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Non-asymptotic Oracle Inequalities for the Lasso and Group Lasso in high dimensional logistic model

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Abstract

We consider the problem of estimating a function f_0 in logistic regression model. We propose to estimate this function f_0 by a sparse approximation build as a linear combination of elements of a given dictionary of p functions. This sparse approximation is selected by the Lasso or Group Lasso procedure. In this context, we state non asymptotic oracle inequalities for Lasso and Group Lasso under restricted eigenvalues assumption as introduced in [3]. Those theoretical results are illustrated through a simulation study.

keywords Logistic model, Lasso, Group Lasso, High-dimensional.

1 Introduction

During the last few years, logistic regression problems with more and more high-dimensional data occur in a wide variety of scientific fields, especially in studies that attempt to find risk factors for disease and clinical outcomes. For example in gene expression data analysis or in genome wide association analysis the number p of predictors may be of the same order or largely higher than the sample size n (thousands p of predictors for only a few dozens of individuals n , see for instance [9] or [28]). In this context the considered model is often what we call here “usual” logistic regression. It is given by

$$\mathbb{P}(Y_i = 1|Z = z_i) = \pi(z_i^T \beta_0) = \frac{\exp(z_i^T \beta_0)}{1 + \exp(z_i^T \beta_0)}, \quad (1)$$

where one observes n independent and identically distributed (i.i.d) random couples $(Z_1, Y_1), \dots, (Z_n, Y_n)$ from the joint distribution of $(Z, Y) \in \mathbb{R}^d \times \{0, 1\}$ and β_0 is the parameter to estimate.

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In this paper, we consider a more general logistic model described by

$$\mathbb{P}(Y_i = 1|Z_i = z_i) = \frac{\exp(f_0(z_i))}{1 + \exp(f_0(z_i))}, \quad (2)$$

where f_0 (not necessarily linear) is an unknown function. We aim at estimating f_0 by constructing a suitable approximation. More precisely we estimate f_0 by a sparse approximation of linear combination of elements of a given dictionary of functions $\mathbb{D} = \{\phi_1, \dots, \phi_p\}$: $\hat{f}(\cdot) := \sum_{j=1}^p \hat{\beta}_j \phi_j(\cdot)$. Our purpose expresses the belief that, in many instances, even if p is large, only a subset of \mathbb{D} may be needed to approximate f_0 well. This construction can be done by minimizing the empirical risk. However, it is well-known that with a large number of parameters in high dimensional data situations, direct minimization of empirical risk can lead to *Overfitting*: the classifier can only behave well in training set, and can be bad in test set. The procedure would also be unstable: since empirical risk is data dependent, hence random, small change in the data can lead to very different estimators. Penalization is used to overcome those drawbacks. One could use ℓ_0 penalization, *ie* penalize by the number of non zero coefficients (see for instance AIC, BIC [1, 23]). Such a penalization would produce interpretable models, but leads to non convex optimization and there is not efficient algorithm to solve this problem in high dimensional framework. Tibshirani [25] proposes to use ℓ_1 penalization, which is a regularization technique for simultaneous estimation and selection. This penalization leads to convex optimization and is important from computational point of view (as well as from theoretical point of view). As the consequence of the optimality conditions, regularization by the ℓ_1 penalty tends to produce some coefficients that are exactly zero and shrink others, thus the name of Lasso (Least Absolute Shrinkage and Selection Operator). There exist some algorithms to solve this convex problem, *glmnet* (see [8]), *predictor-corector* (see [22]) among the others.

A related Lasso-type procedure is the Group Lasso, where the covariates are assumed to be clustered in groups, and instead of ℓ_1 -penalty (summing the absolute values of each individual loading) the sum of Euclidean norms of the loadings in each group is used. It shares the same kind of properties as the Lasso, but encourages predictors to be selected in groups. This is useful when the set of predictors is partitioned into prescribed groups, only few being relevant in the estimation process. Group Lasso has numerous applications: when categorical predictors (factors) are present, the Lasso solution is not adequate since it only selects individual dummy variables instead of whole factors. In this case, categorical variables are usually represented as groups of dummy variables. In speech and signal processing for example, the groups may represent different frequency bands (see [16]).

Previously known results. Recently, a great deal of attention has been focused on ℓ_1 -penalized based estimators. Most of this attention concerns regression models and ℓ_1 -penalized least squares estimator of parameters in high dimensional linear and non linear additive regression. Among them one can cite [4, 6, 5, 15], who have studied the Lasso for linear model in nonparametric setting and proved sparsity oracle inequalities. Similar sparsity oracle inequalities are proved in [3], and those results hold under the so-called *restricted eigenvalues assumption* on the Gram matrix. Those kind of results have been recently

stated for the variants of the Lasso. For instance Lounici et al. [13] under a group version of *restricted eigenvalues assumption* stated oracle inequalities in linear gaussian noise model under Group sparsity. Those results lead to the refinements of their previous results for multi-task learning (see [12]). The behavior of the Lasso for linear regression regarding its selection and estimation properties have been studied in [11, 18, 31, 21, 30, 19] among other. Few results on the Lasso and Group Lasso concern logistic regression model. Most of them are asymptotic results and concern the "usual" logistic regression model defined by (1). Zou [32] shows consistency in variable selection for adaptive Lasso in generalized linear models when the number of covariables p is fixed. Huang et al. [10] prove sign consistency and estimation consistency for high-dimensional logistic regression. To our knowledge there are only two non asymptotic results for the Lasso in logistic model: the first one is from Bach [2], who provided bounds for excess risk (generalization performance) and estimation error in the case of "usual" logistic regression model under *restricted eigenvalues assumption* on the weighted Gram matrix. The second one is from van de Geer [26], who established non asymptotic oracle inequality for Lasso in high dimensional generalized linear models with Lipschitz loss functions. Meir et al. [17] shown consistency for the Group Lasso in "usual" logistic model.

In this paper, we state general non asymptotic oracle inequalities for the Lasso and Group Lasso in logistic model within the framework of high-dimensional statistics. We first state "slow" oracle inequalities (see Theorem 3.1 and Theorem 4.1) without any assumption on the Gram matrix. Secondly we provide "fast" oracle inequalities (see Theorem 3.2 and Theorem 4.2) under *restricted eigenvalues assumption*. In each case, we give, as a consequence, the bounds for excess risk, prediction and estimation errors for Lasso and Group Lasso in the "usual" logistic regression. Our non asymptotic results lead to an adaptive data-driven weighting of the ℓ_1 -norm. Simulation study is given to illustrate the numerical performance of Lasso with such weights.

This paper is organized as follows. In Section 2, we describe our estimation procedure base ℓ_1 -penalized logistic loss. In Section 3 and 4 we state our main results on oracle inequalities for Lasso and Group Lasso. In each case, we give as a consequence the bounds for excess risk, prediction and estimation errors for Lasso and Group Lasso in the "usual" logistic regression. Section 4.3 is devoted to simulation study. The proofs are gathered in Section 5 and Appendix.

2 L_1 -penalized logistic regression

Our aim is to estimate f_0 in Model (2) by a linear combination of the function of a dictionary

$$\mathbb{D} = \{\phi_1, \dots, \phi_p\},$$

where $\phi_j : \mathbb{R}^d \rightarrow \mathbb{R}$ and $p \gg n$. The functions ϕ_j can be viewed as estimators of f_0 constructed from independent training sample, or estimators computed using p different values of the tuning parameter of the same method. They can also be a collection of basis functions, that can approximate f_0 , like wavelets,

splines, kernels, etc... We implicitly assume that f_0 can be well approximated by a linear combination

$$f_\beta(\cdot) = \sum_{j=1}^p \beta_j \phi_j(\cdot),$$

where β has to be estimated.

We consider empirical risk (logistic loss) for logistic model

$$\hat{R}(f) = \frac{1}{n} \sum_{i=1}^n \log(1 + \exp(f(z_i))) - Y_i f(z_i). \quad (3)$$

The ℓ_1 -penalized logistic regression estimator $f_{\hat{\beta}_L}$ is defined as a minimizer of the following ℓ_1 -penalized empirical risk

$$f_{\hat{\beta}_L} := \underset{f_\beta \in \Gamma}{\operatorname{argmin}} \left\{ \hat{R}(f_\beta) + \sum_{j=1}^p \omega_j |\beta_j| \right\}, \quad (4)$$

where the minimum is taken over the set

$$\Gamma \subseteq \left\{ f_\beta(\cdot) = \sum_{j=1}^p \beta_j \phi_j(\cdot), \beta = (\beta_1, \dots, \beta_p) \in \mathbb{R}^p \right\}$$

and ω_j are positive weights to be precised later. The "classical" Lasso penalization corresponds to $\omega_j = \lambda$, where λ is the tuning parameter which makes balance between goodness-of-fit and sparsity. The Lasso estimator has the property that it does predictors selection and estimation at the same time. Indeed for large values of ω_j , the related components $\hat{\beta}_j$ are set exactly to 0 and the other are shrunk toward zero.

2.1 Definitions and notation

Throughout the paper, we consider a fixed design setting (i.e z_1, \dots, z_n are consider deterministic). Consider the matrix $X = (\phi_j(z_i))_{1 \leq i \leq n, 1 \leq j \leq p}$. For any $\beta = (\beta_1, \dots, \beta_p)^T \in \mathbb{R}^p$, define $f_\beta(\cdot) = \sum_{j=1}^p \beta_j \phi_j(\cdot)$. Using this notation

$$(f_\beta(z_1), \dots, f_\beta(z_n))^T = X\beta.$$

Let $K(\beta) = \{j \in \{1, \dots, p\} : \beta_j \neq 0\}$ and $|K(\beta)|$ the cardinality of $K(\beta)$, which characterizes the sparsity of the vector β . For all $\delta \in \mathbb{R}^p$ and a subset $I \subset \{1, \dots, p\}$, we denote by δ_I the vector in \mathbb{R}^p that has the same coordinates as δ on I and zero coordinates on the complement I^c of I . For all $h, f, g : \mathbb{R}^d \rightarrow \mathbb{R}$, we define the scalar products

$$\langle f, h \rangle_n = \frac{1}{n} \sum_{i=1}^n h(z_i) f(z_i),$$

and

$$\langle f, h \rangle_g = \frac{1}{n} \sum_{i=1}^n h(z_i) f(z_i) \pi(g(z_i)) (1 - \pi(g(z_i))), \quad \text{where } \pi(t) = \frac{\exp(t)}{1 + \exp(t)}.$$

We use the notation

$$q_f(h) = \frac{1}{n} \sum_{i=1}^n h(z_i)(Y_i - \pi(f(z_i))),$$

$\|h\|_\infty = \max_i |h(z_i)|$ and $\|h\|_n = \sqrt{\langle h, h \rangle_n} = \sqrt{\frac{1}{n} \sum_{i=1}^n h^2(z_i)}$ which denote the empirical norm. We denote by R the conditionnal expectation of \hat{R} given z_1, \dots, z_n , ie

$$R(f) = \mathbb{E}(\hat{R}(f)) = \frac{1}{n} \sum_{i=1}^n \log(1 + \exp(f(z_i))) - \mathbb{E}(Y_i)f(z_i).$$

It is clear that $R(\cdot)$ is a convex function and f_0 is a minimum of $R(\cdot)$ when the model is well-specified (ie when (2) is satisfied). Note that with our notations

$$R(f) = \mathbb{E}(\hat{R}(f)) = \hat{R}(f) + q_{f_0}(f). \quad (5)$$

We shall use both the excess risk of $f_{\hat{\beta}}$, $R(f_{\hat{\beta}}) - R(f_0)$ and the prediction loss $\|f_{\hat{\beta}} - f_0\|_n^2$ to evaluate the quality of the estimator. Note that $R(f_{\hat{\beta}})$ corresponds to the average Kullback-Leibler divergence to the best model when the model is well-specified, and is common for the study of logistic regression.

3 Oracle inequalities for the Lasso

Consider the following assumptions:

$$\text{There exists a constant } 0 < c_1 < \infty \text{ such that } \max_{1 \leq i \leq n} |f_0(z_i)| \leq c_1. \quad (\mathbf{B}_1)$$

$$\text{There exists a constant } c_2 > 0 \text{ such that } \max_{1 \leq i \leq n} \max_{1 \leq j \leq p} |\phi_j(z_i)| < c_2. \quad (\mathbf{B}_2)$$

$$\text{For all } f_\beta \in \mathcal{F}, \text{ there is some universal constant } C_0 \text{ such that } \max_{1 \leq i \leq n} |f_\beta(z_i)| \leq C_0. \quad (\mathbf{B}_3)$$

Assumptions (\mathbf{B}_1) and (\mathbf{B}_3) are technical Assumptions useful to connect the excess risk and the prediction loss (see Lemma 5.1 and 5.2). An assumption similar to (\mathbf{B}_1) have been used in [7] to prove oracle inequality in gaussian regression model. The same kind of assumption as (\mathbf{B}_3) have been made in [24] to prove oracle inequality for support vector machine type with ℓ_1 complexity regularization.

