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# Non-parametric adaptive estimation of the drift for a jump diffusion process

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## Abstract

In this article, we consider a jump diffusion process  $(X_t)_{t \geq 0}$  observed at discrete times  $t = 0, \Delta, \dots, n\Delta$ . The sampling interval  $\Delta$  tends to 0 and  $n\Delta$  tends to infinity. We assume that  $(X_t)_{t \geq 0}$  is ergodic, strictly stationary and exponentially  $\beta$ -mixing. We use a penalized least-square approach to compute two adaptive estimators of the drift function  $b$ . We provide bounds for the risks of the two estimators.

## 1 Introduction

We consider a general diffusion with jumps:

$$dX_t = b(X_t)dt + \sigma(X_t)dW_t + \xi(X_{t-})dL_t \quad \text{and} \quad X_0 = \eta \quad (1)$$

where  $L_t$  is a centred pure jump Levy process:

$$dL_t = \int_{z \in \mathbb{R}} z (\mu(dt, dz) - dt\nu(dz))$$

with  $\mu$  a random Poisson measure with intensity measure  $\nu(dz)dt$  such that  $\int_{z \in \mathbb{R}} z^2 \nu(dz) < \infty$ . The compensated Poisson measure  $\tilde{\mu}$  is defined by  $\tilde{\mu}(dt, dz) = \mu(dt, dz) - \nu(dz)dt$ . The random variable  $\eta$  is independant of  $(W_t, L_t)_{t \geq 0}$ . Moreover,  $(W_t)_{t \geq 0}$  and  $(L_t)_{t \geq 0}$  are independant.

This process is observed with high frequency (at times  $t = 0, \Delta, \dots, n\Delta$  where  $\Delta \rightarrow 0$  and  $n\Delta \rightarrow \infty$ ). It is assumed to be ergodic, stationary and exponentially  $\beta$ -mixing (see Masuda (2007) for sufficient conditions). Our aim is to construct a non-parametric estimator of  $b$  on a compact set  $A$ .

The non-parametric estimation of  $b$  and  $\sigma$  for a diffusion process observed with high-frequency is well-known (see for instance Hoffmann (1999) and Comte *et al.* (2007)). Diffusion processes with jumps are used in various fields, for instance in finance, for modelling the growth of a population, in hydrology, in

medical science, . . . , but there exist few results for the non-parametric estimation of  $b$  and  $\sigma$ . Shimizu and Yoshida (2006) construct maximum-likelihood estimators of parameters of  $b$  and  $\sigma$ . Their estimators reach the standard rates of convergence:  $\sqrt{n\Delta}$  for the estimator of  $b$ , and  $\sqrt{n}$  for the estimator of  $\sigma$ . Shimizu (2008) and Mancini and Renò (2011) use a kernel estimator to obtain non parametric threshold estimators of  $\sigma$ . Mancini and Renò (2011) also construct a non-parametric truncated estimator of  $b$ , but only when  $L_t$  is a compound Poisson process. To our knowledge, minimax rates of convergences for non-parametric estimators of  $b$ ,  $\sigma$  or  $\xi$  are not available in the literature.

In this paper, we use model selection to construct two non-parametric estimators of  $b$  under the asymptotic framework  $\Delta \rightarrow 0$  and  $n\Delta \rightarrow \infty$ . This method was introduced by Birgé and Massart (1998).

First, we introduce a sequence of linear subspaces  $S_m \subseteq L^2(A)$  and, for each  $m$ , we construct an estimator  $\hat{b}_m$  of  $b$  by minimising on  $S_m$  the contrast function:

$$\gamma_n(t) = \frac{1}{n} \sum_{k=1}^n (Y_{k\Delta} - t(X_{k\Delta}))^2 \quad \text{where} \quad Y_{k\Delta} = \frac{X_{(k+1)\Delta} - X_{k\Delta}}{\Delta}.$$

We obtain a collection of estimators of the drift function  $b$  and we bound their risks (Theorem 1). Then, we introduce a penalty function to select the “best” dimension  $m$  and we deduce an adaptive estimator  $\hat{b}_{\hat{m}}$ . Under the assumption that  $\nu$  is sub-exponential, that is if there exist two positive constants  $C, \lambda$  such that, for  $z$  large enough,  $\nu([-z, z]^c) \leq Ce^{-\lambda z}$ , the risk bound of  $\hat{b}_{\hat{m}}$  is exactly the same as for a diffusion without jumps (Theorem 2) (see Comte *et al.* (2007) or Hoffmann (1999)).

In a second part, we do not assume that  $\nu$  is subexponential and we construct a truncated estimator  $\tilde{b}_m$  of  $b$ . We minimise the contrast function

$$\tilde{\gamma}_n(t) = \frac{1}{n} \sum_{k=1}^n (Y_{k\Delta} \mathbb{1}_{|Y_{k\Delta}| \leq C_\Delta} - t(X_{k\Delta}))^2 \quad \text{where} \quad C_\Delta \propto \sqrt{\Delta} \ln(n)$$

in order to obtain a new estimator  $\tilde{b}_m$ . As in the first part, we introduce a penalty function to obtain an adaptive estimator  $\tilde{b}_{\hat{m}}$ . The risk bound of this adaptive estimator depends on the Blumenthal-Gettoor index of  $\nu$  (Theorems 3 and 4).

In Section 2, we present the model and its assumptions. In Sections 3 and 4, we construct the estimators and bound their risks. Some simulations are presented in Section 5. Proofs are gathered in Sections 6 and 7.

## 2 Assumptions

### 2.1 Assumptions on the model

We consider the following assumptions:

A 1. *The functions  $b, \sigma$  and  $\xi$  are Lipschitz.*

A 2. 1. The function  $\sigma$  is bounded from below and above:

$$\exists \sigma_0, \sigma_1, \forall x \in \mathbb{R}, \quad 0 < \sigma_1 \leq \sigma(x) \leq \sigma_0.$$

2. The function  $\xi$  is bounded:  $\exists \xi_0, \forall x \in \mathbb{R}, \quad 0 \leq \xi(x) \leq \xi_0$ .

3. The drift function  $b$  is elastic: there exists a constant  $M$  such that, for any  $x \in \mathbb{R}, |x| > M$ :  $xb(x) \lesssim -|x|^2$ .

4. The Lévy measure  $\nu$  satisfies:

$$\nu(\{0\}) = 0, \quad \int_{-\infty}^{\infty} z^2 \nu(dz) = 1 \quad \text{and} \quad \int_{-\infty}^{\infty} z^4 \nu(dz) < \infty.$$

Under Assumption A1, the stochastic differential equation (1) admits a unique strong solution. According to Masuda (2007), under Assumptions A1 and A2, the process  $(X_t)$  admits a unique invariant probability  $\varpi$  and satisfies the ergodic theorem: for any measurable function  $g$  such that  $\int |g(x)| \varpi(dx) < \infty$ , when  $T \rightarrow \infty$ ,

$$\frac{1}{T} \int_0^T g(X_s) ds \rightarrow \int g(x) \varpi(dx).$$

This distribution has moments of order 4. Moreover, Masuda (2007) also ensures that under these assumptions, the process  $(X_t)$  is exponentially  $\beta$ -mixing. Furthermore, if there exist two constants  $c$  and  $n_0$  such that, for any  $x \in \mathbb{R}$ ,  $\xi^2(x) \geq c(1 + |x|)^{-n_0}$ , then Ishikawa and Kunita (2006) ensure that a smooth transition density exists.

A 3. 1. The stationary measure  $\varpi$  admits a density  $\pi$  which is bounded from below and above on the compact interval  $A$ :

$$\exists \pi_0, \pi_1, \forall x \in A, \quad 0 < \pi_1 \leq \pi(x) \leq \pi_0.$$

2. The process  $(X_t)_{t \geq 0}$  is stationary ( $\eta \sim \varpi(dx) = \pi(x)dx$ ).

The following proposition very useful for the proofs is proved later.

**Proposition 1.**

Under Assumptions A1-A3, for any  $p \geq 1$ , there exists a constant  $c(p)$  such that, if  $\int_{\mathbb{R}} z^{2p} \nu(dz) < \infty$ :

$$\mathbb{E} \left( \sup_{s \in [t, t+h]} (X_s - X_t)^{2p} \right) \leq c(p)h.$$

## 2.2 Assumptions on the approximation spaces

In order to construct an adaptive estimator of  $b$ , we use model selection: we compute a collection of estimators  $\hat{b}_m$  of  $b$  by minimising a contrast function  $\gamma_n(t)$  on a vectorial subspace  $S_m \subset L^2(A)$ , then we choose the best possible estimator using a penalty function  $pen(m)$ . The collection of vectorial subspaces  $(S_m)_{m \in \mathcal{M}_n}$  has to satisfy the following assumption:

A 4.

1. The subspaces  $S_m$  have finite dimension  $D_m$ .
2. The sequence of vectorial subspaces  $(S_m)_{m \geq 0}$  is increasing: for any  $m$ ,  $S_m \subseteq S_{m+1}$ .
3. Norm connexion: there exists a constant  $\phi_1$  such that, for any  $m \geq 0$ , any  $t \in S_m$ ,

$$\|t\|_\infty^2 \leq \phi_1 D_m \|t\|_{L^2}^2$$

where  $\|\cdot\|_{L^2}$  is the  $L^2$ -norm and  $\|\cdot\|_\infty$  is the sup-norm on  $A$ .

4. For any  $m \in \mathbb{N}$ , there exists an orthonormal basis  $(\psi_\lambda)_{\lambda \in \Lambda_m}$  of  $S_m$  such that

$$\forall \lambda, \quad \text{card}(\lambda', \|\psi_\lambda \psi_{\lambda'}\|_\infty \neq 0) \leq \phi_0$$

where  $\phi_0$  does not depend on  $m$ .

5. For any function  $t$  belonging to the unit ball of the Besov space  $\mathcal{B}_{2,\infty}^\alpha$ ,

$$\exists C, \forall m \quad \|t - t_m\|_{L^2}^2 \leq C 2^{-2m\alpha}$$

where  $t_m$  is the  $L^2$  orthogonal projection of  $t$  on  $S_m$ .

The subspaces generated by piecewise polynomials, compactly supported wavelets or spline functions satisfy A4 (see DeVore and Lorentz (1993) and Meyer (1990) for instance).

## 3 Estimation of the drift

By analogy with Comte *et al.* (2007), we decompose  $Y_{k\Delta}$  in the following way:

$$Y_{k\Delta} = \frac{X^{(k+1)\Delta} - X_{k\Delta}}{\Delta} = b(X_{k\Delta}) + I_{k\Delta} + Z_{k\Delta} + T_{k\Delta} \quad (2)$$

where

$$I_{k\Delta} = \frac{1}{\Delta} \int_{k\Delta}^{(k+1)\Delta} (b(X_s) - b(X_{k\Delta})) ds, \quad Z_{k\Delta} = \frac{1}{\Delta} \int_{k\Delta}^{(k+1)\Delta} \sigma(X_s) dW_s$$

$$T_{k\Delta} = \frac{1}{\Delta} \int_{k\Delta}^{(k+1)\Delta} \xi(X_{s-}) dL_s.$$

The terms  $Z_{k\Delta}$  and  $T_{k\Delta}$  are martingale increments. Let us introduce the mean square contrast function

$$\gamma_n(t) = \frac{1}{n} \sum_{k=1}^n (Y_{k\Delta} - t(X_{k\Delta}))^2 \quad (3)$$

and the empirical risk

$$\mathcal{R}_n(t) = \|t - b_A\|_n^2 \quad \text{where} \quad \|t\|_n^2 = \frac{1}{n} \sum_{k=1}^n t^2(X_{k\Delta}) \quad \text{and} \quad t_A = t\mathbb{1}_A. \quad (4)$$

We consider the asymptotic framework:

$$\Delta \rightarrow 0, \quad n\Delta \rightarrow \infty.$$

For any  $m \in \mathcal{M}_n = \{m, D_m \leq \mathcal{D}_n\}$  where  $\mathcal{D}_n^2 \leq n\Delta / \ln^2(n)$ , we construct the regression-type estimator:

$$\hat{b}_m = \arg \min_{t \in S_m} \gamma_n(t).$$

**Theorem 1.**

*Under Assumptions A1-A4, the risk of the estimator with fixed  $m$  satisfies:*

$$\mathcal{R}_n(\hat{b}_m) \leq 3\pi_1 \|b_m - b_A\|_{L^2}^2 + 48(\sigma_0^2 + \xi_0^2) \frac{D_m}{n\Delta} + c\Delta$$

where  $b_m$  is the orthogonal ( $L^2$ ) projection of  $b_A$  over the vectorial subspace  $S_m$ . The constant  $c$  is independent of  $m$ ,  $n$  and  $\Delta$ .

Except for the constant  $(\sigma_0^2 + \xi_0^2)$  in the variance term, this is exactly the bound of the risk that Comte *et al.* (2007) found for a diffusion process without jumps.

The bias term,  $\|b_m - b_A\|_{L^2}^2$ , decreases when the dimension  $D_m$  increases whereas the variance term  $(\sigma_0^2 + \xi_0^2)D_m/(n\Delta)$  is proportional to the dimension. Under the classical assumption  $n\Delta^2 = O(1)$ , the remainder term  $\Delta$  is negligible. Thus we need to find a good compromise between the bias and the variance term.

*Remark 1. If the regularity of the drift function is known, that is, if  $b$  belongs to a ball of a Besov space  $\mathcal{B}_{2,\infty}^\alpha$ , then the bias term  $\|b_m - b_A\|_{L^2}^2$  is smaller than  $D_m^{-2\alpha}$ . The best estimator is obtained for  $D_{m_{opt}} = (n\Delta)^{1/(1+2\alpha)}$  and the estimator risk satisfies:*

$$\mathcal{R}_n(\hat{b}_{m_{opt}}) \lesssim (n\Delta)^{-2\alpha/(2\alpha+1)} + \Delta.$$

Let us introduce a penalty function  $pen$  such that :

$$pen(m) \geq \kappa(\sigma_0^2 + \xi_0^2) \frac{D_m}{n\Delta}$$

and set:

$$\hat{m} = \arg \min_{m \in \mathcal{M}_n} \left\{ \gamma_n(\hat{b}_m) + \text{pen}(m) \right\}.$$

We will chose  $\kappa$  later. We denote by  $\hat{b}_{\hat{m}}$  the resulting estimator. To bound the risk of the adaptive estimator, an additional assumption is needed:

- A 5. 1. *The Lévy measure  $\nu$  is symmetric or the function  $\xi$  is constant.*  
 2. *The Lévy measure  $\nu$  is sub exponential: there exist  $\lambda, C > 0$  such that, for any  $|z| > 1$ ,  $\nu(\cdot - z, z)^c \leq Ce^{-\lambda|z|}$ .*

**Theorem 2.**

*Under Assumptions A1-A5, there exists a constant  $\kappa$  (depending only on  $\nu$ ) such that, if  $\mathcal{D}_n^2 \leq n\Delta / \ln^2(n)$ :*

$$\mathbb{E} \left( \left\| \hat{b}_{\hat{m}} - b_A \right\|_n^2 \right) \lesssim \inf_{m \in \mathcal{M}_n} \left( \|b_m - b_A\|_{L^2}^2 + \text{pen}(m) \right) + \left( \Delta + \frac{1}{n\Delta} \right).$$

We can bound  $\kappa$  theoretically, however, this bound is in practice too large for the simulations. In Section 5, we calibrate  $\kappa$  by simulations (see Comte *et al.* (2007) for instance). The adaptive estimator automatically realises the bias-variance compromise. Moreover, this is the same oracle inequality as for a diffusion process without jumps.

