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## DEFINABLE ENVELOPES OF NILPOTENT SUBGROUPS OF GROUPS WITH CHAIN CONDITIONS ON CENTRALIZERS

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(Communicated by Julia Knight)

ABSTRACT. An  $\mathfrak{M}_C$  group is a group in which all chains of centralizers have finite length. In this article, we show that every nilpotent subgroup of an  $\mathfrak{M}_C$  group is contained in a definable subgroup which is nilpotent of the same nilpotence class. Definitions are uniform when the lengths of chains are bounded.

### 1. INTRODUCTION

Chain conditions have played a central role in modern infinite group theory and one of the most natural chain conditions is the one on centralizers. A group is said to be  $\mathfrak{M}_C$  if all chains of centralizers of arbitrary subsets are finite. If there is a uniform bound  $d$  on the lengths of such chains, then  $G$  has **finite centralizer dimension (fcd)** and the least such bound  $d$  is known as the  **$c$ -dimension** of  $G$ .

The  $\mathfrak{M}_C$  property has been studied by group theorists since many natural classes of groups possess this property. See [3] for a classic paper on the properties of  $\mathfrak{M}_C$  groups. Many groups possess the stronger property of fcd, including abelian groups, free groups, linear groups, and torsion-free hyperbolic groups. Khukhro’s article on the solvability properties of torsion fcd groups [8] compiles a lengthy list of groups with fcd. Khukhro’s article, as well as several other foundational papers (see, for example, [2, 4, 8, 9, 16]), have demonstrated that  $\mathfrak{M}_C$  groups and groups with fcd are fairly well-behaved, for example by having Engel conditions closely linked to nilpotence.

For model theorists, the interest in these groups derives from the well-studied model-theoretic property of stability. A stable group must have fcd; in fact, it possesses uniform chain conditions on all uniformly definable families of subsets. Stable groups have an extensive literature in model theory (see [12] or [15]), however the properties of  $\mathfrak{M}_C$  and fcd are appearing in other areas of the model theory of groups, such as rosy groups with NIP [5] or Pınar Uğurlu’s recent work on pseudofinite groups with fcd [14].

The results of this paper reinforce Wagner’s work [4, 15, 16] in showing that several basic properties of (sub)stable groups derive purely from these simple group-theoretic chain conditions, which force the left-Engel elements to be well-behaved. In contrast to Wagner’s generalizations, which revealed that  $\mathfrak{M}_C$  suffices for many

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group-theoretic properties of stable groups, we shall show that  $\mathfrak{M}_C$  suffices for a *logical* property of stable groups, asserting the existence of certain definable groups.

It has been known for some time [12, Theorem 3.17], that if  $G$  is a stable group and  $H$  is a nilpotent (or solvable) subgroup, then there exists a definable subgroup  $d(H)$  of  $G$  which contains  $H$  and has the same nilpotence class (derived length) as  $H$ . Such a subgroup  $d(H)$  is called a **definable envelope** of  $H$ . The existence of definable envelopes allowed logicians to approximate arbitrary nilpotent subgroups of stable groups with slightly “larger” nilpotent subgroups which were definable, i.e. manipulable with model-theoretic techniques.

Our main theorem asserts the existence of definable envelopes of nilpotent subgroups in  $\mathfrak{M}_C$  groups and uniformly definable envelopes for groups with fcd. Definability here always refers to formulas in the language  $\mathcal{L}_G$  of groups. These envelopes are  $N_G(H)$ -normal, meaning that if an element normalizes  $H$ , it also normalizes the envelope.