Theorem 3.1. *Let $f_{\hat{\beta}_L}$ be the ℓ_1 -penalized minimum defined in (4).*

A-) *Let $x > 0$ be fixed. For $j = \{1, \dots, p\}$, let*

$$\omega_j = \frac{2}{n} \sqrt{\frac{1}{2} \sum_{i=1}^n \phi_j^2(z_i)(x + \log p)} + \frac{2c_2(x + \log p)}{3n}. \quad (6)$$

Thus with probability at least $1 - \exp(-x)$ we have

$$R(f_{\hat{\beta}_L}) - R(f_0) \leq \inf_{\beta \in \mathbb{R}^p} \left\{ R(f_\beta) - R(f_0) + 2\|\beta\|_1 \max_{1 \leq j \leq p} \omega_j \right\}.$$

B-) Let $A > 2\sqrt{c_2}$. For $j = \{1, \dots, p\}$, let

$$w_j = \lambda = A\sqrt{\frac{\log p}{n}}.$$

Thus with probability at least $1 - 2p^{1-A^2/4c_2}$ we have

$$R(f_{\hat{\beta}_L}) - R(f_0) \leq \inf_{\beta \in \mathbb{R}^p} \left\{ R(f_\beta) - R(f_0) + 2A\|\beta\|_1 \sqrt{\frac{\log p}{n}} \right\}.$$

In each inequality the first part of the right hand corresponds to the approximation error (bias). The selection of the dictionary can be very important to minimize this approximation error. It is recommended to choose a dictionary \mathbb{D} as f_0 could well be approximated by a linear combination of the functions of \mathbb{D} . The second part of the right hand in each inequality is the variance term and is usually referred as the rate of the oracle inequality. In Theorem 3.1, we speak about "slow" oracle inequalities, with the rate at the order $\|\beta\|_1 \sqrt{\log p/n}$ for any β . Those results are obtained without any assumption on the Gram matrix $(\Phi_n = X^T X/n)$. In order to obtain oracle inequalities with a fast rate of order $\log p/n$ we need additional assumption on the restricted eigenvalues of the Gram matrix, namely the *restricted eigenvalues assumption*.

For some integer s such that $1 \leq s \leq p$ and a positive number a_0 , the following condition holds **(RE₁)**

$$\mu(s, a_0) := \min_{K \subseteq \{1, \dots, p\}: |K| \leq s} \min_{\Delta \neq 0: \|\Delta_{K^c}\|_1 \leq a_0 \|\Delta_K\|_1} \frac{\|X\Delta\|_2}{\sqrt{n} \|\Delta_K\|_2} > 0.$$

This assumption has been introduced in [3], where several sufficient conditions for this assumption are described. This condition is known to be one of the weakest to derive fast rates for the Lasso. For instance conditions on the Gram matrix used to prove oracle inequality in [4, 6, 5] are more restrictive than *restricted eigenvalues assumption*. In those papers either Φ_n is positive definite, or mutual coherence condition is imposed. We refer to [27] for a complete comparison of the assumptions used to prove oracle inequality for the Lasso. Especially it is proved that *restricted eigenvalues assumption* is weaker than the neighborhood stability or irrepresentable condition. To emphasize the dependency of Assumption **(RE₁)** on s and a_0 we will sometimes refer to it as $RE(s, a_0)$.

Theorem 3.2. Let $f_{\hat{\beta}_L}$ be the ℓ_1 -penalized minimum defined in (4). Fix $\eta > 0$ and $1 \leq s \leq p$. Assume that Assumptions **(B₁)**, **(B₂)**, **(B₃)** and **(RE₁)** are satisfied, with $a_0 = 3 + 4/\eta$.

A-) Let $x > 0$ be fixed. For $j = \{1, \dots, p\}$, ω_j defined as in (6). Thus with probability at least $1 - \exp(-x)$ we have

$$R(f_{\hat{\beta}_L}) - R(f_0) \leq (1 + \eta) \inf_{f_\beta \in \Gamma} \left\{ R(f_\beta) - R(f_0) + \frac{c(\eta)|K(\beta)| \left(\max_{1 \leq j \leq p} \omega_j \right)^2}{c_0 \epsilon_0 \mu^2(s, 3 + 4/\eta)} \right\}, \quad (7)$$

and

$$\|f_{\hat{\beta}_L} - f_0\|_n^2 \leq \frac{c'_0}{4c_0\epsilon_0}(1+\eta) \inf_{f_\beta \in \Gamma} \left\{ \|f_\beta - f_0\|_n^2 + \frac{4c(\eta)|K(\beta)| \left(\max_{1 \leq j \leq p} \omega_j \right)^2}{c'_0 c_0 \epsilon_0^2 \mu^2(s, 3 + 4/\eta)} \right\}. \quad (8)$$

B-) Let $A > 2\sqrt{c_2}$. For $j = \{1, \dots, p\}$, let

$$w_j = \lambda = A \sqrt{\frac{\log p}{n}}.$$

Thus with probability at least $1 - 2p^{1-A^2/4c_2}$ we have

$$R(f_{\hat{\beta}_L}) - R(f_0) \leq (1+\eta) \inf_{f_\beta \in \Gamma} \left\{ R(f_\beta) - R(f_0) + \frac{A^2 c(\eta)}{c_0 \epsilon_0 \mu^2(s, 3 + 4/\eta)} \frac{|K(\beta)| \log p}{n} \right\}, \quad (9)$$

and

$$\|f_{\hat{\beta}_L} - f_0\|_n^2 \leq \frac{c'_0}{4c_0\epsilon_0}(1+\eta) \inf_{f_\beta \in \Gamma} \left\{ \|f_\beta - f_0\|_n^2 + \frac{4c(\eta)A^2}{c'_0 c_0 \epsilon_0^2 \mu^2(s, 3 + 4/\eta)} \frac{|K(\beta)| \log p}{n} \right\}. \quad (10)$$

In both cases $c(\eta)$ is a constant depending only on η ; $c_0 = c_0(C_0, c_1)$ and $c'_0 = c'_0(C_0, c_1)$ are constants depending on C_0 and c_1 ; and $\epsilon_0 = \epsilon_0(c_1)$ is a constant depending on c_1 .

In this theorem the variance term is always of order $|K(\beta)| \log p/n$. Such order in sparse oracle inequalities usually refer to “fast rate”. This rate is of same kind of the one obtain in [3] for linear regression model. For the best of our knowledge this is a first non asymptotic oracle inequality for prediction loss in logistic model. Some non asymptotic oracle inequality for excess risk has been established in [26] under different assumptions. Indeed, she stated oracle inequality for high dimensional generalized linear model with Lipschitz loss function, where logistic regression is a particular case. Her result assumes to be hold in the “neighborhood” of the target function, while our result is true for all bounded functions. Note also that our result holds under *RE* condition and we use more explicit weight (or parameter λ) on the penalty term.

Remark 3.1. It is important to note that our results remain true if we assume that we are in the “neighborhood” of the target function. It suffices to note that if we suppose that there exists ζ such that $\max_{1 \leq i \leq n} |f_\beta(z_i) - f_0(z_i)| \leq \zeta$, then Lemma 5.1 is still true.

Remark 3.2. The choice of the weights ω_j in part A of the Theorem 3.2 comes from Bernstein’s inequality. It is important to note that we could have used the following weights

$$\omega'_j = \frac{2}{n} \sqrt{2 \sum_{i=1}^n \mathbb{E}(\phi_j^2(z_i) \epsilon_i^2)(x + \log p)} + \frac{2 \max_{1 \leq i \leq n} |\phi_j(z_i)| (x + \log p)}{3n}.$$

Our results still hold with such weights ω'_j . But these weights depend on the unknown function f_0 to be estimated. Indeed $\mathbb{E}(\epsilon_i^2) = \pi(f_0(z_i))(1 - \pi(f_0(z_i)))$, that is the reason for using weights ω_j slightly greater than ω'_j .

3.1 Special case: variable selection in logistic regression model

In this section we assume that f_0 is a linear function that is $f_0(z_i) = f_{\beta_0}(z_i) = \sum_{j=1}^p \beta_{0j} z_{ij} = z_i^T \beta_0$, where $z_i = (z_{i1}, \dots, z_{ip})^T$. Denote $X = (z_{ij})_{1 \leq i \leq n, 1 \leq j \leq p}$ the design matrix. Thus for $i = 1, \dots, n$

$$\mathbb{P}(Y_i = 1 | Z = z_i) = \pi(z_i^T \beta_0) = \frac{\exp(z_i^T \beta_0)}{1 + \exp(z_i^T \beta_0)}. \quad (11)$$

This corresponds to the usual logistic regression *ie* logistic model that allows linear dependency between Z_i and the distribution of Y_i . The Lasso estimator of β_0 is thus defined as

$$\hat{\beta}_L := \underset{\beta: f_\beta \in \Gamma}{\operatorname{argmin}} \left\{ \frac{1}{n} \sum_{i=1}^n \{ \log(1 + \exp(z_i^T \beta)) - Y_i z_i^T \beta \} + \lambda \sum_{j=1}^p \omega_j |\beta_j| \right\}. \quad (12)$$

This infimum is achieved and might not be unique. When the design matrix X has full rank, the solution of optimization problem 12 is usually unique.

Corollary 3.1. *Let assumption $RE(s, 3)$ be satisfied and $|K(\beta_0)| \leq s$, where $1 \leq s \leq p$. Consider the Lasso estimator $f_{\hat{\beta}_L}$ defined by (12) with*

$$\omega_j = \frac{2}{n} \sqrt{\frac{1}{2} \sum_{i=1}^n z_{ij}^2 (x + \log p)} + \frac{2c_2(x + \log p)}{3n}$$

Under the assumptions of Theorem 3.2 with probability at least $1 - \exp(-x)$ we have

$$R(f_{\hat{\beta}_L}) - R(f_{\beta_0}) \leq \frac{8s \left(\max_{1 \leq j \leq p} \omega_j \right)^2}{\mu^2(s, 3) c_0 \epsilon_0} \quad (13)$$

$$\|f_{\hat{\beta}_L} - f_{\beta_0}\|_n^2 \leq \frac{8s \left(\max_{1 \leq j \leq p} \omega_j \right)^2}{\mu^2(s, 3) c_0^2 \epsilon_0^2} \quad (14)$$

$$\|\hat{\beta}_L - \beta_0\|_1 \leq \frac{8\sqrt{2}s \left(\max_{1 \leq j \leq p} \omega_j \right)}{\mu^2(s, 3) c_0 \epsilon_0} \quad (15)$$

$$\|\hat{\beta}_L - \beta_0\|_q^q \leq \left(\frac{8\sqrt{2}s \left(\max_{1 \leq j \leq p} \omega_j \right)}{\mu^2(s, 3) c_0 \epsilon_0} \right)^q \quad \text{for all } 1 < q \leq 2. \quad (16)$$

If $\omega_j = \lambda = A\sqrt{\log p/n}$ we have the same results with probability at least $1 - 2p^{1-A^2/4c_2}$.

The first and third line of the corollary are similar to those of Theorem 5 in [2] except that we assume that $f_{\hat{\beta}_L}$ is bounded whereas he assume that λ is bounded. Note that, up to differences in constant factors,

the rates obtained in this corollary are the same as those obtained in Theorem 7.2 in [3] for linear model with an s -sparse vector.

Remark 3.3. In logistic regression model (11), if vector β_0 is sparse, ie $|K(\beta_0)| \leq s$, then Assumption (RE₁) implies that β_0 is uniquely defined. Indeed, if there exists β^* such that for $i = 1, \dots, n$, $\pi(z_i^T \beta_0) = \pi(z_i^T \beta^*)$, it follows that $X\beta_0 = X\beta^*$ and $|K(\beta^*)| \leq s$. Then according to assumption $RE(s, a_0)$ with $a_0 \geq 1$, we necessarily have $\beta_0 = \beta^*$. Indeed if $RE(s, a_0)$ is satisfied with $a_0 \geq 1$, then $\min\{\|X\beta\|_2 : |K(\beta)| \leq 2s, \beta \neq 0\} > 0$.

4 Oracle inequalities for the Group Lasso

In this section we assume that prior knowledge is available. More precisely the set of relevant predictors have known group structure, and we wish to achieve sparsity at the level of groups. This group sparsity assumption suggests us to use the Group Lasso method. We consider the Group Lasso for logistic regression (see [17, 29]), where predictors are included or excluded in groups. The logistic Group Lasso is the minimizer of the following optimization problem, defined as

$$f_{\hat{\beta}_{GL}} := \operatorname{argmin}_{\beta \in \Gamma_1} \left\{ \hat{R}(f_\beta) + \sum_{l=1}^g \omega_l \|\beta^l\|_2 \right\}, \quad (17)$$

where $\omega_l > 0$ are the weights used to control the amount of penalization, and where

$$\Gamma_1 \subseteq \left\{ f_\beta(\cdot) = \sum_{l=1}^g \sum_{j \in G_l} \beta_{lj} \phi_j(\cdot), \beta \in \mathbb{R}^{\sum_{l=1}^g |G_l|} \right\}.$$

This penalty can be viewed as an intermediate between ℓ_1 and ℓ_2 type penalty. It has the attractive property that it does variables selection at the group level. Denote $X = (\phi_j(z_i))_{1 \leq i \leq n, 1 \leq j \in G_l, l=1, \dots, g}$ the design matrix and $\beta = (\beta^1, \dots, \beta^g)^T$, where $\beta^l = (\beta_{lj})_{j \in G_l}$ for $l = 1, \dots, g$. Let $f_\beta(\cdot) = \sum_{l=1}^g \sum_{j \in G_l} \beta_{lj} \phi_j(\cdot)$. With our notations

$$(f_\beta(z_1), \dots, f_\beta(z_n))^T = X\beta.$$

We define the group norm of β as

$$\|\beta\|_{2,1} = \sum_{l=1}^g \sqrt{\sum_{j \in G_l} \beta_{lj}^2} = \sum_{l=1}^g \|\beta^l\|_2.$$

Finally we set $J(\beta) = \{l : \beta^l \neq 0\}$, the set of relevant groups and $|J(\beta)|$ the number of such groups. Consider the following assumptions:

$$\text{There exists a constant } a_2 > 0 \text{ such that, } \max_{1 \leq i \leq n} \max_{1 \leq l \leq g} \max_{j \in G_l} |\phi_j(z_i)| < a_2. \quad (\mathbf{B}_4)$$

For all $f_\beta \in \Gamma_1$, there is some universal constant C_0 such that, $\max_{1 \leq i \leq n} |f_\beta(z_i)| \leq C_0$. (B₅)

Assumption (B₄) could be viewed as a natural extension of Assumption B₂ for the Lasso.