## 4 Truncated estimator of the drift

Truncated estimators are widely used for the estimation of the diffusion coefficient of a jump diffusion (see for instance Mancini and Renò (2011) and Shimizu (2008)). Our aim is to construct an adaptive estimator of  $b$  even if Assumption A5 is not fulfilled. To this end, we cut off the big jumps. Let us introduce the set

$$\Omega_{X,k} = \left\{ \omega, \left| X_{(k+1)\Delta} - X_{k\Delta} \right| \leq C_\Delta \right\}$$

where  $C_\Delta = (b_{max} + 3)\Delta + (\sigma_0 + 4\xi_0)\sqrt{\Delta} \ln(n)$  (with  $b_{max} = \sup_{x \in A} |b(x)|$ ). Let us consider the random variables

$$\tilde{Y}_{k\Delta} = \frac{X_{(k+1)\Delta} - X_{k\Delta}}{\Delta} \mathbb{1}_{\Omega_{X,k}} \mathbb{1}_{X_{k\Delta} \in A}.$$

We recall here the definition of the Blumenthal-Gettoor index:

**Definition 1.**

*The Blumenthal-Gettoor index of a Lévy measure is*

$$\beta = \inf \left\{ \alpha \geq 0, \int_{|z| \leq 1} |z|^\alpha \nu(dz) < \infty \right\}.$$

A compound Poisson process has  $\beta = 0$ .

We assume that the following assumption is fulfilled.

A 6. 1. The Lévy measure  $\nu$  is symmetric.

2. For  $|x|$  small,  $\nu(dx)$  is absolutely continuous with respect to the Lebesgue measure ( $\nu(x) = n(x)dx$ ) and:

$$\exists \beta \in [0, 2], \exists a_0, \forall x \in [-a_0, a_0], \quad n(x) \leq Cx^{-\beta-1}.$$

This implies that the Blumenthal-Gettoor index is equal to  $\beta$ .

3. The function  $\xi$  is bounded from below: there exists  $\xi_1 > 0$  such that, for any  $z \in \mathbb{R}$ ,  $0 < \xi_1 \leq \xi(z)$ .

4. The functions  $\sigma$  and  $\xi$  are  $\mathcal{C}^2$ , and  $\xi'$  and  $\sigma'$  are Lipschitz.

We consider the following asymptotic framework:

$$\frac{n\Delta}{\ln^2(n)} \rightarrow 0, \quad \Delta^{1-\beta/2} \ln^2(n) \rightarrow 0.$$

The truncated estimator  $\tilde{b}_m$  is obtained by minimising the contrast function:

$$\tilde{b}_m = \arg \min_{t \in S_m} \tilde{\gamma}_n(t) \quad \text{where} \quad \tilde{\gamma}_n(t) = \frac{1}{n} \sum_{k=1}^n \left( \tilde{Y}_{k\Delta} - t(X_{k\Delta}) \right)^2.$$

**Theorem 3 : Risk of the non adaptive truncated estimator.**

Under Assumptions A1-A4 and A6, for any  $m$  such that  $D_m \leq \mathcal{D}_n$  where  $\mathcal{D}_n^2 \leq n\Delta / \ln^2(n)$ :

$$\mathbb{E} \left( \left\| \tilde{b}_m - b_A \right\|_n^2 \right) \lesssim \|b_m - b_A\|_{L^2}^2 + (\sigma_0^2 + \xi_0^2) \frac{D_m}{n\Delta} + \Delta^{1-\beta/2} \ln^2(n) + \frac{1}{n\Delta}.$$

The terms of the rest depend on the Blumenthal-Gettoor index and are larger than for the first estimator. Nevertheless, if  $L_t$  is a compound Poisson process, then  $\beta = 0$  and we obtain (up to a logarithm factor) the same inequality as for the non-truncated estimator.

*Remark 2.* If  $\nu$  is not absolutely continuous, we can prove the weaker inequality:

$$\mathbb{E} \left( \left\| \tilde{b}_m - b_A \right\|_n^2 \right) \lesssim \|b_m - b_A\|_{L^2}^2 + (\sigma_0^2 + \xi_0^2) \frac{D_m}{n\Delta} + \Delta^{1-\beta} \ln^2(n) + \frac{1}{n\Delta}.$$

In that case,  $\tilde{b}_m$  converges towards  $b_A$  only if  $\beta < 1$ , which implies that  $\nu$  has finite density ( $\int_{\mathbb{R}} |z| \nu(dz) < \infty$ ).

*Remark 3.* Assume that  $b_A$  belongs to the Besov space  $\mathcal{B}_{2,\infty}^\alpha$  and that  $\|b_A\|_{\mathcal{B}_{2,\infty}^\alpha} \leq 1$ .

The bias-variance compromise  $\|b_m - b_A\|_{L^2}^2 + D_m/n\Delta$  is minimum when  $m = \log_2(n\Delta)/(1 + 2\alpha)$ , and the risk satisfies:

$$\mathbb{E} \left( \left\| \tilde{b}_m - b_A \right\|_n^2 \right) \lesssim (n\Delta)^{-2\alpha/(1+2\alpha)} + \Delta^{1-\beta/2} \ln^2(n)$$

Let us set  $\Delta \sim n^{-\gamma}$  with  $\gamma > 0$ . We have the following convergence rates:



$\gamma$	first estimator	truncated estimator
$0 < \gamma \leq \frac{2\alpha}{4\alpha+1} \leq \frac{1}{2}$	$\Delta$	$\Delta^{1-\beta/2} \ln^2(n)$
$\frac{2\alpha}{4\alpha+1} \leq \gamma \leq \frac{2\alpha}{4\alpha+1-\beta\alpha-\beta/2} \leq \frac{1}{2(1-\beta/4)}$	$(n\Delta)^{-2\alpha/(2\alpha+1)}$	$\Delta^{1-\beta/2} \ln^2(n)$
$\frac{2\alpha+1}{4\alpha+1-\beta\alpha-\beta/2} \leq \gamma < 1$	$(n\Delta)^{-2\alpha/(2\alpha+1)}$	$(n\Delta)^{-2\alpha/(2\alpha+1)}$

If we have sufficiently high frequency data ( $n\Delta^{2(1-\beta/4)} = O(1)$ ), then the rate of convergence is the same for the two estimators.

To construct the adaptive estimator, we use the same penalty function as in the previous section:

$$\text{pen}(m) \geq \kappa (\sigma_0^2 + \xi_0^2) \frac{D_m}{n\Delta}$$

and define the adaptive estimator:

$$\tilde{m} = \arg \min_{m \in \mathcal{M}_n} \left\{ \tilde{\gamma}_n(\tilde{b}_m) + \text{pen}(m) \right\}.$$

**Theorem 4 : Risk of the adaptive truncated estimator.**

If Assumptions A1-A4 and A6 are satisfied, then there exists  $\kappa$  such that, if  $\mathcal{D}_n^2 \leq n\Delta / \ln^2(n)$ :

$$\mathbb{E} \left( \left\| \tilde{b}_{\tilde{m}} - b_A \right\|_n^2 \right) \lesssim \min_{m \in \mathcal{M}_n} \left( \|b_m - b_A\|_n^2 + \text{pen}(m) \right) + \Delta^{1-\beta/2} \ln^2(n) + \frac{1}{n\Delta}.$$

The adaptive estimator  $\tilde{b}_{\tilde{m}}$  automatically realises the bias/variance compromise if the frequency of data is sufficiently high.

## 5 Numerical simulations and examples

### 5.1 Compound Poisson models

We consider the stochastic differential equation:

$$dX_t = b(X_t)dt + \sigma(X_t)dW_t + \xi(X_{t-})dL_t$$

where  $L_t$  is a compound Poisson process of intensity 1:  $L_t = \sum_{j=1}^{N_t} \zeta_j$ , with  $N_t$  a Poisson process of intensity 1 and  $(\zeta_1, \dots, \zeta_n)$  are independent and identically distributed random variables independent of  $(N_t)$ . We denote by  $f$  the probability law of  $\zeta_i$ .

**Model 1:**

$$b(x) = -2x, \quad \sigma(x) = \xi(x) = 1 \quad \text{and} \quad f(dz) = \nu(dz) = \frac{1}{2}\delta_1 + \frac{1}{2}\delta_{-1}.$$

**Model 2:**

$$b(x) = -(x - 1/4)^3 - (x + 1/4)^3, \quad \sigma(x) = \xi(x) = 1 \quad \text{and} \quad f(dz) = \nu(dz) = \frac{e^{-\lambda|z|} dz}{2}.$$

We can remark that the function  $b$  is not Lipschitz and therefore does not satisfy Assumption A1.

**Model 3:**

We consider the stochastic process of parameters

$$b(x) = -2x + \sin(3x), \quad \sigma(x) = \xi(x) = \sqrt{\frac{3+x^2}{1+x^2}}$$

and

$$f(dz) = \nu(dz) = \frac{1}{4} \sqrt{\frac{\sqrt{24}}{|z|}} e^{-\sqrt{\sqrt{24}|z|}} dz.$$

Let us remark that  $\nu = f$  is not sub-exponential and does not satisfy A5.

**5.2 Simulation algorithm (Compound Poisson case)**

We estimate  $b$  on the compact interval  $A = [-1, 1]$ .

1. Simulate random variables  $(X_0, X_\Delta, \dots, X_{n\Delta})$  thanks to a Euler scheme with sampling interval  $\delta = \Delta/5$ . To this end, we use the same simulation scheme as Rubenthaler (2010). We simulate the times of the jumps  $(\tau_1, \dots, \tau_N, \tau_{N+1})$  with  $\tau_N < n\Delta \leq \tau_{N+1}$  and we fix  $X_0 = 0$ . If  $\delta < \tau_1$ , we compute

$$X_\delta = \delta b(X_0) + \sqrt{\delta} \sigma(X_0) N \quad \text{with} \quad N \sim \mathcal{N}(0, 1)..$$

If  $\tau_1 < \delta$ , we first compute

$$X_{\tau_1} = \tau_1 b(X_0) + \sqrt{\tau_1} \sigma(X_0) N + \xi(X_0) \zeta_1$$

with  $N \sim \mathcal{N}(0, 1)$  and  $\zeta_1 \sim f$  is independent of  $N$ . If  $\delta < \tau_2$ , we compute

$$X_\delta = (\delta - \tau_1) b(X_{\tau_1}) + \sqrt{\delta - \tau_1} \sigma(X_{\tau_1}) N'$$

else we compute

$$X_{\tau_2} = (\tau_2 - \tau_1) b(X_{\tau_1}) + \sqrt{\tau_2 - \tau_1} \sigma(X_{\tau_1}) N' + \xi(X_{\tau_1}) \zeta_2$$

where  $N' \sim \mathcal{N}(0, 1)$  and  $\zeta_2$  has distribution  $f$ .  $N, N', \zeta_1$  and  $\zeta_2$  are independent.

2. Construct the random variables

$$Y_{k\Delta} = \frac{X_{(k+1)\Delta} - X_{k\Delta}}{\Delta} \quad \text{and} \quad \tilde{Y}_{k\Delta} = \frac{X_{(k+1)\Delta} - X_{k\Delta}}{\Delta} \mathbb{1}_{\Omega_{X,k}} \mathbb{1}_{X_{k\Delta} \in A}.$$

3. We consider the vectorial subspaces  $S_{m,r}$  generated by the spline functions of degree  $r$  (see for instance Schmisser (2009a)). In that case  $D_{m,r} = \dim(S_{m,r}) = 2^m + r$ . For  $r \in \{1, 2, 3\}$  and  $m \in \mathcal{M}_n(r) = \{m, D_{m,r} \leq \mathcal{D}_n\}$ , we compute the estimators  $\hat{b}_{m,r}$  and  $\tilde{b}_{m,r}$  by minimising the contrast functions  $\gamma_n$  and  $\tilde{\gamma}_n$  on the vectorial subspaces  $S_{m,r}$ .
4. For the estimation algorithm, we make a selection of  $m$  and  $r$  as follows. Using the penalty function  $pen(m, r) := pen(m) = \kappa(\sigma_0^2 + \xi_0^2)(2^m + r)/n\Delta$ , we select the adaptive estimators  $\hat{b}_{\hat{m},r}$  and  $\tilde{b}_{\tilde{m},r}$ , and then choose the best  $r$  by minimizing  $\gamma_n(\hat{b}_{\hat{m},r}) + pen(\hat{m}, r)$  and  $\tilde{\gamma}_n(\tilde{b}_{\tilde{m},r}) + pen(\tilde{m}, r)$ .

### 5.3 Results

In Figures 1-3, we simulate 5 times the process  $(X_0, \dots, X_{n\Delta})$  for  $\Delta = 10^{-1}$  and  $n = 10^4$  and draw the obtained estimators. The two adaptive estimators are nearly superposed, moreover, they are close to the true function.

In Tables 1-3, for each value of  $(n, \Delta)$ , we simulate 50 trajectories of  $(X_0, X_\Delta, \dots, X_{n\Delta})$ . For each path, we construct the two adaptive estimators  $\hat{b}_{\hat{m},\hat{r}}$  and  $\tilde{b}_{\tilde{m},\tilde{r}}$  and we compute the empirical errors:

$$err_1 = \left\| \hat{b}_{\hat{m},\hat{r}} - b_A \right\|_n^2 \quad \text{and} \quad err_2 = \left\| \tilde{b}_{\tilde{m},\tilde{r}} - b_A \right\|_n^2.$$

In order to check that our algorithm is adaptive, we also compute the minimal errors

$$emin_1 = \min_{m,r} \left\| \hat{b}_{m,r} - b_A \right\|_n^2 \quad \text{and} \quad emin_2 = \min_{m,r} \left\| \tilde{b}_{m,r} - b_A \right\|_n^2$$

and the oracles  $oracle_i = err_i / emin_i$ . We give the means  $\hat{m}_a, \hat{r}_a, \tilde{m}_a$  and  $\tilde{r}_a$  of the selected values  $\hat{m}, \hat{r}, \tilde{m}$  and  $\tilde{r}$ . The value  $risk_i$  is the mean of  $err_i$  over the 50 simulations and  $or_i$  is the mean of  $oracle_i$ .

The empirical risk is decreasing when the product  $n\Delta$  is increasing, which is coherent with the theoretical model. For Model 1, the two estimators are equivalent. When the tails of  $\nu$  become larger (Models 2 and 3), the truncated estimator is better. The improvement is also more significant when the discretisation path is smaller. As on the three models, the processes  $L_t$  are compound Poisson processes, these results were expected. The truncated estimator seems also more robust: we don't observe aberrant values (like for the first estimator in Table 2). This aberrant value may be due to the fact that  $b$  is not Lipschitz and then  $b(X_{k\Delta})$  may be quite large.

## 6 Proofs

Let us introduce the filtration

$$\mathcal{F}_t = \sigma \left( \eta, (W_s)_{0 \leq s \leq t}, (L_s)_{0 \leq s \leq t} \right).$$

The following result is very useful. It comes from Dellacherie and Meyer (1980) (Theorem 92 Chapter VII) and Applebaum (2004), Theorem 4.4.23 p265 (Kunita's first inequality).

