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Frédéric Faure, Masato Tsujii

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Prequantum transfer operator for symplectic Anosov diffeomorphism

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2013 May 8

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Abstract

We define the prequantization of a symplectic Anosov diffeomorphism $f : M \rightarrow M$, which is a $\mathbf{U}(1)$ extension of the diffeomorphism f preserving an associated specific connection, and study the spectral properties of the associated transfer operator, called *prequantum transfer operator*. This is a model for the transfer operators associated to geodesic flows on negatively curved manifolds (or contact Anosov flows).

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We restrict the prequantum transfer operator to the N -th Fourier mode with respect to the $\mathbf{U}(1)$ action and investigate the spectral property in the limit $N \rightarrow \infty$, regarding the transfer operator as a Fourier integral operator and using semi-classical analysis. In the main result, we show a “*band structure*” of the spectrum, that is, the spectrum is contained in a few separated annuli and a disk concentric at the origin.

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We show that, with the special (Hölder continuous) potential $V_0 = \frac{1}{2} \log |\det Df_x|_{E_u}|$, the outermost annulus is the unit circle and separated from the other parts. For this, we use an extension of the transfer operator to the Grassmanian bundle. Using Atiyah-Bott trace formula, we establish the Gutzwiller trace formula with exponentially small reminder for large time. We show also that, for a potential V such that the outermost annulus is separated from the other parts, most of the eigenvalues

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in the outermost annulus concentrate on a circle of radius $\exp(\langle V - V_0 \rangle)$ where $\langle \cdot \rangle$ denotes the spatial average on M . The number of these eigenvalues is given by the “Weyl law”, that is, $N^d \text{Vol}M$ with $d = \frac{1}{2} \dim M$ in the leading order.

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We develop a semiclassical calculus associated to the prequantum operator by defining quantization of observables $\text{Op}_\hbar(\psi)$ in an intrinsic way. We obtain that the semiclassical Egorov formula of quantum transport is exact. We interpret all these results from a physical point of view as the emergence of quantum dynamics in the classical correlation functions for large time. We compare these results with standard quantization (geometric quantization) in quantum chaos.

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1 Introduction and results

1.1 Introduction

We consider a smooth symplectic Anosov diffeomorphism $f : M \rightarrow M$ on a $2d$ -dimensional closed symplectic manifold (M, ω) as a standard model of "chaotic" dynamical system. Following the geometric quantization procedure introduced by Kostant, Souriau and Kirillov in 1970s', we consider the prequantum bundle $P \rightarrow M$. This is the $\mathbf{U}(1)$ -principal bundle over M equipped with a connection whose curvature is $(-2\pi i) \cdot \omega$. Then we introduce the prequantum map $\tilde{f} : P \rightarrow P$ as the $\mathbf{U}(1)$ -equivariant extension of the map f preserving the connection. The prequantum map \tilde{f} thus defined is known to be exponentially mixing¹, that is, any smooth probability density which evolves under the iteration of \tilde{f} converges weakly towards the uniform equilibrium distribution on P and the speed of convergence is exponentially fast if it is measured by a smooth observable. We study the fluctuations in this convergence to the equilibrium by investigating spectral properties of the transfer operator \hat{F} associated to the prequantum map \tilde{f} . Following the approach taken by David Ruelle in his study of expanding dynamical systems[43], we first show that the transfer operator displays discrete spectrum, which is sometimes called *Ruelle-Pollicott resonances*. Precisely we consider the restriction \hat{F}_N of the transfer operator \hat{F} to the N -th Fourier mode with respect to the $\mathbf{U}(1)$ action on P and show that its natural extension to appropriate generalized Sobolev spaces of distributions has discrete spectrum. This result concerning discrete spectrum is already known in the preceding works [44],[10, 25],[6, theorem 1.1],[19, theorem 1] and will be recalled in Theorem 1.15. In this paper we are mainly concerned with the limit $N \rightarrow \infty$ of small wavelength (high Fourier modes). We will use the standard notation of semiclassical analysis and put throughout this paper:

$$\hbar := \frac{1}{2\pi N}$$

The new result of this paper is in Theorem 1.17, where we show that the spectrum of \hat{F}_N has a particular "band" structure: for every N large enough, there is an annulus that contains finitely many (but increasing to infinity as $N \rightarrow \infty$) eigenvalues; they are separated from the rest of the internal spectrum by a gap under some pinching conditions. This means that the convergence to the equilibrium mentioned above, restricted to the N -th Fourier mode, is described by a finite rank operator denoted $\mathcal{F}_\hbar : \mathcal{H}_\hbar \rightarrow \mathcal{H}_\hbar$, up to relatively small exponentially decaying errors. The finite rank operator \mathcal{F}_\hbar is the spectral restriction of the prequantum transfer operator \hat{F}_N on the external annulus. We show, in Theorem 1.21, that the dimension of \mathcal{H}_\hbar is proportional to N^d asymptotically as² $N \rightarrow \infty$. These results

¹Exponential mixing of the map \tilde{f} is already known [14] but is also a direct consequence of results presented in this paper.

²The precise value of $\dim \mathcal{H}_\hbar$ is given by an index formula of Atiyah-Singer in Th. 1.21.

are generalizations of the results in [17] for the linear Arnold cat map to the case of general non-linear symplectic Anosov diffeomorphisms.

155 **Motivations of the study** From the construction above, the prequantum map $\tilde{f} : P \rightarrow P$ is partially hyperbolic, that is, hyperbolic in the directions transverse to the fibers but is neutral (because of equivariance) in the direction of the fibers. Also note that \tilde{f} preserves the connection one form on the prequantum bundle P which is a contact form on P (See Remark 1.7 page 13). These properties of the prequantum map are very similar to those of
 160 the time- t -map of the geodesic flow $\phi_1 : T_1^*\mathcal{M} \rightarrow T_1^*\mathcal{M}$ on a closed Riemannian manifold \mathcal{M} with negative curvature, acting on the unit cotangent bundle $T_1^*\mathcal{M}$: In the latter case the time- t -map of the geodesic flow is partially hyperbolic and preserves the canonical Liouville contact one form ξdx on $T_1^*\mathcal{M}$. (See [32, 52, 53, 20]). With this point of view, the prequantum transfer operator can be considered as a model of the transfer operators
 165 for the geodesic flows on negatively curved manifolds. One of our objective behind the present work is to show some band structure of the spectrum for the case of geodesic flow and extend other results presented in this paper to that case [21, 22]. In the special case of manifolds with constant curvature, such a band structure is readily observed from the classical theorem of Selberg on zeta functions [47].

170 Another motivation already discussed in [17] in a special case is the following observation: The finite rank operator \mathcal{F}_\hbar which describes the long time classical correlation functions of the map \tilde{f} has the properties of a "quantum map" i.e. a "quantization of f " but with additional interesting properties. It satisfies the Gutzwiller trace formula with an error term which decreases exponentially fast in large time, an exact Egorov theorem, etc.
 175 Surprisingly this "quantization" or quantum behavior, appears here dynamically (after long time) in the classical correlation functions of the "classical" map \tilde{f} : the finite dimensional "quantum space" \mathcal{H}_\hbar in which \mathcal{F}_\hbar acts is defined from the dynamics. There are many open questions in "quantum chaos" for example related to "unique quantum ergodicity" or "random matrix theory"[39]. These questions can be posed for the family quantum operators
 180 $(\mathcal{F}_\hbar)_\hbar$ considered here and may be their special properties with respect to the dynamics may help.

Semiclassical approach The general method that we use to obtain the main results is *semiclassical analysis*. We regard the prequantum transfer operator as a Fourier Integral Operator (FIO), which means that we consider its action on wave packets in the high
 185 frequency limit $N \rightarrow \infty$. From the general idea in semiclassical analysis, this action is effectively described by the associated canonical map $(Df^*)^{-1}$ on the cotangent space T^*M equipped with the symplectic structure $\Omega = dx \wedge d\zeta + \pi^*\omega$ (where $dx \wedge d\zeta$ stands for the canonical symplectic structure on T^*M and $\pi^*\omega$ is the pull-back of ω on T^*M). For the action of the canonical map $(Df^*)^{-1}$, the non-wandering set is the zero section
 190 $K \subset T^*M$ and is called the *trapped set*. The additional term $\pi^*\omega$ in Ω makes K a symplectic submanifold. The trapped set is therefore symplectic and normally hyperbolic. We will see that these facts are the core of our argument and give the band structure of the spectrum

in the main theorem.

Structure of the paper In Section 1.2 we define precisely the prequantum map \tilde{f} and the prequantum transfer operator \hat{F} that are uniquely associated to the Anosov map f . In Section 1.3 we present the main results concerning the discrete spectrum of \hat{F}_N (after Fourier decomposition of \hat{F} on Fourier component $N = 1/(2\pi\hbar) \in \mathbb{Z}$) acting on a Hilbert space \mathcal{H}_\hbar^r called the anisotropic Sobolev space. These results are summarized on Figure 1.4. The spectral restriction of the operator \hat{F}_N on this external band will be called the **quantum operator** and denoted $\hat{\mathcal{F}}_\hbar$ in Definition 1.19. The associated spectral projector is denoted by Π_\hbar . In Section 1.4 we show how to extend the results so that the potential function V that enters in the definition 1.9 of the transfer operator \hat{F} can be chosen such that the external band of resonances concentrates on the unit circle in the limit $\hbar \rightarrow 0$, although this requires that V is only Hölder continuous. In Section 1.5 we show that the quantum operator $\hat{\mathcal{F}}_\hbar^n$ satisfies the Gutzwiller trace formula with an error that decays exponentially fast as $n \rightarrow \infty$. We discuss the fact that this property determines the spectrum of $\hat{\mathcal{F}}_\hbar$ and shows somehow that the family of operators $\left(\hat{\mathcal{F}}_\hbar\right)_\hbar$ is a kind of “natural quantization” of the Anosov symplectic map f . In Section 1.6 we pursue the exploration of the properties of this quantum operator $\hat{\mathcal{F}}_\hbar$ obtained by the spectral restriction of \hat{F}_N by the spectral projector Π_\hbar . In Theorem 1.47 it is shown that $\hat{\mathcal{F}}_\hbar$ describes the exponential decay of correlations of the Anosov map f . In Section 1.7 we show in which sense the quantum operator $\hat{\mathcal{F}}_\hbar$ is a kind of “quantum map”: it satisfies an exact Egorov formula with respect to an algebra of quantum observables $\text{Op}_\hbar(\psi)$. For this, we define a new kind of quantization procedure $\text{Op}_\hbar : \psi \in C^\infty(M) \rightarrow \text{Op}_\hbar(\psi) \in \text{End}(\mathcal{H}_\hbar)$ which satisfies most of the usual “axioms of quantization”. In particular the spectral projector on the external band is $\Pi_\hbar = \text{Op}_\hbar(1)$. In Theorem 1.55, $\text{Op}_\hbar(\psi)$ is expressed as an integral over $x \in M$ of $\psi(x) \cdot \pi_x$ where π_x is a rank one projector over a “localized wave packet”³ at position $x \in M$. In Section 1.8 we consider the usual “geometric quantization” of the map f in the sense of Toeplitz quantization and compare it with the quantum operator $\hat{\mathcal{F}}_\hbar$ or “natural quantization”. We show that both quantization coincide up a to small error in the limit $\hbar \rightarrow 0$. In Section 2 we present the main ideas of semiclassical analysis used in the proofs. In Section 3 we present additional results concerning the spectrum of the rough Laplacian operator. This is used to discuss geometric (Toeplitz) quantization. Subsequent sections contain the proofs.

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³This localization property is with respect to the anisotropic Sobolev space \mathcal{H}_\hbar^r .

1.2 Definitions

1.2.1 Symplectic Anosov map

Let M be a C^∞ closed connected symplectic manifold of dimension $2d$ with symplectic two form ω . Let $f : M \rightarrow M$ be a C^∞ symplectic Anosov diffeomorphism, i.e. a C^∞ Anosov diffeomorphism such that $f^*\omega = \omega$. We recall the definition of an Anosov diffeomorphism:

Definition 1.1. [30, p.263] A diffeomorphism $f : M \rightarrow M$ is said to be *Anosov* if there exists a C^∞ Riemannian metric g on M , an f -invariant continuous decomposition of TM ,

$$T_x M = E_u(x) \oplus E_s(x), \quad \forall x \in M \quad (1.1)$$

and a constant $\lambda > 1$, such that, for any $x \in M$, hold

$$\begin{aligned} |D_x f(v_s)|_g &\leq \frac{1}{\lambda} |v_s|_g \quad \forall v_s \in E_s(x), \quad \text{and} \\ |D_x f^{-1}(v_u)|_g &\leq \frac{1}{\lambda} |v_u|_g \quad \forall v_u \in E_u(x). \end{aligned} \quad (1.2)$$

The subbundle E_s (resp. E_u) in which f is uniformly contracting (resp. expanding) is called the stable (resp. unstable) sub-bundle. See figure 1.1.

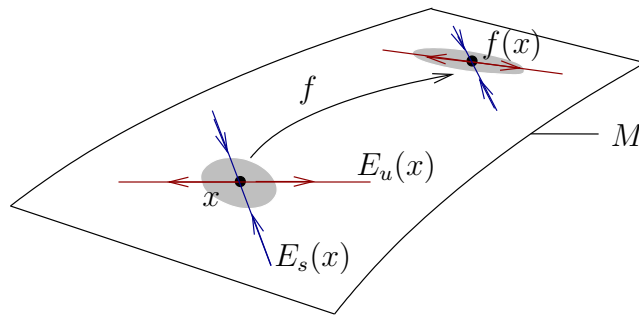


Figure 1.1: A symplectic Anosov map f

Remark 1.2. (1) The subspaces $E_u(x)$ and $E_s(x)$ do not depend smoothly on the point x in general. However it is known that they are Hölder continuous in x with some Hölder exponent [41]. In what follows, we assume that the Hölder exponent is

$$0 < \beta < 1. \quad (1.3)$$

The subspaces $E_u(x)$ and $E_s(x)$ are Lagrangian⁴ linear subspace of $T_x M$ and both have dimension d .

⁴To prove that $E_s(x)$ is Lagrangian, let $u, v \in E_s(x)$; we have $\omega(u, v) = \omega(D_x f^n(u), D_x f^n(v)) \xrightarrow{n \rightarrow +\infty} 0$. Similarly for E_u . So E_s and E_u are isotropic subspaces, $E_u \oplus E_s = TM$, hence they are Lagrangian.

(2) The *Arnold cat map*[3] is a simple example of a symplectic Anosov diffeomorphism on the torus $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$,

$$f_0 \begin{pmatrix} q \\ p \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} q \\ p \end{pmatrix} \quad \text{mod } \mathbb{Z}^2. \quad (1.4)$$

245 It preserves the symplectic form $\omega = dq \wedge dp$. If $h : M \rightarrow M$ is a diffeomorphism close enough to identity in the C^1 norm and preserves the symplectic form ω , the *perturbed cat map*

$$f(x) := h(f_0(x)) \quad (1.5)$$

is also a (probably non-linear) symplectic Anosov diffeomorphism[30, p.266]. Similarly, we get examples of symplectic Anosov diffeomorphisms on \mathbb{T}^{2d} from any symplectic linear map $f_0 \in \text{Sp}_{2d}(\mathbb{Z})$ with no eigenvalues on the unit circle.
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1.2.2 The prequantum bundle and the lift map \tilde{f}

A prequantum bundle is a $\mathbf{U}(1)$ -principal bundle P equipped with a specific connection. In a few paragraphs below, we recall the definition of a $\mathbf{U}(1)$ -principal bundle and that of a connection on it. (For the detailed account, we refer [55].) The one-dimensional unitary group $\mathbf{U}(1)$ is the multiplicative group of complex numbers of the form $e^{i\theta}$, $\theta \in \mathbb{R}$. A $\mathbf{U}(1)$ -principal bundle P over M is a manifold with a free action of $\mathbf{U}(1)$, written

$$p \in P \rightarrow (e^{i\theta}p) \in P, \quad (1.6)$$

such that the quotient space is $M = P/\mathbf{U}(1)$. We write $\pi : P \rightarrow M$ for the projection map. From the definition, the $\mathbf{U}(1)$ -principal bundle P has a local product structure over M : There exist a finite cover of M by simply connected open subsets $U_\alpha \subset M$, $\alpha \in I$, and smooth sections $\tau_\alpha : U_\alpha \rightarrow P$ on each of U_α , called a *local smooth section*; A *local trivialization* of P over U_α is defined as the diffeomorphism

$$T_\alpha : \begin{cases} U_\alpha \times \mathbf{U}(1) & \rightarrow \pi^{-1}(U_\alpha) \\ (x, e^{i\theta}) & \rightarrow e^{i\theta}\tau_\alpha(x) \end{cases} \quad (1.7)$$

A *connection* on P is a differential one form $A \in C^\infty(P, \Lambda^1 \otimes (i\mathbb{R}))$ on P with values in the Lie algebra $\mathfrak{u}(1) = i\mathbb{R}$ which is invariant by the action of $\mathbf{U}(1)$ and normalized so that

$$A \left(\frac{\partial}{\partial \theta} \right) = i \quad (1.8)$$

where $\frac{\partial}{\partial \theta}$ denotes the vector field on P generating the action of $\mathbf{U}(1)$. Consequently the pull-back of the connection A on P by the trivialization map (1.7) is written as

$$T_\alpha^* A = id\theta - i2\pi\eta_\alpha \quad (1.9)$$

where $\eta_\alpha \in C^\infty(U_\alpha, \Lambda^1)$ is a one-form on U_α which depends on the choice of the local section τ_α . A different local section $\tau_\beta : U_\beta \rightarrow P$ with $U_\alpha \cap U_\beta \neq \emptyset$ is written as $\tau_\beta = e^{i\chi}\tau_\alpha$

with using a function $\chi : U_\alpha \cap U_\beta \rightarrow \mathbb{R}$ and hence the connection A pulled-back by the corresponding trivialization T_β is written as (1.9) with

$$\eta_\beta = \eta_\alpha - \frac{1}{2\pi} d\chi \quad \text{on } U_\alpha \cap U_\beta. \quad (1.10)$$

270 The *curvature* of the connection A is the two form $\Theta = dA$ on P . In the local trivialization (1.7), we have $T_\alpha^* \Theta = -i2\pi (d\eta_\alpha)$ and (1.10) implies that $d\eta_\alpha = d\eta_\beta$. Therefore the curvature two form is written as

$$\Theta = -i(2\pi) (\pi^* \tilde{\omega})$$

where $\tilde{\omega} = d\eta_\alpha$ is a closed two form on M independent of the trivialization.

Since there is a given symplectic two form ω on M in our setting, we naturally require
275 below in (1.13) that the two form $\tilde{\omega}$ coincides with ω and then

$$\omega = d\eta_\alpha. \quad (1.11)$$

For the construction of the prequantum bundle and prequantum transfer operator, we will need the following two assumptions:

Assumption 1: The cohomology class $[\omega] \in H^2(M, \mathbb{R})$ represented by the symplectic form ω is integral, that is,

$$[\omega] \in H^2(M, \mathbb{Z}) \quad (1.12)$$

Assumption 2: The integral homology group $H_1(M, \mathbb{Z})$ has no torsion part and 1 is not an eigenvalue of the linear map $f_* : H_1(M, \mathbb{R}) \rightarrow H_1(M, \mathbb{R})$ induced by $f : M \rightarrow M$.

280 *Remark 1.3.* The second assumption above is not restrictive and may not be necessary. In fact this hypothesis is conjectured to be true in general. For the case $M = \mathbb{T}^{2d}$, this is always satisfied.

Theorem 1.4. Under Assumption 1 above, there exists a $\mathbf{U}(1)$ -principal bundle $\pi : P \rightarrow M$ and a connection $A \in C^\infty(P, \Lambda^1 \otimes (i\mathbb{R}))$ on P such that the curvature two form $\Theta = dA$ satisfies

$$\Theta = -i(2\pi)(\pi^*\omega). \quad (1.13)$$

If we put Assumption 2 in addition, we can choose the connection A as above so that there exists an equivariant lift $\tilde{f} : P \rightarrow P$ of the map $f : M \rightarrow M$ preserving the connection A , that is:

$$(\pi \circ \tilde{f})(p) = (f \circ \pi)(p), \quad \forall p \in P \quad : \tilde{f} \text{ is a lift of } f. \quad (1.14)$$

$$\tilde{f}(e^{i\theta}p) = e^{i\theta}\tilde{f}(p), \quad \forall p \in P, \forall \theta \in \mathbb{R} \quad : \tilde{f} \text{ is equivariant w.r.t. the } \mathbf{U}(1)\text{ action.} \quad (1.15)$$

$$\tilde{f}^*A = A \quad : \tilde{f} \text{ preserves the connection } A. \quad (1.16)$$

(See Figure 1.2.)

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The proof of Theorem 1.4 is given in Section A.1.

Definition 1.5. The $\mathbf{U}(1)$ -principal bundle $\pi : P \rightarrow M$ equipped with the connection $A \in C^\infty(P, \Lambda^1 \otimes (i\mathbb{R}))$ satisfying (1.13) is called *prequantum bundle* over the symplectic manifold (M, ω) . The map $\tilde{f} : P \rightarrow P$ satisfying the conditions (1.14), (1.15) and (1.16) is called *prequantum map*.

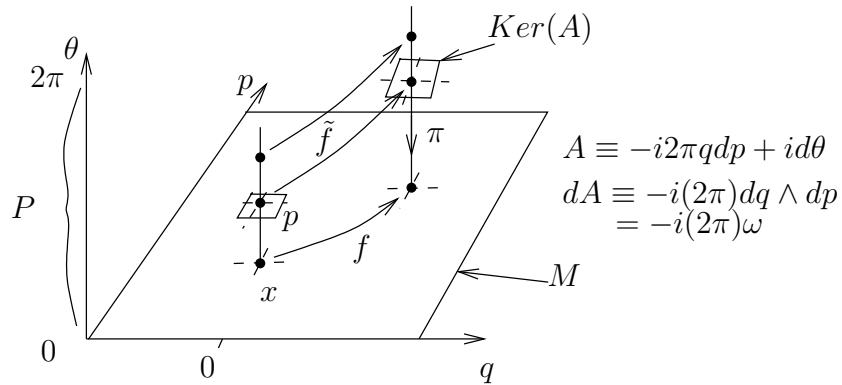


Figure 1.2: A picture of the prequantum bundle $P \rightarrow M$ in the case of $M = \mathbb{T}^2$, with connection one form A and the prequantum map $\tilde{f} : P \rightarrow P$ which is a lift of $f : M \rightarrow M$. A fiber $P_x \equiv U(1)$ over $x \in M$ is represented here as a segment $\theta \in [0, 2\pi[$. The plane at a point p represents the horizontal space $H_p P = \text{Ker}(A_p)$ which is preserved by f . These plane form a non integrable distribution with curvature given by the symplectic form ω .

290
Remark 1.6. (Uniqueness of the prequantum bundle and the prequantum map) The prequantum bundle P is unique (as a smooth manifold) if it exists, because it is determined by its first Chern class $c_1(P) = [\omega] \in H^2(M, \mathbb{Z}^2)$. However the connection A on the prequantum bundle P is not unique. In the proof of the theorem above, we will explicitly show that there may be finitely many connections A which satisfy the condition (1.13) and they differ from each other by a flat connection. Once the prequantum bundle P and the connection A on it is given, the lifted map \tilde{f} is unique up to a global phase $e^{i\theta_0} \in \mathbf{U}(1)$,
 295
 i.e. another map \tilde{g} satisfies the conditions in (2) of Theorem 1.4 if and only if $\tilde{g} = e^{i\theta_0} \tilde{f}$ for some $e^{i\theta_0} \in \mathbf{U}(1)$.

Remark 1.7. Let $\alpha := \frac{i}{2\pi}A$. Then the differential $(2d + 1)$ -form

$$\mu_P := \frac{1}{d!} \alpha \wedge (d\alpha)^d \tag{1.17}$$

is a non-degenerate volume form on P . This means that α is a *contact one form* on P preserved by \tilde{f} .

300
Remark 1.8. Suppose that $x \in M$ is a periodic point of the map f with period $n \in \mathbb{N}$, $n \geq 1$, i.e $x = f^n(x)$. Then if $p \in \pi^{-1}(x)$ is in the fiber, the condition (1.14) implies that $(\tilde{f})^n(p) \in \pi^{-1}(x)$ lies in the same fiber and therefore differ by a phase:

$$(\tilde{f})^n(p) = e^{i2\pi S_{n,x}} p \tag{1.18}$$

305
 with $S_{n,x} \in \mathbb{R}/\mathbb{Z}$ called the **action of the periodic point** x , see Figure 1.3. These actions are important quantities in semiclassical analysis and will appear in the Gutzwiller trace formula in (27).

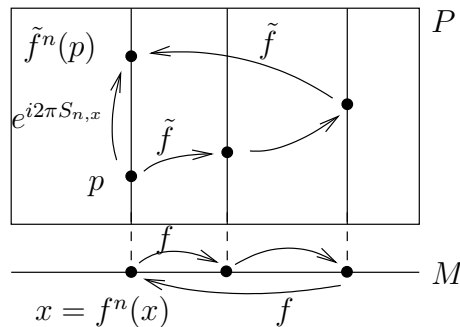


Figure 1.3: Action of a periodic point $x = f^n(x)$.

1.2.3 The prequantum transfer operator \hat{F} and the reduced operator \hat{F}_N

As usual in dynamical system theory, we consider the transfer operator associated to the prequantum map \tilde{f} :

Definition 1.9. Let $V \in C^\infty(M)$ be a real-valued smooth function, called *potential*. The *prequantum transfer operator* is defined as

$$\hat{F} : \begin{cases} C^\infty(P) & \rightarrow C^\infty(P) \\ u & \rightarrow \hat{F}(u) = e^{V \circ \pi} (u \circ \tilde{f}^{-1}) \end{cases} \quad (1.19)$$

where $V \circ \pi \in C^\infty(P)$ is the function V lifted on P .

Remark 1.10. The fact that \tilde{f}^{-1} appears instead of \tilde{f} in (1.19) is a matter of choice. In our choice, \tilde{f} maps the support of u to that of $\hat{F}u$.

From the equivariance property (1.15), the prequantum transfer operator commutes with the action of $U(1)$ on functions on P and therefore is naturally decomposed into each Fourier mode with respect to the $U(1)$ action:

Definition 1.11. For a given $N \in \mathbb{Z}$, we consider the space of functions in the N -th Fourier mode

$$C_N^\infty(P) := \{u \in C^\infty(P) \mid \forall p \in P, \forall \theta \in \mathbb{R}, \quad u(e^{i\theta}p) = e^{iN\theta}u(p)\}. \quad (1.20)$$

The prequantum transfer operator \hat{F} restricted to $C_N^\infty(P)$ is denoted by:

$$\hat{F}_N := \hat{F}|_{C_N^\infty(P)} : C_N^\infty(P) \rightarrow C_N^\infty(P). \quad (1.21)$$

Remark 1.12. The complex conjugation maps $C_N^\infty(P)$ to $C_{-N}^\infty(P)$ and commutes with \hat{F} . It is therefore enough to study \hat{F}_N with $N \geq 0$.

Remark 1.13. The space of equivariant functions $C_N^\infty(P)$ defined in (1.20) can be identified with the space of smooth sections of an associated Hermitian complex line bundle $L^{\otimes N}$ over M (i.e. the N tensor power of a line bundle $L \rightarrow M$) with covariant derivative D , called the *prequantum line bundle* i.e. we have

$$C_N^\infty(P) \cong C^\infty(M, L^{\otimes N}).$$

See [51, p.502, eq.(6.1)]. In order to simplify the presentation we will not use this identification in this paper although it will be present implicitly. Notice however that most of references about geometric quantization use the ‘‘line bundle terminology’’.

In this paper the main object of study is the resonance spectrum of the operator \hat{F}_N , (1.21), in the limit $N \rightarrow \infty$. For $N > 0$, we set

$$\hbar = \frac{1}{2\pi N}. \quad (1.22)$$

This new variable \hbar is in one-to-one correspondence to N , and $\hbar \rightarrow +0$ as $N \rightarrow \infty$. We
 330 introduce it for convenience in referring some argument in semi-classical analysis where \hbar
 is regarded as the Plank constant and the limit $\hbar \rightarrow +0$ is considered.

Remark 1.14. In the following, we will confuse the parameters N and \hbar in the notation.
 For instance, the operator \hat{F}_N will be written \hat{F}_\hbar sometimes.

1.3 Results on the spectrum of the prequantum operator \hat{F}_N

335 The following theorem has been obtained essentially in the works of Rugh [44], Liverani
 et al.[10, 25], Baladi et al.[6, Theorem 1.1], Faure et al. [19, theorem 1]. The method
 employed in the present paper is close to the semiclassical approach given in [19, Theorem
 1]. Before giving the theorem, let us mention that the transfer operator \hat{F}_N has been
 340 defined on the space of smooth functions $C_N^\infty(P)$ and can be extended by duality to the
 distributions space $\mathcal{D}'_N(P)$.

Theorem 1.15. “Discrete spectrum of prequantum transfer operators”. *For any
 $N \in \mathbb{Z}$, there exists a family of Hilbert spaces $\mathcal{H}_N^r(P)$ for arbitrarily large $r > 0$, called
 anisotropic Sobolev space such that*

$$C_N^\infty(P) \subset \mathcal{H}_N^r(P) \subset \mathcal{D}'_N(P)$$

and such that the operator \hat{F}_N extends to a bounded operator

$$\hat{F}_N : \mathcal{H}_N^r(P) \rightarrow \mathcal{H}_N^r(P),$$

and its essential spectral radius $r_{ess}(\hat{F}_N)$ is bounded by $\varepsilon_r := \frac{1}{\lambda^r} \max e^V$, which shrinks
 to zero if $r \rightarrow +\infty$. The discrete eigenvalues of \hat{F}_N on the domain $|z| \geq \varepsilon_r$ (and their
 associated eigenspaces) are independent on the choice of r and are therefore intrinsic to
 the Anosov diffeomorphism f . These discrete eigenvalues $\text{Res}(\hat{F}_N) := \{\lambda_i\}_i \subset \mathbb{C}^*$ are
 called **Ruelle-Pollicott resonances**. The definition of the space $\mathcal{H}_N^r(P)$ depends on the
 diffeomorphism Anosov f but do not depend on the potential function V .

Remark 1.16. See [7],[19, Cor. 1.3] for a general argument about this independence of
 $\text{Res}(\hat{F}_N)$ on the choice of r .

345 The main new result of this paper is the following Theorem . It is illustrated in Figure
 1.4. Let us prepare some notations. For a linear invertible map L we will use the notation

$$\|L\|_{\max} := \|L\|, \quad \|L\|_{\min} := \|L^{-1}\|^{-1}. \quad (1.23)$$

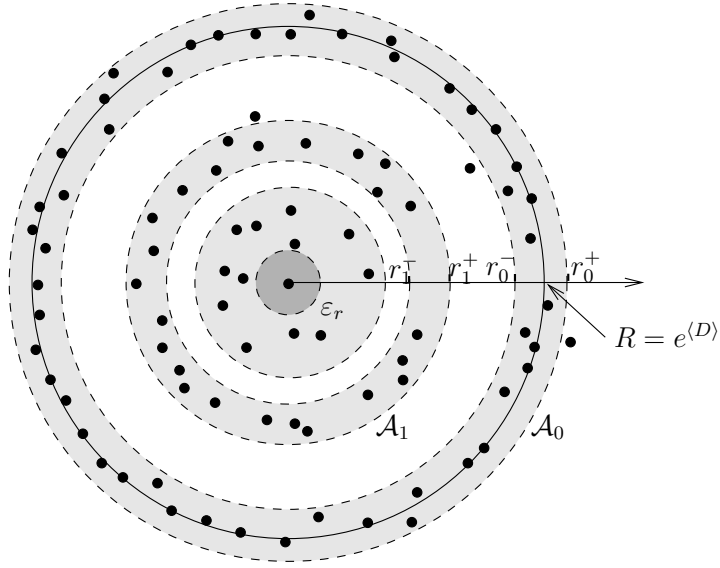


Figure 1.4: Theorem 1.15 shows that the spectrum of \hat{F}_N has discrete eigenvalues called Ruelle-Pollicott resonances. Theorem 1.17 shows that for N large enough it is structured in bands and that the resolvent is bounded between the bands, uniformly with respect to $\hbar = 1/(2\pi N)$. In the external band \mathcal{A}_0 , the number of resonances is given by the Weyl formula $\frac{1}{(2\pi\hbar)^d} \text{Vol}(M)$ at leading order. The precise number is given by the Atiyah-Singer formula in Theorem 1.21. Theorem 1.23 shows that in this external band \mathcal{A}_0 almost all the resonances are distributed uniformly on the circle of radius $R = e^{\langle D \rangle}$ in the limit $N \rightarrow \infty$. The spectral restriction of the operator \hat{F}_N on this external band will be called the **quantum operator** and denoted $\hat{\mathcal{F}}_\hbar$ in Definition 1.19. The spectral projector is Π_\hbar .

We define the special “potential of reference”

$$V_0(x) := \frac{1}{2} \log |\det Df_x|_{E_u(x)}| \quad (1.24)$$

Notice that the unstable foliation $E_u(x)$ is not smooth in x in general (see Remark 1.2(1)) which implies that this function V_0 is Hölder continuous but not smooth in x . We then

350 consider the difference

$$D := V - V_0 \in C^\beta(M) \quad (1.25)$$

which is also a Hölder continuous function on M and that will be called the “**effective damping function**”. It will appear in many results below. Finally we denote by

$$D_n(x) := \sum_{j=1}^n D(f^j(x)) \quad (1.26)$$

the Birkhoff sum of the damping function.

Theorem 1.17. “*Band structure of the spectrum of \hat{F}_N* ”. For any $\varepsilon > 0$, there exists $C_\varepsilon > 0$, $N_\varepsilon \geq 1$ such that for any $N \geq N_\varepsilon$

- (1) the Ruelle-Pollicott resonances of \hat{F}_N is contained in a small neighborhood of the union of annuli $(\mathcal{A}_k := \{r_k^- \leq |z| \leq r_k^+\})_{k \geq 0}$:

$$\text{Res}(\hat{F}_N) \subset \bigcup_{k \geq 0} \underbrace{\{r_k^- - \varepsilon \leq |z| \leq r_k^+ + \varepsilon\}}_{\varepsilon\text{-neighborhood of } \mathcal{A}_k} \quad (1.27)$$

with

$$\begin{aligned} r_k^- &:= \liminf_{n \rightarrow \infty} \inf_{x \in M} \left(e^{\frac{1}{n} D_n(x)} \|Df_x^n|_{E_u}\|_{\max}^{-k/n} \right), \\ r_k^+ &:= \limsup_{n \rightarrow \infty} \sup_{x \in M} \left(e^{\frac{1}{n} D_n(x)} \|Df_x^n|_{E_u}\|_{\min}^{-k/n} \right) \end{aligned} \quad (1.28)$$

355

- (2) Suppose that $r_k^+ < r_{k-1}^-$ for some $k \geq 1$. For any $z \in \mathbb{C}$ such that $r_k^+ + \varepsilon < |z| < r_{k-1}^- - \varepsilon$, i.e. such that z is in a “gap”, the resolvent of \hat{F}_N on $\mathcal{H}_N^r(P)$ is controlled uniformly with respect to N :

$$\left\| \left(z - \hat{F}_N \right)^{-1} \right\| \leq C_\varepsilon \quad (1.29)$$

This is true also for $|z| > r_0^+ + \varepsilon$.

- (3) If $r_1^+ < r_0^-$, i.e. if the outmost annulus \mathcal{A}_0 is isolated from other annuli, then the number of resonances in its neighborhood satisfies the estimate called “**Weyl formula**”

$$\#\left\{ \text{Res}(\hat{F}_N) \cap \{r_0^- - \varepsilon \leq |z| \leq r_0^+ + \varepsilon\} \right\} = N^d \text{Vol}_\omega(M) (1 + O(N^{-\delta})). \quad (1.30)$$

with $\text{Vol}_\omega(M) := \int_M \frac{1}{d!} \omega^{\wedge d}$ being the symplectic volume of M and $\delta > 0$.

The proof of Theorem 1.17 is given in Section 7. There in Theorem 7.1, we will provide a more detailed but more technical result about the operators \hat{F}_N , without any assumption on existence of gaps $r_k^+ < r_{k-1}^-$. The Weyl law (1.30) is obtained in Corollary 1.64. We will give a more precise value in Theorem 1.21 below.

360 *Remark 1.18.*

- (1) Since $\|Df_x^n|_{E_u}\|_{\max}^{1/n} \geq \|Df_x^n|_{E_u}\|_{\min}^{1/n} > \lambda > 1$, from (1.2), we have obviously $r_{k+1}^- < r_k^-$ and $r_{k+1}^+ < r_k^+$ for every $k \geq 0$. However we don’t always have $r_{k+1}^+ < r_k^-$ therefore the annuli \mathcal{A}_k may intersect each other.

365 (2) From Theorem 1.17 it is tempting to take the potential $V = V_0$ defined in (1.24) which would indeed give $D = 0$ hence $r_0^+ = r_0^- = 1$ in (1.28). In that case the external band \mathcal{A}_0 would be the unit circle, separated from the internal band \mathcal{A}_1 by a spectral gap r_1^+ given by

$$r_1^+ = \limsup_{n \rightarrow \infty} \sup_{x \in M} \left(\|Df_x^n|_{E_u}\|_{\min}^{-1/n} \right) < \frac{1}{\lambda} < 1$$

370 However Theorem 1.17 does not apply in this case because the function V_0 is not smooth in x as required. This is the purpose of the next Section 1.4 to show how to handle this non smooth potential V_0 and still get spectral results similar to Theorem 1.17. For the moment let us remark that if $\tilde{E}_u \subset TM$ is a smooth approximation of the unstable sub-bundle $E_u \subset TM$ in C^0 norm and if one chooses the potential:

$$V_0(x) = \frac{1}{2} \log \left| \det Df_x|_{\tilde{E}_u(x)} \right| \tag{1.31}$$

then one can have r_0^-, r_0^+ (arbitrarily) closed to one and the annulus \mathcal{A}_0 of the external band get isolated from the other ones ($\mathcal{A}_0 \cap \mathcal{A}_k = \emptyset$ for $k \neq 0$).

375 (3) In the simple case of the linear hyperbolic map on the torus \mathbb{T}^2 in (1.4) with $V(x) = 0$, we have $r_k^+ = r_k^- = \lambda^{-k-\frac{1}{2}}$ with $\lambda = Df_{0/E_u} = \frac{3+\sqrt{5}}{2} \simeq 2.6$ (constant), and each annulus \mathcal{A}_k is a circle. In this case Theorem 1.17 has been obtained in [17] and is depicted on Figure (1-b) in [17]. If one chooses $V(x) = \frac{1}{2} \log |\det Df_x|_{E_u}| = \frac{1}{2} \log \lambda$ the external band \mathcal{A}_0 is the unit circle and it is shown in [17] that the Ruelle-Pollicott resonances on the external band coincide with the spectrum of the quantized map called the “quantum cat map”.

(4) The estimate (1.29) will be useful in Section 1.6 to express dynamical correlation functions.

(5) For a given $\varepsilon > 0$, in (1.27), only a finite number of annuli \mathcal{A}_k can be distinguished.

385 From now on, we suppose that $r_1^+ < r_0^-$, i.e. that the external annulus \mathcal{A}_0 defined in (1.27) is isolated from other annuli $\bigcup_{k \geq 1} \mathcal{A}_k$. We have seen in (1.31), how to achieve this situation by a suitable choice of the potential $V(x)$.

Definition 1.19. Assume $r_1^+ < r_0^-$. For $N = 1/(2\pi\hbar)$ large enough let

$$\Pi_h : \mathcal{H}_N^r(P) \rightarrow \mathcal{H}_N^r(P) \quad (1.32)$$

be the spectral projector of the operator \hat{F}_N on its external band, i.e. on the spectral domain $\{r_0^- - \varepsilon \leq |z| \leq r_0^+ + \varepsilon\}$. Let

$$\mathcal{H}_h := \text{Im}(\Pi_h)$$

that we call the “**quantum space**” which is finite dimensional from (1.30) and let

$$\hat{\mathcal{F}}_h : \mathcal{H}_h \rightarrow \mathcal{H}_h \quad (1.33)$$

be the finite dimensional spectral restriction of \hat{F}_N on the exterior annulus \mathcal{A}_0 . We call $\hat{\mathcal{F}}_h$ the “**quantum operator**”.

390 *Remark 1.20.* We will justify in Section 1.6 this name of “quantum operator”.

In (1.30), Weyl law gives the leading order for the value of $\dim \mathcal{H}_h$. The next Theorem gives its exact value.

