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# ON SUPERSIMPLE GROUPS

CÉDRIC MILLIET

ABSTRACT. An infinite group with supersimple theory has a finite series of definable groups whose factors are infinite and either virtually- $FC$  or virtually-simple modulo a finite  $FC$ -centre. We deduce that a group which is type-definable in a supersimple theory has a finite series of relatively definable groups whose factors are either abelian or simple groups. In this decomposition, the non-abelian simple factors are unique up to isomorphism.

## 1. INTRODUCTION

Model theory is the study of definable sets. It can be thought as a generalisation of algebraic geometry where the objects under study are algebraic varieties : sets defined by systems of polynomial equations. The existence of a notion of dimension on a class of definable sets strongly restricts the behavior of this class. In a linear algebraic group over an algebraically closed field, the Zariski dimension of an algebraic variety arises from the Zariski topology. A *supersimple group* is a group whose definable sets are equipped with a notion of dimension, the so-called  $SU$ -rank, arising from the logic topology and taking ordinal values. The  $SU$ -rank generalises the Lascar  $U$ -rank of superstable groups, and the Morley rank of groups of finite Morley rank. It is thus a far reaching generalisation of the Zariski dimension for algebraic groups. Example of supersimple groups include :

- Finite groups.
- Linear algebraic groups over algebraically closed fields.
- Groups of finite Morley rank.
- Abelian groups which are divisible, or of bounded exponent.
- $\aleph_0$ -stable groups
- Superstable groups.
- Simple linear algebraic groups over pseudofinite fields.
- Simple pseudofinite groups.

On the other hand, the infinite cyclic group  $\mathbf{Z}$ , non-abelian free groups on  $n$  generators, and more generally non elementary hyperbolic groups are not supersimple (see Corollary 4.10).

In spite of the strong analogy between  $U$ -rank and  $SU$ -rank, the theory of supersimple groups is not nearly as developed as it is for its superstable analogues. For instance, Berline and Lascar [2] have shown every superstable group to have a definable abelian subgroup of the same cardinality. The original argument has

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been much simplified since (see [26, Remark 5.4.11]). Still, it is unknown whether a supersimple group has an infinite abelian subgroup. An abelian subgroup of a (super)simple group is a subgroup of a definable finite-by-abelian one (see [5, Elwes Jaligot Macpherson Ryten] or [15, Milliet]), so this is equivalent with asking whether a supersimple group of rank  $\omega^\alpha \cdot n + \beta$  has a definable finite-by-abelian subgroup of rank at least  $\omega^\alpha$ . Finite-by-abelian groups are *FC-groups*, i.e. groups whose conjugacy classes are finite. In [1], Baudisch has shown every superstable group  $G$  to have a finite series with abelian or simple factors. Our main result is:

**Theorem 1.1.**  *$G$  is an infinite supersimple group. There is a finite chain of definable subgroups  $1 = H_0 \triangleleft H_1 \triangleleft \dots \triangleleft H_n = G$  such that every quotient  $H_{i+1}/H_i$  is infinite and either virtually-FC or virtually-simple modulo a finite FC-centre.*

Consequently, Baudisch's result can be extended to supersimple groups. His arguments however do not adapt to the supersimple context though, mainly for two reasons: the proof is based on a transfinite induction on the rank of  $G$ , the induction basis being Berline and Lascar's result that a superstable group of rank 1 is virtually abelian; in a supersimple group, there is a notion of connected component, but it depends heavily on the specific parameter set over which it is defined. So we have to do without Berline and Lascar and without connected components.

To replace Berline and Lascar, the basic idea is provided by [26, Remark 5.4.11]: a supersimple group of rank one either has an infinite abelian subgroup or is virtually-simple modulo a finite FC-centre; this makes a transfinite induction still possible. To avoid the use of connected component, we study just-infinite supersimple groups. These turn out to be either virtually FC or virtually simple.

Important tools are also Lascar's additive properties of the rank, Wagner's version of Zilber's indecomposability Theorem for supersimple groups, and the observation that the FC-centre of a supersimple group is definable. It should be mentioned that these methods mainly come from Berline and Lascar's investigation of superstable groups [2], later extended by Wagner [26] to supersimple groups.

The main theorem has several consequences. We outline three of them. Sela has shown recently that a torsion-free hyperbolic group is stable. With a theorem of Delzant [4], we derive:

**Corollary 1.2.** *A (non virtually-cyclic) hyperbolic group is not supersimple.*

Jaligot, Elwes, Macpherson, and Ryten have recently proved that the soluble radical of a supersimple group of finite rank such that  $G^{eq}$  eliminates  $\exists^\infty$  is soluble and definable [5]. This has been generalised for an arbitrary supersimple groups in [17]. A group  $G$  is said to be *type-definable* if it is defined by a conjunction of infinitely many formulae. A *relatively definable* subgroup of  $G$  is the trace over  $G$  of a definable set. We show the following:

**Corollary 1.3.** *The soluble radical of a type-definable supersimple group is solvable and relatively definable.*

Pseudofinite supersimple groups of finite rank have been investigated in [5]. Pseudofinite fields are supersimple of rank one, and pseudofinite simple groups are supersimple of finite rank by [28]. Using the work of Wilson [28] and Ryten [20] on pseudofinite simple groups:

**Corollary 1.4.** *A pseudofinite supersimple group either interprets a pseudofinite field or is virtually-solvable.*

## 2. PRELIMINARIES

If  $G$  is a group and  $x$  an element of  $G$ , we write  $x^G$  for the conjugacy class  $\{g^{-1}xg : g \in G\}$  of  $x$ , and  $C(x)$  for its centraliser  $\{g \in G : g^{-1}xg = x\}$  in  $G$ . If  $y$  is another element of  $G$ , we write  $[x, y]$  for the commutator  $x^{-1}y^{-1}xy$ . The element  $x$  is said to be *FC* in  $G$  if  $x^G$  is finite. The *FC-centre* of  $G$  is the subgroup of every of its *FC* elements. We write it  $FC(G)$ . We say that  $G$  is an *FC-group* if every of its elements is *FC* in  $G$ .