*Result 1 (Burkholder-Davis-Gundy inequality). We have that, for any  $p \geq 2$ ,*

$$\mathbb{E} \left[ \sup_{s \in [t, t+h]} \left| \int_t^s \sigma(X_u) dW_u \right|^p \middle| \mathcal{F}_t \right] \leq C_p \left( \mathbb{E} \left[ \left| \int_t^{t+h} \sigma^2(X_u) du \right|^{p/2} \middle| \mathcal{F}_t \right] \right)$$

and, if  $\int_{\mathbb{R}} |z|^p \nu(dz) < \infty$ , as  $\int_{\mathbb{R}} z^2 \nu(dz) = 1$ :

$$\begin{aligned} \mathbb{E} \left[ \sup_{s \in [t, t+h]} \left| \int_t^s \xi(X_{u-}) dL_u \right|^p \middle| \mathcal{F}_t \right] &\leq C_p \mathbb{E} \left[ \left( \int_t^{t+h} \xi^2(X_u) du \right)^{p/2} \middle| \mathcal{F}_t \right] \\ &+ C_p \mathbb{E} \left[ \left( \int_t^{t+h} |\xi(X_u)|^p du \right) \middle| \mathcal{F}_t \right] \int_{\mathbb{R}} |z|^p \nu(dz). \end{aligned}$$

## 6.1 Proof of Proposition 1

By Result 1, there exists a constant  $c_p$  such that:

$$\begin{aligned} \mathbb{E} \left[ \sup_{s \in [t, t+h]} (X_s - X_t)^{2p} \middle| \mathcal{F}_t \right] &\leq c(p) \left( \left[ \left( \int_t^{t+h} |b(X_s)| ds \right)^{2p} \middle| \mathcal{F}_t \right] \right) \\ &+ c(p) \mathbb{E} \left[ \left( \int_t^{t+h} \sigma^2(X_s) ds \right)^p \middle| \mathcal{F}_t \right] \\ &+ c(p) \left( \mathbb{E} \left[ \left( \int_t^{t+h} \xi^2(X_s) ds \right)^p + \int_t^{t+h} \xi^{2p}(X_s) ds \middle| \mathcal{F}_t \right] \right). \end{aligned}$$

Then, as  $\xi$  and  $\sigma$  are bounded and  $b$  Lipschitz (and thus sub-linear), there exists a constant  $C_b$  such that:

$$\mathbb{E} \left[ \sup_{s \in [t, t+h]} (X_s - X_t)^{2p} \middle| \mathcal{F}_t \right] \leq c(p) \left( \sigma_0^{2p} h^p + \xi_0^{2p} (h + h^p) \right) + c(p) h^{2p-1} C_b \int_t^{t+h} \mathbb{E} [X_s^{2p} | \mathcal{F}_t] ds.$$

As  $(X_t)$  is stationary, we obtain the expected result.

## 6.2 Proof of Theorem 1

By (3) and (4), we get:

$$\begin{aligned} \gamma_n(t) = \frac{1}{n} \sum_{k=1}^n (Y_{k\Delta} - t(X_{k\Delta}))^2 &= \frac{1}{n} \sum_{k=1}^n (Y_{k\Delta} - b(X_{k\Delta}))^2 + \|b - t\|_n^2 \\ &+ \frac{2}{n} \sum_{k=1}^n (Y_{k\Delta} - b(X_{k\Delta})) (b(X_{k\Delta}) - t(X_{k\Delta})). \end{aligned}$$

As, by definition,  $\gamma_n(\hat{b}_m) \leq \gamma_n(b_m)$ , we obtain:

$$\left\| \hat{b}_m - b \right\|_n^2 \leq \|b_m - b\|_n^2 + \frac{2}{n} \sum_{k=1}^n (Y_{k\Delta} - b(X_{k\Delta})) \left( \hat{b}_m(X_{k\Delta}) - b_m(X_{k\Delta}) \right).$$

By (2), and as  $\hat{b}_m$  and  $b_m$  are supported by  $A$ ,

$$\left\| \hat{b}_m - b_A \right\|_n^2 \leq \|b_m - b_A\|_n^2 + \frac{2}{n} \sum_{k=1}^n (I_{k\Delta} + Z_{k\Delta} + T_{k\Delta}) \left( \hat{b}_m(X_{k\Delta}) - b_m(X_{k\Delta}) \right).$$

Let us set introduce the unit ball

$$\mathcal{B}_m = \{t \in S_m, \|t\|_{\varpi} \leq 1\} \quad \text{where} \quad \|t\|_{\varpi}^2 = \int_A t^2(x) \varpi(dx)$$

and the englobing space  $\mathcal{S}_n = \bigcup_{m \in \mathcal{M}_n} S_m$ . Let us consider the set

$$\Omega_n = \left\{ \omega, \forall t \in \mathcal{S}_n, \left| \frac{\|t\|_n^2}{\|t\|_{\varpi}^2} - 1 \right| \leq \frac{1}{2} \right\}$$

where the norms  $\|\cdot\|_{\varpi}$  and  $\|\cdot\|_n$  are equivalent.

**Step 1: bound of the risk on  $\Omega_n$**  Thanks to the Cauchy-Schwartz inequality, we obtain that, on  $\Omega_n$ :

$$\left\| \hat{b}_m - b_A \right\|_n^2 \leq \|b_m - b_A\|_n^2 + \frac{1}{12} \left\| \hat{b}_m - b_m \right\|_n^2 + 12 \sum_{k=1}^n I_{k\Delta}^2 + \frac{1}{12} \left\| \hat{b}_m - b_m \right\|_{\varpi}^2 + 12 \sup_{t \in \mathcal{B}_m} \nu_n^2(t)$$

where

$$\nu_n(t) = \frac{1}{n} \sum_{k=1}^n (Z_{k\Delta} + T_{k\Delta}) t(X_{k\Delta}). \quad (5)$$

On  $\Omega_n$ , by definition, we have:

$$\left\| \hat{b}_m - b_m \right\|_n^2 \leq 2 \left\| \hat{b}_m - b_A \right\|_n^2 + 2 \|b_m - b_A\|_n^2 \quad \text{and} \quad \left\| \hat{b}_m - b_m \right\|_{\varpi}^2 \leq 2 \left\| \hat{b}_m - b_m \right\|_n^2.$$

Thus we obtain:

$$\left\| \hat{b}_m - b_A \right\|_n^2 \leq 3 \|b_m - b_A\|_n^2 + 24 \sum_{k=1}^n I_{k\Delta}^2 + 24 \sup_{t \in \mathcal{B}_m} \nu_n^2(t).$$

The following lemma is very useful. It is proved later.

**Lemma 1.** 1.  $\mathbb{E}(I_{k\Delta}^2) \leq c\Delta$  and  $\mathbb{E}(I_{k\Delta}^4) \leq c\Delta$ .

2.  $\mathbb{E}(Z_{k\Delta} | \mathcal{F}_{k\Delta}) = 0$ ,  $\mathbb{E}(Z_{k\Delta}^2 | \mathcal{F}_{k\Delta}) \leq \sigma_0^2/\Delta$  and  $\mathbb{E}(Z_{k\Delta}^4 | \mathcal{F}_{k\Delta}) \leq c/\Delta^2$ .

3.  $\mathbb{E}(T_{k\Delta} | \mathcal{F}_{k\Delta}) = 0$ ,  $\mathbb{E}(T_{k\Delta}^2 | \mathcal{F}_{k\Delta}) \leq \xi_0^2/\Delta$  and  $\mathbb{E}(T_{k\Delta}^4 | \mathcal{F}_{k\Delta}) \leq c/\Delta^3$ .

By Lemma 1,  $\mathbb{E}[I_{k\Delta}^2] \leq \Delta$ . It remains to bound  $\mathbb{E}[\sup_{t \in \mathcal{B}_m} \nu_n^2(t)]$ . We consider an orthonormal basis  $(\varphi_\lambda)_{\lambda \in \Lambda_m}$  of  $S_m$  for the  $L^2_{\varpi}$ -norm with  $|\Lambda_m| = D_m$ . Any function  $t \in S_m$  can be written  $t = \sum_{\lambda \in \Lambda_m} a_\lambda \varphi_\lambda$  and  $\|t\|_{\varpi}^2 = \sum_{\lambda \in \Lambda_m} a_\lambda^2$ . Then:

$$\begin{aligned} \sup_{t \in \mathcal{B}_m} \nu_n^2(t) &= \sup_{\sum_{\lambda} a_\lambda^2 \leq 1} \left( \sum_{\lambda \in \Lambda_m} a_\lambda \nu_n(\varphi_\lambda) \right)^2 \\ &\leq \sup_{\sum_{\lambda} a_\lambda^2 \leq 1} \left( \sum_{\lambda \in \Lambda_m} a_\lambda^2 \right) \left( \sum_{\lambda \in \Lambda_m} \nu_n^2(\varphi_\lambda) \right) \\ &= \sum_{\lambda \in \Lambda_m} \nu_n^2(\varphi_\lambda). \end{aligned}$$

It remains to bound  $\mathbb{E}(\nu_n^2(\varphi_\lambda))$ . By (5),

$$\begin{aligned} \mathbb{E}[\nu_n^2(\varphi_\lambda)] &= \frac{1}{n^2} \sum_{k=1}^n \mathbb{E}[\varphi_\lambda^2(X_{k\Delta}) \mathbb{E}[(Z_{k\Delta} + T_{k\Delta})^2 | \mathcal{F}_{k\Delta}]] \\ &\quad + \frac{2}{n^2} \sum_{k < l}^n \mathbb{E}[(Z_{k\Delta} + T_{k\Delta}) \varphi_\lambda(X_{k\Delta}) \varphi_\lambda(X_{l\Delta}) \mathbb{E}[Z_{l\Delta} + T_{l\Delta} | \mathcal{F}_{l\Delta}]] \end{aligned}$$

Thanks to Lemma 1, the second term of this inequality is null and we obtain, as  $\int_{\mathbb{R}} \varphi_\lambda^2(x) \varpi(dx) = 1$ :

$$\mathbb{E}[\nu_n^2(\varphi_\lambda)] \leq \frac{2(\sigma_0^2 + \xi_0^2)}{n^2 \Delta} \sum_{k=1}^n \mathbb{E}[\varphi_\lambda^2(X_{k\Delta})] = \frac{2(\sigma_0^2 + \xi_0^2)}{n \Delta}.$$

Therefore:

$$\mathbb{E} \left[ \left\| \hat{b}_m - b_A \right\|_n^2 \mathbf{1}_{\Omega_n^c} \right] \leq 3 \|b_m - b_A\|_n^2 + 48(\sigma_0^2 + \xi_0^2) \frac{D_m}{n \Delta} + C \Delta.$$

**Step 2: bound of the risk on  $\Omega_n^c$ .** The process  $(X_t)_{t \geq 0}$  is exponentially  $\beta$ -mixing,  $\pi$  is bounded from below and above and  $n \Delta \rightarrow \infty$ . The following result is proved for  $\xi = 0$  for instance in Comte *et al.* (2007), but as it relies only on the  $\beta$ -mixing property, we can apply it.

*Result 2.*

$$\mathbb{P}[\Omega_n^c] \leq \frac{1}{n^3}.$$

Let us set  $e = (e_\Delta, \dots, e_{n\Delta})^*$  where  $e_{k\Delta} := Y_{k\Delta} - b(X_{k\Delta}) = I_{k\Delta} + Z_{k\Delta} + T_{k\Delta}$  and  $\Pi_m Y = \Pi_m (Y_\Delta, \dots, Y_{n\Delta})^* = (\hat{b}_m(X_0), \dots, \hat{b}_m(X_{n\Delta}))^*$  where  $\Pi_m$  is the

Euclidian orthogonal projection over  $S_m$ . Then

$$\begin{aligned} \left\| \hat{b}_m - b_A \right\|_n^2 &= \left\| \Pi_m Y - b_A \right\|_n^2 = \left\| \Pi_m b_A - b_A \right\|_n^2 + \left\| \Pi_m Y - \Pi_m b_A \right\|_n^2 \\ &\leq \left\| b_A \right\|_n^2 + \left\| e \right\|_n^2. \end{aligned}$$

According to Lemma 1, Result 2 and the Cauchy-Schwarz inequality,

$$\mathbb{E} \left[ \left\| e \right\|_n^2 \mathbb{1}_{\Omega_n^c} \right] \leq \left( \mathbb{E} \left[ \left\| e \right\|_n^4 \right] \right)^{1/2} \left( \mathbb{P}(\Omega_n^c) \right)^{1/2} \leq \frac{C}{(\Delta^3 n^3)^{1/2}} \leq \frac{C}{n\Delta}$$

and, as  $b$  is bounded on the compact set  $A$ ,

$$\mathbb{E} \left[ \left\| b_A \right\|_n^2 \mathbb{1}_{\Omega_n^c} \right] \leq \left( \mathbb{E} \left[ \left\| b_A \right\|_n^4 \right] \mathbb{P}(\Omega_n^c) \right)^{1/2} \lesssim \frac{1}{n^{3/2}}.$$

Collecting the results, we get:

$$\mathbb{E} \left[ \left\| \hat{b}_m - b_A \right\|_n^2 \mathbb{1}_{\Omega_n^c} \right] \lesssim \frac{1}{n\Delta}$$

which ends the proof of Theorem 1.

### 6.2.1 Proof of Lemma 1

By Proposition 1, as  $b$  is Lipschitz,

$$\begin{aligned} \mathbb{E} [I_{k\Delta}^2] &= \frac{1}{\Delta^2} \mathbb{E} \left[ \left( \int_{k\Delta}^{(k+1)\Delta} (b(X_s) - b(X_{k\Delta})) ds \right)^2 \right] \\ &\leq \frac{1}{\Delta} \mathbb{E} \left[ \int_{k\Delta}^{(k+1)\Delta} (b(X_s) - b(X_{k\Delta}))^2 ds \right] \\ &\leq \frac{c}{\Delta} \int_{k\Delta}^{(k+1)\Delta} \mathbb{E} [(X_s - X_{k\Delta})^2] ds \\ &\leq c\Delta. \end{aligned}$$

In the same way, we prove that  $\mathbb{E} [I_{k\Delta}^4] \leq c\Delta$ . We have that

$$\mathbb{E} [Z_{k\Delta}^2 | \mathcal{F}_{k\Delta}] = \mathbb{E} \left[ \left( \frac{1}{\Delta^2} \int_{k\Delta}^{(k+1)\Delta} \sigma^2(X_s) ds \right) \middle| \mathcal{F}_{k\Delta} \right] \leq \frac{\sigma_0^2}{\Delta}.$$

Moreover, by the Burkholder-Davis-Gundy inequality, we get

$$\mathbb{E} [Z_{k\Delta}^4 | \mathcal{F}_{k\Delta}] \leq \frac{C}{\Delta^4} \mathbb{E} \left[ \left( \int_{k\Delta}^{(k+1)\Delta} \sigma^2(X_s) ds \right)^2 \middle| \mathcal{F}_{k\Delta} \right] \leq \frac{C}{\Delta^2}.$$

According to Applebaum (2004), Theorem 4.2.3 p224,

$$\mathbb{E} [T_{k\Delta} | \mathcal{F}_{k\Delta}] = 0$$

and, as  $\int_{\mathbb{R}} z^2 \nu(dz) = 1$ :

$$\mathbb{E} [T_{k\Delta}^2 | \mathcal{F}_{k\Delta}] = \frac{1}{\Delta^2} \int_{k\Delta}^{(k+1)\Delta} \xi^2(X_s) \int_{\mathbb{R}} z^2 \nu(dz) ds \leq \frac{\xi_0^2}{\Delta}.$$

By Result 1, we have

$$\begin{aligned} \mathbb{E} [T_{k\Delta}^4 | \mathcal{F}_{k\Delta}] &\leq \frac{C_p}{\Delta^4} \mathbb{E} \left[ \left( \int_{k\Delta}^{(k+1)\Delta} \xi^2(X_s) \int_{\mathbb{R}} z^2 \nu(dz) ds \right)^2 \right] \\ &\quad + \frac{C_p}{\Delta^4} \mathbb{E} \left[ \left( \int_{k\Delta}^{(k+1)\Delta} \xi^4(X_s) ds \right) \int_{\mathbb{R}} z^4 \nu(dz) \right] \\ &\lesssim \frac{1}{\Delta^3} \end{aligned}$$

### 6.3 Proof of Theorem 2

The bound of the risk on  $\Omega_n^c$  is done exactly in the same way as for the non adaptive estimator. It remains thus to bound the risk on  $\Omega_n$ . As in the previous proof, we get:

$$\begin{aligned} \left\| \hat{b}_{\hat{m}} - b_A \right\|_n^2 \mathbb{1}_{\Omega_n} &\leq 3 \|b_m - b_A\|_n^2 + \frac{24}{n} \sum_{k=1}^n I_{k\Delta}^2 + 2pen(m) - 2pen(\hat{m}) \\ &\quad + 24 \sup_{t \in \mathcal{B}_{m, \hat{m}}} \nu_n^2(t) \end{aligned}$$

where  $\mathcal{B}_{m, m'}$  is the unit ball (for the  $L_{\infty}^2$ -norm) of the subspace  $S_m + S_{m'}$ :  $\mathcal{B}_{m, m'} = \{t \in S_m + S_{m'}, \|t\|_{\infty} \leq 1\}$ . Let us introduce a function  $p(m, m')$  such that  $12p(m, m') = pen(m) + pen(m')$ . We obtain that, on  $\Omega_n$ , for any  $m \in \mathcal{M}_n$ :

$$\begin{aligned} \left\| \hat{b}_{\hat{m}} - b_A \right\|_n^2 &\leq 3 \|b_m - b_A\|_n^2 + \frac{24}{n} \sum_{k=1}^n I_{k\Delta}^2 + 4pen(m) \\ &\quad + 24 \sup_{t \in \mathcal{B}_{m, \hat{m}}} (\nu_n^2(t) - p(m, \hat{m})). \end{aligned}$$

It remains to bound

$$\mathbb{E} \left[ \sup_{t \in \mathcal{B}_{m, \hat{m}}} \nu_n^2(t) - p(m, \hat{m}) \right] \leq \sum_{m'} \mathbb{E} \left[ \sup_{t \in \mathcal{B}_{m, m'}} \nu_n^2(t) - p(m, m') \right]_+.$$

For this purpose, we use the following proposition proved in Applebaum (2004) (Corollary 5.2.2).



