte *et al.* (2007), but as it relies only on the  $\beta$ -mixing property, we can apply it.

*Result 2.*

$$\mathbb{P}[\Omega_n^c] \leq \frac{1}{n^3}.$$

Let us set  $e = (e_\Delta, \dots, e_{n\Delta})^*$  where  $e_{k\Delta} := Y_{k\Delta} - b(X_{k\Delta}) = I_{k\Delta} + Z_{k\Delta} + T_{k\Delta}$  and  $\Pi_m Y = \Pi_m (Y_\Delta, \dots, Y_{n\Delta})^* = (\hat{b}_m(X_0), \dots, \hat{b}_m(X_{n\Delta}))^*$  where  $\Pi_m$  is the

Euclidian orthogonal projection over  $S_m$ . Then

$$\begin{aligned} \left\| \hat{b}_m - b_A \right\|_n^2 &= \left\| \Pi_m Y - b_A \right\|_n^2 = \left\| \Pi_m b_A - b_A \right\|_n^2 + \left\| \Pi_m Y - \Pi_m b_A \right\|_n^2 \\ &\leq \left\| b_A \right\|_n^2 + \left\| e \right\|_n^2. \end{aligned}$$

According to Lemma 1, Result 2 and the Cauchy-Schwarz inequality,

$$\mathbb{E} \left[ \left\| e \right\|_n^2 \mathbf{1}_{\Omega_n^c} \right] \leq \left( \mathbb{E} \left[ \left\| e \right\|_n^4 \right] \right)^{1/2} \left( \mathbb{P}(\Omega_n^c) \right)^{1/2} \leq \frac{C}{(\Delta^3 n^3)^{1/2}} \leq \frac{C}{n\Delta}$$

and, as  $b$  is bounded on the compact set  $A$ ,

$$\mathbb{E} \left[ \left\| b_A \right\|_n^2 \mathbf{1}_{\Omega_n^c} \right] \leq \left( \mathbb{E} \left[ \left\| b_A \right\|_n^4 \right] \mathbb{P}(\Omega_n^c) \right)^{1/2} \lesssim \frac{1}{n^{3/2}}.$$

Collecting the results, we get:

$$\mathbb{E} \left[ \left\| \hat{b}_m - b_A \right\|_n^2 \mathbf{1}_{\Omega_n^c} \right] \lesssim \frac{1}{n\Delta}$$

which ends the proof of Theorem 1.

### 6.2.1 Proof of Lemma 1

By Proposition 1, as  $b$  is Lipschitz,

$$\begin{aligned} \mathbb{E} [I_{k\Delta}^2] &= \frac{1}{\Delta^2} \mathbb{E} \left[ \left( \int_{k\Delta}^{(k+1)\Delta} (b(X_s) - b(X_{k\Delta})) ds \right)^2 \right] \\ &\leq \frac{1}{\Delta} \mathbb{E} \left[ \int_{k\Delta}^{(k+1)\Delta} (b(X_s) - b(X_{k\Delta}))^2 ds \right] \\ &\leq \frac{c}{\Delta} \int_{k\Delta}^{(k+1)\Delta} \mathbb{E} [(X_s - X_{k\Delta})^2] ds \\ &\leq c\Delta. \end{aligned}$$

In the same way, we prove that  $\mathbb{E} [I_{k\Delta}^4] \leq c\Delta$ . We have that

$$\mathbb{E} [Z_{k\Delta}^2 | \mathcal{F}_{k\Delta}] = \mathbb{E} \left[ \left( \frac{1}{\Delta^2} \int_{k\Delta}^{(k+1)\Delta} \sigma^2(X_s) ds \right) \middle| \mathcal{F}_{k\Delta} \right] \leq \frac{\sigma_0^2}{\Delta}.$$

Moreover, by the Burkholder-Davis-Gundy inequality, we get

$$\mathbb{E} [Z_{k\Delta}^4 | \mathcal{F}_{k\Delta}] \leq \frac{C}{\Delta^4} \mathbb{E} \left[ \left( \int_{k\Delta}^{(k+1)\Delta} \sigma^2(X_s) ds \right)^2 \middle| \mathcal{F}_{k\Delta} \right] \leq \frac{C}{\Delta^2}.$$

According to Applebaum (2004), Theorem 4.2.3 p224,

$$\mathbb{E}[T_{k\Delta} | \mathcal{F}_{k\Delta}] = 0$$

and, as  $\int_{\mathbb{R}} z^2 \nu(dz) = 1$ :

$$\mathbb{E}[T_{k\Delta}^2 | \mathcal{F}_{k\Delta}] = \frac{1}{\Delta^2} \int_{k\Delta}^{(k+1)\Delta} \xi^2(X_s) \int_{\mathbb{R}} z^2 \nu(dz) ds \leq \frac{\xi_0^2}{\Delta}.$$

By Result 1, we have

$$\begin{aligned} \mathbb{E}[T_{k\Delta}^4 | \mathcal{F}_{k\Delta}] &\leq \frac{C_p}{\Delta^4} \mathbb{E} \left[ \left( \int_{k\Delta}^{(k+1)\Delta} \xi^2(X_s) \int_{\mathbb{R}} z^2 \nu(dz) ds \right)^2 \right] \\ &\quad + \frac{C_p}{\Delta^4} \mathbb{E} \left[ \left( \int_{k\Delta}^{(k+1)\Delta} \xi^4(X_s) ds \right) \int_{\mathbb{R}} z^4 \nu(dz) \right] \\ &\lesssim \frac{1}{\Delta^3} \end{aligned}$$

### 6.3 Proof of Theorem 2

The bound of the risk on  $\Omega_n^c$  is done exactly in the same way as for the non adaptive estimator. It remains thus to bound the risk on  $\Omega_n$ . As in the previous proof, we get:

$$\begin{aligned} \left\| \hat{b}_{\hat{m}} - b_A \right\|_n^2 1_{\Omega_n} &\leq 3 \|b_m - b_A\|_n^2 + \frac{24}{n} \sum_{k=1}^n I_{k\Delta}^2 + 2pen(m) - 2pen(\hat{m}) \\ &\quad + 24 \sup_{t \in \mathcal{B}_{m, \hat{m}}} \nu_n^2(t) \end{aligned}$$

where  $\mathcal{B}_{m, m'}$  is the unit ball (for the  $L_{\infty}^2$ -norm) of the subspace  $S_m + S_{m'}$ :  $\mathcal{B}_{m, m'} = \{t \in S_m + S_{m'}, \|t\|_{\infty} \leq 1\}$ . Let us introduce a function  $p(m, m')$  such that  $12p(m, m') = pen(m) + pen(m')$ . We obtain that, on  $\Omega_n$ , for any  $m \in \mathcal{M}_n$ :

$$\begin{aligned} \left\| \hat{b}_{\hat{m}} - b_A \right\|_n^2 &\leq 3 \|b_m - b_A\|_n^2 + \frac{24}{n} \sum_{k=1}^n I_{k\Delta}^2 + 4pen(m) \\ &\quad + 24 \sup_{t \in \mathcal{B}_{m, \hat{m}}} (\nu_n^2(t) - p(m, \hat{m})). \end{aligned}$$

It remains to bound

$$\mathbb{E} \left[ \sup_{t \in \mathcal{B}_{m, \hat{m}}} \nu_n^2(t) - p(m, \hat{m}) \right] \leq \sum_{m'} \mathbb{E} \left[ \sup_{t \in \mathcal{B}_{m, m'}} \nu_n^2(t) - p(m, m') \right]_+.$$

For this purpose, we use the following proposition proved in Applebaum (2004) (Corollary 5.2.2).



**Proposition 2 : exponential martingale.**

Let  $(Y_t)_{t \geq 0}$  satisfy:

$$Y_t = \int_0^t F_s dW_s + \int_0^t K_s dL_s - \int_0^t \left[ \frac{F_s^2}{2} + \int_{\mathbb{R}} (e^{K_s z} - 1 - K_s z) \nu(dz) \right] ds$$

where  $F_s$  and  $K_s$  are locally integrable and previsible processes. If for any  $t > 0$ ,

$$\mathbb{E} \left[ \int_0^t \int_{|z| > 1} |e^{K_s z} - 1| \nu(dz) ds \right] < \infty,$$

then  $e^{Y_t}$  is a  $\mathcal{G}_t$ -local martingale where  $\mathcal{G}_t = \sigma(W_s, L_s, 0 \leq s \leq t)$ .

For any  $\varepsilon \leq \varepsilon_1 := (\lambda \wedge 1)/(2 \|t\|_{\infty} \xi_0)$  where  $\lambda$  is defined in Assumption A5, for any  $t \geq 0$

$$\int_0^t \int_{|z| \geq 1} (\exp(\varepsilon t(X_{k\Delta})\xi(X_s)z) - 1) \nu(dz) 1_{s \in ]k\Delta, (k+1)\Delta]} ds < \infty.$$

Let us introduce the two Markov processes

$$A_{\varepsilon, t} := \varepsilon^2 \sum_{k=0}^n t^2(X_{k\Delta}) \int_0^t \sigma^2(X_s) 1_{s \in ]k\Delta, (k+1)\Delta]} ds$$

and

$$B_{\varepsilon, t} := \sum_{k=0}^n \int_0^t \int_{\mathbb{R}} (\exp(\varepsilon t(X_{k\Delta})\xi(X_s)z) - \varepsilon t(X_{k\Delta})\xi(X_s)z - 1) 1_{s \in ]k\Delta, (k+1)\Delta]} \nu(dz) ds$$

and the following martingale:

$$M_t = \int_0^t \sum_{k=0}^n 1_{s \in ]k\Delta, (k+1)\Delta]} t(X_{k\Delta-}) (\sigma(X_s) dW_s + \xi(X_{s-}) dL_s).$$

By Proposition 2,

$$Y_{\varepsilon, s} := \varepsilon M_s - A_{\varepsilon, s} - B_{\varepsilon, s}$$

is such that  $e^{Y_{\varepsilon, s}}$  is a local martingale.