Theorem 4.1. Let $f_{\hat{\beta}_{GL}}$ be the Group Lasso solution defined in (17) with

$$\omega_l = \frac{2|G_l|}{n} \sqrt{\frac{1}{2} \max_{j \in G_l} \sum_{i=1}^n \phi_j^2(z_i) \left(x + \log \sum_{l=1}^g |G_l| \right)} + \frac{2a_2|G_l|}{3n} \left(x + \log \sum_{l=1}^g |G_l| \right) \quad (18)$$

where $x > 0$. Thus with probability at least $1 - \exp(-x)$ we have

$$R(f_{\hat{\beta}_{GL}}) - R(f_0) \leq \inf_{\beta \in \mathbb{R}^{\sum_{l=1}^g |G_l|}} \left\{ R(f_\beta) - R(f_0) + 2\|\beta\|_{2,1} \max_{1 \leq l \leq g} \omega_l \right\}.$$

As previously the weight being of order $\|\beta\|_{2,1} \sqrt{\log g/n}$ for any β , this is a "slow" oracle inequality. Those results are obtained without any assumption on the Gram matrix. To obtain oracle inequalities with a fast rate, of order $\log g/n$, we need some group version of the restricted eigenvalues of the Gram matrix.

For some integer s such that $1 \leq s \leq g$ and a positive number a_0 , the following condition holds (RE₂)

$$\mu_1(s, a_0) := \min_{K \subseteq \{1, \dots, p\}: |K| \leq s} \min_{\Delta \neq 0: \|\Delta_{K^c}\|_{2,1} \leq a_0 \|\Delta_K\|_{2,1}} \frac{\|X\Delta\|_2}{\sqrt{n} \|\Delta_K\|_2} > 0.$$

This is a natural extension to the Group Lasso of Assumption (RE₁) for the usual Lasso. The only main difference lies on the set where the minimum is taken: for the Lasso the minimum is taken over $\{\Delta \neq 0 : \|\Delta_{K^c}\|_1 \leq a_0 \|\Delta_K\|_1\}$ whereas for the Group Lasso the minimum is over $\{\Delta \neq 0 : \|\Delta_{K^c}\|_{2,1} \leq a_0 \|\Delta_K\|_{2,1}\}$. This assumption has already been used in [12, 13] to prove oracle inequality for linear gaussian noise model under Group sparsity and for multi-task learning.

Theorem 4.2. Let $f_{\hat{\beta}_{GL}}$ be the Group Lasso solution defined in (17) with ω_l defined as in (18). Fix $\eta > 0$ and $1 \leq s \leq g$, assume that Assumptions (B₁), (B₄), (B₅) and (RE₂) are satisfied, with $a_0 = 3 + 4/\eta$. Thus with probability at least $1 - \exp(-x)$ we have

$$R(f_{\hat{\beta}_{GL}}) - R(f_0) \leq (1 + \eta) \inf_{f_\beta \in \Gamma_1} \left\{ R(f_\beta) - R(f_0) + \frac{c(\eta)|J(\beta)| \left(\max_{1 \leq l \leq g} \omega_l \right)^2}{c_0 \epsilon_0 \mu_1(s, 3 + 4/\eta)^2} \right\},$$

and

$$\|f_{\hat{\beta}_{GL}} - f_0\|_n^2 \leq \frac{c'_0}{4c_0 \epsilon_0} (1 + \eta) \inf_{f_\beta \in \Gamma_1} \left\{ \|f_\beta - f_0\|_n^2 + \frac{4c(\eta)|J(\beta)| \left(\max_{1 \leq l \leq g} \omega_l \right)^2}{c'_0 c_0 \epsilon_0^2 \mu_1(s, 3 + 4/\eta)^2} \right\},$$

Where $c(\eta)$ is a constant depending only on η ; $c_0 = c_0(C_0, c_1)$ and $c'_0 = c'_0(C_0, c_1)$ are constants depending on C_0 and c_1 ; and $\epsilon_0 = \epsilon_0(c_1)$ is a constant depending on c_1 .

Once again the variance term are of order $\log g/n$, which refers to a "fast rate".

4.1 Special case: variable selection in the usual logistic model

In this section we assume that f_0 is a linear function ie $f_0(z_i) = f_{\beta_0}(z_i) = \sum_{l=1}^g \sum_{j \in G_l} \beta_j z_{ij}$. Denote by $X = (z_{ij})_{1 \leq i \leq n, 1 \leq j \in G_l, l=1, \dots, g}$, the design matrix. Let $z_i = (z_{ij})_{j \in G_l, l=1, \dots, g}$ be the i th row of the matrix X and $z^{(j)} = (z_{1j}, \dots, z_{nj})^T$ is j th column. For $i = 1, \dots, n$

$$\mathbb{P}(Y_i = 1/Z = z_i) = \frac{\exp(z_i^T \beta_0)}{1 + \exp(z_i^T \beta_0)}. \quad (19)$$

The Group Lasso estimator of β_0 is defined by

$$\hat{\beta}_{GL} := \underset{\beta: f_{\beta} \in \Gamma_1}{\operatorname{argmin}} \frac{1}{n} \sum_{i=1}^n \{\log(1 + \exp(z_i^T \beta)) - Y_i z_i^T \beta\} + \sum_{l=1}^g \omega_l \|\beta^l\|_2. \quad (20)$$

Corollary 4.1. *Let assumption $RE(s, 3)$ be satisfied and $|J(\beta_0)| \leq s$, where $1 \leq s \leq g$. Consider the Group Lasso estimator $f_{\hat{\beta}_{GL}}$ defined by (20) with*

$$\omega_l = \frac{2|G_l|}{n} \sqrt{\frac{1}{2} \max_{j \in G_l} \sum_{i=1}^n z_{ij}^2 \left(x + \log \sum_{l=1}^g |G_l| \right)} + \frac{2a_2 |G_l|}{3n} \left(x + \log \sum_{l=1}^g |G_l| \right)$$

where $x > 0$. Under the assumptions of Theorem 4.2, with probability at least $1 - \exp(-x)$ we have

$$R(f_{\hat{\beta}_{GL}}) - R(f_{\beta_0}) \leq \frac{8s \left(\max_{1 \leq l \leq g} \omega_l \right)^2}{\mu^2(s, 3) c_0 \epsilon_0} \quad (21)$$

$$\|f_{\hat{\beta}_{GL}} - f_{\beta_0}\|_n^2 \leq \frac{8s \left(\max_{1 \leq l \leq g} \omega_l \right)^2}{\mu^2(s, 3) c_0^2 \epsilon_0^2} \quad (22)$$

$$\|\hat{\beta}_{GL} - \beta_0\|_{2,1} \leq \frac{8\sqrt{2}s \left(\max_{1 \leq l \leq g} \omega_l \right)}{\sqrt{3}\mu^2(s, 3) c_0 \epsilon_0} \quad (23)$$

$$\|\hat{\beta}_{GL} - \beta_0\|_{2,q}^q \leq \left(\frac{8\sqrt{2}s \left(\max_{1 \leq l \leq g} \omega_l \right)}{\sqrt{3}\mu^2(s, 3) c_0 \epsilon_0} \right)^q \quad \text{for all } 1 < q \leq 2 \quad (24)$$

Remark 3.3 is still true in this case.

4.2 Non bounded functions

In this section we consider the Group Lasso estimator (20). We wish to establish the same result as Corollary 4.1 without assuming (B_1) or (B_5) ie neither f_{β_0} nor f_{β} is bounded. Let us consider the following

assumption.

For some integer s such that $1 \leq s \leq g$ and a positive number a_0 , the following condition holds **(RE₃)**

$$\mu_2(s, a_0) := \min_{K \subseteq \{1, \dots, p\}: |K| \leq s} \min_{\Delta \neq 0: \|\Delta_{K^c}\|_{2,1} \leq a_0 \|\Delta_K\|_{2,1}} \frac{\Delta^T X^T D X \Delta}{n \|\Delta_K\|_2^2} > 0,$$

where $D = \text{Diag}(\text{var}(Y_i))$.

This is an extension of the Assumption **RE₂** to the weighted Gram matrix $X^T D X/n$.

Theorem 4.3. Consider the Group Lasso estimator $f_{\hat{\beta}_{GL}}$ defined by (20) with

$$\omega_l = \frac{2|G_l|}{n} \sqrt{\frac{1}{2} \max_{j \in G_l} \sum_{i=1}^n z_{ij}^2 \left(x + \log \sum_{l=1}^g |G_l| \right)} + \frac{2a_2|G_l|}{3n} \left(x + \log \sum_{l=1}^g |G_l| \right),$$

where $x > 0$. Set $v = \max_{1 \leq i \leq n} \max_{1 \leq l \leq g} \|z_i^l\|_2$. Let Assumptions **(B₄)** and **(RE₃)** be satisfied with

$$a_0 = \frac{3 \max_{1 \leq l \leq g} \omega_l}{\min_{1 \leq l \leq g} \omega_l}.$$

If $\max_{1 \leq l \leq g} \omega_l \leq \mu_2/48v|J(\beta_0)|$, with probability at least $1 - \exp(-x)$ we have

$$R(f_{\hat{\beta}_{GL}}) - R(f_{\beta_0}) \leq \frac{144|J(\beta_0)| \left(\max_{1 \leq l \leq g} \omega_l \right)^2}{\mu_2^2(s, 3)} \quad (25)$$

$$\|\hat{\beta}_{GL} - \beta_0\|_{2,1} \leq \frac{96|J(\beta_0)| \left(\max_{1 \leq l \leq g} \omega_l \right)}{\mu_2^2(s, 3)} \quad (26)$$

$$\|\hat{\beta}_{GL} - \beta_0\|_{2,q}^q \leq \left(\frac{96|J(\beta_0)| \left(\max_{1 \leq l \leq g} \omega_l \right)}{\mu_2^2(s, 3)} \right)^q \quad \text{for all } 1 < q \leq 2. \quad (27)$$

Moreover if we assume that there exists $0 < \epsilon_0 \leq 1/2$ such that

$$\epsilon_0 \leq \pi(f_{\beta_0}(z_i))(1 - \pi(f_{\beta_0}(z_i))) \quad \text{for all } i = 1, \dots, n$$

then,

$$\|X\hat{\beta}_{GL} - X\beta_0\|_n^2 \leq \frac{144|J(\beta_0)| \left(\max_{1 \leq l \leq g} \omega_l \right)^2}{\mu^2(s, 3)\epsilon_0}. \quad (28)$$

4.3 Simulation study

To illustrate the theoretical part of this paper we provide in this section some experimental results on simulated data. Our aim is to compare the Lasso using the weights we proposed to the “classical” Lasso (without weights). We consider the Lasso estimator defined as

$$\hat{\beta}_L := \underset{\beta: f_\beta \in \Gamma}{\operatorname{argmin}} \left\{ \frac{1}{n} \sum_{i=1}^n \{ \log(1 + \exp(z_i^T \beta)) - Y_i z_i^T \beta \} + \lambda \sum_{j=1}^p \omega_j |\beta_j| \right\}, \quad (29)$$

where λ is a tuning parameter. Note that this is equivalent to the Lasso estimator defined in (12). For all the simulations λ will be estimated by cross-validation. According to Remark 3.2 we will use the weights ω'_j which seem more realistic than ω_j . We will proceed in two steps: (i) estimate the variance $\mathbb{E}(\epsilon_i^2)$, $\hat{\sigma}_i^2$; (ii) for each $j \in \{1, \dots, p\}$, calculate the estimator $\hat{\omega}'_j$ of the weight ω'_j , by replacing in ω'_j the variance $\mathbb{E}(\epsilon_i^2)$ by its estimator calculated in step (i). The great challenge is to find an estimator of $\mathbb{E}(\epsilon_i^2)$. We propose two estimators of $\mathbb{E}(\epsilon_i^2) = \pi(z_i^T \beta_0)(1 - \pi(z_i^T \beta_0))$:

1. estimate by $\hat{\sigma}_i^2 = \pi(z_i^T \hat{\beta}_L)(1 - \pi(z_i^T \hat{\beta}_L))$ where $\hat{\beta}_L$ is the “classical” Lasso estimator of β_0 (without weight);
2. the second estimator is $\hat{\sigma}_i^2 = \pi(z_i^T \hat{\beta}_{Logit})(1 - \pi(z_i^T \hat{\beta}_{Logit}))$, where $\hat{\beta}_{Logit}$ is an estimator of β_0 obtained after successively using the Lasso to screen coefficients and a logistic model which take into account coefficients different to zero in the Lasso.

Data generation. We simulate 500 datasets consisting of n observations from logistic model (11) with $\beta_0 = (1.5, -1, 2, 0, 0, \dots, 0) \in \mathbb{R}^p$ where $p = |K(\beta_0)| + k$, ($|K(\beta_0)| = 3$). This corresponds to the model with k null coefficients, where z_i are standard normal. For each data set we calculate the prediction error, estimation error, the rate of true selection and the rate of false relevant or irrelevant coefficient. The true selection is when the procedure select exactly the true relevant coefficients. The rate of relevant or irrelevant coefficient is the rate of bad selection in the estimation (the procedure declares relevant yet it is irrelevant or declares irrelevant yet it is relevant). The results for the four methods are presented in the Figure 2, Figure 3 and Figure 4. Lasso represents the “classical” Lasso (without weight); weight.Logit is the method (29) with weights estimated using procedure (2); weight.Lasso is the method (29) with weights estimated by the procedure (1); weight.theoretical is method (29) with theoretical weights.