**Main Theorem** (Corollary 3.8). *Let  $G$  be an  $\mathfrak{M}_C$  group and  $H \leq G$  a nilpotent subgroup. Then there exists a subgroup  $D \leq G$  containing  $H$ , which is definable in the language of groups with parameters from  $G$ , is nilpotent of the same nilpotence class as  $H$ , and is  $N_G(H)$ -normal.*

*Moreover, in the setting of groups of finite centralizer dimension, the definition of  $D$  becomes uniform. Specifically, for every pair of positive integers  $d$  and  $n$ , there exists a formula  $\phi_{d,n}(x, \bar{y})$ , where  $\ell(\bar{y}) = dn$ , such that for any group  $G$  of dimension  $d$  and any  $H \leq G$  nilpotent of class  $n$ , there exists a tuple  $\bar{a} \in G$  such that  $\phi_{d,n}(G, \bar{a})$  is a nilpotent subgroup of  $G$  of class  $n$  which contains  $H$  and is  $N_G(H)$ -normal.*

We hope that this result will prove useful in some of the current areas in logic where  $\mathcal{M}_C$  groups are appearing. Our main theorem may also open the door to studying some of the logical properties of the non-elementary classes of groups with fcd listed in [8]. In general, the Compactness Theorem from model theory shows that the class of  $\mathfrak{M}_C$  groups is not elementary. On the other hand, the class of fcd groups of a given dimension is elementary. It is worth mentioning that [1] and [10] contain results on definable envelopes in elementary classes of groups whose theories are NIP or simple, respectively.

We assume only a rudimentary knowledge of model theory and logic, namely the notion of “definability”. Readers may consult any introductory text, such as [6] or [11], for explanations of these notions. Otherwise, the material will be primarily group-theoretic and self-contained.

In the next section, we will define relevant terms from group theory and prove some fundamental lemmas about groups in general. In the following section, we restrict our focus to  $\mathfrak{M}_C$  groups and prove our main theorem and some corollaries.

## 2. PRELIMINARIES

We write  $A \leq G$  to denote that  $A$  is a subgroup of  $G$  and  $A \triangleleft G$  to denote  $A$  is normal in  $G$ . If  $A \subseteq G$  then  $\langle A \rangle$  denotes the subgroup generated by  $A$ . For any subset  $A$  of  $G$ , the centralizer of  $A$  is  $C_G(A) = \{g \in G \mid \forall a \in A \ ga = ag\}$ , while the normalizer of  $A$  is  $N_G(A) = \{g \in G \mid \forall a \in A \ g^{-1}ag \in A\}$ . If  $A$  and  $B$  are subgroups of a group  $G$ , then  $A$  is  $N_G(B)$ -normal if  $N_G(B) \leq N_G(A)$ .

Given  $g, h \in G$ , the commutator of  $g$  and  $h$  is  $[g, h] := g^{-1}h^{-1}gh$ . Iterated commutators are interpreted as left-normed, i.e.,  $[x, y, z]$  will denote  $[[x, y], z]$ . When

$A, B \subseteq G$ , then we write  $[A, B] := \langle \{[a, b] \mid a \in A, b \in B\} \rangle$ . We define the lower central series of  $G$  as  $\gamma_1(G) := G$  and  $\gamma_{k+1}(G) := [\gamma_k(G), G]$ . A group  $G$  is nilpotent if  $\gamma_n(G) = 1$  for some  $n < \omega$ ; the least  $n \geq 0$  for which  $\gamma_{n+1}(G) = 1$  is the nilpotence class of  $G$ . It is clear that a subgroup of a nilpotent group is nilpotent of equal or lesser nilpotence class.

The Hall-Witt identity relates the commutators of three elements: For all  $x, y, z \in G$ ,

$$(2.1) \quad 1 = [x, y^{-1}, z]^y [y, z^{-1}, x]^z [z, x^{-1}, y]^x = [x, y, z^x][z, x, y^z][y, z, x^y]$$

The Hall-Witt identity is used to prove the well-known Three Subgroup Lemma, which we state in the needed level of generality.

**Lemma 2.1.** [13, Three Subgroup Lemma, 5.1.10] *Let  $G$  be a group,  $N$  a subgroup, and  $K, L$ , and  $M$  subgroups of  $N_G(N)$ . Then  $[K, L, M] \leq N$  and  $[L, M, K] \leq N$  together imply  $[M, K, L] \leq N$ .*

This article shall be concerned with chains of centralizers. However, in order to analyze them fully, we shall need a more general definition of *iterated* centralizers.