**Bound of  $A_{\varepsilon, s}$  and  $B_{\varepsilon, s}$ .** We obtain easily that  $A_{\varepsilon, s} \leq A_{\varepsilon, (n+1)\Delta} \leq \varepsilon^2 n \Delta \|t\|_n^2 \sigma_0^2$ . Under Assumption A5,  $\xi$  is constant or  $\nu$  is symmetric, and therefore

$$B_{\varepsilon, s} \leq B_{\varepsilon, (n+1)\Delta} \leq \Delta \sum_{k=0}^n \int_{\mathbb{R}} (\exp(\varepsilon t(X_{k\Delta})\xi_0 z) - \varepsilon t(X_{k\Delta-})\xi_0 z - 1) \nu(dz).$$

As  $\int_{\mathbb{R}} z^2 \nu(dz) = 1$ , for any  $\alpha \leq 1$ ,

$$\int_{-1}^1 (\exp(\alpha z) - \alpha z - 1) \nu(dz) \leq \alpha^2 \int_{-1}^1 z^2 \nu(dz) \leq \alpha^2.$$

Moreover, by integration by parts, for any  $\alpha \leq (1 \wedge \lambda)/2$ ,

$$\begin{aligned} \int_{[-1,1]^c} (\exp(\alpha z) - \alpha z - 1) \nu(dz) &\leq (e^\alpha - \alpha - 1) \nu([1, +\infty]) + (e^{-\alpha} + \alpha - 1) \nu(]-\infty, -1]) \\ &\quad + \int_1^{+\infty} \alpha (e^{\alpha z} - 1) \nu([-z, z]^c) dz \\ &\leq 2\alpha^2 \nu([-1, 1]^c) + Ce^{-\lambda} \frac{\alpha}{\lambda} \left( \frac{e^\alpha}{1 - \alpha/\lambda} - 1 \right) \leq C' \alpha^2. \end{aligned}$$

Then  $B_{\varepsilon, s} \lesssim n\Delta\varepsilon^2 \xi_0^2 \|t\|_n^2$ . There exists a constant  $c$  such that, for any  $\varepsilon < \varepsilon_1$ ,

$$A_{\varepsilon, s} + B_{\varepsilon, s} \leq c \frac{n\Delta\varepsilon^2 (\sigma_0^2 + \xi_0^2) \|t\|_n^2}{(1 - \varepsilon/\varepsilon_1)}.$$

**Bound of  $\mathbb{P}(\nu_n(t) \geq \eta, \|t\|_n^2 \leq \zeta^2)$ .** The process  $\exp(Y_{\varepsilon, t})$  is a local martingale, then there exists an increasing sequence  $(\tau_N)$  of stopping times such that  $\lim_{N \rightarrow \infty} \tau_N = \infty$  and  $\exp(Y_{\varepsilon, t \wedge \tau_N})$  is a  $\mathcal{F}_t$ -martingale. For any  $\varepsilon < \varepsilon_1$ , and all  $N$ ,

$$\begin{aligned} E &:= \mathbb{P}\left(M_{(n+1)\Delta \wedge \tau_N} \geq n\Delta\eta, \|t\|_n^2 \leq \zeta^2\right) \\ &\leq \mathbb{P}\left(M_{(n+1)\Delta \wedge \tau_N} \geq n\Delta\eta, A_{(n+1)\Delta \wedge \tau_N} + B_{(n+1)\Delta \wedge \tau_N} \leq \frac{cn\Delta\varepsilon^2 (\sigma_0^2 + \xi_0^2) \zeta^2}{(1 - \varepsilon/\varepsilon_1)}\right) \\ &\leq \mathbb{E}\left(\exp(Y_{\varepsilon, (n+1)\Delta \wedge \tau_N}) \exp\left(-n\Delta\eta\varepsilon + \frac{cn\Delta\varepsilon^2 (\xi_0^2 + \sigma_0^2) \zeta^2}{(1 - \varepsilon/\varepsilon_1)}\right)\right). \end{aligned}$$

As  $\exp(Y_{\varepsilon, t \wedge \tau_N})$  is a martingale,  $\mathbb{E}(\exp(Y_{\varepsilon, t \wedge \tau_N})) = 1$  and

$$E \leq \exp\left(-n\Delta\eta\varepsilon + \frac{cn\Delta\varepsilon^2 (\xi_0^2 + \sigma_0^2) \zeta^2}{(1 - \varepsilon/\varepsilon_1)}\right).$$

Letting  $N$  tends to infinity, by dominated convergence, and as  $\nu_n(t) = n\Delta M_{(n+1)\Delta}$ , we obtain that

$$\mathbb{P}\left(\nu_n(t) \geq \eta, \|t\|_n^2 \leq \zeta^2\right) \leq \exp\left(-n\Delta\eta\varepsilon + \frac{cn\Delta\varepsilon^2 (\xi_0^2 + \sigma_0^2) \zeta^2}{(1 - \varepsilon/\varepsilon_1)}\right).$$

It remains to minimise this inequality in  $\varepsilon$ . Let us set

$$\varepsilon = \frac{\eta}{2c(\sigma_0^2 + \xi_0^2) \zeta^2 / \Delta + \eta / \varepsilon_1} < \varepsilon_1.$$

We get:

$$\mathbb{P}\left(\nu_n(t) \geq \eta, \|t\|_n^2 \leq \zeta^2\right) \leq \exp\left(-\frac{\eta^2 n \Delta}{4c((\sigma_0^2 + \xi_0^2) \zeta^2 + c' \eta \xi_0 \|t\|_\infty)}\right).$$

The following lemma concludes the proof. It is proved thanks to a  $L^2_{\varpi} - L^\infty$  chaining technique. See Comte (2001), proof of Proposition 4, and Schmisser (2010), Appendix D.3.

**Lemma 2.**

There exists a constant  $\kappa$  such that:

$$\mathbb{E} \left[ \sup_{t \in \mathcal{B}_{m,m'}} \nu_n^2(t) - p(m, m') \right] \lesssim \kappa (\xi_0^2 + \sigma_0^2) \frac{D^{3/2}}{n\Delta} e^{-\sqrt{D}}$$

where  $D = \dim(S_m + S_{m'})$ .

As  $\sum_D D^{3/2} e^{-\sqrt{D}} \leq \sum_{k=0}^{+\infty} k^{3/2} e^{-\sqrt{k}} < \infty$ , we obtain that

$$\mathbb{E} \left[ \sup_{t \in \mathcal{B}_{m,\hat{m}}} \nu_n^2(t) - p(m, \hat{m}) \right] \leq \sum_{m' \in \mathcal{M}_n} \mathbb{E} \left[ \sup_{t \in \mathcal{B}_{m,m'}} \nu_n^2(t) - p(m, m') \right] \lesssim \kappa \frac{\xi_0^2 + \sigma_0^2}{n\Delta}.$$

## 6.4 Proof of Theorem 3

We recall that

$$\Omega_{X,k} = \left\{ \omega, |X_{(k+1)\Delta} - X_{k\Delta}| \leq C_\Delta = (b_{max} + 3) \Delta + (\sigma_0 + 4\xi_0) \sqrt{\Delta} \ln(n) \right\}.$$

Let us introduce the set

$$\Omega_{N,k} = \left\{ \omega, N'_{k\Delta} = 0 \right\}$$

where  $N'_{k\Delta}$  is the number of jumps of size larger than  $\Delta^{1/4}$  occurring in the time interval  $]k\Delta, (k+1)\Delta]$ :

$$N'_{k\Delta} = \mu \left( ]k\Delta, (k+1)\Delta], \left[ -\Delta^{1/4}, \Delta^{1/4} \right]^c \right).$$

We have that

$$\begin{aligned} \tilde{Y}_{k\Delta} &= Y_{k\Delta} 1_{\Omega_{X,k}} 1_{X_{k\Delta} \in A} \\ &= b_A(X_{k\Delta}) - b_A(X_{k\Delta}) 1_{\Omega_{X,k}^c \cap (X_{k\Delta} \in A)} + I_{k\Delta} 1_{\Omega_{X,k} \cap (X_{k\Delta} \in A)} + \tilde{Z}_{k\Delta} + \tilde{T}_{k\Delta} \\ &\quad + (Z_{k\Delta} + T_{k\Delta}) 1_{\Omega_{X,k} \cap \Omega_{N,k}^c \cap (X_{k\Delta} \in A)} + \mathbb{E} \left( (Z_{k\Delta} + T_{k\Delta}) 1_{\Omega_{X,k} \cap \Omega_{N,k} \cap (X_{k\Delta} \in A)} \mid \mathcal{F}_{k\Delta} \right). \end{aligned}$$

where

$$\tilde{Z}_{k\Delta} = Z_{k\Delta} 1_{\Omega_{X,k} \cap \Omega_{N,k} \cap (X_{k\Delta} \in A)} - \mathbb{E} \left( Z_{k\Delta} 1_{\Omega_{X,k} \cap \Omega_{N,k} \cap (X_{k\Delta} \in A)} \mid \mathcal{F}_{k\Delta} \right)$$

and

$$\tilde{T}_{k\Delta} = T_{k\Delta} 1_{\Omega_{X,k} \cap \Omega_{N,k} \cap (X_{k\Delta} \in A)} - \mathbb{E} \left( T_{k\Delta} 1_{\Omega_{X,k} \cap \Omega_{N,k} \cap (X_{k\Delta} \in A)} \mid \mathcal{F}_{k\Delta} \right).$$

As previously, we only bound the risk on  $\Omega_n$ . Let us set

$$\tilde{v}_n(t) := \frac{1}{n} \sum_{k=1}^n t(X_{k\Delta}) \left( \tilde{Z}_{k\Delta} + \tilde{T}_{k\Delta} \right).$$

We have that

$$\begin{aligned} \left\| \tilde{b}_m - b_A \right\|_n^2 1_{\Omega_n} &\leq 3 \|b_m - b_A\|_n^2 + 24 \sup_{t \in \mathcal{B}_m} \tilde{v}_n^2(t) + \frac{224}{n} \sum_{k=1}^n \left( I_{k\Delta}^2 + b_A^2(X_{k\Delta}) 1_{\Omega_{X,k}^c} \right) \\ &+ \frac{224}{n} \sum_{k=1}^n (Z_{k\Delta}^2 + T_{k\Delta}^2) 1_{\Omega_{X,k} \cap \Omega_{N,k}^c \cap (X_{k\Delta} \in A)} \\ &+ \frac{224}{n} \sum_{k=1}^n \left( \mathbb{E} \left[ (Z_{k\Delta} + T_{k\Delta}) 1_{\Omega_{X,k} \cap \Omega_{N,k} \cap (X_{k\Delta} \in A)} \mid \mathcal{F}_{k\Delta} \right] \right)^2. \end{aligned}$$

The following lemma is proved later.

**Lemma 3.** 1.  $\mathbb{P}(\Omega_{X,k}^c \cap (X_{k\Delta} \in A)) \lesssim \Delta^{1-\beta/2}$ .

2.  $\mathbb{P}(\Omega_{X,k} \cap \Omega_{N,k}^c \cap (X_{k\Delta} \in A)) \lesssim \Delta^{2-\beta/2}$ .

3.  $\left( \mathbb{E} \left[ (Z_{k\Delta} + T_{k\Delta}) 1_{\Omega_{X,k} \cap \Omega_{N,k} \cap (X_{k\Delta} \in A)} \mid \mathcal{F}_{k\Delta} \right] \right)^2 \lesssim \ln^2(n) \Delta^{1-\beta/2}$ .