Theorem 1.21. “Index formula for the number of resonances”. *If the external annulus \mathcal{A}_0 is isolated, i.e. $r_1^+ < r_0^-$, then the number of resonances in the external annulus \mathcal{A}_0 is given by the Atiyah-Singer index formula: for N large enough,*

$$\dim \mathcal{H}_h = \int_M [e^{N\omega} \text{Todd}(TM)]_{2d} \quad (1.34)$$

where

$$e^{N\omega} = 1 + N\omega + \dots + \frac{N^d \omega^d}{d!}$$

is the Chern character and

$$\text{Todd}(TM) = \det \left(\frac{\Omega(TM)}{1 - \exp(-\Omega(TM))} \right) = 1 + \frac{\Omega(TM)}{2} + \dots \in H_{DR}^\bullet(M)$$

is the Todd class of the tangent bundle defined from the Riemannian curvature $\Omega(TM)$ and $[\cdot]_{2d}$ denotes the restriction to volume $2d$ -forms. Consequently we recover “Weyl formula” of (1.30) and the remainder is also better:

$$\dim \mathcal{H}_h = N^d \text{Vol}_\omega(M) + \mathcal{O}(N^{d-1}) \quad (1.35)$$

395 Theorem 1.21 above follows from Theorem 3.5 where we will introduce a differential operator $\Delta = D^*D$ acting in $C_N^\infty(P)$ called the *rough Laplacian*. In Theorem 3.5, we will show that its low energy spectrum has band spectrum and that the cardinality of the eigenvalues in the first (i.e. the lowest) band equals the quantity on the right hand side of the formula (1.34). The latter is actually a consequence of a theorem in geometry. We will
 400 also show that the rank of the projector Π_0 coincides with the rank of the spectral projector for eigenvalues in the first band. We thus obtain the formula (1.34). Then the Weyl formula (1.35) is a direct consequence. Indeed we have $[e^{N\omega}\text{Todd}(TM)]_{2d} = \frac{N^d \omega^d}{d!} + O(N^{d-1})$ and hence

$$\int_M [e^{N\omega}\text{Todd}(TM)]_{2d} = N^d \text{Vol}_\omega(M) + \mathcal{O}(N^{d-1}).$$

Remark 1.22. In the case of $M = \mathbb{T}^2$ which correspond to example (1.4) and treated in
 405 [17], the projector Π_0 has exactly rank $(\Pi_0) = N$. Indeed, for Riemann surfaces M of genus g , we have $\text{Todd}(TM) = 1 + \frac{c_1(TM)}{2}$ with first Chern number $\int_M c_1(TM) = 2 - 2g$ (the Gauss-Bonnet integral formula). Hence $\text{rank}(\Pi_0) = \int_M (N\omega) + \int_M c_1(TM) = N$ for $M = \mathbb{T}^2$ with genus $g = 1$.

For the next Theorem, recall the definition of the damping function $D(x) = V(x) -$
 410 $V_0(x)$ in (1.25).

Theorem 1.23. “Distribution of resonances”. Assume $r_1^+ < r_0^-$. In the limit $\hbar \rightarrow 0$, most of eigenvalues of $\hat{\mathcal{F}}_\hbar$ concentrate and equidistribute on the circle of radius

$$R := e^{\langle D \rangle}, \quad \text{with } \langle D \rangle := \frac{1}{\text{Vol}_\omega(M)} \int_M D(x) dx \quad (1.36)$$

(See Figure 1.4). More precisely, for any $\varepsilon > 0$, we have

$$\lim_{N \rightarrow \infty} \frac{\#\left\{\text{Res}(\hat{F}_N) \cap \{|z| - R| < \varepsilon\}\right\}}{\#\{\text{Res}(\hat{F}_N)\}} = 1$$

and for any $0 \leq \theta_1 < \theta_2 \leq 2\pi$,

$$\lim_{N \rightarrow \infty} \frac{\#\left\{\text{Res}(\hat{F}_N) \cap \{\theta_1 < \arg(z) < \theta_2\}\right\}}{\#\{\text{Res}(\hat{F}_N)\}} = \frac{\theta_2 - \theta_1}{2\pi}.$$

The proof of Theorem 1.23 will be given in Section 10 .

Remark 1.24. With some pinching conditions, it is possible to show that the resonances in
 415 the internal bands \mathcal{A}_k also concentrate uniformly on circles.

Remark 1.25. The proof of Theorem 1.23 uses ergodicity of the map $f : M \rightarrow M$ and follows a techniques presented by J. Sjöstrand in [49] for the damped wave equation. Using mixing and large deviations properties of the map f it may be possible to improve the results as in [2].

1.4 Spectral results with extended models on the Grassmanian bundle

In this Section we extend the previous results for a family of prequantum transfer operators more general than that considered in Theorem 1.17 in the sense that we will admit some functions V for the potential, defined in (1.39) below that may be only Hölder continuous. This is the case of V_0 given in (1.24). The trick is to consider transfer operators associated to the dynamics of $f : M \rightarrow M$ lifted on the d -dimensional Grassmanian bundle $p : G_d(TM) \rightarrow M$ so that the potential function V on M is derived from a smooth potential function \tilde{V} on $G_d(TM)$. We explain now this construction.

1.4.1 The Grassmanian bundle $G \rightarrow M$ and the lifted map f_G

At a given point $x \in M$ of the manifold M , recall from Remark 1.2(1) that the stable and unstable linear space $E_s(x), E_u(x) \subset T_xM$, $x \in M$ have dimension $d = \dim M/2$ each. For that reason we consider all the d -dimensional linear subspaces of T_xM :

Definition 1.26. At a given point $x \in M$, the d -dimensional linear subspaces of T_xM form a compact manifold of dimension d^2 called the **Grassmanian**⁵ $G_d(T_xM)$. The **Grassmanian bundle** is the bundle

$$G_d(TM) \xrightarrow{p} M \tag{1.37}$$

whose base space is M and the fiber over point $x \in M$ is the Grassmanian $G_d(T_xM)$. For simplicity we will denote $p : G \rightarrow M$ this bundle and $G_x := G_d(T_xM)$ the fiber.

Definition 1.27. The diffeomorphism $f : M \rightarrow M$ induces a natural lifted map

$$f_G = Df : G \rightarrow G \tag{1.38}$$

where (abusively) Df_x denotes the induced action of the differential Df_x on linear subspaces $l \in G_x$. By definition we have a commutative diagram:

$$\begin{array}{ccc} G & \xrightarrow{f_G} & G \\ \downarrow p & & \downarrow p \\ M & \xrightarrow{f} & M \end{array}$$

⁵For $x \in M$, the Grassmanian is a homogeneous space

$$G_x = G_d(T_xM) \cong \frac{O(2d)}{O(d) \times O(d)}, \quad \dim G_x = d^2.$$

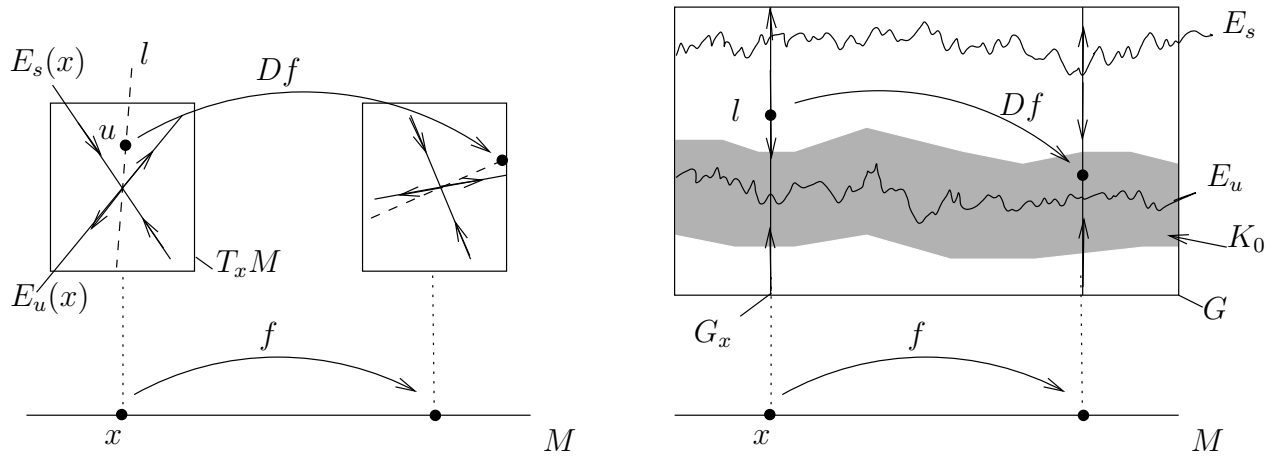


Figure 1.5:

Remark 1.28. The stable d -dimensional bundle $E_s \rightarrow M$ defines a Hölder continuous section E_s of the bundle $G \rightarrow M$ and from Definition 1.1, E_s is a repeller for the map f_G . Similarly the unstable bundle E_u is a Hölder continuous section of the bundle $G \rightarrow M$. The image of E_u (for which we also write E_u) is an attractor for the map f_G . See Figure 1.5.

1.4.2 Prequantum transfer operator \tilde{F}_N on $P_G \rightarrow G$

Let

$$\tilde{V} \in C^\infty(G)$$

be a smooth real valued function called **potential function** and its restriction to the unstable bundle will be denoted by:

$$V := \tilde{V} \circ E_u \in C^\beta(M) \tag{1.39}$$

which is Hölder continuous with Hölder exponent $0 < \beta \leq 1$.

Remark 1.29.

(1) In Definition 1.9 we have considered the particular case of a smooth function $V \in C^\infty(M)$ which can be derived here from the function $\tilde{V} := V \circ p \in C^\infty(G)$ (i.e. constant on the fibers). So results presented in this Section will also apply to the prequantum operator (1.19).

(2) The special potential function $V_0 := \tilde{V}_0 \circ E_u$ in (1.24) is derived as in (1.39) from the smooth function $\tilde{V}_0 \in C^\infty(G)$ given by

$$\tilde{V}_0(l) = \frac{1}{2} \log(\det Df_x | l), \quad l \in G, \quad x = p(l) \tag{1.40}$$

Recall the principal $U(1)$ bundle $\pi : P \rightarrow M$ and the prequantum map $\tilde{f} : P \rightarrow P$ defined in Theorem 1.4.

Definition 1.30. The principal bundle $\pi : P \rightarrow M$ with connection A can be pulled back by $p : G \rightarrow M$ on G and gives a principal bundle

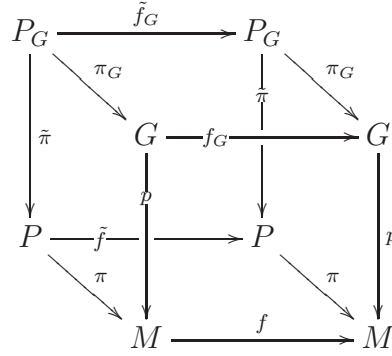
$$\pi_G : P_G \rightarrow G$$

with connection. Let

$$\tilde{f}_G : P_G \rightarrow P_G \tag{1.41}$$

be the lift of the map $\tilde{f} : P \rightarrow P$ on $P_G \rightarrow P$.

Remark 1.31. Of course \tilde{f}_G is also a lift of the map $f_G : G \rightarrow G$, i.e $\pi_G \circ \tilde{f}_G = f_G \circ \pi_G$. By construction, for any $x \in M$, the restricted bundle $P_G \rightarrow G_x$ is trivial. In fact we have a 3 dimensional commutative diagram:



Definition 1.32. Let $\tilde{V} \in C^\infty(G)$. The **prequantum transfer operator** is defined by

$$\tilde{F} : \begin{cases} C^\infty(P_G) & \rightarrow C^\infty(P_G) \\ u & \mapsto e^{\tilde{V}} \cdot |\det(Df_G|_{\ker p})|^{-1} \cdot u \circ \tilde{f}_G^{-1} \end{cases} \tag{1.42}$$

It preserves the space of the N -th Fourier modes for every $N \in \mathbb{Z}$:

$$C_N^\infty(P_G) := \{u \in C^\infty(P_G) \mid \forall p \in P_G, \forall e^{i\theta} \in U(1), u(e^{i\theta}p) = e^{iN\theta}u(p)\} \tag{1.43}$$

and its restriction is denoted

$$\tilde{F}_N := \tilde{F}|_{C_N^\infty(P_G)} : C_N^\infty(P_G) \rightarrow C_N^\infty(P_G). \tag{1.44}$$

Remark 1.33. In the definition (1.42), the additional “potential function” $|\det(Df_G|_{\ker p})|^{-1}$ has been put in order to compensate the attraction effect of the attractor E_u on the extended space G . (See the next subsection.)

1.4.3 Truncation in a neighborhood of E_u

In order to define the discrete spectrum of resonances we first have to consider a specific truncation of the operator. Let $K_0 \subset G$ be an open absorbing neighborhood of the attractor E_u and

$$K_1 := f_G(K_0) \Subset K_0 \quad (1.45)$$

i.e. $\overline{K_1}$ is a proper subset of K_0 . See Figure 1.5. For any $n \geq 1$, let

$$K_n := f_G^n(K_0). \quad (1.46)$$

Then, by definition (of the absorbing neighborhood), we have that

$$E_u = \bigcap_{n=1}^{\infty} K_n. \quad (1.47)$$

Let $\chi \in C^\infty(G)$ be a function such that $\chi(l) = 0$ for $l \notin K_0$, $\chi(l) = 1$ for $l \in K_1$. We denote

$$\hat{\chi} : C^\infty(P_G) \rightarrow C^\infty(P_G) \quad (1.48)$$

the multiplication operator by the function $\chi \circ p$, where $p : G \rightarrow M$ is the projection. For any $n \geq 1$ we have from (1.45) that

$$\left(\tilde{F}\hat{\chi}\right)^n = \tilde{F}^n\hat{\chi} \quad (1.49)$$

Also $\hat{\chi}$ preserves the space of equivariant functions $C_N^\infty(P_G)$ defined in (1.43). By duality the operator $\tilde{F}_N\hat{\chi}$ extends to equivariant distributions $\mathcal{D}'_N(P_G)$. From definition (1.46) and (1.49) we have that, for any $n \geq 1$ and $u \in \mathcal{D}'(P_G)$,

$$\text{supp} \left(\left(\tilde{F}_N\hat{\chi}\right)^n u \right) \subset \pi_G^{-1}(K_n). \quad (1.50)$$

1.4.4 Results on the spectrum of the prequantum operator \tilde{F}_N

The following theorem (and its proof) is similar to Theorem 1.15 but concerns the transfer operator \tilde{F}_N defined in (1.44).

Theorem 1.34. “Discrete spectrum”. For every $N \in \mathbb{Z}$, there exists a family of Hilbert spaces $\mathcal{H}_N^r(P_G)$ for arbitrarily large $r > 0$, such that $C_N^\infty(P_G) \subset \mathcal{H}_N^r(P_G) \subset \mathcal{D}'_N(P_G)$ and such that the operator $\tilde{F}_N \hat{\chi}$ extends to a bounded operator

$$\tilde{F}_N \hat{\chi} : \mathcal{H}_N^r(P_G) \rightarrow \mathcal{H}_N^r(P_G),$$

490 and its essential spectral radius $r_{ess}(\hat{F}_N \hat{\chi})$ is bounded by $\varepsilon_r := \frac{1}{\lambda^r} \max e^{\tilde{V}}$, which shrinks to zero if $r \rightarrow +\infty$. The discrete eigenvalues of $\hat{F}_N \hat{\chi}$ on the domain $|z| \geq \varepsilon_r$ (and their associated eigenspaces) are independent on the choice of χ and r . The support of an eigendistribution is contained in the attractor $\pi_G^{-1}(E_u)$. These discrete eigenvalues are called Ruelle-Pollicott resonances and are denoted $\text{Res}(\tilde{F}_N) := \{\lambda_i\}_i \subset \mathbb{C}^*$.

The fact that the support of an eigendistribution is contained in the attractor $\pi_G^{-1}(E_u)$ is a direct consequence of (1.50) and (1.47).

The next Theorem is similar to Theorem 1.17 but here we restrict ourselves to the description of the external band

$$\mathcal{A}_0 := \{z \in \mathbb{C}, |z| \in [r_0^-, r_0^+]\}$$

495 although description of internal bands may be possible also. Recall that $V(x)$ is defined in (1.39), $D(x) := V(x) - V_0(x)$ is the damping function and $D_n(x) = \sum_{j=1}^n D(f_G^j(x))$ is the Birkhoff sum of $D(x)$.

Theorem 1.35. “External band”. For any $\varepsilon > 0$, there exists $C_\varepsilon > 0$, $N_\varepsilon \geq 1$ such that, for any $N \geq N_\varepsilon$, we have

$$\text{Res}(\tilde{F}_N) \subset \left\{ z \in \mathbb{C}, |z| \in [0, r_1^+ + \varepsilon] \cup [r_0^- - \varepsilon, r_0^+ + \varepsilon] \right\}$$

with

$$r_0^- := \liminf_{n \rightarrow \infty} \inf_{x \in M} \left(e^{\frac{1}{n} D_n(x)} \right), \quad r_0^+ := \limsup_{n \rightarrow \infty} \sup_{x \in M} \left(e^{\frac{1}{n} D_n(x)} \right), \quad (1.51)$$

$$r_1^+ := \limsup_{n \rightarrow \infty} \sup_{x \in M} \left(e^{\frac{1}{n} D_n(x)} \|Df_x^n|_{E_u}\|_{\min}^{-1/n} \right)$$

For any $z \in \mathbb{C}$ such that $r_1^+ + \varepsilon < |z| < r_0^- - \varepsilon$ or $|z| > r_0^+ + \varepsilon$ we have:

$$\left\| \left(z - \tilde{F}_N \right)^{-1} \right\| \leq C_\varepsilon. \quad (1.52)$$

500 In the rest of this section we will assume that the potential \tilde{V} is such that the external annulus \mathcal{A}_0 is isolated i.e. $r_1^+ < r_0^-$, giving a “spectral gap”. We take the same definition of \mathcal{H}_h and $\hat{\mathcal{F}}_h$ as in Definition 1.19, for the spectral restriction of \tilde{F}_N to the external band \mathcal{A}_0 .

Theorem 1.36. Index formula and Weyl law:

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$$\dim \mathcal{H}_h = \int_M [e^{N\omega} \text{Todd}(TM)]_{2d} = N^d \text{Vol}_\omega(M) + \mathcal{O}(N^{d-1}) \quad (1.53)$$

The next Theorem is a particular case of Theorem 1.35, but there, compared to 1.17, we emphasize again that the main interest is the case of the particular smooth potential \tilde{V}_0 in ((1.40)) giving $V_0(x) = \frac{1}{2} \log |\det Df_x|_{E_u(x)}|$, $D = 0$ hence $r_0^+ = r_0^- = 1$ in (1.51), so that the external annulus \mathcal{A}_0 coincides with the unit circle.

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Theorem 1.37. *Let \tilde{F}_N be the transfer operator defined in (1.42) with the special choice of the smooth potential $\tilde{V}_0(l) = \frac{1}{2} \log(\det Df_x|_l)$ on G . Then the operator $\tilde{F}_N \hat{\chi}$ extends to a bounded operator on $\mathcal{H}_N^r(P_G)$. The Ruelle spectrum of $\tilde{F}_N \hat{\chi}$ **concentrates on the unit circle** for $N = 1/(2\pi\hbar) \rightarrow \infty$ and is separated from the internal resonances by a non vanishing asymptotic spectral gap ($r_1^+ < r_0^- = r_0^+ = 1$). That is, for any given $\varepsilon > 0$, its (discrete) spectrum is contained in*

$$\{|z| - 1| < \varepsilon\} \cup \{|z| < r_1^+ + \varepsilon\}$$

for sufficiently large N . The spectrum in $\{|z| - 1| < \varepsilon\}$ obeys the Weyl law and the angular equidistribution law stated in Theorem 1.23. (See Figure 1.6).

We will call the potential function \tilde{V}_0 above the “**potential of reference**”.

1.5 Gutzwiller trace formula

515 In this Section we continue to consider the prequantum transfer operators $\tilde{F}_N \hat{\chi}$ on the Grassmanian bundle P_G , defined in (1.42). We assume the condition $r_1^+ < r_0^-$. (This condition holds if we consider the potential of reference \tilde{V}_0 .) As in Definition 1.19, let $\Pi_h : \mathcal{H}_N^r(P_G) \rightarrow \mathcal{H}_N^r(P_G)$ be the spectral projector for the external band and let \mathcal{H}_h be its image. Let $\hat{\mathcal{F}}_h : \mathcal{H}_h \rightarrow \mathcal{H}_h$ be the restriction of $\tilde{F}_N \hat{\chi}$ to \mathcal{H}_h .

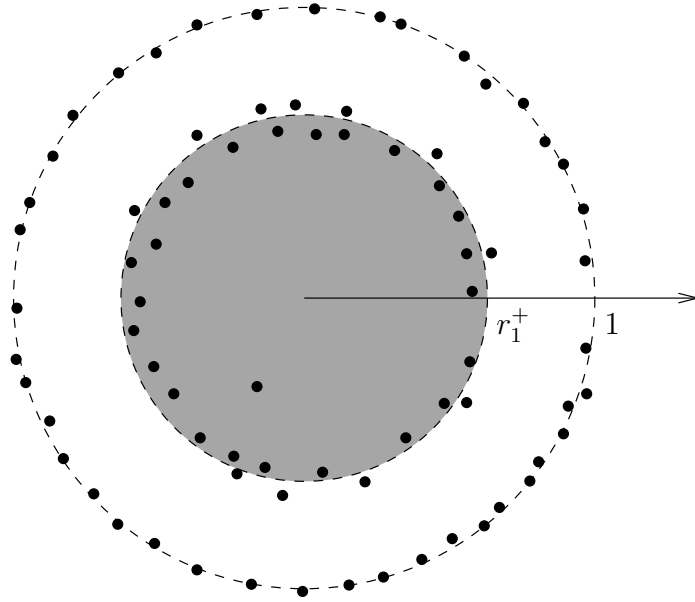


Figure 1.6: With the particular potential $V_0 = \frac{1}{2} \log |\det Df_x|_{E_u(x)}|$ Corollary 1.37 shows that the external spectrum of the transfer operator concentrates uniformly on the unit circle as $N = 1/(2\pi\hbar) \rightarrow \infty$. (We have not represented here the structure of the internal bands inside the disc of radius r_1^+).

Theorem 1.38. “Gutzwiller trace formula for large time”. Let $\varepsilon > 0$. For any $\hbar = 1/(2\pi N)$ small enough, in the limit $n \rightarrow \infty$, we have

$$520 \quad \left| \text{Tr} \left(\hat{\mathcal{F}}_{\hbar}^n \right) - \sum_{x=f^n(x)} \frac{e^{D_n(x)} e^{iS_{n,x}/\hbar}}{\sqrt{|\text{Det}(1 - Df_x^n)|}} \right| < CN^d (r_1^+ + \varepsilon)^n \quad (1.54)$$

where $e^{i2\pi S_{n,x}}$ is the **action** of a periodic point defined in (1.18) and D_n is the Birkhoff sum (1.26) of the effective damping function $D(x) = V(x) - V_0(x)$.

The proof of Theorem 1.38 will be given in Section 10. It is based on the general and remarkable flat trace formula of Atiyah-Bott that we recall in Lemma 11.2.

Remark 1.39. We will see in Proposition 1.42 below, the simple but remarkable fact that the formula (1.54) determines the spectrum of $\hat{\mathcal{F}}_{\hbar}$ with multiplicities.

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Remark 1.40. For large time n we have the equivalence

$$|\text{Det}(1 - Df_x^n)|^{-1/2} \sim |\text{Det}(Df_x^n|_{E_u(x)})|^{-1/2} = e^{-(V_0)_n(x)}$$

Therefore one can expect that

$$\mathrm{Tr} \left(\hat{\mathcal{F}}_h^n \right) = \sum_{x=f^n(x)} e^{D_n(x)} e^{iS_{n,x}/\hbar} |\mathrm{Det} (Df_x^n |_{E_u(x)})|^{-1/2} + \mathcal{O}(1) N^d e^{\sup_x D_n} \lambda^{-n} \quad (1.55)$$

$$= \sum_{x=f^n(x)} e^{V_n(x)} e^{iS_{n,x}/\hbar} |\mathrm{Det} (Df_x^n |_{E_u(x)})|^{-1} + \mathcal{O}(1) N^d e^{\sup_x D_n} \lambda^{-n} \quad (1.56)$$

This is indeed true and this will be obtained in the proof. Let us remark however that (1.55) is not an immediate consequence of (1.54) and vice-versa due to the control of the remainder. Indeed, suppose for example that (1.54) holds, with $D = V - V_0 = 0$ (for simplicity). We have for individual periodic points x and for $n \rightarrow \infty$,

$$\frac{1}{\sqrt{|\mathrm{Det} (1 - Df_x^n)|}} = |\mathrm{Det} (Df_x^n |_{E_u(x)})|^{-1/2} + \mathcal{O}(1) \lambda^{-\frac{dn}{2}-n}$$

but the number of periodic points grows like $\# \{x = f^n(x)\} \geq \lambda^{dn}$ in general with $d = \frac{1}{2} \dim M$. Hence we can only deduce that

$$\left| \sum_{x=f^n(x)} \frac{e^{iS_{n,x}/\hbar}}{\sqrt{|\mathrm{Det} (1 - Df_x^n)|}} - \sum_{x=f^n(x)} e^{iS_{n,x}/\hbar} |\det (Df_x^n |_{E_u(x)})|^{-1/2} \right| = \mathcal{O} \left(\lambda^{dn} \lambda^{-\frac{dn}{2}-n} \right) \\ = \mathcal{O} \left(\left(\lambda^{\frac{d}{2}-1} \right)^n \right)$$

and if $d \geq 2$ the term in the remainder is $\left(\lambda^{\frac{d}{2}-1} \right) \geq 1$, this is not enough to deduce (1.55).

1.5.1 The question of existence of a “natural quantization”

The following problem is a recurrent question in mathematics and physics in the field of quantum chaos, since the discovery of the Gutzwiller trace formula. For simplicity of the discussion we consider $V = V_0$ i.e. no effective damping, as in Figure 1.6.

Problem 1.41. Does there exist a sequence $\hbar_j > 0$, $\hbar_j \rightarrow 0$ with $j \rightarrow \infty$, such that for every $\hbar = \hbar_j$,

- (1) there exists a space \mathcal{H}_\hbar of **finite dimension**, an operator $\hat{\mathcal{F}}_\hbar : \mathcal{H}_\hbar \rightarrow \mathcal{H}_\hbar$ which is **quasi unitary** in the sense that there exists $\varepsilon_\hbar \geq 0$ with $\varepsilon_\hbar \rightarrow 0$, with $j \rightarrow \infty$ and

$$\forall u \in \mathcal{H}_\hbar, (1 - \varepsilon_\hbar) \|u\| \leq \left\| \hat{\mathcal{F}}_\hbar u \right\| \leq (1 + \varepsilon_\hbar) \|u\| \quad (1.57)$$

- (2) The operator $\hat{\mathcal{F}}_\hbar$ satisfies the **asymptotic Gutzwiller Trace formula** for large time; i.e. there exists $0 < \theta < 1$ independent on \hbar and some $C_\hbar > 0$ which may depend on \hbar , such that for \hbar small enough (such that $\theta < 1 - \varepsilon_\hbar$):

$$\forall n \in \mathbb{N}, \left| \text{Tr} \left(\hat{\mathcal{F}}_\hbar^n \right) - \sum_{x=f^n(x)} \frac{e^{iS_{x,n}/\hbar}}{\sqrt{|\det(1 - Df_x^n)|}} \right| \leq C_\hbar \theta^n \quad (1.58)$$

Let us notice first that Theorem 1.38 (for the case $V = V_0$) provides a solution to Problem 1.41: this is the quantum operator $\hat{\mathcal{F}}_\hbar : \mathcal{H}_\hbar \rightarrow \mathcal{H}_\hbar$ defined in (1.33) obtained with the choice of potential $\tilde{V} = \tilde{V}_0$, Eq.(1.40), giving $V = V_0$. Indeed (1.57) holds true from Corollary 1.37 and (1.58) holds true from (1.54) and because $\theta := r_1^+ + \varepsilon < 1$.

Some importance of the Gutzwiller trace formula (1.58) comes from the following property which shows uniqueness of the solution to the problem:

Proposition 1.42. *If $\hat{\mathcal{F}}_\hbar : \mathcal{H}_\hbar \rightarrow \mathcal{H}_\hbar$ is a solution of Problem 1.41 then the spectrum of $\hat{\mathcal{F}}_\hbar$ is **uniquely defined** (with multiplicities). In particular $\dim(\mathcal{H}_\hbar)$ is uniquely defined.*

Proof. This is consequence of the following lemma.

Lemma 1.43. *If A, B are matrices and for any $n \in \mathbb{N}$, $|\text{Tr}(A^n) - \text{Tr}(B^n)| < C\theta^n$ with some $C > 0$, $\theta \geq 0$ then A and B have the same spectrum with same multiplicities on the spectral domain $|z| > \theta$.*

Proof of Lemma 1.43. From the formula⁶:

$$\det(1 - \mu A) = \exp \left(- \sum_{n \geq 1} \frac{\mu^n}{n} \text{Tr}(A^n) \right)$$

⁶This formula is easily proved by using eigenvalues λ_j of A and the Taylor series of $\log(1 - x) =$

The sum on the right is convergent if $1/|\mu| > \|A\|$. Notice that we have (with multiplicities): μ is a zero of $d_A(\mu) = \det(1 - \mu A)$ if and only if $z = \frac{1}{\mu}$ is a (generalized) eigenvalue of A . Using the formula we get that if $1/|\mu| > \theta$ then

$$\begin{aligned} \left| \frac{\det(1 - \mu A)}{\det(1 - \mu B)} \right| &\leq \exp \left(\sum_{n \geq 1} \frac{|\mu|^n}{n} |\operatorname{Tr}(A^n) - \operatorname{Tr}(B^n)| \right) \\ &< \exp \left(C \sum_{n \geq 1} \frac{(|\mu|\theta)^n}{n} \right) = (1 - \theta|\mu|)^{-C} =: \mathcal{B} \end{aligned}$$

Similarly $\left| \frac{\det(1 - \mu A)}{\det(1 - \mu B)} \right| > \frac{1}{\mathcal{B}}$, hence $d_A(\mu)$ and $d_B(\mu)$ have the same zeroes on $1/|\mu| > \theta$. Equivalently A and B have the same spectrum on $|z| > \theta$. \square

If \hat{G}_h is another solution of the problem 1.41 then (1.58) implies that $\left| \operatorname{Tr}(\hat{\mathcal{F}}_h^n) - \operatorname{Tr}(\hat{G}_h^n) \right| \leq 2C\theta^n$ and Lemma 1.43 tells us that \hat{G}_h and $\hat{\mathcal{F}}_h$ have the same spectrum on $|z| > \theta$. But by hypothesis (1.57) their spectrum is in $|z| > 1 - \varepsilon_h > \theta$. Therefore all their spectrum coincides. This finishes the proof of Proposition 1.42. \square

Remark 1.44. Previous results in the literature concerning the “semiclassical Gutzwiller formula” for “quantum maps” do not provide an answer to the problem 1.41 above. We explain why. For any reasonable quantization of the Anosov map $f : M \rightarrow M$, e.g. the Weyl quantization or geometric quantization, one obtains a family of unitary operators $\hat{\mathcal{F}}_h : \mathcal{H}_h \rightarrow \mathcal{H}_h$ acting in some finite dimensional (family of) Hilbert spaces. So this answer to (1.57). Using semiclassical analysis it is possible to show a Gutzwiller formula like (1.58) but with an error term on the right hand side of the form $\mathcal{O}(\hbar\theta^n)$ with $\theta = e^{h_0/2} > 1$ where $h_0 > 0$ is the topological entropy which represents the exponential growing number of periodic orbits ([18] and references therein). Using more refined semiclassical analysis at higher orders, the error can be made

$$\mathcal{O}(\hbar^M \theta^n) \tag{1.59}$$

with any $M > 0$ [18], but nevertheless one has a total error which gets large after the so-called Ehrenfest time: $n \gg M \frac{\log(1/\hbar)}{\lambda_0}$. So all these results obtained from any quantization scheme do not provide an answer to the problem 1.41. We may regard the operator in (1.33) as the only “quantization procedure” for which (1.58) holds true. For that reason we may call it a **natural quantization** of the Anosov map f .

$-\sum_{n \geq 1} \frac{x^n}{n}$ which converges for $|x| < 1$:

$$\begin{aligned} \det(1 - \mu A) &= \prod_j (1 - \mu \lambda_j) = \exp \left(\sum_j \log(1 - \mu \lambda_j) \right) \\ &= \exp \left(- \sum_j \sum_{n \geq 1} \frac{(\mu \lambda_j)^n}{n} \right) = \exp \left(- \sum_{n \geq 1} \frac{\mu^n}{n} \operatorname{Tr}(A^n) \right) \end{aligned}$$

1.6 Dynamical correlation functions and emergence of quantum dynamics

As explained in [19, cor. 1.3] for example, the Ruelle-Pollicott spectrum of the transfer operator \hat{F} has an important meaning in terms of time evolution of correlation functions. If $u, v \in C^\infty(P)$ the time correlation function is defined by

$$C_{v,u}(n) := \int_P \overline{v(p)} \left(\hat{F}^n u \right) (p) d\mu_P = \left(v, \hat{F}^n u \right)_{L^2(P)}.$$

In this Section we show that $C_{v,u}(n)$ can be expressed as an asymptotic over the Ruelle resonances, up to exponentially small error term. In particular we emphasize the role of the external band in the spectrum as a manifestation of “quantum behavior” in the fluctuations of the correlation functions. In this subsection and the next, we consider the transfer operators on P (not the Grassmanian extension P_G).

1.6.1 Use of the resolvent estimate for dynamical control

Before giving results about dynamical correlation functions, we give a proposition which expresses differently (but equivalently) the estimate (1.52) about the resolvent. Recall from (1.33) that if $r_1^+ < r_0^+$, then for N large enough, the transfer operator has a spectral decomposition $\tilde{F}_N = \hat{\mathcal{F}}_h + \left(\hat{F}_N - \hat{\mathcal{F}}_h \right)$ into a finite rank operator $\hat{\mathcal{F}}_h$ for the external band and $\left(\hat{F}_N - \hat{\mathcal{F}}_h \right)$ is for the internal structure.

Proposition 1.45. *For any $\varepsilon > 0$, there exists $C_\varepsilon > 0$ and $N_\varepsilon \geq 1$ such that for any $N \geq N_\varepsilon$ and for any $n \geq 0$,*

$$\left\| \hat{F}_N^n \right\| \leq C_\varepsilon (r_0^+ + \varepsilon)^n \quad (1.60)$$

Moreover if $r_1^+ < r_0^-$ then

$$\left\| \hat{\mathcal{F}}_h^n \right\|_{\max} \leq C_\varepsilon (r_0^+ + \varepsilon)^n, \quad \left\| \hat{\mathcal{F}}_h^n \right\|_{\min} \geq \frac{1}{C_\varepsilon} (r_0^- - \varepsilon)^n \quad (1.61)$$

and

$$\left\| \left(\hat{F}_N - \hat{\mathcal{F}}_h \right)^n \right\| \leq C_\varepsilon (r_1^+ + \varepsilon)^n. \quad (1.62)$$

Remark 1.46. More generally, if there is some internal isolated band k , i.e. if $r_{k+1}^+ < r_k^-, r_k^+ < r_{k-1}^-$ for some $k \geq 1$, then we can consider the corresponding spectral decomposition $\hat{F}_N = \dots + \hat{\mathcal{F}}_{k,h} + \dots$ isolating some finite rank operator $\hat{\mathcal{F}}_{k,h}$ and similarly we can show that (1.29) is equivalent to

$$\frac{1}{C_\varepsilon} (r_k^- - \varepsilon)^n \leq \left\| \hat{\mathcal{F}}_{k,h}^n \right\|_{\min} \leq \left\| \hat{\mathcal{F}}_{k,h}^n \right\|_{\max} \leq C_\varepsilon (r_k^+ + \varepsilon)^n.$$

595 *Proof.* Let γ be the closed path in \mathbb{C} made by the union of the circle of radius $r_0^+ + \varepsilon$ in the direct sense and the circle of radius $r_0^- - \varepsilon$ in the indirect sense. From Cauchy formula one has

$$\hat{\mathcal{F}}_h^n = \frac{1}{2\pi i} \oint_{\gamma} z^n (z - \hat{F}_N)^{-1} dz$$

which implies

$$\left\| \hat{\mathcal{F}}_h^n \right\|_{\max} \leq (r_0^+ + \varepsilon)^{n+1} \max_{z \in \gamma} \left\| (z - \hat{F}_N)^{-1} \right\|$$

Then the uniform bound on the resolvent (1.52) implies

$$\left\| \hat{\mathcal{F}}_h^n \right\| \leq C_{\varepsilon} (r_0^+ + \varepsilon)^n$$

which is the first equation of (1.61). Reversing the sign of n one gets

$$\left\| \hat{\mathcal{F}}_h^{-n} \right\| \leq C_{\varepsilon} (r_0^- - \varepsilon)^{-n}.$$

600 Hence

$$\left\| \hat{\mathcal{F}}_h^n \right\|_{\min} = \left\| \hat{\mathcal{F}}_h^{-n} \right\|^{-1} \geq \frac{1}{C_{\varepsilon}} (r_0^- - \varepsilon)^n$$

which is the second equation of (1.61). The bound (1.60) is obtained similarly but more simply with the closed path γ being only the circle of radius $r_0^+ + \varepsilon$ in the direct sense. The estimate (1.62) is obtained with the closed path γ being the circle of radius $r_1^+ + \varepsilon$ in the direct sense. \square

605 1.6.2 Decay of correlations expressed with the quantum operator

We first introduce a notation: for a given $N \in \mathbb{Z}$, we have seen that the prequantum transfer operator \hat{F}_N has a discrete spectrum of resonances. For $\rho > 0$ such that there is no eigenvalue on the circle $|z| = \rho$ for any N , we denote by $\Pi_{\rho, N}$ the projector on the Fourier space of mode N composed with the spectral projector of the operator \hat{F}_N on the domain $\{z \in \mathbb{C}, |z| > \rho\}$. This is a finite rank operator (which obviously commutes with \hat{F}_N). Recall that $\hbar = \frac{1}{2\pi N}$.

Theorem 1.47. *Suppose that $r_1^+ < r_0^-$ (i.e. the external band is isolated). For any $\varepsilon > 0$, there exists $N_{\varepsilon} \geq 1$ such that for any $N \geq N_{\varepsilon}$, for any $u, v \in C^{\infty}(P)$ such that $\text{supp}(u) \subset K_0$, and for $n \rightarrow \infty$, one has*

$$\underbrace{(v, \hat{F}_h^n u)_{L^2}}_{\text{"classical"}} = \sum_{|N| \leq N_{\varepsilon}} (v_N, (\hat{F}_N \Pi_{\rho, N})^n u_N) + \sum_{|N| > N_{\varepsilon}} \underbrace{(v_N, \hat{\mathcal{F}}_h^n u_N)}_{\text{"quantum"}} + O((r_1^+ + \varepsilon)^n) \quad (1.63)$$

with $\rho = r_1^+ + \varepsilon$ and where $u_N, v_N \in C_N^{\infty}(P)$ are the Fourier components of the functions u and v . In the right hand side of (1.63), the first sum is a finite sum and involves finite rank operators. The second sum is an infinite but convergent sum.

Remark 1.48. The asymptotic formula (1.63) has a nice interpretation: the classical correlation functions $(v, \hat{F}^n u)$ are governed by the quantum correlation functions $(v_N, \hat{\mathcal{F}}_h^n u_N)$ for large time (up to the first finite rank expression), or equivalently the “**quantum dynamics emerge dynamically from the classical dynamics**”.

Remark 1.49. From (1.61) one has that

$$\left| (v_N, \hat{\mathcal{F}}_h^n u_N) \right| \leq \|u_N\|_{(\mathcal{H}_N^r)'} \|v_N\|_{\mathcal{H}_N^r} \left\| \hat{\mathcal{F}}_h^n \right\|_{\mathcal{H}_N^r} \leq O(N^{-\infty}) O((r_0^+ + \varepsilon)^n)$$

uniformly in n and N . We have used the fact that for smooth functions one has fast decay in N : $\|u_N\|, \|v_N\| = O(N^{-\infty})$. (This is also because the weight in the space \mathcal{H}_N^r is polynomially bounded in frequencies ξ).

Remark 1.50. It is known that for $n \rightarrow \infty$,

$$(v, \hat{F}^n u) = \lambda_0^n (v, \Pi_{\lambda_0} u) + O(|\lambda_1|^n)$$

where $\lambda_0 > 0$ is the leading and simple eigenvalue of \tilde{F} (in the space $\mathcal{H}_{N=0}^r$) and λ_1 is the second eigenvalue with $|\lambda_1| < \lambda_0$. The case $V = 0$ for which $\lambda_0 = 1$ gives that the map $\tilde{f} : P \rightarrow P$ is mixing with exponential decay of correlations.