Two subgroups of a given group  $G$  are *commensurable* if the index of their intersection is finite in both of them. The following theorem comes originally from [22], with some precisions coming from [26, Theorem 4.2.4] and its proof.

**Fact 2.1** (Schlichting [22]).  *$G$  is group and  $H$  a subgroup of  $G$  such that  $H/H \cap H^g$  remains finite and bounded by a natural number for all  $g$  in  $G$ . There exists a normal subgroup  $N$  of  $G$  such that  $H/H \cap N$  and  $N/N \cap H$  are finite. Moreover,  $N$  is a finite extension of a finite intersection of  $G$ -conjugates of  $H$ . In particular, if  $H$  is definable then so is  $N$ . More precisely, there are four elements  $g_1, \dots, g_4$  in  $G$  such that  $N$  is contained in  $H^{g_1}H^{g_2}H^{g_3}H^{g_4}$ .*

We recall the properties of a supersimple group that will be needed in the sequel. A *supersimple group*  $G$  is a group equipped with a rank function taking values in the ordinals, and ranking any definable subset  $X$  of  $G$ . We write  $SU(X)$  for the rank of  $X$ . We refer to [26, Definition 5.1.1] for the precise definition of this rank and more generally to [26, Chapter 5] for precisions on supersimple theories. We shall try to avoid technicalities and use only basic properties of this rank that we recall now.

- (1) *Monotonicity.* The rank is increasing : if  $X \subset Y$  are two definable subsets of  $G$ , then  $SU(X)$  is smaller or equal to  $SU(Y)$ .
- (2) *Definable invariance.* The rank is preserved by definable isomorphisms : if there is a definable isomorphism between two definable sets  $X$  and  $Y$ , then  $SU(X)$  equals  $SU(Y)$ .
- (3) *Interpretation and additivity.* If  $G$  is supersimple, then so is  $G^{eq}$ , meaning that any quotient  $G/H$  of left (or right) cosets of a definable subgroup  $H$  has an ordinal  $SU$ -rank. The rank  $SU(G/H)$  can be controlled in terms of  $SU(H)$  and  $SU(G)$  by Lascar inequalities (see Fact 2.2).
- (4) *Rank zero sets.* A definable set (in  $G^{eq}$ ) has rank zero if and only if it is finite. In particular, if  $H$  is a definable subgroup of  $G$ , then  $SU(G/H)$  equals zero if and only if  $H$  has finite index in  $G$ .
- (5) *Preservation by elementary extension.* If  $G$  is supersimple, then so is any of its elementary extension.

We shall use a little ordinal arithmetic : any ordinal  $\alpha$  decomposes in base  $\omega$ , which means that there are unique ordinals  $\alpha_1 > \dots > \alpha_n$  and non-zero natural numbers  $k_1, \dots, k_n$  such that  $\alpha$  equals  $\omega^{\alpha_1} \cdot k_1 + \dots + \omega^{\alpha_n} \cdot k_n$ . If  $\alpha$  and  $\beta$  are two ordinals, we may assume that  $\alpha$  equals  $\omega^{\alpha_1} \cdot k_1 + \dots + \omega^{\alpha_n} \cdot k_n$  and  $\beta$  equals  $\omega^{\alpha_1} \cdot \ell_1 + \dots + \omega^{\alpha_n} \cdot \ell_n$  for the same  $\alpha_1, \dots, \alpha_n$ , adding some additional possibly zero  $k_i$  and  $\ell_i$  if necessary.

We write  $\alpha \oplus \beta$  for their *Cantor sum* defined by

$$\alpha \oplus \beta = \omega^{\alpha_1} \cdot (k_1 + \ell_1) + \cdots + \omega^{\alpha_n} \cdot (k_n + \ell_n)$$

We say that an ordinal is *monomial* if it is of the form  $\omega^\alpha \cdot n$ . Note that for monomials, the two operations  $+$  and  $\oplus$  coincide.

The following inequalities are due to Lascar and their analogue in the supersimple context can be found in [26, Theorem 5.1.6].

**Fact 2.2** (Lascar inequalities).  *$G$  is a supersimple group, and  $H$  a definable subgroup of  $G$ . Then*

$$SU(H) + SU(G/H) \leq SU(G) \leq SU(H) \oplus SU(G/H)$$

The following Lemma will be much used in the sequel :

**Fact 2.3** ([16, 18]). *The FC-centre of a supersimple group is definable and has a finite derived subgroup.*

We say that a group is *FC-soluble* if it has a finite series  $1 \triangleleft \cdots \triangleleft G$  whose factors are FC-groups. Soluble-by-finite groups are FC-soluble groups. In a supersimple theory, the converse is also true :

**Fact 2.4** (Milliet [16]). *In a supersimple group, any FC-soluble subgroup has a soluble subgroup of finite index. Any definable FC-soluble subgroup has a definable soluble subgroup of finite index.*

The following Proposition is the definable version of [25, Corollary 4.2] which is stated for hyperdefinable (hence type-definable) groups. As we could not find any precise reference, we give a proof here for the sake of completeness.