According to Lemma 1,  $\mathbb{E}(I_{k\Delta}^2) \leq \Delta k$ . As  $b$  is bounded on the compact set  $A$ ,  $\mathbb{E} \left[ b_A^2(X_{k\Delta}) 1_{\Omega_{X,k}^c} \right] \lesssim \mathbb{P}(\Omega_{X,k}^c) \lesssim \Delta^{1-\beta/2}$ . Moreover, on  $\Omega_{X,k}$ ,

$$\begin{aligned} (Z_{k\Delta} + T_{k\Delta})^2 1_{\Omega_{X,k} \cap (X_{k\Delta} \in A)} &= \left( \frac{X_{(k+1)\Delta} - X_{k\Delta}}{\Delta} - b_A(X_{k\Delta}) - I_{k\Delta} \right)^2 1_{\Omega_{X,k}} 1_{X_{k\Delta} \in A} \\ &\lesssim \frac{\ln^2(n)}{\Delta} + b_A^2(X_{k\Delta}) + I_{k\Delta}^2 \end{aligned}$$

and then

$$\begin{aligned} E &:= \mathbb{E} \left[ (Z_{k\Delta} + T_{k\Delta})^2 1_{\Omega_{X,k} \cap \Omega_{N,k}^c \cap (X_{k\Delta} \in A)} \right] \\ &\lesssim \left( \frac{\ln^2(n)}{\Delta} + b_{max}^2 \right) \mathbb{P}(\Omega_{X,k} \cap \Omega_{N,k}^c \cap (X_{k\Delta} \in A)) + \mathbb{E}(I_{k\Delta}^2) \\ &\lesssim \ln^2(n) \Delta^{1-\beta/2}. \end{aligned}$$

It remains to bound  $\mathbb{E}(\sup_{t \in \mathcal{B}_m} \tilde{v}_n^2(t))$ . In the same way as in Subsection 6.2, we get:

$$\begin{aligned} \mathbb{E} \left( \sup_{t \in \mathcal{B}_m} \tilde{v}_n^2(t) \right) &\leq \sum_{\lambda \in \Lambda_m} \mathbb{E}(\tilde{v}_n^2(\varphi_\lambda)) \leq \frac{2D_m}{n} \mathbb{E}(\tilde{Z}_\Delta^2 + \tilde{T}_\Delta^2) \\ &\leq \frac{2D_m}{n} \mathbb{E}(Z_\Delta^2 + T_\Delta^2) \leq 2(\sigma_0^2 + \xi_0^2) \frac{D_m}{n\Delta}. \end{aligned}$$

### 6.4.1 Proof of Lemma 3

*Result 3.* Let  $\beta$  be the Blumenthal-Gettoor index of  $L_t$ . Then:

$$\nu([-z, z]^c) \lesssim z^{-\beta} \quad , \quad \int_{|x| \leq z \wedge a_0} x^2 \nu(dx) \lesssim z^{2-\beta} \quad \text{and} \quad \int_{|x| \leq z \wedge a_0} x^4 \nu(dx) \lesssim z^{4-\beta}.$$

The constant  $a_0$  is defined in A6.

**Bound of  $\mathbb{P}(\Omega_{X,k}^c \cap (X_{k\Delta} \in A))$ .** We have:

$$\mathbb{P}(\Omega_{X,k}^c \cap (X_{k\Delta} \in A)) = \mathbb{P}(\{|X_{(k+1)\Delta} - X_{k\Delta}| > C_\Delta\} \cap (X_{k\Delta} \in A)).$$

We know that  $X_{(k+1)\Delta} - X_{k\Delta} = b(X_{k\Delta}) + I_{k\Delta} + Z_{k\Delta} + T_{k\Delta}$ . Then

$$\mathbb{P}(\Omega_{X,k}^c \cap (X_{k\Delta} \in A)) \leq \mathbb{P}(|\Delta I_{k\Delta}| \geq \Delta) + \mathbb{P}(|\Delta Z_{k\Delta}| \geq \sigma_0 \sqrt{\Delta} \ln(n)) + \mathbb{P}(|\Delta T_{k\Delta}| \geq \xi_0 \sqrt{\Delta} \ln(n)). \quad (6)$$

By a Markov inequality and Lemma 1, we obtain:

$$\mathbb{P}(|\Delta I_{k\Delta}| \geq \Delta) \leq \frac{\mathbb{E}(\Delta^2 I_{k\Delta}^2)}{\Delta^2} \lesssim \Delta. \quad (7)$$

By Proposition 2, the process  $\exp\left(c \int_0^t \sigma(X_{s-}) dW_s - c^2 \int_0^t \sigma^2(X_s) ds\right)$  is a local martingale (as  $\sigma$  is bounded, it is in fact a martingale, see Liptser and Shiryaev (2001), pp 229-232). Then, by a Markov inequality:

$$\mathbb{P}(|\Delta Z_{k\Delta}| \geq \sigma_0 \sqrt{\Delta} \ln(n)) \leq \frac{2}{n} \mathbb{E} \left[ \exp \left( \frac{\sqrt{\Delta} Z_{k\Delta}}{\sigma_0} \right) \right] \lesssim \frac{1}{n}. \quad (8)$$

To bound inequality (6), it remains to bound  $\mathbb{P}(|\Delta T_{k\Delta}| \geq \xi_0 \sqrt{\Delta} \ln(n))$ . Let us set

$$T_{k\Delta} = T_{k\Delta}^{(1)} + T_{k\Delta}^{(2)} + T_{k\Delta}^{(3)} \quad \text{where} \quad T_{k\Delta}^{(i)} = \frac{1}{\Delta} \int_{k\Delta}^{(k+1)\Delta} \xi(X_{s-}) dL_s^{(i)}$$

with

$$L_t^{(1)} = \int_0^t \int_{[-\sqrt{\Delta}, \sqrt{\Delta}]} z \tilde{\mu}(ds, dz) \quad , \quad L_t^{(2)} = \int_0^t \int_{[-\Delta^{1/4}, -\sqrt{\Delta}] \cup [\sqrt{\Delta}, \Delta^{1/4}]} z \tilde{\mu}(ds, dz)$$

$$L_t^{(3)} = \int_0^t \int_{[-\Delta^{1/4}, \Delta^{1/4}]^c} z \tilde{\mu}(ds, dz).$$

Let us set  $N_{k\Delta}'' = \mu\left(]k\Delta, (k+1)\Delta], [-\sqrt{\Delta}, \sqrt{\Delta}]^c\right)$ . By Result 3, we have:

$$\mathbb{P}\left(|T_{k\Delta}^{(2)} + T_{k\Delta}^{(3)}| > 0\right) = \mathbb{P}\left(N_{k\Delta}'' \geq 1\right) \lesssim \Delta \nu\left([- \sqrt{\Delta}, \sqrt{\Delta}]^c\right) \lesssim \Delta^{1-\beta/2}.$$

It remains to bound  $\mathbb{P} \left[ \left| \Delta T_{k\Delta}^{(1)} \right| \geq 2\xi_0 \sqrt{\Delta} \ln(n) \right]$ . We have that:

$$\mathbb{P} \left[ \left| \Delta T_{k\Delta}^{(1)} \right| \geq 2\xi_0 \sqrt{\Delta} \ln(n) \right] \leq 2\mathbb{P} \left[ \exp \left( \varepsilon \int_{k\Delta}^{(k+1)\Delta} \xi(X_{s-}) dL_s^{(1)} \right) \geq n^{2\varepsilon\xi_0\sqrt{\Delta}} \right].$$

By Proposition 2, for any  $\varepsilon$ ,

$$D_t := \exp \left( \varepsilon \int_{k\Delta}^t \xi(X_{s-}) dL_s^{(1)} - \int_{k\Delta}^t \int_{|z| \leq \sqrt{\Delta}} (\exp(\varepsilon z \xi(X_{s-}) - 1 - \varepsilon z \xi(X_{s-})) \nu(dz) \right)$$

is a local martingale. Let us set  $\varepsilon = 1/(2\xi_0\Delta^{1/2})$ . There exists an increasing sequence of stopping times  $\tau_N$  such that, for any  $N$ ,

$$\begin{aligned} F &:= \mathbb{P} \left[ \exp \left( \frac{1}{2\xi_0\Delta^{1/2}} \int_{k\Delta}^{(k+1)\Delta \wedge \tau_N} \xi(X_{s-}) dL_s^{(1)} \right) \geq n \right] \\ &\leq n^{-1} \exp \left( \int_{k\Delta}^{(k+1)\Delta \wedge \tau_N} \int_{|z| \leq \sqrt{\Delta}} \left( \exp \left( \frac{z\xi(X_{s-})}{2\xi_0\Delta^{1/2}} \right) - 1 - \frac{z\xi(X_{s-})}{2\xi_0\Delta^{1/2}} \right) \nu(dz) \right) \\ &\leq n^{-1} \exp \left( 2\Delta \int_{|z| \leq \sqrt{\Delta}} \frac{\xi_0^2 z^2}{4\xi_0^2 \Delta} \nu(dz) \right) \leq n^{-1} \exp \left( \int_{\mathbb{R}} z^2 \nu(dz) \right) \leq n^{-1}. \end{aligned}$$

When  $N \rightarrow \infty$ , by dominated convergence, we obtain:

$$\mathbb{P} \left( \left| \Delta T_{k\Delta}^{(1)} \right| \geq \xi_0 \sqrt{\Delta} \ln(n) \right) \lesssim n^{-1}. \quad (9)$$

**Bound of  $\mathbb{P} \left( \Omega_{X,k} \cap \Omega_{N,k}^c \cap (X_{k\Delta} \in A) \right)$ .** We recall that  $N'_{k\Delta} = \mu \left( ]k\Delta, (k+1)\Delta], [-\Delta^{1/4}, \Delta^{1/4}]^c \right)$ . We have:

$$\Omega_{N,k}^c = \left\{ N'_{k\Delta} = 1 \right\} \cup \left\{ N'_{k\Delta} \geq 2 \right\}$$

with

$$\mathbb{P} \left( N'_{k\Delta} = 1 \right) \lesssim \Delta^{1-\beta/4} \quad \text{and} \quad \mathbb{P} \left( N'_{k\Delta} \geq 2 \right) \lesssim \Delta^{2-\beta/2}.$$

