Non-parametric adaptive estimation of the drift for a jump diffusion process

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Abstract

In this article, we consider a jump diffusion process $(X_t)_{t\geq 0}$ observed at discrete times $t = 0, \Delta, \ldots, n\Delta$. The sampling interval Δ tends to 0 and $n\Delta$ tends to infinity. We assume that $(X_t)_{t\geq 0}$ is ergodic, strictly stationary and exponentially β-mixing. We use a penalized least-square approach to compute two adaptive estimators of the drift function b. We provide bounds for the risks of the two estimators.

1 Introduction

We consider a general diffusion with jumps:

$$dX_t = b(X_t)dt + \sigma(X_t)dW_t + \xi(X_{t^-})dL_t \quad \text{and} \quad X_0 = \eta \tag{1}$$

where L_t is a centred pure jump Levy process:

$$dL_t = \int_{z\in\mathbb{R}} z\,(\mu(dt, dz) - dt\nu(dz))$$

with μ a random Poisson measure with intensity measure $\nu(dz)dt$ such that $\int_{z\in\mathbb{R}} z^2\nu(dz) < \infty$. The compensated Poisson measure $\tilde{\mu}$ is defined by $\tilde{\mu}(dt, dz) = \mu(dt, dz) - \nu(dz)dt$. The random variable η is independant of $(W_t, L_t)_{t\geq 0}$. Moreover, $(W_t)_{t\geq 0}$ and $(L_t)_{t\geq 0}$ are independant.

This process is observed with high frequency (at times $t = 0, \Delta, \ldots, n\Delta$ where $\Delta \to 0$ and $n\Delta \to \infty$). It is assumed to be ergodic, stationary and exponentially β-mixing (see Masuda (2007) for sufficient conditions). Our aim is to construct a non-parametric estimator of b on a compact set A.

The non-parametric estimation of b and σ for a diffusion process observed with high-frequency is well-known (see for instance Hoffmann (1999) and Comte *et al.* (2007)). Diffusion processes with jumps are used in various fields, for instance in finance, for modelling the growth of a population, in hydrology, in

medical science, ..., but there exist few results for the non-parametric estimation of b and σ. Shimizu and Yoshida (2006) construct maximum-likelihood estimators of parameters of b and σ. Their estimators reach the standard rates of convergence: $\sqrt{n\Delta}$ for the estimator of b, and $\sqrt{n}$ for the estimator of σ. Shimizu (2008) and Mancini and Renò (2011) use a kernel estimator to obtain non parametric threshold estimators of σ. Mancini and Renò (2011) also construct a non-parametric thruncated estimator of b, but only when L_t is a compound Poisson process. To our knowledge, minimax rates of convergences for non-parametric estimators of b, σ or ξ are not available in the literature.

In this paper, we use model selection to construct two non-parametric estimators of b under the asymptotic framework $\Delta \to 0$ and $n\Delta \to \infty$. This method was introduced by Birgé and Massart (1998).

First, we introduce a sequence of linear subspaces $S_m \subseteq L^2(A)$ and, for each m, we construct an estimator $\hat{b}_m$ of b by minimising on S_m the contrast function:

$$\gamma_n(t) = \frac{1}{n}\sum_{k=1}^{n} \left(Y_{k\Delta} - t(X_{k\Delta})\right)^2 \quad \text{where} \quad Y_{k\Delta} = \frac{X_{(k+1)\Delta} - X_{k\Delta}}{\Delta}.$$

We obtain a collection of estimators of the drift function b and we bound their risks (Theorem 1). Then, we introduce a penalty function to select the "best" dimension m and we deduce an adaptive estimator $\hat{b}_{\hat{m}}$. Under the assumption that ν is sub-exponential, that is if there exist two positive constants C, λ such that, for z large enough, $\nu([-z,z]^c) \leq Ce^{-\lambda z}$, the risk bound of $\hat{b}_{\hat{m}}$ is exactly the same as for a diffusion without jumps (Theorem 2) (see Comte *et al.* (2007) or Hoffmann (1999)).

In a second part, we do not assume that ν is subexponential and we construct a truncated estimator $\tilde{b}_m$ of b. We minimise the contrast function

$$\tilde{\gamma}_n(t) = \frac{1}{n}\sum_{k=1}^{n} \left(Y_{k\Delta}\mathbb{1}_{|Y_{k\Delta}|\leq C_\Delta} - t(X_{k\Delta})\right)^2 \quad \text{where} \quad C_\Delta \propto \sqrt{\Delta}\ln(n)$$

in order to obtain a new estimator $\tilde{b}_m$. As in the first part, we introduce a penalty function to obtain an adaptive estimator $\tilde{b}_{\tilde{m}}$. The risk bound of this adaptive estimator depends on the Blumenthal-Getoor index of ν (Theorems 3 and 4).

In Section 2, we present the model and its assumptions. In Sections 3 and 4, we construct the estimators and bound their risks. Some simulations are presented in Section 5. Proofs are gathered in Sections 6 and 7.

2 Assumptions

2.1 Assumptions on the model

We consider the following assumptions:

A 1. *The functions b, σ and ξ are Lipschitz.*

A 2. *1. The function σ is bounded from below and above:*

$$\exists \sigma_0, \sigma_1, \ \forall x \in \mathbb{R}, \quad 0 < \sigma_1 \leq \sigma(x) \leq \sigma_0.$$

2. The function ξ is bounded: $\exists \xi_0, \ \forall x \in \mathbb{R}, \quad 0 \leq \xi(x) \leq \xi_0$.

3. The drift function b is elastic: there exists a constant M such that, for any $x \in \mathbb{R}$, $|x| > M$: $xb(x) \lesssim -|x|^2$.

4. The Lévy measure ν satisfies:

$$\nu(\{0\}) = 0, \quad \int_{-\infty}^{\infty} z^2 \nu(dz) = 1 \quad \textit{and} \quad \int_{-\infty}^{\infty} z^4 \nu(dz) < \infty.$$

Under Assumption A1, the stochastic differential equation (1) admits a unique strong solution. According to Masuda (2007), under Assumptions A1 and A2, the process (X_t) admits a unique invariant probability ϖ and satisfies the ergodic theorem: for any measurable function g such that $\int |g(x)| \varpi(dx) < \infty$, when $T \to \infty$,

$$\frac{1}{T} \int_0^T g(X_s) ds \to \int g(x) \varpi(dx).$$

This distribution has moments of order 4. Moreover, Masuda (2007) also ensures that under these assumptions, the process (X_t) is exponentially β-mixing. Furtermore, if there exist two constants c and n_0 such that, for any $x \in \mathbb{R}$, $\xi^2(x) \geq c(1 + |x|)^{-n_0}$, then Ishikawa and Kunita (2006) ensure that a smooth transition density exists.

A 3. *1. The stationary measure ϖ admits a density π which is bounded from below and above on the compact interval A:*

$$\exists \pi_0, \pi_1, \ \forall x \in A, \quad 0 < \pi_1 \leq \pi(x) \leq \pi_0.$$

2. The process $(X_t)_{t \geq 0}$ is stationary ($\eta \sim \varpi(dx) = \pi(x)dx$).

The following proposition very useful for the proofs is proved later.

Proposition 1.
Under Assumptions A1-A3, for any $p \geq 1$, there exists a constant $c(p)$ such that, if $\int_{\mathbb{R}} z^{2p} \nu(dz) < \infty$:

$$\mathbb{E}\left(\sup_{s \in [t, t+h]} (X_s - X_t)^{2p} \right) \leq c(p) h.$$

2.2 Assumptions on the approximation spaces

In order to construct an adaptive estimator of b, we use model selection: we compute a collection of estimators $\hat{b}_m$ of b by minimising a contrast function $\gamma_n(t)$ on a vectorial subspace $S_m \subset L^2(A)$, then we choose the best possible estimator using a penalty function $pen(m)$. The collection of vectorial subspaces $(S_m)_{m\in\mathscr{M}_n}$ has to satisfy the following assumption:

A 4.

1. *The subspaces S_m have finite dimension D_m.*
2. *The sequence of vectorial subpsaces $(S_m)_{m\geq 0}$ is increasing: for any m, $S_m \subseteq S_{m+1}$.*
3. *Norm connexion: there exists a constant ϕ_1 such that, for any $m \geq 0$, any $t \in S_m$,*
$$\|t\|_\infty^2 \leq \phi_1 D_m \|t\|_{L^2}^2$$
where $\|.\|_{L^2}$ is the L^2-norm and $\|.\|_\infty$ is the sup-norm on A.
4. *For any $m \in \mathbb{N}$, there exists an orthonormal basis $(\psi_\lambda)_{\lambda\in\Lambda_m}$ of S_m such that*
$$\forall\lambda, \quad card\,(\lambda',\, \|\psi_\lambda\psi_{\lambda'}\|_\infty \neq 0) \leq \phi_0$$
where ϕ_0 does not depend on m.
5. *For any function t belonging to the unit ball of the Besov space $\mathscr{B}^\alpha_{2,\infty}$,*
$$\exists C,\, \forall m \quad \|t - t_m\|_{L^2}^2 \leq C 2^{-2m\alpha}$$
where t_m is the L^2 orthogonal projection of t on S_m.

The subspaces generated by piecewise polynomials, compactly supported wavelets or spline functions satisfy A4 (see DeVore and Lorentz (1993) and Meyer (1990) for instance).

3 Estimation of the drift

By analogy with Comte *et al.* (2007), we decompose $Y_{k\Delta}$ in the following way:

$$Y_{k\Delta} = \frac{X_{(k+1)\Delta} - X_{k\Delta}}{\Delta} = b(X_{k\Delta}) + I_{k\Delta} + Z_{k\Delta} + T_{k\Delta} \tag{2}$$

where

$$I_{k\Delta} = \frac{1}{\Delta}\int_{k\Delta}^{(k+1)\Delta} (b(X_s) - b(X_{k\Delta}))\, ds, \quad Z_{k\Delta} = \frac{1}{\Delta}\int_{k\Delta}^{(k+1)\Delta} \sigma(X_s) dW_s$$
$$T_{k\Delta} = \frac{1}{\Delta}\int_{k\Delta}^{(k+1)\Delta} \xi(X_{s^-}) dL_s.$$

The terms $Z_{k\Delta}$ and $T_{k\Delta}$ are martingale increments. Let us introduce the mean square contrast function

$$\gamma_n(t) = \frac{1}{n}\sum_{k=1}^{n}\left(Y_{k\Delta} - t\left(X_{k\Delta}\right)\right)^2 \tag{3}$$

and the empirical risk

$$\mathscr{R}_n(t) = \|t - b_A\|_n^2 \quad \text{where} \quad \|t\|_n^2 = \frac{1}{n}\sum_{k=1}^{n} t^2\left(X_{k\Delta}\right) \quad \text{and} \quad t_A = t\mathbb{1}_A. \tag{4}$$

We consider the asymptotic framework:

$$\Delta \to 0, \quad n\Delta \to \infty.$$

For any $m \in \mathscr{M}_n = \{m,\ D_m \leq \mathscr{D}_n\}$ where $\mathscr{D}_n^2 \leq n\Delta/\ln^2(n)$, we construct the regression-type estimator:

$$\hat{b}_m = \arg\min_{t \in S_m} \gamma_n(t).$$

Theorem 1.
Under Assumptions A1-A4, *the risk of the estimator with fixed m satisfies:*

$$\mathscr{R}_n(\hat{b}_m) \leq 3\pi_1 \|b_m - b_A\|_{L^2}^2 + 48(\sigma_0^2 + \xi_0^2)\frac{D_m}{n\Delta} + c\Delta$$

where b_m is the orthogonal (L^2) projection of b_A over the vectorial subspace S_m. The constant c is independent of m, n and Δ.

Except for the constant $(\sigma_0^2 + \xi_0^2)$ in the variance term, this is exactly the bound of the risk that Comte *et al.* (2007) found for a diffusion process without jumps.

The bias term, $\|b_m - b_A\|_{L^2}^2$, decreases when the dimension D_m increases whereas the variance term $(\sigma_0^2 + \xi_0^2)D_m/(n\Delta)$ is proportional to the dimension. Under the classical assumption $n\Delta^2 = O(1)$, the remainder term Δ is negligible. Thus we need to find a good compromise between the bias and the variance term.

Remark 1. If the regularity of the drift function is known, that is, if b belongs to a ball of a Besov space $\mathscr{B}_{2,\infty}^{\alpha}$, then the bias term $\|b_m - b_A\|_{L^2}^2$ is smaller than $D_m^{-2\alpha}$. The best estimator is obtained for $D_{m_{opt}} = (n\Delta)^{1/(1+2\alpha)}$ and the estimator risk satisfies:

$$\mathscr{R}_n(\hat{b}_{m_{opt}}) \lesssim (n\Delta)^{-2\alpha/(2\alpha+1)} + \Delta.$$

Let us introduce a penalty function *pen* such that :

$$pen(m) \geq \kappa(\sigma_0^2 + \xi_0^2)\frac{D_m}{n\Delta}$$

and set:

$$\hat{m} = \arg \min_{m \in \mathscr{M}_n} \left\{ \gamma_n(\hat{b}_m) + pen(m) \right\}.$$

We will chose κ later. We denote by $\hat{b}_{\hat{m}}$ the resulting estimator. To bound the risk of the adaptive estimator, an additional assumption is needed:

A 5. *1. The Lévy measure ν is symmetric or the function ξ is constant.*

2. The Lévy measure ν is sub exponential: there exist $\lambda, C > 0$ such that, for any $|z| > 1$, $\nu(]-z, z[^c) \leq Ce^{-\lambda|z|}$.

Theorem 2.
Under Assumptions A1-A5, there exists a constant κ (depending only on ν) such that, if $\mathscr{D}_n^2 \leq n\Delta/\ln^2(n)$:

$$\mathbb{E}\left(\left\| \hat{b}_{\hat{m}} - b_A \right\|_n^2 \right) \lesssim \inf_{m \in \mathscr{M}_n} \left(\|b_m - b_A\|_{L^2}^2 + pen(m) \right) + \left(\Delta + \frac{1}{n\Delta} \right).$$

We can bound κ theoretically, however, this bound is in practice too large for the simulations. In Section 5, we calibrate κ by simulations (see Comte *et al.* (2007) for instance). The adaptive estimator automatically realises the bias-variance compromise. Moreover, this is the same oracle inequality as for a diffusion process without jumps.

4 Truncated estimator of the drift

Truncated estimators are widely used for the estimation of the diffusion coefficient of a jump diffusion (see for instance Mancini and Renò (2011) and Shimizu (2008)). Our aim is to construct an adaptive estimator of b even if Assumption A5 is not fulfilled. To this end, we cut off the big jumps. Let us introduce the set

$$\Omega_{X,k} = \left\{ \omega,\ \left| X_{(k+1)\Delta} - X_{k\Delta} \right| \leq C_\Delta \right\}$$

where $C_\Delta = (b_{max} + 3)\Delta + (\sigma_0 + 4\xi_0)\sqrt{\Delta}\ln(n)$ (with $b_{max} = \sup_{x \in A} |b(x)|$). Let us consider the random variables

$$\tilde{Y}_{k\Delta} = \frac{X_{(k+1)\Delta} - X_{k\Delta}}{\Delta} \mathbb{1}_{\Omega_{X,k}} \mathbb{1}_{X_{k\Delta} \in A}.$$

We recall here the definition of the Blumenthal-Getoor index:

Definition 1.
The Blumenthal-Getoor index of a Lévy measure is

$$\beta = \inf \left\{ \alpha \geq 0,\ \int_{|z| \leq 1} |z|^\alpha \nu(dz) < \infty \right\}.$$

A compound Poisson process has $\beta = 0$.
We assume that the following assumption is fulfilled.