**Proposition 2 : exponential martingale.**

Let  $(Y_t)_{t \geq 0}$  satisfy:

$$Y_t = \int_0^t F_s dW_s + \int_0^t K_s dL_s - \int_0^t \left[ \frac{F_s^2}{2} + \int_{\mathbb{R}} (e^{K_s z} - 1 - K_s z) \nu(dz) \right] ds$$

where  $F_s$  and  $K_s$  are locally integrable and previsible processes. If for any  $t > 0$ ,

$$\mathbb{E} \left[ \int_0^t \int_{|z| > 1} |e^{K_s z} - 1| \nu(dz) ds \right] < \infty,$$

then  $e^{Y_t}$  is a  $\mathcal{G}_t$ -local martingale where  $\mathcal{G}_t = \sigma(W_s, L_s, 0 \leq s \leq t)$ .

For any  $\varepsilon \leq \varepsilon_1 := (\lambda \wedge 1)/(2 \|t\|_{\infty} \xi_0)$  where  $\lambda$  is defined in Assumption A5, for any  $t \geq 0$

$$\int_0^t \int_{|z| \geq 1} (\exp(\varepsilon t(X_{k\Delta})\xi(X_s)z) - 1) \nu(dz) \mathbb{1}_{s \in ]k\Delta, (k+1)\Delta]} ds < \infty.$$

Let us introduce the two Markov processes

$$A_{\varepsilon, t} := \varepsilon^2 \sum_{k=0}^n t^2(X_{k\Delta}) \int_0^t \sigma^2(X_s) \mathbb{1}_{s \in ]k\Delta, (k+1)\Delta]} ds$$

and

$$B_{\varepsilon, t} := \sum_{k=0}^n \int_0^t \int_{\mathbb{R}} (\exp(\varepsilon t(X_{k\Delta})\xi(X_s)z) - \varepsilon t(X_{k\Delta})\xi(X_s)z - 1) \mathbb{1}_{s \in ]k\Delta, (k+1)\Delta]} \nu(dz) ds$$

and the following martingale:

$$M_t = \int_0^t \sum_{k=0}^n \mathbb{1}_{s \in ]k\Delta, (k+1)\Delta]} t(X_{k\Delta-}) (\sigma(X_s) dW_s + \xi(X_{s-}) dL_s).$$

By Proposition 2,

$$Y_{\varepsilon, s} := \varepsilon M_s - A_{\varepsilon, s} - B_{\varepsilon, s}$$

is such that  $e^{Y_{\varepsilon, s}}$  is a local martingale.

**Bound of  $A_{\varepsilon, s}$  and  $B_{\varepsilon, s}$ .** We obtain easily that  $A_{\varepsilon, s} \leq A_{\varepsilon, (n+1)\Delta} \leq \varepsilon^2 n \Delta \|t\|_n^2 \sigma_0^2$ . Under Assumption A5,  $\xi$  is constant or  $\nu$  is symmetric, and therefore

$$B_{\varepsilon, s} \leq B_{\varepsilon, (n+1)\Delta} \leq \Delta \sum_{k=0}^n \int_{\mathbb{R}} (\exp(\varepsilon t(X_{k\Delta})\xi_0 z) - \varepsilon t(X_{k\Delta-})\xi_0 z - 1) \nu(dz).$$

As  $\int_{\mathbb{R}} z^2 \nu(dz) = 1$ , for any  $\alpha \leq 1$ ,

$$\int_{-1}^1 (\exp(\alpha z) - \alpha z - 1) \nu(dz) \leq \alpha^2 \int_{-1}^1 z^2 \nu(dz) \leq \alpha^2.$$

Moreover, by integration by parts, for any  $\alpha \leq (1 \wedge \lambda)/2$ ,

$$\begin{aligned} \int_{[-1,1]^c} (\exp(\alpha z) - \alpha z - 1) \nu(dz) &\leq (e^\alpha - \alpha - 1) \nu([1, +\infty[) + (e^{-\alpha} + \alpha - 1) \nu(]-\infty, -1]) \\ &\quad + \int_1^{+\infty} \alpha (e^{\alpha z} - 1) \nu([-z, z]^c) dz \\ &\leq 2\alpha^2 \nu([-1, 1]^c) + Ce^{-\lambda} \frac{\alpha}{\lambda} \left( \frac{e^\alpha}{1 - \alpha/\lambda} - 1 \right) \leq C' \alpha^2. \end{aligned}$$

Then  $B_{\varepsilon,s} \lesssim n\Delta\varepsilon^2 \xi_0^2 \|t\|_n^2$ . There exists a constant  $c$  such that, for any  $\varepsilon < \varepsilon_1$ ,

$$A_{\varepsilon,s} + B_{\varepsilon,s} \leq c \frac{n\Delta\varepsilon^2 (\sigma_0^2 + \xi_0^2) \|t\|_n^2}{(1 - \varepsilon/\varepsilon_1)}.$$

**Bound of  $\mathbb{P}(\nu_n(t) \geq \eta, \|t\|_n^2 \leq \zeta^2)$ .** The process  $\exp(Y_{\varepsilon,t})$  is a local martingale, then there exists an increasing sequence  $(\tau_N)$  of stopping times such that  $\lim_{N \rightarrow \infty} \tau_N = \infty$  and  $\exp(Y_{\varepsilon,t \wedge \tau_N})$  is a  $\mathcal{F}_t$ -martingale. For any  $\varepsilon < \varepsilon_1$ , and all  $N$ ,

$$\begin{aligned} E &:= \mathbb{P}\left(M_{(n+1)\Delta \wedge \tau_N} \geq n\Delta\eta, \|t\|_n^2 \leq \zeta^2\right) \\ &\leq \mathbb{P}\left(M_{(n+1)\Delta \wedge \tau_N} \geq n\Delta\eta, A_{(n+1)\Delta \wedge \tau_N} + B_{(n+1)\Delta \wedge \tau_N} \leq \frac{cn\Delta\varepsilon^2 (\sigma_0^2 + \xi_0^2) \zeta^2}{(1 - \varepsilon/\varepsilon_1)}\right) \\ &\leq \mathbb{E}\left(\exp(Y_{\varepsilon,(n+1)\Delta \wedge \tau_N}) \exp\left(-n\Delta\eta\varepsilon + \frac{cn\Delta\varepsilon^2 (\xi_0^2 + \sigma_0^2) \zeta^2}{(1 - \varepsilon/\varepsilon_1)}\right)\right). \end{aligned}$$

As  $\exp(Y_{\varepsilon,t \wedge \tau_N})$  is a martingale,  $\mathbb{E}(\exp(Y_{\varepsilon,t \wedge \tau_N})) = 1$  and

$$E \leq \exp\left(-n\Delta\eta\varepsilon + \frac{cn\Delta\varepsilon^2 (\xi_0^2 + \sigma_0^2) \zeta^2}{(1 - \varepsilon/\varepsilon_1)}\right).$$

Letting  $N$  tends to infinity, by dominated convergence, and as  $\nu_n(t) = n\Delta M_{(n+1)\Delta}$ , we obtain that

$$\mathbb{P}\left(\nu_n(t) \geq \eta, \|t\|_n^2 \leq \zeta^2\right) \leq \exp\left(-n\Delta\eta\varepsilon + \frac{cn\Delta\varepsilon^2 (\xi_0^2 + \sigma_0^2) \zeta^2}{(1 - \varepsilon/\varepsilon_1)}\right).$$

It remains to minimise this inequality in  $\varepsilon$ . Let us set

$$\varepsilon = \frac{\eta}{2c(\sigma_0^2 + \xi_0^2) \zeta^2 / \Delta + \eta / \varepsilon_1} < \varepsilon_1.$$

We get:

$$\mathbb{P}\left(\nu_n(t) \geq \eta, \|t\|_n^2 \leq \zeta^2\right) \leq \exp\left(-\frac{\eta^2 n \Delta}{4c((\sigma_0^2 + \xi_0^2) \zeta^2 + c' \eta \xi_0 \|t\|_\infty)}\right).$$

The following lemma concludes the proof. It is proved thanks to a  $L^2_{\varpi} - L^\infty$  chaining technique. See Comte (2001), proof of Proposition 4, and Schmisser (2010), Appendix D.3.

**Lemma 2.**

There exists a constant  $\kappa$  such that:

$$\mathbb{E} \left[ \sup_{t \in \mathcal{B}_{m,m'}} \nu_n^2(t) - p(m, m') \right] \lesssim \kappa (\xi_0^2 + \sigma_0^2) \frac{D^{3/2}}{n\Delta} e^{-\sqrt{D}}$$

where  $D = \dim(S_m + S_{m'})$ .

As  $\sum_D D^{3/2} e^{-\sqrt{D}} \leq \sum_{k=0}^{+\infty} k^{3/2} e^{-\sqrt{k}} < \infty$ , we obtain that

$$\mathbb{E} \left[ \sup_{t \in \mathcal{B}_{m,\hat{m}}} \nu_n^2(t) - p(m, \hat{m}) \right] \leq \sum_{m' \in \mathcal{M}_n} \mathbb{E} \left[ \sup_{t \in \mathcal{B}_{m,m'}} \nu_n^2(t) - p(m, m') \right] \lesssim \kappa \frac{\xi_0^2 + \sigma_0^2}{n\Delta}.$$

### 6.4 Proof of Theorem 3

We recall that

$$\Omega_{X,k} = \left\{ \omega, |X_{(k+1)\Delta} - X_{k\Delta}| \leq C_\Delta = (b_{max} + 3) \Delta + (\sigma_0 + 4\xi_0) \sqrt{\Delta} \ln(n) \right\}.$$

Let us introduce the set

$$\Omega_{N,k} = \left\{ \omega, N'_{k\Delta} = 0 \right\}$$

where  $N'_{k\Delta}$  is the number of jumps of size larger than  $\Delta^{1/4}$  occurring in the time interval  $]k\Delta, (k+1)\Delta[$ :

$$N'_{k\Delta} = \mu \left( ]k\Delta, (k+1)\Delta[ , \left[ -\Delta^{1/4}, \Delta^{1/4} \right]^c \right).$$

We have that

$$\begin{aligned} \tilde{Y}_{k\Delta} &= Y_{k\Delta} \mathbb{1}_{\Omega_{X,k}} \mathbb{1}_{X_{k\Delta} \in A} \\ &= b_A(X_{k\Delta}) - b_A(X_{k\Delta}) \mathbb{1}_{\Omega_{X,k}^c \cap (X_{k\Delta} \in A)} + I_{k\Delta} \mathbb{1}_{\Omega_{X,k} \cap (X_{k\Delta} \in A)} + \tilde{Z}_{k\Delta} + \tilde{T}_{k\Delta} \\ &\quad + (Z_{k\Delta} + T_{k\Delta}) \mathbb{1}_{\Omega_{X,k} \cap \Omega_{N,k}^c \cap (X_{k\Delta} \in A)} + \mathbb{E} \left( (Z_{k\Delta} + T_{k\Delta}) \mathbb{1}_{\Omega_{X,k} \cap \Omega_{N,k} \cap (X_{k\Delta} \in A)} \middle| \mathcal{F}_{k\Delta} \right). \end{aligned}$$

where

$$\tilde{Z}_{k\Delta} = Z_{k\Delta} \mathbb{1}_{\Omega_{X,k} \cap \Omega_{N,k} \cap (X_{k\Delta} \in A)} - \mathbb{E} \left( Z_{k\Delta} \mathbb{1}_{\Omega_{X,k} \cap \Omega_{N,k} \cap (X_{k\Delta} \in A)} \middle| \mathcal{F}_{k\Delta} \right)$$

and

$$\tilde{T}_{k\Delta} = T_{k\Delta} \mathbb{1}_{\Omega_{X,k} \cap \Omega_{N,k} \cap (X_{k\Delta} \in A)} - \mathbb{E} \left( T_{k\Delta} \mathbb{1}_{\Omega_{X,k} \cap \Omega_{N,k} \cap (X_{k\Delta} \in A)} \middle| \mathcal{F}_{k\Delta} \right).$$

As previously, we only bound the risk on  $\Omega_n$ . Let us set

$$\tilde{\nu}_n(t) := \frac{1}{n} \sum_{k=1}^n t(X_{k\Delta}) \left( \tilde{Z}_{k\Delta} + \tilde{T}_{k\Delta} \right).$$

We have that

$$\begin{aligned} \left\| \tilde{b}_m - b_A \right\|_n^2 \mathbb{1}_{\Omega_n} &\leq 3 \|b_m - b_A\|_n^2 + 24 \sup_{t \in \mathcal{B}_m} \tilde{\nu}_n^2(t) + \frac{224}{n} \sum_{k=1}^n \left( I_{k\Delta}^2 + b_A^2(X_{k\Delta}) \mathbb{1}_{\Omega_{X,k}^c} \right) \\ &+ \frac{224}{n} \sum_{k=1}^n (Z_{k\Delta}^2 + T_{k\Delta}^2) \mathbb{1}_{\Omega_{X,k} \cap \Omega_{N,k}^c \cap (X_{k\Delta} \in A)} \\ &+ \frac{224}{n} \sum_{k=1}^n \left( \mathbb{E} \left[ (Z_{k\Delta} + T_{k\Delta}) \mathbb{1}_{\Omega_{X,k} \cap \Omega_{N,k} \cap (X_{k\Delta} \in A)} \middle| \mathcal{F}_{k\Delta} \right] \right)^2. \end{aligned}$$

The following lemma is proved later.

**Lemma 3.** 1.  $\mathbb{P}(\Omega_{X,k}^c \cap (X_{k\Delta} \in A)) \lesssim \Delta^{1-\beta/2}$ .

2.  $\mathbb{P}(\Omega_{X,k} \cap \Omega_{N,k}^c \cap (X_{k\Delta} \in A)) \lesssim \Delta^{2-\beta/2}$ .

3.  $\left( \mathbb{E} \left[ (Z_{k\Delta} + T_{k\Delta}) \mathbb{1}_{\Omega_{X,k} \cap \Omega_{N,k} \cap (X_{k\Delta} \in A)} \middle| \mathcal{F}_{k\Delta} \right] \right)^2 \lesssim \ln^2(n) \Delta^{1-\beta/2}$ .