**Fact 2.5** (Wagner [25, Corollary 4.2]).  *$G$  is a supersimple group of rank  $\omega^{\alpha_1} \cdot k_1 + \cdots + \omega^{\alpha_n} \cdot k_n$  with  $\alpha_1 > \cdots > \alpha_n$ . For every natural number  $i$  such that  $1 \leq i \leq n$ , there is a definable normal subgroup  $H$  of  $G$  of rank  $\omega^{\alpha_1} \cdot k_1 + \cdots + \omega^{\alpha_i} \cdot k_i$ . The group  $H$  is unique up to commensurability.*

*Proof.* We may assume that  $G$  is saturated and we write  $\beta_i$  for  $\omega^{\alpha_1} \cdot k_1 + \cdots + \omega^{\alpha_i} \cdot k_i$ . By [25, Corollary 4.2], there is a type-definable normal subgroup  $H$  of  $G$  having rank  $\beta_i$ . Recall that  $\beta_i$  is by definition the rank of any of the generic types of  $H$ . By [25, Theorem 4.4], the group  $H$  is the conjunction of definable groups  $H_i$  for  $i$  in  $I$ . We may close this family by finite intersections, remove the members that do not have minimal rank and assume that every  $H_i$  has rank  $\beta$  say and are commensurable. It follows that for every  $i$ , the group  $H$  has bounded index in  $H_i$  so  $H$  is a generic type of  $H_i$  by [26, Lemma 4.1.15]. Thus  $\beta$  equals  $\beta_i$ . Take any  $H_i$ . As  $H$  is normal in  $G$ , the groups  $H_i^g$  and  $H_i$  are commensurable for every  $g$  in  $G$ . By compactness and saturation, the cardinal of  $H_i/H_i \cap H_i^g$  must remain bounded by some natural number when  $g$  ranges over  $G$ . By Theorem 2.1, there is a definable normal subgroup  $N$  of  $G$  commensurable with  $H$  hence of rank  $\beta_i$ . If  $K$  is another definable group satisfying the desired requirements, then  $K/K \cap N$  and  $N/K \cap N$  are bounded by [25, Corollary 4.2]. By compactness,  $K$  and  $N$  must be commensurable.  $\square$

## 3. A JORDAN-HÖLDER LIKE DECOMPOSITION

**Lemma 3.1.**  *$G$  is an infinite supersimple group.  $G$  has a definable series  $1 = H_0 \triangleleft \cdots \triangleleft H_n = G$  such that  $H_{i+1}/H_i$  has non-zero monomial rank  $\omega^{\alpha_i}.k_i$  and every definable normal subgroup of  $H_{i+1}/H_i$  has rank either  $\omega^{\alpha_i}.k_i$  or strictly less than  $\omega^{\alpha_i}$ .*

*Proof.* We proceed by induction on the rank of  $G$ .

If  $G$  has rank 1, we take the series  $1 \triangleleft G$ .

Assume that  $G$  has rank  $\alpha$  and that the Lemma is proved for every group of rank less than  $\alpha$ .

If  $\alpha$  is monomial, say  $\omega^\beta.n$ . Either the chain  $1 \triangleleft G$  satisfies the requirement of Lemma 3.1, or there is a definable normal subgroup  $H$  of rank  $\omega^\beta.k + \gamma$  with  $1 \leq k < n$  and  $\gamma < \omega^\beta$ . By Lascar inequalities, both  $H$  and  $G/H$  have rank strictly less than  $\alpha$ . We may apply the induction hypothesis to  $H$  and  $G/H$  and find two series  $1 = H_0 \triangleleft \cdots \triangleleft H_n = H$  and  $1 = H_n/H \triangleleft \cdots \triangleleft H_{n+r}/H = G/H$  with the required properties. As  $H_{n+i+1}/H/H_{n+i}/H$  is definably isomorphic to  $H_{n+i+1}/H_{n+i}$ , they must have same rank ; moreover their definable normal subgroups are in a (rank preserving) one-to-one definable corespondance, so that the series  $1 = H_0 \triangleleft \cdots \triangleleft H_{n+r} = G$  fullfills our purpose.

If  $\alpha$  is not monomial, it equals  $\omega^\beta.n + \gamma$  with  $0 < \gamma < \omega^\beta$ . By Proposition 2.5, the group  $G$  has a definable normal subgroup  $H$  of rank  $\omega^\beta.n$ . By Lascar inequalities, both  $H$  and  $G/H$  have rank strictly less than  $\alpha$  so we conclude again by induction hypothesis as in the monomial case.  $\square$

For the next Lemma, we shall need Zilber's Indecomposibility Theorem formulated by Wagner for hyperdefinable groups in a supersimple theory (see [26, Theorem 5.4.5]). We give the appropriate version for definable groups here. Again, we could not find a precise reference for this particular version, so we provide a proof bellow.