A 6. *1. The Lévy measure ν is symmetric.*

2. For $|x|$ small, $\nu(dx)$ is absolutely continuous with respect to the Lebesgue measure ($\nu(x) = n(x)dx$) and:

$$\exists \beta \in [0, 2[, \exists a_0, \forall x \in [-a_0, a_0], \quad n(x) \leq Cx^{-\beta-1}.$$

This implies that the Blumenthal-Getoor index is equal to β.

3. The function ξ is bounded from below: there exists $\xi_1 > 0$ such that, for any $z \in \mathbb{R}$, $0 < \xi_1 \leq \xi(z)$.

4. The functions σ and ξ are $\mathscr{C}^2$, and ξ' and σ' are Lipschitz.

We consider the following asymptotic framework:

$$\frac{n\Delta}{\ln^2(n)} \to 0, \quad \Delta^{1-\beta/2} \ln^2(n) \to 0.$$

The truncated estimator $\tilde{b}_m$ is obtained by minimising the contrast function:

$$\tilde{b}_m = \arg\min_{t \in S_m} \tilde{\gamma}_n(t) \quad \text{where} \quad \tilde{\gamma}_n(t) = \frac{1}{n} \sum_{k=1}^{n} \left(\tilde{Y}_{k\Delta} - t(X_{k\Delta}) \right)^2.$$

Theorem 3 : Risk of the non adaptive truncated estimator.
Under Assumptions A1-A4 and A6, for any m such that $D_m \leq \mathscr{D}_n$ where $\mathscr{D}_n^2 \leq n\Delta/\ln^2(n)$:

$$\mathbb{E}\left(\left\| \tilde{b}_m - b_A \right\|_n^2 \right) \lesssim \|b_m - b_A\|_{L^2}^2 + (\sigma_0^2 + \xi_0^2) \frac{D_m}{n\Delta} + \Delta^{1-\beta/2} \ln^2(n) + \frac{1}{n\Delta}.$$

The terms of the rest depend on the Blumenthal-Getoor index and are larger than for the first estimator. Nevertheless, if L_t is a compound Poisson process, then $\beta = 0$ and we obtain (up to a logarithm factor) the same inequality as for the non-truncated estimator.

Remark 2. If ν is not absolutely continuous, we can prove the weaker inequality:

$$\mathbb{E}\left(\left\| \tilde{b}_m - b_A \right\|_n^2 \right) \lesssim \|b_m - b_A\|_{L^2}^2 + (\sigma_0^2 + \xi_0^2) \frac{D_m}{n\Delta} + \Delta^{1-\beta} \ln^2(n) + \frac{1}{n\Delta}.$$

In that case, $\tilde{b}_m$ converges towards b_A only if $\beta < 1$, which implies that ν has finite density ($\int_{\mathbb{R}} |z| \nu(dz) < \infty$).

Remark 3. Assume that b_A belongs to the Besov space $\mathscr{B}^{\alpha}_{2,\infty}$ and that $\|b_A\|_{\mathscr{B}^{\alpha}_{2,\infty}} \leq 1$.
The bias-variance compromise $\|b_m - b_A\|_{L^2}^2 + D_m/n\Delta$ is minimum when $m = \log_2(n\Delta)/(1+2\alpha)$, and the risk satisfies:

$$\mathbb{E}\left(\left\| \tilde{b}_m - b_A \right\|_n^2 \right) \lesssim (n\Delta)^{-2\alpha/(1+2\alpha)} + \Delta^{1-\beta/2} \ln^2(n)$$

Let us set $\Delta \sim n^{-\gamma}$ with $\gamma > 0$. We have the following convergence rates:

γ	*first estimator*	*truncated estimator*
$0 < \gamma \leq \frac{2\alpha}{4\alpha+1} \leq \frac{1}{2}$	Δ	$\Delta^{1-\beta/2} \ln^2(n)$
$\frac{2\alpha}{4\alpha+1} \leq \gamma \leq \frac{2\alpha}{4\alpha+1-\beta\alpha-\beta/2} \leq \frac{1}{2(1-\beta/4)}$	$(n\Delta)^{-2\alpha/(2\alpha+1)}$	$\Delta^{1-\beta/2} \ln^2(n)$
$\frac{2\alpha+1}{4\alpha+1-\beta\alpha-\beta/2} \leq \gamma < 1$	$(n\Delta)^{-2\alpha/(2\alpha+1)}$	$(n\Delta)^{-2\alpha/(2\alpha+1)}$

If we have sufficiently high frequency data ($n\Delta^{2(1-\beta/4)} = O(1)$), then the rate of convergence is the same for the two estimators.

To construct the adaptive estimator, we use the same penalty function as in the previous section:

$$pen(m) \geq \kappa \left(\sigma_0^2 + \xi_0^2\right) \frac{D_m}{n\Delta}$$

and define the adaptive estimator:

$$\tilde{m} = \arg \min_{m \in \mathscr{M}_n} \left\{ \tilde{\gamma}_n(\tilde{b}_m) + pen(m) \right\}.$$

Theorem 4 : Risk of the adaptive truncated estimator.
If Assumptions A1-A4 and A6 are satisfied, then there exists κ such that, if $\mathscr{D}_n^2 \leq n\Delta / \ln^2(n)$:

$$\mathbb{E}\left(\left\| \tilde{b}_{\tilde{m}} - b_A \right\|_n^2 \right) \lesssim \min_{m \in \mathscr{M}_n} \left(\|b_m - b_A\|_n^2 + pen(m) \right) + \Delta^{1-\beta/2} \ln^2(n) + \frac{1}{n\Delta}.$$

The adaptive estimator $\tilde{b}_{\tilde{m}}$ automatically realises the bias/variance compromise if the frequency of data is sufficiently high.

5 Numerical simulations and examples

5.1 Compound Poisson models

We consider the stochastic differential equation:

$$dX_t = b(X_t)dt + \sigma(X_t)dW_t + \xi(X_{t^-})dL_t$$

where L_t is a compound Poisson process of intensity 1: $L_t = \sum_{j=1}^{N_t} \zeta_i$, with N_t a Poisson process of intensity 1 and $(\zeta_1, \ldots, \zeta_n)$ are independent and identically distributed random variables independent of (N_t). We denote by f the probability law of ζ_i .

Model 1:

$$b(x) = -2x, \quad \sigma(x) = \xi(x) = 1 \quad \text{and} \quad f(dz) = \nu(dz) = \frac{1}{2}\delta_1 + \frac{1}{2}\delta_{-1}.$$

Model 2:

$$b(x) = -(x-1/4)^3 - (x+1/4)^3, \quad \sigma(x) = \xi(x) = 1 \quad \text{and} \quad f(dz) = \nu(dz) = \frac{e^{-\lambda|z|}dz}{2}.$$

We can remark that the function b is not Lipschitz and therefore does not satisfy Assumption A1.

Model 3:

We consider the stochastic process of parameters

$$b(x) = -2x + \sin(3x), \quad \sigma(x) = \xi(x) = \sqrt{\frac{3+x^2}{1+x^2}}$$

and

$$f(dz) = \nu(dz) = \frac{1}{4}\sqrt{\frac{\sqrt{24}}{|z|}}e^{-\sqrt{\sqrt{24}|z|}}dz.$$

Let us remark that $\nu = f$ is not sub-exponential and does not satisfy A5.

5.2 Simulation algorithm (Compound Poisson case)

We estimate b on the compact interval $A = [-1, 1]$.

1. Simulate random variables $(X_0, X_\Delta, \ldots, X_{n\Delta})$ thanks to a Euler scheme with sampling interval $\delta = \Delta/5$. To this end, we use the same simulation scheme as Rubenthaler (2010). We simulate the times of the jumps $(\tau_1, \ldots, \tau_N, \tau_{N+1})$ with $\tau_N < n\Delta \leq \tau_{N+1}$ and we fix $X_0 = 0$.
   If $\delta < \tau_1$, we compute

   $$X_\delta = \delta b(X_0) + \sqrt{\delta}\sigma(X_0)N \quad \text{with} \quad N \sim \mathcal{N}(0,1)..$$

   If $\tau_1 < \delta$, we first compute

   $$X_{\tau_1} = \tau_1 b(X_0) + \sqrt{\tau_1}\sigma(X_0)N + \xi(X_0)\zeta_1$$

   with $N \sim \mathcal{N}(0,1)$ and $\zeta_1 \sim f$ is independant of N. If $\delta < \tau_2$, we compute

   $$X_\delta = (\delta - \tau_1)b(X_{\tau_1}) + \sqrt{\delta - \tau_1}\sigma(X_{\tau_1})N'$$

   else we compute

   $$X_{\tau_2} = (\tau_2 - \tau_1)b(X_{\tau_1}) + \sqrt{\tau_2 - \tau_1}\sigma(X_{\tau_1})N' + \xi(X_{\tau_1})\zeta_2$$

   where $N' \sim \mathcal{N}(0,1)$ and ζ_2 has distribution f. N, N', ζ_1 and ζ_2 are independent.

2. Construct the random variables

   $$Y_{k\Delta} = \frac{X_{(k+1)\Delta} - X_{k\Delta}}{\Delta} \quad \text{and} \quad \tilde{Y}_{k\Delta} = \frac{X_{(k+1)\Delta} - X_{k\Delta}}{\Delta}\mathbb{1}_{\Omega_{X,k}}\mathbb{1}_{X_{k\Delta}\in A}.$$

3. We consider the vectorial subspaces $S_{m,r}$ generated by the spline functions of degree r (see for instance Schmisser (2009a)). In that case $D_{m,r} = \dim(S_{m,r}) = 2^m + r$. For $r \in \{1,2,3\}$ and $m \in \mathscr{M}_n(r) = \{m, D_{m,r} \leq \mathscr{D}_n\}$, we compute the estimators $\hat{b}_{m,r}$ and $\tilde{b}_{m,r}$ by minimising the contrast functions γ_n and $\tilde{\gamma}_n$ on the vectorial subspaces $S_{m,r}$.

4. For the estimation algorithm, we make a selection of m and r as follows. Using the penalty function $pen(m,r) := pen(m) = \kappa(\sigma_0^2+\xi_0^2)(2^m+r)/n\Delta$, we select the adaptive estimators $\hat{b}_{\hat{m},r}$ and $\tilde{b}_{\tilde{m},r}$, and then choose the best r by minimizing $\gamma_n(\hat{b}_{\hat{m},r}) + pen(\hat{m},r)$ and $\tilde{\gamma}_n(\tilde{b}_{\tilde{m},r}) + pen(\tilde{m},r)$.

5.3 Results

In Figures 1-3, we simulate 5 times the process $(X_0,\dots,X_{n\Delta})$ for $\Delta = 10^{-1}$ and $n = 10^4$ and draw the obtained estimators. The two adaptive estimators are nearly superposed, moreover, they are close to the true function.

In Tables 1-3, for each value of (n,Δ), we simulate 50 trajectories of $(X_0, X_\Delta, \dots, X_{n\Delta})$. For each path, we construct the two adaptive estimators $\hat{b}_{\hat{m},\hat{r}}$ and $\tilde{b}_{\tilde{m},\tilde{r}}$ and we compute the empirical errors:

$$err_1 = \left\|\hat{b}_{\hat{m},\hat{r}} - b_A\right\|_n^2 \quad \text{and} \quad err_2 = \left\|\tilde{b}_{\tilde{m},\tilde{r}} - b_A\right\|_n^2.$$

In order to check that our algorithm is adaptive, we also compute the minimal errors

$$emin_1 = \min_{m,r}\left\|\hat{b}_{m,r} - b_A\right\|_n^2 \quad \text{and} \quad emin_2 = \min_{m,r}\left\|\tilde{b}_{m,r} - b_A\right\|_n^2$$

and the oracles $oracle_i = err_i/emin_i$. We give the means $\hat{m}_a$, $\hat{r}_a$, $\tilde{m}_a$ and $\tilde{r}_a$ of the selected values $\hat{m}$, $\hat{r}$, $\tilde{m}$ and $\tilde{r}$. The value $risk_i$ is the mean of err_i over the 50 simulations and or_i is the mean of $oracle_i$.

The empirical risk is decreasing when the product $n\Delta$ is increasing, which is coherent with the theoretical model. For Model 1, the two estimators are equivalent. When the tails of ν become larger (Models 2 and 3), the truncated estimator is better. The improvement is also more significant when the discretisation path is smaller. As on the three models, the processes L_t are compound Poisson processes, these results were expected. The truncated estimator seems also more robust: we don't observe aberrant values (like for the first estimator in Table 2). This aberrant value may be due to the fact that b is not Lipschitz and then $b(X_{k\Delta})$ may be quite large.

6 Proofs

Let us introduce the filtration

$$\mathscr{F}_t = \sigma\left(\eta, (W_s)_{0\leq s\leq t}, (L_s)_{0\leq s\leq t}\right).$$

The following result is very useful. It comes from Dellacherie and Meyer (1980) (Theorem 92 Chapter VII) and Applebaum (2004), Theorem 4.4.23 p265 (Kunita's first inequality).

Result 1 (Burkholder-Davis-Gundy inequality). We have that, for any $p \geq 2$,

$$\mathbb{E}\left[\sup_{s\in[t,t+h]}\left|\int_t^s \sigma(X_u)dW_u\right|^p\middle|\mathscr{F}_t\right] \leq C_p\left(\mathbb{E}\left[\left|\int_t^{t+h}\sigma^2(X_u)du\right|^{p/2}\middle|\mathscr{F}_t\right]\right)$$

and, if $\int_{\mathbb{R}}|z|^p\,\nu(dz)<\infty$, as $\int_{\mathbb{R}}z^2\nu(dz)=1$:

$$\begin{aligned}\mathbb{E}\left[\sup_{s\in[t,t+h]}\left|\int_t^s \xi(X_{u^-})dL_u\right|^p\middle|\mathscr{F}_t\right] &\leq C_p\mathbb{E}\left[\left(\int_t^{t+h}\xi^2(X_u)du\right)^{p/2}\middle|\mathscr{F}_t\right]\\ &+ C_p\mathbb{E}\left[\left(\int_t^{t+h}|\xi(X_u)|^p\,du\right)\middle|\mathscr{F}_t\right]\int_{\mathbb{R}}|z|^p\,\nu(dz).\end{aligned}$$

6.1 Proof of Proposition 1

By Result 1, there exists a constant c_p such that:

$$\begin{aligned}\mathbb{E}\left[\sup_{s\in[t,t+h]}(X_s-X_t)^{2p}\middle|\mathscr{F}_t\right] &\leq c(p)\left(\left[\left(\int_t^{t+h}|b(X_s)|\,ds\right)^{2p}\middle|\mathscr{F}_t\right]\right)\\ &+ c(p)\mathbb{E}\left[\left(\int_t^{t+h}\sigma^2(X_s)ds\right)^p\middle|\mathscr{F}_t\right]\\ &+ c(p)\left(\mathbb{E}\left[\left(\int_t^{t+h}\xi^2\left(X_s\right)ds\right)^p+\int_t^{t+h}\xi^{2p}(X_s)ds\middle|\mathscr{F}_t\right]\right).\end{aligned}$$

Then, as ξ and σ are bounded and b Lipschitz (and thus sub-linear), there exists a constant C_b such that:

$$\mathbb{E}\left[\sup_{s\in[t,t+h]}(X_s-X_t)^{2p}\middle|\mathscr{F}_t\right] \leq c(p)\left(\sigma_0^{2p}h^p+\xi_0^{2p}(h+h^p)\right)+c(p)h^{2p-1}C_b\int_t^{t+h}\mathbb{E}\left[X_s^{2p}\middle|\mathscr{F}_t\right]ds.$$

As (X_t) is stationary, we obtain the expected result.

