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Abstract

In this paper, we present Lyapunov-based robust and adaptive Higher Order Sliding Mode (HOSM) controllers for nonlinear SISO systems with bounded uncertainty. The proposed controllers can be designed for any arbitrary sliding mode order. The uncertainty bounds are known in the robust control problem whereas they are partially known in the adaptive control problem. Both these problems are formulated as the finite time stabilization of a chain of integrators with bounded uncertainty. The controllers are developed from a class of nonlinear controllers which guarantee finite time stabilization of pure integrator chains. The robust controller establishes ideal HOSM i.e. the sliding variable and its $r-1$ time derivatives converge exactly to the origin in finite time. The adaptive controller establishes real HOSM, which means that the sliding variable and its $r-1$ time derivatives converge to a neighborhood of the origin. Saturation functions are used for gain adaptation, which do not let the states exit the neighborhood after convergence. The effectiveness of these controllers is illustrated through simulations.

Key words: Higher Order Sliding Mode; Robust Control; Adaptive Control; Lyapunov Analysis; Finite-time Stabilization.

1 INTRODUCTION

Nonlinear dynamic physical systems suffer from parametric uncertainty and are difficult to characterize. Parametric uncertainty arises from varying operating conditions and external perturbations that affect the physical characteristics of systems. The variation limits or the bounds of this uncertainty might be known or unknown. This needs to be considered during control design so that the controller counteracts the effect of variations and guarantees performance under different operating conditions. Sliding mode control (SMC) [1, 2] is a well-known method for control of nonlinear systems, renowned for its insensitivity to parametric uncertainty and external disturbance. This technique is based on applying discontinuous control on a system which ensures convergence of the output function (sliding variable) in finite time to a manifold of the state-space, called the sliding manifold [3]. In practice, SMC suffers from *chattering*; the phenomenon of finite-frequency, finite-amplitude oscillations in the output which appear because the high-frequency switching excites unmodeled dynamics of the closed loop system [4]. Higher Order Sliding Mode Control (HOSMC) is an effective method for chattering attenuation [5]. In this method the discontinuous control is applied on a higher time derivative of the sliding variable, such that not only the sliding variable converges to the origin, but also its higher time derivatives. As the discontinuous control does not act upon the system input directly, chattering is automatically reduced.

Many HOSMC algorithms exist in contemporary literature for control of uncertain nonlinear systems, where the bounds on uncertainty are known. These are robust because they preserve the insensitivity of classical sliding mode, and maintain the performance characteristics of the closed loop system so long as it remains inside the defined uncertainty bounds. Levant for example, has presented a method of designing arbitrary order sliding mode controllers for Single Input Single Output (SISO) systems in [6]. In his recent works [7, 8], homogeneity approach has been used to demonstrate finite time stabilization of the proposed method. Laghrouche et al. [9] have proposed a two part integral sliding mode based

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control to deal with the finite time stabilization problem and uncertainty rejection problem separately. Dinuzzo et al. have proposed another method in [10], where the problem of HOSM has been treated as Robust Fuller's problem. Defoort et al. [11] have developed a robust Multi Input Multi Output (MIMO) HOSM controller, using a constructive algorithm with geometric homogeneity based finite time stabilization of an integrator chain. Harmouche et al. have also presented their homogeneous controller in [12] based on the works of Hong [13].

The case where the bounds on uncertainty exist, but are unknown, is still an open problem in the field of arbitrary HOSMC. In this problem, the controller must possess two essential properties, (a) non-requirement of the uncertainty bounds and (b) avoidance of gain overestimation [14]. In recent years, adaptive sliding mode controllers have attracted the interest of many researchers for this case. Adaptive gains have been used with success in the past for chattering suppression. For example, Bartolini et al. [15] have extended Utkin's concept of equivalent control for second order sliding mode control gain adaptation, to suppress residual oscillations due to digital controllers with time delay. Similarly, an equivalent control based adaptive controller is described in [16], in which the equivalent control estimation is improved, using double low pass filters. A concise survey of these methods can be found in [17]. Huang et al. [18] were the first to use dynamic gain adaptation in SMC for the problem of unknown uncertainty bound. They presented an adaptation law for first order SMC, which depends directly upon the sliding variable; the control gains increase until sliding mode is achieved. Once the sliding variable has converged to zero, the gains become constant. This method works without a-priori knowledge of uncertainty bounds, however it does not solve the gain overestimation problem as the gains stabilize at unnecessarily large values. Plestan et al. [14, 19] have overcome this problem by slowly decreasing the gains once sliding mode is achieved. This method yields convergence to a neighborhood of the sliding surface. However it does not guarantee that the states would remain inside the neighborhood after convergence; the states actually overshoot in a known region around the neighborhood. In the field of HOSMC, Shtessel et al. [20] have presented a Second Order adaptive gain SMC for non-overestimation of the control gains, based on supertwisting algorithm. The states overshoot in this case as well, but unlike [14], the magnitude is unknown. A Lyapunov-based variable gains super twisting algorithm has also been presented in [21]. Glumineau et al. [22] have presented a different approach, based on impulsive sliding mode adaptive control of a double integrator system. The gain of the impulsive control is adapted to minimize the convergence time of the double integrator dynamics. To the best of our knowledge, no contemporary work on adaptive HOSMC has been published for orders greater than two.

In this paper, we present Lyapunov-based robust and adaptive Higher Order Sliding Mode Controllers for nonlinear SISO systems with bounded uncertainty. This problem has been formulated as the stabilization of a chain of integrators with bounded uncertainties which is equivalent to the stabilization of the following r^{th} differential equation in finite time [6],

$$s^{(r)} = \varphi(t) + \gamma(t)u,$$

where $s(x,t)$ is a smooth output-feedback function (sliding variable) and $s^{(r)}$ is its r^{th} time derivative. The term u is the control input and $\varphi(t)$ and $\gamma(t)$ are bounded uncertain functions.