**Fact 3.2** (Wagner's version of Zilber's Indecomposibility Theorem).  *$G$  is a supersimple group of rank strictly less than  $\omega^{\alpha+1}$  and  $\mathfrak{X}$  a collection of definable subsets of  $G$ . There is a definable subgroup  $H$  of  $G$  contained in  $X_1^{\pm 1} \cdots X_n^{\pm 1}$  for some  $X_1, \dots, X_n$  in  $\mathfrak{X}$  such that  $SU(XH) < SU(H) + \omega^\alpha$  for all  $X$  in  $\mathfrak{X}$ . Moreover, one has  $H^g = H$  for every  $g$  in  $G$  stabilising  $\mathfrak{X}$  setwise by conjugation.*

*Proof.* By [26, Theorem 5.4.5], there is a type-definable subgroup  $T$  of  $G$  contained in  $X_1^{\pm 1} \cdots X_n^{\pm 1}$  for some  $X_1, \dots, X_n$  in  $\mathfrak{X}$  such that  $SU(XT) < SU(T) + \omega^\alpha$  for all  $X$  in  $\mathfrak{X}$  with  $T$  normalised by every  $g$  in  $G$  stabilising  $\mathfrak{X}$  setwise. By [26, Corollary 5.5.4] and compactness, the group  $T$  is contained in a definable subgroup  $H$  of  $G$  having same rank as  $T$  such that  $H$  is contained in  $X_1^{\pm 1} \cdots X_n^{\pm 1}$ . Let  $A$  be the subset of  $G$  stabilising  $\mathfrak{X}$  setwise by conjugation. As  $H$  contains  $T$  and because  $T$  is invariant under conjugation by elements of  $A$ , the set of  $A$ -conjugates of  $H$  consists of commensurable uniformly definable subgroups : they must be uniformly commensurable. By Theorem 2.1, there is a definable subgroup  $N$  of  $G$  commensurable with  $H$  wich is invariant under conjugation by  $A$ , and contained in a finite product of elements of  $\mathfrak{X} \cup \mathfrak{X}^{-1}$ . As  $N/T \cap N$  and  $T/T \cap N$  are bounded, the groups  $T$  and  $N$  have the same  $SU$  rank by [26, Lemma 4.1.15]. It follows that

$SU(XT) = SU(XN)$  for any  $X$  in  $\mathfrak{X}$ , so that the inequality  $SU(XN) < SU(N) + \omega^\alpha$  holds.  $\square$

Let  $G$  be a group. We say that  $G$  is *virtually- $P$*  if it has a definable normal subgroup of finite index with property  $P$ . We say that  $G$  is *finite-by- $P$*  if it has a finite normal subgroup  $N$  such that  $G/N$  has property  $P$ . We say that  $G$  is *finite-by- $P$ -by-finite* if  $G$  has a series  $1 \triangleleft N \triangleleft H \triangleleft G$  such that  $N$  and  $G/H$  are finite and  $H/N$  has property  $P$ .

**Lemma 3.3.**  *$G$  is a supersimple group which is not virtually-FC. If every non-trivial definable normal subgroup of  $G$  has finite index, then every normal subgroup of  $G$  is definable (hence either trivial or of finite index).*

*Proof.*  $G$  must have monomial rank  $\omega^\alpha \cdot n$  for otherwise Proposition 2.5 together with Lascar inequalities would yield a normal definable subgroup of infinite index. If some  $g^G$  has rank strictly less than  $\omega^\alpha$ , then so does  $G/C(g)$ . By Lascar inequalities,  $G/C(g)$  has rank zero so  $C(g)$  has finite index in  $G$ . If  $g \neq 1$ , then  $FC(G)$  contains  $g$  hence is a non-trivial normal subgroup. It is definable by Lemma 2.3 so it must have finite index in  $G$ . It follows that  $G$  is virtually-FC, a contradiction. So every non-trivial conjugacy class has rank at least  $\omega^\alpha$ . If  $N$  is a normal proper subgroup of  $G$  containing some  $n \neq 1$ , by Theorem 3.2 there is definable normal subgroup  $H$  with  $H \leq \langle n^G \rangle \leq N$  and  $SU(n^G H) < SU(H) + \omega^\alpha$ . As  $SU(n^G) \geq \omega^\alpha$ , the group  $H$  is non-trivial. It follows that  $H$  has finite index in  $G$  so  $N$  is a finite union of cosets of  $H$ : it is definable.  $\square$

**Definition 3.4** (McCarthy, Magnus). A group is *just-infinite* if it is infinite and every of its normal subgroup is either trivial or has finite index.

**Lemma 3.5.**  *$G$  is a supersimple group. If  $G$  is just-infinite, it is either virtually-FC or virtually-simple.*

*Proof.* Let us suppose that  $G$  is not virtually-FC. Let  $G^0$  be the intersection of every definable subgroup of  $G$  of finite index. It is a normal subgroup of  $G$  hence either trivial or of finite index. If  $G^0$  has finite index, it is simple by Lemma 3.3. Suppose for a contradiction that  $G^0$  is trivial. We may use [27, Wagner]:  $G$  is residually finite hence virtually-nilpotent, so has a minimal nilpotent subgroup  $N$  of finite index, which is a characteristic subgroup.  $N'$  is a proper normal subgroup of  $G$  and equals  $\{1\}$ : the group  $G$  is virtually abelian, a contradiction. We may also use a compactness argument: by Lemma 3.3, being just-infinite and being virtually-simple are properties of the theory of  $G$  so we may assume that  $G$  is  $\aleph_0$ -saturated. There is a countable chain of proper normal subgroup having finite index whose intersection is normal hence trivial. This contradicts compactness and saturation.  $\square$

**Lemma 3.6.**  *$G$  is a supersimple group of monomial rank  $\omega^\alpha \cdot n$ . Assume that every definable normal subgroup of  $G$  has rank either  $\omega^\alpha \cdot n$  or strictly less than  $\omega^\alpha$ . Then the quotient  $G/FC(G)$  is either finite or just-infinite.*