6.2 Proof of Theorem 1

By (3) and (4), we get:

$$\begin{aligned}\gamma_n(t)=\frac{1}{n}\sum_{k=1}^n\left(Y_{k\Delta}-t(X_{k\Delta})\right)^2 &= \frac{1}{n}\sum_{k=1}^n\left(Y_{k\Delta}-b(X_{k\Delta})\right)^2+\|b-t\|_n^2\\ &+ \frac{2}{n}\sum_{k=1}^n\left(Y_{k\Delta}-b(X_{k\Delta})\right)\left(b(X_{k\Delta})-t(X_{k\Delta})\right).\end{aligned}$$

As, by definition, $\gamma_n(\hat{b}_m) \leq \gamma_n(b_m)$, we obtain:

$$\left\|\hat{b}_m - b\right\|_n^2 \leq \|b_m - b\|_n^2 + \frac{2}{n}\sum_{k=1}^{n}\left(Y_{k\Delta} - b(X_{k\Delta})\right)\left(\hat{b}_m(X_{k\Delta}) - b_m(X_{k\Delta})\right).$$

By (2), and as $\hat{b}_m$ and b_m are supported by A,

$$\left\|\hat{b}_m - b_A\right\|_n^2 \leq \|b_m - b_A\|_n^2 + \frac{2}{n}\sum_{k=1}^{n}\left(I_{k\Delta} + Z_{k\Delta} + T_{k\Delta}\right)\left(\hat{b}_m(X_{k\Delta}) - b_m(X_{k\Delta})\right).$$

Let us set introduce the unit ball

$$\mathscr{B}_m = \{t \in S_m,\ \|t\|_\varpi \leq 1\} \quad \text{where} \quad \|t\|_\varpi^2 = \int_A t^2(x)\varpi(dx)$$

and the englobing space $\mathscr{S}_n = \bigcup_{m\in\mathscr{M}_n} S_m$. Let us consider the set

$$\Omega_n = \left\{\omega,\ \forall t \in \mathscr{S}_n\ ,\ \left|\frac{\|t\|_n^2}{\|t\|_\varpi^2} - 1\right| \leq \frac{1}{2}\right\}$$

where the norms $\|.\|_\varpi$ and $\|.\|_n$ are equivalent.

Step 1: bound of the risk on Ω_n Thanks to the Cauchy-Schwartz inequality, we obtain that, on Ω_n:

$$\left\|\hat{b}_m - b_A\right\|_n^2 \leq \|b_m - b_A\|_n^2 + \frac{1}{12}\left\|\hat{b}_m - b_m\right\|_n^2 + 12\sum_{k=1}^{n} I_{k\Delta}^2 + \frac{1}{12}\left\|\hat{b}_m - b_m\right\|_\varpi^2 + 12\sup_{t\in\mathscr{B}_m}\nu_n^2(t)$$

where

$$\nu_n(t) = \frac{1}{n}\sum_{k=1}^{n}(Z_{k\Delta} + T_{k\Delta})t(X_{k\Delta}). \tag{5}$$

On Ω_n, by definition, we have:

$$\left\|\hat{b}_m - b_m\right\|_n^2 \leq 2\left\|\hat{b}_m - b_A\right\|_n^2 + 2\|b_m - b_A\|_n^2 \quad \text{and} \quad \left\|\hat{b}_m - b_m\right\|_\varpi^2 \leq 2\left\|\hat{b}_m - b_m\right\|_n^2.$$

Thus we obtain:

$$\left\|\hat{b}_m - b_A\right\|_n^2 \leq 3\|b_m - b_A\|_n^2 + 24\sum_{k=1}^{n} I_{k\Delta}^2 + 24\sup_{t\in\mathscr{B}_m}\nu_n^2(t).$$

The following lemma is very useful. It is proved later.

Lemma 1. *1.* $\mathbb{E}\left(I_{k\Delta}^2\right) \leq c\Delta$ *and* $\mathbb{E}\left(I_{k\Delta}^4\right) \leq c\Delta$.

2. $\mathbb{E}\left(Z_{k\Delta}\middle|\mathscr{F}_{k\Delta}\right) = 0$, $\mathbb{E}\left(Z_{k\Delta}^2\middle|\mathscr{F}_{k\Delta}\right) \leq \sigma_0^2/\Delta$ *and* $\mathbb{E}\left(Z_{k\Delta}^4\middle|\mathscr{F}_{k\Delta}\right) \leq c/\Delta^2$.

3. $\mathbb{E}\left(T_{k\Delta}\middle|\mathscr{F}_{k\Delta}\right)=0$, $\mathbb{E}\left(T_{k\Delta}^2\middle|\mathscr{F}_{k\Delta}\right)\leq\xi_0^2/\Delta$ *and* $\mathbb{E}\left(T_{k\Delta}^4\middle|\mathscr{F}_{k\Delta}\right)\leq c/\Delta^3$.

By Lemma 1, $\mathbb{E}\left[I_{k\Delta}^2\right]\leq\Delta$. It remains to bound $\mathbb{E}\left[\sup_{t\in\mathscr{B}_m}\nu_n^2(t)\right]$. We consider an orthonormal basis $(\varphi_\lambda)_{\lambda\in\Lambda_m}$ of S_m for the L^2_ϖ-norm with $|\Lambda_m|=D_m$. Any function $t\in S_m$ can be written $t=\sum_{\lambda\in\Lambda_m}a_\lambda\varphi_\lambda$ and $\|t\|_\varpi^2=\sum_{\lambda\in\Lambda_m}a_\lambda^2$. Then:

$$\begin{aligned}\sup_{t\in\mathscr{B}_m}\nu_n^2(t) &= \sup_{\sum_\lambda a_\lambda^2\leq 1}\left(\sum_{\lambda\in\Lambda_m}a_\lambda\nu_n\left(\varphi_\lambda\right)\right)^2\\ &\leq \sup_{\sum_\lambda a_\lambda^2\leq 1}\left(\sum_{\lambda\in\Lambda_m}a_\lambda^2\right)\left(\sum_{\lambda\in\Lambda_m}\nu_n^2\left(\varphi_\lambda\right)\right)\\ &= \sum_{\lambda\in\Lambda_m}\nu_n^2\left(\varphi_\lambda\right).\end{aligned}$$

It remains to bound $\mathbb{E}\left(\nu_n^2\left(\varphi_\lambda\right)\right)$. By (5),

$$\begin{aligned}\mathbb{E}\left[\nu_n^2(\varphi_\lambda)\right] &= \frac{1}{n^2}\sum_{k=1}^n\mathbb{E}\left[\varphi_\lambda^2(X_{k\Delta})\mathbb{E}\left[\left(Z_{k\Delta}+T_{k\Delta}\right)^2\middle|\mathscr{F}_{k\Delta}\right]\right]\\ &+ \frac{2}{n^2}\sum_{k<l}^n\mathbb{E}\left[(Z_{k\Delta}+T_{k\Delta})\varphi_\lambda(X_{k\Delta})\varphi_\lambda(X_{l\Delta})\mathbb{E}\left[Z_{l\Delta}+T_{l\Delta}\middle|\mathscr{F}_{l\Delta}\right]\right]\end{aligned}$$

Thanks to Lemma 1, the second term of this inequality is null and we obtain, as $\int_{\mathbb{R}}\varphi_\lambda^2(x)\varpi(dx)=1$:

$$\mathbb{E}\left[\nu_n^2(\varphi_\lambda)\right]\leq\frac{2(\sigma_0^2+\xi_0^2)}{n^2\Delta}\sum_{k=1}^n\mathbb{E}\left[\varphi_\lambda^2(X_{k\Delta})\right]=\frac{2(\sigma_0^2+\xi_0^2)}{n\Delta}.$$

Therefore:

$$\mathbb{E}\left[\left\|\hat{b}_m-b_A\right\|_n^2\mathbb{1}_{\Omega_n}\right]\leq 3\left\|b_m-b_A\right\|_n^2+48(\sigma_0^2+\xi_0^2)\frac{D_m}{n\Delta}+C\Delta.$$

Step 2: bound of the risk on Ω_n^c. The process $(X_t)_{t\geq 0}$ is exponentially β-mixing, π is bounded from below and above and $n\Delta\to\infty$. The following result is proved for $\xi=0$ for instance in Comte *et al.* (2007), but as it relies only on the β-mixing property, we can apply it.

Result 2.

$$\mathbb{P}\left[\Omega_n^c\right]\leq\frac{1}{n^3}.$$

Let us set $e=(e_\Delta,\ldots,e_{n\Delta})^*$ where $e_{k\Delta}:=Y_{k\Delta}-b(X_{k\Delta})=I_{k\Delta}+Z_{k\Delta}+T_{k\Delta}$ and $\Pi_mY=\Pi_m\left(Y_\Delta,\ldots,Y_{n\Delta}\right)^*=\left(\hat{b}_m(X_0),\ldots,\hat{b}_m(X_{n\Delta})\right)^*$ where Π_m is the

Euclidian orthogonal projection over S_m. Then

$$\begin{aligned}\left\|\hat{b}_m - b_A\right\|_n^2 &= \|\Pi_m Y - b_A\|_n^2 = \|\Pi_m b_A - b_A\|_n^2 + \|\Pi_m Y - \Pi_m b_A\|_n^2 \\ &\leq \|b_A\|_n^2 + \|e\|_n^2 .\end{aligned}$$

According to Lemma 1, Result 2 and the Cauchy-Schwarz inequality,

$$\mathbb{E}\left[\|e\|_n^2 \mathbb{1}_{\Omega_n^c}\right] \leq \left(\mathbb{E}\left[\|e\|_n^4\right]\right)^{1/2} \left(\mathbb{P}\left(\Omega_n^c\right)\right)^{1/2} \leq \frac{C}{\left(\Delta^3 n^3\right)^{1/2}} \leq \frac{C}{n\Delta}$$

and, as b is bounded on the compact set A,

$$\mathbb{E}\left[\|b_A\|_n^2 \mathbb{1}_{\Omega_n^c}\right] \leq \left(\mathbb{E}\left[\|b_A\|_n^4\right] \mathbb{P}\left(\Omega_n^c\right)\right)^{1/2} \lesssim \frac{1}{n^{3/2}}.$$

Collecting the results, we get:

$$\mathbb{E}\left[\left\|\hat{b}_m - b_A\right\|_n^2 \mathbb{1}_{\Omega_n^c}\right] \lesssim \frac{1}{n\Delta}$$

which ends the proof of Theorem 1.

6.2.1 Proof of Lemma 1

By Proposition 1, as b is Lipschitz,

$$\begin{aligned}\mathbb{E}\left[I_{k\Delta}^2\right] &= \frac{1}{\Delta^2}\mathbb{E}\left[\left(\int_{k\Delta}^{(k+1)\Delta} \left(b(X_s) - b(X_{k\Delta})\right) ds\right)^2\right] \\ &\leq \frac{1}{\Delta}\mathbb{E}\left[\int_{k\Delta}^{(k+1)\Delta} \left(b(X_s) - b(X_{k\Delta})\right)^2 ds\right] \\ &\leq \frac{c}{\Delta}\int_{k\Delta}^{(k+1)\Delta} \mathbb{E}\left[\left(X_s - X_{k\Delta}\right)^2 ds\right] \\ &\leq c\Delta.\end{aligned}$$

In the same way, we prove that $\mathbb{E}\left[I_{k\Delta}^4\right] \leq c\Delta$. We have that

$$\mathbb{E}\left[\left. Z_{k\Delta}^2 \right| \mathscr{F}_{k\Delta}\right] = \mathbb{E}\left[\left(\frac{1}{\Delta^2}\int_{k\Delta}^{(k+1)\Delta} \sigma^2(X_s) ds\right)\middle| \mathscr{F}_{k\Delta}\right] \leq \frac{\sigma_0^2}{\Delta}.$$

Moreover, by the Burkholder-Davis-Gundy inequality, we get

$$\mathbb{E}\left[\left. Z_{k\Delta}^4 \right| \mathscr{F}_{k\Delta}\right] \leq \frac{C}{\Delta^4}\mathbb{E}\left[\left(\int_{k\Delta}^{(k+1)\Delta} \sigma^2(X_s) ds\right)^2\middle| \mathscr{F}_{k\Delta}\right] \leq \frac{C}{\Delta^2}.$$

According to Applebaum (2004), Theorem 4.2.3 p224,

$$\mathbb{E}\left[T_{k\Delta}\middle|\mathscr{F}_{k\Delta}\right]=0$$

and, as $\int_{\mathbb{R}} z^2\nu(dz)=1$:

$$\mathbb{E}\left[T_{k\Delta}^2\middle|\mathscr{F}_{k\Delta}\right]=\frac{1}{\Delta^2}\int_{k\Delta}^{(k+1)\Delta}\xi^2(X_s)\int_{\mathbb{R}}z^2\nu(dz)ds\leq\frac{\xi_0^2}{\Delta}.$$

By Result 1, we have

$$\begin{aligned}\mathbb{E}\left[T_{k\Delta}^4\middle|\mathscr{F}_{k\Delta}\right] &\leq \frac{C_p}{\Delta^4}\mathbb{E}\left[\left(\int_{k\Delta}^{(k+1)\Delta}\xi^2(X_s)\int_{\mathbb{R}}z^2\nu(dz)ds\right)^2\right]\\ &+ \frac{C_p}{\Delta^4}\mathbb{E}\left[\left(\int_{k\Delta}^{(k+1)\Delta}\xi^4(X_s)ds\right)\right]\int_{\mathbb{R}}z^4\nu(dz)\\ &\lesssim \frac{1}{\Delta^3}\end{aligned}$$

6.3 Proof of Theorem 2

The bound of the risk on Ω_n^c is done exactly in the same way as for the non adaptive estimator. It remains thus to bound the risk on Ω_n. As in the previous proof, we get:

$$\begin{aligned}\left\|\hat{b}_{\hat{m}}-b_A\right\|_n^2\mathbb{1}_{\Omega_n} &\leq 3\left\|b_m-b_A\right\|_n^2+\frac{24}{n}\sum_{k=1}^n I_{k\Delta}^2+2pen(m)-2pen(\hat{m})\\ &+ 24\sup_{t\in\mathscr{B}_{m,\hat{m}}}\nu_n^2(t)\end{aligned}$$

where $\mathscr{B}_{m,m'}$ is the unit ball (for the L^2_ϖ-norm) of the subspace $S_m+S_{m'}$: $\mathscr{B}_{m,m'}=\{t\in S_m+S_{m'},\ \|t\|_\varpi\leq 1\}$. Let us introduce a function $p(m,m')$ such that $12p(m,m')=pen(m)+pen(m')$. We obtain that, on Ω_n, for any $m\in\mathscr{M}_n$:

$$\begin{aligned}\left\|\hat{b}_{\hat{m}}-b_A\right\|_n^2 &\leq 3\left\|b_m-b_A\right\|_n^2+\frac{24}{n}\sum_{k=1}^n I_{k\Delta}^2+4pen(m)\\ &+ 24\sup_{t\in\mathscr{B}_{m,\hat{m}}}\left(\nu_n^2(t)-p(m,\hat{m})\right).\end{aligned}$$

It remains to bound

$$\mathbb{E}\left[\sup_{t\in\mathscr{B}_{m,\hat{m}}}\nu_n^2(t)-p(m,\hat{m})\right]\leq\sum_{m'}\mathbb{E}\left[\sup_{t\in\mathscr{B}_{m,m'}}\nu_n^2(t)-p(m,m')\right]_+.$$

For this purpose, we use the following proposition proved in Applebaum (2004) (Corollary 5.2.2).