Proof. Let N_ε given by Theorem 1.15 or Proposition 1.45 and write

$$(v, \hat{F}^n u) = \sum_{|N| \leq N_\varepsilon} (v_N, (\hat{F}_N \Pi_{\rho, N})^n u_N) + \sum_{|N| \leq N_\varepsilon} (v_N, (\hat{F}_N (1 - \Pi_{\rho, N}))^n u_N) + \sum_{|N| > N_\varepsilon} (v_N, \hat{F}_N^n u_N) \quad (1.64)$$

Let us consider the second term on the right hand side. From definition of $\Pi_{\rho, N}$ one has $r_s(\hat{F}_N (1 - \Pi_{\rho, N})) \leq (r_0^+ + \varepsilon)$ so for the finite number of terms $|N| \leq N_\varepsilon$ one has $\left\| (\hat{F}_N (1 - \Pi_{\rho, N}))^n \right\| \leq C_\varepsilon (r_0^+ + \varepsilon)^n$ and we deduce the estimate

$$\left| \sum_{|N| \leq N_\varepsilon} (v_N, (\hat{F}_N (1 - \Pi_{\rho, N}))^n u_N) \right| = O((r_0^+ + \varepsilon)^n).$$

As in Proposition 1.45 we consider the spectral decomposition $\hat{F}_N = \hat{\mathcal{F}}_h + (\hat{F}_N - \hat{\mathcal{F}}_h)$ and decompose accordingly the last term of (1.64) as

$$\sum_{|N| > N_\varepsilon} (v_N, \hat{F}_N^n u_N) = \sum_{|N| > N_\varepsilon} (v_N, \hat{\mathcal{F}}_h^n u_N) + \sum_{|N| > N_\varepsilon} (v_N, (\hat{F}_N - \hat{\mathcal{F}}_h)^n u_N)$$

From (1.62), one has then

$$\begin{aligned} \left| \left(v_N, \left(\hat{F}_N - \hat{\mathcal{F}}_h \right)^n u_N \right) \right| &\leq \|u_N\|_{(\mathcal{H}_N^r)'} \|v_N\|_{\mathcal{H}_N^r} \left\| \left(\hat{F}_N - \hat{\mathcal{F}}_h \right)^n \right\|_{\mathcal{H}_N^r} \\ &\leq O(N^{-\infty}) O((r_1^+ + \varepsilon)^n) \end{aligned}$$

This implies that $\left| \sum_{|N| > N_\varepsilon} \left(v_N, \left(\hat{F}_N - \hat{\mathcal{F}}_h \right)^n u_N \right) \right| = O((r_1^+ + \varepsilon)^n)$ and we get (1.63). \square

635 1.7 Semiclassical calculus on the quantum space

In Definition 1.19 we have defined the quantum space \mathcal{H}_h for every $\hbar = \frac{1}{2\pi N}$ (small enough), the quantum operator $\hat{\mathcal{F}}_h : \mathcal{H}_h \rightarrow \mathcal{H}_h$ and the finite rank spectral projector $\Pi_h : \mathcal{H}_N^r(P) \rightarrow \mathcal{H}_h$. In this section we introduce the definition of “quantization of symbols” on this quantum space and give some properties of them. In semiclassical analysis these properties are considered as “standard” or “basic properties” for defining “a good semiclassical calculus”. We will comment on them at the end of the Section. Beware that the quantum space \mathcal{H}_h defined here depends on the given Anosov diffeomorphism $f : M \rightarrow M$.

Remark 1.51. The results below extend readily to the case of Grassmanian extension considered in Subsection 1.4 and 1.5.

Definition 1.52. For $0 \leq \delta < 1/2$ and for some family of constant $C_\alpha > 0, \alpha \in \mathbb{N}^{2d}$, we define the **class of symbols**

$$S_\delta := \{ \psi \in C^\infty(M) \text{ s.t. } \forall \alpha \in \mathbb{N}^{2d}, |\partial_x^\alpha \psi| < C_\alpha \hbar^{-\delta|\alpha|} \}$$

A symbol $\psi \in S_\delta$ is therefore a family of smooth functions $(\psi_\hbar)_\hbar$.

Definition 1.53. For any symbol $\psi \in S_\delta$ we define its quantization as the operator

$$\text{Op}_\hbar(\psi) := \Pi_h \circ \mathcal{M}(\psi) \circ \Pi_h \quad : \mathcal{H}_h \rightarrow \mathcal{H}_h \quad (1.65)$$

where $\mathcal{M}(\psi)$ is the multiplication operator by the function ψ in $\mathcal{H}_N^r(P)$.

As it is usual in quantum mechanics, we call the operator $\text{Op}_\hbar(\psi)$ a “**quantum observable**”.

Remark 1.54. Definition 1.53 is very similar to the definition of Toeplitz (or anti-Wick) quantization of a symbol. The difference is that the quantum space \mathcal{H}_h considered here is attached to a given Anosov diffeomorphism $f : M \rightarrow M$.

In this Section beware that the operator norms $\|\cdot\|$ and trace norms $\|\cdot\|_{\text{Tr}}$ are define
 655 with respect to the norm on the Hilbert space $\mathcal{H}_N^r(P)$.

Theorem 1.55. *For any class of symbols S_δ , there exist constants $C > 0$ and $\varepsilon > 0$ such that the following holds: For every \hbar , there exists a smooth family of rank one projectors $\pi_x : \mathcal{H}_N^r(P) \rightarrow \mathcal{H}_\hbar \subset \mathcal{H}_N^r(P)$ with $\|\pi_x\| \leq C$, parametrized by $x \in M$, such that for any symbol $\psi \in S_\delta$, we have*

$$\left\| \text{Op}_\hbar(\psi) - \frac{1}{(2\pi\hbar)^d} \int_M \psi(x) \pi_x dx \right\| \leq C\hbar^\varepsilon \quad (1.66)$$

and

$$\|[\Pi_\hbar, \mathcal{M}(\psi)]\| \leq C\hbar^\varepsilon. \quad (1.67)$$

The proof of Theorem 1.55 is given in Section 7.7.

Remark 1.56. In particular for the choice $\psi = 1$ we get an approximate expression of
 660 $\Pi_\hbar \equiv \text{Id}|_{\mathcal{H}_\hbar}$ as $\left\| \Pi_\hbar - \frac{1}{(2\pi\hbar)^d} \int_M \pi_x dx \right\| \leq C\hbar^\varepsilon$ sometimes called “resolution of identity”.

Remark 1.57. Since $\dim \mathcal{H}_\hbar \leq C \cdot \hbar$ we have obviously that an estimate in norm operator implies an estimate in trace class norm by $\|\cdot\|_{\text{Tr}} \leq C \cdot \hbar^{-d} \|\cdot\|$.

Corollary 1.58. “Composition formula”. *There exist $C > 0$ and $\varepsilon > 0$ such that for any \hbar and any $\psi_1, \psi_2 \in S_\delta$*

$$\|\text{Op}_\hbar(\psi_1) \circ \text{Op}_\hbar(\psi_2) - \text{Op}_\hbar(\psi_1\psi_2)\| \leq C\hbar^\varepsilon \quad (1.68)$$

Proof. Below the notation $O(\hbar^\varepsilon)$ means that this is a term with norm less than $C\hbar^\varepsilon$. We have

$$\begin{aligned} \text{Op}_\hbar(\psi_1) \circ \text{Op}_\hbar(\psi_2) &= \Pi_\hbar \mathcal{M}(\psi_1) \Pi_\hbar \mathcal{M}(\psi_2) \Pi_\hbar \\ &\stackrel{(1.67)}{=} \Pi_\hbar \mathcal{M}(\psi_1) \mathcal{M}(\psi_2) \Pi_\hbar + O(\hbar^\varepsilon) \\ &= \Pi_\hbar \mathcal{M}(\psi_1\psi_2) \Pi_\hbar + O(\hbar^\varepsilon) = \text{Op}_\hbar(\psi_1\psi_2) + O(\hbar^\varepsilon) \end{aligned}$$

□

Remark 1.59. Composition formula (1.68) expressed the property that the operator $\text{Op}_\hbar(\psi)$ has the so-called “microlocal property”. This is seen by taking ψ_1 and ψ_2 with disjoint supports giving that $\|\text{Op}_\hbar(\psi_1) \circ \text{Op}_\hbar(\psi_2)\| \leq C\hbar^\varepsilon$.

Remark 1.60. In Proposition 1.55 the operator $\text{Op}_\hbar(\psi)$ is decomposed into rank one operators π_x . Each operator π_x is a projector and can be written $\pi_x(\cdot) = (\varphi_x, \cdot)_{\mathcal{H}_\hbar^r} \cdot \psi_x$ with $\psi_x, \varphi_x \in \mathcal{H}_\hbar^r$. From the “microlocal property” given in the previous remark, we can think of ψ_x, φ_x as “microlocal wave packets” and π_x as a projection over these wave packets. In the proof of Proposition 1.66, (6.11), we can find a very explicit expression of π_x on local coordinate charts where ψ_x, φ_x are respectively along the unstable and stable directions in the orthogonal space $(K)^\perp$ (in variables ζ) and are localized wave packets within the trapped set K (in the variables ν).

Corollary 1.61. “Adjoint of observables”. *There exist $C > 0, \varepsilon > 0$, such that for any \hbar and any $\psi \in S_\delta$*

$$\left\| (\text{Op}_\hbar(\psi))_{\mathcal{H}_\hbar}^\dagger - \text{Op}_\hbar(\bar{\psi}) \right\| \leq C\hbar^\varepsilon \quad (1.69)$$

The adjoint operator is defined here in the space \mathcal{H}_\hbar by the relation $(u, (\text{Op}_\hbar(\psi))^\dagger v)_{\mathcal{H}_\hbar} = (\text{Op}_\hbar(\psi)u, v)_{\mathcal{H}_\hbar}$ for any $u, v \in \mathcal{H}_\hbar$.

Proof. For any $u, v \in \mathcal{H}_\hbar$ we have

$$\begin{aligned} (u, (\text{Op}_\hbar(\psi))^\dagger v)_{\mathcal{H}_\hbar} &= (\text{Op}_\hbar(\psi)u, v)_{\mathcal{H}_\hbar} = (\Pi_\hbar \mathcal{M}(\psi)u, v)_{\mathcal{H}_N^r(P)} \\ &\stackrel{(1.67)}{=} (\mathcal{M}(\psi)\Pi_\hbar u, v)_{\mathcal{H}_N^r(P)} + O(\hbar^\varepsilon \|u\| \cdot \|v\|) \\ &= (u, \mathcal{M}(\bar{\psi})v)_{\mathcal{H}_N^r(P)} + O(\hbar^\varepsilon \|u\| \cdot \|v\|) \\ &\stackrel{(1.67)}{=} (u, \Pi_\hbar \mathcal{M}(\bar{\psi})v)_{\mathcal{H}_N^r(P)} + O(\hbar^\varepsilon \|u\| \cdot \|v\|) \\ &= (u, \text{Op}_\hbar(\bar{\psi})v)_{\mathcal{H}_\hbar} + O(\hbar^\varepsilon \|u\| \cdot \|v\|) \end{aligned}$$

(The equality in the middle will be given in Corollary 6.4.) □

Proposition 1.62. “Exact Egorov formula”. *For any \hbar and any $\psi \in S_\delta$, we have*

$$\hat{\mathcal{F}}_\hbar \circ \text{Op}_\hbar(\psi) = \text{Op}_\hbar(\psi \circ f^{-1}) \circ \hat{\mathcal{F}}_\hbar \quad (1.70)$$

Proof. We use that Π_\hbar is a spectral projector of $\hat{\mathcal{F}}_\hbar = \Pi_\hbar \hat{F}_N \Pi_\hbar$ and that from its definition (1.19), \hat{F}_N is a transfer operator hence $\hat{F}_N \circ \mathcal{M}(\psi) = \mathcal{M}(\psi \circ f^{-1}) \circ \hat{F}_N$:

$$\begin{aligned} \hat{\mathcal{F}}_\hbar \circ \text{Op}_\hbar(\psi) &= \Pi_\hbar \hat{F}_N \Pi_\hbar \mathcal{M}(\psi) \Pi_\hbar = \Pi_\hbar \hat{F}_N \mathcal{M}(\psi) \Pi_\hbar = \Pi_\hbar \mathcal{M}(\psi \circ f^{-1}) \hat{F}_N \Pi_\hbar \\ &= \Pi_\hbar \mathcal{M}(\psi \circ f^{-1}) \Pi_\hbar \hat{F}_N \Pi_\hbar = \text{Op}_\hbar(\psi \circ f^{-1}) \circ \hat{\mathcal{F}}_\hbar \end{aligned}$$

□

Proposition 1.63. “Trace of observables”. *There exists $C > 0$, $\varepsilon > 0$, such that for any \hbar and any $\psi \in S_\delta$ we have*

$$\left| (2\pi\hbar)^d \operatorname{Tr}(\operatorname{Op}_\hbar(\psi)) - \int_M \psi dx \right| \leq C\hbar^\varepsilon \quad (1.71)$$

Proof. Since $\operatorname{Tr}(\pi_x) = 1$, we have

$$\operatorname{Tr} \left(\int_M \psi(x) \pi_x dx \right) = \int_M \psi dx.$$

Then

$$\begin{aligned} \left| (2\pi\hbar)^d \operatorname{Tr}(\operatorname{Op}_\hbar(\psi)) - \int_M \psi dx \right| &\leq (2\pi\hbar)^d \left\| \operatorname{Op}_\hbar(\psi) - \frac{1}{(2\pi\hbar)^d} \int_M \psi(x) \pi_x dx \right\|_{\operatorname{Tr}} \\ &\leq C \left\| \operatorname{Op}_\hbar(\psi) - \frac{1}{(2\pi\hbar)^d} \int_M \psi(x) \pi_x dx \right\| \\ &\stackrel{(1.66)}{\leq} C'\hbar^\varepsilon \end{aligned}$$

In the second line we have used (1.30) to get that $\|\cdot\|_{\operatorname{Tr}} \leq C\hbar^{-d} \|\cdot\|$. \square

Taking $\psi = 1$ in (1.71) and because $\operatorname{Op}_\hbar(1) = \Pi_\hbar$ hence $\dim \mathcal{H}_\hbar = \operatorname{Tr}(\Pi_\hbar) = \operatorname{Tr}(\operatorname{Op}_\hbar(1))$ we obtain the Weyl law (1.30):

Corollary 1.64. *We have*

$$\dim \mathcal{H}_\hbar = \frac{1}{(2\pi\hbar)^d} \operatorname{Vol}_\omega(M) (1 + O(\hbar^\varepsilon))$$

Remark 1.65. Eq.(1.70) expresses transport properties of the operator $\hat{\mathcal{F}}_\hbar$. For usual quantization scheme of non-linear map $f : M \rightarrow M$ the Egorov formula has some error $O(\hbar)$ in operator norm. It is therefore remarkable that the formula is exact here. We can iterate it for any time $n \geq 1$ and obtain $\hat{\mathcal{F}}_\hbar^n \circ \operatorname{Op}_\hbar(\psi) = \operatorname{Op}_\hbar(\psi \circ f^{-n}) \circ \hat{\mathcal{F}}_\hbar^n$.

For Schrodinger equation, it is a difficult task to express long time dynamics of initial states using the classical underlying dynamics (relevant in the limit of small wavelength). This is due to interferences effects, semiclassical corrections that grows rapidly and that are difficult to control. In our situation, the prequantum operator \hat{F} is a transfer operator

for which transport properties are exact. Since the quantum operator \mathcal{F}_\hbar is a spectral restriction of \tilde{F} we have immediately that for any initial state $u \in \mathcal{H}_\hbar$, any time $n \geq 0$,

$$\mathcal{F}_\hbar^n u = \left(\tilde{F}^n u \right)_{|\mathcal{H}_\hbar}$$

so that quantum evolution and classical transport coincide for any time.

705 *Remark 1.66.* In their paper [35], J. Marklof and S. O’Keefe propose some axioms for quantum observables associated to quantum maps. Their Axioms [35, Axiom2.1,(a),(b),(c) page282] correspond respectively to Proposition 1.61, 1.58 and 1.63 above. Their Axiom [35, Axiom2.2, page282] corresponds to Proposition 1.62 (with no remainder here).

1.8 Geometric quantization of the symplectic map

710 We have discussed above the “natural quantization” of the map f as the operator $\mathcal{F}_\hbar : \mathcal{H}_\hbar \rightarrow \mathcal{H}_\hbar$ and showed its nice properties. There is however a “standard quantization” of the map f defined in the literature in the framework of geometric quantization. In this subsection we recall first the definition of “geometric quantization” or “Toeplitz quantization” of the symplectic map $f : M \rightarrow M$. Then we compare it with the “natural quantization” $\mathcal{F}_\hbar : \mathcal{H}_\hbar \rightarrow \mathcal{H}_\hbar$ introduced above.

715 In Section 3.2 we introduce the rough Laplacian operator Δ . In Theorem 3.5, for each N large enough, we define \mathfrak{P}_0 the spectral projector of the rough Laplacian on its first band $k = 0$. See also Figure 3.1. The projector \mathfrak{P}_0 has finite rank, the same rank as the spectral projector Π_\hbar on the external band of \hat{F}_N . There is some major difference between these projectors even if they have same rank: Π_\hbar depends on the map f and its image $\text{Im}(\Pi_\hbar) \subset \mathcal{D}'_N(P)$ consists of distributions whereas the projector \mathfrak{P}_0 does not depend on the map f and its image $\text{Im}(\mathfrak{P}_0) \subset C^\infty_N(P)$ contains smooth sections. The projector \mathfrak{P}_0 depends only on the symplectic space (M, ω) together with an additional compatible metric g . The following definition is standard in “geometric quantization” [34].

725 **Definition 1.67.** For every N large enough, the **Toeplitz quantum space** \mathcal{H}_T is the finite dimensional space

$$\mathcal{H}_T := \text{Im}(\mathfrak{P}_0) \tag{1.72}$$

The **Toeplitz quantum operator** is

$$\hat{\mathcal{F}}_{T,V} := \mathfrak{P}_0 \hat{F}_N \mathfrak{P}_0 : \mathcal{H}_T \rightarrow \mathcal{H}_T \tag{1.73}$$

In the notation we have emphasized the dependence on the potential V which enters in the definition 1.33 of \hat{F}_N .

In (3.9) we obtain the following Lemma:

Lemma 1.68. For $N = \frac{1}{2\pi\hbar}$ large enough we have that

$$\Phi := \mathfrak{P}_0 : \mathcal{H}_\hbar \rightarrow \mathcal{H}_T \tag{1.74}$$

is a finite rank isomorphism.

730 The previous Lemma allows to consider $\Phi \hat{\mathcal{F}}_\hbar \Phi^{-1} : \mathcal{H}_T \rightarrow \mathcal{H}_T$ and thus to “compare” the quantum operator $\hat{\mathcal{F}}_\hbar : \mathcal{H}_\hbar \rightarrow \mathcal{H}_\hbar$ with the Toeplitz quantum operator $\hat{\mathcal{F}}_T : \mathcal{H}_T \rightarrow \mathcal{H}_T$. The next Theorem shows that they are close to each other under the condition that one adds a correction in the potential function. For that reason we will emphasize the dependence on the potential V which enters in the definition of $\hat{\mathcal{F}}_\hbar$ by noting it $\hat{\mathcal{F}}_{\hbar,V}$.

735

Theorem 1.69. “The quantum operator $\hat{\mathcal{F}}_\hbar$ is close to a Toeplitz quantum operator $\hat{\mathcal{F}}_T$ ”. There exists $\delta > 0$, there exists a function $\mathcal{M} \in C(M)$ such that for $\hbar \rightarrow 0$,

$$\left\| \Phi \hat{\mathcal{F}}_{\hbar,V} \Phi^{-1} - \hat{\mathcal{F}}_{T,V'} \right\| \leq \mathcal{O}(\hbar^\delta), \tag{1.75}$$

where $\hat{\mathcal{F}}_{T,V'} : \mathcal{H}_T \rightarrow \mathcal{H}_T$ is the Toeplitz operator (1.73) but constructed with the potential $V' = V + \mathcal{M}$.

The proof of Theorem 1.69 is given in Section 12.

740 *Remark 1.70.* We have already said but we repeat here, that Toeplitz quantization (or geometric quantization⁷) gives a family of “quantum operators” $\hat{\mathcal{F}}_{T,V'}$ that have nice semiclassical properties as any quantization gives. For example they satisfy Egorov formula (1.70) but with an error of order $O(\hbar)$. They satisfy the Gutzwiller trace formula (1.58) but with a larger error $O(\theta^n)$ with $\theta > 1$. The “natural quantization” of the Anosov map f presented in this paper has the unique additional property that the errors in these formula vanish (or are improved).

745 2 Preliminary results and sketch of the proofs

In this Section we begin with establishing some preliminary results which will be used in the rest of the paper. Then we sketch the proofs of the main theorems presented in Section 1.

2.1 Semiclassical description of the prequantum operator \hat{F}_N

750 2.1.1 The associated canonical map $F : T^*M \rightarrow T^*M$

We first give a local expression of the transfer operator \hat{F}_N defined in (1.21) with respect to local charts and local trivialization of the bundle P . These local expressions will be useful in the sequel of the paper.

⁷or Weyl quantization when available e.g. if $M = \mathbb{T}^{2d}$ is a torus.

As in Section 1.2.2, let $(U_\alpha)_{\alpha \in I}$ be a finite collection of simply connected open subsets which cover M and, for every open set $U_\alpha \subset M$, let

$$\tau_\alpha : U_\alpha \rightarrow P$$

be a local section of the bundle. Recall the local trivialization T_α defined in (1.7) and the one forms η_α in the local expression (1.9) of the connection A .

Lemma 2.1. “Local expression of the prequantum map \tilde{f} ”. Suppose that $V \subset U_\alpha \cap f^{-1}(U_\beta)$ is a simply connected open set. Then for $x \in V$

$$\tilde{f}(\tau_\alpha(x)) = e^{i2\pi\mathcal{A}_{\beta,\alpha}(x)}\tau_\beta(f(x)) \quad (2.1)$$

with the “action function” given by

$$\mathcal{A}_{\beta,\alpha}(x) = \int_{f(\gamma)} \eta_\beta - \int_\gamma \eta_\alpha + c(x_0) = \int_\gamma (f^*(\eta_\beta) - \eta_\alpha) + c(x_0). \quad (2.2)$$

In the last integral, $x_0 \in V$ is any point of reference, $\gamma \subset V$ is a path from x_0 to x and $c(x_0)$ does not depend on x . See figure 2.1.

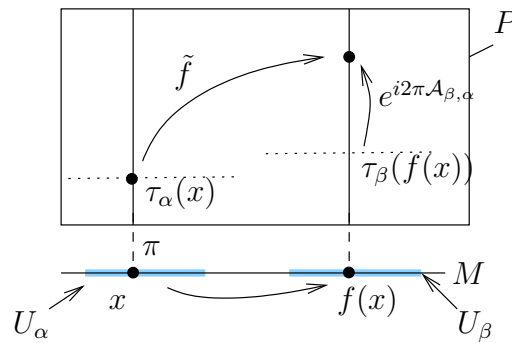


Figure 2.1: Illustrates the expression (2.1) of the prequantum map \tilde{f} with respect to local trivialization. It is characterized by the action function $\mathcal{A}_{\beta,\alpha}(x)$.

Remark 2.2. In other terms, if $p \in P$ is such that $\pi(p) = x$, let $\theta_\alpha, \theta_\beta$ such that

$$p = e^{i\theta_\alpha}\tau_\alpha(x), \quad \tilde{f}(p) = e^{i\theta_\beta}\tau_\beta(f(x))$$

then

$$\theta_\beta = \theta_\alpha + 2\pi\mathcal{A}_{\beta,\alpha}(x)$$

765 *Remark 2.3.* Notice that the integral (2.2) does not depend on the path γ from x_0 to x because the one form $f^*(\eta_\beta) - \eta_\alpha$ is closed. Indeed $d(f^*(\eta_\beta) - \eta_\alpha) = f^*\omega - \omega = 0$ since f is symplectic.

Proof. Let $\gamma \subset V$ be a path from x_0 to x . Let $\tilde{\gamma} : t \rightarrow \tilde{\gamma}(t)$ be the lifted path parallel transported above γ starting from $\tau_\alpha(x_0)$ and ending at point p . (See Figure 2.2.) Since the connection one form vanishes along the path $\tilde{\gamma}$, we have

$$0 = (T_\alpha^*A) \left(\frac{d\tilde{\gamma}_\alpha}{dt} \right) = (id\theta - i2\pi\eta_\alpha) \left(\frac{d\tilde{\gamma}_\alpha}{dt} \right)$$

770 with $\tilde{\gamma}_\alpha = T_\alpha^{-1}(\tilde{\gamma})$. From the construction of the lifted map \tilde{f} in the proof of Lemma A.1, we have

$$p = e^{i\theta_\alpha(x)} \tau_\alpha(x) \tag{2.3}$$

with

$$\theta_\alpha(x) = \int_{\tilde{\gamma}} d\theta = 2\pi \int_\gamma \eta_\alpha.$$

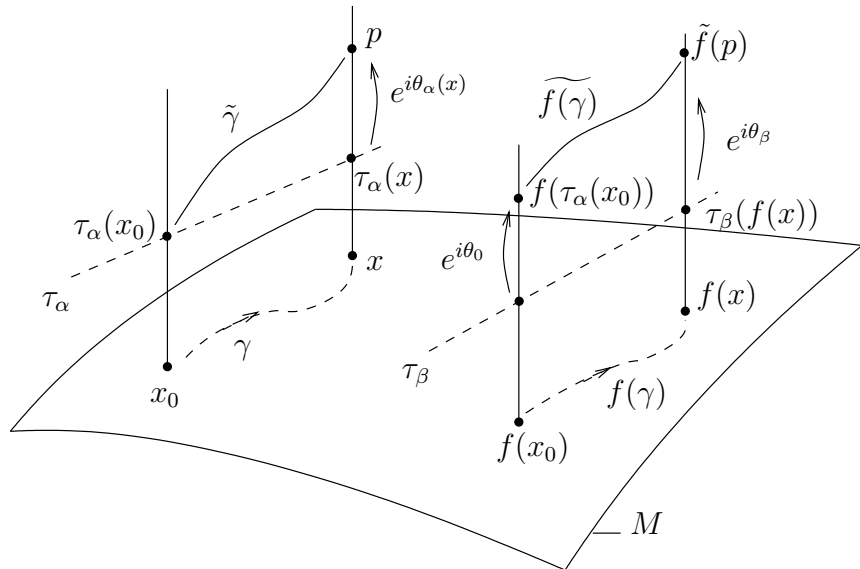


Figure 2.2:

Let θ_0 given by $\tilde{f}(\tau_\alpha(x_0)) = e^{i\theta_0} \tau_\beta(f(x_0))$. Similarly we have

$$\tilde{f}(p) = e^{i\theta_0} e^{ib(x)} \tau_\beta(f(x))$$

with

$$b(x) = 2\pi \int_{f(\gamma)} \eta_\beta.$$

From equivariance of \tilde{f} and (2.3), we have $\tilde{f}(p) = e^{i\theta_\alpha(x)} \tilde{f}(\tau_\alpha(x))$. Therefore

$$\tilde{f}(\tau_\alpha(x)) = e^{-i\theta_\alpha(x)} \tilde{f}(p) = e^{-i\theta_\alpha(x)} e^{i\theta_0} e^{ib(x)} \tau_\beta(f(x)) = e^{i2\pi A_{\beta,\alpha}(x)} \tau_\beta(f(x))$$

775 with

$$A_{\beta,\alpha}(x) = \int_{f(\gamma)} \eta_\beta - \int_\gamma \eta_\alpha + \frac{\theta_0}{2\pi}.$$

□

For a given $N \in \mathbb{Z}$, an equivariant function $u \in C_N^\infty(P)$ defines the set of *associated functions*⁸ $u_\alpha : U_\alpha \rightarrow \mathbb{C}$, $\alpha \in I$, defined by

$$u_\alpha(x) := u(\tau_\alpha(x)) \quad (2.4)$$

for $x \in U_\alpha$. Conversely one reconstructs u from $(u_\alpha)_{\alpha \in I}$ by the relation

$$u(p) = u(e^{i\theta} \tau_\alpha(x)) = e^{iN\theta} u(\tau_\alpha(x)) = e^{iN\theta} u_\alpha(x) \quad \text{for } p = e^{i\theta} \tau_\alpha(x) \text{ and } x \in U_\alpha.$$

780

Proposition 2.4. “Local expression of \hat{F}_N ”. Let $u \in C_N^\infty(P)$ and $u' := \hat{F}_N u \in C_N^\infty(P)$. Let the respective associated functions be $u_\alpha = u \circ \tau_\alpha$ and $u'_\alpha = u' \circ \tau_\alpha$ for any $\alpha \in I$. Then

$$u'_\beta = e^V \cdot e^{-i2\pi N A_{\beta,\alpha} \circ f^{-1}} (u_\alpha \circ f^{-1}) \quad (2.5)$$

Proof. From definition (1.19) of the transfer operator

$$u'_\beta(x) = u'(\tau_\beta(x)) = (\hat{F}u)(\tau_\beta(x)) = e^{V(x)} u(\tilde{f}^{-1}(\tau_\beta(x)))$$

From (2.1) we have

$$\tilde{f}^{-1}(\tau_\beta(x)) = e^{-i2\pi A_{\beta,\alpha}(f^{-1}(x))} \tau_\alpha(f^{-1}(x))$$

hence

$$\begin{aligned} u'_\beta(x) &= e^{V(x)} u\left(e^{-i2\pi A_{\beta,\alpha}(f^{-1}(x))} \tau_\alpha(f^{-1}(x))\right) \\ &= e^{V(x)} e^{-i2\pi N A_{\beta,\alpha}(f^{-1}(x))} u(\tau_\alpha(f^{-1}(x))) \\ &= e^{V(x)} e^{-i2\pi N A_{\beta,\alpha}(f^{-1}(x))} u_\alpha(f^{-1}(x)) \end{aligned}$$

□

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⁸In the language of associate line bundle, these functions u_α are sections of a line bundle $L^{\otimes N}$ expressed with respect to the local trivializations.

The next few results will concern some semiclassical aspects of the transfer operator. The first easy result but of fundamental importance is the following Proposition. (It is not used in the proofs in this paper and the explanation below may be a bit sloppy. But it gives the background of the following argument.)

790 *Remark 2.5.* For the general definition of a Fourier integral operator, we refer to Martinez [36], Zworski [57] or Duistermaat [15]. If the reader is not familiar with Fourier integral operator, it is enough for the reading of this paper to understand the rough idea of a Fourier integral operator \hat{F} , which is quite simple as explained in [12]: if a function ψ is localized at point $x \in M$ and its Fourier transform is localized at point $\xi \in T_x^*M$ (which
 795 means that these functions decay fast outside these points and we say that ψ is *micro-localized* at $(x, \xi) \in T^*M$) then the operator \hat{F} transforms this function ψ to a function ψ' micro-localized in another point $(x', \xi') \in T^*M$. The map $F : T^*M \rightarrow T^*M$, giving $(x', \xi') = F(x, \xi)$ is called the associated **canonical map**. Note that we are interested in the situation of high frequencies $|\xi| \gtrsim N = (2\pi\hbar)^{-1} \gg 1$ and, in the limit $N \rightarrow \infty$ (or
 800 $\hbar \rightarrow +0$), we will normalize ξ by multiplying \hbar .

Proposition 2.6. *The prequantum transfer operator \hat{F}_N is a Fourier Integral Operator if we view it in the local trivializations as in (2.5). The associated canonical map on the cotangent space is given by*

$$F_{\alpha,\beta} : \begin{cases} T^*U_\alpha & \rightarrow T^*U_\beta \\ (x, \xi) & \rightarrow (x', \xi') = (f(x), {}^t(Df_{x'}^{-1})(\xi + \eta_\alpha(x)) - \eta_\beta(x')) \end{cases} \quad (2.6)$$

where $x \in U_\alpha$, $f(x) \in U_\beta$ and $\xi \in T_x^*U_\alpha$. The map $F_{\alpha,\beta}$ preserves the canonical symplectic structure

$$\Omega := \sum_{j=1}^{2d} dx_j \wedge d\xi_j \quad (2.7)$$

Proof. The transfer operator is given in local chart by (2.5). This expression shows that it is the composition of two operators $\hat{F}_2 \circ \hat{F}_1$ where \hat{F}_1 is the pull-back by the diffeomorphism f^{-1} and \hat{F}_2 is the multiplication by a phase function. Both operators are basic examples of F.I.O as explained in [36, chap.5] and the Proposition 2.6 follows. \square
 805

Remark 2.7. We give here a more detailed explanation of the associated canonical map (2.6). From (2.5), the transfer operator \hat{F}_N can be decomposed as elementary operators. Let

$$\hat{F}_1 : u(x) \rightarrow u(f^{-1}(x)).$$

810 This is a Fourier integral operator with canonical map

$$F_1 \left(\begin{array}{c} x \\ \xi \end{array} \right) = \left(\begin{array}{c} x' \\ \xi' \end{array} \right) = \left(\begin{array}{c} f(x) \\ {}^t(Df_{x'}^{-1})\xi \end{array} \right) \quad (2.8)$$

Indeed, it is clear that $\text{supp}(\psi)$ is transported to $f(\text{supp}(\psi))$ hence $x' = f(x)$. Also an oscillating function $u(x) = e^{\frac{i}{\hbar}\xi \cdot x}$ is transformed to $u'(y) = \left(\hat{F}_1 u\right)(y) = e^{\frac{i}{\hbar}\xi \cdot f^{-1}(y)}$ and, for $y = f(x) + y'$ with $|y'| \ll 1$, we have $f^{-1}(y) = x + Df_y^{-1} \cdot y' + o(|y'|)$, hence

$$u'(y) \simeq e^{\frac{i}{\hbar}\xi \cdot (x + Df_{x'}^{-1} \cdot y')} = C e^{\frac{i}{\hbar} {}^t(Df_{x'}^{-1})\xi \cdot y} = C e^{\frac{i}{\hbar}\xi' \cdot y}$$

with $\xi' = {}^t(Df_{x'}^{-1})\xi$ and $C = e^{\frac{i}{\hbar}\xi \cdot (x - Df_{x'}^{-1} \cdot f(x))}$. We deduced (2.8). Next we consider a multiplication operator by a “fast oscillating phase” (recall that $\hbar \ll 1$):

$$\hat{F}_2 : \psi(x) \rightarrow e^{iS(x)/\hbar} \psi(x)$$

For the same reasons, it is a F.I.O. and its canonical map is

$$F_2 \left(\begin{array}{c} x \\ \xi \end{array} \right) = \left(\begin{array}{c} x \\ \xi + dS(x) \end{array} \right)$$

Indeed $u(x) = e^{\frac{i}{\hbar}\xi \cdot x}$ is transformed to $u'(y) = \left(\hat{F}_2 u\right)(y) = e^{\frac{i}{\hbar}(\xi \cdot y + S(y))}$ and for $y = x + y'$ with $|y'| \ll 1$, we have

$$u'(x') \simeq C e^{\frac{i}{\hbar}(\xi \cdot y + dS \cdot y)} = C e^{\frac{i}{\hbar}\xi' \cdot y}$$

with $\xi' = \xi + dS$ and $C = e^{\frac{i}{\hbar}(S(x) - dS_x \cdot x)}$. From these two previous examples and (2.5), we can deduce (2.6). Notice that the multiplication operator by e^V does not appear in the canonical map because it is not a “fast oscillating function” (in the limit $\hbar \rightarrow 0$).

Notice that the canonical maps $F_{\alpha, \beta}$ in the last proposition are given with respect to local trivializations of P . The following proposition gives a global and geometric description of the canonical map (2.6).

Proposition 2.8. Consider the following change of variable on T^*U_α for every $\alpha \in I$:

$$(x, \xi) \in T^*U_\alpha \rightarrow (x, \zeta) = (x, \xi + \eta_\alpha(x)) \in T^*M \quad (2.9)$$

Then the **canonical map** (2.6) get the simpler and global expression (independent on the set U_α) on the phase space T^*M

$$F : \begin{cases} T^*M & \rightarrow T^*M \\ (x, \zeta) & \rightarrow (x', \zeta') = (f(x), {}^t(Df_{x'}^{-1})\zeta) \end{cases} \quad (2.10)$$

The symplectic form Ω in (2.7) preserved by F is expressed:

$$\Omega = \sum_{j=1}^{2d} (dx_j \wedge d\zeta_j) + \tilde{\pi}^*(\omega) \quad (2.11)$$

with the canonical projection map $\tilde{\pi} : T^*M \rightarrow M$.

Remark 2.9. We will see in Remark 3.3 that the variables ζ can be interpreted as the symbol of the covariant derivative operator D . The change of variables (2.9) and the global geometric description (2.10) is standard in mathematical physics for semiclassical problems involving large magnetic fields on manifolds.

Proof. The relation (2.10) is obvious from (2.6) and (2.9). To prove (2.11), we write in coordinates

$$\eta_\alpha = \sum_{j=1}^{2d} \eta_j dx_j$$

and from (1.11),

$$\omega = d\eta_\alpha = \sum_{i,j} \left(\frac{\partial \eta_j}{\partial x_i} \right) (dx_i \wedge dx_j).$$

Then from (2.7)

$$\begin{aligned} \Omega &= \sum_{j=1}^{2d} dx_j \wedge d\xi_j = \sum_{j=1}^{2d} dx_j \wedge (d\zeta_j - d\eta_j) = \sum_{j=1}^{2d} dx_j \wedge \left(d\zeta_j - \sum_{i=1}^{2d} \left(\frac{\partial \eta_j}{\partial x_i} \right) dx_i \right) \\ &= \sum_{j=1}^{2d} (dx_j \wedge d\zeta_j) + \omega \end{aligned}$$

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□

2.1.2 The trapped set K

In relation to the global expression (2.10) of the canonical map associated to the prequantum transfer operator \hat{F}_N , it should be natural to introduce the following definition.

Definition 2.10. The *trapped set* $K \Subset T^*M$ is the set of points $(x, \xi) \in T^*M$ which do not escape to infinity in the past neither in the future $n \rightarrow \pm\infty$ under the dynamics of the canonical map F :

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$$K := \{(x, \xi), \exists C > 0, |\xi(n)| \leq C, \forall n \in \mathbb{Z}, \text{ with } (x(n), \xi(n)) := F^n(x, \xi)\}.$$

Remark 2.11. In terms of the theory of dynamical systems, the trapped set K is the *non-wandering set* for the dynamical system generated by F .

At every point $\rho \in K$ of the trapped set we have the decomposition

$$T_\rho(T^*M) = E_u^*(\rho) \oplus E_s^*(\rho)$$

into the unstable and stable subspaces with respect to the action of DF defined by:

$$E_u^*(\rho) := \{v \in T_\rho(T^*M) \mid |DF_\rho^{-n}(v)| \rightarrow 0 \text{ as } n \rightarrow +\infty\}$$

845 and

$$E_s^*(\rho) := \{v \in T_\rho(T^*M) \mid |DF_\rho^n(v)| \rightarrow 0 \text{ as } n \rightarrow +\infty\}.$$

This decomposition is dual to the decomposition (1.1) in the sense that in the fiber T_ρ^*M we have $(E_u^* \cap T_\rho^*M)(E_u) = 0$, $(E_s^* \cap T_\rho^*M)(E_s) = 0$.

Proposition 2.12. “*Description of the trapped set K* ”. The trapped set $K \subset T^*M$ is the zero section:

$$K = \{(x, \zeta) \in T^*M \mid x \in M, \zeta = 0\}. \quad (2.12)$$

This is a symplectic submanifold of (T^*M, Ω) isomorphic to (M, ω) . For every point $\rho \in K$, the tangent space is decomposed as an Ω -orthogonal sum of symplectic linear subspaces

$$T_\rho(T^*M) = T_\rho K \overset{\perp}{\oplus} (T_\rho K)^\perp \quad (2.13)$$

Moreover each part is decomposed into isotropic unstable/stable linear spaces

$$T_\rho K = E_u^{(1)}(\rho) \oplus E_s^{(1)}(\rho), \quad (T_\rho K)^\perp = E_u^{(2)}(\rho) \oplus E_s^{(2)}(\rho) \quad (2.14)$$

where the subspaces $E_\sigma^{(i)}(\rho)$ for $i = 1, 2$ and $\sigma = s, u$ are d -dimensional subspaces defined by

$$E_\sigma^{(1)}(\rho) = T_\rho K \cap E_\sigma^*(\rho), \quad E_\sigma^{(2)}(\rho) = (T_\rho K)^\perp \cap E_\sigma^*(\rho) \quad \text{for } \sigma = s, u.$$

All the decompositions above are invariant by the map F . See Figure 2.3.