There are two main contributions in this paper. First, a Lyapunov-based approach for arbitrary HOSMC is developed. The controller establishes sliding mode of any arbitrary order under the condition that the bounds of $\varphi(t)$ and $\gamma(t)$ are known. The advantage of our method is that robust HOSM controllers are developed from a class of finite-time controllers for pure integrator chains. To the best of our knowledge, this is the first work on a Lyapunov based approach for arbitrary HOSMC. Lyapunov design is a powerful tool for control system design that allows to estimate an upper bound on convergence time [23]. The second contribution in this paper is the extension of the robust controller to an adaptive controller for the case where the bounds on $\varphi(t)$ are unknown. The proposed adaptive controller guarantees finite time convergence to an adjustable arbitrary neighborhood of origin, i.e. it establishes real HOSM. The gain adaptation dynamics are based on a saturation function [24, 25, 26], which results in rapid increase of gains when the sliding variable and its derivatives are outside the neighborhood, and rapid decrease when they are inside the neighborhood. The advantage of this adaptive controller design, compared to other algorithms mentioned before, is that this controller can be extended to any arbitrary order and the adaptation rates are fast in both directions. Therefore, the amplitude of the discontinuous control decreases faster as compared to [14]. In addition, the states are confined inside the neighborhood after convergence and cannot escape. As a result, there are no state overshoots and no gain overestimation in this controller, and the neighborhood of convergence can be chosen as small as possible. The limiting factor of this design as compared to [14, 18, 20] is that the bounds of $\gamma(t)$ still need to be known. We therefore propose an initial step for the open problem of a complete adaptive arbitrary HOSMC controller.

The paper has been organized as follows: the problem formulation has been presented in section 2. The design of the robust controller has been presented in section 3, and that of adaptive controller in section 4. Simulation results have been presented and discussed in section 5 and some concluding remarks have been given in section

6.

2 Problem Formulation

Let us consider an uncertain nonlinear system:

$$\begin{cases} \dot{x} = f(x,t) + g(x,t)u, \\ y = s(x,t), \end{cases} \tag{1}$$

where $x \in \mathbb{R}^n$ is the state vector and $u \in \mathbb{R}$ is the input control. The sliding variable s is a measured smooth output-feedback function and $f(x,t)$ and $g(x,t)$ are uncertain smooth functions. Let us assume:

H1. The relative degree r of the System (1) with respect to s is constant and known, and the associated zero dynamics are stable.

The control objective is to fulfil the constraint $s(x,t) = 0$ in finite time and to keep it exact by discontinuous feedback control. The r^{th}-order sliding mode is defined as follows:

Definition 2.1 [27]. Consider the nonlinear System (1), and let the system be closed by some possibly-dynamical discontinuous feedback. Then, provided that $s, \dot{s}, ..., s^{(r-1)}$ are continuous functions, and the set $S^r = \{x | s(x,t) = \dot{s}(x,t) = ... = s^{(r-1)}(x,t) = 0\}$, called "$r^{th}$-order sliding set", is non-empty and is locally an integral set in the Filippov sense [28], the motion on S^r is called "r^{th}-order sliding mode" with respect to the sliding variable s.

The r^{th}-order SMC approach allows the finite time stabilization to zero of the sliding variable s and its $r-1$ first time derivatives by defining a suitable discontinuous control function. If the system (1) is extended by the introduction of a fictitious variable $x_{n+1} = t, \dot{x}_{n+1} = 1$, and $f_e = \left(f^T 1\right)^T, g_e = \left(g^T 0\right)^T$ (where the last component corresponds to x_{n+1}), then the output s satisfies the equation [6]:

$$s^{(r)} = \tilde{\varphi}(s) + \tilde{\gamma}(s)u, \text{ with } \tilde{\gamma}(s) = L_{g_e} L_{f_e}^{r-1} s \text{ and } \tilde{\varphi}(s) = L_{f_e}^{r} s.$$

For $x \in X \subset R^n$, X being a bounded open subset of $\mathbb{R}^n$ containing the origin, the functions $\tilde{\varphi}$, $\tilde{\gamma}$ are bounded and it is also customary to assume that $\tilde{\gamma}$ has a positive lower bound that depends only on X. Then the r^{th}-order SMC of (1) with respect to the sliding variable s becomes equivalent to the finite time stabilization of the following system,

$$\begin{cases} \dot{z}_i = z_{i+1}, \ i = 1, ..., r-1, \\ \dot{z}_r = \varphi(t) + \gamma(t)u, \end{cases} \tag{2}$$

where $z = [z_1 \ z_2 \ ... z_r]^T := [s \ \dot{s} ... \ s^{(r-1)}]^T$. The functions φ and γ are subject to the following hypothesis:

H2. The functions $\varphi(t)$ and $\gamma(t)$ are bounded uncertain functions i.e. there exist constants $K_m, K_M > 0$ and $\varphi_0 \geq 0$ such that

$$0 < K_m \leq \gamma(t) \leq K_M, \quad |\varphi(t)| \leq \varphi_0, \quad \forall t \geq 0,$$

This hypothesis implies that results in the proceeding sections of the paper can be only be local to the origin when applied to the r^{th}-order SMC approach, unless appropriate global boundedness assumptions are imposed on $L_{g_e} L_{f_e}^{r-1} s$ and $L_{f_e}^{r} s$.

Remark 1 *r^{th}-order sliding mode can also be established on a system with a relative degree $\rho < r$ by increasing the length of integrator chain by $r - \rho$ integrators [29]. In other words, r^{th}-order sliding mode can be established by applying the discontinuous control on the $(r-\rho)^{th}$ time derivative of the control input. For the sake of clarity, we will consider $r = \rho$ in all further sections.*

In the following sections, a robust controller is developed first for the system (2) under hypothesis **H2**, considering that the uncertainty bounds presented in **H2** are known. Then, an adaptive controller is developed to extend the functionality of the robust controller to the case where the bounds of φ are not known.