**Definition 2.2.** Let  $A$  be a subset of  $G$ . We define the **iterated centralizers of  $A$  in  $G$**  as follows. Set  $C_G^0(A) = 1$  and for  $n \geq 1$ , let

$$C_G^n(A) = \left\{ x \in \bigcap_{k < n} N_G(C_G^k(A)) \mid [x, A] \subseteq C_G^{n-1}(A) \right\}$$

When  $A = G$ , the  $n$ th iterated centralizer of  $G$  is more commonly known as  $Z_n(G)$ , the  **$n$ th center** of  $G$ .

The groups  $Z_n(G) = C_G^n(G)$  are all characteristic in  $G$ , so that their definition simplifies to  $Z_0(G) = \{1\}$  and  $Z_{n+1}(G) = \{g \in G \mid [g, G] \subseteq Z_n(G)\}$  for all  $n \geq 0$ . The subgroup  $Z_1(G) = Z(G)$  is the center of  $G$ . This series is known as the upper central series; a group is nilpotent of class  $n$  if and only if  $Z_n(G) = G$ .

It is easy to show that each  $C_G^n(A)$  is a subgroup of  $G$  since its elements normalize  $C_G^{n-1}(A)$ . The set  $A$  also normalizes each  $C_G^n(A)$ . If  $H$  is the subgroup generated by  $A$ , then one can easily conclude that for any  $n < \omega$

$$C_G^n(H) = \left\{ x \in \bigcap_{k < n} N_G(C_G^k(H)) \mid [x, A] \subseteq C_G^{n-1}(H) \right\},$$

from which it follows by induction that  $C_G^n(H) = C_G^n(A)$  for all  $n < \omega$ . If  $H$  is a subgroup of  $G$ , the intersections with  $H$  are well-behaved:  $C_G^n(H) \cap H = Z_n(H)$ . If  $H$  is a nilpotent subgroup of  $G$  of nilpotence class  $n$ , then  $H \leq C_G^n(H)$ . These results may all be proven easily by induction, as can the following lemma due to P. Hall which relates iterated centralizers of  $H$  to the lower central series of  $H$ .

**Lemma 2.3.** [7, Satz III.2.8] *Let  $G$  be a group and  $H$  a subgroup of  $G$ . Then*

$$[\gamma_i(H), C_G^k(H)] \leq C_G^{k-i}(H)$$

for all positive integers  $i$  and  $k$  such that  $i \leq k$ . In particular,

$$[\gamma_i(G), Z_k(G)] \leq Z_{k-i}(G).$$

Bryant (Lemma 2.5 in [3]) used Hall's lemma to determine conditions under which one could conclude a group and a subgroup have the same iterated centralizer. We shall pursue the same goal and restructure Bryant's argument for our purposes. The following technical lemma is the heart of the proof of our main theorem. Its proof almost reproduces Bryant's subtle argument after some streamlining for which we thank the referee. We include it not only for completeness, but also to clarify how our lemma and Bryant's relate to each other, despite statements that differ considerably.

**Lemma 2.4.** *Let  $k \geq 2$  be an integer,  $G$  be a group, and  $H \leq E$  be two subgroups of  $G$  satisfying the following conditions:*

- (1)  $C_G^i(H) = C_G^i(E)$  for all  $i < k$ ;
- (2)  $[\gamma_k(E), C_G^k(H)] = 1$ ;

*Then  $C_G^k(H) = C_G^k(E)$ .*

*Proof.* The inclusion  $C_G^k(E) \leq C_G^k(H)$  follows immediately from the hypotheses, so we will deal with the reverse inclusion. By hypothesis (1),  $C_G^i(H) = C_G^i(E)$  for all  $i < k$ .