Then  $\mathbb{P} \left( \Omega_{N,k}^c \cap \left\{ N'_{k\Delta} \geq 2 \right\} \right) \lesssim \Delta^{2-\beta/2}$ . We can write:

$$\begin{aligned} G &:= \mathbb{P} \left( \Omega_{X,k} \cap (X_{k\Delta} \in A) \cap (N'_{k\Delta} = 1) \right) \\ &\leq \mathbb{P} \left( N'_{k\Delta} = 1 \right) \mathbb{P} \left( \left| \Delta T_{k\Delta}^{(2)} + \Delta T_{k\Delta}^{(3)} \right| \leq 2C\Delta \mid N'_{k\Delta} = 1 \right) \\ &\quad + \mathbb{P} \left( N'_{k\Delta} = 1 \right) \mathbb{P} \left( \left\{ \left| \Delta T_{k\Delta}^{(2)} + \Delta T_{k\Delta}^{(3)} \right| \geq 2C\Delta \mid N'_{k\Delta} = 1 \right\} \cap \Omega_{X,k} \cap (X_{k\Delta} \in A) \right). \end{aligned}$$

By (7), (8) and (9), we obtain:

$$\begin{aligned} H &:= \mathbb{P} \left( \left\{ \left| \Delta T_{k\Delta}^{(2)} + \Delta T_{k\Delta}^{(3)} \right| \geq 2C\Delta \mid N'_{k\Delta} = 1 \right\} \cap \Omega_{X,k} \cap (X_{k\Delta} \in A) \right) \\ &\leq \mathbb{P} \left( \Delta \left| b_A(X_{k\Delta}) + I_{k\Delta} + Z_{k\Delta} + T_{k\Delta}^{(1)} \right| > C\Delta \right) \\ &\lesssim \Delta + n^{-1}. \end{aligned}$$

It remains to bound  $J := \mathbb{P}\left(\left|\Delta T_{k\Delta}^{(2)} + \Delta T_{k\Delta}^{(3)}\right| \leq 2C_\Delta |N'_{k\Delta} = 1\right)$ . If  $N'_{k\Delta} = 1$ , then  $\left|\Delta T_{k\Delta}^{(3)}\right| = \left|\int_{k\Delta}^{(k+1)\Delta} \xi(X_{s-}) dL_s^{(3)}\right| \geq \xi_1 \Delta^{1/4}$ . Then  $J \leq \mathbb{P}\left(\Delta \left|T_{k\Delta}^{(2)}\right| \geq \xi_1 \Delta^{1/4} - 2C_\Delta\right)$ . Let us set  $n_0 = \left\lceil \frac{1}{1-\beta/2} \right\rceil$  and  $a = (\xi_0 n_0)^{-1} (\xi_1 \Delta^{1/4} - 2C_\Delta)$ . We have:

$$\begin{aligned} J &\leq \mathbb{P}[\mu(|k\Delta, (k+1)\Delta|, [-a, a]^c) \geq 1] + \mathbb{P}\left[\mu(|k\Delta, (k+1)\Delta|, [-a, -\Delta^{1/2}] \cup [\Delta^{1/2}, a]) \geq n_0\right] \\ &\leq \Delta \nu([-a, a]^c) + \Delta^{n_0} \nu([-\Delta^{1/2}, \Delta^{1/2}]^c)^{n_0} \\ &\lesssim \Delta^{1-\beta/4} + \Delta. \end{aligned}$$

Then  $\mathbb{P}(\Omega_{X,k} \cap \Omega_{N,k}^c) \leq \mathbb{P}(N'_{k\Delta} = 1) \Delta^{1-\beta/4} + \mathbb{P}(N'_{k\Delta} = 2) \lesssim \Delta^{2-\beta/2}$ .

**Bound of**  $\left(\mathbb{E}\left[(Z_{k\Delta} + T_{k\Delta}) 1_{\Omega_{X,k} \cap \Omega_{N,k} \cap (X_{k\Delta} \in A)} \middle| \mathcal{F}_{k\Delta}\right]\right)^2$ .

**If  $\sigma$  and  $\xi$  are constants.** Let us set  $E := \left(\mathbb{E}\left[(Z_{k\Delta} + T_{k\Delta}) 1_{\Omega_{X,k} \cap \Omega_{N,k} \cap (X_{k\Delta} \in A)} \middle| \mathcal{F}_{k\Delta}\right]\right)^2$  and

$$\Omega_{I,k} = \left\{\omega, |I_{k\Delta}| \leq 1, \cap |\Delta Z_{k\Delta}| \leq \sigma_0 \sqrt{\Delta} \ln(n), \cap \left|\Delta T_{k\Delta}^{(1)}\right| \leq 2\xi_0 \sqrt{\Delta} \ln(n)\right\}.$$

By (7), (8) and (9),  $\mathbb{P}\left(\Omega_{I,k}^c\right) \leq \Delta + n^{-1}$ . Then, by a Markov inequality:

$$E \lesssim \Delta \ln^2(n) + \left(\mathbb{E}\left[(Z_{k\Delta} + T_{k\Delta}) 1_{\Omega_{X,k} \cap \Omega_{N,k} \cap \Omega_{I,k} \cap (X_{k\Delta} \in A)} \middle| \mathcal{F}_{k\Delta}\right]\right)^2.$$

Let us introduce the set  $\Omega_{ZT,k} := \{\omega, |Z_{k\Delta} + T_{k\Delta}| \leq C_\Delta \Delta^{-1} - b_{max} - 1\}$ . On  $\Omega_{I,k}$ ,  $|I_{k\Delta}| \leq 1$  and therefore:

$$\Omega_{ZT,k} \cap \Omega_{I,k} \subseteq \Omega_{X,k} \cap \Omega_{I,k} \subseteq \{\omega, |Z_{k\Delta} + T_{k\Delta}| \leq C_\Delta \Delta^{-1} + b_{max} + 1\} \cap \Omega_{I,k}.$$

Then

$$E \lesssim \Delta \ln^2(n) + F^2 + G^2$$

where  $F = \mathbb{E}\left[(Z_{k\Delta} + T_{k\Delta}) 1_{\Omega_{ZT,k} \cap \Omega_{N,k} \cap \Omega_{I,k} \cap (X_{k\Delta} \in A)} \middle| \mathcal{F}_{k\Delta}\right]$  and  $G = \mathbb{E}\left[(Z_{k\Delta} + T_{k\Delta}) 1_{\Omega_{ZT,k}^c \cap \Omega_{X,k} \cap \Omega_{N,k} \cap \Omega_{I,k} \cap (X_{k\Delta} \in A)} \middle| \mathcal{F}_{k\Delta}\right]$ . As  $\sigma$  and  $\xi$  are constants, the terms

$$Z_{k\Delta} = \frac{\sigma_0}{\Delta} \int_{k\Delta}^{(k+1)\Delta} dW_s \quad \text{and} \quad T_{k\Delta} = \frac{\xi_0}{\Delta} \int_{k\Delta}^{(k+1)\Delta} dL_s$$

are centered and independent. Then  $F = 0$ . Moreover, on  $\Omega_{N,k}$ ,  $T_{k\Delta}^{(3)} = 0$ . Then

$$|G| \lesssim \left|\mathbb{E}\left[\left(Z_{k\Delta} + T_{k\Delta}^{(1)} + T_{k\Delta}^{(2)}\right) 1_{\Omega_{X,k} \cap \Omega_{ZT,k}^c \cap \Omega_{N,k} \cap \Omega_{I,k} \cap (X_{k\Delta} \in A)} \middle| \mathcal{F}_{k\Delta}\right]\right|.$$

On  $\Omega_{I,k} \cap \Omega_{X,k}$ ,  $|Z_{k\Delta} + T_{k\Delta}^{(1)} + T_{k\Delta}^{(2)}| \lesssim \ln(n)\Delta^{-1/2}$ , and

$$\begin{aligned} |G| &\lesssim \frac{\ln(n)}{\sqrt{\Delta}} \left( \mathbb{P} \left( |Z_{k\Delta} + T_{k\Delta}^{(1)} + T_{k\Delta}^{(2)}| \in [C_\Delta \Delta^{-1} - b_{max} - 1, C_\Delta \Delta^{-1} + b_{max} + 1] 1_{\Omega_{I,k}} \right) \right) \\ &= 2 \frac{\ln(n)}{\sqrt{\Delta}} \int_{\mathbb{R}} \mathbb{P} \left( T_{k\Delta}^{(2)} \in [C_\Delta \Delta^{-1} - b_{max} - 1 - x, C_\Delta \Delta^{-1} + b_{max} + 1 - x] 1_{\Omega_{I,k}} \right) \\ &\quad \times \mathbb{P} \left( Z_{k\Delta} + T_{k\Delta}^{(1)} \in dx \mid T_{k\Delta}^{(2)} \in [C_\Delta \Delta^{-1} - b_{max} - 1 - x, C_\Delta \Delta^{-1} + b_{max} + 1 - x] 1_{\Omega_{I,k}} \right). \end{aligned}$$

On  $\Omega_{I,k}$ ,  $|Z_{k\Delta} + T_{k\Delta}^{(1)}| \leq (\sigma_0 + 2\xi_0) \ln(n)\Delta^{-1/2}$ . Then

$$|G| \lesssim \frac{\ln(n)}{\sqrt{\Delta}} \left[ \sup_{C \geq \xi_0 \ln(n)\Delta^{-1/2}} \mathbb{P} \left( T_{k\Delta}^{(2)} \in [C, C + 2b_{max} + 2] \right) \right]. \quad (10)$$

We recall that  $L_t^{(2)}$  is a compound Poisson process in which all the jumps are greater than  $\sqrt{\Delta}$  and smaller than  $\Delta^{1/4}$ . Let us denote by  $\tau_i$  the times of the jumps of size in  $[\sqrt{\Delta}, \Delta^{1/4}]$  and by  $\zeta_i$  the size of the jumps. We set  $a_j = \xi_0^{-1} C \Delta - \sum_{i=1}^{j-1} \zeta_i$  and  $c := \xi_0^{-1} (2b_{max} + 2)$ . Then, as  $\xi$  is constant equal to  $\xi_0$ :

$$\begin{aligned} H &:= \mathbb{P} \left( T_{k\Delta}^{(2)} \in [C, C + 2b_{max} + 2] \right) \\ &\leq \sum_{j=1}^{\infty} \mathbb{P} \left( j \text{ jumps} \geq \sqrt{\Delta}, \text{ last jump} \in [a_j, a_j + c\Delta] \right) \\ &\lesssim 2 \sup_{a \geq \sqrt{\Delta}} \mathbb{P} (1 \text{ jump} \in [a, a + c\Delta]) = 2\Delta \sup_{a \geq \sqrt{\Delta}} \nu([a, a + c\Delta]). \end{aligned}$$

By A6,

$$H \lesssim \Delta \sup_{a \geq \sqrt{\Delta}} \left[ \frac{1}{a^\beta} - \frac{1}{(a + c\Delta)^\beta} \right] \lesssim \sqrt{\Delta} \Delta^{1-\beta/2} \quad (11)$$

and, by (10) and (11),

$$E \lesssim \Delta \ln^2(n) + \frac{\ln^2(n)}{\Delta} \Delta \Delta^{2-\beta} \lesssim \Delta \ln^2(n) + \Delta^{2-\beta} \ln^2(n).$$

*Remark 4.* If  $\nu$  is not absolutely continuous, we obtain:

$$E \leq \Delta \ln^2(n) + (\Delta \nu([-a + c\Delta, a - c\Delta]^c))^2 \lesssim \Delta \ln^2(n) + \Delta^{2-2\beta} \ln^2(n).$$

**If  $\sigma$  or  $\xi$  are not constants.** The problem is that  $Z_{k\Delta}$  and  $T_{k\Delta}$  are not symmetric and we can't apply directly the previous method. We replace them by two centred terms. The following lemma is very useful.