According to Lemma 1,  $\mathbb{E}(I_{k\Delta}^2) \leq \Delta k$ . As  $b$  is bounded on the compact set  $A$ ,  $\mathbb{E} \left[ b_A^2(X_{k\Delta}) \mathbb{1}_{\Omega_{X,k}^c} \right] \lesssim \mathbb{P}(\Omega_{X,k}^c) \lesssim \Delta^{1-\beta/2}$ . Moreover, on  $\Omega_{X,k}$ ,

$$\begin{aligned} (Z_{k\Delta} + T_{k\Delta})^2 \mathbb{1}_{\Omega_{X,k} \cap (X_{k\Delta} \in A)} &= \left( \frac{X_{(k+1)\Delta} - X_{k\Delta}}{\Delta} - b_A(X_{k\Delta}) - I_{k\Delta} \right)^2 \mathbb{1}_{\Omega_{X,k}} \mathbb{1}_{X_{k\Delta} \in A} \\ &\lesssim \frac{\ln^2(n)}{\Delta} + b_A^2(X_{k\Delta}) + I_{k\Delta}^2 \end{aligned}$$

and then

$$\begin{aligned} E &:= \mathbb{E} \left[ (Z_{k\Delta} + T_{k\Delta})^2 \mathbb{1}_{\Omega_{X,k} \cap \Omega_{N,k}^c \cap (X_{k\Delta} \in A)} \right] \\ &\lesssim \left( \frac{\ln^2(n)}{\Delta} + b_{max}^2 \right) \mathbb{P}(\Omega_{X,k} \cap \Omega_{N,k}^c \cap (X_{k\Delta} \in A)) + \mathbb{E}(I_{k\Delta}^2) \\ &\lesssim \ln^2(n) \Delta^{1-\beta/2}. \end{aligned}$$

It remains to bound  $\mathbb{E}(\sup_{t \in \mathcal{B}_m} \tilde{\nu}_n^2(t))$ . In the same way as in Subsection 6.2, we get:

$$\begin{aligned} \mathbb{E} \left( \sup_{t \in \mathcal{B}_m} \tilde{\nu}_n^2(t) \right) &\leq \sum_{\lambda \in \Lambda_m} \mathbb{E}(\tilde{\nu}_n^2(\varphi_\lambda)) \leq \frac{2D_m}{n} \mathbb{E}(\tilde{Z}_\Delta^2 + \tilde{T}_\Delta^2) \\ &\leq \frac{2D_m}{n} \mathbb{E}(Z_\Delta^2 + T_\Delta^2) \leq 2(\sigma_0^2 + \xi_0^2) \frac{D_m}{n\Delta}. \end{aligned}$$

### 6.4.1 Proof of Lemma 3

*Result 3.* Let  $\beta$  be the Blumenthal-Gettoor index of  $L_t$ . Then:

$$\nu([-z, z]^c) \lesssim z^{-\beta} \quad , \quad \int_{|x| \leq z \wedge a_0} x^2 \nu(dx) \lesssim z^{2-\beta} \quad \text{and} \quad \int_{|x| \leq z \wedge a_0} x^4 \nu(dx) \lesssim z^{4-\beta}.$$

The constant  $a_0$  is defined in A6.

**Bound of  $\mathbb{P}(\Omega_{X,k}^c \cap (X_{k\Delta} \in A))$ .** We have:

$$\mathbb{P}(\Omega_{X,k}^c \cap (X_{k\Delta} \in A)) = \mathbb{P}(\{|X_{(k+1)\Delta} - X_{k\Delta}| > C_\Delta\} \cap (X_{k\Delta} \in A)).$$

We know that  $X_{(k+1)\Delta} - X_{k\Delta} = b(X_{k\Delta}) + I_{k\Delta} + Z_{k\Delta} + T_{k\Delta}$ . Then

$$\mathbb{P}(\Omega_{X,k}^c \cap (X_{k\Delta} \in A)) \leq \mathbb{P}(|\Delta I_{k\Delta}| \geq \Delta) + \mathbb{P}(|\Delta Z_{k\Delta}| \geq \sigma_0 \sqrt{\Delta} \ln(n)) + \mathbb{P}(|\Delta T_{k\Delta}| \geq \xi_0 \sqrt{\Delta} \ln(n)). \quad (6)$$

By a Markov inequality and Lemma 1, we obtain:

$$\mathbb{P}(|\Delta I_{k\Delta}| \geq \Delta) \leq \frac{\mathbb{E}(\Delta^2 I_{k\Delta}^2)}{\Delta^2} \lesssim \Delta. \quad (7)$$

By Proposition 2, the process  $\exp\left(c \int_0^t \sigma(X_{s-}) dW_s - c^2 \int_0^t \sigma^2(X_s) ds\right)$  is a local martingale (as  $\sigma$  is bounded, it is in fact a martingale, see Liptser and Shiryaev (2001), pp 229-232). Then, by a Markov inequality:

$$\mathbb{P}(|\Delta Z_{k\Delta}| \geq \sigma_0 \sqrt{\Delta} \ln(n)) \leq \frac{2}{n} \mathbb{E} \left[ \exp \left( \frac{\sqrt{\Delta} Z_{k\Delta}}{\sigma_0} \right) \right] \lesssim \frac{1}{n}. \quad (8)$$

To bound inequality (6), it remains to bound  $\mathbb{P}(|\Delta T_{k\Delta}| \geq \xi_0 \sqrt{\Delta} \ln(n))$ . Let us set

$$T_{k\Delta} = T_{k\Delta}^{(1)} + T_{k\Delta}^{(2)} + T_{k\Delta}^{(3)} \quad \text{where} \quad T_{k\Delta}^{(i)} = \frac{1}{\Delta} \int_{k\Delta}^{(k+1)\Delta} \xi(X_{s-}) dL_s^{(i)}$$

with

$$L_t^{(1)} = \int_0^t \int_{[-\sqrt{\Delta}, \sqrt{\Delta}]} z \tilde{\mu}(ds, dz) \quad , \quad L_t^{(2)} = \int_0^t \int_{[-\Delta^{1/4}, -\sqrt{\Delta}] \cup [\sqrt{\Delta}, \Delta^{1/4}]} z \tilde{\mu}(ds, dz)$$

$$L_t^{(3)} = \int_0^t \int_{[-\Delta^{1/4}, \Delta^{1/4}]^c} z \tilde{\mu}(ds, dz).$$

Let us set  $N_{k\Delta}'' = \mu\left(]k\Delta, (k+1)\Delta], [-\sqrt{\Delta}, \sqrt{\Delta}]^c\right)$ . By Result 3, we have:

$$\mathbb{P}\left(|T_{k\Delta}^{(2)} + T_{k\Delta}^{(3)}| > 0\right) = \mathbb{P}\left(N_{k\Delta}'' \geq 1\right) \lesssim \Delta \nu\left([- \sqrt{\Delta}, \sqrt{\Delta}]^c\right) \lesssim \Delta^{1-\beta/2}.$$

It remains to bound  $\mathbb{P} \left[ \left| \Delta T_{k\Delta}^{(1)} \right| \geq 2\xi_0 \sqrt{\Delta} \ln(n) \right]$ . We have that:

$$\mathbb{P} \left[ \left| \Delta T_{k\Delta}^{(1)} \right| \geq 2\xi_0 \sqrt{\Delta} \ln(n) \right] \leq 2\mathbb{P} \left[ \exp \left( \varepsilon \int_{k\Delta}^{(k+1)\Delta} \xi(X_{s-}) dL_s^{(1)} \right) \geq n^{2\varepsilon\xi_0\sqrt{\Delta}} \right].$$

By Proposition 2, for any  $\varepsilon$ ,

$$D_t := \exp \left( \varepsilon \int_{k\Delta}^t \xi(X_{s-}) dL_s^{(1)} - \int_{k\Delta}^t \int_{|z| \leq \sqrt{\Delta}} (\exp(\varepsilon z \xi(X_{s-}) - 1 - \varepsilon z \xi(X_{s-})) \nu(dz) \right)$$

is a local martingale. Let us set  $\varepsilon = 1/(2\xi_0\Delta^{1/2})$ . There exists an increasing sequence of stopping times  $\tau_N$  such that, for any  $N$ ,

$$\begin{aligned} F &:= \mathbb{P} \left[ \exp \left( \frac{1}{2\xi_0\Delta^{1/2}} \int_{k\Delta}^{(k+1)\Delta \wedge \tau_N} \xi(X_{s-}) dL_s^{(1)} \right) \geq n \right] \\ &\leq n^{-1} \exp \left( \int_{k\Delta}^{(k+1)\Delta \wedge \tau_N} \int_{|z| \leq \sqrt{\Delta}} \left( \exp \left( \frac{z\xi(X_{s-})}{2\xi_0\Delta^{1/2}} \right) - 1 - \frac{z\xi(X_{s-})}{2\xi_0\Delta^{1/2}} \right) \nu(dz) \right) \\ &\leq n^{-1} \exp \left( 2\Delta \int_{|z| \leq \sqrt{\Delta}} \frac{\xi_0^2 z^2}{4\xi_0^2 \Delta} \nu(dz) \right) \leq n^{-1} \exp \left( \int_{\mathbb{R}} z^2 \nu(dz) \right) \leq n^{-1}. \end{aligned}$$

When  $N \rightarrow \infty$ , by dominated convergence, we obtain:

$$\mathbb{P} \left( \left| \Delta T_{k\Delta}^{(1)} \right| \geq \xi_0 \sqrt{\Delta} \ln(n) \right) \lesssim n^{-1}. \quad (9)$$

**Bound of  $\mathbb{P} \left( \Omega_{X,k} \cap \Omega_{N,k}^c \cap (X_{k\Delta} \in A) \right)$ .** We recall that  $N'_{k\Delta} = \mu \left( ]k\Delta, (k+1)\Delta], [-\Delta^{1/4}, \Delta^{1/4}]^c \right)$ . We have:

$$\Omega_{N,k}^c = \left\{ N'_{k\Delta} = 1 \right\} \cup \left\{ N'_{k\Delta} \geq 2 \right\}$$

with

$$\mathbb{P} \left( N'_{k\Delta} = 1 \right) \lesssim \Delta^{1-\beta/4} \quad \text{and} \quad \mathbb{P} \left( N'_{k\Delta} \geq 2 \right) \lesssim \Delta^{2-\beta/2}.$$

Then  $\mathbb{P} \left( \Omega_{N,k}^c \cap \left\{ N'_{k\Delta} \geq 2 \right\} \right) \lesssim \Delta^{2-\beta/2}$ . We can write:

$$\begin{aligned} G &:= \mathbb{P} \left( \Omega_{X,k} \cap (X_{k\Delta} \in A) \cap (N'_{k\Delta} = 1) \right) \\ &\leq \mathbb{P} \left( N'_{k\Delta} = 1 \right) \mathbb{P} \left( \left| \Delta T_{k\Delta}^{(2)} + \Delta T_{k\Delta}^{(3)} \right| \leq 2C\Delta \mid N'_{k\Delta} = 1 \right) \\ &\quad + \mathbb{P} \left( N'_{k\Delta} = 1 \right) \mathbb{P} \left( \left\{ \left| \Delta T_{k\Delta}^{(2)} + \Delta T_{k\Delta}^{(3)} \right| \geq 2C\Delta \mid N'_{k\Delta} = 1 \right\} \cap \Omega_{X,k} \cap (X_{k\Delta} \in A) \right). \end{aligned}$$

By (7), (8) and (9), we obtain:

$$\begin{aligned} H &:= \mathbb{P} \left( \left\{ \left| \Delta T_{k\Delta}^{(2)} + \Delta T_{k\Delta}^{(3)} \right| \geq 2C\Delta \mid N'_{k\Delta} = 1 \right\} \cap \Omega_{X,k} \cap (X_{k\Delta} \in A) \right) \\ &\leq \mathbb{P} \left( \Delta \left| b_A(X_{k\Delta}) + I_{k\Delta} + Z_{k\Delta} + T_{k\Delta}^{(1)} \right| > C\Delta \right) \\ &\lesssim \Delta + n^{-1}. \end{aligned}$$

It remains to bound  $J := \mathbb{P}\left(\left|\Delta T_{k\Delta}^{(2)} + \Delta T_{k\Delta}^{(3)}\right| \leq 2C_\Delta |N'_{k\Delta} = 1\right)$ . If  $N'_{k\Delta} = 1$ , then  $\left|\Delta T_{k\Delta}^{(3)}\right| = \left|\int_{k\Delta}^{(k+1)\Delta} \xi(X_{s-}) dL_s^{(3)}\right| \geq \xi_1 \Delta^{1/4}$ . Then  $J \leq \mathbb{P}\left(\Delta \left|T_{k\Delta}^{(2)}\right| \geq \xi_1 \Delta^{1/4} - 2C_\Delta\right)$ . Let us set  $n_0 = \left\lceil \frac{1}{1-\beta/2} \right\rceil$  and  $a = (\xi_0 n_0)^{-1} (\xi_1 \Delta^{1/4} - 2C_\Delta)$ . We have:

$$\begin{aligned} J &\leq \mathbb{P}[\mu(|k\Delta, (k+1)\Delta], [-a, a]^c) \geq 1] + \mathbb{P}\left[\mu(|k\Delta, (k+1)\Delta], [-a, -\Delta^{1/2}] \cup [\Delta^{1/2}, a] \geq n_0\right] \\ &\leq \Delta \nu([-a, a]^c) + \Delta^{n_0} \nu([-\Delta^{1/2}, \Delta^{1/2}]^c)^{n_0} \\ &\lesssim \Delta^{1-\beta/4} + \Delta. \end{aligned}$$

Then  $\mathbb{P}(\Omega_{X,k} \cap \Omega_{N,k}^c) \leq \mathbb{P}(N'_{k\Delta} = 1) \Delta^{1-\beta/4} + \mathbb{P}(N'_{k\Delta} = 2) \lesssim \Delta^{2-\beta/2}$ .

**Bound of**  $\left(\mathbb{E}\left[(Z_{k\Delta} + T_{k\Delta}) \mathbb{1}_{\Omega_{X,k} \cap \Omega_{N,k} \cap (X_{k\Delta} \in A)} \middle| \mathcal{F}_{k\Delta}\right]\right)^2$ .