As in the proof of [3, Lemma 2.5], we first show the following containment by induction on  $i < k$ .

$$(*) \quad [\gamma_{k-i}(E), C_G^k(H)] \leq C_G^i(H) = C_G^i(E) .$$

Note that  $i = 0$  is precisely hypothesis (2), so we suppose  $i \geq 1$ . Set  $N = C_G^{i-1}(E) = C_G^{i-1}(H)$ . It follows from the general properties of iterated centralizers and hypothesis (1) that  $\gamma_{k-i}(E)$  and  $C_G^k(H)$  both normalize  $C_G^j(E)$  for all  $j < k$ , so it suffices to check the condition

$$[\gamma_{k-i}(E), C_G^k(H), H] \leq C_G^{i-1}(H) = N .$$

We appeal to the Three Subgroups Lemma. We have:

$$\begin{aligned} [C_G^k(H), H, \gamma_{k-i}(E)] &\leq [C_G^{k-1}(H), \gamma_{k-i}(E)] = [C_G^{k-1}(E), \gamma_{k-i}(E)] \leq N , \\ [H, \gamma_{k-i}(E), C_G^k(H)] &\leq [\gamma_{k-i+1}(E), C_G^k(H)] \leq N , \end{aligned}$$

where Lemma 2.3 was used in the first line, while induction was used in the second. Therefore the Three Subgroups Lemma applies and (\*) holds for  $i < k$ .

With  $i = k - 1$ , the formula (\*) becomes

$$[E, C_G^k(H)] \leq C_G^{k-1}(H) = C_G^{k-1}(E) ,$$

and as  $C_G^k(H)$  normalizes all  $C_G^j(E) = C_G^j(H)$  for  $j < k$ , we conclude that  $C_G^k(H) \leq C_G^k(E)$ .  $\square$

We shall also need a lemma relating the iterated centralizers of three nested groups.

**Lemma 2.5.** *Let  $A \leq B \leq C$  be groups and suppose that for all  $j \leq k$  we have*

$$C_C^j(A) = Z_j(C) .$$

Then

- (1)  $C_C^j(A) = C_C^j(B) = Z_j(C)$  for all  $j \leq k$
- (2)  $C_B^j(A) = Z_j(B) = Z_j(C) \cap B$  for all  $j \leq k$
- (3)  $C_B^{k+1}(A) = C_C^{k+1}(A) \cap B$

*Proof.* We proceed by induction on  $j \leq k$ . For  $j = 0$ , claims (1) and (2) are trivial, so we now assume (1) and (2) hold for  $j$ . Then

$$C_C^{j+1}(B) = \{b \in C \mid [b, B] \leq C_C^j(A) = Z_j(C)\},$$

and thus  $Z_{j+1}(C) = C_C^{j+1}(C) \leq C_C^{j+1}(B) \leq C_C^{j+1}(A)$ . For  $j < k$ , we obtain equality by the hypothesis and thus (1) holds. We also find

$$\begin{aligned} C_B^{j+1}(A) &= \{b \in B \mid [b, A] \leq Z_j(B)\} \\ &= \{b \in B \mid [b, A] \leq Z_j(C)\} \\ &= C_C^{j+1}(A) \cap B. \end{aligned}$$

For  $j = k$ , this establishes (3), while for  $j < k$ , we see that  $C_C^{j+1}(A) \cap B = C_C^{j+1}(B) \cap B = Z_{j+1}(B)$ , so (2) is established.  $\square$

### 3. PROOF OF THE MAIN THEOREM

Before proving our main theorem, we find it useful to restate the definitions of the relevant chain conditions precisely.

**Definition 3.1.** A group  $G$  has the **chain condition on centralizers**, denoted  $\mathfrak{M}_C$ , if there exists no infinite sequence of subsets  $A_n \subseteq G$  such that  $C_G(A_n) > C_G(A_{n+1})$  for all  $n < \omega$ .

