

New Robust and Adaptive Higher Order Sliding mode controllers

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Abstract

In this paper, we have presented a Lyapunov-based design for higher order sliding mode robust and adaptive controllers for nonlinear SISO systems with bounded uncertainties. In the robust control problem, the uncertainty bounds are known, whereas these bounds are unknown in the adaptive control problem. These problems have been formulated as the finite time stabilization of a chain of integrators with bounded uncertainties. The controllers have been developed from a class of nonlinear controllers which guarantee finite time stabilization of pure integrator chains. In the adaptive case, the controller design uses saturation functions. For the first controller, we prove finite time convergence to zero as for the controller in the adaptive case, we show finite time convergence to an arbitrary small neighborhood of the origin. The proposed controllers can be designed for any arbitrary order sliding mode. The simulation results demonstrate the effectiveness of the proposed techniques.

1. INTRODUCTION

Higher order nonlinear dynamic systems are difficult to characterize, and their models contain uncertainty due to uncertain parameters or perturbations. For finite time stability of such systems, the controller must be designed to assure finite time stabilization under a certain range of uncertainty, where the bounds of the uncertainty might be known or unknown. Sliding mode control (SMC) [1, 2] is a well-known method for the control of high-order nonlinear dynamic systems operating under uncertainty conditions. The technique is based on applying discontinuous control on a system to converge to a "sliding surface" (a surface comprising of the system trajectories) in finite time [3]. In practice, classic SMC suffers from high frequency *chattering*; as the infinite switching frequency required by ideal sliding mode is not achievable. Higher order sliding mode control (HOSMC) is

the most effective method for chattering suppression [4]. In this method the discontinuous control is applied on a higher time derivative of the sliding surface, therefore not only the sliding variable, but its higher time derivatives converge in finite time as well.

Many HOSMC algorithms exist in contemporary literature for finite time stabilization of uncertain nonlinear systems, where the bounds on uncertainty are known. These are referred to as robust algorithms. Levant for example, has presented a method to design arbitrary order sliding mode controllers with finite time convergence for Single Input Single Output (SISO) systems in [5, 6, 7]. Laghrouche et al. [8] have proposed a two part integral sliding mode based control to deal with the finite time stabilization problem and uncertainty rejection problem separately. Dinuzzo et al. have proposed another method in [3], where the problem of HOSM has been treated as Robust Fuller's problem. Defoort et al. [9] have developed a robust MIMO HOSM controller, using a constructive algorithm with geometric homogeneity based finite time stabilization of an integrator chain. Harmouche et al. have also presented their controller in [10] based on the works of Hong [11].

In the systems where uncertainty bounds are unknown, SMC with adaptive gains is required as simply setting the gains too high elevates chattering. Huang et al. first addressed this issue for first order SMC in [12]. The main drawback of their technique is that the adaptation algorithm can only increase gains, which leads to gain overestimation and chattering. Plestan et al. [13] have overcome this problem by slowly decreasing the gains once sliding mode is achieved. In HOSMC, Shtessel et al. [14] have presented a Second Order adaptive gain SMC, based on super twisting algorithm for non-overestimation of the control gains. A Lyapunov-based variable gains super twisting algorithm has also been presented in [15]. Glumineau et al. [16] have presented a different approach, based on impulsive sliding mode adaptive control of a second order system. To the best of our knowledge, no contemporary work on adaptive HOSMC has been published for orders greater than two.

In this paper, we have presented new robust and adaptive higher order sliding mode controllers for finite time stabilization of higher order SISO nonlinear systems. This problem has been developed as the finite time stabilization of a chain of integrators with bounded uncertainties. There are two main contributions of this paper. Firstly, a robust controller has been developed from a class of

Lyapunov-based controllers, which stabilizes an integrator chain with known bounded uncertainties. Secondly, the robust controller has been extended as an adaptive controller to handle systems where bounds on uncertainty are not known. This controller establishes real sliding mode [13], i.e., once the system states arrive at the sliding surface, they can leave as well. In other words, we prove convergence in finite time of the states to any arbitrarily small neighborhood of the origin. The gain adaptation dynamics are based on a saturation function [17, 18, 19], which results in rapid increase of gains when the error is large, and rapid decrease when the error is reduced.

The main advantage of our work is that our controllers can be extended to any arbitrary order sliding mode. Another advantage is that the adaptive gain control developed in this paper is focused on prohibiting overestimation of gains. As mentioned before, in the controller presented in [12], the gains can only increase, whereas in [13] the gains decrease very slowly.

The paper has been organized as follows: the problem formulation has been presented in section 2. The design of the robust controller has been presented in section 3, and that of adaptive controller in section 4. Simulation results have been presented and discussed in section 5 and some concluding remarks have been given in section 6.

2. Problem Formulation

Let us consider an uncertain nonlinear system:

$$\begin{cases} \dot{x} = f(x,t) + g(x,t)u, \\ y = s(x,t), \end{cases} \quad (1)$$

where $x \in \mathbb{R}^n$ and $u \in \mathbb{R}$ is the input control, s is a measured smooth output-feedback function (sliding variable) and $f(x,t)$ and $g(x,t)$ are uncertain smooth functions. We assume the following,

H1. The relative degree r of the system (1) with respect to s is constant and known, and the associated zero dynamics are stable.