**Lemma 4.**

Let  $f$  be a  $\mathcal{C}^2$  function such that  $f$  and  $f'$  are Lipschitz. Let us set, for any  $t \in ]k\Delta, (k+1)\Delta[$ :

$$\psi_f(X_{k\Delta}, t) = f'(X_{k\Delta}) \left( \sigma(X_{k\Delta}) \int_{k\Delta}^t dW_s + \xi(X_{k\Delta}) \int_{k\Delta}^t z \tilde{\mu}(ds, dz) \right).$$

We have:

$$\mathbb{E} \left[ (f(X_t) - f(X_{k\Delta}) - \psi_f(X_{k\Delta}, t))^2 1_{\Omega_{N,k}} 1_{X_{k\Delta} \in A} \right] \lesssim \Delta^{2-\beta/4}.$$

Lemma 4 is proved below. Let us set

$$\bar{Z}_{k\Delta} = \frac{1}{\Delta} \int_{k\Delta}^{(k+1)\Delta} (\sigma(X_{k\Delta}) + \psi_\sigma(X_{k\Delta}, s)) dW_s,$$

$$\bar{T}_{k\Delta}^{(i)} = \frac{1}{\Delta} \int_{k\Delta}^{(k+1)\Delta} (\xi(X_{k\Delta}) + \psi_\xi(X_{k\Delta}, s)) dL_s^{(i)} \quad \text{and} \quad \bar{T}_{k\Delta} = \bar{T}_{k\Delta}^{(1)} + \bar{T}_{k\Delta}^{(2)} + \bar{T}_{k\Delta}^{(3)}.$$

The terms  $\bar{Z}_{k\Delta}$  and  $\bar{T}_{k\Delta}$  are symmetric. By lemma 4,

$$\begin{aligned} \mathbb{E} \left[ (\bar{Z}_{k\Delta} - Z_{k\Delta})^2 1_{\Omega_{N,k}} 1_{X_{k\Delta} \in A} \right] &= \frac{1}{\Delta^2} \mathbb{E} \left[ \int_{k\Delta}^{(k+1)\Delta} (\sigma(X_s) - \sigma(X_{k\Delta}) - \psi_\sigma(X_{k\Delta}, s))^2 ds \right] \\ &\lesssim \Delta^{1-\beta/4}. \end{aligned} \tag{12}$$

We prove in the same way that

$$\mathbb{E} \left[ (\bar{T}_{k\Delta} - T_{k\Delta})^2 1_{\Omega_{N,k}} 1_{X_{k\Delta} \in A} \right] \leq \Delta^{1-\beta/4}. \tag{13}$$

Let us set  $U_{k\Delta} = \Delta^{-1} \xi(X_{k\Delta-}) \int_{k\Delta}^{(k+1)\Delta} dL_s^{(2)}$ . By Result 1 and Proposition 1,

$$\mathbb{E} \left[ \Delta^2 (\bar{T}_{k\Delta}^{(2)} - U_{k\Delta})^2 \right] = \mathbb{E} \left[ \int_{k\Delta}^{(k+1)\Delta} \int_{\mathbb{R}} (\xi(X_s) - \xi(X_{k\Delta}))^2 z^2 \nu(dz) ds \right] \leq \Delta^2. \tag{14}$$

Let us introduce the set

$$\begin{aligned} \bar{\Omega}_{I,k} &= \{ \omega, |I_{k\Delta}| + |Z_{k\Delta} - \bar{Z}_{k\Delta}| + |T_{k\Delta} - \bar{T}_{k\Delta}| \leq 3 \} \\ &\cap \left\{ |\Delta \bar{Z}_{k\Delta}| \leq \sigma_0 \sqrt{\Delta} \ln(n) + \Delta, \left| \Delta \bar{T}_{k\Delta}^{(1)} \right| \leq 2\xi_0 \sqrt{\Delta} \ln(n) + \Delta \right\} \\ &\cap \left\{ \left| \Delta (\bar{T}_{k\Delta}^{(2)} - U_{k\Delta}) \right| \leq \xi_0 \sqrt{\Delta} \right\}. \end{aligned}$$

By (7), (8), (9), (12), (13), (14) and Markov inequalities, we obtain:

$$\mathbb{P}(\bar{\Omega}_{I,k}^c) \lesssim \Delta^{1-\beta/4} + \frac{1}{n}. \tag{15}$$

Then

$$\begin{aligned} E &:= \left( \mathbb{E} \left[ (Z_{k\Delta} + T_{k\Delta}) 1_{\Omega_{X,k} \cap \Omega_{N,k} \cap (X_{k\Delta} \in A)} \middle| \mathcal{F}_{k\Delta} \right] \right)^2 \\ &\lesssim \Delta^{1-\beta/2} \ln^2(n) + \left( \mathbb{E} \left[ (\bar{Z}_{k\Delta} + \bar{T}_{k\Delta}) 1_{\Omega_{X,k} \cap \Omega_{N,k} \cap (X_{k\Delta} \in A) \cap \bar{\Omega}_{I,k}} \middle| \mathcal{F}_{k\Delta} \right] \right)^2. \end{aligned} \quad (16)$$

Let us introduce the set:

$$\bar{\Omega}_{ZT,k} := \{ \omega, |\bar{Z}_{k\Delta} + \bar{T}_{k\Delta}| \leq C_\Delta \Delta^{-1} - b_{max} - 3 \}.$$

We have that

$$\bar{\Omega}_{ZT,k} \cap \bar{\Omega}_{I,k} \subseteq \Omega_{X,k} \cap \bar{\Omega}_{I,k} \subseteq \{ \omega, |\bar{Z}_{k\Delta} + \bar{T}_{k\Delta}| \leq C_\Delta \Delta^{-1} + b_{max} + 3 \} \cap \bar{\Omega}_{I,k}.$$

Given the filtration  $\mathcal{F}_{k\Delta}$ , the sum  $\bar{Z}_{k\Delta} + \bar{T}_{k\Delta}$  is symmetric. Then

$$\mathbb{E} \left[ (\bar{Z}_{k\Delta} + \bar{T}_{k\Delta}) 1_{\bar{\Omega}_{ZT,k} \cap \Omega_{N,k} \cap (X_{k\Delta} \in A)} \middle| \mathcal{F}_{k\Delta} \right] = 0.$$

Moreover, on  $\Omega_{N,k}$ ,  $\bar{T}_{k\Delta}^{(3)} = 0$ . Then, by (16),

$$E \lesssim \Delta^{1-\beta/2} \ln^2(n) + G^2 + H^2$$

where  $G := \mathbb{E} \left[ \left( \bar{Z}_{k\Delta} + \bar{T}_{k\Delta}^{(1)} + \bar{T}_{k\Delta}^{(2)} \right) 1_{\Omega_{X,k} \cap \Omega_{ZT,k}^c \cap \Omega_{N,k} \cap \Omega_{I,k} \cap (X_{k\Delta} \in A)} \middle| \mathcal{F}_{k\Delta} \right]$  and  $H := \mathbb{E} \left[ \left( \bar{Z}_{k\Delta} + \bar{T}_{k\Delta}^{(1)} + \bar{T}_{k\Delta}^{(2)} \right) 1_{\Omega_{X,k} \cap \Omega_{ZT,k} \cap \Omega_{N,k} \cap \Omega_{I,k}^c \cap (X_{k\Delta} \in A)} \middle| \mathcal{F}_{k\Delta} \right]$ . We have that  $H^2 \lesssim \Delta^{-1} \ln^2(n) \mathbb{P}^2(\Omega_{I,k}^c) \lesssim \Delta^{1-\beta/2} \ln^2(n)$ . The end of the proof is the same as in the case of  $\sigma$  and  $\xi$  constants. We obtain that

$$|G| \lesssim \frac{\ln(n)}{\sqrt{\Delta}} \sup_{C \geq \kappa_0 \ln(n) \Delta^{-1/2}} \mathbb{P}(U_{k\Delta} \in [C, C + 2b_{max} + 6]) \lesssim \sqrt{\Delta} \Delta^{1-\beta/2}.$$

#### 6.4.2 Proof of Lemma 4

According to the Ito formula (see for instance Applebaum (2004), Theorem 4.4.7 p251), we have that

$$f(X_t) - f(X_{k\Delta}) = I_1 + I_2 + I_3 + I_4$$

where

$$\begin{aligned} I_1 &= \int_{k\Delta}^t f'(X_s) \sigma(X_s) dW_s & I_2 &= \int_{k\Delta}^t \int_{\mathbb{R}} (f(X_{s-} + z\xi(X_{s-})) - f(X_{s-})) \tilde{\mu}(ds, dz) \\ I_3 &= \int_{k\Delta}^t \int_{z \in \mathbb{R}} [f(X_s + z\xi(X_s)) - f(X_s) - z\xi(X_s) f'(X_s)] \nu(dz) ds \\ I_4 &= \int_{k\Delta}^t [f'(X_s) b(X_s) + f''(X_s) \sigma^2(X_s)/2] ds. \end{aligned}$$

By Proposition 1, for any  $t \leq (k+1)\Delta$ , we have:

$$\begin{aligned} \mathbb{E} \left[ \left( I_1 - f'(X_{k\Delta})\sigma(X_{k\Delta}) \int_{k\Delta}^t dW_s \right)^2 \right] &= \mathbb{E} \left[ \left( \int_{k\Delta}^t (\sigma(X_s)f'(X_s) - \sigma(X_{k\Delta})f'(X_{k\Delta})) dW_s \right)^2 \right] \\ &= \int_{k\Delta}^t (\sigma(X_s)f'(X_s) - \sigma(X_{k\Delta})f'(X_{k\Delta}))^2 ds \lesssim \Delta^2. \end{aligned}$$