A group  $G$  has **finite centralizer dimension (fcd)** if there is a uniform bound  $n \geq 1$  on any chain  $G = C_G(1) > C_G(A_1) > \dots > C_G(A_n)$  of centralizers of subsets  $A_i$  of  $G$ . The least bound (i.e. the length of the longest chain of centralizers) is known as the  **$c$ -dimension** of  $G$ .

Note that since  $C_G(C_G(C_G(A))) = C_G(A)$  for all  $A \subseteq G$ , all descending chains of centralizers are finite if and only if all ascending chains are finite. An immediate well-known consequence of the finite chain condition is the following observation:

**Lemma 3.2.** *Let  $G$  be an  $\mathfrak{M}_C$  group. If  $A \subseteq G$ , then there is an  $A' \subseteq A$  finite such that  $C_G(A) = C_G(A')$ . If  $G$  has centralizer dimension  $d$ , then  $A'$  can be chosen such that  $|A'| \leq d$ .*

The next lemma, though not used in the proof of the main theorem, illustrates the ‘‘central’’ intuitions underlying the proof.

**Lemma 3.3.** *Let  $G$  be a group and  $H \leq G$ . Then one of the following is true:*

- (1)  $H \leq Z(G)$ ;
- (2) *there exists a subset  $A \subseteq G$  such that  $H \leq C_G(A) < G$ ; or*
- (3)  $C_G(H) = Z(G)$ , and hence  $Z(H) = Z(G) \cap H$ , i.e.  $Z(H) \leq Z(G)$ .

*Proof.* Assume (1) does not hold, so  $C_G(H) < G$ . If  $C_G(H) > Z(G)$ , then  $H \leq C_G(C_G(H)) < G$ , so  $A = C_G(H)$  witnesses (2). Thus, if (1) and (2) both do not hold for  $H$ , then  $C_G(H) = Z(G)$  and so clearly  $Z(H) = C_G(H) \cap H = Z(G) \cap H$  and  $Z(H) \leq Z(G)$ .  $\square$

While both  $\mathfrak{M}_C$  and  $\text{fcd}$  are preserved under subgroups and finite direct products [9], they behave poorly under quotients. The quotient of an  $\mathfrak{M}_C$  group, even by its center, may fail to be  $\mathfrak{M}_C$  (see [3]). This is the principal complication in the proof of our main theorem. The next lemma demonstrates how we can use the lemmas of Section 2 to sidestep this obstacle.

**Lemma 3.4.** *Let  $G$  be an  $\mathfrak{M}_C$  group and  $H \leq G$ . For each  $n < \omega$ , there exists a finite subset  $A_n$  of  $H$  such that  $C_G^k(A_n) = C_G^k(H)$  for all  $k \leq n$ . Consequently the iterated centralizers  $C_G^n(H)$  are definable in the language of groups with parameters from  $G$  for all  $n < \omega$ .*

*Proof.* The proof is based on the proof of [3, Lemma 2.1]. For  $n = 0$ , the claim is trivial, so assume  $n > 0$ .

By Lemma 3.2, for each  $k$  with  $1 \leq k \leq n$ , we may choose a finite subset  $T_k \subseteq \gamma_k(H)$  such that  $C_G(T_k) = C_G(\gamma_k(H))$ . It follows that there exists a finite subset  $B_k$  of  $H$  for each  $k \leq n$  such that  $T_k$  is a subset of  $\gamma_k(\langle B_k \rangle)$ .