3 Design of robust Higher Order Sliding Mode Controller

In this section, we present a robust controller which stabilizes System (2), considering that the bounds on φ and γ are known. This controller has been derived from a class of Lyapunov-based controllers that guarantee finite time stabilization of pure integrator chains, and satisfy certain additional geometric conditions. The pure integrator chain ($\varphi \equiv 0$ and $\gamma \equiv 1$) is represented as follows:

$$\begin{cases} \dot{z}_i = z_{i+1}, \ i = 1, ..., r-1, \\ \dot{z}_r = u. \end{cases} \tag{3}$$

Let us recall the theorem:

Theorem 1 *[23] Consider System* (3). *Suppose there exists a continuous state-feedback control law $u = u_0(z)$, a positive definite C^1 function V_1 defined on a neighborhood $\hat{U} \subset \mathbb{R}^r$ of the origin and real numbers $c > 0$ and*

$0 < \alpha < 1$, *such that the following condition is true for every trajectory z of system* (3),

$$\dot{V}_1 + c{V_1}^{\alpha}(z(t)) \leqslant 0, \ \textit{if}\, z(t) \in \hat{U}, \tag{4}$$

where $\dot{V}_1$ is the time derivative of $V_1(z)$.
Then all trajectories of System (3) *with the feedback $u_0(z)$ which stay in $\hat{U}$ converge to zero in finite time. If $\hat{U} = \mathbb{R}^r$ and V_1 is radially unbounded, then System* (3) *with the feedback $u_0(z)$ is globally finite time stable with respect to the origin.*

Based on this theorem, we now present a robust controller for System (2).

Theorem 2 *Consider System* (2) *subject to Hypothesis **H2**. Then the following control law establishes Higher Order Sliding Mode with respect to s in finite time:*

$$u = \frac{1}{K_m}(u_0 + \varphi_0 sign(u_0)), \tag{5}$$

where $u_0(z)$ is any state-feedback control law that satisfies Theorem 1 and obeys the following further conditions:

$$\frac{\partial V_1}{\partial z_r}(z)u_0(z) \leq 0, \ \textit{and}\ u_0(z) = 0 \Rightarrow \frac{\partial V_1}{\partial z_r}(z) = 0, \ \forall z \in \hat{U}. \tag{6}$$

If $\hat{U} = \mathbb{R}^r$ and V_1 is radially unbounded, then System (2) *with the feedback $u(z)$ is globally finite time stable with respect to the origin.*

For $a \in \mathbb{R}$, the function $sign(a)$ is defined as follows,

$$sign(a) \begin{cases} = \dfrac{a}{|a|} & , \ a \neq 0, \\ \in \ [-1, 1] & , \ a = 0. \end{cases} \tag{7}$$

Therefore the solutions of System (2) with the feedback law (5) are the solutions of a differential inclusion and are understood in Filippov's sense. Note that u is discontinuous only when $u_0 = 0$.

Proof. Consider System (2) and the control law u defined in (5):

$$\begin{cases} \dot{z}_i &= z_{i+1}, \ i = 1, ..., r-1, \\ \dot{z}_r &= \varphi + \gamma u \\ &= \dfrac{\gamma u_0(z)}{K_m} + \dfrac{\gamma \varphi_0}{K_m} sign(u_0(z)) + \varphi. \end{cases} \tag{8}$$

The time derivative of the Lyapunov function V_1 provided by Theorem 1 is calculated along a trajectory of System (8) inside $\hat{U}$. Let us assume first that $u_0(z(t)) \neq 0$. We obtain:

$$\begin{aligned} \dot{V}_1 &= \sum_{i=1}^{r-1} \frac{\partial V_1}{\partial z_i} z_{i+1} + \frac{\partial V_1}{\partial z_r}\left(\varphi + \gamma u\right), \\ &= \sum_{i=1}^{r-1} \frac{\partial V_1}{\partial z_i} z_{i+1} + \frac{\partial V_1}{\partial z_r}\left(\frac{\gamma}{K_m}u_0 + \frac{\gamma}{K_m}\varphi_0 sign(u_0) + \varphi\right) \\ &\leqslant \sum_{i=1}^{r-1} \frac{\partial V_1}{\partial z_i} z_{i+1} + \frac{\partial V_1}{\partial z_r}u_0 + \frac{\partial V_1}{\partial z_r} sign(u_0)\left(\varphi_0 - |\varphi|\right) \\ &\leqslant \sum_{i=1}^{r-1} \frac{\partial V_1}{\partial z_i} z_{i+1} + \frac{\partial V_1}{\partial z_r}u_0 \leq -c{V_1}^{\alpha}. \end{aligned} \tag{9}$$

This inequality still holds true if $u_0(z(t)) = 0$, since according to condition (6), $\dfrac{\partial V_1}{\partial z_r} sign\left(u_0(z)\right) = 0$ is not dependant upon the value of $sign(u_0) \in [-1, 1]$. This implies that if the trajectory z reaches zero, it must stay there. Moreover the Lyapunov function V_1 strictly decreases along any non trivial trajectory of (6) and reaches zero in finite time according to (9).

■

The previous result becomes non empty if controllers satisfying Theorem 1 and Condition (6) can be identified. It can be verified that the controllers proposed by Hong [30] and Huang [31] fulfill these conditions. Let us consider Hong's controller as an example.
For simplicity, let us introduce $\lfloor a \rceil^{\kappa}$ to denote $|a|^{\kappa}\, sign(a)$ for $a \in \mathbb{R}$ and $\kappa > 0$. Then Hong's controller [30] is defined as follows.