4.4 Comments

For estimation or prediction error the performance of all the methods are almost the same. When the number of sample n increases, the performance of all the methods also increases. The strength of the methods decreases with the number k of null coefficients. The real difference is in rate of true selection

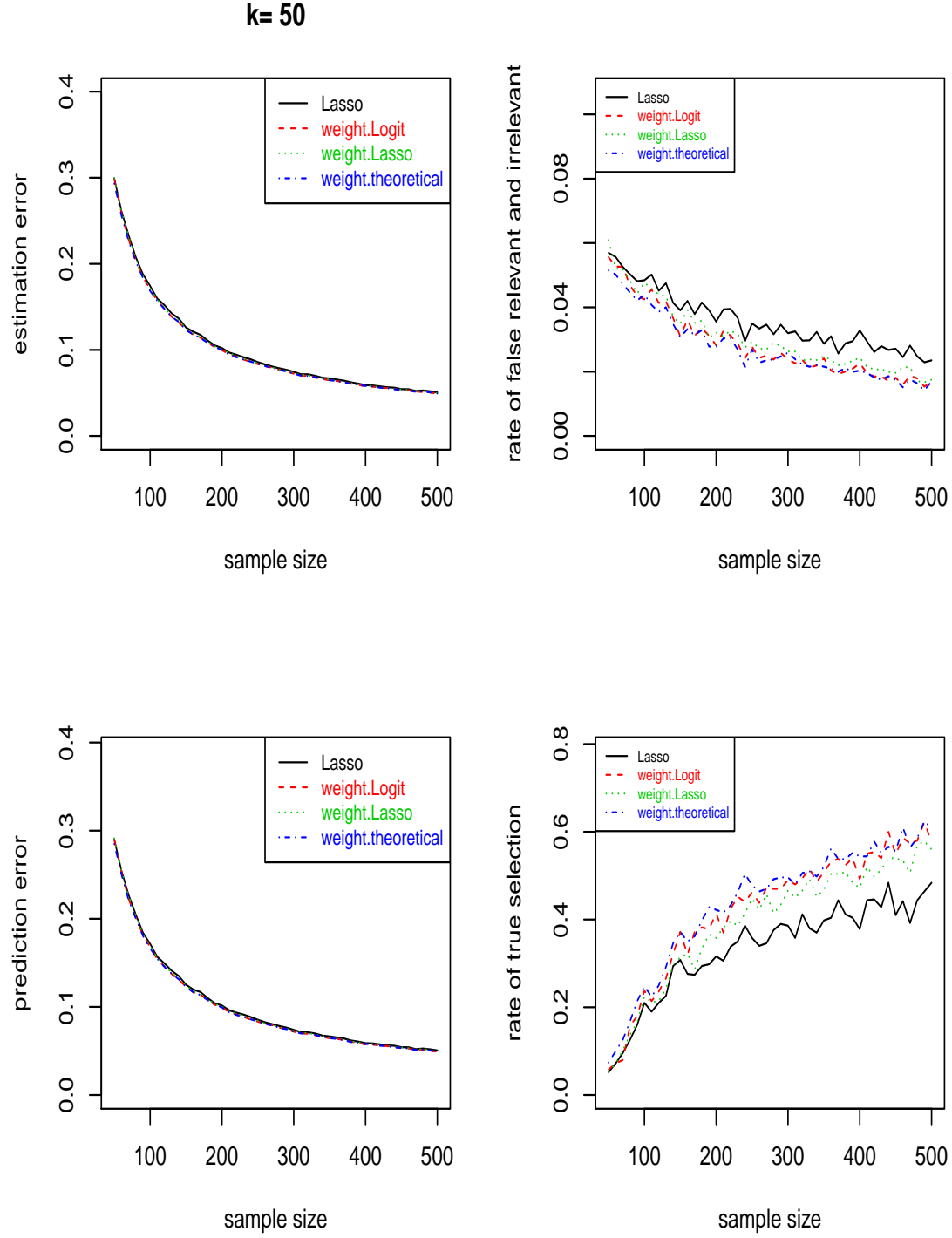


Figure 1: *Evolution of estimation error, prediction error, rate of true selection, and the rate of false relevant or irrelevant coefficient. $k=50$ from the setup described in Section 4.3*

and the rate of false relevant and irrelevant where the weight.theoretical, weight.Logit and weight.Lasso outperform the Lasso. And weight.Logit seem to be better than weight.Lasso.

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5 Proofs of main results

5.1 Proof of Theorem 3.1

Fix an arbitrary $\beta \in \mathbb{R}^p$ and set $\delta = W(\hat{\beta}_L - \beta)$, where $W = \text{Diag}(w_1, \dots, w_p)$. Short calculation shows that

$$R(f_\beta) = \hat{R}(f_\beta) + \frac{1}{n} \varepsilon^T X \beta$$

where $\varepsilon_i = Y_i - \mathbb{E}(Y_i)$, for $i \in 1, \dots, n$. Since $\hat{\beta}_L$ is the minimizer of $\hat{R}(f_\beta) + \sum_{j=1}^p \omega_j |\beta_j|$,

$$R(f_{\hat{\beta}_L}) - \frac{1}{n} \varepsilon^T X \hat{\beta}_L + \sum_{j=1}^p \omega_j |\hat{\beta}_{\lambda,j}| \leq R(f_\beta) - \frac{1}{n} \varepsilon^T X \beta + \sum_{j=1}^p \omega_j |\beta_j|,$$

thus

$$R(f_{\hat{\beta}_L}) - R(f_0) \leq R(f_\beta) - R(f_0) + \frac{1}{n} \varepsilon^T X (\hat{\beta}_L - \beta) + \sum_{j=1}^p \omega_j |\beta_j| - \sum_{j=1}^p \omega_j |\hat{\beta}_{\lambda,j}|.$$

This implies that

$$R(f_{\hat{\beta}_L}) - R(f_0) \leq R(f_\beta) - R(f_0) + \sum_{j=1}^p \left| \frac{1}{n} \sum_{i=1}^n \phi_j(z_i) \varepsilon_i \right| |\hat{\beta}_{\lambda,j} - \beta_j| + \sum_{j=1}^p \omega_j |\beta_j| - \sum_{j=1}^p \omega_j |\hat{\beta}_{\lambda,j}|.$$

For $1 \leq j \leq p$, set $S_j = \sum_{i=1}^n \phi_j(z_i) \varepsilon_i$ and let us denote by E , the event

$$E = \bigcap_{j=1}^p \{|S_j| \leq n\omega_j/2\}.$$

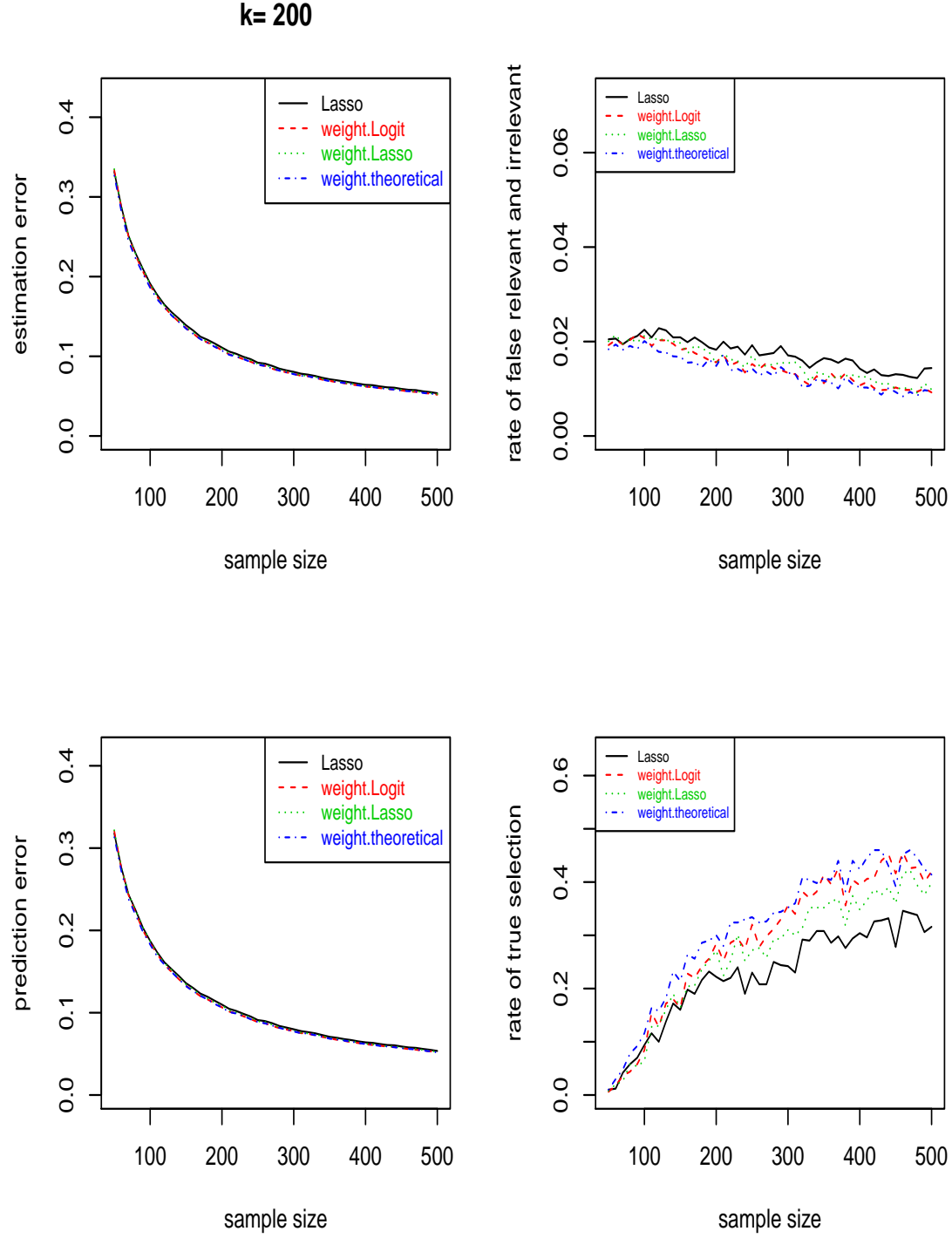


Figure 2: Evolution of estimation error, prediction error, rate of true selection, and the rate of false relevant or irrelevant coefficient. $k=200$ from the setup described in Section 4.3

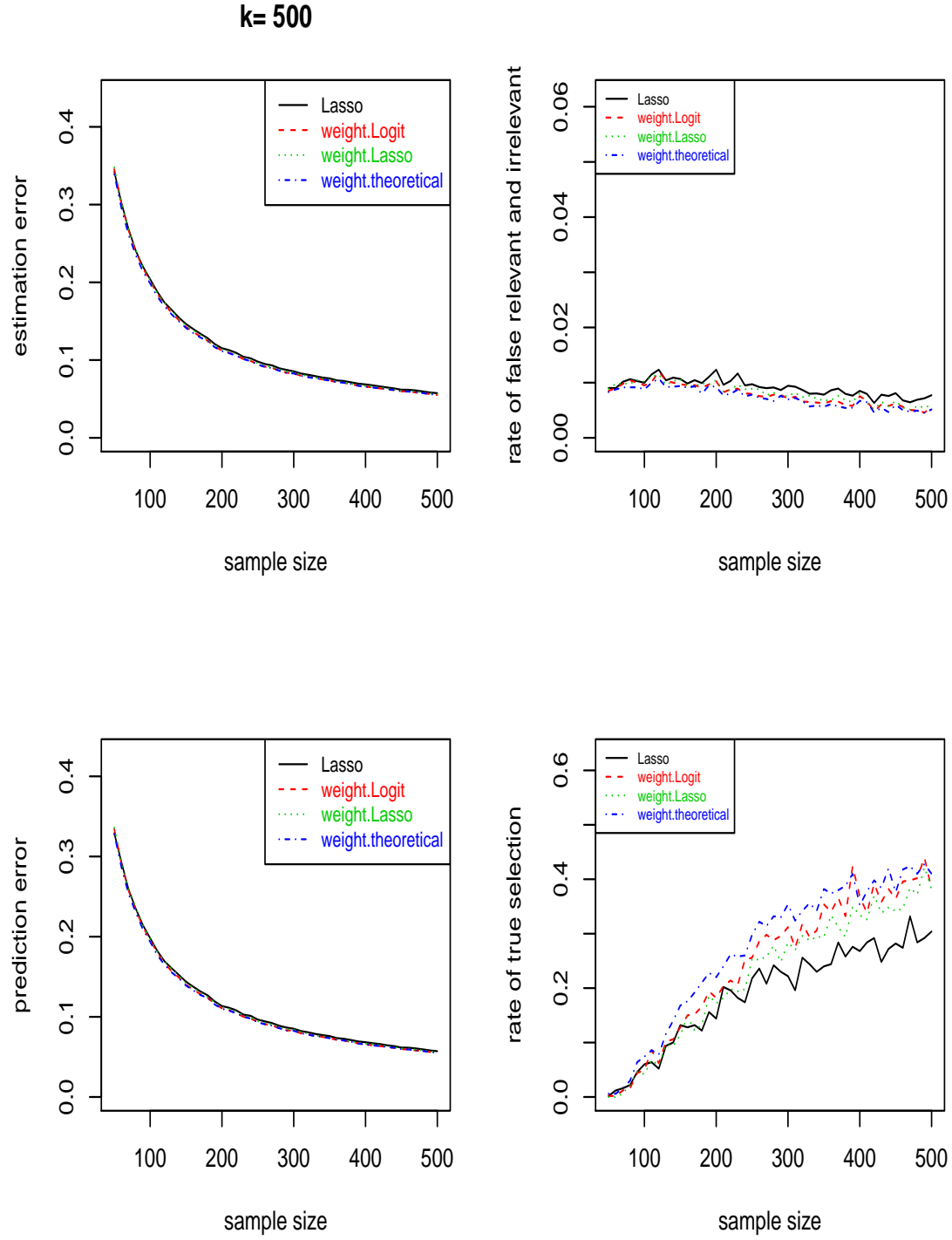


Figure 3: Evolution of estimation error, prediction error, rate of true selection, and the rate of false relevant or irrelevant coefficient. $k=500$ from the setup described in Section 4.3