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Remark 2.13. There is another F -invariant decomposition:

$$\underbrace{T_\rho(T^*M)}_{4d} = \underbrace{T_\rho K}_{2d} \oplus \underbrace{T_x^*M}_{2d}$$

with $x = \tilde{\pi}(\rho)$ and T_x^*M the fiber of the cotangent space. However this sum is not Ω -orthogonal and moreover T_x^*M is Ω -Lagrangian.

855 *Remark 2.14.* It is shown in [9, appendix] that the cotangent space (T^*M, Ω) is an “affine cotangent space” and can be geometrically defined as the space of connections on the principal bundle $P \rightarrow M$. The trapped set K is the section defined by connection itself [9, appendix].

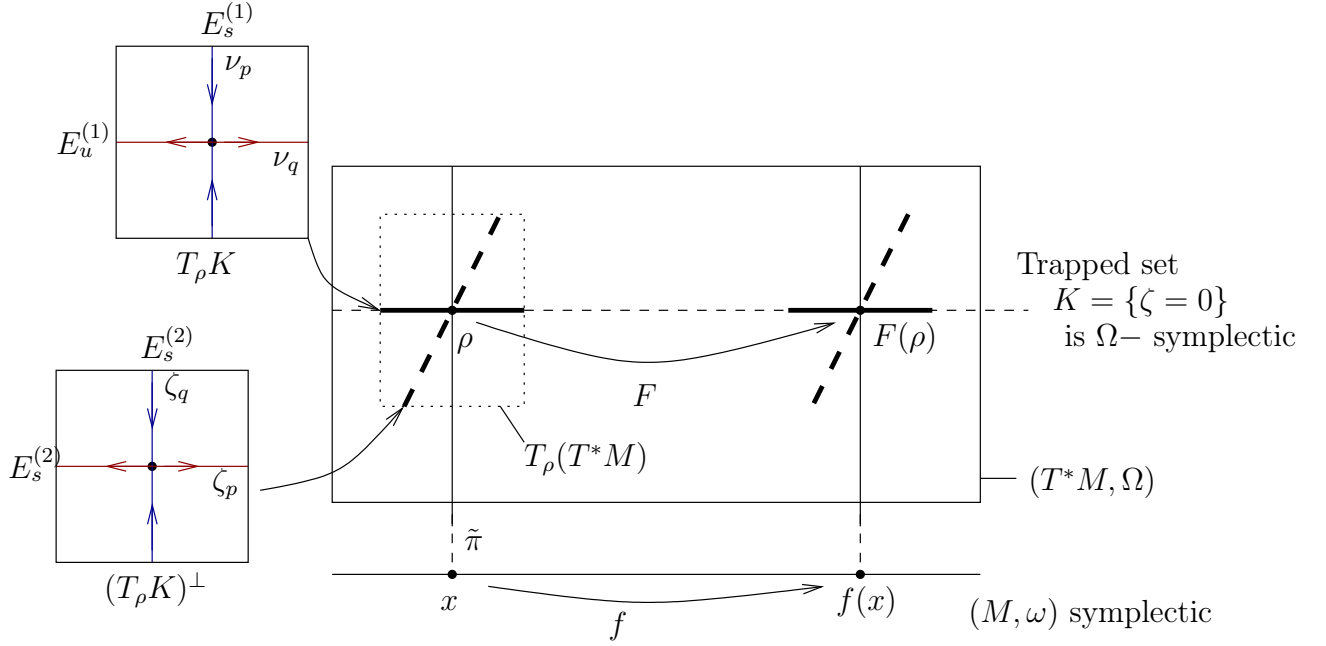


Figure 2.3: The decompositions of the tangent space $T_\rho(T^*M)$.

Proof. From (2.10) it is clear that the trapped set is the zero section $\{(x, \zeta), \zeta = 0\}$ since
860 it is invariant and, if $\zeta \neq 0$, we have $|F^n(x, \zeta)| \rightarrow \infty$ at least either as $n \rightarrow \infty$ or $n \rightarrow -\infty$.
From (2.11) $\Omega|_K = \tilde{\pi}^*\omega$ therefore $\tilde{\pi} : (K, \Omega) \rightarrow (M, \omega)$ is a symplectomorphism. The
symplectic maps $f : M \rightarrow M$ and $F : K \rightarrow K$ are conjugated by $\tilde{\pi}$. For every point
865 $\rho \in K$, $T_\rho K$ is a linear symplectic subspace of the symplectic linear space $T_\rho(T^*M)$ and
therefore admits a unique symplectic orthogonal $(T_\rho K)^\perp$. The decomposition (2.13) is
invariant under the map F because the trapped set K is invariant and because F preserves
the symplectic form Ω . \square

In the next proposition, we introduce convenient local coordinates, called normal coordinates or Darboux coordinates. We will use them later in the proof.

Proposition 2.15. “*Normal coordinates*”. On a sufficiently small neighborhood U of every point $x \in M$, there exist coordinates

$$x = (q, p) = (q^1, \dots, q^d, p^1, \dots, p^d)$$

and a trivialization of P such that the connection one-form in (1.9) is given by

$$\eta = \sum_{i=1}^d \left(\frac{1}{2} q^i dp^i - \frac{1}{2} p^i dq^i \right) \quad (2.15)$$

and consequently the symplectic form ω is given by

$$\omega = d\eta = \sum_{i=1}^d dq^i \wedge dp^i. \quad (2.16)$$

On the cotangent bundle T^*U , there is a change of coordinates $\Phi : (x, \xi) \rightarrow (\nu, \zeta)$ where the variables $\zeta = (\zeta_p^j, \zeta_q^j)_{j=1, \dots, d}$ are already defined in (2.9) as

$$\begin{aligned} \zeta_p^i &: = \xi_p^i + \eta_p^i = \xi_p^i + \frac{1}{2} q^i \\ \zeta_q^i &: = \xi_q^i + \eta_q^i = \xi_q^i - \frac{1}{2} p^i \end{aligned} \quad (2.17)$$

while $\nu = (\nu_q^j, \nu_p^j)_{j=1, \dots, d}$ are given by

$$\begin{aligned} \nu_q^j &= q^j - \zeta_p^j = \frac{1}{2} q^j - \xi_p^j \\ \nu_p^j &= p^j + \zeta_q^j = \frac{1}{2} p^j + \xi_q^j \end{aligned} \quad (2.18)$$

This change of variables transforms the symplectic form Ω in (2.11) to the normal form:

$$\Omega = \sum_{j=1}^d (d\nu_q^j \wedge d\nu_p^j) + (d\zeta_p^j \wedge d\zeta_q^j). \quad (2.19)$$

Remark 2.16. Recall from (2.12) that $T_\rho K = \{(\nu, \zeta), \zeta = 0\}$. Then (2.19) implies that its symplectic orthogonal is given by $(T_\rho K)^\perp = \{(\nu, \zeta), \nu = 0\}$. In other words, (ν, ζ) are symplectic coordinates related to the decomposition (2.13). These coordinates were introduced in the paper [17] treating the linear case, under the different names $(Q_1, P_1) \equiv (\nu_q, \nu_p)$ and $(Q_2, P_2) \equiv (\zeta_p, \zeta_q)$.

Proof. Darboux theorem on symplectic structure (see [1]) tells that, if we take sufficiently

small neighborhood U of x , there exist coordinates $x = (q, p) = (q^1, \dots, q^d, p^1, \dots, p^d)$ such that the symplectic form is expressed in the normal form $\omega = dq \wedge dp = \sum_{i=1}^d dq^i \wedge dp^i$. Take any local smooth section $\tau' : U \rightarrow P$ and let η' be the local connection one form (see (1.9)) with respect to the corresponding local trivialization of P . Let η be given in (2.15). Since we have $d(\eta' - \eta) = \omega - \omega = 0$ there is a smooth function $\chi : U \rightarrow \mathbb{R}$ such that $\eta' - \eta = \frac{1}{2\pi}d\chi$. Setting $\tau = e^{i\chi}\tau'$ and recalling the formula (1.10), we see that the former statement of the proposition holds for the coordinates $x = (q, p) = (q^1, \dots, q^d, p^1, \dots, p^d)$ and the trivialization of P associated to the local smooth section τ thus taken.

We prove now the latter statement. (2.9) and (2.15) imply (2.17). Clearly (2.17) and (2.18) are coordinates on U as we can give the inverse explicitly. Starting from (2.11) we get

$$\begin{aligned} \Omega &= \sum_{j=1}^d (dq^j \wedge d\zeta_q^j + dp^j \wedge d\zeta_p^j + dq^j \wedge dp^j) \\ &= \sum_{j=1}^d ((d\nu_q^j + d\zeta_p^j) \wedge d\zeta_q^j + (d\nu_p^j - d\zeta_q^j) \wedge d\zeta_p^j + d(\nu_q^j + \zeta_p^j) \wedge (d\nu_p^j - d\zeta_q^j)) \\ &= \sum_{j=1}^d d\nu_q^j \wedge d\nu_p^j - d\zeta_q^j \wedge d\zeta_p^j = \sum_{j=1}^d d\nu_q^j \wedge d\nu_p^j + d\zeta_p^j \wedge d\zeta_q^j. \end{aligned}$$

This completes the proof. \square

Remark 2.17. Since the symplectic 2-form $\omega = \sum_j dq^j \wedge dp^j$ on $T_x^*M \cong \mathbb{R}^{2d}$ is non degenerate, it defines an isomorphism, called *flat operator*,

$$\flat : \mathbb{R}^{2d} \rightarrow (\mathbb{R}^{2d})^*, \quad v^\flat = \omega(v, \cdot).$$

Its inverse is called the *sharp operator*.

$$\sharp = \flat^{-1} : (\mathbb{R}^{2d})^* \rightarrow \mathbb{R}^{2d}.$$

For a one-form $\alpha \in (\mathbb{R}^{2d})^*$, α^\sharp is defined by the relation $\alpha = \omega(\alpha^\sharp, \cdot)$. In coordinates, for

$$v = \sum_{j=1}^n v_q^j \frac{\partial}{\partial q^j} + v_p^j \frac{\partial}{\partial p^j} \equiv (v_q, v_p) \in \mathbb{R}^{2d} \quad \text{and} \quad \alpha = \sum_{j=1}^n \alpha_q^j dq^j + \alpha_p^j dp^j \equiv (\alpha_q, \alpha_p) \in (\mathbb{R}^{2d})^*$$

we have

$$v^\flat = \sum_{j=1}^n -v_p^j dq^j + v_q^j dp^j \equiv (-v_p, v_q) \in (\mathbb{R}^{2d})^* \tag{2.20}$$

$$\alpha^\sharp = \sum_{j=1}^n \alpha_p^j \frac{\partial}{\partial q^j} - \alpha_q^j \frac{\partial}{\partial p^j} \equiv (\alpha_p, -\alpha_q) \in \mathbb{R}^{2d} \tag{2.21}$$

We also have

$$\alpha(v) = -v^b(\alpha^\#) \in \mathbb{R}, \quad b^t = -b, \quad \#^t = -\#. \quad (2.22)$$

895 where $b^t, \#^t$ denote the transposed maps. Using these notation, the relation (2.18) can be written in a more intrinsic manner:

$$\nu := x - \zeta^\#. \quad (2.23)$$

With these notations (2.19) get also a more geometric expression:

$$\Omega((\nu_1, \zeta_1), (\nu_2, \zeta_2)) = \omega(\nu_1, \nu_2) + \omega(\#\zeta_2, \#\zeta_1) \quad (2.24)$$

Proposition 2.18. *The normal coordinates $(\nu_q, \nu_p, \zeta_p, \zeta_q)$ introduced in the last proposition can be chosen so that the differential of the coordinate map $\Phi : \tilde{\pi}^{-1}(U) \rightarrow \mathbb{R}^{4d}$ at any point $\rho \in K$ such that $\tilde{\pi}(\rho) = x \in U$ carries the subspaces $E_u^{(1)}(\rho), E_s^{(1)}(\rho), E_u^{(2)}(\rho), E_s^{(2)}(\rho)$ in (2.14) to the subspaces*

$$\begin{aligned} \mathbb{R}_{\nu_q}^d &:= \{(\nu_q, 0, 0, 0) \mid \nu_q \in \mathbb{R}^d\}, & \mathbb{R}_{\nu_p}^d &:= \{(0, \nu_p, 0, 0) \mid \nu_p \in \mathbb{R}^d\}, \\ \mathbb{R}_{\zeta_p}^d &:= \{(0, 0, \zeta_p, 0) \mid \zeta_p \in \mathbb{R}^d\}, & \mathbb{R}_{\zeta_q}^d &:= \{(0, 0, 0, \zeta_q) \mid \zeta_q \in \mathbb{R}^d\} \end{aligned}$$

respectively. That is to say,

$$\begin{array}{ccc} T_\rho(T^*M) & = & \underbrace{E_u^{(1)}(\rho) \oplus E_s^{(1)}(\rho)}_{T_\rho K} \overset{\perp}{\oplus} \underbrace{E_u^{(2)}(\rho) \oplus E_s^{(2)}(\rho)}_{(T_\rho K)^\perp} \\ D\Phi \downarrow & & \downarrow \qquad \qquad \downarrow \\ T^*\mathbb{R}_{(q,p)}^{2d} & = & \underbrace{(\mathbb{R}_{\nu_q}^d \oplus \mathbb{R}_{\nu_p}^d)}_{T^*\mathbb{R}_q^d} \overset{\perp}{\oplus} \underbrace{(\mathbb{R}_{\zeta_p}^d \oplus \mathbb{R}_{\zeta_q}^d)}_{T^*\mathbb{R}_p^d} \end{array}$$

With respect to these coordinates the differential of the canonical map $DF_\rho : T_\rho(T^*M) \rightarrow T_{F(\rho)}(T^*M)$ is expressed as

$$D\Phi \circ DF_\rho \circ D\Phi^{-1} = F^{(1)} \oplus F^{(2)}, \quad F^{(1)} \equiv \begin{pmatrix} A_x & 0 \\ 0 & {}^t A_x^{-1} \end{pmatrix}, \quad F^{(2)} \equiv \begin{pmatrix} A_x & 0 \\ 0 & {}^t A_x^{-1} \end{pmatrix} \quad (2.25)$$

where $A_x \equiv Df|_{E_u(x)} : \mathbb{R}^d \rightarrow \mathbb{R}^d$ is an expanding linear map.

900 *Proof.* We can take the coordinates $x = (q, p) = (q^1, \dots, q^d, p^1, \dots, p^d)$ in the beginning of the proof of the last proposition so that the stable and unstable subspaces, $E_s(x)$ and $E_u(x)$, correspond to the subspaces given by the equations $p = 0$ and $q = 0$ respectively, because they are Lagrangian subspaces. Then the coordinates and the trivialization constructed in

the proof have the required property. The differential of the Anosov map $Df_x : T_x M \rightarrow T_{f(x)} M$ splits according to the invariant decomposition $T_x M = E_u(x) \oplus E_s(x)$ as $Df_x = (A_x, B_x)$ with

$$A_x := Df|_{E_u(x)} : E_u(x) \rightarrow E_u(f(x))$$

and

$$B_x := Df|_{E_s(x)} : E_s(x) \rightarrow E_s(f(x))$$

But since $E_u(x)$ and $E_s(x)$ are Lagrangian subspaces, ω provides an isomorphism $\flat : E_s(x) \rightarrow E_u(x)^*$ defined by $\flat(S) : U \in E_u(x) \rightarrow \omega(S, U) \in \mathbb{R}$ for $S \in E_s(x)$. Since Df_x is symplectic, i.e. preserves ω , then $B_x = \flat \circ {}^t A_x^{-1} \circ \flat^{-1}$ i.e. B_x is isomorphic to

$${}^t A_x^{-1} : E_u^*(x) \rightarrow E_u^*(f(x))$$

So with this identification we have

$$Df_x \equiv \begin{pmatrix} A_x & 0 \\ 0 & {}^t A_x^{-1} \end{pmatrix} \quad (2.26)$$

□

2.1.3 Microlocal description of the prequantum transfer operator near the trapped set

In order to focus on the action of the canonical map $F : T^*M \rightarrow T^*M$ on the vicinity of the trapped set K and relate it to the spectral properties of the prequantum transfer operator \hat{F}_N , we use an *escape function* (or a *weight function*) $W_h(x, \xi)$ on the phase space, which decreases strictly along the flow outside a vicinity of the trapped set K , and use it to define some associate norm and associated *anisotropic Sobolev spaces*⁹. From the fact that the trapped set is compact in phase space and from the property of the escape function mentioned above, we deduce that the spectrum of the prequantum transfer operator \hat{F}_N is discrete in these anisotropic Sobolev spaces [19]. The eigenvalues are called “**resonances**” (from the physical meaning in scattering theory).

Here is how we will proceed in the proofs:

- (1) In Section 7.2 we will consider a system of local charts on the manifold M depending on \hbar (or on N), which is of small size $\hbar^{1/2-\theta} \ll 1$ in the semiclassical limit $\hbar \rightarrow 0$, and then consider the local trivializations of the prequantum bundle $P \rightarrow M$ on each chart, as in Proposition 2.15 and 2.18. On each of such charts, the map f is approximated at first order by its linear approximation, and correspondingly the canonical map F is approximated on the trapped set by its differential $D_\rho F$ given in

⁹This idea of defining a generalized Sobolev space using an escape (or weight) function on the phase space has been used several times before and is not new in this paper. For instance, in the context of semiclassical scattering theory in the phase space, such an idea was developed by B. Helffer and J. Sjöstrand in [28]. In the context of transfer operators for hyperbolic dynamical systems, it was developed in [5, 6] more recently.

(2.25). In Section 7.3, we show how to decompose the global prequantum operator into “prequantum operators on charts” and how to recompose it, i.e. passing from global to local and *vice versa*.

(2) In view of the decomposition above called “microlocal decomposition”, we study first the prequantum operator associated to a linear hyperbolic map. This is done in Section 5. The prequantum operator for a linear hyperbolic map turns out to be the tensor product of two operators, according to the decomposition (2.25): one operator is a prequantum operator associated to the linear map tangent to the trapped set $T_\rho K$ and the other one is associated to the linear map in its (symplectic) orthogonal $(T_\rho K)^\perp$. The first part is a unitary operator, while the second part is treated by using the property of the escape function $W_h(x, \xi)$ as described above and shown to display a discrete decomposition (a discrete spectrum) in Proposition 4.19. For rigorous argument, we will present some technical tools first:

- (a) the Bargmann transform \mathcal{B}_h which represents functions and operators in phase space, is explained in Section 4.1.
- (b) definition of the escape function in phase space and the associated anisotropic Sobolev space, for the hyperbolic dynamics orthogonal to the trapped set, in Section 4.3.

Let us now give a quick description of the tensor product decomposition of point (2) above in order to explain how the band structure estimates (1.28) may be obtained. Eq.(2.25) gives a description of the differential DF_ρ of the canonical map F at the point $\rho \in K$ of the trapped set. It is therefore natural to consider similarly the transfer operator “microlocally” at point ρ and express it as an operator \mathcal{L}_ρ which somehow “quantizes” the symplectic map DF_ρ . We will see in Section 5 that this operator has the following form (which can easily be guessed from (2.5))

$$\mathcal{L}_\rho = e^{V_x} e^{-i2\pi N \mathcal{A}_x} \hat{F}^{(1)} \otimes \hat{F}^{(2)}$$

with constants V_x, \mathcal{A}_x with $x = \tilde{\pi}(\rho) \in M$ and where $\hat{F}^{(1)}, \hat{F}^{(2)}$ are the metaplectic operators associated to the linear symplectic map $F^{(1)}, F^{(2)}$ respectively, given by:

$$\hat{F}^{(1)} = \hat{F}^{(2)} = \hat{F} : \begin{cases} C^\infty(\mathbb{R}^d) & \rightarrow C^\infty(\mathbb{R}^d) \\ u & \rightarrow |\det A_x|^{-1/2} \cdot u \circ A_x^{-1} \end{cases}$$

with $A_x = Df|_{E_u(x)}: \mathbb{R}^d \rightarrow \mathbb{R}^d$ being the linear expanding map. Recall that the linear map $F^{(1)}: T_\rho K \rightarrow T_\rho K$ acts on the tangent space of the trapped set. Observe that $\hat{F}^{(1)}$ is unitary in $L^2(\mathbb{R}^d)$ with spectrum on the unit circle. On the other hand $F^{(2)}: (T_\rho K)^\perp \rightarrow (T_\rho K)^\perp$ acts on the orthogonal symplectic, transverse to the trapped set. It is a hyperbolic linear map and we will see in Section 4, Theorem 4.19, that correspondingly the operator $\hat{F}^{(2)}: H^r(\mathbb{R}^d) \rightarrow H^r(\mathbb{R}^d)$ in anisotropic Sobolev spaces has a discrete spectrum contained in bands indexed by $k \in \mathbb{N}$:

$$|\det A_x|^{-1/2} \|A_x\|_{\max}^{-k} \leq |z| \leq |\det A_x|^{-1/2} \|A_x\|_{\min}^{-k}.$$

965 (We get in fact a more precise description of the operator $\hat{F}^{(2)}$ as a sum of invertible operators with controlled norms). This implies that the operator $\mathcal{L}_\rho : L^2(\mathbb{R}^d) \otimes H^r(\mathbb{R}^d) \rightarrow L^2(\mathbb{R}^d) \otimes H^r(\mathbb{R}^d)$ has its spectrum contained in bands $r_k^-(x) \leq |z| \leq r_k^+(x)$ with

$$r_k^\pm(x) = \exp(D(x)) \|Df|_{E_u(x)}\|_{\max/\min}^{-k}, \quad D(x) := V_x - \frac{1}{2} \log \det Df|_{E_u(x)}$$

After averaging along the trajectory starting from x during time n and taking the min/max over $x \in M$ we see that we obtain the estimates r_k^\pm of (1.28). In fact the precise justification of this last step in the purpose of Sections 5 and 6 and proceed as follows:

- (1) In Section 6 we develop some results that will be used in Section 7 in order to show that non-linearity of the map f can be neglected in the reconstruction of the global prequantum operator from its local parts.
- (2) In Section 7, we assemble all these previous results and obtain the proof of the main Theorem 7.1.

Finally let us mention that Sections 4 and 5, where we study the resonances of the linear model, as explained above, are the core of our argument because they reveal the main mechanism responsible for the band structure of the spectrum described in Theorem 1.17. This mechanism was discovered in the paper [17] originally in the study of the prequantum linear cat map.

3 Covariant derivative D and spectrum of the rough Laplacian Δ

In this Section we provide additional results concerning the rough Laplacian operator Δ_N and its spectrum in the semiclassical limit $N \rightarrow \infty$. The rough Laplacian is defined from the covariant derivative D as $\Delta = D^*D$. The main result presented in Theorem 3.5 shows that the bottom of the spectrum of Δ_N in $L^2_N(P)$ has discrete spectrum with a band structure on the positive real line (beware that Δ is positive and self-adjoint). We show that the spectral projector Π_0 of the prequantum operator \hat{F}_N on its external band \mathcal{A}_0 has the same rank as the spectral projector \mathfrak{P}_0 of Δ_N on its first band. Since it is known from [26, th. 2][33, cor. 1.2] that this rank is given by an index formula, we deduce (1.34) and (1.53). The spectral projector \mathfrak{P}_0 of the Laplacian Δ_N is also used in definition 1.67 page 38 to define the Toeplitz quantum space and the Toeplitz quantization.

3.1 The covariant derivative D

The connection one form A on P given in Theorem 1.4 induces a differential operator D called the covariant derivative. We recall its general definition and give its expression in local coordinates. We will use it for the definition of the rough Laplacian operator Δ in the Section 3.2 and also to treat the affine models on \mathbb{R}^{2d} in Section 5.2.

Recall that the exterior derivative du of a function $u \in C^\infty(P)$ (i.e. the differential of u) is defined at point $p \in P$ by

$$(du)_p(V) = V(u)(p)$$

1000 where $V \in T_pP$ is any tangent vector. The connection one form $A \in C^\infty(P, \Lambda^1 \otimes (i\mathbb{R}))$ on the principal bundle $\pi : P \rightarrow M$ defines a splitting of the tangent space at every point $p \in P$:

$$T_pP = V_pP \oplus H_pP$$

with $H_pP = \text{Ker}(A(p))$ and $V_pP = \text{Ker}((D\pi)(p))$. The subspaces V_pP and H_pP are called respectively vertical subspace and horizontal subspace. We will denote

$$H : T_pP \rightarrow H_pP \tag{3.1}$$

1005 the projection onto the horizontal space with $\text{Ker}(H) = V_pP$. Explicitly if $V \in T_pP$ then from (1.8) its horizontal component is

$$HV = V + iA(V) \frac{\partial}{\partial \theta}. \tag{3.2}$$

This can be checked easily from the requirements that $A(HV) = 0$ and $V - HV \in V_pP$.

Definition 3.1. If $u \in C^\infty(P)$ is a smooth function, its *exterior covariant derivative* $Du \in C^\infty(P; \Lambda^1)$ is a one form on P defined by

$$(Du)_p(V) = ((HV)(u))(p) = (du(HV))(p) \quad \text{for } p \in P, V \in T_pP \tag{3.3}$$

The operator D is equivariant with respect to (or commutes with) the $\mathbf{U}(1)$ action (1.6) in P and therefore restricts naturally to the space (1.20):

$$D : C_N^\infty(P) \rightarrow C_N^\infty(P, \Lambda^1)$$

1010 for every $N \in \mathbb{Z}$.

3.1.1 Expression of D in local charts

Proposition 3.2. *With respect to the local trivialization (1.7) of the bundle P over open sets $U_\alpha \subset M$, if $u \in C_N^\infty(P)$ then Du is expressed as the first order differential operator $D_\alpha : C^\infty(U_\alpha) \rightarrow C^\infty(U_\alpha, \Lambda^1)$ given by:*

$$D_\alpha u_\alpha = du_\alpha + \frac{i}{\hbar} u_\alpha \eta_\alpha \quad (3.4)$$

with $u_\alpha := (u \circ \tau_\alpha) \in C^\infty(U_\alpha)$ as in (2.4) and $D_\alpha u_\alpha := (Du) \circ \tau_\alpha \in C^\infty(U_\alpha, \Lambda^1)$. More specifically, in the normal coordinates $x = (q, p) = (q^1, \dots, q^d, p^1, \dots, p^d)$ and the local trivialization on U_α in Proposition 2.15, it is expressed as

$$D_\alpha u_\alpha = \frac{i}{\hbar} \left(\sum_{j=1}^d \left(\widehat{\zeta}_q^j u_\alpha \right) dq^j + \left(\widehat{\zeta}_p^j u_\alpha \right) dp^j \right) \quad (3.5)$$

with the basis (dq, dp) of Λ_x^1 , where $\widehat{\zeta}_q^j$ and $\widehat{\zeta}_p^j$ are the differential operators on $C^\infty(\mathbb{R}^{2d})$ defined respectively by

$$\begin{aligned} \widehat{\zeta}_q^j &:= \widehat{\xi}_q^j - \frac{1}{2} p^j & \text{with } \widehat{\xi}_q^j &:= -i\hbar \frac{\partial}{\partial q^j} \\ \widehat{\zeta}_p^j &:= \widehat{\xi}_p^j + \frac{1}{2} q^j & \text{with } \widehat{\xi}_p^j &:= -i\hbar \frac{\partial}{\partial p^j}. \end{aligned} \quad (3.6)$$

Remark 3.3. Notice that the canonical variables (ζ_q, ζ_p) defined in (2.17) are the symbol of the operators (3.6) as a pseudodifferential operator. In more geometrical terms, the symbol of the covariant derivative $-i\hbar D$ is the one form $\sigma(-i\hbar D) = \zeta dx = \sum_j \zeta_q^j dq^j + \zeta_p^j dp^j$ on T^*U_α . This can be understood as a generalization of the simpler case of the exterior derivative $d : C^\infty(M, \Lambda^p) \rightarrow C^\infty(M, \Lambda^{p+1})$ (by taking $\eta = 0$, i.e. a connection with zero curvature), for which the principal symbol is known to be $\sigma(d) = \frac{i}{\hbar} (\xi dx) \wedge \cdot$, [50, (10.12) on p.162].

Proof. Consider local coordinates $x = (x^1, \dots, x^{2d}) \in U_\alpha \subset M$ and a local trivialization of P giving local coordinates (x, θ) on P . Let $V = V^x \frac{\partial}{\partial x} + V^\theta \frac{\partial}{\partial \theta}$ be a vector field on P expressed in these local coordinates. From (3.2) and (1.9) we have

$$\begin{aligned} HV &= V + iA(V) \frac{\partial}{\partial \theta} = V + (-d\theta(V) + 2\pi\eta_\alpha(V)) \frac{\partial}{\partial \theta} \\ &= V^x \frac{\partial}{\partial x} + 2\pi\eta_\alpha(V) \cdot \frac{\partial}{\partial \theta}. \end{aligned}$$

Then, from the definition (3.3),

$$(Du)(V) = (HV)(u) = V^x \frac{\partial u}{\partial x} + 2\pi\eta_\alpha(V) \frac{\partial u}{\partial \theta}.$$

1025 Suppose now that $u \in C_N^\infty(P)$ and write $p = e^{i\theta}\tau_\alpha(x) \in P$. Then

$$u(p) = u(e^{i\theta}\tau_\alpha(x)) = e^{iN\theta}u(\tau_\alpha(x)) = e^{iN\theta}u_\alpha(x)$$

and

$$\begin{aligned} (Du)_p(V) &= e^{iN\theta} \left(V^x \frac{\partial u_\alpha}{\partial x} + iN2\pi\eta_\alpha(V)u_\alpha \right) \\ &= e^{iN\theta} (du_\alpha(V) + iN2\pi\eta_\alpha(V)u_\alpha) \\ &= e^{iN\theta} \left(du_\alpha + \frac{i}{\hbar}u_\alpha\eta_\alpha \right) (V). \end{aligned}$$

Hence

$$D_\alpha u_\alpha = (Du)_p \circ \tau_\alpha = du_\alpha + \frac{i}{\hbar}u_\alpha\eta_\alpha$$

We obtain the rest of the claims by simple calculation. \square

3.2 The rough Laplacian Δ

1030 In order to define the adjoint operator D^* and the Laplacian $\Delta = D^*D$ which is used in *geometric quantization* we need an additional structure on the manifold M , namely a metric g compatible with ω . References are [54, p.400],[8, p.168],[51, p.504].

3.2.1 Compatible metrics and Laplacian

1035 We recall [45, p.72] that on a symplectic manifold (M, ω) , there exists a Riemannian metric g compatible with ω in the sense that there exists an *almost complex structure*¹⁰ J on M such that

$$\omega(Ju, Jv) = \omega(u, v) \text{ and } g(u, v) = \omega(u, Jv) \text{ for all } x \in M \text{ and } u, v \in T_x M. \quad (3.7)$$

In general J is not integrable, i.e. it is not a complex structure. In the rest of this Section we suppose given such a metric g and an almost complex structure J on M .

1040 The metric g on M induces an equivariant metric g_P on P by declaring that [51, ex.1,ex.2 p.508]:

- (1) for every point $p \in P$, $V_p P \perp H_p P$ are orthogonal,
- (2) on the horizontal space $H_p P$, g_P is the pull back of g by π : $(g_P)_{/H_p P} = \pi^*(g)$
- (3) on the vertical space $V_p P$, g_P is the canonical (Killing) metric on $\mathfrak{u}(1)$ i.e. $\left\| \frac{\partial}{\partial \theta} \right\|_{g_P} = 1$.

¹⁰ An almost complex structure J on M is a section of the bundle $\text{Hom}(TM, TM)$ such that $J \circ J = -\text{Id}$.

1045 This metric g_P induces a L^2 scalar product $\langle \alpha | \beta \rangle_{\Lambda^1(p)}$ in the space of one forms $\Lambda^1(p)$ at point $p \in P$. Using the volume form μ_P on P in (1.17), we define a L^2 scalar product in the space of differential one forms $C^\infty(P, \Lambda^1)$ by

$$\langle \alpha | \beta \rangle_{L^2(P, \Lambda^1)} := \int \langle \alpha(p) | \beta(p) \rangle_{\Lambda^1(p)} d\mu_P(p) \quad \text{for } \alpha, \beta \in C^\infty(P, \Lambda^1)$$

The L^2 product of functions is of course defined by

$$\langle u | v \rangle_{L^2(P)} := \int \overline{u(p)} \cdot v(p) d\mu_P(p) \quad \text{for } u, v \in C^\infty(P).$$

Then the operators D^* and Δ are defined as follows.

Definition 3.4. The **adjoint covariant derivative** $D^* : C^\infty(P, \Lambda^1) \rightarrow C^\infty(P)$ is defined by the relation

$$\langle u | D^* \alpha \rangle_{L^2(P, \Lambda^1)} = \langle Du | \alpha \rangle_{L^2(P)} \quad \text{for all } u \in C^\infty(P) \quad \text{and } \alpha \in C^\infty(P, \Lambda^1).$$

The *rough Laplacian* $\Delta : C^\infty(P) \rightarrow C^\infty(P)$ is defined as the composition

$$\Delta = D^* D.$$

1050 The operators introduced above are equivariant, i.e. $D^* R_\theta = R_\theta D^*$ and $\Delta R_\theta = R_\theta \Delta$ with $R_\theta : p \rightarrow e^{i\theta} p$, because so is the metric g_P . Hence D^* and Δ restrict naturally to

$$D^* : C_N^\infty(P, \Lambda^1) \rightarrow C_N^\infty(P) \quad \text{and} \quad \Delta_N : C_N^\infty(P) \rightarrow C_N^\infty(P)$$

for each $N \in \mathbb{Z}$. We have denoted Δ_N for the restriction of Δ to $C_N^\infty(P)$.

1055 It is known that, for every $N \in \mathbb{Z}$, the operator Δ_N is an essentially self-adjoint positive operator with compact resolvent. Hence its spectrum is discrete and consists of real positive eigenvalues. The next theorem shows that these eigenvalues form some “clusters” (also called “bands”) in the lower part of this spectrum. Precisely the eigenvalues of $\frac{1}{2\pi N} \Delta_N$ concentrate around the specific integer values $d + 2k$ with $d = \frac{1}{2} \dim M$, $k \in \mathbb{N}$. (These half-integer values correspond essentially to the eigenvalues of a harmonic oscillator model as we will see later.) See Figure 3.1. These clusters of eigenvalues are called *Landau levels* or *Landau bands* in physics. The existence of the first band is given in various papers, see 1060 [33, Cor 1.2] and reference therein.¹¹

¹¹ The result for all the bands seems to be known to specialists although it does not appear explicitly in the literature to the best of our knowledge.

Theorem 3.5. “*The bottom spectrum of Δ_N has band structure*”. For any $\alpha > 0$, the spectral set of the rough Laplacian $\frac{1}{2\pi N}\Delta_N$ in the interval $[0, \alpha]$ is contained in the $N^{-\epsilon}$ -neighborhood of the subset $\{d + 2k, k \in \mathbb{N}\}$ for sufficiently large N and any $0 < \epsilon < 1/2$. The number of eigenvalues in the $N^{-\epsilon}$ -neighborhood of $d + 2k$ (or in the k -th band) is proportional to N^d , that is, if we write \mathfrak{P}_k for the spectral projector for the eigenvalues on the k -th band, we have

$$C^{-1}N^d < \text{rank } \mathfrak{P}_k < CN^d$$

for some constant C independent of N . In particular, for the spectral projector \mathfrak{P}_0 for the first band, we have

$$\text{rank } (\mathfrak{P}_0) = \int_M [e^{N\omega} \text{Todd}(TM)]_{2d}. \quad (3.8)$$

Further, for the relation to the prequantum transfer operator \hat{F}_N , we have

$$\text{rank } \mathfrak{P}_k = \text{rank } \tau^{(k)} = \dim \mathcal{H}_k \quad \text{for } 0 \leq k \leq n$$

for sufficiently large N , where n , $\tau^{(k)}$ and \mathcal{H}_k are those in Theorem 7.1. In particular

$$\mathfrak{P}_0 : \text{Im } (\tau^{(0)}) \rightarrow \text{Im } (\mathfrak{P}_0) \quad (3.9)$$

is a finite rank isomorphism.

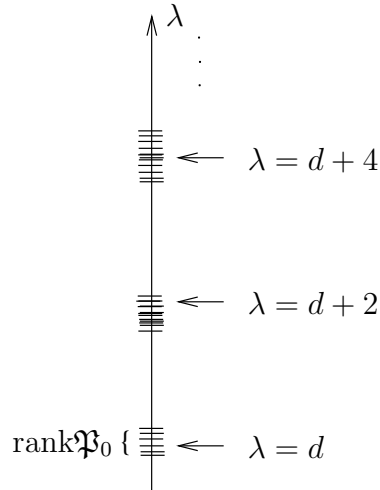


Figure 3.1: The Landau levels of the spectrum of the rough Laplacian $\frac{1}{2\pi N}\Delta : L_N^2(P) \rightarrow L_N^2(P)$ for $N \gg 1$.

In the separate Section 8 we give the proof for Theorem 3.5 concerning the spectrum of the rough Laplacian $\Delta = D^*D$. This proof follows the same strategy as for the proof of Theorem 7.1 namely the Laplacian is decomposed into local charts and approximated by an Euclidean Laplacian. We use the Harmonic oscillator described in Section 8.1. In Section 8.2, we obtain the spectrum of the rough Euclidean Laplacian. In Section 8.3 we establish some lemmas in order to show that non-linearity can be neglected. The proof of the index formula (3.8) is given in [26, Th. 2], see also [33, Cor. 1.2] and references therein.

3.2.2 Expression of D^* and Δ in local charts

Let us see the expression of differential operators D^* and Δ introduced above in the local trivialization. Consider a local trivialization (1.7) of the bundle P over an open set $U_\alpha \subset M$. As explained in Proposition 3.2, the operator $D : C_N^\infty(P) \rightarrow C_N^\infty(P, \Lambda^1)$ in such local trivialization is represented by the differential operator

$$D_\alpha : C^\infty(U_\alpha) \rightarrow C^\infty(U_\alpha, \Lambda^1), \quad D_\alpha u_\alpha = (Du) \circ \tau_\alpha$$

where $u \in C_N^\infty(P)$ and $u_\alpha := (u \circ \tau_\alpha) \in C^\infty(U_\alpha)$. Similarly $D^* : C_N^\infty(P, \Lambda^1) \rightarrow C_N^\infty(P)$ and $\Delta_N = D^*D$ are represented by operators

$$D_\alpha^* : C^\infty(U_\alpha, \Lambda^1) \rightarrow C^\infty(U_\alpha) \quad \text{and} \quad \Delta_\alpha : C^\infty(U_\alpha) \rightarrow C^\infty(U_\alpha).$$

The next proposition gives the explicit expression of the differential operators D_α , D_α^* and Δ_α using local coordinates (x^1, \dots, x^{2d}) on U_α (we have already obtain such an expression for D_α in Proposition 3.2). Note that the operators D_α^* and Δ_α depend on the Riemann metric g on M and also on N (or \hbar) though it is not explicit in the notation. We write $g = \sum_{j,k} g_{j,k} dx^j \otimes dx^k$ for the metric tensor and $g^{j,k} = (g_{j,k})_{j,k}^{-1}$ for the entries of the inverse matrix.