We can write:

$$\begin{aligned} E &:= \mathbb{E} \left[ \left( I_2 - f'(X_{k\Delta})\xi(X_{k\Delta-}) \int_{k\Delta}^t dL_s^{(1)} + dL_s^{(2)} \right)^2 1_{\Omega_{N,k}} \right] \\ &\leq 2 \int_{k\Delta}^t \int_{|z| \leq \Delta^{1/4}} \mathbb{E} \left[ (f(X_s + z\xi(X_s)) - f(X_s) - z\xi(X_s)f'(X_s))^2 \right] \nu(dz) ds \\ &\quad + 2 \int_{k\Delta}^t \int_{|z| \leq \Delta^{1/4}} \mathbb{E} \left[ z^2 (\xi(X_s)f'(X_s) - \xi(X_{k\Delta})f'(X_{k\Delta}))^2 \right] \nu(dz) ds. \end{aligned}$$

The function  $f$  is  $\mathcal{C}^2$ , then, by the Taylor formula, for any  $s \in [k\Delta, t]$ ,  $z \in \mathbb{R}$ , there exists  $\zeta_{s,z}$  in  $[X_s, X_s + z\xi(X_s)]$  such that:

$$f(X_s + z\xi(X_s)) - f(X_s) - z\xi(X_s)f'(X_s) = \frac{z^2\xi^2(X_s)}{2} f''(\zeta_{s,z}).$$

Then, as  $\xi$  and  $f''$  are bounded:

$$\mathbb{E} \left[ (f(X_s + z\xi(X_s)) - f(X_s) - z\xi(X_s)f'(X_s))^2 \right] = \frac{z^4}{4} \mathbb{E} \left[ (\xi(X_s)f''(\zeta_{s,z}))^2 \right] \lesssim z^4$$

and, by Result 3, for any  $t \leq (k+1)\Delta$ ,

$$\begin{aligned} F &:= \int_{k\Delta}^t \int_{|z| \leq \Delta^{1/4}} \mathbb{E} \left[ (f(X_s + z\xi(X_s)) - f(X_s) - z\xi(X_s)f'(X_s))^2 \right] \nu(dz) ds \\ &\lesssim \Delta \int_{|z| \leq \Delta^{1/4}} z^4 \nu(dz) \lesssim \Delta^{2-\beta/4}. \end{aligned}$$

The functions  $\xi$  and  $f'$  are Lipschitz, then by Proposition 1,

$$\mathbb{E} \left[ z^2 (\xi(X_s)f'(X_s) - \xi(X_{k\Delta})f'(X_{k\Delta}))^2 \right] \lesssim z^2 \mathbb{E} \left[ (X_s - X_{k\Delta})^2 \right] \lesssim \Delta z^2$$

and consequently, for any  $t \leq (k+1)\Delta$ :

$$\int_{k\Delta}^t \int_{|z| \leq \Delta^{1/4}} \mathbb{E} \left[ z^2 (\xi(X_s)f'(X_s) - \xi(X_{k\Delta})f'(X_{k\Delta}))^2 \right] \nu(dz) ds \lesssim \Delta^{3-\beta/2}$$

then  $E \lesssim \Delta^{2-\beta/4}$ . By the same way, we obtain that

$$\mathbb{E} [I_3^2] \leq \mathbb{E} \left[ \int_{k\Delta}^t \int_{|z| \leq \Delta^{1/4}} \left( \frac{z^2\xi^2(X_s)}{2} f''(\zeta_{s,z}) \right)^2 \nu(dz) ds \right] \leq \Delta^{2-\beta/4}.$$

The functions  $b$  and  $f'$  are Lipschitz and  $f''$  and  $\sigma$  are bounded, then, for any  $t \leq (k+1)\Delta$  :

$$\mathbb{E} [I_4^2] \lesssim \Delta \int_{k\Delta}^t (1 + \mathbb{E} [X_s^4]) ds \lesssim \Delta^2.$$

Then, for any  $t \leq (k+1)\Delta$ :

$$\mathbb{E} [(f(X_t) - f(X_{k\Delta}) - \psi_f(X_{k\Delta}, t))] \leq \Delta^{2-\beta/4}.$$

## 6.5 Proof of Theorem 4

As previously, we only bound the risk on  $\Omega_n$ . As in Subsection 6.3, we introduce the function  $p(m, m')$  such that  $p(m, m') = 12(\text{pen}(m) + \text{pen}(m'))$ . On  $\Omega_n$ , for any  $m \in \mathcal{M}_n$ , we have:

$$\begin{aligned} \left\| \tilde{b}_{\tilde{m}} - b_A \right\|_n^2 &\leq 3 \|b_m - b_A\|_n^2 + \frac{224}{n} \sum_{k=1}^n b_A^2(X_{k\Delta}) 1_{\Omega_{X,k}^c} + I_{k\Delta}^2 + 2(Z_{k\Delta}^2 + T_{k\Delta}^2) 1_{\Omega_{X,k} \cap \Omega_{Z,k}^c} \\ &+ \frac{224}{n} \sum_{k=1}^n (\mathbb{E} [(Z_{k\Delta} + T_{k\Delta}) 1_{\Omega_{X,k} \cap \Omega_{Z,k}} | \mathcal{F}_{k\Delta}])^2 \\ &+ 24 \sup_{t \in \mathcal{B}_{m, \tilde{m}}} (\tilde{\nu}_n^2(t) - p(m, \tilde{m})) + 4\text{pen}(m). \end{aligned}$$

It remains only to bound  $\mathbb{E} \left[ \sup_{t \in \mathcal{B}_{m, \tilde{m}}} (\tilde{\nu}_n^2(t) - p(m, \tilde{m})) \right] \leq \sum_{m'} \mathbb{E} \left[ \sup_{t \in \mathcal{B}_{m, m'}} (\tilde{\nu}_n^2(t) - p(m, \tilde{m})) \right]$ . As in the proof of Theorem 2, we bound the quantity

$$\mathbb{E} \left[ \exp \left( \varepsilon t(X_{k\Delta}) \left( \tilde{Z}_{k\Delta} + \tilde{T}_{k\Delta} \right) \right) \middle| \mathcal{F}_{k\Delta} \right].$$

We have that

$$\mathbb{E} \left[ \exp \left( \varepsilon t(X_{k\Delta}) Z_{k\Delta} \right) 1_{\Omega_{N,k}} \middle| \mathcal{F}_{k\Delta} \right] \leq \exp \left( \frac{\varepsilon^2 \sigma_0^2 t^2(X_{k\Delta})}{2\Delta} \right).$$

The truncated Lévy process  $\tilde{L}_t = \int_0^t \int_{|z| \leq \Delta^{1/4}} z \tilde{\mu}(ds, dz)$  satisfies Assumption A5 and then there exists a constant  $c$  such that:

$$\mathbb{E} \left[ \exp \left( \varepsilon t(X_{k\Delta}) T_{k\Delta} \right) 1_{\Omega_{N,k}} \middle| \mathcal{F}_{k\Delta} \right] \leq \exp \left( \frac{c\varepsilon^2 \xi_0^2 t^2(X_{k\Delta})}{\Delta(1 - \varepsilon/\varepsilon_1)} \right).$$

As  $Z_{k\Delta} 1_{\Omega_{N,k}}$  and  $T_{k\Delta} 1_{\Omega_{N,k}}$  are centred, we obtain:

$$\mathbb{E} \left[ \exp \left( \varepsilon |t(X_{k\Delta}) (Z_{k\Delta} + T_{k\Delta})| \right) 1_{\Omega_{N,k}} \middle| \mathcal{F}_{k\Delta} \right] \leq 2 \exp \left( \frac{c\varepsilon^2 (\sigma_0^2 + \xi_0^2) t^2(X_{k\Delta})}{\Delta(1 - \varepsilon/\varepsilon_1)} \right)$$

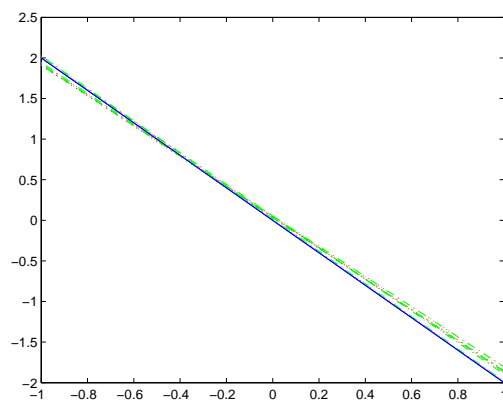
and then

$$\mathbb{E} \left[ \exp \left( \varepsilon \left| t(X_{k\Delta}) \left( \tilde{Z}_{k\Delta} + \tilde{T}_{k\Delta} \right) \right| \right) 1_{\Omega_{N,k} \cap \Omega_{X,k}} \middle| \mathcal{F}_{k\Delta} \right] \leq 2 \exp \left( \frac{c\varepsilon^2 (\sigma_0^2 + \xi_0^2) t^2(X_{k\Delta})}{\Delta(1 - \varepsilon/\varepsilon_1)} \right).$$

We conclude as in the proof of Theorem 2.

Figure 1: Model 1: Ornstein-Uhlenbeck and binomial law

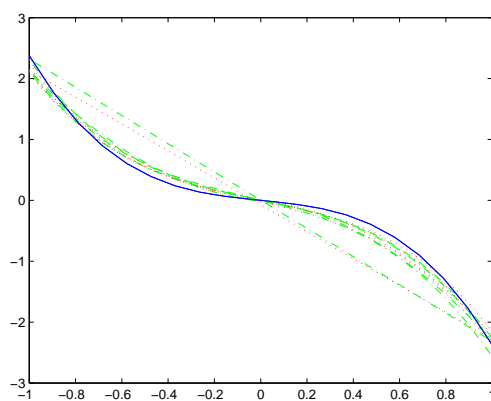
$$b(x) = -2x, \sigma(x) = \xi(x) = 1 \text{ and binomial law}$$



- : true function    -.-: first estimator    ...: truncated estimator  
 $n = 10^4$  et  $\Delta = 10^{-1}$

Figure 2: Model 2: Double well and Laplace law

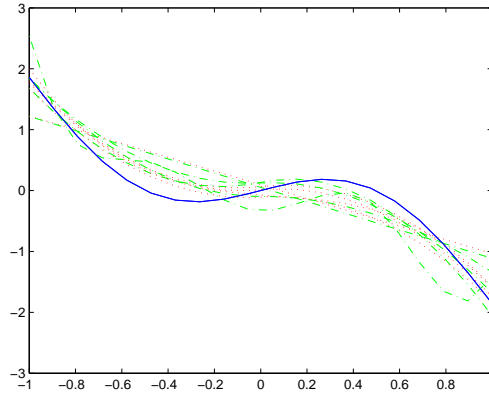
$$b(x) = -(x - 1/4)^3 - (x + 1/4)^3, \quad \sigma = \xi = 1 \text{ and Laplace law}$$



- : true function    -.-: first estimator    ...: truncated estimator  
 $n = 10^4$  et  $\Delta = 10^{-1}$