Let $k < 0$ and $l_1, \cdots, l_r$ positive real numbers. For $z = (z_1, \cdots, z_r)$, we define for $i = 0, ..., r-1$:

$$\begin{aligned} & p_i = 1 + (i-1)k, \\ & v_0 = 0, \ v_{i+1} = -l_{i+1} \lfloor \lfloor z_{i+1} \rceil^{\beta_i} - \lfloor v_i \rceil^{\beta_i} \rceil^{(\alpha_{i+1}/(\beta_i)}, \end{aligned} \tag{10}$$

where $\alpha_i = \dfrac{p_{i+1}}{p_i}$, for $i = 1, ..., r$, and, for $k < 0$ sufficiently small,

$$\beta_0 = p_2, \ \ (\beta_i + 1)p_{i+1} = \beta_0 + 1 > 0, \quad i = 1, ..., r-1. \tag{11}$$

Consider the positive definite radially unbounded function $V_1 : \mathbb{R}^r \to \mathbb{R}^+$ given by

$$V_1 = \sum_{j=1}^{r} \int_{v_{j-1}}^{z_j} \lfloor s \rceil^{\beta_{j-1}} - \lfloor v_{j-1} \rceil^{\beta_{j-1}} ds. \tag{12}$$

It has been proved in [30] that for a sufficiently small k, there exist $l_i > 0$, $i = 1, ..., r$, such that the control law $u_0 = v_r$ defined above stabilizes System (3) in finite time and there exists $c > 0$ and $0 < \alpha < 1$ such that u_0 and V_1 fulfill the conditions of Theorem 1. Moreover,

$$\begin{aligned} \frac{\partial V_1}{\partial z_r} &= \lfloor z_r \rceil^{\beta_{r-1}} - \lfloor v_{r-1} \rceil^{\beta_{r-1}}, \\ u_0(z) &= v_r = -l_r \left\lfloor \lfloor z_r \rceil^{\beta_{r-1}} - \lfloor v_{r-1} \rceil^{\beta_{r-1}} \right\rceil^{\frac{\alpha_r}{\beta_{r-1}}}. \end{aligned} \tag{13}$$

It can be verified that
$\frac{\partial V_1}{\partial z_r} u_0(z) \leq 0$ and $u_0(z) = 0 \Rightarrow \frac{\partial V_1}{\partial z_r} = 0$.
The feedback law of [30] can be simplified by choosing all $\beta_i = 1$ in (10) as explained below:

Proposition 1 *For System (3), there exist a sufficiently small $k < 0$ and real numbers $l_i > 0$, such that the control law $u_0 = v_r$ defined below stabilizes System (3) in finite time.*
For $i = 0, ..., r-1$,

$$v_0 = 0, \quad v_{i+1} = -l_{i+1} \lfloor z_{i+1} - v_i \rceil^{\frac{1+(i+2)k}{1+(i+1)k}}. \tag{14}$$

Proof. The proof of Theorem 1 can be developed simply by adapting the proof presented in [30] to the parameter choice of (14). Let $\lambda = \frac{1+(r+2)k}{1+(r+1)k}$ and f_λ be the closed-loop vector field obtained by using the feedback (14) in (3). For each $\lambda > 0$, the vector field f_λ is continuous and homogeneous of degree $k < 0$ with respect to the family of dilations $(p_1, ..., p_r)$, where $p_i = 1 + (i-1)k$, $i = 1, ..., r$. Let l_i, $i = 1, ..., r$ be positive constants such that the polynomial $y^r + l_r(y^{r-1} + l_{r-1}(y^{r-2} + ... + l_2(y + l_1)))...))$ is Hurwitz. If $k = 0$ the vector field is linear and therefore $\lambda = 1$. Therefore, there exists a positive-definite, radially unbounded, Lyapunov function $V : \mathbb{R}^r \to \mathbb{R}$ such that $L_{f_1}V$ is continuous and negative definite.

Let $\mathcal{A} = V^{-1}([0,1])$ and $\mathcal{S} = bd\mathcal{A} = V^{-1}(\{1\})$, where $bd\mathcal{A}$ is the boundary of the set $\mathcal{A}$, i.e. $\mathcal{A} = \{z \in \mathbb{R}^r | V(z) \in [0,1]\}$ and $\mathcal{S} = \{z \in \mathbb{R}^r | V(z) = 1\}$. Then $\mathcal{A}$ and $\mathcal{S}$ are compact since V is proper. Also, $0 \notin \mathcal{S}$ as V is positive definite. Defining $\phi : (0,1] \times \mathcal{S} \to \mathbb{R}$ by $\phi(\lambda, z) = L_{f_\lambda} V(z)$. Then V is continuous and satisfies $\phi(\lambda, z) < 0$ for all $z \in \mathcal{S}$, i.e. $\varphi(\{1\} \times \mathcal{S}) \subset (-\infty, 0)$. Since $\mathcal{S}$ is compact, by continuity there exists $\epsilon > 0$ such that $\phi((1-\epsilon, 1] \times \mathcal{S}) \subset (-\infty, 0)$. It follows that for $\lambda \in (1-\epsilon, 1]$, $L_{f_\lambda} V$ takes negative values on $\mathcal{S}$. Thus, $\mathcal{A}$ is strictly positively invariant under f_λ for every $\lambda \in (1-\epsilon, 1]$. Therefore the origin is global asymptotic stable under f_λ, for $\lambda \in (1-\epsilon, 1]$. Finally, for $\lambda \in (1-\epsilon, 1)$ i.e $|k|$ small enough, by homogeneity, the origin is globally finite time stable.

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4 Adaptive Controller

We shall now consider the case for System (2) where the uncertainty bound on φ is unknown. More precisely, according to the definitions given in **H2**, the control design requires the a-priori knowledge of K_m only whereas φ_0 and K_M need not be known.

For any $a \in \mathbb{R}$, let $\sigma(a)$ be the standard saturation function defined by

$$\sigma(a) = \frac{a}{\max(1, |a|)}. \tag{15}$$

For $\varepsilon > 0$, $a \in \mathbb{R}$, we define $\nu_\varepsilon(a)$ by

$$\nu_\varepsilon(a) = \frac{1}{2} + \frac{1}{2} \sigma \left(\frac{|a| - \frac{3}{4}\varepsilon}{\frac{1}{4}\varepsilon} \right). \tag{16}$$

Our main result for the adaptive case is given by the following theorem.