Proposition 2 : exponential martingale.
Let $(Y_t)_{t\geq 0}$ satisfy:

$$Y_t = \int_0^t F_s dW_s + \int_0^t K_s dL_s - \int_0^t \left[\frac{F_s^2}{2} + \int_{\mathbb{R}} \left(e^{K_s z} - 1 - K_s z \right) \nu(dz) \right] ds$$

where F_s and K_s are locally integrable and previsible processes. If for any $t > 0$,

$$\mathbb{E}\left[\int_0^t \int_{|z|>1} \left| e^{K_s z} - 1 \right| \nu(dz) ds \right] < \infty,$$

then e^{Y_t} is a $\mathscr{G}_t$-local martingale where $\mathscr{G}_t = \sigma(W_s, L_s, 0 \leq s \leq t)$.

For any $\varepsilon \leq \varepsilon_1 := (\lambda \wedge 1)/(2 \|t\|_\infty \xi_0)$ where λ is defined in Assumption A5, for any $t \geq 0$

$$\int_0^t \int_{|z|\geq 1} \left(\exp(\varepsilon t(X_{k\Delta})\xi(X_s)z) - 1 \right) \nu(dz) \mathbb{1}_{s\in]k\Delta,(k+1)\Delta]} ds < \infty.$$

Let us introduce the two Markov processes

$$A_{\varepsilon,t} := \varepsilon^2 \sum_{k=0}^{n} t^2(X_{k\Delta}) \int_0^t \sigma^2(X_s) \mathbb{1}_{s\in]k\Delta,(k+1)\Delta]} ds$$

and

$$B_{\varepsilon,t} := \sum_{k=0}^{n} \int_0^t \int_{\mathbb{R}} \left(\exp\left(\varepsilon t(X_{k\Delta})\xi(X_s)z \right) - \varepsilon t(X_{k\Delta})\xi(X_s)z - 1 \right) \mathbb{1}_{s\in]k\Delta,(k+1)\Delta]} \nu(dz) ds$$

and the following martingale:

$$M_t = \int_0^t \sum_{k=0}^{n} \mathbb{1}_{s\in]k\Delta,(k+1)\Delta]} t(X_{k\Delta^-}) \left(\sigma(X_s) dW_s + \xi(X_{s^-}) dL_s \right).$$

By Proposition 2,

$$Y_{\varepsilon,s} := \varepsilon M_s - A_{\varepsilon,s} - B_{\varepsilon,s}$$

is such that $e^{Y_{\varepsilon,s}}$ is a local martingale.

Bound of $A_{\varepsilon,s}$ and $B_{\varepsilon,s}$. We obtain easily that $A_{\varepsilon,s} \leq A_{\varepsilon,(n+1)\Delta} \leq \varepsilon^2 n\Delta \|t\|_n^2 \sigma_0^2$. Under Assumption A5, ξ is constant or ν is symmetric, and therefore

$$B_{\varepsilon,s} \leq B_{\varepsilon,(n+1)\Delta} \leq \Delta \sum_{k=0}^{n} \int_{\mathbb{R}} \left(\exp\left(\varepsilon t(X_{k\Delta})\xi_0 z \right) - \varepsilon t(X_{k\Delta^-})\xi_0 z - 1 \right) \nu(dz).$$

As $\int_{\mathbb{R}} z^2 \nu(dz) = 1$, for any $\alpha \leq 1$,

$$\int_{-1}^{1} \left(\exp(\alpha z) - \alpha z - 1 \right) \nu(dz) \leq \alpha^2 \int_{-1}^{1} z^2 \nu(dz) \leq \alpha^2.$$

Moreover, by integration by parts, for any $\alpha \leq (1 \wedge \lambda)/2$,

$$\begin{aligned}\int_{[-1,1]^c} (\exp(\alpha z) - \alpha z - 1)\,\nu(dz) &\leq (e^{\alpha} - \alpha - 1)\,\nu([1,+\infty[) + \left(e^{-\alpha} + \alpha - 1\right)\nu(]-\infty,-1]) \\ &+ \int_1^{+\infty} \alpha\,(e^{\alpha z} - 1)\,\nu([-z,z]^c)dz \\ &\leq 2\alpha^2\nu\left([-1,1]^c\right) + Ce^{-\lambda}\frac{\alpha}{\lambda}\left(\frac{e^{\alpha}}{1-\alpha/\lambda} - 1\right) \leq C'\alpha^2.\end{aligned}$$

Then $B_{\varepsilon,s} \lesssim n\Delta\varepsilon^2\xi_0^2\left\|t\right\|_n^2$. There exists a constant c such that, for any $\varepsilon < \varepsilon_1$,

$$A_{\varepsilon,s} + B_{\varepsilon,s} \leq c\frac{n\Delta\varepsilon^2\left(\sigma_0^2 + \xi_0^2\right)\left\|t\right\|_n^2}{(1-\varepsilon/\varepsilon_1)}.$$

Bound of $\mathbb{P}\left(\nu_n(t) \geq \eta,\ \|t\|_n^2 \leq \zeta^2\right)$. The process $\exp(Y_{\varepsilon,t})$ is a local martingale, then there exists an increasing sequence (τ_N) of stopping times such that $\lim_{N\to\infty}\tau_N = \infty$ and $\exp(Y_{\varepsilon,t\wedge\tau_N})$ is a $\mathscr{F}_t$-martingale. For any $\varepsilon < \varepsilon_1$, and all N,

$$\begin{aligned}E &:= \mathbb{P}\left(M_{(n+1)\Delta\wedge\tau_N} \geq n\Delta\eta,\ \|t\|_n^2 \leq \zeta^2\right) \\ &\leq \mathbb{P}\left(M_{(n+1)\Delta\wedge\tau_N} \geq n\Delta\eta,\ A_{(n+1)\Delta\wedge\tau_N} + B_{(n+1)\Delta\wedge\tau_N} \leq \frac{cn\Delta\varepsilon^2\left(\sigma_0^2+\xi_0^2\right)\zeta^2}{(1-\varepsilon/\varepsilon_1)}\right) \\ &\leq \mathbb{E}\left(\exp(Y_{\varepsilon,(n+1)\Delta\wedge\tau_N})\right)\exp\left(-n\Delta\eta\varepsilon + \frac{cn\Delta\varepsilon^2\left(\xi_0^2+\sigma_0^2\right)\zeta^2}{(1-\varepsilon/\varepsilon_1)}\right).\end{aligned}$$

As $\exp(Y_{\varepsilon,t\wedge\tau_N})$ is a martingale, $\mathbb{E}(\exp(Y_{\varepsilon,t\wedge\tau_N}) = 1$ and

$$E \leq \exp\left(-n\Delta\eta\varepsilon + \frac{cn\Delta\varepsilon^2\left(\xi_0^2+\sigma_0^2\right)\zeta^2}{(1-\varepsilon/\varepsilon_1)}\right).$$

Letting N tends to infinity, by dominated convergence, and as $\nu_n(t) = n\Delta M_{(n+1)\Delta}$, we obtain that

$$\mathbb{P}\left(\nu_n(t) \geq \eta,\ \|t\|_n^2 \leq \zeta^2\right) \leq \exp\left(-n\Delta\eta\varepsilon + \frac{cn\Delta\varepsilon^2\left(\xi_0^2+\sigma_0^2\right)\zeta^2}{(1-\varepsilon/\varepsilon_1)}\right).$$

It remains to minimise this inequality in ε. Let us set

$$\varepsilon = \frac{\eta}{2c\left(\sigma_0^2+\xi_0^2\right)\zeta^2/\Delta + \eta/\varepsilon_1} < \varepsilon_1.$$

We get:

$$\mathbb{P}\left(\nu_n(t) \geq \eta,\ \|t\|_n^2 \leq \zeta^2\right) \leq \exp\left(-\frac{\eta^2 n\Delta}{4c\left(\left(\sigma_0^2+\xi_0^2\right)\zeta^2 + c'\eta\xi_0\left\|t\right\|_\infty\right)}\right).$$

The following lemma concludes the proof. It is proved thanks to a $L^2_\varpi - L^\infty$ chaining technique. See Comte (2001), proof of Proposition 4, and Schmisser (2010), Appendix D.3.

Lemma 2.
There exists a constant κ such that:

$$\mathbb{E}\left[\sup_{t\in\mathscr{B}_{m,m'}} \nu_n^2(t) - p(m,m')\right] \lesssim \kappa(\xi_0^2+\sigma_0^2)\frac{D^{3/2}}{n\Delta}e^{-\sqrt{D}}$$

where $D = \dim(S_m + S_{m'})$.

As $\sum_D D^{3/2}e^{-\sqrt{D}} \leq \sum_{k=0}^{+\infty} k^{3/2}e^{-\sqrt{k}} < \infty$, we obtain that

$$\mathbb{E}\left[\sup_{t\in\mathscr{B}_{m,\hat{m}}} \nu_n^2(t) - p(m,\hat{m})\right] \leq \sum_{m'\in\mathscr{M}_n} \mathbb{E}\left[\sup_{t\in\mathscr{B}_{m,m'}} \nu_n^2(t) - p(m,m')\right] \lesssim \kappa\frac{\xi_0^2+\sigma_0^2}{n\Delta}.$$

6.4 Proof of Theorem 3

We recall that

$$\Omega_{X,k} = \left\{\omega,\ \left|X_{(k+1)\Delta} - X_{k\Delta}\right| \leq C_\Delta = (b_{max}+3)\,\Delta + (\sigma_0 + 4\xi_0)\,\sqrt{\Delta}\ln(n)\right\}.$$

Let us introduce the set

$$\Omega_{N,k} = \left\{\omega,\ N'_{k\Delta} = 0\right\}$$

where $N'_{k\Delta}$ is the number of jumps of size larger than $\Delta^{1/4}$ occuring in the time interval $]k\Delta,(k+1)\Delta]$:

$$N'_{k\Delta} = \mu\left(]k\Delta,(k+1)\Delta]\ ,\ \left[-\Delta^{1/4},\Delta^{1/4}\right]^c\right).$$

We have that

$$\begin{aligned}
\tilde{Y}_{k\Delta} &= Y_{k\Delta}\mathbb{1}_{\Omega_{X,k}}\mathbb{1}_{X_{k\Delta}\in A} \\
&= b_A(X_{k\Delta}) - b_A(X_{k\Delta})\mathbb{1}_{\Omega^c_{X,k}\cap(X_{k\Delta}\in A)} + I_{k\Delta}\mathbb{1}_{\Omega_{X,k}\cap(X_{k\Delta}\in A)} + \tilde{Z}_{k\Delta} + \tilde{T}_{k\Delta} \\
&+ \left(Z_{k\Delta}+T_{k\Delta}\right)\mathbb{1}_{\Omega_{X,k}\cap\Omega^c_{N,k}\cap(X_{k\Delta}\in A)} + \mathbb{E}\left(\left(Z_{k\Delta}+T_{k\Delta}\right)\mathbb{1}_{\Omega_{X,k}\cap\Omega_{N,k}\cap(X_{k\Delta}\in A)}\middle|\mathscr{F}_{k\Delta}\right).
\end{aligned}$$

where

$$\tilde{Z}_{k\Delta} = Z_{k\Delta}\mathbb{1}_{\Omega_{X,k}\cap\Omega_{N,k}\cap(X_{k\Delta}\in A)} - \mathbb{E}\left(Z_{k\Delta}\mathbb{1}_{\Omega_{X,k}\cap\Omega_{N,k}\cap(X_{k\Delta}\in A)}\middle|\mathscr{F}_{k\Delta}\right)$$

and

$$\tilde{T}_{k\Delta} = T_{k\Delta}\mathbb{1}_{\Omega_{X,k}\cap\Omega_{N,k}\cap(X_{k\Delta}\in A)} - \mathbb{E}\left(T_{k\Delta}\mathbb{1}_{\Omega_{X,k}\cap\Omega_{N,k}\cap(X_{k\Delta}\in A)}\middle|\mathscr{F}_{k\Delta}\right).$$

As previously, we only bound the risk on Ω_n. Let us set

$$\tilde{\nu}_n(t) := \frac{1}{n}\sum_{k=1}^{n} t(X_{k\Delta})\left(\tilde{Z}_{k\Delta} + \tilde{T}_{k\Delta}\right).$$

We have that

$$\begin{aligned}
\left\|\tilde{b}_m - b_A\right\|_n^2 \mathbb{1}_{\Omega_n} &\leq 3\left\|b_m - b_A\right\|_n^2 + 24\sup_{t\in\mathscr{B}_m}\tilde{\nu}_n^2(t) + \frac{224}{n}\sum_{k=1}^{n}\left(I_{k\Delta}^2 + b_A^2(X_{k\Delta})\mathbb{1}_{\Omega_{X,k}^c}\right) \\
&+ \frac{224}{n}\sum_{k=1}^{n}\left(Z_{k\Delta}^2 + T_{k\Delta}^2\right)\mathbb{1}_{\Omega_{X,k}\cap\Omega_{N,k}^c\cap(X_{k\Delta}\in A)} \\
&+ \frac{224}{n}\sum_{k=1}^{n}\left(\mathbb{E}\left[\left(Z_{k\Delta} + T_{k\Delta}\right)\mathbb{1}_{\Omega_{X,k}\cap\Omega_{N,k}\cap(X_{k\Delta}\in A)}\middle|\mathscr{F}_{k\Delta}\right]\right)^2.
\end{aligned}$$

The following lemma is proved later.

Lemma 3. *1.* $\mathbb{P}(\Omega_{X,k}^c \cap (X_{k\Delta}\in A)) \lesssim \Delta^{1-\beta/2}$.

2. $\mathbb{P}(\Omega_{X,k}\cap\Omega_{N,k}^c\cap(X_{k\Delta}\in A)) \lesssim \Delta^{2-\beta/2}$.