The control objective consists of fulfilling the constraint $s(x, t) = 0$ in finite time and to keep it exact by discontinuous feedback control. The r^{th} -order sliding mode is defined as follows:

Definition 2.1 [20, 21]. Consider the nonlinear system (1), and let the system be closed by some possibly-dynamical discontinuous feedback. Then, provided that $s, \dot{s}, \dots, s^{(r-1)}$ are continuous functions, and the set $S^r = \{x | s(x, t) = \dot{s}(x, t) = \dots = s^{(r-1)}(x, t) = 0\}$, called " r^{th} -order sliding set", is non-empty and is locally an integral set in the Filippov sense [22], the motion on S^r is called " r^{th} -order sliding mode" with respect to the sliding variable s .

The r^{th} -order SMC approach allows the finite time stabilization to zero of the sliding variable s and its $r - 1$ first time derivatives by defining a suitable discontinuous control function. If the system (1) is extended by the introduction of a fictitious variable $x_{n+1} = t, \dot{x}_{n+1} = 1$, and $f_e = (f^T 1)^T, g_e = (g^T 0)^T$ (where the last component corresponds to x_{n+1}), then the output s satisfies the equation [21]:

$$s^{(r)} = \varphi(\cdot) + \gamma(\cdot)u, \text{ with } \gamma = L_{g_e} L_{f_e}^{r-1} s \text{ and } \varphi = L_{f_e}^r s.$$

We also assume the following.

H2. The solutions are understood in the Filippov sense [22], and system trajectories are infinitely extendible in time for any bounded Lebesgue measurable input.

H3. Functions $\varphi(\cdot)$ and $\gamma(\cdot)$ are bounded uncertain functions i.e. there exist constants $K_m, K_M > 0$ and $\varphi_0 \geq 0$ such that

$$0 \leq K_m \leq \gamma(x) \leq K_M, \quad |\varphi(x)| \leq \varphi_0,$$

for $x \in X \subset \mathbb{R}^n$, X being a bounded open subset of \mathbb{R}^n within which the boundedness of the system dynamics is ensured.

Assumption **H3** implies that results in the following sections of the paper can be considered as local. Then the r^{th} -order SMC of (1) with respect to the sliding variable s becomes equivalent to

the finite time stabilization of

$$\begin{cases} \dot{z}_i = z_{i+1}, & i = 1, \dots, r-1, \\ \dot{z}_r = \varphi(\cdot) + \gamma(\cdot)u, \end{cases} \quad (2)$$

where $z = [z_1 z_2 \dots z_r]^T := [s \dot{s} \dots s^{(r-1)}]^T$.

In the following sections, we will first develop a robust controller for the system (2) under hypothesis **H3**, for the case where the uncertainty bounds presented in **H3** are known. Then, an adaptive controller will be developed to extend the functionality of the robust controller to the case where the bounds are not known.

Remark 1. *Considering the system (1) with relative degree ρ [23] with respect to sliding variable s , the problem of higher order SMC for $r > \rho$ can be extended naturally by increasing the length of integrator chain by $r - \rho$. For the sake of clarity, we will consider $r = \rho$ in all further sections.*

3. Design of robust higher order sliding mode controller

In this section, we will present a robust controller which stabilizes System (2), considering that the bounds on φ and γ are known. This controller has been derived from a class of Lyapunov-based controllers that guarantee finite time stabilization of pure integrator chains, and satisfy certain additional geometric conditions. The pure integrator chain ($\varphi \equiv 0$ and $\gamma \equiv 1$) is represented as follows:

$$\begin{cases} \dot{z}_i = z_{i+1}, & i = 1, \dots, r-1, \\ \dot{z}_r = u. \end{cases} \quad (3)$$

Let us recall the following theorem:

Theorem 1. [24]. *Consider System (3). Suppose there exists a state-feedback control law $u = u_0(z)$ and a C^1 function V_1 defined on a neighborhood $\hat{U} \subset \mathbb{R}^r$ of the origin, and real numbers $c > 0$ and $0 < \alpha < 1$, such that*

1. V_1 is positive definite on \hat{U} ;
2. $\dot{V}_1 + cV_1^\alpha(z(t)) \leq 0$, if $z(t) \in \hat{U}$,

where $z(\cdot)$ is a trajectory of System (3) with the feedback $u_0(z)$ and \dot{V}_1 is the time derivative of $V_1(z(\cdot))$ at time $t \geq 0$.

Then all trajectories of System (3) with the feedback $u_0(z)$ which stay in \hat{U} converge to zero in finite time. If $\hat{U} = \mathbb{R}^r$ and V_1 is radially unbounded, then System (3) with the feedback $u_0(z)$ is globally finite time stable with respect to the origin.

Based on this theorem, we now present a robust controller for System (2).