*Proof.* Let  $H$  be a definable normal subgroup of  $G$  having rank less than  $\omega^\alpha$ , and  $h$  in  $H$ . Then  $H$  contains  $h^G$  so  $SU(h^G) < \omega^\alpha$ , hence  $SU(G/C(h)) < \omega^\alpha$ . By Lascar inequalities,  $G/C(h)$  is finite. It follows that  $FC(G)$  contains any normal

subgroup having rank less than  $\omega^\alpha$ . If  $FC(G)$  has maximum  $SU$ -rank,  $G/FC(G)$  is finite. So we may assume that  $FC(G)$  has rank less than  $\omega^\alpha$ , so that  $G/FC(G)$  has maximal rank  $\omega^\alpha.n$ . Note that if  $H/FC(G)$  is a definable normal subgroup of  $G/FC(G)$ , then  $H$  is a definable normal subgroup of  $G$ . Thus every definable normal subgroup of  $G/FC(G)$  must be either  $\{1\}$  or have maximal rank hence be of finite index. The group  $G/FC(G)$  is just-infinite by Lemma 3.3.  $\square$

**Theorem 3.7.**  *$G$  is an infinite supersimple group. There is a finite chain of definable subgroups  $1 = H_0 \triangleleft H_1 \triangleleft \cdots \triangleleft H_n = G$  such that every quotient  $H_{i+1}/H_i$  is infinite and either virtually- $FC$  or virtually-simple modulo a finite  $FC$ -centre.*

*Proof.* Lemma 3.1 provides a series  $1 = H_0 \triangleleft H_1 \triangleleft \cdots \triangleleft H_n = G$ . Lemma 3.6 and 3.5 ensure that there is a refinement of the series whose factors have the required properties.  $\square$

#### 4. COROLLARIES

A supersimple  $FC$  group is finite-by-abelian according to Lemma 2.3. As a finite group  $F$  decomposes into a finite series  $1 = F_0 \triangleleft F_1 \triangleleft \cdots \triangleleft F_n = F$  whose factors are abelian or simple, we may state:

**Corollary 4.1.**  *$G$  is a supersimple group. There is a finite series of definable subgroups  $1 = H_0 \triangleleft H_1 \triangleleft \cdots \triangleleft H_n = G$  such that every factor  $H_{i+1}/H_i$  is either abelian or simple. Moreover, in this decomposition, the non-abelian simple factors (in particular the infinite simple ones) are unique up to definable group isomorphism.*

*Proof.* We need just a proof that the non-abelian simple factors are unique. Let be two series for  $G$  with abelian and simple factors. By Schreier refinement Theorem, in any two normal series of  $G$ , intermediate subgroups can be inserted to yield two new normal series for the group  $G$  whose factor groups coincide up to group isomorphism, counting multiplicity. As neither an abelian group nor a simple one has a non-abelian simple proper factor, the non-abelian simple factors occurring in the two new normal series must already occur in the original ones.  $\square$

Theorem 3.7 has many consequences, about soluble subnormal subgroups, residual properties or pseudofinite groups. We discuss them here.

##### 4.1. About soluble subgroups and local properties.

**Corollary 4.2.**  *$G$  is a supersimple group. Either  $G$  interprets an infinite simple group, or  $G$  is virtually soluble.*

*Proof.* If  $G$  does not interpret an infinite simple group, then  $G$  has a series whose factors are finite or abelian. So  $G$  has a series with  $FC$  factors : it is  $FC$ -soluble. By Theorem 2.4,  $G$  is virtually soluble.  $\square$

Let  $G$  be any group with a series  $1 = G_0 \triangleleft G_1 \triangleleft \cdots \triangleleft G_n = G$ . If  $N$  is a normal subgroup of  $G$ , it has a series

$$1 = G_0 \cap N \triangleleft G_1 \cap N \triangleleft \cdots \triangleleft G_n \cap N = N$$

where each factor  $G_{i+1} \cap N / G_i \cap N$  is isomorphic to  $(G_{i+1} \cap N)G_i / G_i$  hence can be considered as normal subgroups of  $G_{i+1}/G_i$ . We say that  $H$  is a *subnormal*



subgroup of  $G$  if there is a finite series  $H \triangleleft \cdots \triangleleft G$ . For a subnormal subgroup  $H$  of  $G$ , an immediate induction yields a series (of length no greater than  $n$ ) whose factors are isomorphic to subnormal factors of the series for  $G$ . In particular:

**Corollary 4.3.** *In a supersimple group, any subnormal subgroup  $S$  (not necessarily definable) has a series  $1 = H_0 \triangleleft H_1 \triangleleft \cdots \triangleleft H_n = S$  of bounded length with abelian or simple factors.*

**Corollary 4.4.** *In a supersimple group, there is a natural number  $n$  bounding the derived length of any subnormal soluble subgroup.*

**Question 1.** In a supersimple group, or even a superstable one, is there a bound on the derived length of any soluble subgroup? At least on the number of infinite abelian factors? For that it is equivalent to deal with the case where the ambient group is infinite, simple of monomial rank.

**Corollary 4.5.** *A locally soluble subnormal subgroup of a supersimple group is soluble.*

*Proof.* By [21], every locally soluble simple group is finite hence soluble. Corollary 4.3 concludes.  $\square$

We call the *soluble radical* of a group, the subgroup generated by all normal soluble subgroups. It is a locally soluble characteristic subgroup. From Corollary 4.5, it follows that the soluble radical of a supersimple group must be soluble. By an observation of Ould Houcine [12], the soluble radical of any group is definable provided that it is soluble. This shows:

**Corollary 4.6.** *The soluble radical of a supersimple group  $G$  is soluble and definable.*

Wagner and Evans [6] have shown that a supersimple  $\aleph_0$ -categorical group is finite-by-abelian-by-finite. A simple  $\aleph_0$ -categorical group is nilpotent-by-finite [26, Wagner]. Note that  $\aleph_0$ -categorical groups are locally finite groups of bounded exponent. In the same vein :

**Corollary 4.7.** *A locally finite supersimple group of bounded exponent is soluble-by-finite.*

*Proof.* Same argument as Felgner in [7] : by an inspection of the classification of finite simple groups, and a result of Kargapolov [14].  $\square$

**Question 2.** Is a locally finite supersimple group soluble-by-finite? For that it is equivalent to ask whether a supersimple locally finite simple group is finite. Recall from [10] that in an infinite locally finite simple group, the centraliser of every element is infinite.