3. $\left(\mathbb{E}\left[\left(Z_{k\Delta}+T_{k\Delta}\right)\mathbb{1}_{\Omega_{N,k}\cap\Omega_{X,k}\cap(X_{k\Delta}\in A)}\middle|\mathscr{F}_{k\Delta}\right]\right)^2 \lesssim \ln^2(n)\Delta^{1-\beta/2}$.

According to Lemma 1, $\mathbb{E}(I_{k\Delta}^2) \leq \Delta\mathrm{k}$. As b is bounded on the compact set A, $\mathbb{E}\left[b_A^2(X_{k\Delta})\mathbb{1}_{\Omega_{X,k}^c}\right] \lesssim \mathbb{P}(\Omega_{X,k}^c) \lesssim \Delta^{1-\beta/2}$. Moreover, on $\Omega_{X,k}$,

$$\begin{aligned}
\left(Z_{k\Delta}+T_{k\Delta}\right)^2\mathbb{1}_{\Omega_{X,k}\cap(X_{k\Delta}\in A)} &= \left(\frac{X_{(k+1)\Delta}-X_{k\Delta}}{\Delta} - b_A(X_{k\Delta}) - I_{k\Delta}\right)^2\mathbb{1}_{\Omega_{X,k}}\mathbb{1}_{X_{k\Delta}\in A} \\
&\lesssim \frac{\ln^2(n)}{\Delta} + b_A^2(X_{k\Delta}) + I_{k\Delta}^2
\end{aligned}$$

and then

$$\begin{aligned}
E &:= \mathbb{E}\left[\left(Z_{k\Delta}+T_{k\Delta}\right)^2\mathbb{1}_{\Omega_{X,k}\cap\Omega_{N,k}^c\cap(X_{k\Delta}\in A)}\right] \\
&\lesssim \left(\frac{\ln^2(n)}{\Delta} + b_{max}^2\right)\mathbb{P}\left(\Omega_{X,k}\cap\Omega_{N,k}^c\cap(X_{k\Delta}\in A)\right) + \mathbb{E}\left(I_{k\Delta}^2\right) \\
&\lesssim \ln^2(n)\Delta^{1-\beta/2}.
\end{aligned}$$

It remains to bound $\mathbb{E}\left(\sup_{t\in\mathscr{B}_m}\tilde{\nu}_n^2(t)\right)$. In the same way as in Subsection 6.2, we get:

$$\begin{aligned}
\mathbb{E}\left(\sup_{t\in\mathscr{B}_m}\tilde{\nu}_n^2(t)\right) &\leq \sum_{\lambda\in\Lambda_m}\mathbb{E}\left(\tilde{\nu}_n^2(\varphi_\lambda)\right) \leq \frac{2D_m}{n}\mathbb{E}\left(\tilde{Z}_\Delta^2 + \tilde{T}_\Delta^2\right) \\
&\leq \frac{2D_m}{n}\mathbb{E}\left(Z_\Delta^2 + T_\Delta^2\right) \leq 2\left(\sigma_0^2+\xi_0^2\right)\frac{D_m}{n\Delta}.
\end{aligned}$$

6.4.1 Proof of Lemma 3

Result 3. Let β be the Blumenthal-Getoor index of L_t. Then:

$$\nu([-z,z]^c) \lesssim z^{-\beta} \quad , \quad \int_{|x|\leq z\wedge a_0} x^2\nu(dx) \lesssim z^{2-\beta} \quad \textit{and} \quad \int_{|x|\leq z\wedge a_0} x^4\nu(dx) \lesssim z^{4-\beta}.$$

The constant a_0 is defined in A6.

Bound of $\mathbb{P}(\Omega_{X,k}^c \cap (X_{k\Delta} \in A))$. We have:

$$\mathbb{P}\left(\Omega_{X,k}^c \cap (X_{k\Delta} \in A)\right) = \mathbb{P}\left(\left\{\left|X_{(k+1)\Delta} - X_{k\Delta}\right| > C_\Delta\right\} \cap (X_{k\Delta} \in A)\right).$$

We know that $X_{(k+1)\Delta} - X_{k\Delta} = b(X_{k\Delta}) + I_{k\Delta} + Z_{k\Delta} + T_{k\Delta}$. Then

$$\mathbb{P}\left(\Omega_{X,k}^c \cap (X_{k\Delta} \in A)\right) \leq \mathbb{P}\left(|\Delta I_{k\Delta}| \geq \Delta\right) + \mathbb{P}\left(|\Delta Z_{k\Delta}| \geq \sigma_0\sqrt{\Delta}\ln(n)\right) + \mathbb{P}\left(|\Delta T_{k\Delta}| \geq \xi_0\sqrt{\Delta}\ln(n)\right). \tag{6}$$

By a Markov inequality and Lemma 1, we obtain:

$$\mathbb{P}\left(|\Delta I_{k\Delta}| \geq \Delta\right) \leq \frac{\mathbb{E}\left(\Delta^2 I_{k\Delta}^2\right)}{\Delta^2} \lesssim \Delta. \tag{7}$$

By Proposition 2, the process $\exp\left(c\int_0^t \sigma(X_{s^-})dW_s - c^2\int_0^t \sigma^2(X_s)ds\right)$ is a local martingale (as σ is bounded, it is in fact a martingale, see Liptser and Shiryaev (2001), pp 229-232). Then, by a Markov inequality:

$$\mathbb{P}\left(|\Delta Z_{k\Delta}| \geq \sigma_0\sqrt{\Delta}\ln(n)\right) \leq \frac{2}{n}\mathbb{E}\left[\exp\left(\frac{\sqrt{\Delta}Z_{k\Delta}}{\sigma_0}\right)\right] \lesssim \frac{1}{n}. \tag{8}$$

To bound inequality (6), it remains to bound $\mathbb{P}\left(|\Delta T_{k\Delta}| \geq \xi_0\sqrt{\Delta\ln(n)}\right)$. Let us set

$$T_{k\Delta} = T_{k\Delta}^{(1)} + T_{k\Delta}^{(2)} + T_{k\Delta}^{(3)} \quad \text{where} \quad T_{k\Delta}^{(i)} = \frac{1}{\Delta}\int_{k\Delta}^{(k+1)\Delta} \xi(X_{s^-})dL_s^{(i)}$$

with

$$L_t^{(1)} = \int_0^t \int_{[-\sqrt{\Delta},\sqrt{\Delta}]} z\tilde{\mu}(ds,dz) \quad , \quad L_t^{(2)} = \int_0^t \int_{[-\Delta^{1/4},-\sqrt{\Delta}]\cup[\sqrt{\Delta},\Delta^{1/4}]} z\tilde{\mu}(ds,dz)$$

$$L_t^{(3)} = \int_0^t \int_{[-\Delta^{1/4},\Delta^{1/4}]^c} z\tilde{\mu}(ds,dz).$$

Let us set $N_{k\Delta}^{''} = \mu\left(]k\Delta,(k+1)\Delta], \left[-\sqrt{\Delta},\sqrt{\Delta}\right]^c\right)$. By Result 3, we have:

$$\mathbb{P}\left(\left|T_{k\Delta}^{(2)} + T_{k\Delta}^{(3)}\right| > 0\right) = \mathbb{P}\left(N_{k\Delta}^{''} \geq 1\right) \lesssim \Delta\nu\left(\left[-\sqrt{\Delta},\sqrt{\Delta}\right]^c\right) \lesssim \Delta^{1-\beta/2}.$$

It remains to bound $\mathbb{P}\left[\left|\Delta T_{k\Delta}^{(1)}\right| \geq 2\xi_0\sqrt{\Delta}\ln(n)\right]$. We have that:

$$\mathbb{P}\left[\left|\Delta T_{k\Delta}^{(1)}\right| \geq 2\xi_0\sqrt{\Delta}\ln(n)\right] \leq 2\mathbb{P}\left[\exp\left(\varepsilon\int_{k\Delta}^{(k+1)\Delta}\xi(X_{s^-})dL_s^{(1)}\right) \geq n^{2\varepsilon\xi_0\sqrt{\Delta}}\right].$$

By Proposition 2, for any ε,

$$D_t := \exp\left(\varepsilon\int_{k\Delta}^{t}\xi(X_{s^-})dL_s^{(1)} - \int_{k\Delta}^{t}\int_{|z|\leq\sqrt{\Delta}}\left(\exp(\varepsilon z\xi(X_{s^-}) - 1 - \varepsilon z\xi(X_{s^-}))\,\nu(dz)\right)\right)$$

is a local martingale. Let us set $\varepsilon = 1/(2\xi_0\Delta^{1/2})$. There exists an increasing sequence of stopping times τ_N such that, for any N,

$$\begin{aligned}
F &:= \mathbb{P}\left[\exp\left(\frac{1}{2\xi_0\Delta^{1/2}}\int_{k\Delta}^{(k+1)\Delta\wedge\tau_N}\xi(X_{s^-})dL_s^{(1)}\right) \geq n\right] \\
&\leq n^{-1}\exp\left(\int_{k\Delta}^{(k+1)\Delta\wedge\tau_N}\int_{|z|\leq\sqrt{\Delta}}\left(\exp\left(\frac{z\xi(X_{s^-})}{2\xi_0\Delta^{1/2}}\right) - 1 - \frac{z\xi(X_{s^-})}{2\xi_0\Delta^{1/2}}\right)\nu(dz)\right) \\
&\leq n^{-1}\exp\left(2\Delta\int_{|z|\leq\sqrt{\Delta}}\frac{\xi_0^2z^2}{4\xi_0^2\Delta}\nu(dz)\right) \leq n^{-1}\exp\left(\int_{\mathbb{R}}z^2\nu(dz)\right) \leq n^{-1}.
\end{aligned}$$

When $N\to\infty$, by dominated convergence, we obtain:

$$\mathbb{P}\left(\left|\Delta T_{k\Delta}^{(1)}\right| \geq \xi_0\sqrt{\Delta}\ln(n)\right) \lesssim n^{-1}. \tag{9}$$

Bound of $\mathbb{P}\left(\Omega_{X,k}\cap\Omega_{N,k}^c\cap(X_{k\Delta}\in A)\right)$. We recall that $N'_{k\Delta} = \mu\left(]k\Delta,(k+1)\Delta],\,[-\Delta^{1/4},\Delta^{1/4}]^c\right)$. We have:

$$\Omega_{N,k}^c = \left\{N_{k\Delta}^{'}=1\right\}\cup\left\{N_{k\Delta}^{'}\geq 2\right\}$$

with

$$\mathbb{P}\left(N_{k\Delta}^{'}=1\right)\lesssim\Delta^{1-\beta/4} \quad\text{and}\quad \mathbb{P}\left(N_{k\Delta}^{'}\geq 2\right)\lesssim\Delta^{2-\beta/2}.$$

Then $\mathbb{P}\left(\Omega_{N,k}^c\cap\left\{N_{k\Delta}^{'}\geq 2\right\}\right)\lesssim\Delta^{2-\beta/2}$. We can write:

$$\begin{aligned}
G &:= \mathbb{P}\left(\Omega_{X,k}\cap(X_{k\Delta}\in A)\cap(N_{k\Delta}^{'}=1)\right) \\
&\leq \mathbb{P}\left(N_{k\Delta}^{'}=1\right)\mathbb{P}\left(\left|\Delta T_{k\Delta}^{(2)}+\Delta T_{k\Delta}^{(3)}\right|\leq 2C_\Delta\,\middle|\,N'_{k\Delta}=1\right) \\
&+ \mathbb{P}\left(N_{k\Delta}^{'}=1\right)\mathbb{P}\left(\left\{\left|\Delta T_{k\Delta}^{(2)}+\Delta T_{k\Delta}^{(3)}\right|\geq 2C_\Delta\,\middle|\,N'_{k\Delta}=1\right\}\cap\Omega_{X,k}\cap(X_{k\Delta}\in A)\right).
\end{aligned}$$

By (7), (8) and (9), we obtain:

$$\begin{aligned}
H &:= \mathbb{P}\left(\left\{\left|\Delta T_{k\Delta}^{(2)}+\Delta T_{k\Delta}^{(3)}\right|\geq 2C_\Delta\,\middle|\,N'_{k\Delta}=1\right\}\cap\Omega_{X,k}\cap(X_{k\Delta}\in A)\right) \\
&\leq \mathbb{P}\left(\Delta\left|b_A(X_{k\Delta})+I_{k\Delta}+Z_{k\Delta}+T_{k\Delta}^{(1)}\right|>C_\Delta\right) \\
&\lesssim \Delta+n^{-1}.
\end{aligned}$$

It remains to bound $J := \mathbb{P}\left(\left|\Delta T^{(2)}_{k\Delta} + \Delta T^{(3)}_{k\Delta}\right| \leq 2C_\Delta \,|\, N'_{k\Delta} = 1\right)$. If $N'_{k\Delta} = 1$, then $\left|\Delta T^{(3)}_{k\Delta}\right| = |\int_{k\Delta}^{(k+1)\Delta} \xi(X_{s^-}) dL^{(3)}_s| \geq \xi_1 \Delta^{1/4}$. Then $J \leq \mathbb{P}\left(\Delta \left|T^{(2)}_{k\Delta}\right| \geq \xi_1\Delta^{1/4} - 2C_\Delta\right)$. Let us set $n_0 = \left\lceil \frac{1}{1-\beta/2} \right\rceil$ and $a = (\xi_0 n_0)^{-1}\left(\xi_1\Delta^{1/4} - 2C_\Delta\right)$. We have:

$$\begin{aligned} J &\leq \mathbb{P}\left[\mu(]k\Delta,(k+1)\Delta],[-a,a]^c) \geq 1\right] + \mathbb{P}\left[\mu(]k\Delta,(k+1)\Delta],\, [-a,-\Delta^{1/2}] \cup [\Delta^{1/2},a]) \geq n_0\right] \\ &\leq \Delta\nu([-a,a]^c) + \Delta^{n_0}\nu([-\Delta^{1/2},\Delta^{1/2}]^c)^{n_0} \\ &\lesssim \Delta^{1-\beta/4} + \Delta. \end{aligned}$$

Then $\mathbb{P}(\Omega_{X,k} \cap \Omega^c_{N,k}) \leq \mathbb{P}(N'_{k\Delta} = 1)\Delta^{1-\beta/4} + \mathbb{P}(N'_{k\Delta} = 2) \lesssim \Delta^{2-\beta/2}$.

Bound of $\left(\mathbb{E}\left[(Z_{k\Delta} + T_{k\Delta})\,\mathbb{1}_{\Omega_{X,k}\cap\Omega_{N,k}\cap(X_{k\Delta}\in A)}\middle|\mathscr{F}_{k\Delta}\right]\right)^2$.