Proposition 3.6. *With respect to the local trivialization and coordinate system described above, we have the following expressions for D , D^* and $\Delta = D^*D$:*

$$D_\alpha u_\alpha = \frac{i}{\hbar} \sum_{j=1}^{2d} \left(\widehat{\zeta}^j u_\alpha \right) dx^j, \quad \text{with } \widehat{\zeta}^j = -i\hbar \frac{\partial}{\partial x^j} + \eta_j, \quad (3.10)$$

$$D_\alpha^* \left(\sum_{j=1}^{2d} v_j dx^j \right) = -\frac{i}{\hbar} \sum_{j,k=1}^{2d} \left(g^{j,k} \widehat{\zeta}^j - i\hbar (\partial_j g^{j,k}) \right) v_k,$$

$$\Delta_\alpha u_\alpha = \frac{1}{\hbar^2} \sum_{j,k=1}^{2d} \left(g^{jk} \widehat{\zeta}^j \widehat{\zeta}^k - i\hbar (\partial_j g^{jk}) \widehat{\zeta}^k \right) u_\alpha. \quad (3.11)$$

Proof. The expression (3.5) gives (3.10). Let $v = \sum_{j=1}^{2d} v_j dx^j \in C^\infty(U_\alpha, \Lambda^1)$ and $u \in C^\infty(U_\alpha)$. Using integration by parts, we have

$$\begin{aligned} \langle u, D_\alpha^* v \rangle_{L^2(U_\alpha)} &= \langle D_\alpha u, v \rangle_{L^2(U_\alpha, \Lambda^1)} = \int \langle du + \frac{i}{\hbar} \eta u | v \rangle dx \\ &= \sum_{j,k} \int \overline{\left(\partial_j u + \frac{i}{\hbar} \eta_j u \right)} g^{jk} v_k dx = - \sum_{j,k} \int \left(\bar{u} \partial_j (g^{jk} v_k) + \frac{i}{\hbar} \eta_j \bar{u} g^{jk} v_k \right) dx \\ &= - \int \bar{u} \sum_{j,k} \left((\partial_j g^{jk}) v_k + g^{jk} \left(\partial_j v_k + \frac{i}{\hbar} \eta_j v_k \right) \right) dx. \end{aligned}$$

Hence

$$D_\alpha^* v = -\frac{i}{\hbar} \sum_{j,k} \left(g^{jk} \widehat{\zeta}^j - i\hbar (\partial_j g^{jk}) \right) v_k.$$

1085 We deduce (3.11) from the computation

$$\begin{aligned} \Delta_\alpha u_\alpha &= D_\alpha^* D_\alpha u_\alpha = -\frac{i}{\hbar} \sum_{j,k} \left(g^{jk} \widehat{\zeta}^j - i\hbar (\partial_j g^{jk}) \right) \frac{i}{\hbar} \left(\widehat{\zeta}^k u_\alpha \right) \\ &= \frac{1}{\hbar^2} \sum_{j,k} \left(g^{jk} \widehat{\zeta}^j \widehat{\zeta}^k - i\hbar (\partial_j g^{jk}) \widehat{\zeta}^k \right) u_\alpha \end{aligned}$$

□

Corollary 3.7. *In local Darboux coordinates $x = (q, p) = (q^1, \dots, q^d, p^1, \dots, p^d)$ on U_α and in the special case of the Euclidean metric $g = \sum_{j=1}^d dq^j \otimes dq^j + dp^j \otimes dp^j$, we have*

$$\Delta_\alpha = \frac{1}{\hbar^2} \left(\sum_{j=1}^d \widehat{\zeta}_p^j{}^2 + \widehat{\zeta}_q^j{}^2 \right) \quad (3.12)$$

with $\widehat{\zeta}_q^j, \widehat{\zeta}_p^j$ given in (3.6).

The operator (3.12) is called the *Euclidean rough Laplacian*. In Section 8.2, we will deduce the cluster structure (or the Landau levels) of the spectrum of the Euclidean rough Laplacian by identifying it with the *harmonic oscillator*.

1090 4 Resonances of linear expanding maps

This Section is self-contained. The main result of this Section is Proposition 4.19.

4.1 The Bargmann transform

4.1.1 Definitions

In this section, we recall some basic facts related to the so-called Bargmann transform. For more detailed account about the Bargmann transform, we refer to the books [36, Ch.3], [23, p.39], [27], [40, p.19].

Let D be a positive integer. Let $\hbar > 0$. We consider the Euclidean space \mathbb{R}^D with its canonical Euclidean norm written $|\cdot|$. For each point $(x, \xi) \in T^*\mathbb{R}^D = \mathbb{R}^D \oplus \mathbb{R}^D$, we assign the complex-valued smooth function $\phi_{x,\xi} \in \mathcal{S}(\mathbb{R}^D)$ defined by

$$\phi_{x,\xi}(y) = a_D \exp\left(\frac{i}{\hbar}\xi \cdot (y - \frac{x}{2}) - \frac{1}{2\hbar}|y - x|^2\right), \quad y \in \mathbb{R}^D \quad (4.1)$$

with

$$a_D = (\pi\hbar)^{-D/4}. \quad (4.2)$$

We will henceforth consider the measure $dx = dx^1 \dots dx^D$ on \mathbb{R}^D defining the Hilbert spaces $L^2(\mathbb{R}^D)$ and the measure $(2\pi\hbar)^{-D} dx d\xi$ on $T^*\mathbb{R}^D = \mathbb{R}^{2D}$ defining $L^2(\mathbb{R}^{2D})$. Notice that the constant a_D is taken so that $\|\phi_{x,\xi}\|_{L^2(\mathbb{R}^D)} = 1$.

Definition 4.1. The *Bargmann transform* is the continuous operator

$$\mathcal{B}_\hbar : \mathcal{S}(\mathbb{R}^D) \rightarrow \mathcal{S}(\mathbb{R}^{2D}), \quad (\mathcal{B}_\hbar u)(x, \xi) = \int \overline{\phi_{x,\xi}(y)} \cdot u(y) dy \quad (4.3)$$

on the Schwartz space $\mathcal{S}(\mathbb{R}^D)$. Then the (formal) adjoint of \mathcal{B}_\hbar defined by $\langle u, \mathcal{B}_\hbar^* v \rangle = \langle \mathcal{B}_\hbar u, v \rangle$ is given by

$$\mathcal{B}_\hbar^* : \mathcal{S}(\mathbb{R}^{2D}) \rightarrow \mathcal{S}(\mathbb{R}^D), \quad (\mathcal{B}_\hbar^* v)(y) = \int \phi_{x,\xi}(y) \cdot v(x, \xi) \frac{dx d\xi}{(2\pi\hbar)^D}. \quad (4.4)$$

Lemma 4.2. [36, p70, Proposition 3.1.1] We have that

(1) \mathcal{B}_\hbar extends uniquely to an isometric embedding $\mathcal{B}_\hbar : L^2(\mathbb{R}^D) \rightarrow L^2(\mathbb{R}^{2D})$.

(2) \mathcal{B}_\hbar^* extends uniquely to a bounded operator $\mathcal{B}_\hbar^* : L^2(\mathbb{R}^{2D}) \rightarrow L^2(\mathbb{R}^D)$.

(3) $\mathcal{B}_\hbar^* \circ \mathcal{B}_\hbar = \text{Id}$ on $L^2(\mathbb{R}^D)$.

Proof. For any $u \in \mathcal{S}(\mathbb{R}^D)$, we have

$$\begin{aligned} \|\mathcal{B}_\hbar u\|_{L^2(\mathbb{R}^{2D})}^2 &= \frac{a_D^2}{(2\pi\hbar)^D} \int \phi_{x,\xi}(y') \overline{u(y')} \cdot \overline{\phi_{x,\xi}(y)} u(y) dx d\xi dy dy' \\ &= \frac{a_D^2}{(2\pi\hbar)^D} \int u(y) \overline{u(y')} \exp\left(\frac{i}{\hbar}\xi(y' - y) - \frac{1}{2\hbar}(|x - y|^2 + |x - y'|^2)\right) dx d\xi dy dy' \\ &= (\pi\hbar)^{-D/2} \int |u(y)|^2 \exp(-|x - y|^2/\hbar) dx dy \\ &= \int |u(y)|^2 dy = \|u\|_{L^2(\mathbb{R}^D)}^2. \end{aligned}$$

This gives the claims of the lemma by the usual continuity argument. \square

4.1.2 Bargmann projector

Proposition 4.3. *The space $L^2(\mathbb{R}^{2D})$ is orthogonally decomposed as*

$$L^2(\mathbb{R}^{2D}) = \text{Im} \mathcal{B}_h \oplus \ker \mathcal{B}_h^*. \quad (4.5)$$

The Bargmann projector \mathcal{P}_h is the orthogonal projection onto $\text{Im} \mathcal{B}_h \subset L^2(\mathbb{R}^{2D})$ along $\ker \mathcal{B}_h^*$. It is given by

$$\mathcal{P}_h := \mathcal{B}_h \circ \mathcal{B}_h^* : L^2(\mathbb{R}^{2D}) \rightarrow L^2(\mathbb{R}^{2D}). \quad (4.6)$$

It can be expressed as an integral operator $(\mathcal{P}_h v)(z) = \int K_{\mathcal{P},h}(z, z') v(z') dz'$ with the Schwartz kernel:

$$K_{\mathcal{P},h}(z, z') = \exp \left(\frac{i}{2\hbar} \omega(z, z') - \frac{1}{4\hbar} |z - z'|^2 \right) \quad (4.7)$$

with $z = (x, \xi)$, $z' = (x', \xi') \in \mathbb{R}^{2D}$, the measure $dz' = dx' d\xi' / (2\pi\hbar)^D$, the Euclidean norm $|z|^2 := |x|^2 + |\xi|^2$ and the canonical symplectic form on $T^*\mathbb{R}^D$, $\omega(z, z') = x \cdot \xi' - \xi \cdot x'$

Proof. \mathcal{P}_h is an orthogonal projection because $\mathcal{P}_h^* = (\mathcal{B}_h \circ \mathcal{B}_h^*)^* = \mathcal{P}_h$ and $\mathcal{P}_h^2 = \mathcal{B}_h \circ \mathcal{B}_h^* \circ \mathcal{B}_h \circ \mathcal{B}_h^* = \mathcal{P}_h$ from Lemma 4.2. From Definition 4.1, the Schwartz kernel of $\mathcal{P}_h = \mathcal{B}_h \circ \mathcal{B}_h^*$ is

$$\begin{aligned} K_{\mathcal{P},h}(z, z') &= \int dy \overline{\phi_z(y)} \phi_{z'}(y) \\ &= a_D^2 \int_{\mathbb{R}^D} dy \exp \left(-i \frac{\xi}{\hbar} (y - \frac{x}{2}) + i \frac{\xi'}{\hbar} (y - \frac{x'}{2}) - \frac{1}{2\hbar} (|y - x|^2 + |y - x'|^2) \right) \\ &= a_D^2 \exp \left(i \frac{1}{2\hbar} (\xi x - \xi' x') - \frac{1}{2\hbar} (|x|^2 + |x'|^2) \right) \\ &\quad \times \int dy \exp \left(\frac{1}{\hbar} (i(\xi' - \xi) + (x + x')) y - \frac{1}{\hbar} |y|^2 \right). \end{aligned}$$

We use the following formula for Gaussian integral in \mathbb{R}^D :

$$\int_{\mathbb{R}^D} e^{-\frac{1}{2} y \cdot (Ay) + b \cdot y} dy = \sqrt{\frac{(2\pi)^D}{\det A}} \exp \left(\frac{1}{2} b \cdot (A^{-1} b) \right), \quad b \in \mathbb{C}^D, A \in \mathcal{L}(\mathbb{R}^D) \quad (4.8)$$

with $A = (2/\hbar) \cdot \text{Id}$, $b = \frac{i}{\hbar} (\xi' - \xi) + \frac{1}{\hbar} (x + x')$ and get

$$\begin{aligned} K_{\mathcal{P},h}(z, z') &= \frac{1}{(\pi\hbar)^{D/2}} \frac{(2\pi)^{D/2}}{(2/\hbar)^{D/2}} \exp \left(\frac{1}{4\hbar} (i(\xi' - \xi) + (x + x'))^2 \right) \\ &\quad \times \exp \left(i \frac{1}{2\hbar} (\xi x - \xi' x') - \frac{1}{2\hbar} (|x|^2 + |x'|^2) \right) \\ &= \exp \left(\frac{1}{4\hbar} (-|\xi' - \xi|^2 - |x - x'|^2) + \frac{i}{2\hbar} (\xi x - \xi' x' + (\xi' - \xi)(x + x')) \right) \\ &= \exp \left(\frac{i}{2\hbar} \omega(z, z') - \frac{1}{4\hbar} |z - z'|^2 \right). \end{aligned}$$

□

1125 **4.1.3 The Bargmann transform in more general setting**

We have seen that the Bargmann transform gives a phase-space representation, i.e. a unitary isomorphism: $\mathcal{B}_\hbar : L^2(\mathbb{R}^D) \rightarrow \text{Im}(\mathcal{B}_\hbar) \subset L^2(T^*\mathbb{R}^D)$. The next proposition gives the Bargmann transform in a slightly more general setting. We start from a symplectic linear space (E, ω) of dimension $2D$ and a Lagrangian subspace $L \subset E$. Let $g(\cdot, \cdot)$ be a scalar product on E that is compatible with the symplectic form ω on E in the sense that there is a linear map $J : E \rightarrow E$ such that $J \circ J = -\text{Id}$ and holds

$$g(z, z') = \omega(z, Jz') \quad \text{for all } z, z' \in E.$$

This is nothing but the point-wise version of the condition (3.7). Let L^{\perp_g} be the orthogonal complement of L with respect to the inner product g , so that a point $z \in E$ can be decomposed uniquely as $z = x + \xi$, $x \in L$, $\xi \in L^{\perp_g}$.

1135 For each point $z = x + \xi$, $x \in L$, $\xi \in L^{\perp_g}$, we define the wave packet $\phi_z(y) \in \mathcal{S}(L)$ by

$$\phi_z(y) = a_D \exp\left(\frac{i}{\hbar}\omega(y - \frac{x}{2}, \xi) - \frac{1}{2\hbar}|y - x|_g^2\right) \quad \text{for } y \in L.$$

We define the Bargmann transform $\mathcal{B}_\hbar : \mathcal{S}(L) \rightarrow \mathcal{S}(E)$ and its adjoint $\mathcal{B}_\hbar^* : \mathcal{S}(E) \rightarrow \mathcal{S}(L)$ as in Definition 4.1. Then the statement corresponding to Lemma 4.2 and Proposition 4.3 holds. Namely

Proposition 4.4. *For the Bargmann transform \mathcal{B}_\hbar and its adjoint defined as above,*

- 1140 (1) \mathcal{B}_\hbar extends uniquely to an isometric embedding $\mathcal{B}_\hbar : L^2(L) \rightarrow L^2(E)$.
- (2) \mathcal{B}_\hbar^* extends uniquely to a bounded operator $\mathcal{B}_\hbar^* : L^2(E) \rightarrow L^2(L)$.
- (3) $\mathcal{B}_\hbar^* \circ \mathcal{B}_\hbar = \text{Id}$ on $L^2(L)$.
- (4) The Bargmann projector $\mathcal{P}_\hbar : L^2(E) \rightarrow L^2(E)$, defined by $\mathcal{P}_\hbar := \mathcal{B}_\hbar \circ \mathcal{B}_\hbar^*$, is the orthogonal projection onto $\text{Im} \mathcal{B}_\hbar \subset L^2(E)$. It is expressed as an integral operator with kernel (4.7) with $|\cdot|$ replaced by $|\cdot|_g$.

Remark 4.5. Notice that the Bargmann projector $\mathcal{P}_\hbar : L^2(E) \rightarrow L^2(E)$ can be defined directly from its kernel (4.7) and is independent on the choice of the Lagrangian subspace L .

1150 *Proof.* There are linear isomorphisms $\psi : \mathbb{R}^D \rightarrow L$ and $\Psi : \mathbb{R}^{2D} \rightarrow E$ such that the following diagram commutes:

$$\begin{array}{ccc} \mathbb{R}^{2D} & \xrightarrow{\Psi} & E \\ p \downarrow & & p' \downarrow \\ \mathbb{R}^D & \xrightarrow{\psi} & L \end{array}$$

where $p : \mathbb{R}^{2D} = \mathbb{R}^D \oplus \mathbb{R}^D \rightarrow \mathbb{R}^D$ is the projection to the first D components: $p(x, \xi) = x$, and $p' : E \rightarrow L$ is the orthogonal projection to L with respect to g , and moreover that

the pull-back of the symplectic form ω and the inner product g by Ψ coincides with the standard Euclidean inner product $g_0(z, z') = z \cdot z'$ and the standard symplectic form:

$$\omega_0(z, z') = x \cdot \xi' - \xi \cdot x' \quad \text{for } z = (x, \xi), z' = (x', \xi') \in T^*\mathbb{R}^D$$

1155 Through such correspondence by Ψ and ψ , the definition of the Bargmann transform and its adjoint above coincides with those that we made in the last subsection. Therefore the claims are just restatement of Lemma 4.2 and Proposition 4.3. □

4.1.4 Scaling

1160 The operators \mathcal{B}_\hbar and \mathcal{B}_\hbar^* are related to \mathcal{B}_1 and \mathcal{B}_1^* (i.e. with $\hbar = 1$) by the simple scaling $x \mapsto \hbar^{1/2}x$. Though this fact should be obvious, we give the relations explicitly for the later use. Let us introduce the unitary operators

$$s_\hbar : L^2(\mathbb{R}^D) \rightarrow L^2(\mathbb{R}^D), \quad s_\hbar u(x) = \hbar^{-D/4} u(\hbar^{-1/2}x) \quad (4.9)$$

and¹²

$$S_\hbar : L^2\left(\mathbb{R}^{2D}, \frac{dx d\xi}{(2\pi\hbar)^D}\right) \rightarrow L^2\left(\mathbb{R}^{2D}, \frac{dx d\xi}{(2\pi\hbar)^D}\right), \quad S_\hbar u(x, \xi) = u(\hbar^{-1/2}x, \hbar^{-1/2}\xi) \quad (4.10)$$

Then we have

1165 **Lemma 4.6.** *The following diagram commutes:*

$$\begin{array}{ccc} L^2(\mathbb{R}^{2D}) & \xrightarrow{S_\hbar} & L^2(\mathbb{R}^{2D}) \\ \mathcal{B}_\hbar \uparrow & & \mathcal{B}_1 \uparrow \\ L^2(\mathbb{R}^D) & \xrightarrow{s_\hbar} & L^2(\mathbb{R}^D) \end{array}$$

4.2 Action of linear transforms

Definition 4.7. The lift of a bounded operator $L : L^2(\mathbb{R}^D) \rightarrow L^2(\mathbb{R}^D)$ with respect to the Bargmann transform \mathcal{B}_\hbar is defined as the operator

$$L^{\text{lift}} := \mathcal{B}_\hbar \circ L \circ \mathcal{B}_\hbar^* : L^2(\mathbb{R}^{2D}) \rightarrow L^2(\mathbb{R}^{2D}) \quad (4.11)$$

By definition, it makes the following diagram commutes:

$$\begin{array}{ccc} L^2(\mathbb{R}^{2D}) & \xrightarrow{L^{\text{lift}}} & L^2(\mathbb{R}^{2D}) \\ \mathcal{B}_\hbar \uparrow & & \mathcal{B}_\hbar \uparrow \\ L^2(\mathbb{R}^D) & \xrightarrow{L} & L^2(\mathbb{R}^D). \end{array} \quad (4.12)$$

¹²Recall the convention on the norm on $L^2(\mathbb{R}^D)$ and $L^2(\mathbb{R}^{2D})$ made in the beginning of this section.

1170 Since $\mathcal{P}_h \circ L^{\text{lift}} \circ \mathcal{P}_h = L^{\text{lift}}$, the lift L^{lift} is always trivial on the second factor with respect to the decomposition $L^2(\mathbb{R}^{2D}) = \text{Im}\mathcal{B}_h \oplus \ker \mathcal{B}_h^* = \text{Im}\mathcal{P}_h \oplus \ker \mathcal{P}_h$ in (4.5), that is,

$$L^{\text{lift}} = (\mathcal{B}_h \circ L \circ \mathcal{B}_h^*)_{\text{Im}\mathcal{P}_h} \oplus (\mathbf{0})_{\text{Ker}\mathcal{P}_h}. \quad (4.13)$$

Lemma 4.8. *For an invertible linear transformation $A : \mathbb{R}^D \rightarrow \mathbb{R}^D$, we associate a bounded transfer operator defined by*

$$L_A : L^2(\mathbb{R}^D) \rightarrow L^2(\mathbb{R}^D), \quad L_A u = u \circ A^{-1} \quad (4.14)$$

Then we have

$$L_A = d(A) \cdot \mathcal{B}_h^* \circ L_{A \oplus {}^t A^{-1}} \circ \mathcal{B}_h, \quad (4.15)$$

1175 where $L_{A \oplus {}^t A^{-1}} : L^2(\mathbb{R}^{2D}) \rightarrow L^2(\mathbb{R}^{2D})$ is the unitary transfer operator given by

$$(L_{A \oplus {}^t A^{-1}} u)(x, \xi) := u(A^{-1}x, {}^t A \xi) \quad (4.16)$$

and we set

$$d(A) := \det \left(\frac{1}{2} (1 + {}^t A A) \right)^{1/2}.$$

Consequently the lift of L_A ,

$$L_A^{\text{lift}} := \mathcal{B}_h \circ L_A \circ \mathcal{B}_h^* \quad (4.17)$$

is expressed as

$$L_A^{\text{lift}} = d(A) \cdot \mathcal{P}_h \circ L_{A \oplus {}^t A^{-1}} \circ \mathcal{P}_h. \quad (4.18)$$

1180 *Remark 4.9.* (1) The expression (4.15) shows that L_A can be expressed as an operator on the phase space defined in terms of the Bargmann projector and the transfer operator $L_{A \oplus {}^t A^{-1}}$, but with an additional factor $d(A)$, sometimes called the *metaplectic correction*. This may be regarded as a realization of the idea explained in the last section: L_A can be seen as a Fourier integral operator and canonical map is $A \oplus {}^t A^{-1}$ on $T^*\mathbb{R}^D$. But notice that the correction term $d(A)$ will be crucially important for
1185 our argument.

(2) For an orthogonal transform $A \in SO(D)$, we have $d(A) = 1$.

(3) The operator $\frac{1}{\sqrt{|\det A|}} L_A$ is unitary in $L^2(\mathbb{R}^D)$ but because in the main result of this Section, Proposition 4.19, the Hilbert space is not $L^2(\mathbb{R}^D)$, we don't care about this property. It will however have some importance later in Proposition 5.11 where the
1190 factor $\frac{1}{\sqrt{|\det A|}}$ will therefore appear.

Proof. To prove (4.15), we write the operator $\mathcal{B}_h^* \circ L_{A \oplus {}^t A^{-1}} \circ \mathcal{B}_h$ as an integral operator

$$(\mathcal{B}_h^* \circ L_{A \oplus {}^t A^{-1}} \circ \mathcal{B}_h u)(y) = \int K(y, y') u(y') dy'$$

with the kernel (from (4.3) and (4.4))

$$K(y, y') = \int \phi_{x, \xi}(y) \cdot \overline{\phi_{A^{-1}x, {}^t A \xi}(y')} \frac{dx d\xi}{(2\pi\hbar)^D}.$$

Using the formula (12.3) for the Gaussian integral and change of variables, we can calculate the integral on the right hand side as

$$\begin{aligned} K(y, y') &= \int \phi_{Ax', {}^t A^{-1} \xi'}(y) \cdot \overline{\phi_{x', \xi'}(y')} \frac{dx' d\xi'}{(2\pi\hbar)^D}, \quad (x' = A^{-1}x, \xi' = {}^t A \xi) \\ &= a_D^2 \cdot \int e^{\frac{i}{\hbar}(\xi', y' - A^{-1}y) - |y' - x'|^2/(2\hbar) - |y - Ax'|^2/(2\hbar)} \frac{dx' d\xi'}{(2\pi\hbar)^D} \\ &= (\pi\hbar)^{-D/2} \cdot \delta(y' - A^{-1}y) \cdot \int e^{-|A^{-1}y - x'|^2/(2\hbar) - |y - Ax'|^2/(2\hbar)} dx' \\ &= \pi^{-D/2} \cdot \delta(y' - A^{-1}y) \cdot \int e^{-|t|^2/2 - |At|^2/2} dt \quad (t = (x' - A^{-1}y)/\hbar) \\ &= \det((I + {}^t AA)/2)^{-1/2} \cdot \delta(y' - A^{-1}y). \end{aligned}$$

Therefore we have

$$(\mathcal{B}_h^* \circ L_{A \oplus {}^t A^{-1}} \circ \mathcal{B}_h)u(y) = \det((I + {}^t AA)/2)^{-1/2} \cdot u(A^{-1}y) = d(A)^{-1} \cdot (L_A u)(y)$$

and hence the claim (4.15) follows. This implies

$$L_A^{\text{lift}} = \mathcal{B}_h \circ L_A \circ \mathcal{B}_h^* = d(A) \cdot \mathcal{P}_h \circ L_{A \oplus {}^t A^{-1}} \circ \mathcal{P}_h.$$

The other claims follow immediately. \square

1195 The next Lemma introduces operators which realize translation in phase space $T^*\mathbb{R}^D$. In [23, 40] it is shown that this gives a unitary irreducible representation of the Weyl-Heisenberg group.

Lemma 4.10. For $(x_0, \xi_0) \in T^*\mathbb{R}^D = \mathbb{R}^{2D}$, we associate a unitary operator defined by

$$T_{(x_0, \xi_0)} : L^2(\mathbb{R}^D) \rightarrow L^2(\mathbb{R}^D), \quad (T_{(x_0, \xi_0)}v)(y) = e^{\frac{i}{\hbar}\xi_0 \cdot (y + \frac{x_0}{2})} v(y - x_0) \quad (4.19)$$

Then we have

$$T_{(x_0, \xi_0)} = \mathcal{B}_h^* \circ \mathcal{T}_{(x_0, \xi_0)} \circ \mathcal{B}_h, \quad (4.20)$$

1200 where $\mathcal{T}_{(x_0, \xi_0)} : L^2(\mathbb{R}^{2D}) \rightarrow L^2(\mathbb{R}^{2D})$ is the unitary transfer operator given by

$$\mathcal{T}_{(x_0, \xi_0)}u(x, \xi) := e^{\frac{i}{2\hbar}(\xi_0 \cdot x - x_0 \cdot \xi)} u(x - x_0, \xi - \xi_0) \quad (4.21)$$

Consequently the lift of $T_{(x_0, \xi_0)}$ is

$$T_{(x_0, \xi_0)}^{\text{lift}} := \mathcal{B}_h \circ T_{(x_0, \xi_0)} \circ \mathcal{B}_h^* = \mathcal{P}_h \circ \mathcal{T}_{(x_0, \xi_0)} \circ \mathcal{P}_h. \quad (4.22)$$

Proof. The Schwartz kernel of $\mathcal{B}_\hbar^* \circ \mathcal{T}_{(x_0, \xi_0)} \circ \mathcal{B}_\hbar$ is

$$\begin{aligned}
K(y, y') &= \int_{\mathbb{R}^{2D}} e^{\frac{i}{2\hbar}(\xi_0 \cdot x - x_0 \cdot \xi)} \phi_{x, \xi}(y) \cdot \overline{\phi_{x-x_0, \xi-\xi_0}(y')} \frac{dx d\xi}{(2\pi\hbar)^D} \\
&= a_D^2 \cdot \int \frac{dx d\xi}{(2\pi\hbar)^D} e^{\frac{i}{2\hbar}(\xi_0 \cdot x - x_0 \cdot \xi)} e^{\frac{i}{\hbar}(\xi \cdot (y - \frac{x}{2}) - (\xi - \xi_0) \cdot (y' - \frac{x-x_0}{2}))} e^{-\frac{1}{2\hbar}(|y' - (x-x_0)|^2 + |y-x|^2)} \\
&= (\pi\hbar)^{-D/2} \delta(y' - y + x_0) e^{\frac{i}{\hbar}\xi_0 \cdot (y + \frac{x_0}{2})} \int e^{-\frac{1}{\hbar}|y-x|^2} dx \\
&= \delta(y' - y + x_0) \cdot e^{\frac{i}{\hbar}\xi_0 \cdot (y + \frac{x_0}{2})}.
\end{aligned}$$

This is the kernel of the operator $T_{(x_0, \xi_0)}$. □

Corollary 4.11. *The lift of the operator $T_{(x_0, \xi_0)}$ is expressed as*

$$T_{(x_0, \xi_0)}^{\text{lift}} := \mathcal{B}_\hbar \circ T_{(x_0, \xi_0)} \circ \mathcal{B}_\hbar^* = \mathcal{T}_{(x_0, \xi_0)} \circ \mathcal{P}_\hbar = \mathcal{P}_\hbar \circ \mathcal{T}_{(x_0, \xi_0)}. \quad (4.23)$$

Proof. We can check the equality $\mathcal{T}_{(x_0, \xi_0)} \circ \mathcal{P}_\hbar = \mathcal{P}_\hbar \circ \mathcal{T}_{(x_0, \xi_0)}$ on the left by showing that the Schwartz kernels of $\mathcal{T}_{(x_0, \xi_0)} \circ \mathcal{P}_\hbar$ and $\mathcal{P}_\hbar \circ \mathcal{T}_{(x_0, \xi_0)}$ are equal. This an easy computation using the expressions (4.7) and (4.21). The rest of the claim follows from Lemma 4.10. □

4.3 The weighted L^2 spaces : $L^2(\mathbb{R}^{2D}, (W_\hbar^r)^2)$

For each $t > 0$, we define the cones in \mathbb{R}^{2D} :

$$\mathbf{C}_+(t) = \{(x, \xi) \in \mathbb{R}^{2D} \mid |\xi| \leq t \cdot |x|\}, \quad \mathbf{C}_-(t) = \{(x, \xi) \in \mathbb{R}^{2D} \mid |x| \leq t \cdot |\xi|\}.$$

Let $r > 0$. Take and fix a C^∞ function $m : \mathbb{P}(\mathbb{R}^{2D}) \rightarrow [-r, r]$, called *order function*, on the projective space $\mathbb{P}(\mathbb{R}^{2D})$ so that

$$m([x, \xi]) = \begin{cases} -r, & \text{if } (x, \xi) \in \mathbf{C}_+(1/2); \\ +r, & \text{if } (x, \xi) \in \mathbf{C}_-(1/2). \end{cases} \quad (4.24)$$

We then define the *escape function* (or *the weight function*) by

$$W^r : \mathbb{R}^{2D} \rightarrow \mathbb{R}_+, \quad W^r(x, \xi) = \langle |(x, \xi)| \rangle^{m([x, \xi])}$$

where $\langle s \rangle := (1 + s^2)^{1/2}$ for $s \in \mathbb{R}$ and $|(x, \xi)|^2 := |x|^2 + |\xi|^2$. From this definition we have

$$W^r(x, \xi) \sim \langle |(x, \xi)| \rangle^{-r} \quad \text{if } |x| \geq 2|\xi| \text{ and } |(x, \xi)| \gg 1$$

and

$$W^r(x, \xi) \sim \langle |(x, \xi)| \rangle^r \quad \text{if } |x| \leq |\xi|/2 \text{ and } |(x, \xi)| \gg 1.$$

For convenience in a later argument, we also take and fix C^∞ functions

$$m^+, m^- : \mathbb{P}(\mathbb{R}^{2D}) \rightarrow [-r, r]$$

such that

$$m^+([(x, \xi)]) = \begin{cases} -r, & \text{if } (x, \xi) \in \mathbf{C}_+(1/9); \\ +r, & \text{if } (x, \xi) \in \mathbf{C}_-(3), \end{cases}$$

and

$$m^-([(x, \xi)]) = \begin{cases} -r, & \text{if } (x, \xi) \in \mathbf{C}_+(3); \\ +r, & \text{if } (x, \xi) \in \mathbf{C}_-(1/9), \end{cases}$$

and define the functions $W^{r,\pm} : \mathbb{R}^{2D} \rightarrow \mathbb{R}_+$ by

$$W^{r,\pm}(x, \xi) := \langle |(x, \xi)| \rangle^{m^\pm([(x, \xi)])}. \quad (4.25)$$

1215 Obviously we have

$$W^{r,-}(x, \xi) \leq W^r(x, \xi) \leq W^{r,+}(x, \xi). \quad (4.26)$$

These functions, W^r and $W^{r,\pm}$, satisfy the following regularity estimate that we will make use of later on: For any $\epsilon > 0$ and multi-index α , there exists a constant $C_{\alpha,\epsilon} > 0$ such that

$$|\partial_{x,\xi}^\alpha W^r(x, \xi)| \leq C_{\alpha,\epsilon} \langle |(x, \xi)| \rangle^{-(1-\epsilon)|\alpha|} \cdot W^r(x, \xi) \quad \text{for all } (x, \xi) \in \mathbb{R}^{2D} \quad (4.27)$$

and the same inequalities for $W^{r,\pm}(\cdot)$ hold.

Definition 4.12. For $\hbar > 0$, let

$$W_h^r : \mathbb{R}^{2D} \rightarrow \mathbb{R}_+, \quad W_h^r(x, \xi) := W^r(\hbar^{-1/2}x, \hbar^{-1/2}\xi) = (S_h W^r)(x, \xi) \quad (4.28)$$

1220 where S_h is the operator defined in (4.10). We consider the weighted L^2 space defined as

$$L^2(\mathbb{R}^{2D}, (W_h^r)^2) = \{v \in L_{\text{loc}}^2(\mathbb{R}^{2D}) \mid \|W_h^r \cdot v\|_{L^2(\mathbb{R}^{2D})} < \infty\}. \quad (4.29)$$

Likewise, we define the functions $W_h^{r,\pm}$ and the weighted L^2 spaces $L^2(\mathbb{R}^{2D}, (W_h^{r,\pm})^2)$ in the parallel manner, replacing W^r by $W^{r,\pm}$.

Note that the function W^r (and $W^{r,\pm}$) satisfies the condition

$$W^r(x, \xi) \leq C \cdot W^r(y, \eta) \cdot \langle |(x, \xi) - (y, \eta)| \rangle^{2r} \quad \text{for any } x, y \in \mathbb{R}^d \quad (4.30)$$

for some constant $C > 0$. Consequently the function W_h^r (and $W_h^{r,\pm}$) satisfies

$$W_h^r(x, \xi) \leq C \cdot W_h^r(y, \eta) \cdot \langle \hbar^{-1/2} |(x, \xi) - (y, \eta)| \rangle^{2r} \quad \text{for any } x, y \in \mathbb{R}^d. \quad (4.31)$$

1225 The next Lemma characterizes a class of bounded integral operators in $L^2(\mathbb{R}^{2D}, (W_h^r)^2)$ in terms of its kernel.

Lemma 4.13. *If $R : \mathcal{S}(\mathbb{R}^{2D}) \rightarrow \mathcal{S}(\mathbb{R}^{2D})$ is an integral operator of the form*

$$Ru(x, \xi) = \int K_R(x, \xi; x', \xi') u(x', \xi') dx' d\xi'$$

and if the kernel $K_R(\cdot)$ is a continuous function satisfying

$$|K_R(x, \xi; x', \xi')| \leq \langle \hbar^{-1/2} |(x, \xi) - (x', \xi')| \rangle^{-\nu}$$

for some $\nu > 2r + 2D$, then it extends to a bounded operator on $L^2(\mathbb{R}^{2D}, (W_\hbar^r)^2)$ and

$$\|R : L^2(\mathbb{R}^{2D}, (W_\hbar^r)^2) \rightarrow L^2(\mathbb{R}^{2D}, (W_\hbar^r)^2)\| \leq C_\nu$$

1230 *where C_ν is a constant which depends only on ν .*

Proof. From (4.31), we have

$$\left| \frac{W_\hbar^r(x, \xi)}{W_\hbar^r(x', \xi')} \cdot K_R(x, \xi; x', \xi') \right| \leq C \langle \hbar^{-1/2} |(x, \xi) - (x', \xi')| \rangle^{2r-\nu}.$$

Hence, by Schur Lemma¹³ [36, p.50] (or by Young inequality for convolution), the operator norm of $u \mapsto W_\hbar^r \cdot R((W_\hbar^r)^{-1} \cdot u)$ with respect to the L^2 norm is bounded by a constant C_ν . This implies the claim of the lemma. \square

1235 From expression (4.7), the Bargmann projector \mathcal{P}_\hbar satisfies the assumption of the last lemma. Thus we have

Corollary 4.14. *The Bargmann projector \mathcal{P}_\hbar is a bounded operator on $L^2(\mathbb{R}^{2D}, (W_\hbar^r)^2)$.*

4.4 Spectrum of transfer operator for linear expanding map

Below we consider the action of a linear expanding map.

1240 **Lemma 4.15.** *If $A : \mathbb{R}^D \rightarrow \mathbb{R}^D$ is an expanding linear map i.e. satisfying*

$$\|A^{-1}\| \leq \frac{1}{\lambda} \quad \text{for some } \lambda > 1, \quad (4.33)$$

then the lift L_A^{lift} of L_A , defined in (4.17), extends to a bounded operator

$$L_A^{\text{lift}} : L^2(\mathbb{R}^{2D}, (W_\hbar^r)^2) \rightarrow L^2(\mathbb{R}^{2D}, (W_\hbar^r)^2). \quad (4.34)$$

Further, if $\lambda > 1$ is sufficiently large (say $\lambda > 9$), L_A^{lift} extends to a bounded operator

$$L_A^{\text{lift}} : L^2(\mathbb{R}^{2D}, (W_\hbar^{r,-})^2) \rightarrow L^2(\mathbb{R}^{2D}, (W_\hbar^{r,+})^2). \quad (4.35)$$

¹³For an integral bounded operator $A : L^2(\mathbb{R}^n) \rightarrow L^2(\mathbb{R}^n)$ with Schwartz kernel $K_A \in C^0(\mathbb{R}^n \times \mathbb{R}^n)$, i.e. $(Au)(x) = \int K_A(x, y) u(y) dy$ we have

$$\|A\|_{L^2} \leq \max_y \left| \int K_A(x, y) dx \right|^{1/2} \cdot \max_x \left| \int K_A(x, y) dy \right|^{1/2} \quad (4.32)$$

Proof. From (4.18) in Lemma 4.8 and Corollary 4.14, we have only to check boundedness of $L_{A \oplus A^{-1}}$ as an operator on $L^2(\mathbb{R}^{2D}, (W^r)^2)$ (resp. from $L^2(\mathbb{R}^{2D}, (W^{r,-})^2)$ to $L^2(\mathbb{R}^{2D}, (W^{r,+})^2)$). But this is clear from the definitions of W^r and $W^{r,\pm}$. \square

To look into more detailed structure of the operator L_A and L_A^{lift} , we introduce some definitions. For $k \in \mathbb{N}$, let $\text{Polynom}^{(k)}$ be the space of homogeneous polynomial on \mathbb{R}^D of order k . Then we consider the operator¹⁴

$$T^{(k)} : C^\infty(\mathbb{R}^D) \rightarrow \text{Polynom}^{(k)}, \quad (T^{(k)}u)(x) = \sum_{\alpha \in \mathbb{N}^D, |\alpha|=k} \frac{\partial^\alpha u(0)}{\alpha!} \cdot x^\alpha. \quad (4.36)$$

This is a projector which extracts the terms of order k in the Taylor expansion. Clearly the operator $T^{(k)}$ is of finite rank

$$\text{rank}(T^{(k)}) = \binom{D+k-1}{D-1} = \frac{(D+k-1)!}{(D-1)!k!}$$

and satisfies the following relations

$$T^{(k)} \circ T^{(k')} = \begin{cases} T^{(k)}, & \text{if } k = k'; \\ 0 & \text{otherwise,} \end{cases} \quad (4.37)$$

and

$$T^{(k)} \circ L_A = L_A \circ T^{(k)}. \quad (4.38)$$

As in (4.11) we define the lift of the operator $T^{(k)}$ by

$$\mathcal{T}_h^{(k)} := \mathcal{B}_h \circ T^{(k)} \circ \mathcal{B}_h^*. \quad (4.39)$$

Lemma 4.16. *Let $n \in \mathbb{N}$ and $r > 0$ such that*

$$r > n + 2D. \quad (4.40)$$

Then for $0 \leq k \leq n$ the operator $\mathcal{T}_h^{(k)}$ extends naturally to bounded operators

$$\mathcal{T}_h^{(k)} : L^2(\mathbb{R}^{2D}, (W_h^{r,-})^2) \rightarrow L^2(\mathbb{R}^{2D}, (W_h^{r,+})^2) \quad (4.41)$$

and, in particular, from (4.26),

$$\mathcal{T}_h^{(k)} : L^2(\mathbb{R}^{2D}, (W_h^r)^2) \rightarrow L^2(\mathbb{R}^{2D}, (W_h^r)^2). \quad (4.42)$$

Further if we write the operator $\mathcal{T}_h^{(k)}$ as an integral operator

$$(\mathcal{T}_h^{(k)}u)(x, \xi) = \int K(x, \xi; x', \xi') u(x', \xi') dx' d\xi',$$

¹⁴for a multi index $\alpha \in \mathbb{N}^D$, $\alpha = (\alpha_1, \dots, \alpha_D)$, we write $|\alpha| = \alpha_1 + \dots + \alpha_D$.

the kernel $K(\cdot)$ satisfies the estimate

$$\left| \frac{W_h^{r,+}(x, \xi)}{W_h^{r,-}(x', \xi')} \cdot K(x, \xi; x', \xi') \right| \leq C \langle \hbar^{-1/2} |x, \xi| \rangle^{k-r} \cdot \langle \hbar^{-1/2} |x', \xi'| \rangle^{k-r} \quad (4.43)$$

$$\leq C' \langle \hbar^{-1/2} |(x, \xi) - (x', \xi')| \rangle^{k-r} \quad (4.44)$$

for some constants $C, C' > 0$ that do not depend on $\hbar > 0$.