**4.2. About free groups, hyperbolic groups and residual properties.** Non-abelian free groups are not superstable [19, Gibone]. Sela has shown that free groups, and more generally torsion-free hyperbolic groups are stable [23]. By a Theorem of Magnus, an early result of combinatorial group theory, free groups are residually-nilpotent (see [9, chapter 11]). Wagner [27] has shown that a supersimple group which is residually soluble is in fact soluble. It follows that non-abelian free

groups are not supersimple. This can also be derived directly from Corollary 4.1 following Poizat's argument in [1].

**Corollary 4.8.** *A non-abelian free group is not supersimple.*

*Proof.* Suppose for a contradiction that  $F$  is a supersimple non-abelian free group on  $r > 1$  generators. By Corollary 4.1,  $F$  has a series of bounded length  $n$  with abelian or simple factors. So does every quotient of  $F$  by a normal subgroup. In particular, any soluble group generated by  $r$  elements is isomorphic to a quotient of  $F$  and would have derived length bounded by  $n$ .  $\square$

Let  $G$  be a (non virtually-cyclic) hyperbolic group. A theorem of Gromov states that if  $g$  and  $h$  are non commuting elements of  $G$ , then  $g^n$  and  $h^m$  generate a non-abelian free subgroup on 2 generators for sufficiently large natural numbers  $n$  and  $m$  (see [8, Théorème 37 p. 157]). This was generalised by Delzant [4] : if  $g$  is in  $G$ , we write  $\ell(g)$  for the length of  $g$  and  $n(g)$  for the limit of  $\ell(g^n)/n$  when  $n$  goes to infinity. This can be shown to be a natural number.

**Fact 4.9** (Delzant [4]).  *$G$  is a (non virtually cyclic)  $\delta$ -hyperbolic group. There is a natural number  $m$  such that for all  $g_1, \dots, g_n$  in  $G$  with  $n(g_i) = n(g_j) \geq 1000\delta$ , the normal subgroup generated by  $g_1^n, \dots, g_m^n$  is free.*

**Corollary 4.10.** *A (non virtually cyclic) hyperbolic group is not supersimple.*

Let  $\mathcal{C}$  be a *pseudo-variety* of groups *i.e.* a class of groups closed under taking subgroups, quotients and finite cartesian products. A group  $G$  is said to be *residually*  $\mathcal{C}$  if

$$\bigcap_{\substack{H \triangleleft G \\ G/H \in \mathcal{C}}} H = \{1\}$$

It was shown in [27] that a supersimple group which is residually  $\mathcal{C}$  has series  $G = G_0 \triangleright \dots \triangleright G_n$  such that every factor  $G_i/G_{i+1}$  is in  $\mathcal{C}$  and  $G_n$  is nilpotent. In the same spirit :

**Corollary 4.11.**  *$\mathcal{C}$  is a pseudo-variety of groups that does not contain an infinite simple group. A residually  $\mathcal{C}$  supersimple group is virtually soluble. In particular, a residually soluble supersimple group is soluble (Wagner).*

*Proof.* Let  $G$  be residually  $\mathcal{C}$  and supersimple. By 4.6, the soluble radical  $R$  of  $G$  is soluble and definable. It may not be equationnally definable, so we may not apply [11, Lemma 2.2.2], but it is *equationnally type-definable*, that is to say, definable by the infinite set of equations  $f(x, g_1, \dots, g_{2^n}) = 1$  where the  $g_i$  range over all elements of  $G$  and where  $f(x, g_1, \dots, g_{2^n})$  is the word defined inductively on  $n \geq 1$  by

$$f(x, g_1, g_2) = [x^{g_1}, x^{g_2}] \quad \text{and}$$

$$f(x, g_1, g_2, \dots, g_{2^{n+1}}) = [f(x, g_1, g_2, \dots, g_{2^n}), f(x, g_{2^n+1}, g_{2^n+2}, \dots, g_{2^{n+1}})]$$

By [27, Lemma 1], the quotient  $G/R$  is residually  $\mathcal{C}$ . If  $G/R$  is infinite, by Corollary 4.1, it has a finite series  $1 = G_0/R \triangleleft \dots \triangleleft G_n/R = G/R$  such that  $G_1/R$  is infinite and either finite-by-simple, or finite-by-abelian. The first case is impossible as  $\mathcal{C}$  does not contain any infinite simple group. In the second case, there is a finite  $F$  extension of  $R$  such that  $G_1/F$  is abelian, so  $G_1/F$  is a subnormal abelian

subgroup of  $G/F$ , and is contained in the Baer radical. The Baer radical of a group is generated by every subnormal abelian subgroup. It is a characteristic locally nilpotent group. By Corollary 4.5,  $G_1/F$  is contained in a normal soluble subgroup  $S/F$  of  $G/F$ . As  $F$  is soluble-by-finite,  $S$  is  $FC$ -soluble hence soluble-by-finite by Theorem 2.4. It must have a maximal soluble subgroup of finite index which is characteristic in  $S$  hence normal in  $G$ . But  $S/R$  is infinite : this contradicts the maximality of  $R$ . It follows that  $G/R$  is finite.  $\square$

**4.3. About pseudofinite groups.** Wilson has shown in [28] that every pseudofinite simple group is elementary equivalent to a Chevalley group over a pseudofinite field. As pseudofinite fields are supersimple (see [13]) of rank 1 (see [3]), a pseudofinite simple group is supersimple of finite rank.