Theorem 2. Consider System (2) subject to Hypothesis **H3**. Then the following control law establishes higher order sliding mode with respect to s in finite time:

$$\bar{u} = \frac{1}{K_m}(u_0 + \varphi_0 \text{sign}(u_0)), \quad (4)$$

where $u_0(z)$ is any state-feedback law that satisfies Theorem 1 (i.e., u_0 guarantees finite time stabilization of System (3)), and in addition the Lyapunov function V_1 satisfies the following further conditions along with those of Theorem 1:

$$\frac{\partial V_1}{\partial z_r}(z)u_0(z) \leq 0, \text{ and } u_0(z) = 0 \Rightarrow \frac{\partial V_1}{\partial z_r}(z) = 0, \forall z \in \hat{U}, \quad (5)$$

The global finite time stability condition of Theorem 1 exists here as well if $\hat{U} = \mathbb{R}^r$ and V_1 is radially unbounded.

Proof. Consider System (2) and the control law u defined in (4):

$$\begin{cases} \dot{z}_i &= z_{i+1}, \quad i = 1, \dots, r-1, \\ \dot{z}_r &= \varphi + \gamma u \\ &= \frac{\gamma u_0(z)}{K_m} + \frac{\gamma \varphi_0}{K_m} \text{sign}(u_0(z)) + \varphi. \end{cases} \quad (6)$$

The time derivative of the Lyapunov function V_1 provided by Theorem 1 is calculated along a

trajectory of System (3) inside \hat{U} . Assume first that $u_0(z(t)) \neq 0$. We obtain

$$\begin{aligned}
\dot{V}_1 &= \sum_{i=1}^{r-1} \frac{\partial V_1}{\partial z_i} z_{i+1} + \frac{\partial V_1}{\partial z_r} (\varphi + \gamma u), \\
&= \sum_{i=1}^{r-1} \frac{\partial V_1}{\partial z_i} z_{i+1} + \frac{\partial V_1}{\partial z_r} \left(\frac{\gamma}{K_m} u_0 + \frac{\gamma}{K_m} \varphi_0 \text{sign}(u_0) + \varphi \right) \\
&\leq \sum_{i=1}^{r-1} \frac{\partial V_1}{\partial z_i} z_{i+1} + \frac{\partial V_1}{\partial z_r} u_0 + \frac{\partial V_1}{\partial z_r} \text{sign}(u_0) (\varphi_0 - |\varphi|) \\
&\leq \sum_{i=1}^{r-1} \frac{\partial V_1}{\partial z_i} z_{i+1} + \frac{\partial V_1}{\partial z_r} u_0 \leq -cV_1^\alpha.
\end{aligned} \tag{7}$$

If $u_0(z(t)) = 0$, the previous inequality still holds since $\frac{\partial V_1}{\partial z_r} \dot{z}_r = \frac{\partial V_1}{\partial z_r} u_0(z) = 0$. It can be deduced that System (2) is finite time stable with respect to the origin as well.

□

The previous result becomes non empty if controllers can be identified, that verify the conditions of Theorem 1 and condition (5). The controllers proposed by Hong [11] and Huang [25] fulfil these conditions. For $x \in \mathbb{R}$ and $\alpha > 0$, let us use $[x]^\alpha$ to denote $|x|^\alpha \text{sign}(x)$. Hong's controller [11] is expressed as:

Let $k < 0$ and l_1, \dots, l_r positive real numbers. For $z = (z_1, \dots, z_r)$, we define for $i = 0, \dots, r-1$:

$$p_i = 1 + (i-1)k, \quad v_0 = 0, \quad v_{i+1} = -l_{i+1} [|z_{i+1}|^{\beta_i} - |v_i|^{\beta_i}]^{(\alpha_{i+1}/\beta_i)}, \tag{8}$$

where $\alpha_i = \frac{p_{i+1}}{p_i}$, for $i = 1, \dots, r$, and, for $k < 0$ sufficiently small,

$$\beta_0 = p_2, \quad (\beta_i + 1)p_{i+1} = \beta_0 + 1 > 0, \quad i = 1, \dots, r-1. \tag{9}$$

Consider the positive definite radially unbounded function $V : \mathbb{R}^r \rightarrow \mathbb{R}^+$ given by

$$V = \sum_{j=1}^r \int_{v_{j-1}}^{z_j} [s]^{\beta_{j-1}} - [v_{j-1}]^{\beta_{j-1}} ds. \tag{10}$$

It has been proved in [11] that, for a sufficiently small k , there exist $l_i > 0$, $i = 1, \dots, r$, such that the control law $u_0 = v_r$ defined above stabilizes System (3) in finite time and there exists $c > 0$ and $0 < \alpha < 1$ such that u_0 and V fulfill the conditions of Theorem 1. Let us now consider the following proposition:

Proposition 1. *The Lyapunov function V defined in (10) satisfies Condition (5).*

Proof. For $z \in \mathbb{R}^r$, we obtain

$$\frac{\partial V}{\partial z_r} = \frac{\partial W_r}{\partial z_r} = [z_r]^{\beta_{r-1}} - [v_{r-1}]^{\beta_{r-1}}, \quad u_0(z) = v_r = -l_r \left[[z_r]^{\beta_{r-1}} - [v_{r-1}]^{\beta_{r-1}} \right]^{\frac{\alpha_r}{\beta_{r-1}}}.$$