If σ and ξ are constants. Let us set $E := \left(\mathbb{E}\left[(Z_{k\Delta} + T_{k\Delta})\,\mathbb{1}_{\Omega_{X,k}\cap\Omega_{N,k}\cap(X_{k\Delta}\in A)}\middle|\mathscr{F}_{k\Delta}\right]\right)^2$ and

$$\Omega_{I,k} = \left\{\omega, |I_{k\Delta}| \leq 1, \cap\, |\Delta Z_{k\Delta}| \leq \sigma_0\sqrt{\Delta}\ln(n), \cap \left|\Delta T^{(1)}_{k\Delta}\right| \leq 2\xi_0\sqrt{\Delta}\ln(n)\right\}.$$

By (7), (8) and (9), $\mathbb{P}\left(\Omega^c_{I,k}\right) \leq \Delta + n^{-1}$. Then, by a Markov inequality:

$$E \lesssim \Delta\ln^2(n) + \left(\mathbb{E}\left[(Z_{k\Delta} + T_{k\Delta})\,\mathbb{1}_{\Omega_{X,k}\cap\Omega_{N,k}\cap\Omega_{I,k}\cap(X_{k\Delta}\in A)}\middle|\mathscr{F}_{k\Delta}\right]\right)^2.$$

Let us introduce the set $\Omega_{ZT,k} := \left\{\omega, |Z_{k\Delta} + T_{k\Delta}| \leq C_\Delta\Delta^{-1} - b_{max} - 1\right\}$. On $\Omega_{I,k}$, $|I_{k\Delta}| \leq 1$ and therefore:

$$\Omega_{ZT,k} \cap \Omega_{I,k} \subseteq \Omega_{X,k} \cap \Omega_{I,k} \subseteq \left\{\omega, |Z_{k\Delta} + T_{k\Delta}| \leq C_\Delta\Delta^{-1} + b_{max} + 1\right\} \cap \Omega_{I,k}.$$

Then

$$E \lesssim \Delta\ln^2(n) + F^2 + G^2$$

where $F = \mathbb{E}\left[(Z_{k\Delta} + T_{k\Delta})\,\mathbb{1}_{\Omega_{ZT,k}\cap\Omega_{N,k}\cap\Omega_{I,k}\cap(X_{k\Delta}\in A)}\middle|\mathscr{F}_{k\Delta}\right]$ and $G = \mathbb{E}\left[(Z_{k\Delta} + T_{k\Delta})\,\mathbb{1}_{\Omega^c_{ZT,k}\cap\Omega_{X,k}\cap\Omega_{N,k}\cap\Omega_{I,k}\cap(X_{k\Delta}\in A)}\middle|\mathscr{F}_{k\Delta}\right]$. As σ and ξ are constants, the terms

$$Z_{k\Delta} = \frac{\sigma_0}{\Delta}\int_{k\Delta}^{(k+1)\Delta} dW_s \quad \text{and} \quad T_{k\Delta} = \frac{\xi_0}{\Delta}\int_{k\Delta}^{(k+1)\Delta} dL_s$$

are centered and independent. Then $F = 0$. Moreover, on $\Omega_{N,k}$, $T^{(3)}_{k\Delta} = 0$. Then

$$|G| \lesssim \left|\mathbb{E}\left[\left(Z_{k\Delta} + T^{(1)}_{k\Delta} + T^{(2)}_{k\Delta}\right)\mathbb{1}_{\Omega_{X,k}\cap\Omega^c_{ZT,k}\cap\Omega_{N,k}\cap\Omega_{I,k}\cap(X_{k\Delta}\in A)}\middle|\mathscr{F}_{k\Delta}\right]\right|.$$

On $\Omega_{I,k} \cap \Omega_{X,k}$, $\left|Z_{k\Delta} + T^{(1)}_{k\Delta} + T^{(2)}_{k\Delta}\right| \lesssim \ln(n)\Delta^{-1/2}$, and

$$\begin{aligned}
|G| &\lesssim \frac{\ln(n)}{\sqrt{\Delta}} \left(\mathbb{P}\left(\left| Z_{k\Delta} + T^{(1)}_{k\Delta} + T^{(2)}_{k\Delta} \right| \in \left[C_\Delta \Delta^{-1} - b_{max} - 1, C_\Delta \Delta^{-1} + b_{max} + 1 \right] \mathbb{1}_{\Omega_{I,k}} \right) \right) \\
&= 2\frac{\ln(n)}{\sqrt{\Delta}} \int_{\mathbb{R}} \mathbb{P}\left(T^{(2)}_{k\Delta} \in \left[C_\Delta \Delta^{-1} - b_{max} - 1 - x, C_\Delta \Delta^{-1} + b_{max} + 1 - x \right] \mathbb{1}_{\Omega_{I,k}} \right) \\
&\times \mathbb{P}\left(Z_{k\Delta} + T^{(1)}_{k\Delta} \in dx \middle| T^{(2)}_{k\Delta} \in \left[C_\Delta \Delta^{-1} - b_{max} - 1 - x, C_\Delta \Delta^{-1} + b_{max} + 1 - x \right] \mathbb{1}_{\Omega_{I,k}} \right).
\end{aligned}$$

On $\Omega_{I,k}$, $\left|Z_{k\Delta} + T^{(1)}_{k\Delta}\right| \leq (\sigma_0 + 2\xi_0)\ln(n)\Delta^{-1/2}$. Then

$$|G| \lesssim \frac{\ln(n)}{\sqrt{\Delta}} \left[\sup_{C \geq \xi_0 \ln(n)\Delta^{-1/2}} \mathbb{P}\left(T^{(2)}_{k\Delta} \in [C, C + 2b_{max} + 2] \right) \right]. \tag{10}$$

We recall that $L^{(2)}_t$ is a compound Poisson process in which all the jumps are greater than $\sqrt{\Delta}$ and smaller than $\Delta^{1/4}$. Let us denote by τ_i the times of the jumps of size in $[\sqrt{\Delta}, \Delta^{1/4}]$ and by ζ_i the size of the jumps. We set $a_j = \xi_0^{-1} C\Delta - \sum_{i=1}^{j-1} \zeta_i$ and $c := \xi_0^{-1}(2b_{max} + 2)$. Then, as ξ is constant equal to ξ_0:

$$\begin{aligned}
H &:= \mathbb{P}\left(T^{(2)}_{k\Delta} \in [C, C + 2b_{max} + 2] \right) \\
&\leq \sum_{j=1}^{\infty} \mathbb{P}\left(j \text{ jumps } \geq \sqrt{\Delta}, \text{ last jump } \in [a_j, a_j + c\Delta] \right) \\
&\lesssim 2 \sup_{a \geq \sqrt{\Delta}} \mathbb{P}\left(1 \text{ jump } \in [a, a + c\Delta] \right) = 2\Delta \sup_{a \geq \sqrt{\Delta}} \nu\left([a, a + c\Delta] \right).
\end{aligned}$$

By A6,

$$H \lesssim \Delta \sup_{a \geq \sqrt{\Delta}} \left[\frac{1}{a^\beta} - \frac{1}{(a + c\Delta)^\beta} \right] \lesssim \sqrt{\Delta}\Delta^{1-\beta/2} \tag{11}$$

and, by (10) and (11),

$$E \lesssim \Delta \ln^2(n) + \frac{\ln^2(n)}{\Delta}\Delta\Delta^{2-\beta} \lesssim \Delta \ln^2(n) + \Delta^{2-\beta} \ln^2(n).$$

Remark 4. If ν is not absolutely continuous, we obtain:

$$E \leq \Delta \ln^2(n) + \left(\Delta\nu\left([-a + c\Delta, a - c\Delta]^c \right) \right)^2 \lesssim \Delta \ln^2(n) + \Delta^{2-2\beta} \ln^2(n).$$

If σ or ξ are not constants. The problem is that $Z_{k\Delta}$ and $T_{k\Delta}$ are not symmetric and we can't apply directly the previous method. We replace them by two centred terms. The following lemma is very useful.

Lemma 4.
Let f be a $\mathscr{C}^2$ function such that f and f' are Lipschitz. Let us set, for any $t \in]k\Delta,(k+1)\Delta]$:

$$\psi_f(X_{k\Delta},t) = f'(X_{k\Delta})\left(\sigma(X_{k\Delta})\int_{k\Delta}^{t} dW_s + \xi(X_{k\Delta})\int_{k\Delta}^{t} z\tilde{\mu}(ds,dz)\right).$$

We have:

$$\mathbb{E}\left[\left(f(X_t) - f(X_{k\Delta}) - \psi_f(X_{k\Delta},t)\right)^2 \mathbb{1}_{\Omega_{N,k}}\mathbb{1}_{X_{k\Delta}\in A}\right] \lesssim \Delta^{2-\beta/4}.$$

Lemma 4 is proved below. Let us set

$$\bar{Z}_{k\Delta} = \frac{1}{\Delta}\int_{k\Delta}^{(k+1)\Delta} \left(\sigma(X_{k\Delta}) + \psi_\sigma(X_{k\Delta,s})\right) dW_s,$$

$$\bar{T}^{(i)}_{k\Delta} = \frac{1}{\Delta}\int_{k\Delta}^{(k+1)\Delta} \left(\xi(X_{k\Delta}) + \psi_\xi(X_{k\Delta,s})\right) dL^{(i)}_s \quad \text{and} \quad \bar{T}_{k\Delta} = \bar{T}^{(1)}_{k\Delta} + \bar{T}^{(2)}_{k\Delta} + \bar{T}^{(3)}_{k\Delta}.$$

The terms $\bar{Z}_{k\Delta}$ and $\bar{T}_{k\Delta}$ are symmetric. By lemma 4,

$$\begin{aligned}\mathbb{E}\left[\left(\bar{Z}_{k\Delta} - Z_{k\Delta}\right)^2 \mathbb{1}_{\Omega_{N,k}}\mathbb{1}_{X_{k\Delta}\in A}\right] &= \frac{1}{\Delta^2}\mathbb{E}\left[\int_{k\Delta}^{(k+1)\Delta} \left(\sigma(X_s) - \sigma(X_{k\Delta}) - \psi_\sigma(X_{k\Delta,s})\right)^2 ds\right] \\ &\lesssim \Delta^{1-\beta/4}. \end{aligned} \tag{12}$$

We prove in the same way that

$$\mathbb{E}\left[\left(\bar{T}_{k\Delta} - T_{k\Delta}\right)^2 \mathbb{1}_{\Omega_{N,k}}\mathbb{1}_{X_{k\Delta}\in A}\right] \leq \Delta^{1-\beta/4}. \tag{13}$$

Let us set $U_{k\Delta} = \Delta^{-1}\xi(X_{k\Delta^-})\int_{k\Delta}^{(k+1)\Delta} dL^{(2)}_s$. By Result 1 and Proposition 1,

$$\mathbb{E}\left[\Delta^2\left(\bar{T}^{(2)}_{k\Delta} - U_{k\Delta}\right)^2\right] = \mathbb{E}\left[\int_{k\Delta}^{(k+1)\Delta}\int_{\mathbb{R}} \left(\xi(X_s) - \xi(X_{k\Delta})\right)^2 z^2\nu(dz)ds\right] \leq \Delta^2. \tag{14}$$

Let us introduce the set

$$\begin{aligned}\bar{\Omega}_{I,k} &= \left\{\omega, |I_{k\Delta}| + \left|Z_{k\Delta} - \bar{Z}_{k\Delta}\right| + \left|T_{k\Delta} - \bar{T}_{k\Delta}\right| \leq 3\right\} \\ &\bigcap \left\{\left|\Delta\bar{Z}_{k\Delta}\right| \leq \sigma_0\sqrt{\Delta}\ln(n) + \Delta,\ \left|\Delta\bar{T}^{(1)}_{k\Delta}\right| \leq 2\xi_0\sqrt{\Delta}\ln(n) + \Delta\right\} \\ &\bigcap \left\{\left|\Delta(\bar{T}^{(2)}_{k\Delta} - U_{k\Delta})\right| \leq \xi_0\sqrt{\Delta}\right\}.\end{aligned}$$

By (7), (8), (9), (12), (13), (14) and Markov inequalities, we obtain:

$$\mathbb{P}\left(\bar{\Omega}^c_{I,k}\right) \lesssim \Delta^{1-\beta/4} + \frac{1}{n}. \tag{15}$$

Then

$$\begin{aligned} E &:= \left(\mathbb{E}\left[\left(Z_{k\Delta}+T_{k\Delta}\right)\mathbb{1}_{\Omega_{X,k}\cap\Omega_{N,k}\cap(X_{k\Delta}\in A)}\middle|\mathscr{F}_{k\Delta}\right]\right)^2 \qquad (16)\\ &\lesssim \Delta^{1-\beta/2}\ln^2(n)+\left(\mathbb{E}\left[\left(\bar{Z}_{k\Delta}+\bar{T}_{k\Delta}\right)\mathbb{1}_{\Omega_{X,k}\cap\Omega_{N,k}\cap(X_{k\Delta}\in A)\cap\bar{\Omega}_{I,k}}\middle|\mathscr{F}_{k\Delta}\right]\right)^2. \end{aligned}$$

Let us introduce the set:

$$\bar{\Omega}_{ZT,k}:=\left\{\omega,\left|\bar{Z}_{k\Delta}+\bar{T}_{k\Delta}\right|\leq C_\Delta\Delta^{-1}-b_{max}-3\right\}.$$

We have that

$$\bar{\Omega}_{ZT,k}\cap\bar{\Omega}_{I,k}\subseteq\Omega_{X,k}\cap\bar{\Omega}_{I,k}\subseteq\left\{\omega,\left|\bar{Z}_{k\Delta}+\bar{T}_{k\Delta}\right|\leq C_\Delta\Delta^{-1}+b_{max}+3\right\}\cap\bar{\Omega}_{I,k}.$$

Given the filtration $\mathscr{F}_{k\Delta}$, the sum $\bar{Z}_{k\Delta}+\bar{T}_{k\Delta}$ is symmetric. Then

$$\mathbb{E}\left[\left(\bar{Z}_{k\Delta}+\bar{T}_{k\Delta}\right)\mathbb{1}_{\bar{\Omega}_{ZT,k}\cap\Omega_{N,k}\cap(X_{k\Delta}\in A)}\middle|\mathscr{F}_{k\Delta}\right]=0.$$

Moreover, on $\Omega_{N,k}$, $\bar{T}^{(3)}_{k\Delta}=0$. Then, by (16),

$$E\lesssim\Delta^{1-\beta/2}\ln^2(n)+G^2+H^2$$

where $G:=\mathbb{E}\left[\left(\bar{Z}_{k\Delta}+\bar{T}^{(1)}_{k\Delta}+\bar{T}^{(2)}_{k\Delta}\right)\mathbb{1}_{\Omega_{X,k}\cap\Omega^c_{ZT,k}\cap\Omega_{N,k}\cap\Omega_{I,k}\cap(X_{k\Delta}\in A)}\middle|\mathscr{F}_{k\Delta}\right]$ and $H:=\mathbb{E}\left[\left(\bar{Z}_{k\Delta}+\bar{T}^{(1)}_{k\Delta}+\bar{T}^{(2)}_{k\Delta}\right)\mathbb{1}_{\Omega_{X,k}\cap\Omega_{ZT,k}\cap\Omega_{N,k}\cap\Omega^c_{I,k}\cap(X_{k\Delta}\in A)}\middle|\mathscr{F}_{k\Delta}\right]$. We have that $H^2\lesssim\Delta^{-1}\ln^2(n)\mathbb{P}^2(\Omega^c_{I,k})\lesssim\Delta^{1-\beta/2}\ln^2(n)$. The end of the proof is the same as in the case of σ and ξ constants. We obtain that

$$|G|\lesssim\frac{\ln(n)}{\sqrt{\Delta}}\sup_{C\geq\kappa_0\ln(n)\Delta^{-1/2}}\mathbb{P}\left(U_{k\Delta}\in[C,C+2b_{max}+6]\right)\lesssim\sqrt{\Delta}\Delta^{1-\beta/2}.$$