Proof. For each multi-index $\alpha \in \mathbb{N}^D$ with $|\alpha| = k$, we set

$$T^{(\alpha)} : \mathcal{S}(\mathbb{R}^D) \rightarrow \mathcal{S}(\mathbb{R}^D)', \quad (T^{(\alpha)}u)(x) := \frac{\partial^\alpha u(0)}{\alpha!} \cdot x^\alpha \quad (4.45)$$

and

$$\mathcal{T}_h^{(\alpha)} = \mathcal{B}_h \circ T^{(\alpha)} \circ \mathcal{B}_h^* : \mathcal{S}(\mathbb{R}^{2D}) \rightarrow \mathcal{S}(\mathbb{R}^{2D})'.$$

1260 Since $\mathcal{T}_h^{(k)} = \sum_{\alpha: |\alpha|=k} \mathcal{T}_h^{(\alpha)}$, the claims of the lemma follows if one proves the corresponding claim for $\mathcal{T}_h^{(\alpha)}$. From (4.3) and (4.4) the kernel of the operator $\mathcal{T}_h^{(\alpha)}$ is written as

$$K(x, \xi; x', \xi') = \frac{1}{\alpha!} \cdot k_+(x, \xi) \cdot k_-(x', \xi')$$

with

$$k_+(x, \xi) := \hbar^{-D/4} \int \overline{\phi_{x, \xi}(y)} \cdot (\hbar^{-1/2} y)^\alpha dy, \quad k_-(x', \xi') := \hbar^{D/4} \hbar^{k/2} \cdot \partial^\alpha \phi_{x', \xi'}(0).$$

Applying integration by parts to the integral of $k_+(\cdot)$, we see that

$$|k_+(x, \xi)| \leq C_\nu \cdot \langle \hbar^{-1/2} |x| \rangle^k \cdot \langle \hbar^{-1/2} |\xi| \rangle^{-\nu}$$

1265 for arbitrarily large integer ν , where C_ν is a constant depending only on ν . Also a straight-forward computation gives the similar estimate for $k_-(\cdot)$:

$$|k_-(x', \xi')| \leq C_\nu \cdot \langle \hbar^{-1/2} |\xi'| \rangle^k \cdot \langle \hbar^{-1/2} |x'| \rangle^{-\nu}.$$

These estimates for sufficiently large ν imply that

$$W_h^r(x, \xi) \cdot |k_+(x, \xi)| \leq C \langle \hbar^{-1/2} |(x, \xi)| \rangle^{k-r}$$

and

$$\frac{1}{W_h^r(x', \xi')} \cdot |k_-(x', \xi')| \leq C \langle \hbar^{-1/2} |(x', \xi')| \rangle^{k-r}$$

1270 for some constant $C > 0$ independent of $\hbar > 0$. Thus we have obtained (4.43). Since $r - k \geq r - n > 2D$ from the assumption (4.40) on the choice of r , boundedness of the operators follows from Schur Lemma (4.32). \square

The following is a direct consequence of the relation (4.37) and (4.38).

Corollary 4.17. For $0 \leq k, k' \leq n$, we have

$$\mathcal{T}_h^{(k)} \circ \mathcal{T}_h^{(k')} = \begin{cases} \mathcal{T}_h^{(k)}, & \text{if } k = k'; \\ 0, & \text{otherwise,} \end{cases}$$

and

$$L_A^{\text{lift}} \circ \mathcal{T}_h^{(k)} = \mathcal{T}_h^{(k)} \circ L_A^{\text{lift}} = \mathcal{B}_h \circ L_A \circ T^{(k)} \circ \mathcal{B}_h^*.$$

Let us set

$$\tilde{\mathcal{T}}_h = \text{Id} - \sum_{k=0}^n \mathcal{T}_h^{(k)} : L^2(\mathbb{R}^{2D}, (W_h^r)^2) \rightarrow L^2(\mathbb{R}^{2D}, (W_h^r)^2). \quad (4.46)$$

1275 Then the set of operators $\mathcal{T}_h^{(k)}$, $0 \leq k \leq n$, and $\tilde{\mathcal{T}}_h$ form a complete set of mutually commuting projection operators on $L^2(\mathbb{R}^{2D}, (W_h^r)^2)$ such that

$$\text{rank } \mathcal{T}_h^{(k)} = \dim \text{Polynom}^{(k)} = \binom{D+k-1}{D-1} = \frac{(D+k-1)!}{(D-1)!k!}, \quad \text{rank } \tilde{\mathcal{T}}_h = \infty.$$

Let

$$H_k := \text{Im } \mathcal{T}_h^{(k)} \quad \text{and} \quad \tilde{H} = \text{Im } \tilde{\mathcal{T}}_h.$$

Then the Hilbert space $L^2(\mathbb{R}^{2D}, (W_h^r)^2)$ is decomposed as

$$L^2(\mathbb{R}^{2D}, (W_h^r)^2) = H_0 \oplus H_1 \oplus H_2 \oplus \cdots \oplus H_n \oplus \tilde{H} \quad (4.47)$$

1280 Since the operator L_A^{lift} commutes with the projections $\mathcal{T}_h^{(k)}$ and $\tilde{\mathcal{T}}_h$, it preserves this decomposition and therefore the operator L_A^{lift} acting on $L^2(\mathbb{R}^{2D}, (W_h^r)^2)$ is identified with the direct sum of the operators

$$L_A^{\text{lift}} : H_k \rightarrow H_k \quad \text{for } 0 \leq k \leq n, \quad \text{and} \quad L_A^{\text{lift}} : \tilde{H} \rightarrow \tilde{H}.$$

The former is identified with the action of L_A on $\text{Polynom}^{(k)}$, because the diagram

$$\begin{array}{ccc} H_k & \xrightarrow{L_A^{\text{lift}}} & H_k \\ \mathcal{B}_h \uparrow & & \mathcal{B}_h \uparrow \\ \text{Polynom}^{(k)} & \xrightarrow{L_A} & \text{Polynom}^{(k)} \end{array} \quad (4.48)$$

commutes and the operator $\mathcal{B}_h : \text{Polynom}^{(k)} \rightarrow H_k$ in the vertical direction is an isomorphism between finite dimensional linear spaces.

1285 To state the next proposition which is the main result of this section, we introduce the following definition:

Definition 4.18. The Hilbert space $H_h^r(\mathbb{R}^D) \subset \mathcal{S}'(\mathbb{R}^D)$ of distributions is the completion of $\mathcal{S}(\mathbb{R}^D)$ with respect to the norm induced by the scalar product

$$(u, v)_{H_h^r(\mathbb{R}^D)} := (\mathcal{B}_h u, \mathcal{B}_h v)_{L^2(\mathbb{R}^{2D}, (W_h^r)^2)} = \int (W_h^r)^2 \cdot \overline{\mathcal{B}_h u} \cdot \mathcal{B}_h v \frac{dx d\xi}{(2\pi\hbar)^D} \quad \text{for } u, v \in \mathcal{S}(\mathbb{R}^D).$$

The induced norm on $H_h^r(\mathbb{R}^D)$ will be written as $\|u\|_{H_h^r(\mathbb{R}^D)} := \|\mathcal{B}_h u\|_{L^2(\mathbb{R}^{2D}, (W_h^r)^2)}$.

Proposition 4.19. *”Discrete spectrum of the linear expanding map”. Let $A : \mathbb{R}^D \rightarrow \mathbb{R}^D$ be a linear expanding map satisfying $\|A^{-1}\| \leq 1/\lambda$ for some $\lambda > 1$. Let L_A be the unitary transfer operator defined in (4.14): $L_A u = u \circ A^{-1}$. Let $n > 0$ and $r > n + 2D$. Then the Hilbert space $H_h^r(\mathbb{R}^D)$ of definition 4.18 is decomposed into subspaces of homogeneous polynomial of degree k for $0 \leq k \leq n$ and the remainder:*

$$H_h^r(\mathbb{R}^D) = \left(\bigoplus_{k=0}^n \text{Polynom}^{(k)} \right) \oplus \tilde{\mathcal{H}}_h$$

where $\tilde{\mathcal{H}}_h := \tilde{T}(H_h^r(\mathbb{R}^D))$ with setting

$$\tilde{T} := \text{Id} - \sum_{k=0}^n T^{(k)}. \quad (4.49)$$

(The operators $T^{(k)}$ are defined in (4.36).) This decomposition is preserved by L_A . There exists a constant $C_0 > 0$ independent of A and \hbar such that

(1) For $0 \leq k \leq n$ and $0 \neq u \in \text{Polynom}^{(k)}$, we have

$$C_0^{-1} \|A\|_{\max}^{-k} \leq \frac{\|L_A u\|_{H_h^r(\mathbb{R}^D)}}{\|u\|_{H_h^r(\mathbb{R}^D)}} \leq C_0 \|A\|_{\min}^{-k} \quad (4.50)$$

(Recall (1.23) for the definition of $\|\cdot\|_{\max}$ and $\|\cdot\|_{\min}$.)

(2) The operator norm of the restriction of L_A to $\tilde{\mathcal{H}}_h$ is bounded by

$$C_0 \max\{\|A\|_{\min}^{-(n+1)}, \|A\|_{\min}^{-r} \cdot |\det A|\}. \quad (4.51)$$

The following equivalent statements holds for the lifted operator:

$$L_A^{\text{lift}} : L^2(\mathbb{R}^{2D}, (W_h^r)^2) \rightarrow L^2(\mathbb{R}^{2D}, (W_h^r)^2).$$

The operator L_A^{lift} preserves the decomposition of $L^2(\mathbb{R}^{2D}, (W_h^r)^2)$ in (4.47) and

(1) For $0 \leq k \leq n$ and for $0 \neq u \in H_k$, we have

$$C_0^{-1} \|A\|_{\max}^{-k} \leq \frac{\|L_A^{\text{lift}} u\|_{L^2(\mathbb{R}^{2D}, (W_h^r)^2)}}{\|u\|_{L^2(\mathbb{R}^{2D}, (W_h^r)^2)}} \leq C_0 \|A\|_{\max}^{-k}. \quad (4.52)$$

(2) The operator norm of the restriction of L_A^{lift} to \tilde{H} is bounded by (4.51).

Remark 4.20. Proposition 4.19 implies that the spectrum of the transfer operator L_A in

the Hilbert space $H_{\hbar}^r(\mathbb{R}^D)$ is discrete outside the radius given by (4.51). The eigenvalues outside this radius are given by the action of L_A in the finite dimensional space $\text{Polynom}^{(k)}$.
 1295 These eigenvalues can be computed explicitly from the Jordan block decomposition of A . In particular if $A = \text{Diag}(a_1, \dots, a_D)$ is diagonal then the monomials $x^\alpha = x_1^{\alpha_1} \dots x_D^{\alpha_D}$ are obviously eigenvectors of L_A with respective eigenvalues $\prod_j a_j^{-\alpha_j}$.

Proof. For the proof of (4.50) and (4.52), we use the fact that the space $\text{Polynom}^{(k)}$ is identical to the space $\text{Sym}^k(\mathbb{R}^D)$ of totally symmetric tensors of rank k . For the linear
 1300 operator $(A^{-1})^{\otimes k}$ acting on $(\mathbb{R}^D)^{\otimes k}$, we have a commutative diagram:

$$\begin{array}{ccc} (\mathbb{R}^D)^{\otimes k} & \xrightarrow{(A^{-1})^{\otimes k}} & (\mathbb{R}^D)^{\otimes k} \\ \text{Sym} \downarrow & & \text{Sym} \downarrow \\ \text{Sym}^k(\mathbb{R}^D) & \xrightarrow{L_A} & \text{Sym}^k(\mathbb{R}^D) \end{array}$$

where Sym denotes the symmetrization operation. For every $0 \neq \tilde{u} \in (\mathbb{R}^D)^{\otimes k}$ we have

$$\|A\|_{\max}^{-k} \leq \frac{\|(A^{-1})^{\otimes k} \tilde{u}\|}{\|\tilde{u}\|} \leq \|A\|_{\min}^{-k}.$$

Since the spaces are finite dimensional (and hence all norms are equivalent), we deduce (4.50) for some constant $C_0 > 0$ independent of A , and also independent on \hbar because of the scaling invariance (4.28). The proof of the Claim (2) is postponed to Subsection 4.5,
 1305 as it requires more detailed argument. \square

4.5 Proof of Claim (2) in Proposition 4.19

We prove Claim (2) on the lifted operator L_A^{lift} in the latter part of the statement, which is equivalent to Claim (2) in the former part. In the proof below, we may and do assume $\hbar = 1$, because the Bargmann transforms for different parameter \hbar are related by the scaling (4.10), as we noted in Subsection 4.1. Accordingly we will drop the subscript \hbar
 1310 from the notation. Let $\chi : \mathbb{R}^D \rightarrow [0, 1]$ be a smooth function such that

$$\chi(x) = \begin{cases} 1 & \text{if } |x| \leq 1 \\ 0 & \text{if } |x| \geq 2. \end{cases} \quad (4.53)$$

Below we use C_0 as a generic symbol for the constants which do not depend on A (but may depend on r, n and D). Letting λ smaller if necessary, we suppose

$$\lambda = \|A^{-1}\|^{-1} > 1$$

for simplicity. We write $\mathcal{M}(\varphi)$ for the multiplication operator by φ .

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To prove the claim, it is enough to show

$$\|L_A^{\text{lift}} \circ \tilde{\mathcal{T}}\|_{L^2(\mathbb{R}^{2D}, (W^r)^2)} \leq C_0 \cdot \max\{\lambda^{-n-1}, \lambda^{-r} |\det A|\}$$

where $\tilde{\mathcal{T}}$ is the operator defined in (4.46) with $\hbar = 1$ and $\|\cdot\|_{L^2(\mathbb{R}^{2D}, (W^r)^2)}$ denotes the operator norm on $L^2(\mathbb{R}^{2D}, (W^r)^2)$.

Let us consider the operators

$$X = \mathcal{B} \circ \mathcal{M}(\chi) \circ \mathcal{B}^* : L^2(\mathbb{R}^{2D}, W^r) \rightarrow L^2(\mathbb{R}^{2D}, W^r)$$

and

$$\Xi : L^2(\mathbb{R}^{2D}, (W^r)^2) \rightarrow L^2(\mathbb{R}^{2D}, (W^r)^2), \quad (\Xi v)(x, \xi) = \chi \left(\frac{|\xi|}{\lambda} \right) \cdot v(x, \xi).$$

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The next lemma is the main ingredient of the proof.

Lemma 4.21. $\|L_A^{\text{lift}} \circ X \circ \tilde{\mathcal{T}} \circ \Xi\|_{L^2(\mathbb{R}^{2D}, (W^r)^2)} \leq C_0 \cdot \lambda^{-(n+1)}$.

Proof. Let $\mathbf{1}_{\mathbf{C}_-(2)}$ be the indicator function of the cone

$$\mathbf{C}_-(2) := \{(x, \xi) \mid |x| \leq 2|\xi|\} = \overline{\mathbb{R}^{2D} \setminus \mathbf{C}_+(1/2)}$$

and set

$$W_+^r(x, \xi) := \mathbf{1}_{\mathbf{C}_-(2)}(x, \xi) \cdot \langle |\xi| \rangle^r, \quad W_-^r(x, \xi) := \langle |x| \rangle^{-r}.$$

Then the weight function $W^r(x, \xi)$ satisfies

$$W^r(x, \xi) \leq C_0 \cdot W_+^r(x, \xi) + C_0 \cdot W_-^r(x, \xi)$$

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for a constant $C_0 > 0$. Hence, to prove the lemma, it is enough to show for $\sigma = \pm$ the claim

$$\|W_\sigma^r \cdot \mathcal{B} \circ L_A \circ \mathcal{M}(\chi) \circ \tilde{\mathcal{T}} \circ \mathcal{B}^* \circ \Xi u\|_{L^2} \leq C_0 \cdot \lambda^{-(n+1)} \|u\| \quad \text{for any } u \in L^2(\mathbb{R}^{2D}, (W^r)^2) \quad (4.54)$$

where $\tilde{\mathcal{T}}$ is the operator defined in (4.49). Before proceeding to the proof of (4.54), we prepare a few estimates. Take $u \in L^2(\mathbb{R}^{2D}, (W^r)^2)$ arbitrarily and set

$$v(y) := (\mathcal{B}^* \circ \Xi u)(y) = \int \phi_{x, \xi}(y) \chi \left(\frac{|\xi|}{\lambda} \right) u(x, \xi) dx d\xi.$$

Then, for any multi-index $\alpha \in \mathbb{N}^D$ and arbitrarily large ν , we have

$$|\partial_y^\alpha v(y)| \leq C_{\alpha, \nu} \int_{|\xi| \leq 2\lambda} \langle |x - y| \rangle^{-\nu} \cdot \langle |\xi| \rangle^{|\alpha|} \cdot |u(x, \xi)| dx d\xi \quad \text{for any } y \in \mathbb{R}^D. \quad (4.55)$$

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Note that we have

$$\langle |x - y| \rangle^{-r} \cdot \langle |\xi| \rangle^r \leq C_0 \cdot W^r(x, \xi) \quad \text{for any } x, y, \xi \in \mathbb{R}^D \text{ with } |y| \leq 2.$$

Hence

$$\langle |x - y| \rangle^{-\nu} \cdot \langle |\xi| \rangle^{|\alpha|} \leq C_0 (\langle |x - y| \rangle^{-\nu+r} \cdot \langle |\xi| \rangle^{-D/2+1}) \cdot \langle |\xi| \rangle^{|\alpha|+D/2+1-r} \cdot W^r(x, \xi).$$

Putting this estimate in (4.55) with $\nu \geq D/2 + 1 + r$, we obtain, by Cauchy-Schwarz inequality,

$$|\partial_y^\alpha v(y)| \leq C_\alpha \lambda^{\max\{|\alpha|+D/2+1-r, 0\}} \|u\|_{L^2(\mathbb{R}^{2D}, (W^r)^2)} \quad \text{for any } y \in \mathbb{R}^D \text{ with } |y| \leq 2. \quad (4.56)$$

1335 Notice that, if $|\alpha| \leq n + 1$, we have $|\alpha| + D/2 + 1 - r \leq 0$ from the assumption (4.40) and hence the last inequality implies

$$|\partial_y^\alpha v(y)| \leq C_\alpha \|u\|_{L^2(\mathbb{R}^{2D}, (W^r)^2)} \quad \text{for any } y \in \mathbb{R}^D \text{ with } |y| \leq 2. \quad (4.57)$$

Next we consider the function

$$w := \mathcal{M}(\chi) \circ \tilde{T} v = \mathcal{M}(\chi) \circ \tilde{T} \circ \mathcal{B}^* \circ \Xi u.$$

Note that the support of w is contained in that of χ . It follows from (4.56) that, for each multi-index α ,

$$|\partial^\alpha w(y)| \leq C_\alpha \lambda^{\max\{|\alpha|+D/2+1-r, 0\}} \|u\|_{L^2(\mathbb{R}^{2D}, (W^r)^2)} \quad \text{for all } y \in \mathbb{R}^D. \quad (4.58)$$

Further, if $|\alpha| \leq n + 1$, it follows from (4.57) and the definition of \tilde{T} that

$$|\partial_y^\alpha w(y)| \leq C_\alpha |y|^{n+1-|\alpha|} \cdot \max_{|y| \leq 2} |\partial_y^{n+1} v| \leq C_\alpha |y|^{n+1-|\alpha|} \|u\|_{L^2(\mathbb{R}^{2D}, (W^r)^2)} \quad \text{for all } y \in \mathbb{R}^D. \quad (4.59)$$

1340 Now we prove the claim (4.54) in the case $\sigma = +$. Let u, v, w be as above. We are going to estimate the quantity

$$\xi^\alpha \cdot (\mathcal{B} \circ L_A \circ \mathcal{M}(\chi) \circ \tilde{T} \circ \mathcal{B}^* \circ \Xi u)(x, \xi) = \xi^\alpha \int \overline{\phi_{x, \xi}(y)} \cdot w(A^{-1}y) dy.$$

By integration by parts, we see that this is bounded in absolute value by

$$C_{\alpha, \nu} \sum_{\alpha' \leq \alpha} \int \langle |x - y| \rangle^{-\nu} \cdot \lambda^{-|\alpha'|} \cdot |\partial^{\alpha'} w(A^{-1}y)| dy$$

for each $\nu > 0$, where $C_{\alpha, \nu}$ is a constant depending only on α and ν . If ν is sufficiently large, we have from (4.58), (4.59) and then from (4.40) that

$$\begin{aligned} & \sum_{\alpha' \leq \alpha} \int \langle |x - y| \rangle^{-\nu} \cdot \lambda^{-|\alpha'|} \cdot |\partial^{\alpha'} w(A^{-1}y)| dy \\ & \leq C_\alpha \left(\sum_{\alpha': |\alpha'| \leq n+1} \lambda^{-|\alpha'|} \left(\frac{\langle |x| \rangle}{\lambda} \right)^{n+1-|\alpha'|} + \sum_{\alpha': n+2 \leq |\alpha'| \leq |\alpha|} \lambda^{-|\alpha'|} \lambda^{\max\{|\alpha'|+D/2+1-r, 0\}} \right) \|u\|_{L^2(\mathbb{R}^{2D}, (W^r)^2)} \\ & \leq C_\alpha \cdot \lambda^{-(n+1)} \langle |x| \rangle^{n+1} \cdot \|u\|_{L^2(\mathbb{R}^{2D}, (W^r)^2)}. \end{aligned}$$

Therefore we obtain

$$\langle |\xi| \rangle^\nu \cdot |\mathcal{B} \circ L_A \circ \mathcal{M}(\chi) \circ \tilde{T} \circ \mathcal{B}^* \circ \Xi u(x, \xi)| \leq C_\nu \lambda^{-(n+1)} \|u\|_{L^2(\mathbb{R}^{2D}, (W^r)^2)} \cdot \langle |x| \rangle^{n+1}$$

for arbitrarily large ν . For (x, ξ) on $\text{supp } W_+^r = \mathbf{C}_-(2)$, we have $\langle |x| \rangle \leq 2\langle |\xi| \rangle$ and hence

$$W_+^r(x, \xi) \leq \langle |\xi| \rangle^r \leq C_0 \langle |\xi| \rangle^{r+D/2+1+(n+1)} \cdot \langle |x| \rangle^{-D/2-1-(n+1)}.$$

Using this in the last inequality, we get

$$\begin{aligned} W_+^r(x, \xi) |(\mathcal{B} \circ L_A \circ \mathcal{M}(\chi) \circ (\text{Id} - \mathbf{T}_n) \circ \mathcal{B}^* \circ \Xi u)(x, \xi)| \\ \leq C_\nu \lambda^{-(n+1)} \|u\|_{L^2(\mathbb{R}^{2D}, (W^r)^2)} \langle |x| \rangle^{-D/2-1} \langle |\xi| \rangle^{-\nu+r+D/2+1+(n+1)}. \end{aligned}$$

This estimate for sufficiently large ν implies the claim (4.54) in the case $\sigma = +$, by Cauchy-Schwarz inequality. 1345

We prove the claim (4.54) for $\sigma = -$. The proof is easier than the previous case actually. Note that we have

$$\|W_-^r \cdot \mathcal{B}\varphi\|_{L^2(\mathbb{R}^{2D})} \leq C_0 \cdot \|\langle \cdot \rangle^{-r} \cdot \varphi(\cdot)\|_{L^2(\mathbb{R}^D)}.$$

Hence, from (4.59) with $\alpha = \emptyset$, we get

$$\begin{aligned} \|W_-^r \cdot \mathcal{B} \circ L_A \circ \mathcal{M}(\chi) \circ \tilde{T} \circ \mathcal{B}^* \circ \Xi u\|_{L^2(\mathbb{R}^{2D})} &= \|W_-^r \cdot \mathcal{B} \circ L_A \circ w\|_{L^2(\mathbb{R}^{2D})} \\ &\leq C_0 \cdot \left| \int \langle x \rangle^{-2r} \langle x/\lambda \rangle^{2(n+1)} dx \right|^{1/2} \cdot \|u\|_{L^2(\mathbb{R}^{2D}, (W^r)^2)} \\ &\leq C_0 \cdot \lambda^{-n-1} \cdot \|u\|_{L^2(\mathbb{R}^{2D}, (W^r)^2)}. \end{aligned}$$

Clearly this implies (4.54) for $\sigma = -$. □

To finish, it is enough to show

$$\|L_A^{\text{lift}} \circ \tilde{\mathcal{T}} - L_A^{\text{lift}} \circ X \circ \tilde{\mathcal{T}} \circ \Xi\|_{L^2(\mathbb{R}^{2D}, (W^r)^2)} \leq C_0 \lambda^{-r} |\det A|. \quad (4.60)$$

1350 Note the relations

$$L_A^{\text{lift}} \circ \tilde{\mathcal{T}} = \mathcal{B} \circ L_A \circ \tilde{T} \circ \mathcal{B} = \mathcal{B} \circ \tilde{T} \circ L_A \circ \mathcal{B} = \tilde{\mathcal{T}} \circ L_A^{\text{lift}}$$

and

$$\begin{aligned} L_A^{\text{lift}} \circ (\text{Id} - X) &= \mathcal{B} \circ L_A \circ \mathcal{M}(1 - \chi) \circ \mathcal{B}^* \\ &= \mathcal{B} \circ \mathcal{M}(1 - \chi_A) \circ L_A \circ \mathcal{B}^* = (\text{Id} - X_A) \circ L_A^{\text{lift}} \end{aligned}$$

where $\chi_A = \chi \circ A^{-1}$ and $X_A = \mathcal{B} \circ \chi_A \circ \mathcal{B}^*$. Below we will prove the claims

$$\|L_A^{\text{lift}} \circ (\text{Id} - X)\|_{L^2(\mathbb{R}^{2D}, (W^r)^2)} \leq C_0 \lambda^{-r} |\det A| \quad (4.61)$$

and

$$\|L_A^{\text{lift}} \circ (\text{Id} - \Xi)\|_{L^2(\mathbb{R}^{2D}, (W^r)^2)} \leq C_0 \lambda^{-r} |\det A|. \quad (4.62)$$

Since $\tilde{\mathcal{T}} = \text{Id} - \sum_{k=0}^n \mathcal{T}^{(k)}$ is a bounded operator on $L^2(\mathbb{R}^{2D}, W^r)$, these claims would imply

$$\|L_A^{\text{lift}} \circ (\text{Id} - X) \circ \tilde{\mathcal{T}}\|_{L^2(\mathbb{R}^{2D}, (W^r)^2)} \leq C_0 \cdot \lambda^{-r} |\det A|$$

and

$$\|L_A^{\text{lift}} \circ X \circ \tilde{\mathcal{T}} \circ (\text{Id} - \Xi)\|_{L^2(\mathbb{R}^{2D}, (W^r)^2)} = \|X_A \circ \tilde{\mathcal{T}} \circ L_A^{\text{lift}} \circ (\text{Id} - \Xi)\|_{L^2(\mathbb{R}^{2D}, (W^r)^2)} \leq C_0 \lambda^{-r} |\det A|$$

1355 and therefore the conclusion (4.60) would follow.

We can prove (4.61) and (4.62) by straightforward estimate. Writing the kernel of the operator X_A explicitly and applying integration by parts to it, we get the estimate

$$|(\text{Id} - X_A)v(x, \xi)| \leq C_\nu \int K_1^{(\nu)}(x, \xi; x', \xi') \cdot v(x', \xi') dx' d\xi'$$

for arbitrarily large ν , where

$$K_1^{(\nu)}(x, \xi; x', \xi') = \int_{\text{supp}(1-\chi_A)} \langle |x-y| \rangle^{-\nu} \langle |y-x'| \rangle^{-\nu} \langle |\xi-\xi'| \rangle^{-\nu} dy.$$

From the expression (4.18) of the operator L_A^{lift} , we also have

$$|L_A^{\text{lift}}v(x', \xi')| \leq C_\nu d(A) \int K_2^{(\nu)}(x', \xi'; x'', \xi'') v(x'', \xi'') dx'' d\xi''$$

1360 for arbitrarily large ν , where

$$K_2^{(\nu)}(x', \xi'; x'', \xi'') = \int \langle |x' - x_\dagger| \rangle^{-\nu} \langle |\xi' - \xi_\dagger| \rangle^{-\nu} \langle |A^{-1}x_\dagger - x''| \rangle^{-\nu} \langle |{}^t A \xi_\dagger - \xi''| \rangle^{-\nu} dx_\dagger d\xi_\dagger.$$

We have $d(A) \leq |\det A|$ also. From the definition of the function W^r and the expanding property (4.33) of A , we have

$$\frac{W^r(x_\dagger, \xi_\dagger)}{W^r(A^{-1}x_\dagger, {}^t A \xi_\dagger)} \cdot \langle |x_\dagger - y| \rangle^{-2r} \leq C_0 \lambda^{-r} \quad \text{if } y \in \text{supp}(1 - \chi_A).$$

Also note that, from the property (4.31) of W^r , we have

$$W^r(x, \xi) \cdot \langle |x-y| \rangle^{-2r} \langle |y-x'| \rangle^{-2r} \langle |x'-x_\dagger| \rangle^{-2r} \leq C_0 \cdot W^r(x_\dagger, \xi_\dagger)$$

and

$$\frac{1}{W^r(x'', \xi'')} \cdot \langle |A^{-1}x_\dagger - x''| \rangle^{-2r} \langle |{}^t A \xi_\dagger - \xi''| \rangle^{-2r} \leq C_0 \cdot \frac{1}{W^r(A^{-1}x_\dagger, {}^t A \xi_\dagger)}.$$

Summarizing these estimates, we obtain

$$\begin{aligned} & \frac{W^r(x, \xi)}{W^r(x'', \xi'')} \cdot \int K_1^{(\nu)}(x, \xi; x', \xi'; y) \cdot K_2^{(\nu)}(x', \xi'; x'', \xi'') dx' d\xi' \\ & \leq C_0 \lambda^{-r} \cdot \int \int K_1^{(\nu-2r)}(x, \xi; x', \xi'; y) \cdot K_2^{(\nu-4r)}(x', \xi'; x'', \xi'') dx' d\xi'. \end{aligned}$$

1365 By Schur inequality (4.32), the integral operators with the kernels $K_1^{(\nu-4r)}(\cdot)$ and $K_2^{(\nu-2r)}(\cdot)$ are bounded operators on $L^2(\mathbb{R}^{2D})$ and the operator norms are bounded by a constant that does not depend on A , provided ν is sufficiently large. Therefore the last estimate implies (4.61). We can prove the claim (4.62) in the same manner.

5 Resonance of hyperbolic linear prequantum maps

1370 The main result of this section, Proposition 5.11, concerns the spectrum of the prequantum transfer operator for linear hyperbolic symplectic maps.

5.1 Prequantum transfer operator on \mathbb{R}^{2d}

In this section, we study prequantum transfer operators and rough Laplacian in a special and easy case: The manifold M is the linear space \mathbb{R}^{2d} with the coordinates

$$x \equiv (q, p) = (q^1, \dots, q^d, p^1, \dots, p^d) \in \mathbb{R}^{2d}. \quad (5.1)$$

1375 We regard it as a symplectic manifold equipped with the symplectic two form

$$\omega = dq \wedge dp := \sum_{i=1}^d dq^i \wedge dp^i. \quad (5.2)$$

The prequantum bundle P is the trivial $U(1)$ -bundle $\pi : P = \mathbb{R}^{2d} \times U(1) \rightarrow \mathbb{R}^{2d}$ over \mathbb{R}^{2d} equipped with the connection one form $A = id\theta - i(2\pi)\eta$ where

$$\eta = \sum_{i=1}^d \left(\frac{1}{2} q^i dp^i - \frac{1}{2} p^i dq^i \right). \quad (5.3)$$

The corresponding curvature two form is then

$$\Theta = -i(2\pi)(\pi^*\omega)$$

1380 because $\omega = d\eta$. Under these settings, we may rephrase the construction of the prequantum transfer operator for a symplectic diffeomorphism on \mathbb{R}^{2d} .

Let $f : U \rightarrow U'$ be a symplectic diffeomorphism between two domains U and U' in \mathbb{R}^{2d} with respect to the symplectic two form ω . Let $\tilde{f} : U \times \mathbf{U}(1) \rightarrow U' \times \mathbf{U}(1)$ be the equivariant lift of f preserving the connection A , that is, the map satisfying the conditions

(1.14), (1.15) and (1.16) in Theorem 1.4. Then we define as in (1.19) (but for the set $V \equiv 0$) the prequantum operator

$$\hat{F} : C^\infty(U \times \mathbf{U}(1)) \rightarrow C^\infty(U' \times \mathbf{U}(1)), \quad \hat{F}u(x) = u \circ \tilde{f}^{-1}(x)$$

and let

$$\hat{F}_N : C_N^\infty(U \times \mathbf{U}(1)) \rightarrow C_N^\infty(U' \times \mathbf{U}(1))$$

be its restriction to the space of functions in the N -th Fourier mode. Let

$$\mathcal{L}_f : C^\infty(\mathbb{R}^{2d}) \rightarrow C^\infty(\mathbb{R}^{2d})$$

be the expression of the prequantum transfer operator \hat{F}_N with respect to the trivialization using the trivial section $\tau_0 : \mathbb{R}^{2d} \rightarrow P = \mathbb{R}^{2d} \times U(1)$ defined by $\tau_0(x) = (x, 1)$. This operator \mathcal{L}_f is the *prequantum transfer operator* for $f : U \rightarrow U'$. (Note that \mathcal{L}_f depends on the integer $N \in \mathbb{Z}$ and hence on \hbar .) We recall its concrete expression obtained in Proposition 2.4.

Proposition 5.1. *The operator \mathcal{L}_f as above is written*

$$(\mathcal{L}_f u)(x) := e^{-\frac{i}{\hbar} \mathcal{A}_f(f^{-1}(x))} u(f^{-1}(x)) \quad (5.4)$$

with the (action) function

$$\mathcal{A}_f(x) = \int_\gamma f^* \eta - \eta \quad (5.5)$$

where γ is a path from a fixed point $x_0 \in U'$ to x .

5.2 Prequantum operator for a symplectic affine map on \mathbb{R}^{2d}

Let $f : \mathbb{R}^{2d} \rightarrow \mathbb{R}^{2d}$ be an affine map preserving the symplectic form ω , written:

$$f : \mathbb{R}^{2d} \rightarrow \mathbb{R}^{2d}, \quad f(x) = Bx + b \quad (5.6)$$

where $B : \mathbb{R}^{2d} \rightarrow \mathbb{R}^{2d}$ is a linear symplectic map and $b \in \mathbb{R}^{2d}$ a constant vector.

Proposition 5.2. *The prequantum transfer operator \mathcal{L}_f for an affine map f as above is written as*

$$\mathcal{L}_f u(x) := e^{-\frac{i}{\hbar} \mathcal{A}_f(f^{-1}(x))} u(f^{-1}(x)) \quad (5.7)$$

with the (action) function

$$\mathcal{A}_f(x) = \frac{1}{2} \omega(b, x). \quad (5.8)$$

Remark 5.3. Notice that the function \mathcal{A}_f in (5.8) does not depend on the linear map B which enters in (5.6).

1405 *Proof.* Then, for any $x = (q, p) \in \mathbb{R}^{2d}$, using the parametrized path $\gamma(t) = (q'(t), p'(t)) = (tq, tp)$ with $t \in [0, 1]$ we have

$$\int_{\gamma} \eta = \int_{\gamma} \frac{1}{2} (q' dp' - p' dq') = \int_0^1 \frac{1}{2} (tqp - tpq) dt = 0$$

Therefore for a linear symplectic map $f_2(x) = Bx$, the action defined in (5.5) vanishes:

$$\mathcal{A}_{f_2}(x) = \int_{f_2(0)}^{f_2(x)} \eta - \int_0^x \eta = 0 - 0 = 0.$$

For a translation map $f_1(x) = x + b$, using the parametrized path $(q'(t), p'(t)) = (tq + b_q, tp + b_p)$ with $b = (b_q, b_p)$ and $t \in [0, 1]$ we have

$$\begin{aligned} \mathcal{A}_{f_1}(x) &= \int_b^{x+b} \eta - \int_0^x \eta = \int_b^{x+b} \frac{1}{2} (q' dp' - p' dq') \\ &= \int_0^1 \frac{1}{2} ((tq + b_q)p - (tp + b_p)q) dt = \frac{1}{2} (b_q p - b_p q) = \frac{1}{2} \omega(b, x) \end{aligned}$$

Finally for the affine map $f(x) = Bx + b = (f_1 \circ f_2)(x)$, the action is

$$\mathcal{A}_f = \mathcal{A}_{f_1} + \mathcal{A}_{f_2} \circ f_2^1 = \mathcal{A}_{f_1} = \frac{1}{2} \omega(b, x).$$

1410

□

We next consider the lift of the operator \mathcal{L}_f , Eq.(5.7), with respect to the Bargmann transform \mathcal{B}_h . Following the idea explained in Subsection 2.1, we express it with respect to the coordinates (ν, ζ) introduced in Proposition 2.15. Then, in the next Lemma, we will obtain an expression of \mathcal{L}_f as a tensor product of two operators: each of the two operators is associated to the dynamics of the canonical map $F = {}^t Df^{-1} : T^*\mathbb{R}^{2d} \rightarrow T^*\mathbb{R}^{2d}$ of \mathcal{L}_f in the directions along and orthogonal to the trapped set K , defined by

$$K = \{(x, \xi) \in \mathbb{R}^{2d} \mid \zeta = 0\}$$

in the simple setting we are considering. (See Proposition 2.15.)

Let us write the change of variable given in Proposition 2.15 as

$$\Phi : \left(\underbrace{q, p}_x, \underbrace{\xi_q, \xi_p}_\xi \right) \in \mathbb{R}^{2d} \oplus \mathbb{R}^{2d} \rightarrow \left(\underbrace{\nu_q, \nu_p}_\nu, \underbrace{\zeta_p, \zeta_q}_\zeta \right) \in \mathbb{R}^{2d} \oplus \mathbb{R}^{2d}. \quad (5.9)$$

It maps the standard symplectic form $\Omega_0 = dx \wedge d\xi$ on $\mathbb{R}^{2d} \oplus \mathbb{R}^{2d}$ to

$$(D\Phi^*)^{-1}(\Omega_0) = d\nu_q \wedge d\nu_p + d\zeta_p \wedge d\zeta_q$$

1420 and the metric

$$g_0 = \frac{1}{2}dx^2 + 2d\xi^2 \quad (5.10)$$

on $\mathbb{R}^{2d} \oplus \mathbb{R}^{2d}$ (g_0 is the metric induced by g on $T^*\mathbb{R}^{2d}$ as explained in Section 3.2.1) to the standard Euclidean metric on $\mathbb{R}^{2d} \oplus \mathbb{R}^{2d}$:

$$(D\Phi^*)^{-1}(g_0) = d\nu^2 + d\zeta^2.$$

Remark 5.4. With the choice of metric g_0 , in (5.10), the linear subsets K and $(K^{\perp\omega})$ are Ω_0 -symplectic orthogonal but are also g_0 -orthogonal.

1425 The unitary operator associated to the coordinate change Φ is defined as

$$\Phi^* : L^2(\mathbb{R}_{\nu}^{2d} \oplus \mathbb{R}_{\zeta}^{2d}) \rightarrow L^2(\mathbb{R}_x^{2d} \oplus \mathbb{R}_{\xi}^{2d}), \quad (\Phi^*u) := u \circ \Phi.$$

Here (and henceforth) we make the convention that the subscript in the notation such as \mathbb{R}_{ν}^{2d} indicates the name of the coordinates on the space.

We define the operators

$$\mathcal{B}_{\nu_q} : L^2(\mathbb{R}_{\nu_q}^d) \rightarrow L^2(\mathbb{R}_{(\nu_q, \nu_p)}^{2d}) \quad \text{and} \quad \mathcal{B}_{\nu_q}^* : L^2(\mathbb{R}_{(\nu_q, \nu_p)}^{2d}) \rightarrow L^2(\mathbb{R}_{\nu_q}^d)$$

as the Bargmann transform \mathcal{B}_h and its adjoint \mathcal{B}_h^* in Subsection 4.1 for the case $D = d$.