Then,

$$\frac{\partial V}{\partial z_r} u_0(z) = -l_r \left| [z_r]^{\beta_{r-1}} - [v_{r-1}]^{\beta_{r-1}} \right|^{1 + \frac{\alpha_r}{\beta_{r-1}}} \leq 0,$$

and $u_0(z) = 0$ if and only if $\frac{\partial V}{\partial z_r} = 0$. □

The feedback law of [11] can be simplified by choosing all $\beta_i = 1$ in (8). The following theorem presents a formalization of this simplification:

Theorem 3. *For System (3), there exist a sufficiently small $k < 0$ and real numbers $l_i > 0$, such that the control law $u_0 = v_r$ defined below stabilizes System (3) in finite time.*

For $i = 0, \dots, r-1$,

$$v_0 = 0, \quad v_{i+1} = -l_{i+1} [z_{i+1} - v_i]^{\frac{1+(i+2)k}{1+(i+1)k}}. \quad (11)$$

Proof. The proof of theorem 3 can be developed simply by adapting the proof presented in [11] to the parameter choice of (11). Let f_α be the closed-loop vector field obtained by using the feedback (11) in (3). For each $\alpha > 0$, the vector field f_α is continuous and homogeneous of degree $k < 0$ with respect to the family of dilations (p_1, \dots, p_r) , where $p_i = 1 + (i-1)k$, $i = 1, \dots, r$. Let l_i , $i = 1, \dots, r$ be positive constants such that the polynomial $y^r + l_r(y^{r-1} + l_{r-1}(y^{r-2} + \dots + l_2(y + l_1))) \dots$ is Hurwitz. It is clear that for $\alpha = 1$, the vector field f_1 is linear with this characteristic polynomial. Therefore, there exists a positive-definite, radially unbounded, Lyapunov function $V : \mathbb{R}^r \rightarrow \mathbb{R}$ such that $L_{f_1} V$ is continuous and negative definite.

Let $\mathcal{A} = V^{-1}([0, 1])$ and $\mathcal{S} = \text{bd}\mathcal{A} = V^{-1}(\{1\})$. Then \mathcal{A} and \mathcal{S} are compact since V is proper. Also, $0 \notin \mathcal{S}$ as V is positive definite. Defining $\varphi : (0, 1] \times \mathcal{S} \rightarrow \mathbb{R}$ by $\varphi(\alpha, z) = L_{f_\alpha} V(z)$. Then V is continuous and satisfies $\varphi(\alpha, z) < 0$ for all $z \in \mathcal{S}$, i.e. $\varphi(\{1\} \times \mathcal{S}) \subset (-\infty, 0)$. Since \mathcal{S} is compact, by continuity there exists $\varepsilon > 0$ such that $\varphi((1 - \varepsilon, 1] \times \mathcal{S}) \subset (-\infty, 0)$. It follows that for $\alpha \in (1 - \varepsilon, 1]$, $L_{f_\alpha} V$ takes negative values on \mathcal{S} . Thus, \mathcal{A} is strictly positively invariant under f_α for every $\alpha \in (1 - \varepsilon, 1]$. Therefore the origin is global asymptotic stable under f_α , for

$\alpha \in (1 - \varepsilon, 1]$. Finally, for $\alpha \in (1 - \varepsilon, 1)$ i.e $|k|$ small enough, by homogeneity, the origin is globally finite time stable. □

Remark 2. Once u_0 has been determined, u can be chosen as $u = \frac{\delta}{K_m}(u_0 + \mu \varphi_0 \cdot \text{sign}(u_0))$, with $\delta \geq 1$, $\mu \geq 1$.

Remark 3. It is important to note that u is not necessarily a first order sliding mode on u_0 , even though once the states reach the sliding surface $u_0 = 0$, but they can still leave it as well.

Remark 4. When Higher Order Sliding Mode is established ($z_1 = z_2 = \dots = z_r = 0$) the discontinuous control u is defined by the equivalent control method¹ as the solution of the equation $\dot{z}_r = 0$ [26]. The equivalent control here is given by

$$u_{eq} = -\frac{\varphi}{\gamma}.$$

4. Adaptive Controller

We shall now consider the case where the uncertainty bounds on φ and γ are unknown. In addition to the assumptions established for the robust controller, let us now formalize the representation of unknown bounds on system (2).

H4 Let us consider $\varphi = \varphi_N + \Delta\varphi$, $\gamma = \gamma_N + \Delta\gamma$.

where φ_N and γ_N are the nominal, known values of φ and γ , and $\Delta\varphi$ and $\Delta\gamma$ represent the uncertainty in the knowledge of bounds on φ and γ . Then, System (2) becomes:

¹In sliding mode theory, once sliding mode has been attained, the discontinuous control can be replaced by an equivalent control, without altering the system behavior (see, [26]). The equivalent control can be approximated by linear first order filter after (4).

$$\begin{cases} \dot{z}_i = z_{i+1}, & i = 1, \dots, r-1, \\ \dot{z}_r = \varphi_N + \gamma_N u + \underbrace{\Delta\varphi + u\Delta\gamma}_F. \end{cases} \quad (12)$$

The unknown factors have been coupled into a single factor F . This factor has an unknown upper bound \bar{F} i.e.:

$$|F| \leq \bar{F}. \quad (13)$$

Based on the conditions for robust controller presented in the previous section, let us define a new control law v , obtained by the following change of variables:

$$u = \frac{1}{\gamma_N} (v - \varphi_N) \quad (14)$$

then system (12) becomes:

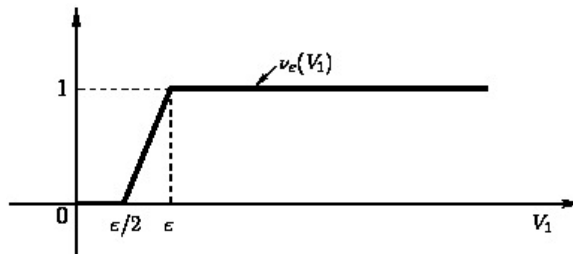
$$\begin{cases} \dot{z}_i = z_{i+1}, & i = 1, \dots, r-1, \\ \dot{z}_r = v + F. \end{cases} \quad (15)$$

The HOSMC problem becomes equivalent to finding v which can stabilize (15) in finite time. The subsequent controllers are defined using saturation functions. Let $\sigma(\cdot)$ be the standard saturation function given by

$$\sigma(x) = \frac{x}{\max(1, |x|)},$$

and, for $\varepsilon > 0$, let $v_\varepsilon(\cdot)$ be the real valued function defined by

$$v_\varepsilon(x) = \frac{1}{2} + \frac{1}{2} \sigma\left(\frac{|x| - \frac{3}{4}\varepsilon}{\frac{1}{4}\varepsilon}\right). \quad (16)$$



Theorem 4. Consider System (15) and let u_0, V_1 be the control law and the Lyapunov function respectively provided by Theorem 2, with the associated global assumptions. Consider the following choice of controller, defined for $\varepsilon > 0$:

$$v = u_0 + \hat{F} \text{sign}(u_0), \quad (17)$$

where \hat{F} is solution of the Cauchy problem defined by:

$$\hat{F}(0) = 0, \quad \dot{\hat{F}} = k_1 v_\varepsilon(V_1) - (1 - v_\varepsilon(V_1)) [\hat{F}]^\alpha + \sigma(V_1). \quad (18)$$

Then the following holds true: there exists $M > 0$ such that, for $\varepsilon > 0$ small enough, there exist $k_1 > 0$ so that for every initial condition $z_0 \neq 0$, if $z(\cdot)$ denotes the trajectory of System (15) starting at z_0 with the above choice of controller:

(i) $\limsup_{t \rightarrow \infty} V_1(z(t)) < \varepsilon$;

(ii) $\limsup_{t \rightarrow \infty} |\hat{F}| \leq \frac{M m_\varepsilon \bar{F}}{\varepsilon^\alpha}$, where $m_\varepsilon = \max_{V_1 \leq \varepsilon} \left| \frac{\partial V_1}{\partial z_r} \right|$.

Proof. The dynamic of \hat{F} can be defined explicitly by:

$$\dot{\hat{F}} = \begin{cases} k_1 + \sigma(V_1), & V_1 \geq \varepsilon \\ \left(V_1 - \frac{\varepsilon}{2}\right) \frac{2k_1}{\varepsilon} - (\varepsilon - V_1) \frac{2}{\varepsilon} [\hat{F}]^\alpha + \sigma(V_1), & \frac{\varepsilon}{2} \leq V_1 \leq \varepsilon, \\ -[\hat{F}]^\alpha + \sigma(V_1), & V_1 \leq \frac{\varepsilon}{2}. \end{cases}$$

We first need the following intermediate result.

Lemma 1. The function \hat{F} is non-negative and is defined as long as the trajectory of z is defined, $\liminf_{t \rightarrow \infty} V_1(z) \leq \frac{3\varepsilon}{4}$ and $\liminf_{t \rightarrow \infty} \hat{F} \leq \bar{F}$.

Proof. It is clear that \hat{F} is strictly positive in an interval of the type $(0, \tau)$, since $\dot{\hat{F}}(0) > 0$. We argue by contradiction. Let us suppose that there exist $\tau_1 > 0$ such that $\hat{F}(\tau_1) < 0$. Since \hat{F} continuous, there exists a time $\tau_0 \geq 0$, $\tau_0 < \tau_1$, such that $\hat{F}(\tau_0) = 0$, and $\hat{F}(t) < 0$, $\forall t \in]\tau_0, \tau_1]$.

In this case, $V_1(\tau_0) = 0$ otherwise $\dot{\hat{F}}(\tau_0) > 0$ and \hat{F} cannot be negative on a right interval at τ_0 .

In that case, there exists a right interval at τ_0 (still denoted $] \tau_0, \tau_1]$) where $V_1 < \frac{\varepsilon}{2}$ and then $\dot{\hat{F}} \geq -[\hat{F}]^\alpha + \sigma(V_1) > 0$. One then gets that

$$\hat{F}(\tau_1) = \int_{\tau_0}^{\tau_1} \dot{\hat{F}} dt > 0,$$

which is a contradiction.