Theorem 3 *Consider System (2), subject to assumption **H1** and **H2**, and let u_0, V_1 be the control law and the Lyapunov function respectively, provided by Theorem 1.Then the following holds true,*
$\exists M > 0$ such that $\forall \varepsilon > 0$ small enough and $0 < \eta < 1$, $\exists k_\varepsilon > 0$ such that, for every non zero initial condition $z_0 \in \hat{U}$; if u_ε is the controller defined by:

$$u_\varepsilon = \frac{1}{K_m} \left(u_0 + \hat{\varphi}_\varepsilon sign(u_0) \right), \tag{17}$$

where $(z, \hat{\varphi}_\varepsilon)$ is the solution of the Cauchy problem defined by (2) starting at $(z_0, 0)$ (i.e. $z(0) = z_0$ and $\hat{\varphi}_\varepsilon(0) = 0$) with the controller u_ε and the dynamics

$$\dot{\hat{\varphi}} = k_\varepsilon \nu_\varepsilon(V_1(z(t))) - (1 - \nu_\varepsilon(V_1(z(t)))) \lfloor \hat{\varphi}_\varepsilon \rceil^\eta + \sigma(V_1), \tag{18}$$

then we have:

(i) $\limsup_{t \to \infty} V_1(z(t)) \leq \varepsilon$*;*

(ii) $\limsup_{t \to \infty} |\hat{\varphi}_\varepsilon| \leq \frac{M m_\varepsilon \varphi_0}{\varepsilon^\alpha}$*, where* $m_\varepsilon = \max_{V_1 \leq \varepsilon} \left| \frac{\partial V_1}{\partial z_r} \right|$.

Remark 2 *As u_ε is discontinuous when $u_0 = 0$, the solutions of System (2) with the controller u_ε are solutions of a differential inclusion and must be understood in Filippov's sense.*

Remark 3 *Inequality (i) of Theorem 3 is equivalent to Levant's concept of real Higher Order Sliding Mode, defined as*

$$\exists t_1 > 0 : \forall t > t_1, |z_i(t)| \leq \mu_i, i = 1, \cdots, r-1,$$

where μ_i is an arbitrarily small positive number. This is equivalent to practical stability of $z_1, \cdots, z_r$. Details on real sliding mode and real HOSM can be found in Section 2 of [27].

Proof. The dynamics of $\hat{\varphi}_\varepsilon$ are defined by:

$$\dot{\hat{\varphi}}_\varepsilon = \begin{cases} k_\varepsilon + \sigma(V_1), & V_1 \geq \varepsilon \\ \left(V_1 - \frac{\varepsilon}{2}\right)\frac{2k_\varepsilon}{\varepsilon} - (\varepsilon - V_1)\frac{2}{\varepsilon}\lfloor\hat{\varphi}_\varepsilon\rceil^\eta + \sigma(V_1), & \frac{\varepsilon}{2} \leq V_1 \leq \varepsilon, \\ -\lfloor\hat{\varphi}_\varepsilon\rceil^\eta + \sigma(V_1), & V_1 \leq \frac{\varepsilon}{2}. \end{cases} \tag{19}$$

We first need the following intermediate result.

Lemma 1 *The function $\hat{\varphi}_\varepsilon$ is non-negative and is defined as long as the trajectory of z is defined, $\liminf_{t\to\infty} V_1(z) \leq \frac{3\varepsilon}{4}$ and $\liminf_{t\to\infty} \hat{\varphi}_\varepsilon \leq \varphi_0$.*

Proof. It is clear that $\hat{\varphi}_\varepsilon$ is strictly positive in an interval of the type $(0, \tau)$, since $\dot{\hat{\varphi}}_\varepsilon(0) > 0$. We argue by contradiction. Let us suppose that there exist $\tau_1 > 0$ such that $\hat{\varphi}_\varepsilon(\tau_1) < 0$. Since $\hat{\varphi}_\varepsilon$ continuous, there exists a time $\tau_0 \geq 0$, $\tau_0 < \tau_1$, such that $\hat{\varphi}_\varepsilon(\tau_0) = 0$, and $\hat{\varphi}_\varepsilon(t) < 0$, $\forall t \in]\tau_0, \tau_1]$.
In this case, $V_1(\tau_0) = 0$ otherwise $\dot{\hat{\varphi}}_\varepsilon(\tau_0) > 0$ and $\hat{\varphi}_\varepsilon$ cannot be negative on a right interval at τ_0. In that case, there exists a right interval at τ_0 (still denoted $]\tau_0, \tau_1]$) where $V_1 < \frac{\varepsilon}{2}$ and then $\dot{\hat{\varphi}}_\varepsilon \geq -\lfloor\hat{\varphi}_\varepsilon\rceil^\eta + \sigma(V_1) > 0$. We therefore obtain

$$\hat{\varphi}_\varepsilon(\tau_1) = \int_{\tau_0}^{\tau_1} \dot{\hat{\varphi}}_\varepsilon dt > 0,$$

which is a contradiction.

To prove the second part of the lemma, we argue again by contradiction. If $\liminf_{t\to\infty} V_1(z) > \frac{3\varepsilon}{4}$, then according to the dynamics of $\hat{\varphi}_\varepsilon$ in Equation (19). After a sufficiently large time t and for k_ε larger than a universal constant, we get $\dot{\hat{\varphi}}_\varepsilon > \frac{k_\varepsilon}{4}$. This implies that $\hat{\varphi}_\varepsilon$ is increasing and after a sufficiently large time t, we obtain $\hat{\varphi}_\varepsilon > \varphi_0$. Since we have:

$$\begin{aligned} \dot{V}_1 &\leq \sum_{i=1}^{r-1} \frac{\partial V_1}{\partial z_i} z_{i+1} + \frac{\partial V_1}{\partial z_r}\frac{\gamma}{K_m}u_0 + \left|\frac{\partial V_1}{\partial z_r}\right|\varphi_0 - \left|\frac{\partial V_1}{\partial z_r}\right|\frac{\gamma}{K_m}\hat{\varphi}_\varepsilon, \\ &\leq \sum_{i=1}^{r-1} \frac{\partial V_1}{\partial z_i} z_{i+1} + \frac{\partial V_1}{\partial z_r}u_0, \\ &\leq -cV_1^\alpha, \end{aligned} \tag{20}$$

then $V_1(z)$ converges to zero in finite time, which contradicts the hypothesis $\liminf_{t\to\infty} V_1(z) > \frac{3\varepsilon}{4}$.