**If  $\sigma$  and  $\xi$  are constants.** Let us set  $E := \left(\mathbb{E}\left[(Z_{k\Delta} + T_{k\Delta}) \mathbb{1}_{\Omega_{X,k} \cap \Omega_{N,k} \cap (X_{k\Delta} \in A)} \middle| \mathcal{F}_{k\Delta}\right]\right)^2$  and

$$\Omega_{I,k} = \left\{\omega, |I_{k\Delta}| \leq 1, \cap |\Delta Z_{k\Delta}| \leq \sigma_0 \sqrt{\Delta} \ln(n), \cap \left|\Delta T_{k\Delta}^{(1)}\right| \leq 2\xi_0 \sqrt{\Delta} \ln(n)\right\}.$$

By (7), (8) and (9),  $\mathbb{P}\left(\Omega_{I,k}^c\right) \leq \Delta + n^{-1}$ . Then, by a Markov inequality:

$$E \lesssim \Delta \ln^2(n) + \left(\mathbb{E}\left[(Z_{k\Delta} + T_{k\Delta}) \mathbb{1}_{\Omega_{X,k} \cap \Omega_{N,k} \cap \Omega_{I,k} \cap (X_{k\Delta} \in A)} \middle| \mathcal{F}_{k\Delta}\right]\right)^2.$$

Let us introduce the set  $\Omega_{ZT,k} := \{\omega, |Z_{k\Delta} + T_{k\Delta}| \leq C_\Delta \Delta^{-1} - b_{max} - 1\}$ . On  $\Omega_{I,k}$ ,  $|I_{k\Delta}| \leq 1$  and therefore:

$$\Omega_{ZT,k} \cap \Omega_{I,k} \subseteq \Omega_{X,k} \cap \Omega_{I,k} \subseteq \{\omega, |Z_{k\Delta} + T_{k\Delta}| \leq C_\Delta \Delta^{-1} + b_{max} + 1\} \cap \Omega_{I,k}.$$

Then

$$E \lesssim \Delta \ln^2(n) + F^2 + G^2$$

where  $F = \mathbb{E}\left[(Z_{k\Delta} + T_{k\Delta}) \mathbb{1}_{\Omega_{ZT,k} \cap \Omega_{N,k} \cap \Omega_{I,k} \cap (X_{k\Delta} \in A)} \middle| \mathcal{F}_{k\Delta}\right]$  and  $G = \mathbb{E}\left[(Z_{k\Delta} + T_{k\Delta}) \mathbb{1}_{\Omega_{ZT,k}^c \cap \Omega_{X,k} \cap \Omega_{N,k} \cap \Omega_{I,k} \cap (X_{k\Delta} \in A)} \middle| \mathcal{F}_{k\Delta}\right]$ . As  $\sigma$  and  $\xi$  are constants, the terms

$$Z_{k\Delta} = \frac{\sigma_0}{\Delta} \int_{k\Delta}^{(k+1)\Delta} dW_s \quad \text{and} \quad T_{k\Delta} = \frac{\xi_0}{\Delta} \int_{k\Delta}^{(k+1)\Delta} dL_s$$

are centered and independent. Then  $F = 0$ . Moreover, on  $\Omega_{N,k}$ ,  $T_{k\Delta}^{(3)} = 0$ . Then

$$|G| \lesssim \left|\mathbb{E}\left[\left(Z_{k\Delta} + T_{k\Delta}^{(1)} + T_{k\Delta}^{(2)}\right) \mathbb{1}_{\Omega_{X,k} \cap \Omega_{ZT,k}^c \cap \Omega_{N,k} \cap \Omega_{I,k} \cap (X_{k\Delta} \in A)} \middle| \mathcal{F}_{k\Delta}\right]\right|.$$

On  $\Omega_{I,k} \cap \Omega_{X,k}$ ,  $|Z_{k\Delta} + T_{k\Delta}^{(1)} + T_{k\Delta}^{(2)}| \lesssim \ln(n)\Delta^{-1/2}$ , and

$$\begin{aligned} |G| &\lesssim \frac{\ln(n)}{\sqrt{\Delta}} \left( \mathbb{P} \left( |Z_{k\Delta} + T_{k\Delta}^{(1)} + T_{k\Delta}^{(2)}| \in [C_\Delta \Delta^{-1} - b_{max} - 1, C_\Delta \Delta^{-1} + b_{max} + 1] \mathbb{1}_{\Omega_{I,k}} \right) \right) \\ &= 2 \frac{\ln(n)}{\sqrt{\Delta}} \int_{\mathbb{R}} \mathbb{P} \left( T_{k\Delta}^{(2)} \in [C_\Delta \Delta^{-1} - b_{max} - 1 - x, C_\Delta \Delta^{-1} + b_{max} + 1 - x] \mathbb{1}_{\Omega_{I,k}} \right) \\ &\quad \times \mathbb{P} \left( Z_{k\Delta} + T_{k\Delta}^{(1)} \in dx \mid T_{k\Delta}^{(2)} \in [C_\Delta \Delta^{-1} - b_{max} - 1 - x, C_\Delta \Delta^{-1} + b_{max} + 1 - x] \mathbb{1}_{\Omega_{I,k}} \right). \end{aligned}$$

On  $\Omega_{I,k}$ ,  $|Z_{k\Delta} + T_{k\Delta}^{(1)}| \leq (\sigma_0 + 2\xi_0) \ln(n)\Delta^{-1/2}$ . Then

$$|G| \lesssim \frac{\ln(n)}{\sqrt{\Delta}} \left[ \sup_{C \geq \xi_0 \ln(n)\Delta^{-1/2}} \mathbb{P} \left( T_{k\Delta}^{(2)} \in [C, C + 2b_{max} + 2] \right) \right]. \quad (10)$$

We recall that  $L_t^{(2)}$  is a compound Poisson process in which all the jumps are greater than  $\sqrt{\Delta}$  and smaller than  $\Delta^{1/4}$ . Let us denote by  $\tau_i$  the times of the jumps of size in  $[\sqrt{\Delta}, \Delta^{1/4}]$  and by  $\zeta_i$  the size of the jumps. We set  $a_j = \xi_0^{-1} C \Delta - \sum_{i=1}^{j-1} \zeta_i$  and  $c := \xi_0^{-1} (2b_{max} + 2)$ . Then, as  $\xi$  is constant equal to  $\xi_0$ :

$$\begin{aligned} H &:= \mathbb{P} \left( T_{k\Delta}^{(2)} \in [C, C + 2b_{max} + 2] \right) \\ &\leq \sum_{j=1}^{\infty} \mathbb{P} \left( j \text{ jumps} \geq \sqrt{\Delta}, \text{ last jump} \in [a_j, a_j + c\Delta] \right) \\ &\lesssim 2 \sup_{a \geq \sqrt{\Delta}} \mathbb{P} (1 \text{ jump} \in [a, a + c\Delta]) = 2\Delta \sup_{a \geq \sqrt{\Delta}} \nu([a, a + c\Delta]). \end{aligned}$$

By A6,

$$H \lesssim \Delta \sup_{a \geq \sqrt{\Delta}} \left[ \frac{1}{a^\beta} - \frac{1}{(a + c\Delta)^\beta} \right] \lesssim \sqrt{\Delta} \Delta^{1-\beta/2} \quad (11)$$

and, by (10) and (11),

$$E \lesssim \Delta \ln^2(n) + \frac{\ln^2(n)}{\Delta} \Delta \Delta^{2-\beta} \lesssim \Delta \ln^2(n) + \Delta^{2-\beta} \ln^2(n).$$

*Remark 4.* If  $\nu$  is not absolutely continuous, we obtain:

$$E \leq \Delta \ln^2(n) + (\Delta \nu([-a + c\Delta, a - c\Delta]^c))^2 \lesssim \Delta \ln^2(n) + \Delta^{2-2\beta} \ln^2(n).$$

**If  $\sigma$  or  $\xi$  are not constants.** The problem is that  $Z_{k\Delta}$  and  $T_{k\Delta}$  are not symmetric and we can't apply directly the previous method. We replace them by two centred terms. The following lemma is very useful.



**Lemma 4.**

Let  $f$  be a  $\mathcal{C}^2$  function such that  $f$  and  $f'$  are Lipschitz. Let us set, for any  $t \in ]k\Delta, (k+1)\Delta[$ :

$$\psi_f(X_{k\Delta}, t) = f'(X_{k\Delta}) \left( \sigma(X_{k\Delta}) \int_{k\Delta}^t dW_s + \xi(X_{k\Delta}) \int_{k\Delta}^t z \tilde{\mu}(ds, dz) \right).$$

We have:

$$\mathbb{E} \left[ (f(X_t) - f(X_{k\Delta}) - \psi_f(X_{k\Delta}, t))^2 \mathbb{1}_{\Omega_{N,k}} \mathbb{1}_{X_{k\Delta} \in A} \right] \lesssim \Delta^{2-\beta/4}.$$

Lemma 4 is proved below. Let us set

$$\bar{Z}_{k\Delta} = \frac{1}{\Delta} \int_{k\Delta}^{(k+1)\Delta} (\sigma(X_{k\Delta}) + \psi_\sigma(X_{k\Delta}, s)) dW_s,$$

$$\bar{T}_{k\Delta}^{(i)} = \frac{1}{\Delta} \int_{k\Delta}^{(k+1)\Delta} (\xi(X_{k\Delta}) + \psi_\xi(X_{k\Delta}, s)) dL_s^{(i)} \quad \text{and} \quad \bar{T}_{k\Delta} = \bar{T}_{k\Delta}^{(1)} + \bar{T}_{k\Delta}^{(2)} + \bar{T}_{k\Delta}^{(3)}.$$

The terms  $\bar{Z}_{k\Delta}$  and  $\bar{T}_{k\Delta}$  are symmetric. By lemma 4,

$$\begin{aligned} \mathbb{E} \left[ (\bar{Z}_{k\Delta} - Z_{k\Delta})^2 \mathbb{1}_{\Omega_{N,k}} \mathbb{1}_{X_{k\Delta} \in A} \right] &= \frac{1}{\Delta^2} \mathbb{E} \left[ \int_{k\Delta}^{(k+1)\Delta} (\sigma(X_s) - \sigma(X_{k\Delta}) - \psi_\sigma(X_{k\Delta}, s))^2 ds \right] \\ &\lesssim \Delta^{1-\beta/4}. \end{aligned} \tag{12}$$

We prove in the same way that

$$\mathbb{E} \left[ (\bar{T}_{k\Delta} - T_{k\Delta})^2 \mathbb{1}_{\Omega_{N,k}} \mathbb{1}_{X_{k\Delta} \in A} \right] \leq \Delta^{1-\beta/4}. \tag{13}$$

Let us set  $U_{k\Delta} = \Delta^{-1} \xi(X_{k\Delta-}) \int_{k\Delta}^{(k+1)\Delta} dL_s^{(2)}$ . By Result 1 and Proposition 1,

$$\mathbb{E} \left[ \Delta^2 \left( \bar{T}_{k\Delta}^{(2)} - U_{k\Delta} \right)^2 \right] = \mathbb{E} \left[ \int_{k\Delta}^{(k+1)\Delta} \int_{\mathbb{R}} (\xi(X_s) - \xi(X_{k\Delta}))^2 z^2 \nu(dz) ds \right] \leq \Delta^2. \tag{14}$$

Let us introduce the set

$$\begin{aligned} \bar{\Omega}_{I,k} &= \{ \omega, |I_{k\Delta}| + |Z_{k\Delta} - \bar{Z}_{k\Delta}| + |T_{k\Delta} - \bar{T}_{k\Delta}| \leq 3 \} \\ &\cap \left\{ |\Delta \bar{Z}_{k\Delta}| \leq \sigma_0 \sqrt{\Delta} \ln(n) + \Delta, \left| \Delta \bar{T}_{k\Delta}^{(1)} \right| \leq 2\xi_0 \sqrt{\Delta} \ln(n) + \Delta \right\} \\ &\cap \left\{ \left| \Delta (\bar{T}_{k\Delta}^{(2)} - U_{k\Delta}) \right| \leq \xi_0 \sqrt{\Delta} \right\}. \end{aligned}$$

By (7), (8), (9), (12), (13), (14) and Markov inequalities, we obtain:

$$\mathbb{P}(\bar{\Omega}_{I,k}^c) \lesssim \Delta^{1-\beta/4} + \frac{1}{n}. \tag{15}$$

Then

$$\begin{aligned} E &:= \left( \mathbb{E} \left[ (Z_{k\Delta} + T_{k\Delta}) \mathbb{1}_{\Omega_{X,k} \cap \Omega_{N,k} \cap (X_{k\Delta} \in A)} \middle| \mathcal{F}_{k\Delta} \right] \right)^2 \\ &\lesssim \Delta^{1-\beta/2} \ln^2(n) + \left( \mathbb{E} \left[ (\bar{Z}_{k\Delta} + \bar{T}_{k\Delta}) \mathbb{1}_{\Omega_{X,k} \cap \Omega_{N,k} \cap (X_{k\Delta} \in A) \cap \bar{\Omega}_{I,k}} \middle| \mathcal{F}_{k\Delta} \right] \right)^2. \end{aligned} \quad (16)$$

Let us introduce the set:

$$\bar{\Omega}_{ZT,k} := \{ \omega, |\bar{Z}_{k\Delta} + \bar{T}_{k\Delta}| \leq C_\Delta \Delta^{-1} - b_{max} - 3 \}.$$

We have that

$$\bar{\Omega}_{ZT,k} \cap \bar{\Omega}_{I,k} \subseteq \Omega_{X,k} \cap \bar{\Omega}_{I,k} \subseteq \{ \omega, |\bar{Z}_{k\Delta} + \bar{T}_{k\Delta}| \leq C_\Delta \Delta^{-1} + b_{max} + 3 \} \cap \bar{\Omega}_{I,k}.$$

Given the filtration  $\mathcal{F}_{k\Delta}$ , the sum  $\bar{Z}_{k\Delta} + \bar{T}_{k\Delta}$  is symmetric. Then

$$\mathbb{E} \left[ (\bar{Z}_{k\Delta} + \bar{T}_{k\Delta}) \mathbb{1}_{\bar{\Omega}_{ZT,k} \cap \Omega_{N,k} \cap (X_{k\Delta} \in A)} \middle| \mathcal{F}_{k\Delta} \right] = 0.$$

Moreover, on  $\Omega_{N,k}$ ,  $\bar{T}_{k\Delta}^{(3)} = 0$ . Then, by (16),

$$E \lesssim \Delta^{1-\beta/2} \ln^2(n) + G^2 + H^2$$

where  $G := \mathbb{E} \left[ \left( \bar{Z}_{k\Delta} + \bar{T}_{k\Delta}^{(1)} + \bar{T}_{k\Delta}^{(2)} \right) \mathbb{1}_{\Omega_{X,k} \cap \Omega_{ZT,k}^c \cap \Omega_{N,k} \cap \Omega_{I,k} \cap (X_{k\Delta} \in A)} \middle| \mathcal{F}_{k\Delta} \right]$  and  $H := \mathbb{E} \left[ \left( \bar{Z}_{k\Delta} + \bar{T}_{k\Delta}^{(1)} + \bar{T}_{k\Delta}^{(2)} \right) \mathbb{1}_{\Omega_{X,k} \cap \Omega_{ZT,k} \cap \Omega_{N,k} \cap \Omega_{I,k}^c \cap (X_{k\Delta} \in A)} \middle| \mathcal{F}_{k\Delta} \right]$ . We have that  $H^2 \lesssim \Delta^{-1} \ln^2(n) \mathbb{P}^2(\Omega_{I,k}^c) \lesssim \Delta^{1-\beta/2} \ln^2(n)$ . The end of the proof is the same as in the case of  $\sigma$  and  $\xi$  constants. We obtain that

$$|G| \lesssim \frac{\ln(n)}{\sqrt{\Delta}} \sup_{C \geq \kappa_0 \ln(n) \Delta^{-1/2}} \mathbb{P}(U_{k\Delta} \in [C, C + 2b_{max} + 6]) \lesssim \sqrt{\Delta} \Delta^{1-\beta/2}.$$

#### 6.4.2 Proof of Lemma 4

According to the Ito formula (see for instance Applebaum (2004), Theorem 4.4.7 p251), we have that

$$f(X_t) - f(X_{k\Delta}) = I_1 + I_2 + I_3 + I_4$$

where

$$\begin{aligned} I_1 &= \int_{k\Delta}^t f'(X_s) \sigma(X_s) dW_s & I_2 &= \int_{k\Delta}^t \int_{\mathbb{R}} (f(X_{s-} + z\xi(X_{s-})) - f(X_{s-})) \tilde{\mu}(ds, dz) \\ I_3 &= \int_{k\Delta}^t \int_{z \in \mathbb{R}} [f(X_s + z\xi(X_s)) - f(X_s) - z\xi(X_s) f'(X_s)] \nu(dz) ds \\ I_4 &= \int_{k\Delta}^t [f'(X_s) b(X_s) + f''(X_s) \sigma^2(X_s)/2] ds. \end{aligned}$$

By Proposition 1, for any  $t \leq (k+1)\Delta$ , we have:

$$\begin{aligned} \mathbb{E} \left[ \left( I_1 - f'(X_{k\Delta})\sigma(X_{k\Delta}) \int_{k\Delta}^t dW_s \right)^2 \right] &= \mathbb{E} \left[ \left( \int_{k\Delta}^t (\sigma(X_s)f'(X_s) - \sigma(X_{k\Delta})f'(X_{k\Delta})) dW_s \right)^2 \right] \\ &= \int_{k\Delta}^t (\sigma(X_s)f'(X_s) - \sigma(X_{k\Delta})f'(X_{k\Delta}))^2 ds \lesssim \Delta^2. \end{aligned}$$

We can write:

$$\begin{aligned} E &:= \mathbb{E} \left[ \left( I_2 - f'(X_{k\Delta})\xi(X_{k\Delta-}) \int_{k\Delta}^t dL_s^{(1)} + dL_s^{(2)} \right)^2 \mathbb{1}_{\Omega_{N,k}} \right] \\ &\leq 2 \int_{k\Delta}^t \int_{|z| \leq \Delta^{1/4}} \mathbb{E} \left[ (f(X_s + z\xi(X_s)) - f(X_s) - z\xi(X_s)f'(X_s))^2 \right] \nu(dz) ds \\ &\quad + 2 \int_{k\Delta}^t \int_{|z| \leq \Delta^{1/4}} \mathbb{E} \left[ z^2 (\xi(X_s)f'(X_s) - \xi(X_{k\Delta})f'(X_{k\Delta}))^2 \right] \nu(dz) ds. \end{aligned}$$