To prove the second part of the lemma, we argue by contradiction. If $\liminf_{t \rightarrow \infty} V_1(z) > \frac{2\varepsilon}{3}$, then $V_1(z(t)) > \frac{\varepsilon}{5}$ and then $\dot{\hat{F}} > \frac{k_1}{4}$ for t large enough and k_1 larger than a universal constant. It implies that $\hat{F} > \bar{F}$ for t large enough. Since one has that

$$\begin{aligned} \dot{V}_1 &\leq \sum_{i=1}^{r-1} \frac{\partial V_1}{\partial z_i} z_{i+1} + \frac{\partial V_1}{\partial z_r} u_0 + \left| \frac{\partial V_1}{\partial z_r} \right| \bar{F} - \left| \frac{\partial V_1}{\partial z_r} \right| \hat{F}, \\ &\leq \sum_{i=1}^{r-1} \frac{\partial V_1}{\partial z_i} z_{i+1} + \frac{\partial V_1}{\partial z_r} u_0, \\ &\leq -c_1 V_1^{\alpha_1}, \end{aligned} \tag{19}$$

then $V_1(z)$ converges to zero in finite time, which is not possible.

Finally we turn to the third part of the lemma and we again argue by contradiction. In that case, $\hat{F} > \bar{F}$ for t large enough and by the previous computation, V_1 converges to zero in finite time, implying the same conclusion for \hat{F} , which is a contradiction. \square

We next turn to the proof of Item (i) of the theorem. We again argue by contradiction and suppose that $\limsup_{t \rightarrow \infty} V_1(z) \geq \varepsilon$. Then, there exists times $t_1 < t_2 < t_3$ arbitrarily large and $\frac{3}{4} < l < L < 1$ so that $V_1(z(t_1)) = \frac{3\varepsilon}{4}$, $V_1(z(t_2)) = l\varepsilon$, $V_1(z(t_3)) = L\varepsilon$, and

$$\frac{3\varepsilon}{4} \leq V_1(z(t)) \leq l\varepsilon \text{ on } [t_1, t_2], \quad l\varepsilon \leq V_1(z(t)) \leq L\varepsilon \text{ on } [t_2, t_3].$$

Note that one has $\dot{V}_1 \leq m_\varepsilon \bar{F}$, which implies that $t_2 - t_1 \geq \frac{(l - \frac{3}{4})\varepsilon}{m_\varepsilon \bar{F}}$. Since $\dot{\hat{F}} \geq \frac{k_1}{4}$ on $[t_1, t_2]$, one has

$$\hat{F}(t_2) \geq \frac{(l - \frac{3}{4})\varepsilon k_1}{4m_\varepsilon \bar{F}}$$

Choosing k_1 so that the right-hand side of the above inequality is equal to \bar{F} , one immediately deduces that $\hat{F} \geq \bar{F}$ on $[t_2, t_3]$ and thus V_1 is strictly decreasing on that interval, which is a contradiction.

We next turn to the proof of Item (ii) of the theorem. Let $[t_1, t_2]$ be an interval of time where $\hat{F}(t_1) = \bar{F}$, $\hat{F}(t_2) = K\bar{F}$ and $\bar{F} \leq \hat{F}(t) \leq K\bar{F}$ on $[t_1, t_2]$. Here $K > 1$ and will be bounded independently of the time. From the inequality $\dot{V}_1 \leq -cV_1^\alpha$, one deduces that $t_2 - t_1 \leq C_0\varepsilon^{1-\alpha}$, for some universal constant C_0 . Moreover, on that interval, $\dot{\hat{F}} \leq k_1$, implying that

$$(K - 1)\bar{F} \leq k_1(t_2 - t_1) \leq \frac{C_0 m_\varepsilon \bar{F}}{\varepsilon^\alpha},$$

for some positive constant C_0 independent of $\varepsilon > 0$ small enough. This ends the proof of the theorem. \square

Remark 5. For both choices of controllers from [11] or Theorem 3, it is not difficult to show that there exists $C_0, \gamma > 0$ such that for sufficiently small $\varepsilon > 0$, we have $m_\varepsilon \leq C_0 \varepsilon^\gamma$ with $\gamma < \alpha$. Then, both upper bounds for the gain value k_1 and the upper bound on \hat{F} tend to infinity as ε tends to zero.

5. Simulation Results

The performance of the control laws presented in the previous sections has been evaluated through simulation. Considering an academic kinematic model of a car [5] (see Fig.1), the system model is given by:

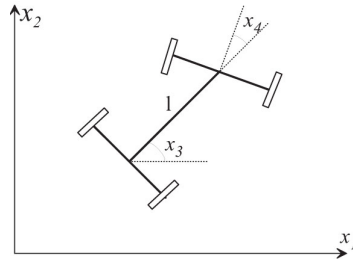


Figure 1: Kinematic car model.

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} w \cos(x_3) \\ w \sin(x_3) \\ w/L \tan(x_4) \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} u \quad (20)$$

where x_1 and x_2 are the cartesian coordinates of the rear axle middle point, x_3 the orientation angle and x_4 the steering angle. u is the control input. w is the longitudinal velocity ($w = 10ms^{-1}$), and L the distance between the two axles ($L = 5m$). The velocity is assumed to be have an error of $\delta w = 5\%$. All the state variables are assumed to be measured.