Set  $A_n = B_1 \cup \dots \cup B_n$  and  $X_n = \langle A_n \rangle$ . By the choice of the  $T_k$ , we find that

$$C_G(T_k) \geq C_G(\gamma_k(X_n)) \geq C_G(\gamma_k(H)) = C_G(T_k),$$

so  $C_G(\gamma_k(X_n)) = C_G(\gamma_k(H))$  for all  $k \leq n$ . By Lemma 2.5 of [3],  $C_G^k(H) = C_G^k(X_n) = C_G^k(A_n)$  for all  $k \leq n$ .  $\square$

Note that if  $H$  is a nilpotent subgroup of class  $n$ , then  $H \leq C_G^n(H)$ , so  $C_G^n(H)$  is a definable subgroup containing  $H$ . However, we have no guarantee that  $C_G^n(H)$  is nilpotent; indeed, the abelian subgroup  $Z(G)$  has  $C_G^1(Z(G)) = G$ . Thus to obtain definable *nilpotent* subgroups containing  $H$ , we must delve deeper.

The observation that  $C_G(C_G(C_G(A))) = C_G(A)$  for all  $A \subseteq G$  allowed us to disregard the directionality of our centralizer chains in the definition of  $\mathfrak{M}_C$ . However, it also provides a springboard for constructing envelopes: the subgroup  $C_G(C_G(A))$  contains  $A$ . We generalize this observation to the iterated centralizers of  $A$  with the following sequence of subgroups.

**Definition 3.5.** Let  $G$  be a group and  $H$  a subgroup. For  $n < \omega$ , we recursively define a descending sequence of subgroups  $E_n(H)$  of  $G$ , where  $E_0(H) = G$  and

$$E_{n+1}(H) = \{g \in E_n(H) \mid [g, C_{E_n(H)}^{n+1}(H)] \leq C_{E_n(H)}^n(H)\}$$

Note that with this definition,  $E_1(H) = C_G(C_G(H))$ . We now prove some basic facts about these  $E_n(H)$ .

**Lemma 3.6.** *Let  $G$  be a group and  $H$  a subgroup. For all  $n \geq 0$ , the following hold:*

- (1)  $E_n(H)$  is a group.
- (2)  $H \leq E_{n+1}(H) \leq E_n(H)$ .
- (3)  $E_{n+1}(H) \leq N_G(C_{E_n(H)}^n(H))$
- (4)  $N_G(H) \leq N_G(E_n(H))$ .

*Proof.* Since  $C_{E_n(H)}^n(H) \leq C_{E_n(H)}^{n+1}(H)$ , the set  $E_{n+1}(H)$  normalizes  $C_{E_n(H)}^n(H)$  for every  $n$ . It immediately follows that each  $E_n(H)$  is a group. From the definition of iterated centralizers, we have  $H \leq E_n(H)$ , and by definition the  $E_n$  are a descending sequence. Claim (4) is proven easily by induction on  $n$ .  $\square$

We shall now prove our main theorem, which asserts that these subgroups  $E_n(H)$  are definable. As a corollary, we will demonstrate the existence of definable envelopes of nilpotent subgroups of  $\mathfrak{M}_C$  groups. The advantage of fcd in this case is a uniformity to the definition of the envelopes, in terms of the dimension of the ambient group and the nilpotence class of the subgroup.

**Theorem 3.7.** *Let  $G$  be an  $\mathfrak{M}_C$  group and  $H \leq G$  a subgroup. Then for all  $n < \omega$*

- (1) *the subgroups  $E_n(H)$  are definable with parameters from  $G$ ; and*
- (2) *for each  $j \leq n$ ,  $C_{E_n(H)}^j(H) = Z_j(E_n(H))$ .*

*In the setting of groups of finite centralizer dimension  $d$ , the definition of the  $E_n(H)$  becomes uniform. Specifically, for every pair of positive integers  $d$  and  $n$ , there exists a formula  $\phi_{d,n}(x, \bar{y})$ , where  $\ell(\bar{y}) = dn$ , such that for any group  $G$  of dimension  $d$  and any  $H \leq G$ , there exists a tuple  $\bar{a} \in G$  such that  $\phi_{d,n}(G, \bar{a}) = E_n(H)$ .*

*Proof.* Let  $G$  be an  $\mathfrak{M}_C$  group and  $H$  be a subgroup of  $G$ . We will denote the various  $E_n(H)$  by  $E_n$ . We must prove the following two conditions for all  $k < \omega$ :

- (1)  $E_k$  is definable with parameters from  $G$ ; and
- (2) for each  $j \leq k$ ,  $C_{E_k}^j(H) = Z_j(E_k)$ .