6.4.2 Proof of Lemma 4

According to the Ito formula (see for instance Applebaum (2004), Theorem 4.4.7 p251), we have that

$$f(X_t)-f(X_{k\Delta})=I_1+I_2+I_3+I_4$$

where

$$I_1=\int_{k\Delta}^t f'(X_s)\sigma(X_s)dW_s\quad I_2=\int_{k\Delta}^t\int_{\mathbb{R}}\left(f\left(X_{s^-}+z\xi(X_{s^-})\right)-f(X_{s^-})\right)\tilde{\mu}(ds,dz)$$

$$I_3=\int_{k\Delta}^t\int_{z\in\mathbb{R}}\left[f(X_s+z\xi(X_s))-f(X_s)-z\xi(X_s)f'(X_s)\right]\nu(dz)ds$$

$$I_4=\int_{k\Delta}^t\left[f'(X_s)b(X_s)+f''(X_s)\sigma^2(X_s)/2\right]ds.$$

By Proposition 1, for any $t \leq (k+1)\Delta$, we have:

$$\begin{aligned}\mathbb{E}\left[\left(I_1 - f'(X_{k\Delta})\sigma(X_{k\Delta})\int_{k\Delta}^{t} dW_s\right)^2\right] &= \mathbb{E}\left[\left(\int_{k\Delta}^{t}\left(\sigma(X_s)f'(X_s) - \sigma(X_{k\Delta})f'(X_{k\Delta})\right)dW_s\right)^2\right] \\ &= \int_{k\Delta}^{t}\left(\sigma(X_s)f'(X_s) - \sigma(X_{k\Delta})f'(X_{k\Delta})\right)^2 ds \lesssim \Delta^2.\end{aligned}$$

We can write:

$$\begin{aligned}E &:= \mathbb{E}\left[\left(I_2 - f'(X_{k\Delta})\xi(X_{k\Delta^-})\int_{k\Delta}^{t} dL_s^{(1)} + dL_s^{(2)}\right)^2 \mathbb{1}_{\Omega_{N,k}}\right] \\ &\leq 2\int_{k\Delta}^{t}\int_{|z|\leq\Delta^{1/4}} \mathbb{E}\left[\left(f(X_s + z\xi(X_s)) - f(X_s) - z\xi(X_s)f'(X_s)\right)^2\right]\nu(dz)ds \\ &+ 2\int_{k\Delta}^{t}\int_{|z|\leq\Delta^{1/4}} \mathbb{E}\left[z^2\left(\xi(X_s)f'(X_s) - \xi(X_{k\Delta})f'(X_{k\Delta})\right)^2\right]\nu(dz)ds.\end{aligned}$$

The function f is $\mathscr{C}^2$, then, by the Taylor formula, for any $s \in [k\Delta, t]$, $z \in \mathbb{R}$, there exists $\zeta_{s,z}$ in $[X_s, X_s + z\xi(X_s)]$ such that:

$$f\left(X_s + z\xi(X_s)\right) - f(X_s) - z\xi(X_s)f'(X_s) = \frac{z^2\xi^2(X_s)}{2}f''(\zeta_{s,z}).$$

Then, as ξ and f'' are bounded:

$$\mathbb{E}\left[\left(f(X_s + z\xi(X_s) - f(X_s) - z\xi(X_s)f'(X_s)\right)^2\right] = \frac{z^4}{4}\mathbb{E}\left[\left(\xi(X_s)f''(\zeta_{s,z})\right)^2\right] \lesssim z^4$$

and, by Result 3, for any $t \leq (k+1)\Delta$,

$$\begin{aligned}F &:= \int_{k\Delta}^{t}\int_{|z|\leq\Delta^{1/4}} \mathbb{E}\left[\left(f(X_s + z\xi(X_s) - f(X_s) - z\xi(X_s)f'(X_s)\right)^2\right]\nu(dz)ds \\ &\lesssim \Delta\int_{|z|\leq\Delta^{1/4}} z^4\nu(dz) \lesssim \Delta^{2-\beta/4}.\end{aligned}$$

The functions ξ and f' are Lipschitz, then by Proposition 1,

$$\mathbb{E}\left[z^2\left(\xi(X_s)f'(X_s) - \xi(X_{k\Delta})f'(X_{k\Delta})\right)^2\right] \lesssim z^2\mathbb{E}\left[\left(X_s - X_{k\Delta}\right)^2\right] \lesssim \Delta z^2$$

and consequently, for any $t \leq (k+1)\Delta$:

$$\int_{k\Delta}^{t}\int_{|z|\leq\Delta^{1/4}} \mathbb{E}\left[z^2\left(\xi(X_s)f'(X_s) - \xi(X_{k\Delta})f'(X_{k\Delta})\right)^2\right]\nu(dz)ds \lesssim \Delta^{3-\beta/2}$$

then $E \lesssim \Delta^{2-\beta/4}$. By the same way, we obtain that

$$\mathbb{E}\left[I_3^2\right] \leq \mathbb{E}\left[\int_{k\Delta}^{t}\int_{|z|\leq\Delta^{1/4}}\left(\frac{z^2\xi^2(X_s)}{2}f''(\zeta_{s,z})\right)^2\nu(dz)ds\right] \leq \Delta^{2-\beta/4}.$$

The functions b and f' are Lipschitz and f'' and σ are bounded, then, for any $t \leq (k+1)\Delta$:

$$\mathbb{E}\left[I_4^2\right] \lesssim \Delta \int_{k\Delta}^{t} \left(1 + \mathbb{E}\left[X_s^4\right]\right) ds \lesssim \Delta^2.$$

Then, for any $t \leq (k+1)\Delta$:

$$\mathbb{E}\left[(f(X_t) - f(X_{k\Delta}) - \psi_f(X_{k\Delta}, t))\right] \leq \Delta^{2-\beta/4}.$$

6.5 Proof of Theorem 4

As previously, we only bound the risk on Ω_n. As in Subsection 6.3, we introduce the function $p(m, m')$ such that $p(m, m') = 12(pen(m) + pen(m'))$. On Ω_n, for any $m \in \mathscr{M}_n$, we have:

$$\begin{aligned}
\left\|\tilde{b}_{\tilde{m}} - b_A\right\|_n^2 &\leq 3\left\|b_m - b_A\right\|_n^2 + \frac{224}{n}\sum_{k=1}^{n} b_A^2(X_{k\Delta})\mathbb{1}_{\Omega_{X,k}^c} + I_{k\Delta}^2 + 2\left(Z_{k\Delta}^2 + T_{k\Delta}^2\right)\mathbb{1}_{\Omega_{X,k}\cap\Omega_{Z,k}^c} \\
&+ \frac{224}{n}\sum_{k=1}^{n}\left(\mathbb{E}\left[\left(Z_{k\Delta} + T_{k\Delta}\right)\mathbb{1}_{\Omega_{X,k}\cap\Omega_{Z,k}}\middle|\mathscr{F}_{k\Delta}\right]\right)^2 \\
&+ 24\sup_{t\in\mathscr{B}_{m,\tilde{m}}}\left(\tilde{\nu}_n^2(t) - p(m,\tilde{m})\right) + 4pen(m).
\end{aligned}$$

It remains only to bound $\mathbb{E}\left[\sup_{t\in\mathscr{B}_{m,\tilde{m}}}\left(\tilde{\nu}_n^2(t) - p(m,\tilde{m})\right)\right] \leq \sum_{m'}\mathbb{E}\left[\sup_{t\in\mathscr{B}_{m,m'}}\left(\tilde{\nu}_n^2(t) - p(m,\tilde{m})\right)\right]$. As in the proof of Theorem 2, we bound the quantity

$$\mathbb{E}\left[\exp\left(\varepsilon t(X_{k\Delta})\left(\tilde{Z}_{k\Delta} + \tilde{T}_{k\Delta}\right)\right)\middle|\mathscr{F}_{k\Delta}\right].$$

We have that

$$\mathbb{E}\left[\exp\left(\varepsilon t(X_{k\Delta})Z_{k\Delta}\right)\mathbb{1}_{\Omega_{N,k}}\middle|\mathscr{F}_{k\Delta}\right] \leq \exp\left(\frac{\varepsilon^2\sigma_0^2 t^2(X_{k\Delta})}{2\Delta}\right).$$

The truncated Lévy process $\tilde{L}_t = \int_0^t\int_{|z|\leq\Delta^{1/4}} z\tilde{\mu}(ds,dz)$ satisfies Assumption A5 and then there exists a constant c such that:

$$\mathbb{E}\left[\exp\left(\varepsilon t(X_{k\Delta})T_{k\Delta}\right)\mathbb{1}_{\Omega_{N,k}}\middle|\mathscr{F}_{k\Delta}\right] \leq \exp\left(\frac{c\varepsilon^2\xi_0^2 t^2(X_{k\Delta})}{\Delta\left(1-\varepsilon/\varepsilon_1\right)}\right).$$

As $Z_{k\Delta}\mathbb{1}_{\Omega_{N,k}}$ and $T_{k\Delta}\mathbb{1}_{\Omega_{N,k}}$ are centred, we obtain:

$$\mathbb{E}\left[\exp\left(\varepsilon\left|t(X_{k\Delta})\left(Z_{k\Delta} + T_{k\Delta}\right)\right|\right)\mathbb{1}_{\Omega_{N,k}}\middle|\mathscr{F}_{k\Delta}\right] \leq 2\exp\left(\frac{c\varepsilon^2\left(\sigma_0^2+\xi_0^2\right)t^2(X_{k\Delta})}{\Delta\left(1-\varepsilon/\varepsilon_1\right)}\right)$$

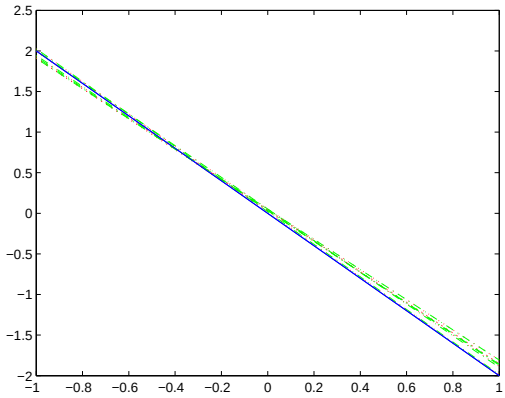
and then

$$\mathbb{E}\left[\exp\left(\varepsilon\left|t(X_{k\Delta})\left(\tilde{Z}_{k\Delta} + \tilde{T}_{k\Delta}\right)\right|\right)\mathbb{1}_{\Omega_{N,k}\cap\Omega_{X,k}}\middle|\mathscr{F}_{k\Delta}\right] \leq 2\exp\left(\frac{c\varepsilon^2\left(\sigma_0^2+\xi_0^2\right)t^2(X_{k\Delta})}{\Delta\left(1-\varepsilon/\varepsilon_1\right)}\right).$$

We conclude as in the proof of Theorem 2.

Figure 1: Model 1: Ornstein-Uhlenbeck and binomial law

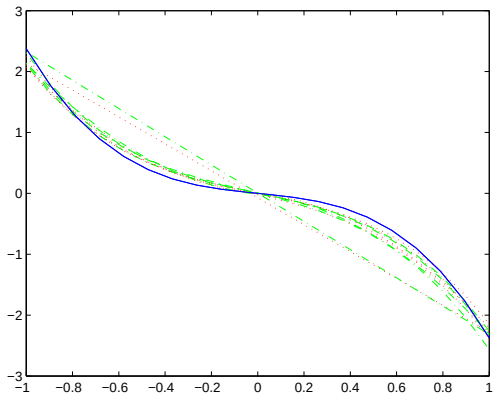
$b(x) = -2x,\ \sigma(x) = \xi(x) = 1$ and binomial law



– : true function -.-: first estimator ...: truncated estimator
$n = 10^4$ et $\Delta = 10^{-1}$

Figure 2: Model 2: Double well and Laplace law

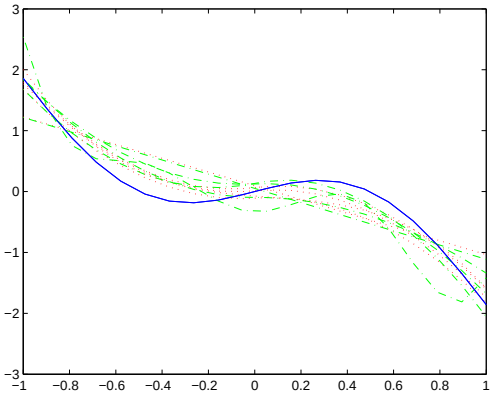
$b(x) = -(x - 1/4)^3 - (x + 1/4)^3, \quad \sigma = \xi = 1$ and Laplace law



– : true function -.-: first estimator ...: truncated estimator
$n = 10^4$ et $\Delta = 10^{-1}$

Figure 3: Model 3: Sine function

$b(x) = -2x+\sin(3x),\ \sigma(x) = \xi(x) = \sqrt{(3+x^2)/(1+x^2)}$ jumps not sub-exponential



– : true function -.-: first estimator ...: truncated estimator
$n = 10^4$ et $\Delta = 10^{-1}$

Table 1: Model 1: Ornstein-Uhlenbeck and binomial law

$b(x) = -2x,\ \sigma(x) = \xi(x) = 1$ and compound Poisson process (binomial law)

		first estimator				truncated estimator			
n	Δ	$\hat{m}_a$	$\hat{r}_a$	$risk_1$	or_1	$\tilde{m}_a$	$\tilde{r}_a$	$risk_2$	or_2
10^3	10^{-1}	0	1.02	0.044	1.3	0	1.02	0.044	1.3
10^4	10^{-1}	0	1.02	0.011	1.3	0	1.02	0.011	1.3
10^3	10^{-2}	0	1.02	0.55	1.04	0	1.02	0.55	1.04
10^4	10^{-2}	0	1	0.047	1	0	1	0.047	1
5.10^4	10^{-2}	0.04	1	0.010	1.4	0	1	0.0053	1

$\hat{m}_a$, $\hat{r}_a$ and $\tilde{m}_a$, $\tilde{r}_a$: average values of $\hat{m}$, $\hat{r}$ and $\tilde{m}$, $\tilde{r}$ on the 50 simulations.
$risk_1$ and $risk_2$: means of the empirical errors of the adaptive estimators.
or_1 and or_2: means of *oracle* =empirical error of the adaptive estimator / empirical error of the best possible estimator.