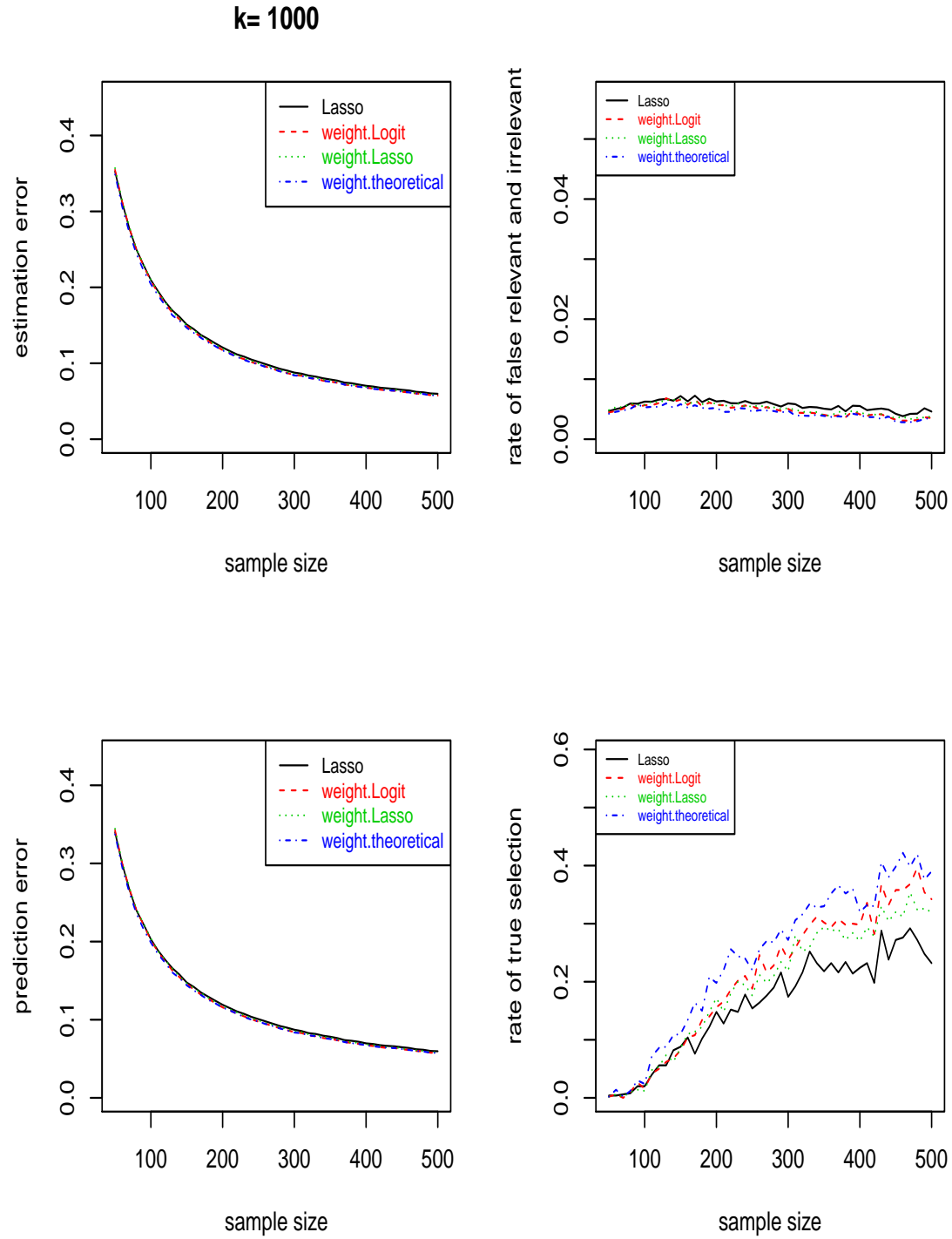


Figure 4: Evolution of estimation error, prediction error, rate of true selection, and the rate of false relevant or irrelevant coefficient. $k=1000$ from the setup described in Section 4.3

We state the results on the event E and then find an upper bound of $\mathbb{P}(E^c)$.

On the event E :

$$\begin{aligned} R(f_{\hat{\beta}_L}) - R(f_0) &\leq R(f_\beta) - R(f_0) + \sum_{j=1}^p \omega_j |\hat{\beta}_{\lambda,j} - \beta_j| + \sum_{j=1}^p \omega_j |\beta_j| - \sum_{j=1}^p \omega_j |\hat{\beta}_{\lambda,j}| \\ &\leq R(f_\beta) - R(f_0) + 2 \sum_{j=1}^p \omega_j |\beta_j|. \end{aligned}$$

We conclude that on the event E we have

$$R(f_{\hat{\beta}_L}) - R(f_0) \leq \inf_{\beta \in \mathbb{R}^p} \left\{ R(f_\beta) - R(f_0) + 2 \|\beta\|_1 \max_{1 \leq j \leq p} \omega_j \right\}.$$

Now we are going to find an upper bound of $\mathbb{P}(E^c)$:

$$\begin{aligned} \mathbb{P}(E^c) &\leq \mathbb{P} \left(\bigcup_{j=1}^p \left\{ \left| \sum_{i=1}^n \phi_j(z_i) (Y_i - \mathbb{E}(Y_i)) \right| > \omega_j n/2 \right\} \right) \\ &\leq \sum_{j=1}^p \mathbb{P} \left(\left| \sum_{i=1}^n \phi_j(z_i) (Y_i - \mathbb{E}(Y_i)) \right| > \omega_j n/2 \right). \end{aligned}$$

For $j \in \{1, \dots, p\}$, set $v_j = \sum_{i=1}^n \mathbb{E}(\phi_j^2 \epsilon_i^2)$. Since $\sum_{i=1}^n \phi_j^2(z_i) \geq 4v_j$, we have

$$\mathbb{P}(|S_j| > n\omega_j/2) \leq \mathbb{P} \left(|S_j| > \sqrt{2v_j(x + \log p)} + \frac{c_2}{3}(x + \log p) \right).$$

By applying Bernstein's inequality (see Lemma A.4) to the right hand side of the previous inequality we get

$$\mathbb{P}(|S_j| > n\omega_j/2) \leq 2 \exp(-x - \log p).$$

It follows that

$$\mathbb{P}(E^c) \leq \sum_{j=1}^p \mathbb{P}(|S_j| > \omega_j n/2) \leq \exp(-x). \quad (30)$$

When $\omega_j = \lambda$, for all $j \in \{1, \dots, p\}$ we apply Hoeffding's inequality (see Lemma A.5). This leads to

$$\begin{aligned} \mathbb{P}(E^c) &= \mathbb{P} \left(\bigcup_{j=1}^p \left\{ \left| \sum_{i=1}^n \phi_j(z_i) (Y_i - \mathbb{E}(Y_i)) \right| > \lambda n/2 \right\} \right) \\ &\leq \sum_{j=1}^p \mathbb{P} \left(\left| \sum_{i=1}^n \phi_j(z_i) (Y_i - \mathbb{E}(Y_i)) \right| > \lambda n/2 \right) \\ &\leq 2p \exp \left(-\frac{2(\lambda n/2)^2}{\sum_{i=1}^n 2c_2} \right) = 2p \exp \left(-\frac{\lambda^2 n}{4c_2} \right) = 2p^{1 - \frac{A^2}{4c_2}}. \end{aligned} \quad (31)$$

This ends the proof of Theorem 3.1. ■

5.2 Proof of Theorem 3.2

Fix an arbitrary $\beta \in \mathbb{R}^p$ such that $f_\beta \in \Gamma$, and set $\delta = W(\hat{\beta}_L - \beta)$, where $W = \text{Diag}(w_1, \dots, w_p)$. Short calculation shows that

$$R(f_\beta) = \hat{R}(f_\beta) + \frac{1}{n} \varepsilon^T X \beta$$

where $\varepsilon_i = Y_i - \mathbb{E}(Y_i)$, for $i \in 1, \dots, n$. Since $\hat{\beta}_L$ is the minimizer of $\hat{R}(f_\beta) + \sum_{j=1}^p \omega_j |\beta_j|$,

$$R(f_{\hat{\beta}_L}) - \frac{1}{n} \varepsilon^T X \hat{\beta}_L + \sum_{j=1}^p \omega_j |\hat{\beta}_{\lambda,j}| \leq R(f_\beta) - \frac{1}{n} \varepsilon^T X \beta + \sum_{j=1}^p \omega_j |\beta_j|,$$

then

$$R(f_{\hat{\beta}_L}) \leq R(f_\beta) + \sum_{j=1}^p \left| \frac{1}{n} \sum_{i=1}^n \phi_j(z_i) \varepsilon_i \right| |\hat{\beta}_{\lambda,j} - \beta_j| + \sum_{j=1}^p \omega_j |\beta_j| - \sum_{j=1}^p \omega_j |\hat{\beta}_{\lambda,j}|. \quad (32)$$

For $1 \leq j \leq p$, set $S_j = \sum_{i=1}^n \phi_j(z_i) \varepsilon_i$ and let us denote by E , the event

$$E = \bigcap_{j=1}^p \{|S_j| \leq n\omega_j/2\}.$$

We state the results on the event E and find an upper bound of $\mathbb{P}(E^c)$.

On the event E : adding the term $\frac{1}{2} \sum_{j=1}^p \omega_j |\hat{\beta}_{\lambda,j} - \beta_j|$ to both sides of Inequality (32) yields to

$$R(f_{\hat{\beta}_L}) + \frac{1}{2} \sum_{j=1}^p \omega_j |\hat{\beta}_{\lambda,j} - \beta_j| \leq R(f_\beta) + \sum_{j=1}^p \omega_j (|\hat{\beta}_{\lambda,j} - \beta_j| + |\beta_j| - |\hat{\beta}_{\lambda,j}|).$$

Since $|\hat{\beta}_{\lambda,j} - \beta_j| + |\beta_j| - |\hat{\beta}_{\lambda,j}| = 0$ for $j \notin K(\beta)$, we have

$$R(f_{\hat{\beta}_L}) - R(f_0) + \frac{1}{2} \sum_{j=1}^p \omega_j |\hat{\beta}_{\lambda,j} - \beta_j| \leq R(f_\beta) - R(f_0) + 2\|\delta_K\|_1. \quad (33)$$

We get from Equation (33)

$$R(f_{\hat{\beta}_L}) - R(f_0) \leq R(f_\beta) - R(f_0) + 2\|\delta_K\|_1. \quad (34)$$

Now, we consider separately two events:

$$E_1 = \{2\|\delta_K\|_1 \leq \eta(R(f_\beta) - R(f_0))\},$$

and

$$E_2 = \{\eta(R(f_\beta) - R(f_0)) < 2\|\delta_K\|_1\}.$$

On the event $E \cap E_1$ we get from (34)

$$R(f_{\hat{\beta}_L}) - R(f_0) \leq (1 + \eta)(R(f_\beta) - R(f_0)), \quad (35)$$

and hence the result follows. On the event $E \cap E_2$ all subsequent inequalities are valid. On one hand, by applying Cauchy-Schwarz inequality, we get from (34) that

$$R(f_{\hat{\beta}_L}) - R(f_0) \leq R(f_\beta) - R(f_0) + 2\sqrt{|K(\beta)|}\|\delta_K\|_2. \quad (36)$$

On the other hand, we get from Equation (33) that

$$\frac{1}{2} \sum_{j=1}^p \omega_j |\hat{\beta}_{\lambda_j} - \beta_j| \leq R(f_\beta) - R(f_0) + 2\|\delta_K\|_1,$$

and using (5.2) we obtain

$$\|\delta\|_1 \leq (4 + 4/\eta)\|\delta_K\|_1,$$

which implies $\|\delta_{K^c}\|_1 \leq (3 + 4/\eta)\|\delta_K\|_1$. We can therefore apply Assumption (RE₁). with $a_0 = 3 + 4/\eta$. This yields,

$$\mu^2 \|\delta_K\|_2^2 \leq \frac{\|X\delta\|_2^2}{n} = \frac{1}{n}(\hat{\beta}_L - \beta)^T W X^T X W (\hat{\beta}_L - \beta) \leq \left(\max_{1 \leq j \leq p} \omega_j \right)^2 \|f_{\hat{\beta}_L} - f_\beta\|_n^2. \quad (37)$$

Gathering Equations (36) and (37) we obtain

$$\begin{aligned} R(f_{\hat{\beta}_L}) - R(f_0) &\leq R(f_\beta) - R(f_0) + 2 \left(\max_{1 \leq j \leq p} \omega_j \right) \sqrt{|K(\beta)|} \mu^{-1} \|f_{\hat{\beta}_L} - f_\beta\|_n \\ &\leq R(f_\beta) - R(f_0) + 2 \left(\max_{1 \leq j \leq p} \omega_j \right) \sqrt{|K(\beta)|} \mu^{-1} (\|f_{\hat{\beta}_L} - f_0\|_n + \|f_\beta - f_0\|_n). \end{aligned}$$

We now apply Lemma 5.1

Lemma 5.1. *Under Assumptions (B₁) and (B₃) we have*

$$c_0 \epsilon_0 \|f_\beta - f_0\|_n^2 \leq R(f_\beta) - R(f_0) \leq \frac{1}{4} c'_0 \|f_\beta - f_0\|_n^2,$$

where c_0 and c'_0 are constants depending on C_0 and c_1 ; ϵ_0 is a constant depending on c_1 and c_2 .