For  $k = 0$ , both conditions are trivially satisfied, so we assume conditions (1) and (2) are true for  $k$  and consider  $k+1$ . The second condition immediately implies

$$(\bullet) \quad E_{k+1} = \{g \in E_k \mid [g, C_{E_k}^{k+1}(H)] \leq Z_k(E_k)\}.$$

Trivially,  $Z_{k+1}(E_k) \leq E_{k+1}$ . By Lemma 3.6 (2), we have  $H \leq E_{k+1} \leq E_k$ . Condition (2) and Lemma 2.5 (2) imply that for all  $j \leq k$ , we have  $C_{E_{k+1}}^j(H) = Z_j(E_{k+1}) = Z_j(E_k) \cap E_{k+1} = Z_j(E_k)$ . Thus the  $C_{E_{k+1}}^j(H)$  are normal in  $E_{k+1}$  for all  $j \leq k$  and  $[Z_{k+1}(E_{k+1}), H] \leq Z_k(E_{k+1}) = C_{E_{k+1}}^k(H)$ , so  $Z_{k+1}(E_{k+1}) \leq C_{E_{k+1}}^{k+1}(H)$ . By part (3) of Lemma 2.5, we have  $C_{E_{k+1}}^{k+1}(H) = C_{E_k}^{k+1}(H) \cap E_{k+1}$ . Yet by  $(\bullet)$ , we find

$$[E_{k+1}, C_{E_{k+1}}^{k+1}(H)] = [E_{k+1}, C_{E_k}^{k+1}(H) \cap E_{k+1}] \leq Z_k(E_k) \cap E_{k+1} = Z_k(E_{k+1}),$$

and thus we have the reverse inclusion  $C_{E_{k+1}}^{k+1}(H) \leq Z_{k+1}(E_{k+1})$ . Condition (2) has now been shown for  $k+1$ . It remains to show that  $E_{k+1}$  is definable with parameters from  $G$ .

By Lemma 3.2, for some finite subset  $A$  of  $C_{E_k}^{k+1}(H)$ , we have

$$C_G(C_{E_k}^{k+1}(H)) = C_G(A).$$

For each  $a \in A$ , we define a group  $E_a$  as

$$E_a = \{g \in E_k \mid [g, a] \in Z_k(E_k)\}.$$

The group  $E_A$  defined as

$$E_A = \bigcap_{a \in A} E_a = \{g \in E_k \mid [g, A] \leq Z_k(E_k)\}$$

is definable over  $A$  and the parameters used to define  $E_k$ . It will suffice to show that  $E_{k+1} = E_A$ . By  $(\bullet)$ , we have  $E_{k+1} \leq E_A$ .

If  $k = 0$ , then we have

$$E_1 = C_G(C_G(H)) = C_G(A) = E_A$$

and our claim holds:  $E_1 = E_A$ . So we suppose  $k \geq 1$ .



We will utilize Lemma 2.4. As  $H \leq E_{k+1} \leq E_A \leq E_k$ , we find by condition (2) and Lemma 2.5 (1) that  $C_{E_k}^j(H) = C_{E_k}^j(E_{k+1}) = C_{E_k}^j(E_A) = Z_j(E_k)$  for all  $j \leq k$ . We also have  $a \in Z_{k+1}(E_a)$  for each  $a \in A$  by the definition of the  $E_a$  (but note that  $A$  need not be contained in  $Z_{k+1}(E_A)$  since we are not even guaranteed  $A \subseteq E_A$ !). By Lemma 2.3 we have  $[\gamma_{k+1}(E_a), a] = 1$  for all  $a \in A$  and thus  $[\gamma_{k+1}(E_A), A] = 1$ . By the choice of  $A$ , we have