The goal is to steer the car from a given initial position to the trajectory $x_{2ref} = 10\sin(0.05x_1) + 5$ in finite time. Considering the sliding variable $s(x) = x_2 - x_{2ref}$: the relative degree of the system w.r.t. the surface is 3. The 3rd time derivative of s is $s^{(3)} = \varphi(\cdot) + \gamma(\cdot)u$, where

$$\begin{aligned}\varphi(\cdot) &= \left[\frac{1}{800} \cos\left(\frac{x_1}{20}\right) \cdot (\cos(x_3))^2 \right. \\ &\quad \left. - \frac{1}{40L} \sin\left(\frac{x_1}{20}\right) \cdot \sin(x_3) \cdot \tan(x_4) \right] w^3 \cos(x_3) \\ &\quad + \left[-\frac{1}{20} \sin\left(\frac{x_1}{20}\right) \cos(x_3) \sin(x_3) \right. \\ \gamma(\cdot) &= \left. \frac{w^2}{L} \left[\frac{1}{2} \cos\left(\frac{x_1}{20}\right) \sin(x_3) + \cos(x_3) \right] \right. \\ &\quad \left. \cdot [1 + \tan^2(x_4)] \right].\end{aligned}$$

For the robust controller, the 3rd-order SMC is developed in two steps:

- Defining the control law u_0 , which stabilizes a three integrator chain in finite time.
- Obtaining the robust control law u via the equation (4).

The robust control law can hence be expressed as:

$$u = \frac{1}{K_m} \cdot (u_0 + \varphi_0 \cdot \text{sign}(u_0)),$$

where u_0 is determined below,

$$\begin{aligned}v_1 &= -l_1 [|z_1|^{\beta_0} - 0]^{\alpha_1/\beta_0}, \\ v_2 &= -l_2 [|z_2|^{\beta_1} - |v_1|^{\beta_1}]^{\alpha_2/\beta_1}, \\ u_0 &= v_3 = -l_3 [|z_3|^{\beta_2} - |v_2|^{\beta_1}]^{\alpha_3/\beta_2}.\end{aligned}\tag{21}$$

In our simulation, the parameters have been tuned to the following values:

$$\begin{aligned}l_1 &= 10, l_2 = 10, l_3 = 70, k = -0.2, \\ \beta_0 &= 0.8, \beta_1 = 1.25, \beta_2 = 2, \\ \alpha_1 &= 4/5, \alpha_2 = 3/4, \alpha_3 = 2/3, \\ K_m &= 500, \varphi_0 = 2000.\end{aligned}$$

Fig.6 displays the convergence of states $(s, \dot{s}$ and $\ddot{s}) = (z_1, z_2$ and $z_3)$ to zero. Fig.3 displays the control u . Fig.9 displays the tracking of the desired trajectory by x_2 without chattering phenomena. Fig.10 displays the steering angle x_4 versus time. These results show the applicability and robustness of the controller.

For the adaptive controller, the 3rd-order SMC is designed in three steps:

- Determining a nominal model of the system
- Defining the control law u_0 that stabilizes a three-integrator chain in finite time.
- Tuning the dynamics of \hat{F} .

The control law can hence be expressed as:

$$u = \frac{1}{\gamma_N} (u_0 + \hat{F} \text{sign}(u_0) - \varphi_N),$$

where u_0 is determined in the same way as in (21). The parameters used in adaptive simulation are as follows:

$$\begin{aligned} l_1 &= 10, l_2 = 10, l_3 = 7, k = -0.2, \\ \beta_0 &= 0.8, \beta_1 = 1.25, \beta_2 = 2, \\ \alpha_1 &= 4/5, \alpha_2 = 3/4, \alpha_3 = 2/3, \\ \gamma_N &= 100, \varphi_N = 0, \\ k_1 &= 100, \alpha = 0.5, \varepsilon = 0.1. \end{aligned}$$

Fig.6 shows the convergence of states $(s, \dot{s}$ and $\ddot{s}) = (z_1, z_2$ and $z_3)$ to zero. Fig.7 shows the dynamics of \hat{F} . It can be seen that the states converge to zero in 4 seconds, and then \hat{F} starts to decrease rapidly. During this time, the states are kept equal to zero, ensuring an ideal sliding mode. After 12 sec real sliding mode is established. Fig.8 shows the control u , the chattering of u is adaptive with the uncertain model. Fig.9 demonstrates the tracking properties of the controller, without chattering. Fig.10 displays the steering angle x_4 versus time. These results show the applicability and robustness of the adaptive proposed controller. Comparing figures 3 and 8, it can be seen that after convergence, the amplitude of the control signal is significantly smaller when adaptive gains are applied.

6. CONCLUSIONS

In this paper, we have presented two new controllers for the stabilization of nonlinear uncertain SISO system, where the problem was formulated by HOSM Approach, by stabilization of perturbed integrator chain. The first controller is based upon the transformation of a function that can stabilize a pure integrator chain. This controller is robust, and depends upon the knowledge of the perturbation bounds. This controller has the minimum chattering level as compared to other controllers in the literature, and is simple to implement. The second controller does not need require any quantitative knowledge of the perturbation bounds, and is adaptive in nature. It only insures finite time convergence to an arbitrarily small neighborhood of the origin. The simulation results illustrate the good performance and effectiveness of both these controllers. The advantage of using adaptive controllers is also evident in the simulation results, as the controller output is much smaller when adaptive gains are used. In real life, this results in significant improvement and longevity of electrical and electromechanical systems in particular.