The function  $f$  is  $\mathcal{C}^2$ , then, by the Taylor formula, for any  $s \in [k\Delta, t]$ ,  $z \in \mathbb{R}$ , there exists  $\zeta_{s,z}$  in  $[X_s, X_s + z\xi(X_s)]$  such that:

$$f(X_s + z\xi(X_s)) - f(X_s) - z\xi(X_s)f'(X_s) = \frac{z^2\xi^2(X_s)}{2} f''(\zeta_{s,z}).$$

Then, as  $\xi$  and  $f''$  are bounded:

$$\mathbb{E} \left[ (f(X_s + z\xi(X_s)) - f(X_s) - z\xi(X_s)f'(X_s))^2 \right] = \frac{z^4}{4} \mathbb{E} \left[ (\xi(X_s)f''(\zeta_{s,z}))^2 \right] \lesssim z^4$$

and, by Result 3, for any  $t \leq (k+1)\Delta$ ,

$$\begin{aligned} F &:= \int_{k\Delta}^t \int_{|z| \leq \Delta^{1/4}} \mathbb{E} \left[ (f(X_s + z\xi(X_s)) - f(X_s) - z\xi(X_s)f'(X_s))^2 \right] \nu(dz) ds \\ &\lesssim \Delta \int_{|z| \leq \Delta^{1/4}} z^4 \nu(dz) \lesssim \Delta^{2-\beta/4}. \end{aligned}$$

The functions  $\xi$  and  $f'$  are Lipschitz, then by Proposition 1,

$$\mathbb{E} \left[ z^2 (\xi(X_s)f'(X_s) - \xi(X_{k\Delta})f'(X_{k\Delta}))^2 \right] \lesssim z^2 \mathbb{E} \left[ (X_s - X_{k\Delta})^2 \right] \lesssim \Delta z^2$$

and consequently, for any  $t \leq (k+1)\Delta$ :

$$\int_{k\Delta}^t \int_{|z| \leq \Delta^{1/4}} \mathbb{E} \left[ z^2 (\xi(X_s)f'(X_s) - \xi(X_{k\Delta})f'(X_{k\Delta}))^2 \right] \nu(dz) ds \lesssim \Delta^{3-\beta/2}$$

then  $E \lesssim \Delta^{2-\beta/4}$ . By the same way, we obtain that

$$\mathbb{E} [I_3^2] \leq \mathbb{E} \left[ \int_{k\Delta}^t \int_{|z| \leq \Delta^{1/4}} \left( \frac{z^2\xi^2(X_s)}{2} f''(\zeta_{s,z}) \right)^2 \nu(dz) ds \right] \leq \Delta^{2-\beta/4}.$$

The functions  $b$  and  $f'$  are Lipschitz and  $f''$  and  $\sigma$  are bounded, then, for any  $t \leq (k+1)\Delta$  :

$$\mathbb{E} [I_4^2] \lesssim \Delta \int_{k\Delta}^t (1 + \mathbb{E} [X_s^4]) ds \lesssim \Delta^2.$$

Then, for any  $t \leq (k+1)\Delta$ :

$$\mathbb{E} [(f(X_t) - f(X_{k\Delta}) - \psi_f(X_{k\Delta}, t))] \leq \Delta^{2-\beta/4}.$$

## 6.5 Proof of Theorem 4

As previously, we only bound the risk on  $\Omega_n$ . As in Subsection 6.3, we introduce the function  $p(m, m')$  such that  $p(m, m') = 12(\text{pen}(m) + \text{pen}(m'))$ . On  $\Omega_n$ , for any  $m \in \mathcal{M}_n$ , we have:

$$\begin{aligned} \left\| \tilde{b}_{\tilde{m}} - b_A \right\|_n^2 &\leq 3 \|b_m - b_A\|_n^2 + \frac{224}{n} \sum_{k=1}^n b_A^2(X_{k\Delta}) \mathbb{1}_{\Omega_{X,k}^c} + I_{k\Delta}^2 + 2(Z_{k\Delta}^2 + T_{k\Delta}^2) \mathbb{1}_{\Omega_{X,k} \cap \Omega_{Z,k}^c} \\ &+ \frac{224}{n} \sum_{k=1}^n (\mathbb{E} [(Z_{k\Delta} + T_{k\Delta}) \mathbb{1}_{\Omega_{X,k} \cap \Omega_{Z,k}} | \mathcal{F}_{k\Delta}])^2 \\ &+ 24 \sup_{t \in \mathcal{B}_{m, \tilde{m}}} (\tilde{\nu}_n^2(t) - p(m, \tilde{m})) + 4\text{pen}(m). \end{aligned}$$

It remains only to bound  $\mathbb{E} \left[ \sup_{t \in \mathcal{B}_{m, \tilde{m}}} (\tilde{\nu}_n^2(t) - p(m, \tilde{m})) \right] \leq \sum_{m'} \mathbb{E} \left[ \sup_{t \in \mathcal{B}_{m, m'}} (\tilde{\nu}_n^2(t) - p(m, \tilde{m})) \right]$ . As in the proof of Theorem 2, we bound the quantity

$$\mathbb{E} \left[ \exp \left( \varepsilon t (X_{k\Delta}) \left( \tilde{Z}_{k\Delta} + \tilde{T}_{k\Delta} \right) \right) \middle| \mathcal{F}_{k\Delta} \right].$$

We have that

$$\mathbb{E} \left[ \exp \left( \varepsilon t (X_{k\Delta}) Z_{k\Delta} \right) \mathbb{1}_{\Omega_{N,k}} \middle| \mathcal{F}_{k\Delta} \right] \leq \exp \left( \frac{\varepsilon^2 \sigma_0^2 t^2 (X_{k\Delta})}{2\Delta} \right).$$

The truncated Lévy process  $\tilde{L}_t = \int_0^t \int_{|z| \leq \Delta^{1/4}} z \tilde{\mu}(ds, dz)$  satisfies Assumption A5 and then there exists a constant  $c$  such that:

$$\mathbb{E} \left[ \exp \left( \varepsilon t (X_{k\Delta}) T_{k\Delta} \right) \mathbb{1}_{\Omega_{N,k}} \middle| \mathcal{F}_{k\Delta} \right] \leq \exp \left( \frac{c\varepsilon^2 \xi_0^2 t^2 (X_{k\Delta})}{\Delta (1 - \varepsilon/\varepsilon_1)} \right).$$

As  $Z_{k\Delta} \mathbb{1}_{\Omega_{N,k}}$  and  $T_{k\Delta} \mathbb{1}_{\Omega_{N,k}}$  are centred, we obtain:

$$\mathbb{E} \left[ \exp \left( \varepsilon |t(X_{k\Delta}) (Z_{k\Delta} + T_{k\Delta})| \right) \mathbb{1}_{\Omega_{N,k}} \middle| \mathcal{F}_{k\Delta} \right] \leq 2 \exp \left( \frac{c\varepsilon^2 (\sigma_0^2 + \xi_0^2) t^2 (X_{k\Delta})}{\Delta (1 - \varepsilon/\varepsilon_1)} \right)$$

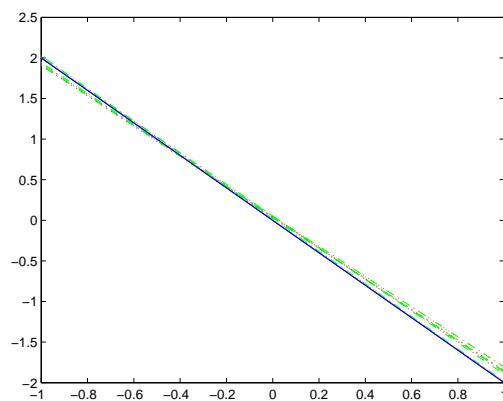
and then

$$\mathbb{E} \left[ \exp \left( \varepsilon \left| t(X_{k\Delta}) \left( \tilde{Z}_{k\Delta} + \tilde{T}_{k\Delta} \right) \right| \right) \mathbb{1}_{\Omega_{N,k} \cap \Omega_{X,k}} \middle| \mathcal{F}_{k\Delta} \right] \leq 2 \exp \left( \frac{c\varepsilon^2 (\sigma_0^2 + \xi_0^2) t^2 (X_{k\Delta})}{\Delta (1 - \varepsilon/\varepsilon_1)} \right).$$

We conclude as in the proof of Theorem 2.

Figure 1: Model 1: Ornstein-Uhlenbeck and binomial law

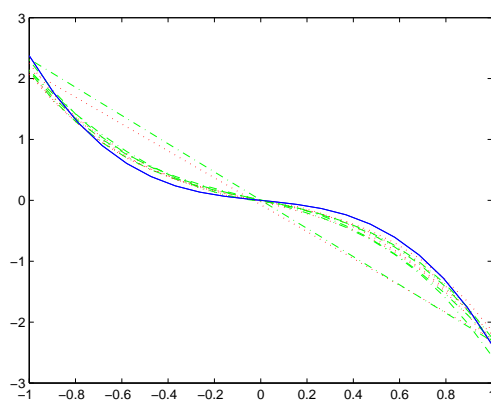
$$b(x) = -2x, \sigma(x) = \xi(x) = 1 \text{ and binomial law}$$



- : true function    -.-: first estimator    ...: truncated estimator  
 $n = 10^4$  et  $\Delta = 10^{-1}$

Figure 2: Model 2: Double well and Laplace law

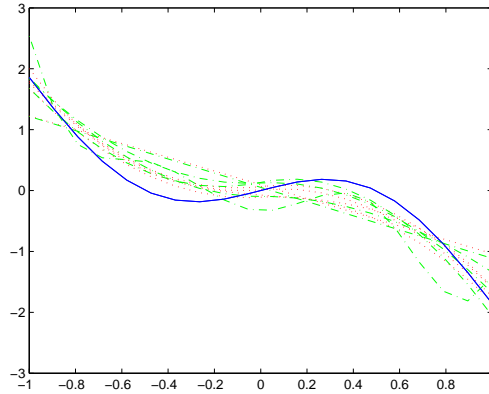
$$b(x) = -(x - 1/4)^3 - (x + 1/4)^3, \quad \sigma = \xi = 1 \text{ and Laplace law}$$



- : true function    -.-: first estimator    ...: truncated estimator  
 $n = 10^4$  et  $\Delta = 10^{-1}$

Figure 3: Model 3: Sine function

$$b(x) = -2x + \sin(3x), \sigma(x) = \xi(x) = \sqrt{(3+x^2)/(1+x^2)} \text{ jumps not sub-exponential}$$



— : true function    -.-: first estimator    ...: truncated estimator  
 $n = 10^4$  et  $\Delta = 10^{-1}$

Table 1: Model 1: Ornstein-Uhlenbeck and binomial law

$$b(x) = -2x, \sigma(x) = \xi(x) = 1 \text{ and compound Poisson process (binomial law)}$$

$n$	$\Delta$	first estimator				truncated estimator			
		$\hat{m}_a$	$\hat{r}_a$	$risk_1$	$or_1$	$\tilde{m}_a$	$\tilde{r}_a$	$risk_2$	$or_2$
$10^3$	$10^{-1}$	0	1.02	0.044	1.3	0	1.02	0.044	1.3
$10^4$	$10^{-1}$	0	1.02	0.011	1.3	0	1.02	0.011	1.3
$10^3$	$10^{-2}$	0	1.02	0.55	1.04	0	1.02	0.55	1.04
$10^4$	$10^{-2}$	0	1	0.047	1	0	1	0.047	1
$5 \cdot 10^4$	$10^{-2}$	0.04	1	0.010	1.4	0	1	0.0053	1

$\hat{m}_a, \hat{r}_a$  and  $\tilde{m}_a, \tilde{r}_a$  : average values of  $\hat{m}, \hat{r}$  and  $\tilde{m}, \tilde{r}$  on the 50 simulations.  
 $risk_1$  and  $risk_2$  : means of the empirical errors of the adaptive estimators.  
 $or_1$  and  $or_2$  : means of *oracle* = empirical error of the adaptive estimator / empirical error of the best possible estimator.

Table 2: Model 2: Double well and Laplace law  
 $b(x) = -(x - 1/4)^3 - (x + 1/4)^3$ ,  $\sigma(x) = \xi(x) = 1$  and Laplace law.

$n$	$\Delta$	first estimator				truncated estimator			
		$\hat{m}_a$	$\hat{r}_a$	$risk_1$	$or_1$	$\tilde{m}_a$	$\tilde{r}_a$	$risk_2$	$or_2$
$10^3$	$10^{-1}$	0.02	1.0	0.12	3.1	0.02	1.0	0.12	3.1
$10^4$	$10^{-1}$	1.7	2.1	$2.10^{96}$	51	0.4	2.1	0.04	1.5
$10^3$	$10^{-2}$	0.26	1.2	1.8	3.1	0.06	1	0.51	1.4
$10^4$	$10^{-2}$	0.12	1.5	0.16	1.8	0.08	1.2	0.13	2.4
$5.10^4$	$10^{-2}$	0.30	2.5	0.035	1.6	0.26	2.5	0.019	1.8

$\hat{m}_a, \hat{r}_a$  and  $\tilde{m}_a, \tilde{r}_a$  : average values of  $\hat{m}, \hat{r}$  and  $\tilde{m}, \tilde{r}$  on the 50 simulations.  
 $risk_1$  and  $risk_2$  : means of the empirical errors of the adaptive estimators.  
 $or_1$  and  $or_2$ : means of *oracle* =empirical error of the adaptive estimator / empirical error of the best possible estimator.

Table 3: Model 3: Sine function and jumps not sub-exponential

$$b(x) = -2x + \sin(3x), \quad \sigma(x) = \xi(x) = \sqrt{(3 + x^2)/(1 + x^2)} \quad \text{and} \quad \nu(z) \propto e^{-\sqrt{az}}/\sqrt{z}$$

$n$	$\Delta$	first estimator				truncated estimator			
		$\hat{m}_a$	$\hat{r}_a$	$risk_1$	$or_1$	$\tilde{m}_a$	$\tilde{r}_a$	$risk_2$	$or_2$
$10^3$	$10^{-1}$	0.34	1.2	0.76	3.6	0.04	1.2	0.28	1.9
$10^4$	$10^{-1}$	0.8	2.2	0.082	1.3	0.68	2.2	0.073	1.2
$10^3$	$10^{-2}$	0.96	1.2	18	6.3	0.02	1.2	1.3	1.2
$10^4$	$10^{-2}$	0.78	1.4	1.5	4.3	0.12	1.4	0.24	3.3
$5.10^4$	$10^{-2}$	0.92	2.3	0.24	4.3	0.70	2.3	0.039	1.3

$\hat{m}_a, \hat{r}_a$  and  $\tilde{m}_a, \tilde{r}_a$  : average values of  $\hat{m}, \hat{r}$  and  $\tilde{m}, \tilde{r}$  on the 50 simulations.  
 $risk_1$  and  $risk_2$  : means of the empirical errors of the adaptive estimators.  
 $or_1$  and  $or_2$ : means of *oracle* =empirical error of the adaptive estimator / empirical error of the best possible estimator.