$$[\gamma_{k+1}(E_A), C_{E_k}^{k+1}(H)] = 1.$$

Now by Lemma 2.4, we obtain

$$C_{E_k}^{k+1}(H) = C_{E_k}^{k+1}(E_A).$$

Thus

$$[E_A, C_{E_k}^{k+1}(H)] = [E_A, C_{E_k}^{k+1}(E_A)] \leq C_{E_k}^k(E_A) = Z_k(E_k)$$

and we conclude  $E_A = E_{k+1}$  by  $(\bullet)$ . Thus  $E_{k+1}$  is definable from  $E_k$  using the parameters  $A$ .

Note that if  $G$  has finite centralizer dimension  $d$ , by Lemma 3.2 the finite subset  $A$  in the induction step may be chosen to have size  $d$ . As there are  $n$  steps to reach  $E_n(H)$ , we find that a total of  $dn$  parameters are needed to define  $E_n(H)$ . It is clear that for each  $n$  and  $d$  the definition of  $E_n(H)$  is uniform across all groups  $G$  of dimension  $d$  and subgroups  $H$  of  $G$ .  $\square$

As a corollary, we obtain our desired result on definable envelopes of nilpotent subgroups.

**Corollary 3.8.** *Let  $G$  be an  $\mathfrak{M}_C$  group and  $H \leq G$  a nilpotent subgroup. Then there exists a subgroup  $D \leq G$  containing  $H$ , which is definable in the language of groups with parameters from  $G$ , is nilpotent of the same nilpotence class as  $H$ , and is  $N_G(H)$ -normal.*

*Moreover, in the setting of groups of finite centralizer dimension, the definition of  $D$  becomes uniform. Specifically, for every pair of positive integers  $d$  and  $n$ , there exists a formula  $\phi_{d,n}(x, \bar{y})$ , where  $\ell(\bar{y}) = dn$ , such that for any group  $G$  of dimension  $d$  and any  $H \leq G$  nilpotent of class  $n$ , there exists a tuple  $\bar{a} \in G$  such that  $\phi_{d,n}(G, \bar{a})$  is a nilpotent subgroup of  $G$  of class  $n$  which contains  $H$  and is  $N_G(H)$ -normal.*

*Proof.* Let  $G$  be an  $\mathfrak{M}_C$  group and  $H$  be a nilpotent subgroup of  $G$  of class  $n$ . By Theorem 3.7 (1), the subgroup  $E_n(H)$  is definable with parameters from  $G$ ; in the case of finite dimension  $d$ , the parameter set can be taken to have size  $dn$ . The group  $E_n(H)$  contains  $H$  and, by condition (2) of that theorem,  $C_{E_n(H)}^n(H) = Z_n(E_n(H))$ . Yet since  $H$  is nilpotent of class  $n$ ,  $H \leq C_{E_n(H)}^n(H)$ . Thus  $Z_n(E_n(H))$  is our definable envelope of  $H$ : it is definable using the same parameters as  $E_n(H)$ , it contains  $H$ , and it is nilpotent of class  $n$ . Since  $Z_n(E_n(H))$  is characteristic in  $E_n(H)$ , its normalizer contains the normalizer of  $E_n(H)$  and thus the normalizer of  $H$ , by Lemma 3.6 (4).  $\square$

At this point, the following question is natural:

*Is the solvable analogue of Theorem 3.8 true in an  $\mathfrak{M}_C$ -group?*

Our construction yields a partial, but very incomplete, answer to this question, which hinges upon the fact that the envelope is  $N_G(H)$ -normal.

**Corollary 3.9.** *Let  $G$  be an  $\mathfrak{M}_C$  group and  $H \leq G$  a solvable subgroup. If there exist nilpotent subgroups  $A, B \leq H$  such that  $A \triangleleft H$  and  $H = AB$ , then  $H$  is contained in a definable solvable subgroup of  $G$ .*

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