Finally we turn to the third part of the lemma and we again argue by contradiction. In that case, $\hat{\varphi}_\varepsilon > \varphi_0$ for t large enough and by the previous computation, V_1 converges to zero in finite time, implying the same conclusion for $\hat{\varphi}_\varepsilon$, which is a contradiction.

■

We next turn to the proof of Item (i) of Theorem (3). We again argue by contradiction and suppose that $\limsup_{t\to\infty} V_1(z) \geq \varepsilon$. Then, there exists times $t_1 < t_2 < t_3$ arbitrarily large and $\frac{3}{4} < l < L < 1$ so that $V_1(z(t_1)) = \frac{3\varepsilon}{4}$, $V_1(z(t_2)) = l\varepsilon$, $V_1(z(t_3)) = L\varepsilon$, and

$$\frac{3\varepsilon}{4} \leq V_1(z(t)) \leq l\varepsilon \text{ on } [t_1, t_2], \;\; l\varepsilon \leq V_1(z(t)) \leq L\varepsilon \text{ on } [t_2, t_3].$$

Note that we have $\dot{V}_1 \leq m_\varepsilon \varphi_0$, which implies that $t_2 - t_1 \geq \frac{(l-\frac{3}{4})\varepsilon}{m_\varepsilon\varphi_0}$. Since $\dot{\hat{\varphi}}_\varepsilon \geq \frac{k_\varepsilon}{4}$ on $[t_1, t_2]$, we have

$$\hat{\varphi}_\varepsilon(t_2) \geq \frac{(l-\frac{3}{4})\varepsilon k_\varepsilon}{4m_\varepsilon\varphi_0}.$$

Choosing k_ε so that the right-hand side of the above inequality is equal to φ_0, it can be immediately deduced that $\hat{\varphi}_\varepsilon \geq \varphi_0$ on $[t_2, t_3]$ and thus V_1 is strictly decreasing on that interval, which is a contradiction.

We now consider the proof of Item (ii) of the theorem (3). Let $[t_1, t_2]$ be an interval of time where $\hat{\varphi}_\varepsilon(t_1) = \varphi_0$, $\hat{\varphi}_\varepsilon(t_2) = K\varphi_0$ and $\varphi_0 \leq \hat{\varphi}_\varepsilon(t) \leq K\varphi_0$ on $[t_1, t_2]$. Here $K > 1$ and will be bounded independently of the time. From the inequality $\dot{V}_1 \leq -cV_1^\alpha$, one deduces that $t_2 - t_1 \leq C_0\varepsilon^{1-\alpha}$, for some universal constant C_0. Moreover, on that interval, $\dot{\hat{\varphi}} \leq k_\varepsilon$, implying that

$$(K-1)\varphi_0 \leq k_\varepsilon(t_2 - t_1) \leq \frac{C_0 m_\varepsilon \varphi_0}{\varepsilon^\alpha},$$

for some positive constant C_0 independent of $\varepsilon > 0$ small enough. This ends the proof of the theorem.

■

Remark 4 *For both choices of controllers from [30] or Theorem 1, it is not difficult to show that there exists $C_0, \gamma > 0$ such that for sufficiently small $\varepsilon > 0$, we have $m_\varepsilon \leq C_0\varepsilon^\gamma$ with $\gamma < \alpha$. Then, both upper bounds for the gain value k_ε and the upper bound on $\hat{\varphi}_\varepsilon$ tend to infinity as ε tends to zero.*

5 Simulation Results

The performance of the control laws presented in the previous sections has been evaluated through simulation. Let us consider an academic kinematic model of a car [6] (see Fig.1), given by:

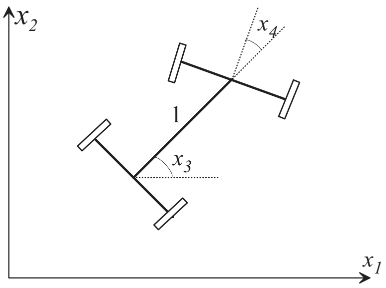

Fig. 1. Kinematic car model.

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} w\cos(x_3) \\ w\sin(x_3) \\ {}^{w}/_{L}\tan(x_4) \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} u, \qquad (21)$$

where x_1 and x_2 are the cartesian coordinates of the rear axle middle point, x_3 is the orientation angle, x_4 is the steering angle and u is the control input. w is the longitudinal velocity ($w = 10ms^{-1}$), and L is the distance between the two axles ($L = 5m$). All the state variables are assumed to be measured and the velocity is assumed to have an uncertainty of $\delta w = 5\%$.

The goal is to steer the car from a given initial position to the trajectory $x_{2ref} = 10sin(0.05x_1) + 5$ in finite time. Considering the sliding variable $s = x_2 - x_{2ref}$: the relative degree of the system w.r.t. s is 3. The 3rd time derivative of s is $s^{(3)} = \varphi + \gamma u$, where

$$\begin{aligned} \varphi &= [\frac{1}{800}cos\left(\frac{x_1}{20}\right).(cos(x_3))^2 \\ &\quad -\frac{1}{40L}sin\left(\frac{x_1}{20}\right).\sin(x_3).tan(x_4)]w^3\cos(x_3) \\ &\quad +[-\frac{1}{20}sin\left(\frac{x_1}{20}\right)cos(x_3)\,sin(x_3)\,, \\ \gamma &= \frac{w^2}{L}\left[\frac{1}{2}cos\left(\frac{x_1}{20}\right)sin(x_3) + cos(x_3)\right] \\ &\quad .\left[1 + tan^2(x_4)\right]. \end{aligned}$$

We first develop a 3^{rd}-order SMC robust controller in two steps:

- Defining the control law u_0, which stabilizes a three integrator chain in finite time.
- Obtaining the robust control law u via the Equation (5).