Table 2: Model 2: Double well and Laplace law
$b(x) = -(x-1/4)^3 - (x+1/4)^3$, $\sigma(x) = \xi(x) = 1$ and Laplace law.

		first estimator				truncated estimator			
n	Δ	$\hat{m}_a$	$\hat{r}_a$	$risk_1$	or_1	$\tilde{m}_a$	$\tilde{r}_a$	$risk_2$	or_2
10^3	10^{-1}	0.02	1.0	0.12	3.1	0.02	1.0	0.12	3.1
10^4	10^{-1}	1.7	2.1	2.10^{96}	51	0.4	2.1	0.04	1.5
10^3	10^{-2}	0.26	1.2	1.8	3.1	0.06	1	0.51	1.4
10^4	10^{-2}	0.12	1.5	0.16	1.8	0.08	1.2	0.13	2.4
5.10^4	10^{-2}	0.30	2.5	0.035	1.6	0.26	2.5	0.019	1.8

$\hat{m}_a$, $\hat{r}_a$ and $\tilde{m}_a$, $\tilde{r}_a$: average values of $\hat{m}$, $\hat{r}$ and $\tilde{m}$, $\tilde{r}$ on the 50 simulations.
$risk_1$ and $risk_2$: means of the empirical errors of the adaptive estimators.
or_1 and or_2: means of *oracle* =empirical error of the adaptive estimator / empirical error of the best possible estimator.

Table 3: Model 3: Sine function and jumps not sub-exponential

$b(x) = -2x + \sin(3x)$, $\quad \sigma(x) = \xi(x) = \sqrt{(3+x^2)/(1+x^2)}$ and $\quad \nu(z) \propto e^{-\sqrt{az}}/\sqrt{z}$

		first estimator				truncated estimator			
n	Δ	$\hat{m}_a$	$\hat{r}_a$	$risk_1$	or_1	$\tilde{m}_a$	$\tilde{r}_a$	$risk_2$	or_2
10^3	10^{-1}	0.34	1.2	0.76	3.6	0.04	1.2	0.28	1.9
10^4	10^{-1}	0.8	2.2	0.082	1.3	0.68	2.2	0.073	1.2
10^3	10^{-2}	0.96	1.2	18	6.3	0.02	1.2	1.3	1.2
10^4	10^{-2}	0.78	1.4	1.5	4.3	0.12	1.4	0.24	3.3
5.10^4	10^{-2}	0.92	2.3	0.24	4.3	0.70	2.3	0.039	1.3

$\hat{m}_a$, $\hat{r}_a$ and $\tilde{m}_a$, $\tilde{r}_a$: average values of $\hat{m}$, $\hat{r}$ and $\tilde{m}$, $\tilde{r}$ on the 50 simulations.
$risk_1$ and $risk_2$: means of the empirical errors of the adaptive estimators.
or_1 and or_2: means of *oracle* =empirical error of the adaptive estimator / empirical error of the best possible estimator.

7 Auxiliary proofs

7.1 Decomposition on a lattice

Proposition 3.
If there exist some constants c_1, c_2 and K independent of D, n, Δ, b and σ and two constants α and β independent of n and D such that, for any function $t \in S_m + S'_m$:

$$\forall \eta, \zeta > 0,\ \forall t \in S_m + S_{m'}\ \|t\|_\infty \leq C\zeta,\ \mathbb{P}\left(f_n(t) \geq \eta, \|t\|_n^2 \leq \zeta^2\right) \leq K \exp\left(-\frac{\eta^2 n\beta}{(c_1\alpha^2\zeta^2 + 2Cc_2\alpha\eta\zeta)}\right),$$

then there exist some constants C and κ depending only of ν such that, if $D \leq n\beta$:

$$\mathbb{E}\left[\sup_{t \in \mathscr{B}_{m,m'}} f_n^2(t) - \frac{\kappa\alpha^2 D}{n\beta}\right]_+ \leq CK\frac{\kappa\alpha^2 D^{3/2} e^{-\sqrt{D}}}{n\beta}.$$

Let us consider an orthonormal (for the L^2_ϖ-norm) basis $(\psi_\lambda)_{\lambda \in \Lambda_{m,m'}}$ of $S_{m,m'} = S_m + S_{m'}$ such that

$$\forall \lambda, \quad \mathrm{card}\left(\{\lambda',\ \|\psi_\lambda \psi_{\lambda'}\| \neq 0\}\right) \leq \phi_2.$$

Let us set

$$\bar{r}_{m,m'} = \frac{1}{\sqrt{D}} \sup_{\beta \neq 0} \frac{\|\sum_\lambda \beta_\lambda \psi_\lambda\|_\infty}{|\beta|_\infty}.$$

We obtain that

$$\left\|\sum_\lambda \beta_\lambda \psi_\lambda\right\|_\infty \leq \phi_2 |\beta|_\infty \sup_\lambda \|\psi_\lambda\|_\infty \quad \text{et} \quad \|\psi_\lambda\|_\infty \leq \sqrt{D}\,\|\psi_\lambda\|_{L^2} \leq \pi_1\sqrt{D}\,\|\psi_\lambda\|_\varpi$$

then

$$\bar{r}_{m,m'} \leq \bar{r} := \phi_2\pi_1.$$

We need a lattice of which the infinite norm is bounded. We use Lemma 9 of Barron *et al.* (1999):

Result 4. There exists a δ_k-lattice T_k of $L^2_\varpi \cap (S_m + S_{m'})$ such that

$$|T_k \cap \mathscr{B}_{m,m'}| \leq \left(5/\delta^k\right)^D$$

where $\delta_k = 2^{-k}/5$. Let us denote by $p_k(u)$ the orthogonal projection of u on T_k. For any $u \in S_{m,m'}$, $\|u - p_k(u)\|_\pi \leq \delta_k$ and

$$\sup_{u \in p_k^{-1}(t)} \|u - t\|_\infty \leq \bar{r}_{m,m'}\delta_k \leq \bar{r}\delta_k.$$

Let us set $H_k = \ln(|T_k \cap \mathscr{B}_{m,m'}|)$. We have that:

$$H_k \leq D \ln(5/\delta_k) = D\left(k\ln(2) + \ln(5/\delta_0)\right) \leq C(k+1)D.$$

The decomposition of u_k on the δ_k-lattice must be done very carefully: the norms $\|u_k - u_{k-1}\|_\varpi$ and $\|u_k - u_{k-1}\|_\infty$ must be controlled. Let us set

$$\mathscr{E}_k = \left\{u_k \in T_k \cap \mathscr{B}_{m,m'}, \quad \|u - u_k\|_\varpi \leq \delta_k \quad \text{et} \quad \|u - u_k\|_\infty \leq \bar{r}\delta_k\right\}.$$

We have that $\ln(|\mathscr{E}_k|) \leq H_k$. For any function $u \in \mathscr{B}_{m,m'}$, there exist a series $(u_k)_{k\geq 0} \in \prod_k \mathscr{E}_k$ such that

$$u = u_0 + \sum_{k=1}^{\infty} (u_k - u_{k-1}).$$

Let us consider $(\eta_k)_{k\geq 0}$ and $\eta \in \mathbb{R}$ such that $\eta_0 + \sum_{k=1}^{\infty} \eta_k \leq \eta$. We obtain:

$$\begin{aligned}
\mathbb{P}\left(\sup_{u\in\mathscr{B}_{m,m'}} |f_n(u)| > \eta\right) &\leq \mathbb{P}\left(\exists (u_k) \in \prod \mathscr{E}_k, \left|f_n(u_0) + \sum_{k=1}^{\infty} f_n(u_k - u_{k-1})\right| > \eta_0 + \sum_{k=1}^{\infty} \eta_k\right) \\
&\leq P_1 + \sum_{k=1}^{\infty} P_{2,k} \qquad (17)
\end{aligned}$$

where

$$P_1 = \sum_{u_0\in\mathscr{E}_0} \mathbb{P}\left(|f_n(u_0)| > \eta_0\right) \quad \text{and} \quad P_{2,k} = \sum_{u_k\in\mathscr{E}_k} \mathbb{P}\left(|f_n(u_k - u_{k-1})| > \eta_k\right).$$

As $u_0 \in T_0$, $\|u_0\|_\varpi \leq 1$ and $\|u_0\|_\infty \leq \bar{r}\sqrt{D}$. Moreover, $\|u_0\|_n^2 \leq 3/2\|u_0\|_\varpi^2 \leq 3\delta_0/2$. Then

$$\mathbb{P}\left(|f_n(u_0)| > \eta_0\right) = \mathbb{P}\left(|f_n(u_0)| > \eta_0,\ \|u_0\|_n^2 \leq 3\delta_0/2\right).$$

There exist two constants c_1' and c_2' depending only on δ_0 and $\bar{r}$ such that

$$\mathbb{P}\left(|f_n(u_0)| > \eta_0\right) \leq K \exp\left(-\frac{n\beta\eta_0^2}{c_1'\alpha^2 + 2c_2'\sqrt{D}\alpha\eta_0}\right).$$

Let us set x_0 such that $\eta_0 = \alpha\left(\sqrt{c_1'\left(x_0/\beta\right)} + c_2'\sqrt{D}\left(x_0/\beta\right)\right)$. Then:

$$x_0 \leq \frac{\beta\eta_0^2}{c_1'\alpha^2 + 2c_2'\sqrt{D}\alpha\eta_0}$$

and

$$\mathbb{P}\left(f_n(u_0) > \eta_0\right) \leq K\exp\left(-nx_0\right).$$

Then

$$P_1 \leq K \sum_{u_0\in\mathscr{E}_0} \exp\left(-nx_0\right) \leq K\exp\left(H_0 - nx_0\right). \qquad (18)$$

We have that

$$\|u_k - u_{k-1}\|_\pi^2 \leq 2\left(\|u - u_{k-1}\|_\pi^2 + \|u - u_k\|_\pi^2\right) \leq 5\delta_{k-1}^2/2$$

then $\|u_k - u_{k-1}\|_n^2 \leq 15\delta_{k-1}^2/4$. As $u_{k-1}, u_k \in \mathscr{E}_{k-1} \times \mathscr{E}_k$, it follows that $\|u_k - u_{k-1}\|_\infty^2 \leq 5\delta_{k-1}^2\bar{r}^2/2$. There exists two constants c_3 and c_4 such that:

$$\begin{aligned}\mathbb{P}_n\left(|f_n(u_k - u_{k-1})| > \eta_k\right) &= \mathbb{P}_n\left(|f_n(u_k - u_{k-1})| > \eta_k,\ \|u_k - u_{k-1}\|_n^2 \leq 15\delta_{k-1}^2/4\right)\\ &\leq K\exp\left(-\frac{n\beta\eta_k^2}{c_3\alpha^2\delta_{k-1}^2 + 2c_4\alpha\delta_{k-1}}\right).\end{aligned}$$

Let us fix x_k such that $\eta_k = \delta_{k-1}a\left(\sqrt{c_3\left(x_k/\beta\right)} + c_4\left(x_k/\beta\right)\right)$. We obtain:

$$x_k \leq \frac{\beta\eta_k^2}{c_3\alpha^2\delta_{k-1}^2 + 2c_4\alpha\delta_{k-1}}$$

and

$$\mathbb{P}\left(|f_n(u_k - u_{k-1})| > \eta_k\right) \leq K\exp\left(-nx_k\right).$$

Then, $P_{2,k} \leq K\exp\left(H_{k-1} + H_k - nx_k\right)$ and

$$P_2 = \sum_{k=1}^{\infty} P_{2,k} \leq K\sum_{k=1}^{\infty}\exp\left(H_{k-1} + H_k - nx_k\right). \tag{19}$$

Let us set $\tau > 0$ and choose (x_k) (and then (η_k)) such that

$$\begin{cases}\sqrt{D}nx_0 = H_0 + D + \tau\\ nx_k = H_{k-1} + H_k + (k+1)D + \tau.\end{cases}$$

Collecting the results, we obtain, by (17), (18) and (19):

$$\mathbb{P}\left(\sup_{u\in\mathscr{B}_{m,m'}}|f_n(u)| > \eta\right) \leq C\left(e^{-D}e^{-\tau} + e^{-\sqrt{D}}e^{-\tau/\sqrt{D}}\right). \tag{20}$$

It remains to compute η^2. We denote by C a constant depending only on δ_0 and $\bar{r}$. This constant may vary from one line to another. We have that:

$$\eta = \sum_{k=0}^{\infty}\eta_k \leq C\alpha\left(\sum_{k=1}^{\infty}\delta_{k-1}\left(\sqrt{\frac{x_k}{\beta}} + \frac{x_k}{\beta}\right)\right) + \alpha\left(\sqrt{\frac{x_0}{\beta}} + \sqrt{D}\frac{x_0}{\beta}\right).$$

Let us recall that $H_k = C(k+1)D$. Then, $nx_k = C(3k+2)D + \tau$, $\sqrt{D}nx_0 = CD + \tau$ and

$$\sum_{k=0}^{\infty}\frac{\delta_{k-1}x_k}{\beta} \leq \frac{1}{n\beta}\sum_{k=0}^{\infty}2^{-(k-1)}(C(3k+2)D + \tau) \leq C\frac{D+\tau}{n\beta}.$$

Moreover,

$$\sum_{k=0}^{\infty} \delta_{k-1} \sqrt{\frac{x_k}{\beta}} \leq C \frac{\sqrt{D} + \sqrt{\tau}}{\sqrt{n\beta}}.$$

As $D/n\beta \leq 1$, there exists a constant κ such that

$$\eta^2 \leq \kappa\alpha^2 \left(\frac{D}{n\beta} + 2\frac{\tau}{n\beta} + \frac{\tau^2}{n^2\beta^2} \right).$$

Then, according to (20):

$$\mathbb{P}\left(\sup_{u \in \mathscr{B}_{m,m'}} f_n^2(u) > \kappa\alpha^2 \left(\frac{D}{n\beta} + 2\frac{\tau}{n\beta} + \frac{\tau^2}{n^2\beta^2} \right) \right) \leq C \left(e^{-D-\tau} + e^{-\sqrt{D}-\tau/\sqrt{D}} \right). \tag{21}$$

Furthermore

$$\begin{aligned} E &:= \mathbb{E}\left(\left[\sup_{u \in \mathscr{B}_{m,m'}} f_n^2(u) - \kappa a^2 \frac{D}{n\beta} \right]_+ \right) \\ &= \int_0^{\infty} \mathbb{P}\left(\sup_{u \in \mathscr{B}_{m,m'}} f_n^2(u) > \kappa a^2 \frac{D}{n\beta} + \tau \right) d\tau \end{aligned}$$

Setting $\tau = \kappa\alpha^2 \left(2y/n\beta + y^2/n^2\beta^2 \right)$, it follows:

$$E = C\gamma^2 \int_0^{\infty} \mathbb{P}\left(\sup_{u \in \mathscr{B}_{m,m'}} f_n^2(u) > \kappa\alpha^2 \left(\frac{D}{n\beta} + 2\frac{y}{n\beta} + \frac{y^2}{n^2\beta^2} \right) \right) \left(\frac{2}{n\beta} + \frac{2y}{n^2\beta^2} \right) dy.$$

By (21),

$$\begin{aligned} E &= C\kappa\alpha^2 \left(e^{-D} + e^{-\sqrt{D}} \right) \left(\frac{1}{n\beta} \int_0^{\infty} y e^{-y/\sqrt{D}} dy \right) \\ &\leq C \frac{\kappa\alpha^2}{n\beta} D^{3/2} e^{-\sqrt{D}}. \end{aligned}$$

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