Consequently

$$\begin{aligned} R(f_{\hat{\beta}_L}) - R(f_0) &\leq R(f_\beta) - R(f_0) + \frac{2 \left(\max_{1 \leq j \leq p} \omega_j \right) \sqrt{|K(\beta)|} \mu^{-1}}{\sqrt{c_0 \epsilon_0}} \sqrt{R(f_{\hat{\beta}_L}) - R(f_0)} \\ &\quad + \frac{2 \left(\max_{1 \leq j \leq p} \omega_j \right) \sqrt{|K(\beta)|} \mu^{-1}}{\sqrt{c_0 \epsilon_0}} \sqrt{R(f_\beta) - R(f_0)}. \end{aligned}$$

Using inequality $2uv < u^2/b + bv^2$ for all $b > 1$, with $u = \left(\max_{1 \leq j \leq p} \omega_j \right) \sqrt{|K(\beta)|} \mu^{-1} / \sqrt{c_0 \epsilon_0}$ and v being either $\sqrt{R(f_{\hat{\beta}_L}) - R(f_0)}$ or $\sqrt{R(f_\beta) - R(f_0)}$ we have

$$\begin{aligned} R(f_{\hat{\beta}_L}) - R(f_0) &\leq R(f_\beta) - R(f_0) + 2b \left(\frac{\omega_{\max} \sqrt{|K(\beta)|} \mu^{-1}}{\sqrt{c_0 \epsilon_0}} \right)^2 \\ &\quad + \frac{R(f_{\hat{\beta}_L}) - R(f_0)}{b} + \frac{R(f_\beta) - R(f_0)}{b}. \end{aligned}$$

It follows that

$$R(f_{\hat{\beta}_L}) - R(f_0) \leq \frac{b+1}{b-1} \left\{ R(f_\beta) - R(f_0) + \frac{2b^2 \left(\max_{1 \leq j \leq p} \omega_j \right)^2 |K(\beta)|}{(b+1)(b-1)\mu^2 c_0 \epsilon_0} \right\}.$$

Now taking $b = 1 + 2/\eta$ leads to

$$R(f_{\hat{\beta}_L}) - R(f_0) \leq (1 + \eta) \left\{ R(f_\beta) - R(f_0) + \frac{c(\eta) \left(\max_{1 \leq j \leq p} \omega_j \right)^2 |K(\beta)|}{\mu^2 c_0 \epsilon_0} \right\}. \quad (38)$$

According to Inequalities (35) and (38), we conclude that on the event E we have

$$R(f_{\hat{\beta}_L}) - R(f_0) \leq (1 + \eta) \left\{ R(f_\beta) - R(f_0) + \frac{c(\eta) \left(\max_{1 \leq j \leq p} \omega_j \right)^2 |K(\beta)|}{\mu^2 c_0 \epsilon_0} \right\}, \quad (39)$$

where $c(\eta) = 2(1 + 2/\eta)^2 / [(2 + 2/\eta)(2/\eta)]$. This ends the proof of (7) of the Theorem 3.2. Equation (8) follows from Lemma 5.1: To prove Inequalities (9) and (10) we just replace ω_j by λ .

This ends the proof of the Theorem 3.2 by using (30) and (31). ■

5.3 Proof of Corollary 3.1

Set $\delta = W(\hat{\beta}_L - \beta_0)$. The result (13) directly comes by taking $\beta = \beta_0$ and $\eta = 2$ in (39). Note that, on the event E we have

$$\|\delta_{K(\beta_0)^c}\|_1 \leq 3\|\delta_{K(\beta_0)}\|_1.$$

Indeed, since $\hat{\beta}_L$ is the minimizer of $\hat{R}(f_\beta) + \sum_{j=1}^p \omega_j |\beta_j|$, then

$$R(f_{\hat{\beta}_L}) - R(f_{\beta_0}) + \sum_{j=1}^p \omega_j |\hat{\beta}_{\lambda,j}| \leq \frac{1}{n} \varepsilon^T X(\hat{\beta}_L - \beta_0) + \sum_{j=1}^p \omega_j |\beta_{0j}|,$$

which implies that

$$\|W\hat{\beta}_L\|_1 \leq \sum_{j=1}^p \left| \frac{1}{n} \sum_{i=1}^n \phi_j(z_i) \epsilon_i \right| |\hat{\beta}_{\lambda,j} - \beta_j| + \|W\beta_0\|_1.$$

On the event E we have

$$\begin{aligned} \|W(\hat{\beta}_L)_{K(\beta_0)}\|_1 + \|W(\hat{\beta}_L)_{K^c(\beta_0)}\|_1 &\leq \frac{1}{2}(\|W(\hat{\beta}_L - \beta_0)_{K(\beta_0)}\|_1 + \|W(\hat{\beta}_L)_{K^c(\beta_0)}\|_1) \\ &\quad + \|W(\beta_0)_{K(\beta_0)}\|_1, \end{aligned}$$

thus $\|\delta_{K(\beta_0)^c}\|_1 \leq 3\|\delta_{K(\beta_0)}\|_1$. Result (14) follows from Equation (13) by applying Lemma 5.1. Equation (15) follows from (14) by applying Equation (37) and $\|\delta\|_1^2 \leq 16s\|\delta_{K(\beta_0)}\|_2^2$. The last line follows from Lemma A.3 in Appendix with $a_j = |\hat{\beta}_{\lambda,j} - \beta_{0j}|$ and $b_1 = 8\sqrt{2}s(\max_{1 \leq j \leq p} \omega_j)/\mu^2(s, 3)c_0\epsilon_0$. ■

5.4 Proof of Theorem 4.1

Fix an arbitrary $\beta \in \mathbb{R}^{\sum_{i=1}^g |G_l|}$. Set $\delta = W(\hat{\beta}_{GL} - \beta)$ where $W = \text{Diag}(W_1, \dots, W_p)$ is a block diagonal matrix, with $W_l = \text{Diag}(\omega_l, \dots, \omega_l)$. Since $\hat{\beta}_{GL}$ is the minimizer of $\hat{R}(f_\beta) + \sum_{l=1}^g \omega_l \|\beta^l\|_2$, we get

$$R(f_{\hat{\beta}_{GL}}) - \frac{1}{n} \varepsilon^T X \hat{\beta}_{GL} + \sum_{l=1}^g \omega_l \|\hat{\beta}^l\|_2 \leq R(f_\beta) - \frac{1}{n} \varepsilon^T X \beta + \sum_{l=1}^g \omega_l \|\beta^l\|_2.$$

By applying Cauchy-Schwarz inequality, this implies

$$\begin{aligned} R(f_{\hat{\beta}_{GL}}) - R(f_0) &\leq R(f_\beta) - R(f_0) + \sum_{l=1}^g \frac{1}{n} \sqrt{\sum_{j \in G_l} \left(\sum_{i=1}^n \phi_j(z_i) \epsilon_i \right)^2} \|(\hat{\beta}_{GL} - \beta)^l\|_2 \\ &\quad + \sum_{l=1}^g \omega_l \|\beta^l\|_2 - \sum_{l=1}^g \omega_l \|\hat{\beta}^l\|_2. \end{aligned}$$

Set $Z_l = n^{-1} \sqrt{\sum_{j \in G_l} (\sum_{i=1}^n \phi_j(z_i) \epsilon_i)^2}$, for $l \in \{1, \dots, g\}$ and the event

$$\mathcal{A} = \bigcap_{l=1}^g \{Z_l \leq \omega_l/2\}.$$

We state the result on event \mathcal{A} and find an upper bound of $\mathbb{P}(\mathcal{A}^c)$.

On the event \mathcal{A} :

$$R(f_{\hat{\beta}_{GL}}) - R(f_0) \leq R(f_\beta) - R(f_0) + \sum_{l=1}^g \omega_l \|(\hat{\beta}_{GL} - \beta)^l\|_2 + \sum_{l=1}^g \omega_l \|\beta^l\|_2 - \sum_{l=1}^g \omega_l \|\hat{\beta}^l\|_2.$$

This implies that

$$R(f_{\hat{\beta}_{GL}}) - R(f_0) \leq R(f_\beta) - R(f_0) + 2 \sum_{l=1}^g \omega_l \|\beta^l\|_2.$$

We conclude that on the event \mathcal{A} we have

$$R(f_{\hat{\beta}_{GL}}) - R(f_0) \leq \inf_{\beta \in \mathbb{R}^{\sum_{l=1}^g |G_l|}} \left\{ R(f_\beta) - R(f_0) + 2\|\beta\|_{2,1} \max_{1 \leq l \leq g} \omega_l \right\}.$$

We now come to the bound on $\mathbb{P}(A^c)$ and write

$$\begin{aligned} \mathbb{P}(A^c) &= \mathbb{P} \left(\bigcup_{l=1}^g \left\{ \sqrt{\sum_{j \in G_l} \left(\sum_{i=1}^n \phi_j(z_i) \epsilon_i \right)^2} > n\omega_l/2 \right\} \right) \\ &\leq \sum_{l=1}^g \mathbb{P} \left(\sqrt{\sum_{j \in G_l} \left(\sum_{i=1}^n \phi_j(z_i) \epsilon_i \right)^2} > n\omega_l/2 \right). \end{aligned}$$

For $j \in G_l$ set $T_j^l = \sum_{i=1}^n \phi_j(z_i) \epsilon_i$, we have

$$\begin{aligned} \mathbb{P}(A^c) &\leq \sum_{l=1}^g \mathbb{P} \left(\sqrt{\sum_{j \in G_l} (T_j^l)^2} > n\omega_l/2 \right) \\ &\leq \sum_{l=1}^g \mathbb{P} \left(\sum_{j \in G_l} |T_j^l| > n\omega_l/2 \right). \end{aligned}$$

Using the fact that, for all $l \in \{1, \dots, g\}$

$$\left\{ \sum_{j \in G_l} |T_j^l| > n\omega_l/2 \right\} \subset \bigcup_{j \in G_l} \left\{ |T_j^l| > \frac{n\omega_l}{2|G_l|} \right\}, \quad (40)$$

it follows that

$$\mathbb{P}(A^c) \leq \sum_{l=1}^g \sum_{j \in G_l} \mathbb{P} \left(|T_j^l| > \frac{n\omega_l}{2|G_l|} \right).$$

For $j \in G_l$, set $v_j^l = \sum_{i=1}^n \mathbb{E}(\phi_j^2 \epsilon_i^2)$. Since $\sum_{i=1}^n \phi_j^2(z_i) \geq 4v_j^l$, we have

$$\mathbb{P}(|T_j^l| > \frac{n\omega_l}{2|G_l|}) \leq \mathbb{P} \left(|T_j^l| > \sqrt{2v_j^l \left(x + \log \sum_{l=1}^g |G_l| \right)} + \frac{c_l}{3} \left(x + \log \sum_{l=1}^g |G_l| \right) \right).$$

By applying Bernstein's inequality (see Lemma A.4) to the right hand side of the previous inequality we get

$$\mathbb{P}(|T_j^l| > \frac{n\omega_l}{2|G_l|}) \leq 2 \exp \left(-x - \log \sum_{l=1}^g |G_l| \right).$$

It follows that

$$\mathbb{P}(A^c) \leq \sum_{l=1}^g \sum_{j \in G_l} \mathbb{P} \left(|T_j^l| > \frac{n\omega_l}{2|G_l|} \right) \leq \exp(-x). \quad (41)$$

this ends the proof of the Theorem 4.1. ■

5.5 Proof of Theorem 4.2

The proof starts as in the proof of Theorem 3.2. Fix an arbitrary $\beta \in \mathbb{R}^{\sum_{l=1}^g |G_l|}$ such that $f_\beta \in \Gamma_1$. Set $\delta = W(\hat{\beta}_{GL} - \beta)$ where $W = \text{Diag}(W_1, \dots, W_p)$ is a block diagonal matrix, with $W_l = \text{Diag}(\omega_l, \dots, \omega_l)$. Since $\hat{\beta}_{GL}$ is the minimizer of $\hat{R}(f_\beta) + \sum_{l=1}^g \omega_l \|\beta^l\|_2$, we get

$$R(f_{\hat{\beta}_{GL}}) - \frac{1}{n} \varepsilon^T X \hat{\beta}_{GL} + \sum_{l=1}^g \omega_l \|\hat{\beta}^l\|_2 \leq R(f_\beta) - \frac{1}{n} \varepsilon^T X \beta + \sum_{l=1}^g \omega_l \|\beta^l\|_2$$

which implies:

$$R(f_{\hat{\beta}_{GL}}) \leq R(f_\beta) + \frac{1}{n} \varepsilon^T X (\hat{\beta}_{GL} - \beta) + \sum_{l=1}^g \omega_l \|\beta^l\|_2 - \sum_{l=1}^g \omega_l \|\hat{\beta}^l\|_2 \quad (42)$$

By applying Cauchy-Schwarz inequality, we obtain

$$R(f_{\hat{\beta}_{GL}}) \leq R(f_\beta) + \sum_{l=1}^g \frac{1}{n} \sqrt{\sum_{j \in G_l} \left(\sum_{i=1}^n \phi_j(z_i) \epsilon_i \right)^2} \|(\hat{\beta}_{GL} - \beta)^l\|_2 + \sum_{l=1}^g \omega_l \|\beta^l\|_2 - \sum_{l=1}^g \omega_l \|\hat{\beta}^l\|_2.$$

Set $Z_l = n^{-1} \sqrt{\sum_{j \in G_l} \left(\sum_{i=1}^n \phi_j(z_i) \epsilon_i \right)^2}$, for $l \in \{1, \dots, g\}$ and the event

$$\mathcal{A} = \bigcap_{l=1}^g \{Z_l \leq \omega_l/2\}.$$

We state the result on event \mathcal{A} and find an upper bound of $\mathbb{P}(A^c)$.