The robust control law can hence be expressed as

$$u = \frac{1}{K_m}(u_0 + \varphi_0 sign(u_0)),$$

where u_0 is determined below,

$$\begin{aligned} v_1 &= -l_1\lfloor\lfloor z_1\rceil^{\beta_0} - 0\rceil^{\alpha_1/\beta_0}, \\ v_2 &= -l_2\lfloor\lfloor z_2\rceil^{\beta_1} - \lfloor v_1\rceil^{\beta_1}\rceil^{\alpha_2/\beta_1}, \\ u_0 &= v_3 = -l_3\lfloor\lfloor z_3\rceil^{\beta_2} - \lfloor v_2\rceil^{\beta_1}\rceil^{\alpha_3/\beta_2}. \end{aligned} \qquad (22)$$

In our simulation, the parameters have been tuned to the following values:

$$\begin{aligned} & l_1 = 5,\ l_2 = 10,\ l_3 = 40,\ k = -0.2, \\ & \beta_0 = 0.8,\ \beta_1 = 1.25,\ \beta_2 = 2, \\ & \alpha_1 = 4/5,\ \alpha_2 = 3/4,\ \alpha_3 = 2/3, \\ & K_m = 200,\ \varphi_0 = 1000. \end{aligned}$$

Figures 2 to 5 illustrate the simulation results of the robust controller on the car model. Fig.2 shows that the sliding variable s and its first two time derivatives $\dot{s}$ and $\ddot{s}$ converge exactly to zero. Fig.3 shows the discontinuous control u. Fig.4 shows that x_2 converges to the desired trajectory. As HOSM has been established, it can be seen that there is no chattering in this figure. Fig.5 displays the steering angle x_4 versus time. These results show the applicability and robustness of the controller. We will now develop a 3^{rd}-order SMC adaptive controller, which requires three steps:

- Determining a nominal model of the system.
- Defining the control law u_0 that stabilizes a three-integrator chain in finite time.
- Tuning the dynamics of $\hat{\varphi}_\varepsilon$.

The control law can hence be expressed as:

$$u = \frac{1}{K_m}\left(u_0 + \hat{\varphi}_\varepsilon sign(u_0)\right),$$

where u_0 is determined in the same way as in (22). The parameters used in adaptive simulation are as follows:

$$\begin{aligned} & l_1 = 5,\ l_2 = 10,\ l_3 = 40,\ k = -0.2, \\ & \beta_0 = 0.8,\ \beta_1 = 1.25,\ \beta_2 = 2, \\ & \alpha_1 = 4/5,\ \alpha_2 = 3/4,\ \alpha_3 = 2/3, \\ & K_m = 200,\ k_\varepsilon = 50,\ \eta = 0.5,\ \varepsilon = 1. \end{aligned}$$

Fig.6 shows the convergence of s, $\dot{s}$ and $\ddot{s}$ under the adaptive controller. Fig.8 shows the dynamics of $\hat{\varphi}_\varepsilon$. It can be seen that the states converge to zero in 4 seconds, and then $\hat{\varphi}_\varepsilon$ starts to decrease rapidly. During this time, the states are kept equal to zero, ensuring an ideal HOSM. This is because the gains are overestimated in this duration. In Fig.7, it can be seen that real HOSM is established after 11 sec because the gains are just sufficient to counteract the uncertainty. The states converge to neighborhood of zero defined by $V_1 \leq \varepsilon$, i.e. $V_1 \leq 1$. Fig.9 shows the control u where the amplitude of the discontinuous part of u adapts to counteract the uncertainty. Fig.10 shows the convergence of x_2 to the desired trajectory x_{2ref}. Fig.11 displays the steering angle x_4 versus time. Comparing Figures 3 and 9, it can be seen that after convergence, the amplitude of the control signal is significantly smaller when adaptive gains are applied.

6 CONCLUSIONS

In this paper, we have presented two HOSMC controllers for the stabilization of nonlinear uncertain SISO systems. The first controller has been developed from a class of Lyapunov based controllers for the stabilization of a pure integrator chain. This controller is robust and depends upon the knowledge of the perturbation bounds. The second controller is adaptive, as it does not require any quantitative knowledge of the perturbation bounds of the uncertain function φ, while the bounds of the uncertain function γ need to be known. It establishes real HOSM. The simulation results illustrate the good performance and effectiveness of both these controllers. The advantage of using adaptive controllers can also be seen in the simulation results, as the controller output is much smaller when adaptive gains are used. Today, arbitrary adaptive Higher Order Sliding Mode Controller is an open problem, where the general problem is to obtain an adaptive controller that ensures establishment of HOSM without the knowledge of the bounds of both φ and γ. In future works, we aim to address this issue and develop adaptive controllers which are completely independent of the knowledge of the bounds.

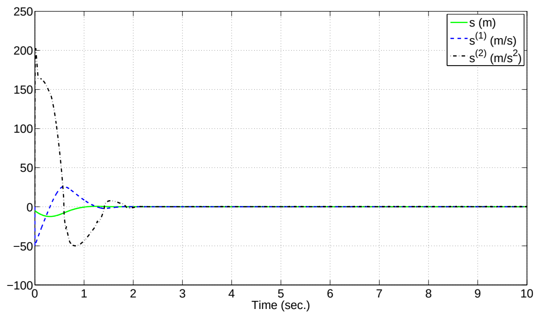

Fig. 2. $s(m)$, $\dot{s}$ (ms^{-1}) and $\ddot{s}$ (ms^{-2}) versus time (s).