**Corollary 4.12.** *A pseudofinite supersimple group has a finite series whose factors are abelian, finite simple groups, or elementary equivalent to Chevalley groups over pseudofinite fields. In particular, if a pseudofinite supersimple group does not interpret a pseudofinite field, it must be virtually soluble.*

*Proof.* If a pseudofinite supersimple group  $G$  interprets an infinite simple group  $S$ , this must be a Chevalley group over a pseudofinite field  $F$ . By [20],  $F$  is interpretable in  $G$ . Otherwise,  $G$  is virtually soluble by Corollary 4.2.  $\square$

In [5] Elwes Jaligot Macpherson and Ryten have shown that a supersimple pseudofinite group of rank 2 is virtually soluble provided that  $T^{eq}$  eliminates  $\exists^\infty$ . With this assumption, they also showed using [28] that there are no supersimple pseudofinite simple groups of rank less than 3, and that one of rank 3 is elementary equivalent to  $PSL_2(F)$  for some pseudofinite field  $F$ .

**Corollary 4.13.** *A pseudofinite supersimple group  $G$  of rank 3 such that  $G^{eq}$  eliminates  $\exists^\infty$  is either virtually soluble or has a series of definable subgroups  $1 \triangleleft F \triangleleft S \triangleleft G$  such that  $F$  and  $G/S$  are finite and  $S/F$  is elementary equivalent to  $PSL_2(F)$  for some pseudofinite field  $F$ .*

## 5. TYPE-DEFINABLE VERSION

In a theory  $T$  equipped with a definable binary law  $\times$ , we call a *type-definable group* a partial type  $\pi$  in  $n$  variables such that for all model  $M$  of  $T$  the set of realisations of  $\pi$  in in the cartesian product  $M^n$  has a group structure for  $\times$ . In a saturated structure, we assimilate the type  $\pi$  and the group that it defines. We now extend the main results of the previous section to type-definable groups in a supersimple theory. We first recall :

**Fact 5.1** (Wagner [26, Theorem 5.5.4]). *In a supersimple theory, a type-definable group is a conjunction of definable groups.*

It is shown in [1, Corollary 1.11] that a type-definable simple group in a superstable theory is definable. We generalise that to supersimple theories :

**Proposition 5.2.** *A simple group  $H$  wich is type-definable in a supersimple theory is definable.*

*Proof.* By [26, Proposition 5.4.9], any elementary extension of  $H$  is still a simple group, so we may assume that  $H$  is infinite and that the ambient structure is saturated. By Theorem 5.1, the type-definable group  $H$  is contained in a definable one  $G$  that we may take of minimal rank possible, so that  $G/H$  is bounded. It follows that  $H$  is normal in  $G$ . Note that both  $G$  and  $H$  have monomial rank say  $\omega^\alpha \cdot n$ . Let  $h \neq 1$  be in  $H$ . If  $h^G$  has rank strictly less than  $\omega^\alpha$ , then  $C(h)$  has finite index in  $G$  so  $h$  is in  $FC(H)$  a contradiction with  $H$  being simple. So  $h^G$  has rank at least  $\omega^\alpha$ . By Theorem 3.2 there is a definable normal subgroup  $N$  of  $G$  with  $N \leq \langle h^G \rangle \leq H$  and  $SU(h^G N) < SU(N) + \omega^\alpha$ . As  $SU(h^G) \geq \omega^\alpha$ , the group  $N$  is non trivial and must equal  $H$  by simplicity.  $\square$

Let  $G$  be a type-definable group and  $H$  a type-definable subgroup of  $G$ . We say that  $H$  is *relatively definable in  $G$*  if it is the trace on  $G$  of definable group. We now extend Theorem 3.7 to type-definable groups :

**Theorem 5.3.**  *$G$  is a type-definable supersimple group. There is a finite chain of relatively definable subgroups  $1 = G_0 \triangleleft G_1 \triangleleft \dots \triangleleft G_n \triangleleft G$  such every quotient  $G_{i+1}/G_i$  is either abelian or simple. Moreover, the non-abelian simple factors are unique up to isomorphism.*

*Proof.* We may add parameters to the language and assume that  $G$  is defined with no parameters. We also may assume that the structure is  $\aleph_0$ -saturated. By Theorem 5.1, the group  $G$  is contained in a  $\emptyset$ -definable group  $H$ . We may assume that  $H$  has minimal rank so that  $H/G$  is bounded. Let  $H^0$  be the connected component over  $\emptyset$ , that is, the intersection of every  $\emptyset$ -definable subgroup of finite index. It is a normal subgroup of  $H$  contained in  $G$ . Let  $1 = H_0 \triangleleft H_1 \triangleleft \dots \triangleleft H_n = H$  be a definable series for  $H$  provided by Theorem 3.7. As  $H_{i+1} \cap H^0 / H_i \cap H^0$  is isomorphic to  $(H^0 \cap H_{i+1})H_i / H_i$ , the groups  $H_{i+1} \cap H^0 / H_i \cap H^0$  can be considered as normal subgroup of  $H_{i+1}/H_i$  : they are again abelian or simple. The series  $1 = H_0 \triangleleft H_1 \cap H^0 \triangleleft \dots \triangleleft H_{n-1} \cap H^0 \triangleleft H^0 \triangleleft G$  has the required properties, but maybe the last factor  $G_n/G$ , which is bounded.