Figure 3: Model 3: Sine function

$$b(x) = -2x + \sin(3x), \sigma(x) = \xi(x) = \sqrt{(3+x^2)/(1+x^2)} \text{ jumps not sub-exponential}$$



— : true function    -.-: first estimator    ...: truncated estimator  
 $n = 10^4$  et  $\Delta = 10^{-1}$

Table 1: Model 1: Ornstein-Uhlenbeck and binomial law

$$b(x) = -2x, \sigma(x) = \xi(x) = 1 \text{ and compound Poisson process (binomial law)}$$

$n$	$\Delta$	first estimator				truncated estimator			
		$\hat{m}_a$	$\hat{r}_a$	$risk_1$	$or_1$	$\tilde{m}_a$	$\tilde{r}_a$	$risk_2$	$or_2$
$10^3$	$10^{-1}$	0	1.02	0.044	1.3	0	1.02	0.044	1.3
$10^4$	$10^{-1}$	0	1.02	0.011	1.3	0	1.02	0.011	1.3
$10^3$	$10^{-2}$	0	1.02	0.55	1.04	0	1.02	0.55	1.04
$10^4$	$10^{-2}$	0	1	0.047	1	0	1	0.047	1
$5 \cdot 10^4$	$10^{-2}$	0.04	1	0.010	1.4	0	1	0.0053	1

$\hat{m}_a, \hat{r}_a$  and  $\tilde{m}_a, \tilde{r}_a$  : average values of  $\hat{m}, \hat{r}$  and  $\tilde{m}, \tilde{r}$  on the 50 simulations.  
 $risk_1$  and  $risk_2$  : means of the empirical errors of the adaptive estimators.  
 $or_1$  and  $or_2$  : means of *oracle* = empirical error of the adaptive estimator / empirical error of the best possible estimator.

Table 2: Model 2: Double well and Laplace law  
 $b(x) = -(x - 1/4)^3 - (x + 1/4)^3$ ,  $\sigma(x) = \xi(x) = 1$  and Laplace law.

$n$	$\Delta$	first estimator				truncated estimator			
		$\hat{m}_a$	$\hat{r}_a$	$risk_1$	$or_1$	$\tilde{m}_a$	$\tilde{r}_a$	$risk_2$	$or_2$
$10^3$	$10^{-1}$	0.02	1.0	0.12	3.1	0.02	1.0	0.12	3.1
$10^4$	$10^{-1}$	1.7	2.1	$2.10^{96}$	51	0.4	2.1	0.04	1.5
$10^3$	$10^{-2}$	0.26	1.2	1.8	3.1	0.06	1	0.51	1.4
$10^4$	$10^{-2}$	0.12	1.5	0.16	1.8	0.08	1.2	0.13	2.4
$5.10^4$	$10^{-2}$	0.30	2.5	0.035	1.6	0.26	2.5	0.019	1.8

$\hat{m}_a, \hat{r}_a$  and  $\tilde{m}_a, \tilde{r}_a$  : average values of  $\hat{m}, \hat{r}$  and  $\tilde{m}, \tilde{r}$  on the 50 simulations.  
 $risk_1$  and  $risk_2$  : means of the empirical errors of the adaptive estimators.  
 $or_1$  and  $or_2$ : means of *oracle* =empirical error of the adaptive estimator / empirical error of the best possible estimator.

Table 3: Model 3: Sine function and jumps not sub-exponential

$$b(x) = -2x + \sin(3x), \quad \sigma(x) = \xi(x) = \sqrt{(3+x^2)/(1+x^2)} \quad \text{and} \quad \nu(z) \propto e^{-\sqrt{az}}/\sqrt{z}$$

$n$	$\Delta$	first estimator				truncated estimator			
		$\hat{m}_a$	$\hat{r}_a$	$risk_1$	$or_1$	$\tilde{m}_a$	$\tilde{r}_a$	$risk_2$	$or_2$
$10^3$	$10^{-1}$	0.34	1.2	0.76	3.6	0.04	1.2	0.28	1.9
$10^4$	$10^{-1}$	0.8	2.2	0.082	1.3	0.68	2.2	0.073	1.2
$10^3$	$10^{-2}$	0.96	1.2	18	6.3	0.02	1.2	1.3	1.2
$10^4$	$10^{-2}$	0.78	1.4	1.5	4.3	0.12	1.4	0.24	3.3
$5.10^4$	$10^{-2}$	0.92	2.3	0.24	4.3	0.70	2.3	0.039	1.3

$\hat{m}_a, \hat{r}_a$  and  $\tilde{m}_a, \tilde{r}_a$  : average values of  $\hat{m}, \hat{r}$  and  $\tilde{m}, \tilde{r}$  on the 50 simulations.  
 $risk_1$  and  $risk_2$  : means of the empirical errors of the adaptive estimators.  
 $or_1$  and  $or_2$ : means of *oracle* =empirical error of the adaptive estimator / empirical error of the best possible estimator.

## 7 Auxiliary proofs

### 7.1 Decomposition on a lattice

**Proposition 3.**

If there exist some constants  $c_1, c_2$  and  $K$  independent of  $D, n, \Delta, b$  and  $\sigma$  and two constants  $\alpha$  and  $\beta$  independent of  $n$  and  $D$  such that, for any function  $t \in S_m + S'_m$ :

$$\forall \eta, \zeta > 0, \forall t \in S_m + S_{m'} \quad \|t\|_\infty \leq C\zeta, \mathbb{P} \left( f_n(t) \geq \eta, \|t\|_n^2 \leq \zeta^2 \right) \leq K \exp \left( - \frac{\eta^2 n \beta}{(c_1 \alpha^2 \zeta^2 + 2C c_2 \alpha \eta \zeta)} \right),$$

then there exist some constants  $C$  and  $\kappa$  depending only of  $\nu$  such that, if  $D \leq n\beta$ :

$$\mathbb{E} \left[ \sup_{t \in \mathcal{B}_{m,m'}} f_n^2(t) - \frac{\kappa \alpha^2 D}{n\beta} \right]_+ \leq CK \frac{\kappa \alpha^2 D^{3/2} e^{-\sqrt{D}}}{n\beta}.$$

Let us consider an orthonormal (for the  $L^2_\sigma$ -norm) basis  $(\psi_\lambda)_{\lambda \in \Lambda_{m,m'}}$  of  $S_{m,m'} = S_m + S_{m'}$  such that

$$\forall \lambda, \quad \text{card}(\{\lambda', \|\psi_\lambda \psi_{\lambda'}\| \neq 0\}) \leq \phi_2.$$

Let us set

$$\bar{r}_{m,m'} = \frac{1}{\sqrt{D}} \sup_{\beta \neq 0} \frac{\|\sum_\lambda \beta_\lambda \psi_\lambda\|_\infty}{|\beta|_\infty}.$$

We obtain that

$$\left\| \sum_\lambda \beta_\lambda \psi_\lambda \right\|_\infty \leq \phi_2 |\beta|_\infty \sup_\lambda \|\psi_\lambda\|_\infty \quad \text{et} \quad \|\psi_\lambda\|_\infty \leq \sqrt{D} \|\psi_\lambda\|_{L^2} \leq \pi_1 \sqrt{D} \|\psi_\lambda\|_\sigma$$

then

$$\bar{r}_{m,m'} \leq \bar{r} := \phi_2 \pi_1.$$

We need a lattice of which the infinite norm is bounded. We use Lemma 9 of Barron *et al.* (1999):

*Result 4.* There exists a  $\delta_k$ -lattice  $T_k$  of  $L^2_\sigma \cap (S_m + S_{m'})$  such that

$$|T_k \cap \mathcal{B}_{m,m'}| \leq (5/\delta^k)^D$$

where  $\delta_k = 2^{-k}/5$ . Let us denote by  $p_k(u)$  the orthogonal projection of  $u$  on  $T_k$ . For any  $u \in S_{m,m'}$ ,  $\|u - p_k(u)\|_\pi \leq \delta_k$  and

$$\sup_{u \in p_k^{-1}(t)} \|u - t\|_\infty \leq \bar{r}_{m,m'} \delta_k \leq \bar{r} \delta_k.$$



Let us set  $H_k = \ln(|T_k \cap \mathcal{B}_{m,m'}|)$ . We have that:

$$H_k \leq D \ln(5/\delta_k) = D(k \ln(2) + \ln(5/\delta_0)) \leq C(k+1)D.$$

The decomposition of  $u_k$  on the  $\delta_k$ -lattice must be done very carefully: the norms  $\|u_k - u_{k-1}\|_{\varpi}$  and  $\|u_k - u_{k-1}\|_{\infty}$  must be controlled. Let us set

$$\mathcal{E}_k = \{u_k \in T_k \cap \mathcal{B}_{m,m'}, \quad \|u - u_k\|_{\varpi} \leq \delta_k \quad \text{et} \quad \|u - u_k\|_{\infty} \leq \bar{r}\delta_k\}.$$

We have that  $\ln(|\mathcal{E}_k|) \leq H_k$ . For any function  $u \in \mathcal{B}_{m,m'}$ , there exist a series  $(u_k)_{k \geq 0} \in \prod_k \mathcal{E}_k$  such that

$$u = u_0 + \sum_{k=1}^{\infty} (u_k - u_{k-1}).$$

Let us consider  $(\eta_k)_{k \geq 0}$  and  $\eta \in \mathbb{R}$  such that  $\eta_0 + \sum_{k=1}^{\infty} \eta_k \leq \eta$ . We obtain:

$$\begin{aligned} \mathbb{P}\left(\sup_{u \in \mathcal{B}_{m,m'}} |f_n(u)| > \eta\right) &\leq \mathbb{P}\left(\exists (u_k) \in \prod \mathcal{E}_k, \left|f_n(u_0) + \sum_{k=1}^{\infty} f_n(u_k - u_{k-1})\right| > \eta_0 + \sum_{k=1}^{\infty} \eta_k\right) \\ &\leq P_1 + \sum_{k=1}^{\infty} P_{2,k} \end{aligned} \quad (17)$$

where

$$P_1 = \sum_{u_0 \in \mathcal{E}_0} \mathbb{P}(|f_n(u_0)| > \eta_0) \quad \text{and} \quad P_{2,k} = \sum_{u_k \in \mathcal{E}_k} \mathbb{P}(|f_n(u_k - u_{k-1})| > \eta_k).$$

As  $u_0 \in T_0$ ,  $\|u_0\|_{\varpi} \leq 1$  and  $\|u_0\|_{\infty} \leq \bar{r}\sqrt{D}$ . Moreover,  $\|u_0\|_n^2 \leq 3/2 \|u_0\|_{\varpi}^2 \leq 3\delta_0/2$ . Then

$$\mathbb{P}(|f_n(u_0)| > \eta_0) = \mathbb{P}\left(|f_n(u_0)| > \eta_0, \|u_0\|_n^2 \leq 3\delta_0/2\right).$$