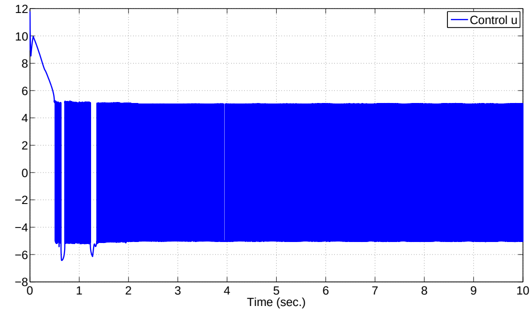

Fig. 3. control law u versus time (s).

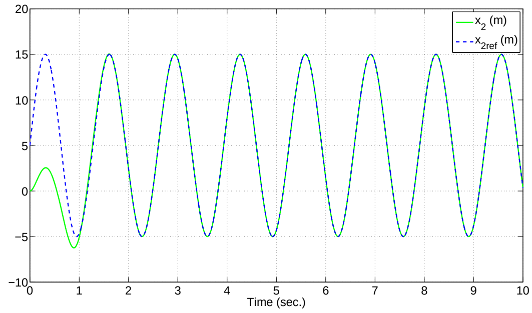

Fig. 4. x_2 (m) and x_{2ref} (m) versus time (s)

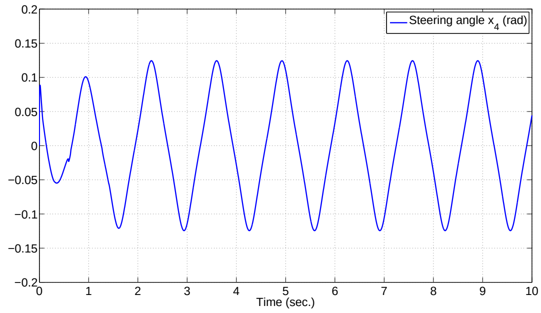

Fig. 5. Steering angle x_4 (rad) versus time (s)

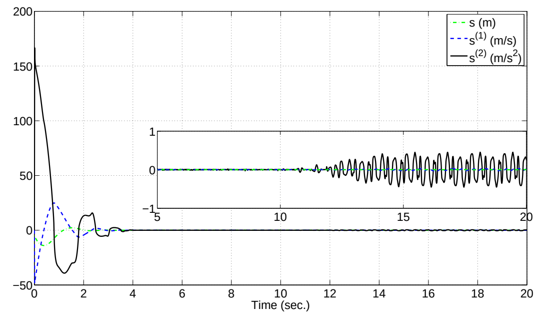

Fig. 6. $s(m)$, $\dot{s}$ (ms^{-1}) and $\ddot{s}$ (ms^{-2}) versus time (s).

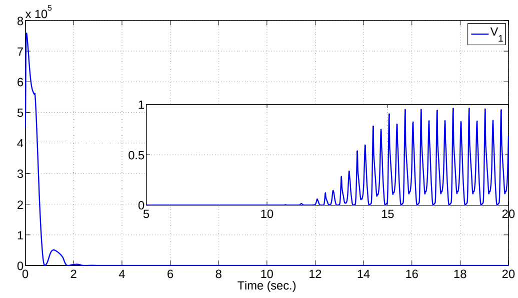

Fig. 7. V_1 versus time (s).

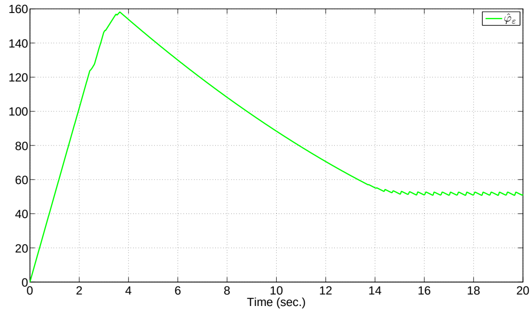

Fig. 8. $\hat{\varphi}_\varepsilon$ versus time (s).

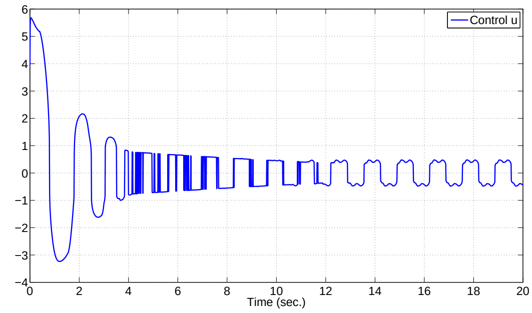

Fig. 9. control law u versus time (s).

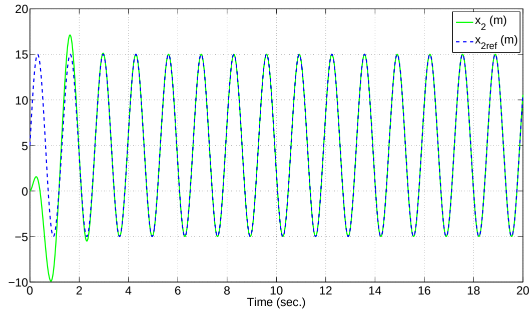

Fig. 10. x_2 (m) and x_{2ref} (m) versus time (s)

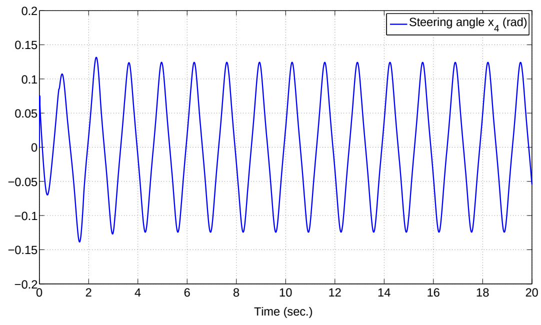

Fig. 11. Steering angle x_4 (rad) versus time (s)

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