1430 We define

$$\mathcal{B}_{\zeta_p} : L^2(\mathbb{R}_{\zeta_p}^d) \rightarrow L^2(\mathbb{R}_{(\zeta_p, \zeta_q)}^{2d}) \quad \text{and} \quad \mathcal{B}_{\zeta_p}^* : L^2(\mathbb{R}_{(\zeta_p, \zeta_q)}^{2d}) \rightarrow L^2(\mathbb{R}_{\zeta_p}^d)$$

similarly. Suppose that \mathcal{P}_{ν_q} and \mathcal{P}_{ζ_p} are defined correspondingly: That is to say, with setting $D = d$, we define

$$\mathcal{B}_{\nu_q} = \mathcal{B}_{\zeta_p} = \mathcal{B}_h, \quad \mathcal{B}_{\nu_q}^* = \mathcal{B}_{\zeta_p}^* = \mathcal{B}_h^* \quad \text{and} \quad \mathcal{P}_{\nu_q} = \mathcal{P}_{\zeta_p} = \mathcal{P}_h.$$

Next we define the operators

$$\mathcal{B}_x : L^2(\mathbb{R}_x^{2d}) \rightarrow L^2(\mathbb{R}_{(x, \xi)}^{4d}) \quad \text{and} \quad \mathcal{B}_x^* : L^2(\mathbb{R}_{(x, \xi)}^{4d}) \rightarrow L^2(\mathbb{R}_x^{2d})$$

by

$$\mathcal{B}_x := \tilde{\sigma}^{-1} \circ \mathcal{B}_h \circ \sigma \quad \text{and} \quad \mathcal{B}_x^* := \sigma^{-1} \circ \mathcal{B}_h^* \circ \tilde{\sigma} \quad (5.11)$$

1435 where \mathcal{B}_h and \mathcal{B}_h^* are now those in the case $D = 2d$, and

$$\sigma : L^2(\mathbb{R}_x^{2d}) \rightarrow L^2(\mathbb{R}_x^{2d}) \quad \text{and} \quad \tilde{\sigma} : L^2(\mathbb{R}_{(x, \xi)}^{4d}) \rightarrow L^2(\mathbb{R}_{(x, \xi)}^{4d})$$

are simple unitary operators defined by

$$\sigma u(x) = 2^{-d}u(2^{-1/2}x) \quad \text{and} \quad \tilde{\sigma}v(x, \xi) = v(2^{-1/2}x, 2^{1/2}\xi)$$

introduced in relation to the additional factor 1/2 in (5.10).

Correspondingly we set

$$\mathcal{P}_x = \mathcal{B}_x \circ \mathcal{B}_x^* = \tilde{\sigma}^{-1} \circ \mathcal{P}_h \circ \tilde{\sigma}. \quad (5.12)$$

1440 *Remark 5.5.* (1) In terms of the (generalized) Bargmann transforms considered in Sub-
 section 4.1.3, the operators \mathcal{B}_x and \mathcal{B}_x^* are the Bargmann transform and its adjoint
 for the combination of the Euclidean space $E = \mathbb{R}^{4d}$, the standard symplectic form
 $\Omega_0 = dx \wedge d\xi$, the metric $g_0 = \frac{1}{2}dx^2 + 2d\xi^2$ and the Lagrangian subspace \mathbb{R}_x^{2d} . Since
 the metric g_0 corresponds to the standard Euclidean metric through Φ , the defini-
 1445 tion of the operators \mathcal{B}_x , \mathcal{B}_x^* and \mathcal{P}_x above is more convenient and natural for our
 argument than those without σ and $\tilde{\sigma}$. Indeed we have

$$\mathcal{P}_x^* \circ \Phi^* = \Phi^* \circ (\mathcal{P}_{\nu_q} \otimes \mathcal{P}_{\zeta_p})^*. \quad (5.13)$$

(2) In the notation introduced above, the subscripts indicate the related coordinates.
 Though this may deviate from the standard usage, it is convenient for our argument.
 Notice that the operators introduced above, such as \mathcal{B}_x , depend on the parameter \hbar
 (and hence on N).

1450 **Lemma 5.6.** *Let \mathcal{L}_f be the prequantum transfer operator (5.7) associated to a symplectic
 affine map in (5.6). Then the following diagram commutes:*

$$\begin{array}{ccc} L^2(\mathbb{R}_x^{2d}) & \xrightarrow{\mathcal{L}_f} & L^2(\mathbb{R}_x^{2d}) \\ \uparrow \mathcal{U} & & \uparrow \mathcal{U} \\ L^2(\mathbb{R}_{\nu_q}^d) \otimes L^2(\mathbb{R}_{\zeta_p}^d) & \xrightarrow{M_\nu(f) \otimes M_\zeta(B)} & L^2(\mathbb{R}_{\nu_q}^d) \otimes L^2(\mathbb{R}_{\zeta_p}^d) \end{array} \quad (5.14)$$

where \mathcal{U} , $M_\nu(f)$ and $M_\zeta(B)$ are the unitary operators defined respectively by

$$\begin{aligned} \mathcal{U} : L^2(\mathbb{R}_{\nu_q}^d) \otimes L^2(\mathbb{R}_{\zeta_p}^d) &\rightarrow L^2(\mathbb{R}_x^{2d}), & \mathcal{U} &= \mathcal{B}_x^* \circ \Phi^* \circ (\mathcal{B}_{\nu_q} \otimes \mathcal{B}_{\zeta_p}), \\ M_\nu(f) : L^2(\mathbb{R}_{\nu_q}^d) &\rightarrow L^2(\mathbb{R}_{\nu_q}^d), & M_\nu(f) &= \sqrt{d(B)} \cdot \mathcal{B}_{\nu_q}^* \circ (e^{\frac{i}{2\hbar}\omega(\nu,b)} \cdot L_f) \circ \mathcal{B}_{\nu_q}, \\ M_\zeta(B) : L^2(\mathbb{R}_{\zeta_p}^d) &\rightarrow L^2(\mathbb{R}_{\zeta_p}^d), & M_\zeta(B) &= \sqrt{d(B)} \cdot \mathcal{B}_{\zeta_p}^* \circ L_B \circ \mathcal{B}_{\zeta_p} \end{aligned} \quad (5.15)$$

with $d(B) = \det((1 + {}^t B \cdot B)/2)^{1/2}$, $(L_f u)(\nu) := (u \circ f^{-1})(\nu)$ and $L_B u = u \circ B^{-1}$ as
 before. Equivalently, in terms of lifted operators, we have the following commuting diagram:

$$\begin{array}{ccc} L^2(\mathbb{R}_x^{2d} \oplus \mathbb{R}_\xi^{2d}) & \xrightarrow{\mathcal{L}_f^{\text{lift}}} & L^2(\mathbb{R}_x^{2d} \oplus \mathbb{R}_\xi^{2d}) \\ \uparrow \Phi^* & & \uparrow \Phi^* \\ L^2(\mathbb{R}_\nu^{2d}) \otimes L^2(\mathbb{R}_\zeta^{2d}) & \xrightarrow{M_\nu^{\text{lift}}(f) \otimes M_\zeta^{\text{lift}}(B)} & L^2(\mathbb{R}_\nu^{2d}) \otimes L^2(\mathbb{R}_\zeta^{2d}) \end{array} \quad (5.16)$$

1455 *Proof.* Recall the operators $\flat : \mathbb{R}^{2d} \rightarrow (\mathbb{R}^{2d})^* = \mathbb{R}^{2d}$ and $\sharp : (\mathbb{R}^{2d})^* = \mathbb{R}^{2d} \rightarrow \mathbb{R}^{2d}$ introduced
 in Remark 2.17. The expression (5.7) shows that \mathcal{L}_f can be written $\mathcal{L}_f = T_{(b, -\frac{1}{2}b^\flat)} \circ L_B$
 where the unitary transfer operator $T_{(b, -\frac{1}{2}b^\flat)}$ is defined in (4.19). We apply Lemma 4.10,
 Lemma 4.8 and corollary 4.11 to obtain

$$\begin{aligned}
\mathcal{L}_f &= \mathcal{B}_x^* \circ \mathcal{T}_{(b, -\frac{1}{2}b^b)} \circ \mathcal{B}_x \circ (d(B) \cdot \mathcal{B}_x^*) \circ L_{B \oplus {}^t B^{-1}} \circ \mathcal{B}_x \\
&= d(B) \cdot \mathcal{B}_x^* \circ \mathcal{T}_{(b, -\frac{1}{2}b^b)} \circ \mathcal{B}_x \circ \mathcal{B}_x^* \circ L_{B \oplus {}^t B^{-1}} \circ \mathcal{B}_x \\
&= d(B) \cdot \mathcal{B}_x^* \circ (e^{\frac{i}{2\hbar}\varphi} \cdot L_F) \circ \mathcal{B}_x
\end{aligned} \tag{5.17}$$

with $F(x, \xi) = (Bx + b, {}^t B^{-1}\xi - \frac{1}{2}b^b)$ and $\varphi(x, \xi) = -\frac{1}{2}b^b \cdot x - b \cdot \xi$. Since we have ${}^t B^{-1} = \flat \circ B \circ \flat^{-1}$ for B symplectic, we get the following expression of F in the new coordinates (ν, ζ) :

$$(\Phi \circ F \circ \Phi^{-1})(\nu, \zeta) = (B\nu + b, {}^t B^{-1}\zeta) = (f(\nu), {}^t Df^{-1}\zeta).$$

This implies

$$L_F = \Phi^* \circ (L_f \otimes L_B) \circ (\Phi^*)^{-1}.$$

From (5.13), we have

$$\mathcal{B}_x^* \circ \Phi^* = \mathcal{B}_x^* \circ \mathcal{P}_x \circ \Phi^* = \mathcal{B}_x^* \circ \Phi^* \circ (\mathcal{P}_{\nu_q} \otimes \mathcal{P}_{\zeta_p}) = \mathcal{U} \circ (\mathcal{B}_{\nu_q} \otimes \mathcal{B}_{\zeta_p})^*$$

and

$$(\Phi^*)^{-1} \circ \mathcal{B}_x = (\Phi^*)^{-1} \circ \mathcal{P}_x \circ \mathcal{B}_x = (\mathcal{P}_{\nu_q} \otimes \mathcal{P}_{\zeta_p}) \circ (\Phi^*)^{-1} \circ \mathcal{B}_x = (\mathcal{B}_{\nu_q} \otimes \mathcal{B}_{\zeta_p}) \circ \mathcal{U}^{-1}.$$

Using these relations to continue (5.17) and noting that $\varphi(x, \xi) = \omega(\nu, b)$, we conclude

$$\begin{aligned}
\mathcal{L}_f &= d(B) \cdot \mathcal{B}_x^* \circ e^{\frac{i}{2\hbar}\varphi} L_F \circ \mathcal{B}_x \\
&= d(B) \cdot \mathcal{B}_x^* \circ \Phi^* \circ \left(e^{\frac{i}{2\hbar}\varphi} L_f \otimes L_B \right) \circ (\Phi^*)^{-1} \circ \mathcal{B}_x \\
&= d(B) \cdot \mathcal{U} \circ \left(\left(\mathcal{B}_{\nu_q}^* \circ (e^{\frac{i}{2\hbar}\varphi} \cdot L_f) \circ \mathcal{B}_{\nu_q} \right) \otimes \left(\mathcal{B}_{\zeta_p}^* \circ L_B \circ \mathcal{B}_{\zeta_p} \right) \right) \circ \mathcal{U}^{-1} \\
&= \mathcal{U} \circ (M_\nu(f) \otimes M_\zeta(B)) \circ \mathcal{U}^{-1}.
\end{aligned}$$

1465

□

Remark 5.7. Eq.(5.14) shows that \mathcal{L}_f is conjugated to the product of two operators $M_\nu(f) \otimes M_\zeta(B)$. This is remarkable but it is due here to the fact that Df is constant. Each of these operators is usually called a metaplectic operator, we refer to [23]. Here we have used a derivation in phase space using the Bargmann transform.

1470

5.3 The prequantum transfer operator for a linear hyperbolic map

In this subsection, we restrict the argument in the last subsection to the case where f in (5.6) is hyperbolic in the sense that f is expanding in $\mathbb{R}^d \oplus \{0\}$ while contracting in $\{0\} \oplus \mathbb{R}^d$ and is linear, i.e. $b = 0$. Since f preserves the symplectic form ω , we may express it as (see last remark in the proof of Proposition 2.18)

$$f(q, p) = B(q, p) = (Aq, {}^t A^{-1}p) \quad \text{where } B = \begin{pmatrix} A & 0 \\ 0 & {}^t A^{-1} \end{pmatrix} \quad (5.18)$$

with $A : \mathbb{R}^d \rightarrow \mathbb{R}^d$ an expanding linear map satisfying $\|A^{-1}\| \leq 1/\lambda$ for some $\lambda > 1$. Notice that, since $b = 0$, the action \mathcal{A} vanishes in (5.7) and the prequantum transfer operator gets the simpler expression:

$$(\mathcal{L}_f u)(x) = u(B^{-1}x) = L_B u(x). \quad (5.19)$$

The next proposition is deduced from Proposition 5.6.

Proposition 5.8. *The following diagram commutes:*

$$\begin{array}{ccc} L^2(\mathbb{R}_x^{2d}) & \xrightarrow{\mathcal{L}_f} & L^2(\mathbb{R}_x^{2d}) \\ \uparrow \mathcal{U} & & \uparrow \mathcal{U} \\ L^2(\mathbb{R}_{\nu_q}^d) \otimes L^2(\mathbb{R}_{\zeta_p}^d) & \xrightarrow{U_A \otimes U_A} & L^2(\mathbb{R}_{\nu_q}^d) \otimes L^2(\mathbb{R}_{\zeta_p}^d) \end{array} \quad (5.20)$$

with the unitary operator \mathcal{U} defined in (5.15) and

$$U_A := \frac{1}{\sqrt{|\det A|}} L_A$$

which is unitary in $L^2(\mathbb{R}^d)$. Equivalently using lifted operators, expressed in Lemma 4.8, we have the following commuting diagram:

$$\begin{array}{ccc} L^2(\mathbb{R}_x^{2d} \oplus \mathbb{R}_\xi^{2d}) & \xrightarrow{\mathcal{L}_f^{\text{lift}}} & L^2(\mathbb{R}_x^{2d} \oplus \mathbb{R}_\xi^{2d}) \\ \uparrow \Phi^* & & \uparrow \Phi^* \\ L^2(\mathbb{R}_\nu^{2d}) \otimes L^2(\mathbb{R}_\zeta^{2d}) & \xrightarrow{U_A^{\text{lift}} \otimes U_A^{\text{lift}}} & L^2(\mathbb{R}_\nu^{2d}) \otimes L^2(\mathbb{R}_\zeta^{2d}) \end{array} \quad (5.21)$$

with $U_A^{\text{lift}} := \frac{1}{\sqrt{|\det A|}} L_A^{\text{lift}}$.

Proof. We have

$$\begin{aligned} d(B) &= \det((1 + {}^t B \cdot B)/2)^{1/2} = \det\left(\frac{1}{2}(B + {}^t B^{-1})\right)^{1/2} \\ &= \det\left(\frac{1}{2}(A + {}^t A^{-1})\right) = (\det A)^{-1} \end{aligned}$$

and $L_f = L_B = L_{A \oplus^t A^{-1}}$. Hence, by the expression (4.15), we get

$$M_\nu(f) := \sqrt{d(B)} \cdot \mathcal{B}_{\nu_q}^* \circ (e^{\frac{i}{2\hbar}\omega(\nu,b)} \cdot L_f) \circ \mathcal{B}_{\nu_q} = (\det A)^{-1/2} d(A) \cdot \mathcal{B}_{\nu_q}^* \circ L_{A \oplus^t A^{-1}} \circ \mathcal{B}_{\nu_q} = U_A$$

and

$$M_\zeta(B) := \sqrt{d(B)} \cdot \mathcal{B}_{\zeta_p}^* \circ L_B \circ \mathcal{B}_{\zeta_p} = (\det A)^{-1/2} d(A) \cdot \mathcal{B}_{\zeta_p}^* \circ L_{A \oplus^t A^{-1}} \circ \mathcal{B}_{\zeta_p} = U_A.$$

Putting these in Proposition 5.6, we obtain the conclusion. \square

5.4 Anisotropic Sobolev space

1490 In order to observe a discrete spectrum of resonances of the prequantum operator \mathcal{L}_f , we have to consider the action of \mathcal{L}_f on an appropriate spaces of functions. As we explained in subsection 2.1.3, we define such space of function, called anisotropic Sobolev space, by changing the norm in the directions transverse to the trapped set K (that is, in the directions of the variables ζ). Below is the precise definition.

1495 **Definition 5.9.** We define the *escape function* or *weight function*

$$\mathcal{W}_h^r : \mathbb{R}_x^{2d} \oplus \mathbb{R}_\xi^{2d} \rightarrow \mathbb{R}_+ \quad \text{and} \quad \mathcal{W}_h^{r,\pm} : \mathbb{R}_x^{2d} \oplus \mathbb{R}_\xi^{2d} \rightarrow \mathbb{R}_+$$

by

$$\mathcal{W}_h^r(x, \xi) := W_h^r(\zeta_p, \zeta_q) \quad \text{and} \quad \mathcal{W}_h^{r,\pm}(x, \xi) := W_h^r(\zeta_p, \zeta_q) \quad (5.22)$$

where the functions W_h^r and $W_h^{r,\pm}$ are defined in Definition 4.12, and (ζ_p, ζ_q) is part of the coordinates introduced in (5.9). The *anisotropic Sobolev space* $\mathcal{H}_h^r(\mathbb{R}^{2d})$ is the Hilbert space obtained as the completion of the Schwartz space $\mathcal{S}(\mathbb{R}^{2d})$ with respect to the norm

$$\|u\|_{\mathcal{H}_h^r} := \|\mathcal{W}_h^r \cdot \mathcal{B}_x u\|_{L^2}$$

1500 where \mathcal{B}_x is the operator defined in (5.11). Similarly, let $\mathcal{H}_h^{r,\pm}(\mathbb{R}^{2d})$ be the Hilbert space defined in the parallel manner by replacing $\mathcal{W}_h^r(\cdot)$ by $\mathcal{W}_h^{r,\pm}(\cdot)$.

By definition, the operator \mathcal{B}_x extends to an isometric embedding

$$\mathcal{B}_x : \mathcal{H}_h^r(\mathbb{R}_x^{2d}) \rightarrow L^2(\mathbb{R}_x^{2d} \oplus \mathbb{R}_\xi^{2d}, (\mathcal{W}_h^r)^2).$$

Since the weight function $\mathcal{W}^r(\cdot)$ can be expressed as

$$\mathcal{W}_h^r = (1 \otimes W_h^r) \circ \Phi,$$

where Φ is given in (5.9), we see that the following diagram commutes:

$$\begin{array}{ccc}
\mathcal{H}_h^r(\mathbb{R}_x^{2d}) & \xrightarrow{\mathcal{L}_f} & \mathcal{H}_h^r(\mathbb{R}_x^{2d}) \\
\downarrow \mathcal{B}_x & & \downarrow \mathcal{B}_x \\
L^2(\mathbb{R}_x^{2d} \oplus \mathbb{R}_\xi^{2d}, (\mathcal{W}_h^r)^2) & \xrightarrow{\mathcal{L}^{\text{lift}}} & L^2(\mathbb{R}_x^{2d} \oplus \mathbb{R}_\xi^{2d}, (\mathcal{W}_h^r)^2) \\
\uparrow \Phi^* & & \uparrow \Phi^* \\
L^2(\mathbb{R}_\nu^{2d}) \otimes L^2(\mathbb{R}_\zeta^{2d}, (W_h^r)^2) & \xrightarrow{U_A^{\text{lift}} \otimes U_A^{\text{lift}}} & L^2(\mathbb{R}_\nu^{2d}) \otimes L^2(\mathbb{R}_\zeta^{2d}, (W_h^r)^2) \\
\uparrow \mathcal{B}_{\nu_q} \otimes \mathcal{B}_{\zeta_p} & & \uparrow \mathcal{B}_{\nu_q} \otimes \mathcal{B}_{\zeta_p} \\
L^2(\mathbb{R}_{\nu_q}^{2d}) \otimes H_h^r(\mathbb{R}_{\zeta_p}^d) & \xrightarrow{U_A \otimes U_A} & L^2(\mathbb{R}_{\nu_q}^{2d}) \otimes H_h^r(\mathbb{R}_{\zeta_p}^d)
\end{array} \tag{5.23}$$

1505 where Φ^* is an isomorphism between Hilbert spaces. Further (5.13) is an isomorphism between the image of \mathcal{B}_x and the image of $\mathcal{B}_{\nu_q} \otimes \mathcal{B}_{\zeta_p}$. $H_h^r(\mathbb{R}_{\zeta_p}^d)$ in the last line is defined in Definition 4.18. Hence, skipping the lines in the middle, we get:

$$\begin{array}{ccc}
\mathcal{H}_h^r(\mathbb{R}_x^{2d}) & \xrightarrow{\mathcal{L}_f} & \mathcal{H}_h^r(\mathbb{R}_x^{2d}) \\
\uparrow \mathcal{U} & & \uparrow \mathcal{U} \\
L^2(\mathbb{R}_{\nu_q}^2) \otimes H_h^r(\mathbb{R}_{\zeta_p}^d) & \xrightarrow{U_A \otimes U_A} & L^2(\mathbb{R}_{\nu_q}^2) \otimes H_h^r(\mathbb{R}_{\zeta_p}^d)
\end{array} \tag{5.24}$$

with \mathcal{U} the unitary operator defined in (5.15). For the operator $U_A \otimes U_A$ on the bottom line, we know that the operator $U_A = \frac{1}{\sqrt{|\det A|}} L_A : L^2(\mathbb{R}_{\nu_q}^{2d}) \rightarrow L^2(\mathbb{R}_{\nu_q}^{2d})$ is unitary and Proposition 4.19 gives a description on the spectral structure of the operator $L_A : H_h^r(\mathbb{R}_{\zeta_p}^d) \rightarrow H_h^r(\mathbb{R}_{\zeta_p}^d)$. Therefore we obtain the next proposition as a consequence. We fix some integer $n \geq 0$ and assume

$$r > n + 2d. \tag{5.25}$$

This assumption on r corresponds to (4.40) in the last section.

Definition 5.10. For $0 \leq k \leq n$, we consider the projection operators

$$t_h^{(k)} := \mathcal{U} \circ (\text{Id} \otimes T^{(k)}) \circ \mathcal{U}^{-1} : \mathcal{H}_h^r(\mathbb{R}_x^{2d}) \rightarrow \mathcal{H}_h^r(\mathbb{R}_x^{2d}) \tag{5.26}$$

1515 and

$$\tilde{t}_h := \text{Id} - \sum_{k=0}^n t_h^{(k)} = \mathcal{U} \circ (\text{Id} \otimes \tilde{T}) \circ \mathcal{U}^{-1} : \mathcal{H}_h^r(\mathbb{R}_x^{2d}) \rightarrow \mathcal{H}_h^r(\mathbb{R}_x^{2d}) \tag{5.27}$$

where $T^{(k)}$ and \tilde{T} are the projection operators introduced in (4.36) and (4.49) respectively.

Proposition 5.11. *The operators $t_h^{(k)}$, $0 \leq k \leq n$, and \tilde{t}_h defined in (5.26), (5.27), form a complete set of mutually commutative projection operators on $\mathcal{H}_h^r(\mathbb{R}_x^{2d})$. These operators also commute with the prequantum transfer operator \mathcal{L}_f defined in (5.19). Consequently the space $\mathcal{H}_h^r(\mathbb{R}_x^{2d})$ has a decomposition invariant under the action of \mathcal{L}_f :*

$$\mathcal{H}_h^r(\mathbb{R}_x^{2d}) = H'_0 \oplus H'_1 \oplus \cdots \oplus H'_n \oplus \tilde{H}' \quad \text{where } H'_k = \text{Im } t_h^{(k)} \text{ and } \tilde{H}' = \text{Im } \tilde{t}_h$$

For this decomposition we have

(1) For every $0 \leq k \leq n$, we have a commuting diagram

$$\begin{array}{ccc} H'_k & \xrightarrow{\mathcal{L}_f} & H'_k \\ \uparrow u & & \uparrow u \\ L^2(\mathbb{R}_{\nu_q}^2) \otimes \text{Polynom}^{(k)} & \xrightarrow{U_A \otimes U_A^{(k)}} & L^2(\mathbb{R}_{\nu_q}^2) \otimes \text{Polynom}^{(k)} \end{array}$$

with $U_A = \frac{1}{\sqrt{|\det A|}} L_A : L^2(\mathbb{R}_{\nu_q}^{2d}) \rightarrow L^2(\mathbb{R}_{\nu_q}^{2d})$ unitary and $U_A^{(k)} := \frac{1}{\sqrt{|\det A|}} L_A : \text{Polynom}^{(k)} \rightarrow \text{Polynom}^{(k)}$ finite rank. From (4.50) we have for every $u \in H'_k$,

$$C_0^{-1} \|A\|_{\max}^{-k} \cdot |\det(A)|^{-1/2} \cdot \|u\|_{\mathcal{H}_h^r} \leq \|\mathcal{L}_f u\|_{\mathcal{H}_h^r} \leq C_0 \|A\|_{\min}^{-k} \cdot |\det(A)|^{-1/2} \cdot \|u\|_{\mathcal{H}_h^r}.$$

(2) The operator norm of $\mathcal{L}_f : \tilde{H}' \rightarrow \tilde{H}'$ is bounded by

$$C_0 \cdot \max\{\|A\|_{\min}^{-n-1} \cdot |\det A|^{-1/2}, \|A\|_{\min}^{-r} \cdot |\det A|^{1/2}\}.$$

The constant C_0 is independent of A and \hbar .

5.5 A few technical lemmas

In this subsection, we collect a few miscellaneous technical lemmas related to the anisotropic Sobolev spaces $\mathcal{H}_h^r(\mathbb{R}^{2d})$ and $\mathcal{H}_h^{r,\pm}(\mathbb{R}^{2d})$, which we will use in the later sections. The following are immediate consequences of Lemma 4.15 and 4.16 respectively.

Lemma 5.12. *Suppose that f is a hyperbolic linear transformation (5.18) defined for an expanding linear map A satisfying (4.33) for some large λ (say $\lambda > 9$). Then the operator \mathcal{L}_f , (5.19), extends to a bounded operator $\mathcal{L}_f : \mathcal{H}_h^{r,-}(\mathbb{R}^{2d}) \rightarrow \mathcal{H}_h^{r,+}(\mathbb{R}^{2d})$ and the operator norm is bounded by a constant independent of $\hbar > 0$ and f .*

1525 **Corollary 5.13.** *The operator $t_h^{(k)}$ for $0 \leq k \leq n$, defined in (5.26), extends to a bounded operator*

$$t_h^{(k)} : \mathcal{H}_h^{r,-}(\mathbb{R}^{2d}) \rightarrow \mathcal{H}_h^{r,+}(\mathbb{R}^{2d})$$

whose operator norm is bounded by a constant independent of \hbar .

For convenience in the later argument, let us put the following definition:

1530 **Definition 5.14.** Let \mathcal{A} be the group of affine transformation on \mathbb{R}^{2d} that preserves the symplectic form ω , the Euclidean norm and its derivative preserves the splitting $\mathbb{R}^{2d} = \mathbb{R}^d \oplus \mathbb{R}^d$ simultaneously.

The function \mathcal{W}_h^r , (5.22), is invariant with respect to the transformation $(a, {}^t Da^{-1})$ on $T^*\mathbb{R}^{2d} = \mathbb{R}^{2d} \times \mathbb{R}^{2d}$ when $a \in \mathcal{A}$. This fact, together with Lemma 4.8, yields

1535 **Lemma 5.15.** *If $a \in \mathcal{A}$, then the prequantum operator \mathcal{L}_a , defined in (5.7) extends to an isometry on $\mathcal{H}_h^r(\mathbb{R}^{2d})$.*

The norm $\|\cdot\|_{\mathcal{H}_h^r}$ on the Hilbert space $\mathcal{H}_h^r(\mathbb{R}^{2d})$ is induced by a (unique) inner product $(\cdot, \cdot)_{\mathcal{H}_h^r(\mathbb{R}^{2d})}$. Notice that even if two distributions u and v in $\mathcal{H}_h^r(\mathbb{R}^{2d})$ have mutually disjoint supports, the inner product $(u, v)_{\mathcal{H}_h^r(\mathbb{R}^{2d})}$ may not vanish. This is somewhat inconvenient. But we have the following “pseudo-local” property. We omit the proof because it can be
1540 given by a straightforward estimate.

Lemma 5.16. *Let $\epsilon > 0$. If $d(\text{supp } u, \text{supp } v) \geq \hbar^{(1-\epsilon)/2}$ for $u, v \in \mathcal{H}_h^r(\mathbb{R}^{2d})$, we have*

$$|(u, v)_{\mathcal{H}_h^r(\mathbb{R}^{2d})}| \leq C_{\nu, \epsilon} \cdot \hbar^\nu \cdot \|u\|_{\mathcal{H}_h^r(\mathbb{R}^{2d})} \|v\|_{\mathcal{H}_h^r(\mathbb{R}^{2d})} \quad \text{for } u, v \in \mathcal{H}_h^r(\mathbb{R}^{2d})$$

for arbitrarily large ν , with $C_{\nu, \epsilon} > 0$ a constant depending on ϵ and ν .

From the definition of the function $\mathcal{W}_h^r(\cdot)$ and (4.30), we have

$$\mathcal{W}_h^r(x, \xi) \leq C \cdot \mathcal{W}_h^r(y, \eta) \cdot \langle \hbar^{-1/2} |(x, \xi) - (y, \eta)| \rangle^{2r}. \quad (5.28)$$

1545 The next lemma and corollary are direct consequences of this estimate. The proof is completely parallel to that of Lemma 4.13.

Lemma 5.17. *If $R_h : \mathcal{S}(\mathbb{R}^{2d} \oplus \mathbb{R}^{2d}) \rightarrow \mathcal{S}(\mathbb{R}^{2d} \oplus \mathbb{R}^{2d})$ is an integral operator of the form*

$$(R_h u)(x, \xi) = \int K_h(x, \xi; x', \xi') u(x', \xi') dx' d\xi'$$

depending on \hbar and if the kernel $K_h(\cdot; \cdot)$ is a continuous function satisfying

$$|K_h(x, \xi; x', \xi')| \leq \langle \hbar^{-1/2} \cdot |(x, \xi) - (x', \xi')| \rangle^{-\nu}$$

for some $\nu > 2r + 4d$, then the operator R_h extends uniquely to a bounded operator on $L^2(\mathbb{R}^{2d} \oplus \mathbb{R}^{2d}, (\mathcal{W}_h^r)^2)$ and

$$\|R_h\|_{L^2(\mathbb{R}^{2d} \oplus \mathbb{R}^{2d}, (\mathcal{W}_h^r)^2)} \leq C_\nu$$

1550 where C_ν is a constant independent of \hbar . The same holds true with \mathcal{W}_h^r replaced by $\mathcal{W}_h^{r, \pm}$ simultaneously.

Corollary 5.18. *The Bargmann projector \mathcal{P}_h extends uniquely to a bounded operator on $L^2(\mathbb{R}^{2d} \oplus \mathbb{R}^{2d}, (\mathcal{W}_h^r)^2)$ and its operator norm is bounded by a constant that does not depend on $\hbar > 0$. The same holds true with \mathcal{W}_h^r replaced by $\mathcal{W}_h^{r, \pm}$ simultaneously.*

1555 **6 Nonlinear prequantum maps on \mathbb{R}^{2d}**

In this section, we prepare some basic estimates on the effect of non-linearity of the Anosov diffeomorphism f on the anisotropic Sobolev space $\mathcal{H}_h^r(\mathbb{R}^{2d})$. Most of the results in this section may be rather obvious at least for those readers who are familiar with Fourier analysis. However we have to be attentive to the following particular situations in our
 1560 argument:

- The escape function \mathcal{W}_h^r in the definition of the anisotropic Sobolev space $\mathcal{H}_h^r(\mathbb{R}^{2d})$ has variable growth order $m(\cdot)$ depending on the directions. This leads to the fact that the (prequantum) transfer operator associated to a non-linear map may be unbounded even if the map is very close to the identity.
- 1565 • The spectral projection operators $t_h^{(k)}$ in Proposition 5.11 is also very anisotropic and rather singular. It is not well-defined (bounded) on any usual (isotropic) Sobolev spaces of positive or negative order.
- The escape function \mathcal{W}_h^r is not very smooth, viewed at the scale ($\sim \hbar^{1/2}$) of the smallest-possible wave packets in the phase space. In terms of the theory of pseudodifferential operators, this implies that the escape function \mathcal{W}_h^r belongs only to the symbol class of “critical order”. We have to avoid carefully the difficulties caused by
 1570 this fact.

For these reasons, we are going to give the argument to some detail. The main result in this section is Proposition 6.19, which concerns the third item above.

1575 Recall that the stable and unstable subspaces $E_s(x)$ and $E_u(x)$ for the Anosov diffeomorphism f depend on the point $x \in M$ not smoothly but only Hölder continuously. We let $0 < \beta < 1$ be the Hölder exponent. (See Remark 1.2(1) page 9) In what follows, we fix a small positive constant θ such that

$$0 < \theta < \beta/8. \tag{6.1}$$

The open ball of radius $c > 0$ on \mathbb{R}^{2d} is denoted by

$$\mathbb{D}(c) = \{x \in \mathbb{R}^{2d} \mid |x| < c\}.$$

1580 **6.1 Truncation operations in the real space**

We first consider the operation of truncating functions in the (real) space \mathbb{R}_x^{2d} by multiplying smooth functions with small supports. Below we consider the following setting:

Setting I: For each $\hbar > 0$, there is a given set \mathcal{X}_\hbar of C^∞ functions on \mathbb{R}^{2d} such that, for all $\psi \in \mathcal{X}_\hbar$ and $\hbar > 0$,

(C1) the support of ψ is contained in the disk $\mathbb{D}(C_*\hbar^{1/2-\theta})$ and

(C2) $|\partial_x^\alpha \psi(x)| < C_\alpha \hbar^{-(\frac{1}{2}-\theta)|\alpha|}$ for each multi-index $\alpha \in \mathbb{N}^{2d}$,

where $C_* > 0$ and $C_\alpha > 0$ are constants independent of $\psi \in \mathcal{X}_\hbar$ and $\hbar > 0$.

In the next section, we will consider a few specific sets of functions as \mathcal{X}_\hbar and apply the argument in this section to them.

Remark 6.1. The condition above on \mathcal{X}_\hbar is equivalent to the condition that the normalized family

$$\widetilde{\mathcal{X}}_\hbar = \{ \varphi(x) = \psi(\hbar^{1/2-\theta}x) \in C^\infty(\mathbb{R}^{2d}) \mid \psi \in \mathcal{X}_\hbar \} \quad \text{for } \hbar > 0 \quad (6.2)$$

are uniformly bounded in the (uniform) C^∞ topology and supported in a fixed bounded subset of \mathbb{R}^{2d} .

Recall the transformations

$$\mathcal{B}_x : L^2(\mathbb{R}_x^{2d}) \rightarrow L^2(\mathbb{R}_x^{2d} \oplus \mathbb{R}_\xi^{2d}), \quad \mathcal{B}_x^* : L^2(\mathbb{R}_x^{2d} \oplus \mathbb{R}_\xi^{2d}) \rightarrow L^2(\mathbb{R}_x^{2d})$$

and

$$\mathcal{P}_x := \mathcal{B}_x^* \circ \mathcal{B}_x : L^2(\mathbb{R}_x^{2d} \oplus \mathbb{R}_\xi^{2d}) \rightarrow L^2(\mathbb{R}_x^{2d} \oplus \mathbb{R}_\xi^{2d}),$$

which are defined in (5.11) and (5.12) as slight modifications of the Bargmann transform \mathcal{B}_\hbar , its adjoint \mathcal{B}_\hbar^* and the Bargmann projector \mathcal{P}_\hbar in the case $D = 2d$. Notice that the operators \mathcal{B}_x , \mathcal{B}_x^* and \mathcal{P}_x depend on the parameter \hbar (and hence on N).

Below we write $\mathcal{M}(\varphi)$ for the multiplication operator by a function φ . Since $\hbar^{1/2-\theta} \gg \hbar^{1/2}$ for small \hbar , the functions in \mathcal{X}_\hbar are very smooth (or flat) viewed in the scale of the wave packet $\phi_{x,\xi}(\cdot)$ used in the Bargmann transform \mathcal{B}_\hbar . This observation naturally leads to the following few statements.

For each $\psi \in \mathcal{X}_\hbar$, let $\mathcal{M}^{\text{lift}}(\psi) = \mathcal{B}_x \circ \mathcal{M}(\psi) \circ \mathcal{B}_x^*$ be the lift of the multiplication operator $\mathcal{M}(\psi)$ with respect to the (modified) Bargmann transform \mathcal{B}_x . Then it is approximated by the multiplication by the function $\psi \circ \pi$ with $\pi(x, \xi) := x$.

Lemma 6.2. *There exists a constant $C > 0$ such that, for any $\hbar > 0$ and $\psi \in \mathcal{X}_\hbar$, we have*

$$\| \mathcal{M}^{\text{lift}}(\psi) - \mathcal{M}(\psi \circ \pi) \circ \mathcal{P}_x \|_{L^2(\mathbb{R}^{2d} \oplus \mathbb{R}^{2d}, (\mathcal{W}_\hbar^r)^2)} < C\hbar^\theta \quad (6.3)$$

and

$$\| \mathcal{M}^{\text{lift}}(\psi) - \mathcal{P}_x \circ \mathcal{M}(\psi \circ \pi) \|_{L^2(\mathbb{R}^{2d} \oplus \mathbb{R}^{2d}, (\mathcal{W}_\hbar^r)^2)} < C\hbar^\theta.$$

Consequently we have

$$\| [\mathcal{P}_x, \mathcal{M}(\psi \circ \pi)] \|_{L^2(\mathbb{R}^{2d} \oplus \mathbb{R}^{2d}, (\mathcal{W}_\hbar^r)^2)} < C\hbar^\theta \quad (6.4)$$

where $[A, B]$ denotes the commutator of two operators: $[A, B] = A \circ B - B \circ A$. The same statement holds true with \mathcal{W}_\hbar^r replaced by $\mathcal{W}_\hbar^{r,\pm}$.

Proof. The kernel of the operator

$$\mathcal{M}^{\text{lift}}(\psi) - \mathcal{M}(\psi \circ \pi) \circ \mathcal{P}_x = \mathcal{B}_x \circ \mathcal{M}(\psi) \circ \mathcal{B}_x^* - \mathcal{M}(\psi \circ \pi) \circ \mathcal{B}_x \circ \mathcal{B}_x^*$$

is written

$$K(x, \xi; x', \xi') = (2\pi\hbar)^{-d} \int e^{(i/\hbar)(\xi(y-x) - \xi'(x'-y))} \cdot e^{-|y-x|^2/4\hbar - |y-x'|^2/4\hbar} (\psi(y) - \psi(x)) dy.$$

1610 We apply integration by parts, using the differential operator

$$L = \frac{1 - i(\xi - \xi')\partial_y}{1 + \hbar^{-1}(\xi - \xi')^2},$$

which satisfies $L^\nu (e^{(i/\hbar)(\xi(y-x) - \xi'(x'-y))}) = e^{(i/\hbar)(\xi(y-x) - \xi'(x'-y))}$ for ν times. Then we get

$$K(x, \xi; x', \xi') = (2\pi\hbar)^d \int e^{(i/\hbar)(\xi(y-x) - \xi'(x'-y))} \cdot ({}^tL)^\nu \left(e^{-|y-x|^2/4\hbar - |y-x'|^2/4\hbar} (\psi(y) - \psi(x)) \right) dy$$

where ${}^tL = (1 - i(\xi - \xi')\partial_y)/(1 + \hbar^{-1}(\xi - \xi')^2)$ is the transpose of L . Using the conditions (C1) and (C2) on the family \mathcal{X}_\hbar and, in particular, the estimate

$$|\psi(x) - \psi(y)| \cdot \langle \hbar^{-1/2}|x - y| \rangle^{-1} < C\hbar^\theta \quad (6.5)$$

that follows from the condition (C2), we see that the integrand is bounded in absolute value by

$$C\hbar^\theta \cdot \langle \hbar^{-1/2}|\xi - \xi'| \rangle^{-\nu} \cdot \langle \hbar^{-1/2}|x - y| \rangle^{-\nu} \cdot \langle \hbar^{-1/2}|x' - y| \rangle^{-\nu}.$$

1615 Hence, letting ν large, we obtain

$$|K(x, \xi; x', \xi')| \leq C\hbar^\theta \cdot \langle \hbar^{-1/2}|\xi - \xi'| \rangle^{-\nu} \cdot \langle \hbar^{-1/2}|x - x'| \rangle^{-\nu}.$$

This estimate for sufficiently large ν and Lemma 5.17 give the first inequality (6.3). We can get the second inequality in the same manner. \square

Corollary 6.3. *The multiplication operator $\mathcal{M}(\psi)$ by $\psi \in \mathcal{X}_\hbar$ extends to a bounded operator on $\mathcal{H}_\hbar^r(\mathbb{R}^{2d})$ and, for the operator norm, we have $\|\mathcal{M}(\psi)\|_{\mathcal{H}_\hbar^r(\mathbb{R}^{2d})} < \|\psi\|_\infty + C\hbar^\theta$ for all $\psi \in \mathcal{X}_\hbar$, with a constant $C > 0$ independent of \hbar and ψ .*

Proof. From the commutative diagram (4.12), the operator norm of $\mathcal{M}(\psi) : \mathcal{H}_\hbar^r(\mathbb{R}^{2d}) \rightarrow \mathcal{H}_\hbar^r(\mathbb{R}^{2d})$ coincides with that of the operator

$$\mathcal{M}^{\text{lift}}(\psi) : L^2(\mathbb{R}^{2d} \oplus \mathbb{R}^{2d}, (\mathcal{W}_\hbar^r)^2) \rightarrow L^2(\mathbb{R}^{2d} \oplus \mathbb{R}^{2d}, (\mathcal{W}_\hbar^r)^2)$$

restricted to the image of $\mathcal{B}_x : \mathcal{H}_\hbar^r(\mathbb{R}^{2d}) \rightarrow L^2(\mathbb{R}^{2d} \oplus \mathbb{R}^{2d}, (\mathcal{W}_\hbar^r)^2)$. Hence the claim follows from Lemma 6.2. \square

1625 **Corollary 6.4.** *There exists $C > 0$, such that for every $\hbar > 0$, for $u, v \in \mathcal{H}_\hbar^r(\mathbb{R}^{2d})$ and $\psi \in \mathcal{X}_\hbar$ we have*

$$\left| (u, \psi \cdot v)_{\mathcal{H}_\hbar^r(\mathbb{R}^{2d})} - (\overline{\psi} \cdot u, v)_{\mathcal{H}_\hbar^r(\mathbb{R}^{2d})} \right| \leq C\hbar^\theta \cdot \|u\|_{\mathcal{H}_\hbar^r(\mathbb{R}^{2d})} \cdot \|v\|_{\mathcal{H}_\hbar^r(\mathbb{R}^{2d})}$$

where $\mathcal{O}(\hbar^\theta)$ denotes a term whose absolute value is bounded by $C\hbar^\theta$ with

Proof. This is a consequence of the equality

$$\begin{aligned} (u, \psi \cdot v)_{\mathcal{H}_\hbar^r(\mathbb{R}^{2d})} &= (\mathcal{B}_x u, \mathcal{M}^{\text{lift}}(\psi) \circ \mathcal{B}_x v)_{L^2(\mathbb{R}^{2d} \oplus \mathbb{R}^{2d}, (\mathcal{W}_\hbar^r)^2)} \\ &= (\mathcal{B}_x u, \mathcal{M}(\psi \circ \pi) \circ \mathcal{B}_x v)_{L^2(\mathbb{R}^{2d} \oplus \mathbb{R}^{2d}, (\mathcal{W}_\hbar^r)^2)} + \mathcal{O}(\hbar^\theta) \cdot \|u\|_{\mathcal{H}_\hbar^r(\mathbb{R}^{2d})} \cdot \|v\|_{\mathcal{H}_\hbar^r(\mathbb{R}^{2d})} \end{aligned}$$

and the parallel estimate for $(\overline{\psi} \cdot u, v)_{\mathcal{H}_\hbar^r(\mathbb{R}^{2d})}$, which follow from Lemma 6.2. \square

1630 *Remark 6.5.* The statements of Corollary 6.3 and Corollary 6.4 above hold true with $\mathcal{H}_\hbar^r(\mathbb{R}^{2d})$ replaced by $\mathcal{H}_\hbar^{r,\pm}(\mathbb{R}^{2d})$ and the proofs are completely parallel. This is the case for a few statements (Lemma 6.15, Proposition 6.19, Lemma 6.22 and Corollary 6.23, precisely) in this section.