On the event \mathcal{A} : adding the term $\frac{1}{2} \sum_{l=1}^g \omega_l \|(\hat{\beta}_{GL} - \beta)^l\|_2$ to both sides of Inequality (42) yields to

$$R(f_{\hat{\beta}_{GL}}) + \frac{1}{2} \sum_{l=1}^g \omega_l \|(\hat{\beta}_{GL} - \beta)^l\|_2 \leq R(f_\beta) + \sum_{l=1}^g \omega_l (\|(\hat{\beta}_{GL} - \beta)^l\|_2 - \|\hat{\beta}_{GL}^l\|_2 + \|\beta^l\|_2).$$

Since $\|(\hat{\beta}_{GL} - \beta)^l\|_2 - \|\hat{\beta}_{GL}^l\|_2 + \|\beta^l\|_2 = 0$ for $l \notin J(\beta) = J$, we have

$$R(f_{\hat{\beta}_{GL}}) - R(f_0) + \frac{1}{2} \sum_{l=1}^g \omega_l \|(\hat{\beta}_{GL} - \beta)^l\|_2 \leq R(f_\beta) - R(f_0) + 2 \sum_{l \in J} \omega_l \|(\hat{\beta}_{GL} - \beta)^l\|_2. \quad (43)$$

we get from Equation (43) that

$$R(f_{\hat{\beta}_{GL}}) - R(f_0) \leq R(f_\beta) - R(f_0) + 2 \sum_{l \in J} \omega_l \|(\hat{\beta}_{GL} - \beta)^l\|_2 \quad (44)$$

Consider separately the two events:

$$\mathcal{A}_1 = \{2 \sum_{l \in J} \omega_l \|(\hat{\beta}_{GL} - \beta)^l\|_2 \leq \eta(R(f_\beta) - R(f_0))\},$$

and

$$\mathcal{A}_2 = \{\eta(R(f_\beta) - R(f_0)) < 2 \sum_{l \in J} \omega_l \|(\hat{\beta}_{GL} - \beta)^l\|_2\}.$$

On the event $\mathcal{A} \cap \mathcal{A}_1$, we get from (44)

$$R(f_{\hat{\beta}_{GL}}) - R(f_0) \leq (1 + \eta)(R(f_\beta) - R(f_0)), \quad (45)$$

and the result follows. On the event $\mathcal{A} \cap \mathcal{A}_1$, all the following inequalities are valid. On one hand, by applying Cauchy Schwarz inequality, we get from (44) that

$$\begin{aligned} R(f_{\hat{\beta}_{GL}}) - R(f_0) &\leq R(f_\beta) - R(f_0) + 2\sqrt{|J(\beta)|} \sqrt{\sum_{l \in J} \omega_l^2 \|(\hat{\beta}_{GL} - \beta)^l\|_2^2} \\ &\leq R(f_\beta) - R(f_0) + 2\sqrt{|J(\beta)|} \|\delta_J\|_2. \end{aligned} \quad (46)$$

On the other hand we get from Equation (43) that

$$\frac{1}{2} \sum_{l=1}^g \omega_l \|(\hat{\beta}_{GL} - \beta)^l\|_2 \leq R(f_\beta) - R(f_0) + 2 \sum_{l \in J} \omega_l \|(\hat{\beta}_{GL} - \beta)^l\|_2,$$

and using (5.5) we obtain

$$\frac{1}{2} \sum_{l \in J} \omega_l \|(\hat{\beta}_{GL} - \beta)^l\|_2 + \frac{1}{2} \sum_{l \in J^c} \omega_l \|(\hat{\beta}_{GL} - \beta)^l\|_2 \leq \frac{2}{\eta} \sum_{l \in J} \omega_l \|(\hat{\beta}_{GL} - \beta)^l\|_2 + 2 \sum_{l \in J} \omega_l \|(\hat{\beta}_{GL} - \beta)^l\|_2,$$

which implies

$$\|\delta_{J^c}\|_{2,1} \leq (3 + 4/\eta) \|\delta_J\|_{2,1}.$$

We can therefore apply Assumption (RE₂) with $a_0 = 3 + 4/\eta$. It follows that,

$$\mu_1^2 \|\delta_J\|_2^2 \leq \frac{\|X\delta\|_2^2}{n} = \frac{1}{n} (\hat{\beta}_{GL} - \beta)^T W X^T X W (\hat{\beta}_{GL} - \beta) \leq (\max_{1 \leq l \leq g} \omega_l)^2 \|f_{\hat{\beta}_{GL}} - f_\beta\|_n^2. \quad (47)$$

Gathering Equation (46) and (47) we get

$$\begin{aligned} R(f_{\hat{\beta}_{GL}}) - R(f_0) &\leq R(f_\beta) - R(f_0) + 2(\max_{1 \leq l \leq g} \omega_l) \sqrt{|K(\beta)|} \mu_1^{-1} \|f_{\hat{\beta}_{GL}} - f_\beta\|_n \\ &\leq R(f_\beta) - R(f_0) + 2(\max_{1 \leq l \leq g} \omega_l) \sqrt{|K(\beta)|} \mu_1^{-1} (\|f_{\hat{\beta}_{GL}} - f_0\|_n + \|f_\beta - f_0\|_n). \end{aligned}$$

We now apply Lemma 5.2

Lemma 5.2. Under assumptions (\mathbf{B}_1) and (\mathbf{B}_5) we have

$$c_0 \epsilon_0 \|f_\beta - f_0\|_n^2 \leq R(f_\beta) - R(f_0) \leq \frac{1}{4} c'_0 \|f_\beta - f_0\|_n^2.$$

where c_0 and c'_0 are constants depending on C_0 and ϵ_0 is a constant depending on c_1 and c_2 .

Consequently

$$\begin{aligned} R(f_{\hat{\beta}_{GL}}) - R(f_0) &\leq R(f_\beta) - R(f_0) + \frac{2(\max_{1 \leq l \leq g} \omega_l) \sqrt{|J(\beta)|} \mu_1^{-1}}{\sqrt{c_0 \epsilon_0}} \sqrt{R(f_{\hat{\beta}_{GL}}) - R(f_0)} \\ &\quad + \frac{2(\max_{1 \leq l \leq g} \omega_l) \sqrt{|J(\beta)|} \mu_1^{-1}}{\sqrt{c_0 \epsilon_0}} \sqrt{R(f_\beta) - R(f_0)}. \end{aligned}$$

Using inequality $2uv < u^2/b + bv^2$ for all $b > 1$, with $u = (\max_{1 \leq l \leq g} \omega_l) \frac{\sqrt{|J(\beta)|} \mu_1^{-1}}{\sqrt{c_0 \epsilon_0}}$ and v being either $\sqrt{R(f_{\hat{\beta}_{GL}}) - R(f_0)}$ or $\sqrt{R(f_\beta) - R(f_0)}$ we have

$$\begin{aligned} R(f_{\hat{\beta}_{GL}}) - R(f_0) &\leq R(f_\beta) - R(f_0) + 2b \left(\frac{(\max_{1 \leq l \leq g} \omega_l) \sqrt{|J(\beta)|} \mu_1^{-1}}{\sqrt{c_0 \epsilon_0}} \right)^2 \\ &\quad + \frac{R(f_{\hat{\beta}_{GL}}) - R(f_0)}{b} + \frac{R(f_\beta) - R(f_0)}{b}. \end{aligned}$$

This implies that

$$R(f_{\hat{\beta}_{GL}}) - R(f_0) \leq \frac{b+1}{b-1} \left\{ R(f_\beta) - R(f_0) + \frac{2b^2 (\max_{1 \leq l \leq g} \omega_l)^2 |J(\beta)|}{(b+1)(b-1) \mu_1^2 c_0 \epsilon_0} \right\}. \quad (48)$$

Now taking $b = 1 + 2/\eta$ leads to

$$R(f_{\hat{\beta}_{GL}}) - R(f_0) \leq (1 + \eta) \left\{ R(f_\beta) - R(f_0) + \frac{c(\eta) (\max_{1 \leq l \leq g} \omega_l)^2 |J(\beta)|}{\mu_1^2 c_0 \epsilon_0} \right\}. \quad (49)$$

According to Inequalities (45) and (49) we conclude that on event \mathcal{A} ,

$$R(f_{\hat{\beta}_{GL}}) - R(f_0) \leq (1 + \eta) \left\{ R(f_\beta) - R(f_0) + \frac{c(\eta) (\max_{1 \leq l \leq g} \omega_l)^2 |J(\beta)|}{\mu_1^2 c_0 \epsilon_0} \right\}, \quad (50)$$

where $c(\eta) = 2(1 + 2/\eta)^2 / (2 + 2/\eta)(2/\eta)$. This ends the proof of (4.2) of the theorem 4.2. Equation 4.2 of the theorem follows from Lemma 5.2.

This ends the proof of the Theorem 4.2 by considering (41). ■

5.6 Proof of Corollary 4.1

Set $\delta = W(\hat{\beta}_{GL} - \beta_0)$, Line (21) of corollary 4.1 follows directly from Equation (50) with $\beta = \beta_0$ and $\eta = 2$. Note that on the event A , we have

$$\|\delta_{J(\beta_0)^c}\|_{2,1} \leq 3\|\delta_{J(\beta_0)}\|_{2,1}. \quad (51)$$

Indeed, since $\hat{\beta}_{GL}$ is the minimizer of $\hat{R}(f_\beta) + \sum_{l=1}^g \omega_l \|\beta^l\|_2$,

$$R(f_{\hat{\beta}_{GL}}) - R(f_{\beta_0}) + \sum_{l=1}^g \omega_l \|\hat{\beta}_{GL}^l\|_2 \leq \frac{1}{n} \varepsilon^T X(\hat{\beta}_{GL} - \beta_0) + \sum_{l=1}^g \omega_l \|\beta_0^l\|_2$$

which implies

$$\|W\hat{\beta}_{GL}\|_{2,1} \leq \sum_{l=1}^g \frac{1}{n} \sqrt{\sum_{j \in G_l} \left(\sum_{i=1}^n (z_{ij} \epsilon_i) \right)^2} \|(\hat{\beta}_{GL} - \beta_0)^l\|_2 + \|W\beta_0\|_{2,1}$$

On the event A we have

$$\begin{aligned} \|W(\hat{\beta}_{GL})_{J(\beta_0)}\|_{2,1} + \|W(\hat{\beta}_{GL})_{J^c(\beta_0)}\|_{2,1} &\leq \frac{1}{2} (\|W(\hat{\beta}_{GL} - \beta_0)_{J(\beta_0)}\|_{2,1} + \|W(\hat{\beta}_{GL})_{J^c(\beta_0)}\|_{2,1}) \\ &\quad + \|W(\beta_0)_{J(\beta_0)}\|_{2,1}, \end{aligned}$$

thus $\|\delta_{J(\beta_0)^c}\|_{2,1} \leq 3\|\delta_{J(\beta_0)}\|_{2,1}$. Line (22) follows from Line 21 by applying Lemma (5.2). Line (23) follows from the second by using Equation (47) and $\|\delta\|_{2,1}^2 \leq 16s\|\delta_{J(\beta_0)}\|_2^2$. Line (24) is the consequence of the Lemma A.3 with $a_l = \|(\hat{\beta}_{GL} - \beta_0)^l\|_2$ and $b_1 = 8\sqrt{2}s(\max_{1 \leq l \leq g} \omega_l)/\mu^2(s, 3)c_0\epsilon_0$. ■

5.7 Proof of Theorem 4.3

Set $\delta' = \hat{\beta}_{GL} - \beta_0$ and $J = J(\beta_0)$. By definition, $\hat{\beta}_{GL}$ is the minimizer of $\hat{R}(f_\beta) + \sum_{l=1}^g \omega_l \|\beta^l\|_2$, and hence

$$R(f_{\hat{\beta}_{GL}}) - R(f_{\beta_0}) + \sum_{l=1}^g \omega_l \|\hat{\beta}_{GL}^l\|_2 \leq \frac{1}{n} \varepsilon^T X(\hat{\beta}_{GL} - \beta_0) + \sum_{l=1}^g \omega_l \|\beta_0^l\|_2.$$

This implies that

$$R(f_{\hat{\beta}_{GL}}) - R(f_{\beta_0}) \leq \frac{1}{n} \varepsilon^T X(\hat{\beta}_{GL} - \beta_0) - \sum_{l=1}^g \omega_l \|\hat{\beta}_{GL}^l\|_2 + \sum_{l=1}^g \omega_l \|\beta_0^l\|_2.$$