## 7 Auxiliary proofs

### 7.1 Decomposition on a lattice

**Proposition 3.**

If there exist some constants  $c_1, c_2$  and  $K$  independent of  $D, n, \Delta, b$  and  $\sigma$  and two constants  $\alpha$  and  $\beta$  independent of  $n$  and  $D$  such that, for any function  $t \in S_m + S'_m$ :

$$\forall \eta, \zeta > 0, \forall t \in S_m + S_{m'} \quad \|t\|_\infty \leq C\zeta, \mathbb{P} \left( f_n(t) \geq \eta, \|t\|_n^2 \leq \zeta^2 \right) \leq K \exp \left( -\frac{\eta^2 n \beta}{(c_1 \alpha^2 \zeta^2 + 2C c_2 \alpha \eta \zeta)} \right),$$

then there exist some constants  $C$  and  $\kappa$  depending only of  $\nu$  such that, if  $D \leq n\beta$ :

$$\mathbb{E} \left[ \sup_{t \in \mathcal{B}_{m,m'}} f_n^2(t) - \frac{\kappa \alpha^2 D}{n\beta} \right]_+ \leq CK \frac{\kappa \alpha^2 D^{3/2} e^{-\sqrt{D}}}{n\beta}.$$

Let us consider an orthonormal (for the  $L^2_\sigma$ -norm) basis  $(\psi_\lambda)_{\lambda \in \Lambda_{m,m'}}$  of  $S_{m,m'} = S_m + S_{m'}$  such that

$$\forall \lambda, \quad \text{card}(\{\lambda', \|\psi_\lambda \psi_{\lambda'}\| \neq 0\}) \leq \phi_2.$$

Let us set

$$\bar{r}_{m,m'} = \frac{1}{\sqrt{D}} \sup_{\beta \neq 0} \frac{\|\sum_\lambda \beta_\lambda \psi_\lambda\|_\infty}{|\beta|_\infty}.$$

We obtain that

$$\left\| \sum_\lambda \beta_\lambda \psi_\lambda \right\|_\infty \leq \phi_2 |\beta|_\infty \sup_\lambda \|\psi_\lambda\|_\infty \quad \text{et} \quad \|\psi_\lambda\|_\infty \leq \sqrt{D} \|\psi_\lambda\|_{L^2} \leq \pi_1 \sqrt{D} \|\psi_\lambda\|_\sigma$$

then

$$\bar{r}_{m,m'} \leq \bar{r} := \phi_2 \pi_1.$$

We need a lattice of which the infinite norm is bounded. We use Lemma 9 of Barron *et al.* (1999):

*Result 4.* There exists a  $\delta_k$ -lattice  $T_k$  of  $L^2_\sigma \cap (S_m + S_{m'})$  such that

$$|T_k \cap \mathcal{B}_{m,m'}| \leq (5/\delta^k)^D$$

where  $\delta_k = 2^{-k}/5$ . Let us denote by  $p_k(u)$  the orthogonal projection of  $u$  on  $T_k$ . For any  $u \in S_{m,m'}$ ,  $\|u - p_k(u)\|_\pi \leq \delta_k$  and

$$\sup_{u \in p_k^{-1}(t)} \|u - t\|_\infty \leq \bar{r}_{m,m'} \delta_k \leq \bar{r} \delta_k.$$



Let us set  $H_k = \ln(|T_k \cap \mathcal{B}_{m,m'}|)$ . We have that:

$$H_k \leq D \ln(5/\delta_k) = D(k \ln(2) + \ln(5/\delta_0)) \leq C(k+1)D.$$

The decomposition of  $u_k$  on the  $\delta_k$ -lattice must be done very carefully: the norms  $\|u_k - u_{k-1}\|_{\varpi}$  and  $\|u_k - u_{k-1}\|_{\infty}$  must be controlled. Let us set

$$\mathcal{E}_k = \{u_k \in T_k \cap \mathcal{B}_{m,m'}, \quad \|u - u_k\|_{\varpi} \leq \delta_k \quad \text{et} \quad \|u - u_k\|_{\infty} \leq \bar{r}\delta_k\}.$$

We have that  $\ln(|\mathcal{E}_k|) \leq H_k$ . For any function  $u \in \mathcal{B}_{m,m'}$ , there exist a series  $(u_k)_{k \geq 0} \in \prod_k \mathcal{E}_k$  such that

$$u = u_0 + \sum_{k=1}^{\infty} (u_k - u_{k-1}).$$

Let us consider  $(\eta_k)_{k \geq 0}$  and  $\eta \in \mathbb{R}$  such that  $\eta_0 + \sum_{k=1}^{\infty} \eta_k \leq \eta$ . We obtain:

$$\begin{aligned} \mathbb{P}\left(\sup_{u \in \mathcal{B}_{m,m'}} |f_n(u)| > \eta\right) &\leq \mathbb{P}\left(\exists (u_k) \in \prod \mathcal{E}_k, \left|f_n(u_0) + \sum_{k=1}^{\infty} f_n(u_k - u_{k-1})\right| > \eta_0 + \sum_{k=1}^{\infty} \eta_k\right) \\ &\leq P_1 + \sum_{k=1}^{\infty} P_{2,k} \end{aligned} \quad (17)$$

where

$$P_1 = \sum_{u_0 \in \mathcal{E}_0} \mathbb{P}(|f_n(u_0)| > \eta_0) \quad \text{and} \quad P_{2,k} = \sum_{u_k \in \mathcal{E}_k} \mathbb{P}(|f_n(u_k - u_{k-1})| > \eta_k).$$

As  $u_0 \in T_0$ ,  $\|u_0\|_{\varpi} \leq 1$  and  $\|u_0\|_{\infty} \leq \bar{r}\sqrt{D}$ . Moreover,  $\|u_0\|_n^2 \leq 3/2 \|u_0\|_{\varpi}^2 \leq 3\delta_0/2$ . Then

$$\mathbb{P}(|f_n(u_0)| > \eta_0) = \mathbb{P}\left(|f_n(u_0)| > \eta_0, \|u_0\|_n^2 \leq 3\delta_0/2\right).$$

There exist two constants  $c'_1$  and  $c'_2$  depending only on  $\delta_0$  and  $\bar{r}$  such that

$$\mathbb{P}(|f_n(u_0)| > \eta_0) \leq K \exp\left(-\frac{n\beta\eta_0^2}{c'_1\alpha^2 + 2c'_2\sqrt{D}\alpha\eta_0}\right).$$

Let us set  $x_0$  such that  $\eta_0 = \alpha\left(\sqrt{c'_1(x_0/\beta)} + c'_2\sqrt{D}(x_0/\beta)\right)$ . Then:

$$x_0 \leq \frac{\beta\eta_0^2}{c'_1\alpha^2 + 2c'_2\sqrt{D}\alpha\eta_0}$$

and

$$\mathbb{P}(f_n(u_0) > \eta_0) \leq K \exp(-nx_0).$$

Then

$$P_1 \leq K \sum_{u_0 \in \mathcal{E}_0} \exp(-nx_0) \leq K \exp(H_0 - nx_0). \quad (18)$$

We have that

$$\|u_k - u_{k-1}\|_\pi^2 \leq 2 \left( \|u - u_{k-1}\|_\pi^2 + \|u - u_k\|_\pi^2 \right) \leq 5\delta_{k-1}^2/2$$

then  $\|u_k - u_{k-1}\|_n^2 \leq 15\delta_{k-1}^2/4$ . As  $u_{k-1}, u_k \in \mathcal{E}_{k-1} \times \mathcal{E}_k$ , it follows that  $\|u_k - u_{k-1}\|_\infty^2 \leq 5\delta_{k-1}^2 \bar{r}^2/2$ . There exists two constants  $c_3$  and  $c_4$  such that:

$$\begin{aligned} \mathbb{P}_n(|f_n(u_k - u_{k-1})| > \eta_k) &= \mathbb{P}_n\left(|f_n(u_k - u_{k-1})| > \eta_k, \|u_k - u_{k-1}\|_n^2 \leq 15\delta_{k-1}^2/4\right) \\ &\leq K \exp\left(-\frac{n\beta\eta_k^2}{c_3\alpha^2\delta_{k-1}^2 + 2c_4\alpha\delta_{k-1}}\right). \end{aligned}$$

Let us fix  $x_k$  such that  $\eta_k = \delta_{k-1}a \left( \sqrt{c_3(x_k/\beta)} + c_4(x_k/\beta) \right)$ . We obtain:

$$x_k \leq \frac{\beta\eta_k^2}{c_3\alpha^2\delta_{k-1}^2 + 2c_4\alpha\delta_{k-1}}$$

and

$$\mathbb{P}(|f_n(u_k - u_{k-1})| > \eta_k) \leq K \exp(-nx_k).$$

Then,  $P_{2,k} \leq K \exp(H_{k-1} + H_k - nx_k)$  and

$$P_2 = \sum_{k=1}^{\infty} P_{2,k} \leq K \sum_{k=1}^{\infty} \exp(H_{k-1} + H_k - nx_k). \quad (19)$$

Let us set  $\tau > 0$  and choose  $(x_k)$  (and then  $(\eta_k)$ ) such that

$$\begin{cases} \sqrt{D}nx_0 = H_0 + D + \tau \\ nx_k = H_{k-1} + H_k + (k+1)D + \tau. \end{cases}$$

Collecting the results, we obtain, by (17), (18) and (19):

$$\mathbb{P}\left(\sup_{u \in \mathcal{B}_{m,m'}} |f_n(u)| > \eta\right) \leq C \left( e^{-D}e^{-\tau} + e^{-\sqrt{D}}e^{-\tau/\sqrt{D}} \right). \quad (20)$$

It remains to compute  $\eta^2$ . We denote by  $C$  a constant depending only on  $\delta_0$  and  $\bar{r}$ . This constant may vary from one line to another. We have that:

$$\eta = \sum_{k=0}^{\infty} \eta_k \leq C\alpha \left( \sum_{k=1}^{\infty} \delta_{k-1} \left( \sqrt{\frac{x_k}{\beta}} + \frac{x_k}{\beta} \right) \right) + \alpha \left( \sqrt{\frac{x_0}{\beta}} + \sqrt{D} \frac{x_0}{\beta} \right).$$

Let us recall that  $H_k = C(k+1)D$ . Then,  $nx_k = C(3k+2)D + \tau$ ,  $\sqrt{D}nx_0 = CD + \tau$  and

$$\sum_{k=0}^{\infty} \frac{\delta_{k-1}x_k}{\beta} \leq \frac{1}{n\beta} \sum_{k=0}^{\infty} 2^{-(k-1)} (C(3k+2)D + \tau) \leq C \frac{D + \tau}{n\beta}.$$

Moreover,

$$\sum_{k=0}^{\infty} \delta_{k-1} \sqrt{\frac{x_k}{\beta}} \leq C \frac{\sqrt{D} + \sqrt{\tau}}{\sqrt{n\beta}}.$$

As  $D/n\beta \leq 1$ , there exists a constant  $\kappa$  such that

$$\eta^2 \leq \kappa \alpha^2 \left( \frac{D}{n\beta} + 2 \frac{\tau}{n\beta} + \frac{\tau^2}{n^2 \beta^2} \right).$$

Then, according to (20):

$$\mathbb{P} \left( \sup_{u \in \mathcal{B}_{m,m'}} f_n^2(u) > \kappa \alpha^2 \left( \frac{D}{n\beta} + 2 \frac{\tau}{n\beta} + \frac{\tau^2}{n^2 \beta^2} \right) \right) \leq C \left( e^{-D-\tau} + e^{-\sqrt{D}-\tau/\sqrt{D}} \right). \quad (21)$$

Furthermore

$$\begin{aligned} E &:= \mathbb{E} \left( \left[ \sup_{u \in \mathcal{B}_{m,m'}} f_n^2(u) - \kappa \alpha^2 \frac{D}{n\beta} \right]_+ \right) \\ &= \int_0^{\infty} \mathbb{P} \left( \sup_{u \in \mathcal{B}_{m,m'}} f_n^2(u) > \kappa \alpha^2 \frac{D}{n\beta} + \tau \right) d\tau \end{aligned}$$

Setting  $\tau = \kappa \alpha^2 (2y/n\beta + y^2/n^2 \beta^2)$ , it follows:

$$E = C \gamma^2 \int_0^{\infty} \mathbb{P} \left( \sup_{u \in \mathcal{B}_{m,m'}} f_n^2(u) > \kappa \alpha^2 \left( \frac{D}{n\beta} + 2 \frac{y}{n\beta} + \frac{y^2}{n^2 \beta^2} \right) \right) \left( \frac{2}{n\beta} + \frac{2y}{n^2 \beta^2} \right) dy.$$

By (21),

$$\begin{aligned} E &= C \kappa \alpha^2 \left( e^{-D} + e^{-\sqrt{D}} \right) \left( \frac{1}{n\beta} \int_0^{\infty} y e^{-y/\sqrt{D}} dy \right) \\ &\leq C \frac{\kappa \alpha^2}{n\beta} D^{3/2} e^{-\sqrt{D}}. \end{aligned}$$

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## References

- Applebaum, D. (2004) *Lévy processes and stochastic calculus*, *Cambridge Studies in Advanced Mathematics*, volume 93. Cambridge University Press, Cambridge.
- Barron, A., Birgé, L. and Massart, P. (1999) Risk bounds for model selection via penalization. *Probab. Theory Related Fields*, 113 (3) pp. 301–413.

- Birgé, L. and Massart, P. (1998) Minimum contrast estimators on sieves: exponential bounds and rates of convergence. *Bernoulli*, 4 (3) pp. 329–375.
- Comte, F. (2001) Adaptive estimation of the spectrum of a stationary gaussian sequence. *Bernoulli*, 7 (2) pp. 267–298.
- Comte, F., Genon-Catalot, V. and Rozenholc, Y. (2007) Penalized nonparametric mean square estimation of the coefficients of diffusion processes. *Bernoulli*, 13 (2) pp. 514–543.
- Dellacherie, C. and Meyer, P.A. (1980) *Probabilités et potentiel. Chapitres V à VIII, Actualités Scientifiques et Industrielles [Current Scientific and Industrial Topics]*, volume 1385. Hermann, Paris, revised edition. Théorie des martingales. [Martingale theory].
- DeVore, R.A. and Lorentz, G.G. (1993) *Constructive approximation, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*, volume 303. Springer-Verlag, Berlin.
- Hoffmann, M. (1999) Adaptive estimation in diffusion processes. *Stochastic Process. Appl.*, 79 (1) pp. 135–163.
- Ishikawa, Y. and Kunita, H. (2006) Malliavin calculus on the Wiener-Poisson space and its application to canonical SDE with jumps. *Stochastic Process. Appl.*, 116 (12) pp. 1743–1769.
- Liptser, R.S. and Shiryaev, A.N. (2001) *Statistics of random processes. I, Applications of Mathematics (New York)*, volume 5. Springer-Verlag, Berlin, expanded edition. General theory, Translated from the 1974 Russian original by A. B. Aries, Stochastic Modelling and Applied Probability.
- Mancini, C. and Renò, R. (2011) Threshold estimation of Markov models with jumps and interest rate modeling. *J. Econometrics*, 160 (1) pp. 77–92.
- Masuda, H. (2007) Ergodicity and exponential  $\beta$ -mixing bounds for multidimensional diffusions with jumps. *Stochastic Process. Appl.*, 117 (1) pp. 35–56.
- Meyer, Y. (1990) *Ondelettes et opérateurs. I. Actualités Mathématiques. [Current Mathematical Topics]*. Hermann, Paris. Ondelettes. [Wavelets].
- Rubenthaler, S. (2010) *Probabilités : aspects théoriques et applications en filtrage non linéaire, systèmes de particules et processus stochastiques..* Habilitation à diriger des recherches, Université de Nice-Sophia Antipolis, France.
- Schmisser, E. (2009a) Penalized nonparametric drift estimation for a multidimensional diffusion process. Preprint 2009-02, MAP5, Université Paris Descartes. accepté pour publication à Statistics.
- Schmisser, E. (2010) *Estimation non paramétrique pour des processus de diffusion.* Ph.D. thesis, Université Paris Descartes.

- Shimizu, Y. (2008) Some remarks on estimation of diffusion coefficients for jump-diffusions from finite samples. *Bull. Inform. Cybernet.*, 40 pp. 51–60.
- Shimizu, Y. and Yoshida, N. (2006) Estimation of parameters for diffusion processes with jumps from discrete observations. *Stat. Inference Stoch. Process.*, 9 (3) pp. 227–277.