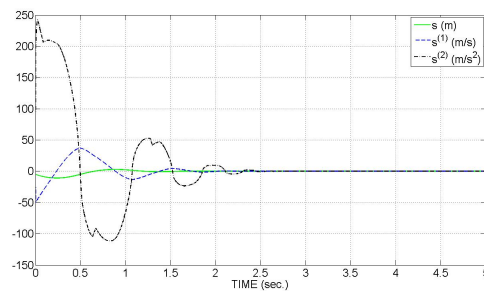


Figure 2: $s(m)$, $\dot{s} (ms^{-1})$ and $\ddot{s} (ms^{-2})$ versus time (s).

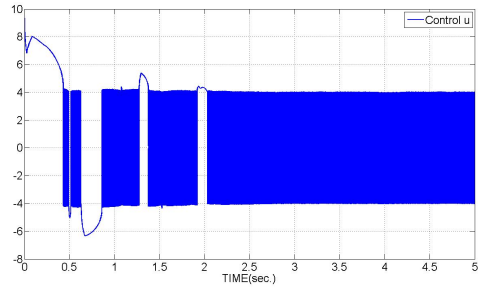


Figure 3: control law u versus time (s).

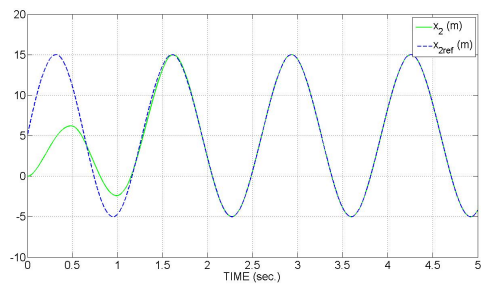


Figure 4: x_2 (m) and x_{2ref} (m) versus time (s)

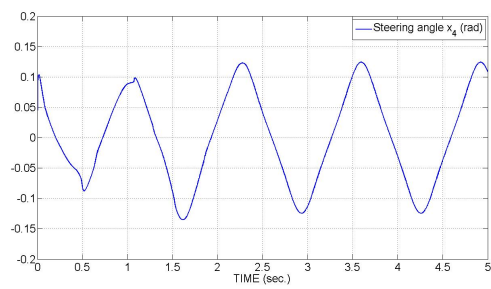


Figure 5: Steering angle x_4 (rad) versus time (s)

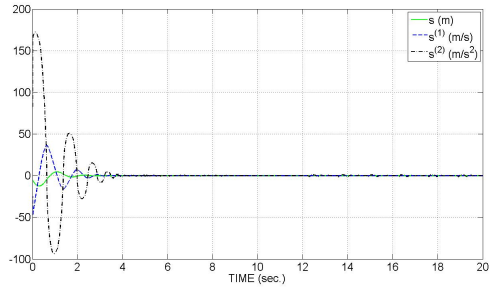


Figure 6: $s(m)$, $\dot{s} (ms^{-1})$ and $\ddot{s} (ms^{-2})$ versus time (s).

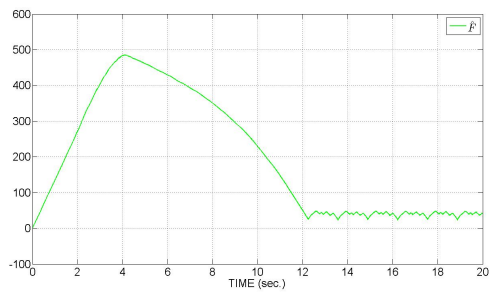


Figure 7: \hat{F} versus time (s).

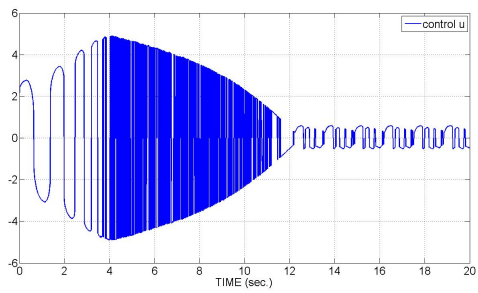


Figure 8: control law u versus time (s).

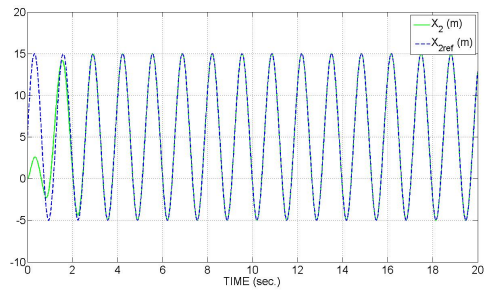


Figure 9: $x_2 (m)$ and $x_{2ref} (m)$ versus time (s)

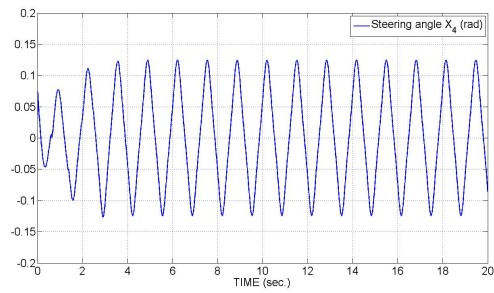


Figure 10: Steering angle $x_4 (rad)$ versus time (s)

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