There exist two constants  $c'_1$  and  $c'_2$  depending only on  $\delta_0$  and  $\bar{r}$  such that

$$\mathbb{P}(|f_n(u_0)| > \eta_0) \leq K \exp\left(-\frac{n\beta\eta_0^2}{c'_1\alpha^2 + 2c'_2\sqrt{D}\alpha\eta_0}\right).$$

Let us set  $x_0$  such that  $\eta_0 = \alpha\left(\sqrt{c'_1(x_0/\beta)} + c'_2\sqrt{D}(x_0/\beta)\right)$ . Then:

$$x_0 \leq \frac{\beta\eta_0^2}{c'_1\alpha^2 + 2c'_2\sqrt{D}\alpha\eta_0}$$

and

$$\mathbb{P}(f_n(u_0) > \eta_0) \leq K \exp(-nx_0).$$

Then

$$P_1 \leq K \sum_{u_0 \in \mathcal{E}_0} \exp(-nx_0) \leq K \exp(H_0 - nx_0). \quad (18)$$

We have that

$$\|u_k - u_{k-1}\|_\pi^2 \leq 2 \left( \|u - u_{k-1}\|_\pi^2 + \|u - u_k\|_\pi^2 \right) \leq 5\delta_{k-1}^2/2$$

then  $\|u_k - u_{k-1}\|_n^2 \leq 15\delta_{k-1}^2/4$ . As  $u_{k-1}, u_k \in \mathcal{E}_{k-1} \times \mathcal{E}_k$ , it follows that  $\|u_k - u_{k-1}\|_\infty^2 \leq 5\delta_{k-1}^2 \bar{r}^2/2$ . There exists two constants  $c_3$  and  $c_4$  such that:

$$\begin{aligned} \mathbb{P}_n(|f_n(u_k - u_{k-1})| > \eta_k) &= \mathbb{P}_n\left(|f_n(u_k - u_{k-1})| > \eta_k, \|u_k - u_{k-1}\|_n^2 \leq 15\delta_{k-1}^2/4\right) \\ &\leq K \exp\left(-\frac{n\beta\eta_k^2}{c_3\alpha^2\delta_{k-1}^2 + 2c_4\alpha\delta_{k-1}}\right). \end{aligned}$$

Let us fix  $x_k$  such that  $\eta_k = \delta_{k-1}a \left( \sqrt{c_3(x_k/\beta)} + c_4(x_k/\beta) \right)$ . We obtain:

$$x_k \leq \frac{\beta\eta_k^2}{c_3\alpha^2\delta_{k-1}^2 + 2c_4\alpha\delta_{k-1}}$$

and

$$\mathbb{P}(|f_n(u_k - u_{k-1})| > \eta_k) \leq K \exp(-nx_k).$$

Then,  $P_{2,k} \leq K \exp(H_{k-1} + H_k - nx_k)$  and

$$P_2 = \sum_{k=1}^{\infty} P_{2,k} \leq K \sum_{k=1}^{\infty} \exp(H_{k-1} + H_k - nx_k). \quad (19)$$

Let us set  $\tau > 0$  and choose  $(x_k)$  (and then  $(\eta_k)$ ) such that

$$\begin{cases} \sqrt{D}nx_0 = H_0 + D + \tau \\ nx_k = H_{k-1} + H_k + (k+1)D + \tau. \end{cases}$$

Collecting the results, we obtain, by (17), (18) and (19):

$$\mathbb{P}\left(\sup_{u \in \mathcal{B}_{m,m'}} |f_n(u)| > \eta\right) \leq C \left( e^{-D}e^{-\tau} + e^{-\sqrt{D}}e^{-\tau/\sqrt{D}} \right). \quad (20)$$

It remains to compute  $\eta^2$ . We denote by  $C$  a constant depending only on  $\delta_0$  and  $\bar{r}$ . This constant may vary from one line to another. We have that:

$$\eta = \sum_{k=0}^{\infty} \eta_k \leq C\alpha \left( \sum_{k=1}^{\infty} \delta_{k-1} \left( \sqrt{\frac{x_k}{\beta}} + \frac{x_k}{\beta} \right) \right) + \alpha \left( \sqrt{\frac{x_0}{\beta}} + \sqrt{D}\frac{x_0}{\beta} \right).$$

Let us recall that  $H_k = C(k+1)D$ . Then,  $nx_k = C(3k+2)D + \tau$ ,  $\sqrt{D}nx_0 = CD + \tau$  and

$$\sum_{k=0}^{\infty} \frac{\delta_{k-1}x_k}{\beta} \leq \frac{1}{n\beta} \sum_{k=0}^{\infty} 2^{-(k-1)} (C(3k+2)D + \tau) \leq C \frac{D + \tau}{n\beta}.$$

Moreover,

$$\sum_{k=0}^{\infty} \delta_{k-1} \sqrt{\frac{x_k}{\beta}} \leq C \frac{\sqrt{D} + \sqrt{\tau}}{\sqrt{n\beta}}.$$

As  $D/n\beta \leq 1$ , there exists a constant  $\kappa$  such that

$$\eta^2 \leq \kappa \alpha^2 \left( \frac{D}{n\beta} + 2 \frac{\tau}{n\beta} + \frac{\tau^2}{n^2 \beta^2} \right).$$

Then, according to (20):

$$\mathbb{P} \left( \sup_{u \in \mathcal{B}_{m,m'}} f_n^2(u) > \kappa \alpha^2 \left( \frac{D}{n\beta} + 2 \frac{\tau}{n\beta} + \frac{\tau^2}{n^2 \beta^2} \right) \right) \leq C \left( e^{-D-\tau} + e^{-\sqrt{D}-\tau/\sqrt{D}} \right). \quad (21)$$

Furthermore

$$\begin{aligned} E &:= \mathbb{E} \left( \left[ \sup_{u \in \mathcal{B}_{m,m'}} f_n^2(u) - \kappa \alpha^2 \frac{D}{n\beta} \right]_+ \right) \\ &= \int_0^{\infty} \mathbb{P} \left( \sup_{u \in \mathcal{B}_{m,m'}} f_n^2(u) > \kappa \alpha^2 \frac{D}{n\beta} + \tau \right) d\tau \end{aligned}$$

Setting  $\tau = \kappa \alpha^2 (2y/n\beta + y^2/n^2 \beta^2)$ , it follows:

$$E = C \gamma^2 \int_0^{\infty} \mathbb{P} \left( \sup_{u \in \mathcal{B}_{m,m'}} f_n^2(u) > \kappa \alpha^2 \left( \frac{D}{n\beta} + 2 \frac{y}{n\beta} + \frac{y^2}{n^2 \beta^2} \right) \right) \left( \frac{2}{n\beta} + \frac{2y}{n^2 \beta^2} \right) dy.$$

By (21),

$$\begin{aligned} E &= C \kappa \alpha^2 \left( e^{-D} + e^{-\sqrt{D}} \right) \left( \frac{1}{n\beta} \int_0^{\infty} y e^{-y/\sqrt{D}} dy \right) \\ &\leq C \frac{\kappa \alpha^2}{n\beta} D^{3/2} e^{-\sqrt{D}}. \end{aligned}$$

**Acknowledgment:** the author wishes to thank M. Reiss and V. Genon-Catalot for helpful discussions.

## References

- Applebaum, D. (2004) *Lévy processes and stochastic calculus*, *Cambridge Studies in Advanced Mathematics*, volume 93. Cambridge University Press, Cambridge.
- Barron, A., Birgé, L. and Massart, P. (1999) Risk bounds for model selection via penalization. *Probab. Theory Related Fields*, 113 (3) pp. 301–413.

- Birgé, L. and Massart, P. (1998) Minimum contrast estimators on sieves: exponential bounds and rates of convergence. *Bernoulli*, 4 (3) pp. 329–375.
- Comte, F. (2001) Adaptive estimation of the spectrum of a stationary gaussian sequence. *Bernoulli*, 7 (2) pp. 267–298.
- Comte, F., Genon-Catalot, V. and Rozenholc, Y. (2007) Penalized nonparametric mean square estimation of the coefficients of diffusion processes. *Bernoulli*, 13 (2) pp. 514–543.
- Dellacherie, C. and Meyer, P.A. (1980) *Probabilités et potentiel. Chapitres V à VIII, Actualités Scientifiques et Industrielles [Current Scientific and Industrial Topics]*, volume 1385. Hermann, Paris, revised edition. Théorie des martingales. [Martingale theory].
- DeVore, R.A. and Lorentz, G.G. (1993) *Constructive approximation, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*, volume 303. Springer-Verlag, Berlin.
- Hoffmann, M. (1999) Adaptive estimation in diffusion processes. *Stochastic Process. Appl.*, 79 (1) pp. 135–163.
- Ishikawa, Y. and Kunita, H. (2006) Malliavin calculus on the Wiener-Poisson space and its application to canonical SDE with jumps. *Stochastic Process. Appl.*, 116 (12) pp. 1743–1769.
- Liptser, R.S. and Shiryaev, A.N. (2001) *Statistics of random processes. I, Applications of Mathematics (New York)*, volume 5. Springer-Verlag, Berlin, expanded edition. General theory, Translated from the 1974 Russian original by A. B. Aries, Stochastic Modelling and Applied Probability.
- Mancini, C. and Renò, R. (2011) Threshold estimation of Markov models with jumps and interest rate modeling. *J. Econometrics*, 160 (1) pp. 77–92.
- Masuda, H. (2007) Ergodicity and exponential  $\beta$ -mixing bounds for multidimensional diffusions with jumps. *Stochastic Process. Appl.*, 117 (1) pp. 35–56.
- Meyer, Y. (1990) *Ondelettes et opérateurs. I. Actualités Mathématiques. [Current Mathematical Topics]*. Hermann, Paris. Ondelettes. [Wavelets].
- Rubenthaler, S. (2010) *Probabilités : aspects théoriques et applications en filtrage non linéaire, systèmes de particules et processus stochastiques..* Habilitation à diriger des recherches, Université de Nice-Sophia Antipolis, France.
- Schmitter, E. (2009a) Penalized nonparametric drift estimation for a multidimensional diffusion process. Preprint 2009-02, MAP5, Université Paris Descartes. accepté pour publication à Statistics.
- Schmitter, E. (2010) *Estimation non paramétrique pour des processus de diffusion.* Ph.D. thesis, Université Paris Descartes.

- Shimizu, Y. (2008) Some remarks on estimation of diffusion coefficients for jump-diffusions from finite samples. *Bull. Inform. Cybernet.*, 40 pp. 51–60.
- Shimizu, Y. and Yoshida, N. (2006) Estimation of parameters for diffusion processes with jumps from discrete observations. *Stat. Inference Stoch. Process.*, 9 (3) pp. 227–277.