We now show inductively on the length  $n$  of the chain that the previous series can be refined into a new series where every factor is abelian, simple or finite. As a finite group has a series whose factor are either simple or abelian, this is sufficient. If  $n$  is 1, then  $H^0 = \{1\}$  so  $H$  is finite by compactness. For the induction step, note that  $H^0/H^0 \cap H_{n-1}$  is isomorphic to  $H^0.H_{n-1}/H_{n-1}$ , which is either abelian or simple. If it simple, then  $H^0.H_{n-1}$  must be definable by Proposition 5.2 for it is of finite index in a definable group. By compactness, there is a definable normal subgroup  $H^1$  of  $H$  of finite index such that  $H^0.H_{n-1}$  equals  $H^1.H_{n-1}$ . It follows that  $H^1/H^1 \cap H_{n-1}$  is isomorphic to  $H^0/H^0 \cap H_{n-1}$  hence simple. Note that  $H^1 \cap G/H^1 \cap H_{n-1}$  has bounded index in  $H^1/H^1 \cap H_{n-1}$  hence is also simple. As  $H^1 \cap H_{n-1}/H^0 \cap H_{n-1}$  is bounded, the series

$$1 = H_0 \triangleleft H_1 \cap H^0 \triangleleft \dots \triangleleft H_{n-1} \cap H^0 \triangleleft H_{n-1} \cap H^1 \triangleleft H^1 \cap G \triangleleft G$$

can be refined by a series with abelian, simple or finite factors by induction hypothesis. Otherwise,  $H^0/H^0 \cap H_{n-1}$  is abelian. By compactness, there is a definable subgroup  $G_n$  of finite index in  $H$ , such that  $G_n/H^0 \cap H_{n-1}$  is abelian. We may replace  $G_n$  by a finite intersection of  $H$ -conjugates of it and assume that  $G_n$  is

normalised by  $H$ , hence by  $G$ . We consider the series

$$1 = H_0 \triangleleft H_1 \cap H^0 \triangleleft \cdots \triangleleft H_{n-1} \cap H^0 \triangleleft G_n \cap G \triangleleft G$$

where only the two last factor groups  $G_n \cap G / H_{n-1} \cap H^0$  and  $G / G \cap G_n$  have been modified. As a subgroup of  $G_n / H^0 \cap H_{n-1}$ , the group  $G_n \cap G / H_{n-1} \cap H^0$  is abelian. We have just build a series of type-definable groups  $1 = G_0 \triangleleft G_0 \triangleleft \cdots \triangleleft G_n = G$  such that every factor is abelian or simple. By the same type of arguments, one can show inductively that each  $G_i$  may be replaced by one which is relatively definable in  $G_{i+1}$ .  $\square$

**Corollary 5.4.** *In a type-definable supersimple group, there is a natural number  $n$  bounding the derived length of any subnormal soluble subgroup.*

**Corollary 5.5.** *A locally soluble subnormal subgroup of a type-definable supersimple group is soluble.*

**Lemma 5.6.**  *$G$  is any type-definable group. Assume that  $G$  is the conjunction of definable groups  $H_i$  where  $i$  ranges over  $I$ . If the soluble radical of  $G$  is soluble, there is a finite conjunction  $H_J$  of groups  $H_i$  such that*

$$R(G) = R(H_J) \cap G$$

*Proof.* For any finite subset  $J$  of  $I$ , we write  $H_J$  for the intersection of  $H_j$  when  $j$  ranges over  $J$ . For any finite subset  $J$  of  $I$ , if  $x$  is in  $R(H_J) \cap G$ , then  $x^{H_J}$  generates a soluble group so  $x^G$  also generates a soluble group and  $x$  is in  $R(G)$ . Reciprocally, if  $x$  is in  $R(G)$ , then  $x^G$  generates a soluble group of derived length  $n$  say. Note that  $n$  does not depend on  $x$  as  $R(G)$  is soluble. It follows that  $(x^G \cup x^{-G})^{(n)}$  equals  $\{1\}$  where  $X^{(n)}$  is defined inductively by... This means that the partial type  $G \times \cdots \times G$  in  $2^n$  variables  $(x_1, \dots, x_{2^n})$  implies the formula  $f(x, x_1, \dots, x_{2^n}) = 1$  where  $f(x, x_1, \dots, x_{2^n})$  is the word defined inductively on  $n \geq 1$  by

$$f(x, x_1, x_2) = [x^{x_1}, x^{x_2}] \quad \text{and}$$

$$f(x, x_1, x_2, \dots, x_{2^{n+1}}) = [f(x, x_1, x_2, \dots, x_{2^n}), f(x, x_{2^n+1}, x_{2^n+2}, \dots, x_{2^{n+1}})]$$

By compactness there is a finite subset  $J$  of  $I$  such that the same formula

$$f(x, x_1, \dots, x_{2^n}) = 1$$

is implied by the formula

$$(x_1, \dots, x_{2^n}) \in H_J \times \cdots \times H_J$$

This means precisely that  $x^{H_J}$  generates a soluble group of derived length at most  $n$ , hence  $x$  is in  $R(H_J) \cap G$ .  $\square$

**Corollary 5.7.** *The soluble radical of a group  $G$  which is type-definable in a supersimple theory is soluble and relatively definable in  $G$ .*

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