By applying Cauchy-Schwartz inequality we have

$$R(f_{\hat{\beta}_{GL}}) - R(f_{\beta_0}) \leq \sum_{l=1}^g \frac{1}{n} \sqrt{\sum_{j \in G_l} \left(\sum_{i=1}^n (z_{ij} \epsilon_i) \right)^2} \|(\hat{\beta}_{GL} - \beta_0)^l\|_2 + \sum_{l=1}^g \omega_l \|(\beta_0 - \hat{\beta}_{GL})^l\|_2,$$

Define the event

$$A = \bigcap_{l=1}^g \left\{ \frac{1}{n} \sqrt{\sum_{j \in G_l} \left(\sum_{i=1}^n (z_{ij} \epsilon_i) \right)^2} \leq \omega_l/2 \right\}.$$

On the event A we have

$$R(f_{\hat{\beta}_{GL}}) - R(f_{\beta_0}) \leq \sum_{l=1}^g \frac{3\omega_l}{2} \|(\hat{\beta}_{GL} - \beta_0)^l\|_2. \quad (52)$$

By Lemma A.1 we have,

$$\frac{\langle h, h \rangle_{f_{\beta_0}}}{\|h\|_\infty^2} (\exp(-\|h\|_\infty) + \|h\|_\infty - 1) \leq R(f_{\hat{\beta}_{GL}}) - R(f_{\beta_0}) \quad (53)$$

where

$$h(z_i) = (f_{\hat{\beta}_{GL}} - f_{\beta_0})(z_i) = \sum_{l=1}^g \sum_{j \in G_l} (\hat{\beta}_{\lambda_j} - \beta_{0j}) z_{ij}$$

and $\|h\|_\infty \leq v\|\delta'\|_{2,1}$, with $v = \max_{1 \leq i \leq n} \max_{1 \leq l \leq g} \|z_i^l\|_2$. Equation (53) and the decreasing of $t \mapsto \frac{\exp(-t) + t - 1}{t^2}$ lead to

$$\frac{\delta'^T X^T D X \delta'}{n(v\|\delta'\|_{2,1})^2} (\exp(-v\|\delta'\|_{2,1}) + v\|\delta'\|_{2,1} - 1) \leq R(f_{\hat{\beta}_{GL}}) - R(f_{\beta_0}).$$

Now, Inequality (51) implies

$$\|\delta'_{J(\beta_0)^c}\|_{2,1} \leq 3 \frac{\left(\max_{1 \leq l \leq g} \omega_l \right)}{\min_{1 \leq l \leq g} \omega_l} \|\delta'_{J(\beta_0)}\|_{2,1}.$$

We can therefore apply Assumption (RE₃) with $a_0 = 3(\max_{1 \leq l \leq g} \omega_l) / \min_{1 \leq l \leq g} \omega_l$ and get that

$$\frac{\mu_2^2 \|\delta'_J\|_2^2}{v^2 \|\delta'\|_{2,1}^2} (\exp(-v\|\delta'\|_{2,1}) + v\|\delta'\|_{2,1} - 1) \leq R(f_{\hat{\beta}_{GL}}) - R(f_{\beta_0}).$$

We can use that $\|\delta'\|_{2,1}^2 \leq 16|J|\|\delta'_J\|_2^2$ to write

$$\frac{\mu_2^2}{16|J|v^2} (\exp(-v\|\delta'\|_{2,1}) + v\|\delta'\|_{2,1} - 1) \leq R(f_{\hat{\beta}_{GL}}) - R(f_{\beta_0}).$$

According to Equation (52) we have

$$\exp(-v\|\delta'\|_{2,1}) + v\|\delta'\|_{2,1} - 1 \leq \frac{24|J|v^2 \left(\max_{1 \leq l \leq g} \omega_l \right)}{\mu_2^2} \|\delta'\|_{2,1}. \quad (54)$$

Now, a short calculation shows that for all $a \in (0, 1]$,

$$e^{\frac{-2a}{1-a}} + (1-a) \frac{2a}{1-a} - 1 \geq 0 \quad (55)$$

Set $a = v\|\delta'\|_{2,1} / (v\|\delta'\|_{2,1} + 2)$. Thus $v\|\delta'\|_{2,1} = 2a/(1-a)$ and we have

$$e^{-v\|\delta'\|_{2,1}} + v\|\delta'\|_{2,1} - 1 \geq \frac{v^2 \|\delta'\|_{2,1}^2}{v\|\delta'\|_{2,1} + 2}. \quad (56)$$

This implies using Equation (54) that

$$v\|\delta'\|_{2,1} \leq \frac{48 \left(\max_{1 \leq l \leq g} \omega_l \right) |J|v/\mu_2^2}{1 - 24 \left(\max_{1 \leq l \leq g} \omega_l \right) |J|v/\mu_2^2}.$$

Now if $\max_{1 \leq l \leq g} \omega_l \leq \frac{\mu_2}{48v|J|}$, we have $v\|\delta'\|_{2,1} \leq 2$ and consequently

$$\frac{\exp(-v\|\delta'\|_{2,1}) + v\|\delta'\|_{2,1} - 1}{v^2\|\delta'\|_{2,1}^2} \geq 1/4.$$

Using Equation (54) we have that

$$\|\delta'\|_{2,1} \leq \frac{96|J| \left(\max_{1 \leq l \leq g} \omega_l \right)}{\mu_2^2},$$

which prove the Line (26). Line (25) follows from (26) by using Equation (52). Line (27) is the consequence of Lemma A.3 Taking $a_l = \|(\hat{\beta}_{\lambda_j} - \beta_0)^l\|_2$ and $b_1 = 144|J|(\min_{1 \leq l \leq g} \omega_l)/\mu_2^2(s, 3)$. Line (28) follows from (25) and Equation (53). ■

A Appendix

The proof of Lemma 5.1 and 5.2 are based on property of self concordant function (see for instance [20]), ie, the functions whose third derivatives are controlled by their second derivatives. A one-dimensional, convex function g is called self concordant if

$$|g'''(x)| \leq Cg''(x)^{3/2}.$$

The function we use ($g(t) = \hat{R}(g + th)$) is not really self concordant but we can bound his third derivative by the second derivative times a constant. Our results on self-concordant functions are based on the ones of [2]. He has used and extended tools from convex optimization and self-concordance to provide simple extensions of theoretical results for the square loss to logistic loss. We use the same kind of arguments and state some relations between excess risk and prediction loss in the context of nonparametric logistic model, where f_0 is not necessarily linear as assumed in [2]. Precisely we extend Proposition 1 in [2] to the functions which are not necessarily linear (see Lemma A.1). This allows us to establish Lemma 5.1 and 5.2.

Lemma A.1. *For all $h, f : \mathbb{R}^d \rightarrow \mathbb{R}$,*

$$\frac{\langle h, h \rangle_f}{\|h\|_\infty^2} (\exp(-\|h\|_\infty) + \|h\|_\infty - 1) \leq R(f + h) - R(f) + (q_f - q_{f_0})(h) \quad (57)$$

$$R(f + h) - R(f) + (q_f - q_{f_0})(h) \leq \frac{\langle h, h \rangle_f}{\|h\|_\infty^2} (\exp(\|h\|_\infty) - \|h\|_\infty - 1) \quad (58)$$

$$\langle h, h \rangle_f e^{-\|h\|_\infty} \leq \langle h, h \rangle_{f+g} \leq \langle h, h \rangle_f e^{\|h\|_\infty}. \quad (59)$$

A.1 Proof of Lemma A.1

We use the following lemma (see [2] Lemma 1) that we recall here:

Lemma A.2. *Let g be a convex three times differentiable function $g : \mathbb{R} \rightarrow \mathbb{R}$ such that for all $t \in \mathbb{R}$ $|g'''(t)| \leq Sg''(t)$, for some $S \geq 0$. Then, for all $t \geq 0$:*

$$\frac{g''(0)}{S^2}(\exp(-St) + St - 1) \leq g(t) - g(0) - g'(0)t \leq \frac{g''(0)}{S^2}(\exp(St) - St - 1). \quad (60)$$

We refer to Appendix A of [2] for the proof of this lemma.

Set

$$g(t) = \hat{R}(f + th) = \frac{1}{n} \sum_{i=1}^n l((f + th)(z_i)) - Y_i(f + th)(z_i), \quad f, h \in H,$$

where $l(u) = \log(1 + \exp(u))$. A short calculation leads to $l'(u) = \pi(u)$, $l''(u) = \pi(u)(1 - \pi(u))$, $l'''(u) = \pi(u)[1 - \pi(u)][1 - 2\pi(u)]$. It follows that

$$g''(t) = \frac{1}{n} \sum_{i=1}^n h^2(z_i) l''((f + th)(z_i)) = \langle h, h \rangle_{f+th},$$

and

$$g'''(t) = \frac{1}{n} \sum_{i=1}^n h^3(z_i) l'''((f + th)(z_i)).$$

Since $l'''(u) \leq l''(u)$ we have,

$$\begin{aligned} |g'''(t)| &= \left| \frac{1}{n} \sum_{i=1}^n h^3(z_i) l'''((f + th)(z_i)) \right| \\ &\leq \frac{1}{n} \sum_{i=1}^n h^2(z_i) l''((f + th)(z_i)) \|h\|_\infty = \|h\|_\infty g''(t). \end{aligned}$$

We now apply Lemma A.2 to $g(t)$ with $S = \|h\|_\infty$, taking $t = 1$. Using Equation (5) we get the first and second inequality of Lemma A.1. Now by considering $g(t) = \langle h, h \rangle_{f+th}$, a short calculation leads to $|g'(t)| \leq \|h\|_\infty g(t)$ which implies $g(0)e^{-\|h\|_\infty t} \leq g(t) \leq g(0)e^{\|h\|_\infty t}$. By applying the last inequality to $g(t)$, and taking $t = 1$ we get the third inequality of Lemma A.1.

A.2 Proof of Lemma 5.1

Set $h_0 = f_\beta - f_0$ from Lemma A.1 below,

$$\frac{\langle h_0, h_0 \rangle_{f_0}}{\|h_0\|_\infty^2} (\exp(-\|h_0\|_\infty) + \|h_0\|_\infty - 1) \leq R(f_\beta) - R(f_0).$$

Using Assumptions **(B₃)**, **(B₁)** and the decreasing of $t \mapsto \frac{\exp(-t)+t-1}{t^2}$, we claim that there exists $c_0 = c_0(C_0, c_1) > 0$ such that

$$c_0 \leq \frac{\exp(-\|h_0\|_\infty) + \|h_0\|_\infty - 1}{\|h_0\|_\infty^2}.$$

According to Assumption **(B₁)**, there exists $0 \leq \epsilon_0 \leq 1/2$ such that for $1 \leq i \leq n$

$$\epsilon_0 \leq \pi(f_0(z_i))(1 - \pi(f_0(z_i))) \leq 1 - \epsilon_0.$$

The proof of the left hand side of Lemma 5.1 follows from the fact that $\epsilon_0 \|h_0\|_n^2 \leq \langle h_0, h_0 \rangle_{f_0}$. From the second line of Lemma A.1 we have

$$R(f_\beta) - R(f_0) \leq \frac{\langle h_0, h_0 \rangle_{f_0}}{\|h_0\|_\infty^2} (\exp(\|h_0\|_\infty) - \|h_0\|_\infty - 1).$$

Using assumption **(B₃)** and increasing of $t \mapsto \frac{\exp(t)-t-1}{t^2}$ thus there exists $c'_0 = c'_0(C_0, c_1) > 0$ such that

$$\begin{aligned} R(f_\beta) - R(f_0) &\leq c'_0 \langle h_0, h_0 \rangle_{f_0} \\ &\leq c'_0 \frac{1}{4} \|h_0\|_n^2. \end{aligned}$$

This end the proof of the right hand side of the Lemma 5.1.

Lemma A.3. *If we assume that $\sum_{i=1}^p a_j \leq b_1$ with $a_j > 0$, this implies that $\sum_{i=1}^p a_j^q \leq b_1^q$, with $1 \leq q \leq 2$*

A.3 Proof of Lemma A.3

$$\begin{aligned} \sum_{i=1}^p a_j^q &= \sum_{i=1}^p a_j^{2-q} a_j^{2q-2} \\ &\leq \left(\sum_{i=1}^p a_j \right)^{2-q} \left(\sum_{i=1}^p a_j^2 \right)^{q-1} \end{aligned}$$

Since $\sum_{i=1}^p a_j^2 \leq (\sum_{i=1}^p a_j)^2 \leq b_1^2$, thus

$$\sum_{i=1}^p a_j^q \leq b_1^{2-q} b_1^{2q-2} = b_1^q \tag{61}$$

A.4 Proof of Lemma 5.2

We act similarly to the proof of Lemma 5.1.

Lemma A.4 (Bernstein’s inequality). *Let X_1, \dots, X_n be independent real valued random variables such that for all $i \leq n$, $X_i \leq b$ almost surely, then we have*

$$\mathbb{P} \left[\left| \sum_{i=1}^n X_i - \mathbb{E}(E_i) \right| \geq \sqrt{2vx} + bx/3 \right] \leq 2 \exp(-x),$$

where $v = \sum_{i=1}^n \mathbb{E}(X_i^2)$.

This lemma is obtain by gattering Proposition 2.9 and inequality (2.23) from [14].

Lemma A.5 (Hoeffding’s inequality). *Let X_1, \dots, X_n be independent random variables such that X_i takes its values in $[a_i, b_i]$ almost surely for all $i \leq n$. Then for any positive x , we have*

$$\mathbb{P} \left[\left| \sum_{i=1}^n X_i - \mathbb{E}(E_i) \right| \geq x \right] \leq 2 \exp\left(-\frac{2x^2}{\sum_{i=1}^n (b_i - a_i)^2}\right).$$

This lemma is a consequence of Proposition 2.7 in [14].

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