Next we recall the projection operators $t_\hbar^{(k)}$ for $0 \leq k \leq n$ in (5.26) and \tilde{t}_\hbar in (5.27). We henceforth assume

$$r > n + 2 + 4d \tag{6.6}$$

1635 for the choice of r . (This is a little more restrictive than (5.25).)

Lemma 6.6. *There exists a constant $C > 0$ such that for any $\hbar > 0$, $\psi \in \mathcal{X}_\hbar$ and $0 \leq k \leq n$,*

$$\left\| \left[\mathcal{M}(\psi), t_\hbar^{(k)} \right] \right\|_{\mathcal{H}_\hbar^{r,-}(\mathbb{R}^{2d}) \rightarrow \mathcal{H}_\hbar^{r,+}(\mathbb{R}^{2d})} < C\hbar^\theta$$

Proof. From (5.13) and the definition of the operator $t_\hbar^{(k)}$, we have

$$\begin{aligned} \mathcal{M}(\psi) \circ t_\hbar^{(k)} &= \mathcal{B}_x^* \circ \mathcal{M}^{\text{lift}}(\psi) \circ \mathcal{B}_x \circ \mathcal{U} \circ (\text{Id} \otimes T^{(k)}) \circ \mathcal{U}^{-1} \\ &= \mathcal{B}_x^* \circ \mathcal{M}^{\text{lift}}(\psi) \circ \Phi^* \circ (\mathcal{B}_{\nu_q} \otimes \mathcal{B}_{\zeta_p}) \circ (\text{Id} \otimes T^{(k)}) \circ (\mathcal{B}_{\nu_q}^* \otimes \mathcal{B}_{\zeta_p}^*) \circ (\Phi^*)^{-1} \circ \mathcal{B}_x \\ &= \mathcal{B}_x^* \circ \mathcal{M}^{\text{lift}}(\psi) \circ \Phi^* \circ (\mathcal{P}_{\nu_q} \otimes \mathcal{T}_\hbar^{(k)}) \circ (\Phi^*)^{-1} \circ \mathcal{B}_x \end{aligned}$$

and, similarly

$$t_\hbar^{(k)} \circ \mathcal{M}(\psi) = \mathcal{B}_x^* \circ \Phi^* \circ (\mathcal{P}_{\nu_q} \otimes \mathcal{T}_\hbar^{(k)}) \circ (\Phi^*)^{-1} \circ \mathcal{M}^{\text{lift}}(\psi) \circ \mathcal{B}_x.$$

Thus, from Lemma 6.2, it is enough to show that

$$\left\| \left[\mathcal{M}(\psi \circ \pi), \Phi^* \circ (\mathcal{P}_{\nu_q} \otimes \mathcal{T}_\hbar^{(k)}) \circ (\Phi^*)^{-1} \right] \right\|_{L^2(\mathbb{R}^{2d} \oplus \mathbb{R}^{2d}, (\mathcal{W}_\hbar^{r,-})^2) \rightarrow L^2(\mathbb{R}^{2d} \oplus \mathbb{R}^{2d}, (\mathcal{W}_\hbar^{r,+})^2)} < C\hbar^\theta. \tag{6.7}$$

From Proposition 4.3 and Lemma 4.16 page 70, if we write $K(x, \xi; x', \xi')$ for the kernel of the operator $\Phi^* \circ (\mathcal{P}_{\nu_q} \otimes \mathcal{T}_h^{(k)}) \circ (\Phi^*)^{-1}$, it satisfies

$$\frac{\mathcal{W}_h^{r,+}(x, \xi)}{\mathcal{W}_h^{r,-}(x', \xi')} |K(x, \xi; x', \xi')| \quad (6.8)$$

$$\leq C_\nu \langle \hbar^{-1/2} |\nu_q - \nu'_q| \rangle^{-\nu} \langle \hbar^{-1/2} |\nu_p - \nu'_p| \rangle^{-\nu} \langle \hbar^{-1/2} |(\zeta_p, \zeta_q)| \rangle^{-(r-k)} \langle \hbar^{-1/2} |(\zeta'_p, \zeta'_q)| \rangle^{-(r-k)} \quad (6.9)$$

$$\leq C'_\nu \langle \hbar^{-1/2} |(\nu_q, \nu_p) - (\nu'_q, \nu'_p)| \rangle^{-\nu} \langle \hbar^{-1/2} |(\zeta_p, \zeta_q) - (\zeta'_p, \zeta'_q)| \rangle^{-(r-k)} \quad (6.10)$$

for arbitrarily large $\nu > 0$, where $C_\nu, C'_\nu > 0$ are constants independent of \hbar . The variables $\nu_q, \nu_p, \zeta_q, \zeta_p$ (resp. $\nu'_q, \nu'_p, \zeta'_q, \zeta'_p$) are the coordinates for (x, ξ) (resp. (x', ξ')) introduced in (5.9) and $|\cdot|$ denotes the Euclidean norms. The kernel $\tilde{K}(x, \xi; x', \xi')$ of the commutator in (6.7) is then

$$\tilde{K}(x, \xi; x', \xi') = (\psi(x) - \psi(x')) \cdot K(x, \xi; x', \xi').$$

By (6.10) with sufficiently large ν and (6.5), we get

$$\frac{\mathcal{W}_h^{r,+}(x, \xi)}{\mathcal{W}_h^{r,-}(x', \xi')} |\tilde{K}(x, \xi; x', \xi')| \leq C \hbar^\theta \cdot \langle \hbar^{-1/2} |(x, \xi) - (y, \eta)| \rangle^{-(r-k-1)}.$$

Hence we obtain the required estimate by Schur inequality (69), noting that $r - k - 1 \geq r - n - 1 > 4d$ from the assumption (6.6). \square

Corollary 6.7. *There exists a constant $C > 0$, such that*

$$\left\| \left[\mathcal{M}(\psi), t_h^{(k)} \right] \right\|_{\mathcal{H}_h^r(\mathbb{R}^{2d})} < C \hbar^\theta \quad \text{for } 0 \leq k \leq n$$

and

$$\left\| \left[\mathcal{M}(\psi), \tilde{t}_h \right] \right\|_{\mathcal{H}_h^r(\mathbb{R}^{2d})} < C \hbar^\theta$$

for any $\psi \in \mathcal{X}_h$.

Proof. The former claim is an immediate consequence of the last lemma. Since $\tilde{t}_h = \text{Id} - \sum_{k=0}^n t_h^{(k)}$ by definition, the latter claim follows. \square

6.2 Localization of the projection operator $t_h^{(k)}$ and estimates on trace norm

We consider the operator $t_h^{(k)} : \mathcal{H}_h^r(\mathbb{R}_x^{2d}) \rightarrow \mathcal{H}_h^r(\mathbb{R}_x^{2d})$ defined in (5.26). We write $t_h^{(k)} = \sum_{|\alpha|=k} t_h^{(\alpha)}$ with setting (as in (5.26)):

$$t_h^{(\alpha)} := \mathcal{U} \circ \left(\text{Id}_{L^2(\mathbb{R}_{\nu_q}^d)} \otimes T^{(\alpha)} \right) \circ \mathcal{U}^{-1} \quad : \mathcal{H}_h^r(\mathbb{R}_x^{2d}) \rightarrow \mathcal{H}_h^r(\mathbb{R}_x^{2d})$$

for a multi-index $\alpha \in \mathbb{N}^d$. Here $\mathcal{U} = \mathcal{B}_x^* \circ \Phi^* \circ (\mathcal{B}_{\nu_q} \otimes \mathcal{B}_{\zeta_p})$ is the operator defined in (5.15) and $T^{(\alpha)} : H_{\hbar}^r(\mathbb{R}_{\zeta_p}^d) \rightarrow H_{\hbar}^r(\mathbb{R}_{\zeta_p}^d)$ is the rank one projector defined in (4.45).

1655 The next Lemma decomposes the projector $t_{\hbar}^{(\alpha)}$ as an integral of localized rank one projectors. Below $\|\cdot\|_{\text{Tr}}$ denotes the trace norm of an operator on $\mathcal{H}_{\hbar}^r(\mathbb{R}_x^{2d})$.

Lemma 6.8. *For $\nu \in \mathbb{R}^{2d}$, let*

$$\hat{\pi}_{\alpha}(\nu) := \mathcal{U} \circ \left((\cdot, \phi_{\nu_q, \nu_p}) \otimes (\phi_{\nu_q, \nu_p}, \cdot) \right) \otimes T^{(\alpha)} \circ \mathcal{U}^{-1} : \mathcal{H}_{\hbar}^r(\mathbb{R}_x^{2d}) \rightarrow \mathcal{H}_{\hbar}^r(\mathbb{R}_x^{2d}) \quad (6.11)$$

where ϕ_{ν_q, ν_p} is the wave packet defined in (4.1) and $(\cdot, \phi_{\nu_q, \nu_p}) \otimes (\phi_{\nu_q, \nu_p}, \cdot) : L^2(\mathbb{R}_{\nu_q}^d) \rightarrow L^2(\mathbb{R}_{\nu_q}^d)$ denotes the rank one projection operator defined by

$$\{(\cdot, \phi_{\nu_q, \nu_p}) \otimes (\phi_{\nu_q, \nu_p}, \cdot)\} u = (\phi_{\nu_q, \nu_p}, u)_{L^2} \cdot \phi_{\nu_q, \nu_p}.$$

1660 The operator $\hat{\pi}_{\alpha}(\nu)$ is a rank one projector satisfying $\|\hat{\pi}_{\alpha}(\nu)\|_{\text{Tr}} \leq \|\hat{\pi}_{\alpha}(\nu)\|_{\mathcal{H}_{\hbar}^r(\mathbb{R}^{2d})} \leq C$ with C independent of \hbar , and depends smoothly on $\nu \in \mathbb{R}^{2d}$. We have (in strong operator topology)

$$t_{\hbar}^{(\alpha)} = \int_{\mathbb{R}^{2d}} \hat{\pi}_{\alpha}(\nu) \frac{d\nu}{(2\pi\hbar)^d} \quad (6.12)$$

Proof. We have

$$\begin{aligned} t_{\hbar}^{(\alpha)} &= \mathcal{U} \circ \left(\text{Id}_{|L^2(\mathbb{R}_{\nu_q}^d)} \otimes T^{(\alpha)} \right) \circ \mathcal{U}^{-1} \\ &= \mathcal{U} \circ \left((\mathcal{B}_{\nu_q}^* \mathcal{B}_{\nu_q}) \otimes T^{(\alpha)} \right) \circ \mathcal{U}^{-1} \end{aligned} \quad (6.13)$$

Since

$$\mathcal{B}_{\nu_q}^* \mathcal{B}_{\nu_q} = \int_{\mathbb{R}^{2d}} (\cdot, \phi_{\nu_q, \nu_p}) \otimes (\phi_{\nu_q, \nu_p}, \cdot) \frac{d\nu}{(2\pi\hbar)^d}$$

1665 we get (6.12). □

The next lemma gives an estimate on the lift of the localized projection operator $\hat{\pi}_{\alpha}(\nu)$ with respect to the Bargmann transform.

Lemma 6.9. *The lifted operator $\mathcal{B}_x \circ \hat{\pi}_{\alpha}(\nu) \circ \mathcal{B}_x^*$ is written as an integral operator*

$$(\mathcal{B}_x \circ \hat{\pi}_{\alpha}(\nu) \circ \mathcal{B}_x^*) u(x_1, \xi_1) = \int K(x_1, \xi_1; x_2, \xi_2) u(x_2, \xi_2) \frac{dx_2 d\xi_2}{(2\pi\hbar)^d}.$$

The kernel satisfies

$$\begin{aligned} &\frac{\mathcal{W}^r(x_1, \xi_1)}{\mathcal{W}^r(x_2, \xi_2)} \cdot |K(x_1, \xi_1; x_2, \xi_2)| \\ &\leq C_m \langle \hbar^{-1/2} |\nu_1 - \nu| \rangle^{-m} \cdot \langle \hbar^{-1/2} |\nu_2 - \nu| \rangle^{-m} \cdot \langle \hbar^{-1/2} |\zeta_1| \rangle^{-(r-k)} \cdot \langle \hbar^{-1/2} |\zeta_2| \rangle^{-(r-k)} \end{aligned}$$

1670 *for arbitrarily large $m > 0$ with a uniform constant $C_m > 0$, where (ν_1, ζ_1) (resp. (ν_2, ζ_2)) is the coordinates of (x_1, ξ_1) (resp. (x_2, ξ_2)) defined in Proposition 2.15. (Note that $r - k \geq 4d + 2$ from the choice of r in (6.6).)*

Proof. Since we have the expression (6.13) of the operator $\hat{\pi}_\alpha(\nu)$, the conclusion readily follows from Lemma 4.16 and Proposition 4.3. \square

Lemma 6.10. For $\psi \in \mathcal{X}_\hbar$ and $\alpha \in \mathbb{N}^d$, the operator

$$\mathcal{M}(\psi) \circ t_\hbar^{(\alpha)} : \mathcal{H}_\hbar^r(\mathbb{R}^{2d}) \rightarrow \mathcal{H}_\hbar^r(\mathbb{R}^{2d})$$

1675 is a trace class operator. We have the following estimates on the operator norm and the trace class norm (as operators on $\mathcal{H}_\hbar^r(\mathbb{R}^{2d})$): There exists a constant $C > 0$, independent of $\psi \in \mathcal{X}_\hbar$, $\hbar > 0$ and $\alpha \in \mathbb{N}^d$, such that

$$\left\| \mathcal{M}(\psi) \circ t_\hbar^{(\alpha)} - \int_{\mathbb{R}^{2d}} \psi(\nu) \hat{\pi}_\alpha(\nu) \frac{d\nu}{(2\pi\hbar)^d} \right\|_{\mathcal{H}_\hbar^r(\mathbb{R}^{2d})} \leq C\hbar^\theta \quad (6.14)$$

$$\left\| \mathcal{M}(\psi) \circ t_\hbar^{(\alpha)} - \int_{\mathbb{R}^{2d}} \psi(\nu) \hat{\pi}_\alpha(\nu) \frac{d\nu}{(2\pi\hbar)^d} \right\|_{\text{Tr}} \leq C\hbar^{-2\theta d + \theta} \quad (6.15)$$

with $\hat{\pi}_\alpha(\nu)$ defined in (6.11). The same statement holds true for $t_\hbar^{(\alpha)} \circ \mathcal{M}(\psi)$.

1680 *Proof.* Let $\psi \in \mathcal{X}_\hbar$. We have

$$\mathcal{M}(\psi) \circ t_\hbar^{(\alpha)} - \int_{\mathbb{R}^{2d}} \psi(\nu) \hat{\pi}_\alpha(\nu) \frac{d\nu}{(2\pi\hbar)^d} \stackrel{(6.12)}{=} \int_{\mathbb{R}^{2d}} (\mathcal{M}(\psi) - \psi(\nu)) \hat{\pi}_\alpha(\nu) \frac{d\nu}{(2\pi\hbar)^d}$$

Let $T(\nu) := (\mathcal{M}(\psi) - \psi(\nu)) \hat{\pi}_\alpha(\nu)$ for every $\nu \in \mathbb{R}^{2d}$. From Lemma 6.9 on the kernel of the lift of $\hat{\pi}_\alpha(\nu)$, we deduce the estimates

$$\|(T(x))^* T(y)\|_{\mathcal{H}_\hbar^r(\mathbb{R}^{2d})} \leq C_m \hbar^\theta \left(\frac{|x-y|}{\sqrt{\hbar}} \right)^{-m}, \quad \|T(x) (T(y))^*\|_{\mathcal{H}_\hbar^r(\mathbb{R}^{2d})} \leq C_m \hbar^\theta \left(\frac{|x-y|}{\sqrt{\hbar}} \right)^{-m} \quad (6.16)$$

for any $m > 0$ with a constant C_m uniform for $x, y \in M$ and $\hbar > 0$.

We have from (6.16) that

$$\sup_x \int_{\mathbb{R}^{2d}} \|(T(x))^* T(y)\|_{\mathcal{H}_\hbar^r(\mathbb{R}^{2d})}^{1/2} d\mu(y) \leq C\hbar^\theta, \quad \sup_x \int_{\mathbb{R}^{2d}} \|T(x) (T(y))^*\|_{\mathcal{H}_\hbar^r(\mathbb{R}^{2d})}^{1/2} d\mu(y) \leq C\hbar^\theta$$

1685 with setting $d\mu(\nu) := \frac{d\nu}{(2\pi\hbar)^d}$. We now apply the integral version of the Cotlar-Stein Lemma¹⁵ to the integral $\int_{\mathbb{R}^{2d}} T(\nu) d\mu(\nu)$ of operators and deduce that

$$\left\| \int_{\mathbb{R}^{2d}} (\mathcal{M}(\psi) - \psi(\nu)) \hat{\pi}_\alpha(\nu) \frac{d\nu}{(2\pi\hbar)^d} \right\|_{\mathcal{H}_\hbar^r(\mathbb{R}^{2d})} = \left\| \int_{\mathbb{R}^{2d}} T(\nu) d\mu(\nu) \right\|_{\mathcal{H}_\hbar^r(\mathbb{R}^{2d})} \leq C\hbar^\theta.$$

15

Lemma 6.11. “Integral version of the Cotlar-Stein Lemma”: If $(T(x))_x$ is a continuous family of bounded operators, if $d\mu(x)$ is a smooth measure, if $A := \sup_x \int T(x) T(y)^* d\mu(y) < \infty$ and $B := \sup_x \int T(x)^* T(y) d\mu(y) < \infty$ then $T(x)u$ is integrable for every u and $\|\int T(x) d\mu(x)\| \leq \sqrt{AB}$.

This gives (6.14). It is easy to get (6.15). Since $(\mathcal{M}(\psi) - \psi(\nu)) \hat{\pi}_\alpha(\nu)$ is a rank one operator, we have $\|(\mathcal{M}(\psi) - \psi(\nu)) \hat{\pi}_\alpha(\nu)\|_{\text{Tr}} = \|(\mathcal{M}(\psi) - \psi(\nu)) \hat{\pi}_\alpha(\nu)\|$ and therefore we get (6.15) from the triangle inequality. \square

1690 **Corollary 6.12.** *There exists a constant $C > 0$, independent of $\psi \in \mathcal{X}_\hbar$, $\hbar > 0$ and $0 \leq k \leq n$, such that*

$$\left| \text{Tr} \left(\mathcal{M}(\psi) \circ t_\hbar^{(k)} \right) - \frac{r(k, d)}{(2\pi\hbar)^d} \int \psi dx \right| \leq C\hbar^{-2\theta d + \theta}$$

$$\|\mathcal{M}(\psi) \circ t_\hbar^{(k)}\|_{\text{Tr}} \leq \frac{C}{(2\pi\hbar)^d} \int |\psi| dx$$

We can get the following statements by slightly modifying the argument in the proof of Lemma 6.10.

1695 **Corollary 6.13.** *There exists a constant $C > 0$, such that, for $0 \leq k \leq n$ and $\psi \in \mathcal{X}_\hbar$,*

$$\|\mathcal{M}(\psi) \circ t_\hbar^{(k)} : \mathcal{H}_\hbar^{r,-}(\mathbb{R}^{2d}) \rightarrow \mathcal{H}_\hbar^{r,+}(\mathbb{R}^{2d})\|_{\text{Tr}} \leq \frac{C}{(2\pi\hbar)^d} \int |\psi| dx + C\hbar^{-\theta d + \theta}$$

Corollary 6.14. *There exists a constant $C > 0$, such that*

$$\left\| \left[\mathcal{M}(\psi), t_\hbar^{(k)} \right] : \mathcal{H}_\hbar^{r,-}(\mathbb{R}^{2d}) \rightarrow \mathcal{H}_\hbar^{r,+}(\mathbb{R}^{2d}) \right\|_{\text{Tr}} < C\hbar^{-\theta d + \theta} \quad \text{for } 0 \leq k \leq n$$

for any $\psi \in \mathcal{X}_\hbar$.

6.3 Truncation operations in the phase space

1700 In order to truncate functions in the phase space $T^*\mathbb{R}^{2d} = \mathbb{R}_x^{2d} \oplus \mathbb{R}_\xi^{2d}$, we consider the smooth function

$$Y_\hbar : T^*\mathbb{R}^{2d} \rightarrow [0, 1], \quad Y_\hbar(x, \xi) = \chi(\hbar^{2\theta-1/2}|(x, \xi)|) \quad (6.17)$$

with $\chi : \mathbb{R} \rightarrow [0, 1]$ a C^∞ function satisfying (4.53), and then introduce the operator

$$\mathcal{Y}_\hbar : L^2(\mathbb{R}^{2d}) \rightarrow L^2(\mathbb{R}^{2d}), \quad \mathcal{Y}_\hbar = \mathcal{B}_x^* \circ \mathcal{M}(Y_\hbar) \circ \mathcal{B}_x. \quad (6.18)$$

Note that the size ($\sim \hbar^{1/2-2\theta}$) of the support of the function Y_\hbar is much larger than the size ($\sim \hbar^{1/2-\theta}$) of the region on which the Bargmann transform of the functions in \mathcal{X}_\hbar concentrates in Setting I page 92, when $\hbar > 0$ is small.

1705 First of all, we show

Lemma 6.15. *The operator \mathcal{Y}_h extends naturally to a bounded operator on $\mathcal{H}_h^r(\mathbb{R}^{2d})$ and we have*

$$\|\mathcal{Y}_h\|_{\mathcal{H}_h^r(\mathbb{R}^{2d})} < 1 + C\hbar^\theta$$

and

$$\|[\mathcal{Y}_h, \mathcal{M}(\psi)]\|_{\mathcal{H}_h^r(\mathbb{R}^{2d})} < C\hbar^\theta \quad \text{for any } \psi \in \mathcal{X}_h$$

1710 *with some positive constants C independent of \hbar and ψ .*

Proof. It is enough to show

$$\|\mathcal{P}_x \circ \mathcal{M}(Y_h) : L^2(\mathbb{R}^{2d} \oplus \mathbb{R}^{2d}, (\mathcal{W}_h^r)^2) \rightarrow L^2(\mathbb{R}^{2d} \oplus \mathbb{R}^{2d}, (\mathcal{W}_h^r)^2)\| < 1 + C\hbar^\theta$$

and

$$\|[\mathcal{P}_x \circ \mathcal{M}(Y_h) \circ \mathcal{P}_x, \mathcal{M}^{\text{lift}}(\psi)] : L^2(\mathbb{R}^{2d} \oplus \mathbb{R}^{2d}, (\mathcal{W}_h^r)^2) \rightarrow L^2(\mathbb{R}^{2d} \oplus \mathbb{R}^{2d}, (\mathcal{W}_h^r)^2)\| < C\hbar^\theta.$$

Note that we have $\|[\mathcal{P}_x, \mathcal{M}(Y_h)]\|_{L^2(\mathbb{R}^{2d} \oplus \mathbb{R}^{2d}, (\mathcal{W}_h^r)^2)} \leq C\hbar^\theta$ by a simple estimate on the kernel. The first claim is a consequence of this estimate. For the second, we use Lemma 6.2. \square

1715 The next lemma tells roughly that the truncation operator \mathcal{Y}_h , (6.18), hardly affect the projection operators $t_h^{(k)}$, $0 \leq k \leq n$, defined in (5.26), if we view it in the anisotropic Sobolev spaces.

Lemma 6.16. *For $0 \leq k \leq n$ and $\psi \in \mathcal{X}_h$, we have*

$$\|(\text{Id} - \mathcal{Y}_h) \circ \mathcal{M}(\psi) \circ t_h^{(k)}\|_{\mathcal{H}_h^{r,-}(\mathbb{R}^{2d}) \rightarrow \mathcal{H}_h^{r,+}(\mathbb{R}^{2d})} < C\hbar^\theta$$

and

$$\|t_h^{(k)} \circ (\text{Id} - \mathcal{Y}_h) \circ \mathcal{M}(\psi)\|_{\mathcal{H}_h^{r,-}(\mathbb{R}^{2d}) \rightarrow \mathcal{H}_h^{r,+}(\mathbb{R}^{2d})} < C\hbar^\theta$$

1720 *with some constant $C > 0$ independent of \hbar and ψ .*

Proof. For the proof of the first inequality, it suffices to show the estimate

$$\|A : L^2(\mathbb{R}^{2d} \oplus \mathbb{R}^{2d}, (\mathcal{W}_h^{r,-})^2) \rightarrow L^2(\mathbb{R}^{2d} \oplus \mathbb{R}^{2d}, (\mathcal{W}_h^{r,+})^2)\| < C\hbar^\theta$$

for the operator

$$A := \mathcal{P}_x \circ (\text{Id} - \mathcal{M}(Y_h)) \circ \mathcal{P}_x \circ \mathcal{M}^{\text{lift}}(\psi) \circ \Phi^* \circ (\mathcal{P}_{\nu_q} \otimes \mathcal{T}_h^{(k)}) \circ (\Phi^*)^{-1}.$$

Recall that we already have the estimates (4.7) and (6.9) respectively for the kernel of the operator \mathcal{P}_x and $\Phi^* \circ (\mathcal{P}_{\nu_q} \otimes \mathcal{T}_h^{(k)}) \circ (\Phi^*)^{-1}$. Using those estimates with the property (5.28) 1725 of the escape function $\mathcal{W}_h^{r,\pm}$ and noting that

$$|(x, \xi)| \geq \hbar^{1/2-2\theta} \quad \text{for } (x, \xi) \in \text{supp}(\text{Id} - Y_h) \quad (\text{resp. } |x| \leq 2\hbar^{1/2-\theta} \quad \text{for } x \in \text{supp}(\psi)),$$

we can estimate the kernel of the operator $\mathcal{M}(\mathcal{W}_h^{r,+}) \circ A \circ \mathcal{M}(\mathcal{W}_h^{r,-})^{-1}$ in absolute value and obtain the required estimate. The second inequality can be proved in the parallel manner. We omit the tedious details. \square

6.4 Prequantum transfer operators for non-linear transformations close to the identity

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In this subsection, we study the Euclidean prequantum transfer operators for diffeomorphisms defined on small open subsets on \mathbb{R}^{2d} and close to the identity map. Roughly we show that the action of those prequantum operators are close to the identity as an operator on $\mathcal{H}_\hbar^r(\mathbb{R}^{2d})$, though this is *not* true in the literal sense.

1735

In this subsection, we consider the following setting in addition to Setting I page 92:

Setting II: For every $\hbar > 0$, there exists a given set \mathcal{G}_\hbar of C^∞ diffeomorphisms

$$g : \mathbb{D}(\hbar^{1/2-2\theta}) \rightarrow g(\mathbb{D}(\hbar^{1/2-2\theta})) \subset \mathbb{R}^{2d}$$

such that every $g \in \mathcal{G}_\hbar$ satisfies

(G1) g is symplectic with respect to the symplectic form ω in (5.2),

(G2) $g(0) = 0$ and $\|Dg(0) - \text{Id}\| < C\hbar^{\beta(1/2-\theta)}$, and

(G3) $|\partial^\alpha g| < C_\alpha$ for any multiindices α .

where C and C_α are positive constants that does not depend on \hbar nor $g \in \mathcal{G}_\hbar$.

Remark 6.17. In the next section we will consider a few different sets of diffeomorphisms as \mathcal{G}_\hbar and apply the argument below. At this moment, the meaning of the bound $C\hbar^{\beta(1/2-\theta)}$ in the condition (G2) may not be clear. This is a consequence of the fact that the hyperbolic splitting (1.1) is β -Hölder continuous. The reason will become clear when we introduce a family of local coordinates on M in the beginning of the next section.

1740

For $g \in \mathcal{G}_\hbar$, we consider the Euclidean prequantum transfer operator \mathcal{L}_g defined in Subsection 5.1. Recall from Proposition 5.1 that this operator is of the form

$$\mathcal{L}_g : C_0^\infty(\mathbb{D}(\hbar^{1/2-2\theta})) \rightarrow C_0^\infty(g(\mathbb{D}(\hbar^{1/2-2\theta}))), \quad \mathcal{L}_g u(x) = e^{-(i/\hbar) \cdot \mathcal{A}_g(g^{-1}(x))} \cdot u(g^{-1}(x)) \quad (6.19)$$

with

$$\mathcal{A}_g(x) = \int_\gamma g^* \eta - \eta \quad (6.20)$$

1745

where γ is a path from the origin 0 to x and η is given in (5.3). For convenience, we take the origin 0 as a fixed point of reference. We first show the following lemma for the (action) function $\mathcal{A}_g(x)$.

Lemma 6.18. *If $g \in \mathcal{G}_\hbar$, we have*

$$|\partial^\alpha \mathcal{A}_g(x)| \leq C_\alpha \cdot \min\{|x|^{3-|\alpha|}, 1\}$$

for any multi-index α with $|\alpha| > 0$, where $C_\alpha > 0$ is a constant independent of \hbar .

1750 *Proof.* From the definition, we have

$$|\partial^\alpha \mathcal{A}_g(x)| \leq C_\alpha$$

for any multi-index α . Hence the conclusion holds obviously in the case $|\alpha| \geq 3$. The first derivatives of \mathcal{A}_g at 0 vanishes from the assumption $g(0) = 0$ in the condition (G2). Let us show that the second derivatives of \mathcal{A}_g also vanish. For this we use the coordinate $x = (p, q)$ in (5.1) and the notation

$$g^{-1}(p, q) = (g_p(p, q), g_q(p, q)),$$

1755 Note that we have $g_p(0, 0) = g_q(0, 0) = 0$ from condition (G2). Condition (G1) that g^{-1} is symplectic i.e. $(g^{-1})^*\omega = \omega$ writes

$$\frac{\partial g_q}{\partial q_i} \cdot \frac{\partial g_p}{\partial q_j} - \frac{\partial g_q}{\partial q_j} \cdot \frac{\partial g_p}{\partial q_i} = 0, \quad \frac{\partial g_q}{\partial p_i} \cdot \frac{\partial g_p}{\partial p_j} - \frac{\partial g_q}{\partial p_j} \cdot \frac{\partial g_p}{\partial p_i} = 0, \quad \frac{\partial g_q}{\partial q_j} \cdot \frac{\partial g_p}{\partial p_i} - \frac{\partial g_q}{\partial p_i} \cdot \frac{\partial g_p}{\partial q_j} = \begin{cases} 1 & (i = j); \\ 0 & (i \neq j). \end{cases}$$

Then we have

$$(g^{-1})^* \eta - \eta = \frac{1}{2} \sum_{i=1}^d \left(g_q \cdot \frac{\partial g_p}{\partial p_i} - g_p \cdot \frac{\partial g_q}{\partial p_i} - q_i \right) dp_i + \frac{1}{2} \sum_{i=1}^d \left(g_q \cdot \frac{\partial g_p}{\partial q_i} - g_p \cdot \frac{\partial g_q}{\partial q_i} + p_i \right) dq_i,$$

and we check that all of the first order partial derivatives of the coefficients of dp_i and dq_i of this one form vanish at the origin $0 \in \mathbb{R}^{2d}$. This implies that the second derivatives of \mathcal{A}_g vanishes at the origin. The claim of the lemma for the case $|\alpha| \leq 2$ then follows immediately. \square

In the next section, we consider the action of the operator \mathcal{L}_g on functions supported on $\mathbb{D}(\hbar^{1/2-\theta})$ (or sometimes on $\mathbb{D}(2\hbar^{1/2-\theta})$). For this reason, we take the C^∞ function

$$\chi_\hbar : \mathbb{R}^{2d} \rightarrow [0, 1], \quad \chi_\hbar(x) = \chi(\hbar^{-1/2+\theta}x/2) \tag{6.21}$$

with letting χ be a C^∞ function satisfying (4.53) and consider the operator

$$\mathcal{L}_g \circ \mathcal{M}(\chi_\hbar) : C^\infty(\mathbb{R}^{2d}) \rightarrow C_0^\infty(\mathbb{R}^{2d})$$

1765 instead of the operator \mathcal{L}_g itself. The next lemma is the main ingredient of this subsection, which tells roughly that the operator \mathcal{L}_g for $g \in \mathcal{G}_\hbar$ is close to the identity, under the effect of truncation by the operator \mathcal{Y}_\hbar .

Proposition 6.19. *There exist constants $C > 0$ and $\epsilon > 0$ such that, for any $\hbar > 0$ and $g \in \mathcal{G}_\hbar$, we have*

$$\|\mathcal{Y}_\hbar \circ (\mathcal{L}_g - \text{Id}) \circ \mathcal{M}(\chi_\hbar)\|_{\mathcal{H}_\hbar^r(\mathbb{R}^{2d})} < C\hbar^\epsilon \quad \text{and} \quad \|(\mathcal{L}_g - \text{Id}) \circ \mathcal{Y}_\hbar \circ \mathcal{M}(\chi_\hbar)\|_{\mathcal{H}_\hbar^r(\mathbb{R}^{2d})} < C\hbar^\epsilon.$$

Proof. The proof below is elementary but a little demanding. We will use the following estimate which follows from Lemma 6.18 and the conditions in Setting II on \mathcal{G}_\hbar : For any $x \in \mathbb{R}^{2d}$ with $|x| \leq \hbar^{1/2-\theta}$, it holds

$$\begin{aligned} |\mathcal{A}_g(x) - \mathcal{A}_g(0)| &\leq C|x|^3 < C\hbar^{3(1/2-\theta)}, & \|D\mathcal{A}_g(x)\| &\leq C|x|^2 < C\hbar^{2(1/2-\theta)}, \\ \|Dg(x) - \text{Id}\| &\leq C\hbar^{\beta(1/2-\theta)} \quad \text{and} & |g(x) - x| &\leq C|x|^{1+\beta} < C\hbar^{(1+\beta)(1/2-\theta)} \end{aligned}$$

1770 with C a constant independent of $\hbar > 0$ and $g \in \mathcal{G}_\hbar$. Also we note that, if $(x, \xi) \in \text{supp } Y_\hbar$, we have $|(x, \xi)| \leq 2\hbar^{1/2-2\theta}$ and, in particular, $|\xi| \leq 2\hbar^{1/2-2\theta}$.

From Corollary 5.18, the first claim follows if we show

$$\|\mathcal{M}(Y_\hbar) \circ \mathcal{B}_x \circ (\mathcal{L}_g - \text{Id}) \circ \mathcal{M}(\chi_\hbar) \circ \mathcal{B}_x^*\|_{L^2(\mathbb{R}^{2d} \oplus \mathbb{R}^{2d}, (\mathcal{W}_\hbar^r)^2)} \leq C\hbar^\epsilon. \quad (6.22)$$

Recalling the definition of the operators \mathcal{B}_x and \mathcal{B}_x^* in (5.11), we write the operator $\mathcal{B}_x \circ (\mathcal{L}_g - \text{Id}) \circ \mathcal{M}(\chi_\hbar) \circ \mathcal{B}_x^*$ as an integral operator of the form

$$(\mathcal{B}_x \circ (\mathcal{L}_g - \text{Id}) \circ \mathcal{M}(\chi_\hbar) \circ \mathcal{B}_x^* u)(x, \xi) = \int K(2^{-1/2}x, 2^{1/2}\xi; 2^{-1/2}x', 2^{1/2}\xi') u(x', \xi') \frac{dx' d\xi'}{(2\pi\hbar)^{2d}},$$

where

$$\begin{aligned} K(x, \xi; x', \xi') &= a_D^2 \int e^{(i/\hbar)\xi((x/2)-y) + (i/\hbar)\xi'(y-(x'/2))} \cdot e^{-|y-x|^2/(2\hbar) - |y-x'|^2/(2\hbar)} \\ &\quad \cdot \chi_\hbar(y) \cdot k(x, \xi, x', \xi', y) dy \end{aligned}$$

1775 and

$$k(x, \xi, x', \xi', y) = e^{(i/\hbar)\mathcal{A}_g(g(y)) - (i/\hbar)\xi(g(y)-y) - (|g(y)-x|^2 - |y-x|^2)/(2\hbar)} - 1.$$

(The factor $2^{\pm 1}$ appears because of the change of variable $\tilde{\sigma}$ in the definition of \mathcal{B}_x and \mathcal{B}_x^* . But this is not important in any sense.) Applying integration by parts to the integral above for ν times, we see

$$\begin{aligned} &K(x, \xi; x', \xi') \\ &= a_D^2 \int L^\nu(e^{(i/\hbar)\xi((x/2)-y) + (i/\hbar)\xi'(y-(x'/2))}) \cdot e^{-|y-x|^2/(2\hbar) - |y-x'|^2/(2\hbar)} \chi_\hbar(y) k(x, \xi, x', \xi', y) dy \\ &= a_D^2 \int e^{(i/\hbar)\xi((x/2)-y) + (i/\hbar)\xi'(y-(x'/2))} \cdot ({}^t L)^\nu(e^{-|y-x|^2/(2\hbar) - |y-x'|^2/(2\hbar)} \chi_\hbar(y) k(x, \xi, x', \xi', y)) dy \end{aligned}$$

where L is the differential operators defined by

$$Lu = \frac{1}{1 + \hbar^{-1}|\xi - \xi'|^2} \cdot \left(1 + i \sum_{j=1}^{2d} (\xi_j - \xi'_j) \frac{\partial}{\partial y_j} \right) u$$

and ${}^t L$ is its transpose:

$${}^tLu = \left(1 - i \sum_{j=1}^{2d} (\xi_j - \xi'_j) \frac{\partial}{\partial y_j} \right) \left(\frac{1}{1 + \hbar^{-1} |\xi - \xi'|^2} \cdot u \right).$$

Using the estimates noted in the beginning in the resulting terms, we can get the estimate

$$|K(x, \xi; x', \xi')| \leq C_\nu \cdot \hbar^\epsilon \cdot \langle \hbar^{-1/2} |(x, \xi) - (x', \xi')| \rangle^{-\nu} \quad \text{for } (x, \xi) \in \text{supp } \mathcal{Y}_\hbar \quad (6.23)$$

1780 for a small constant $\epsilon > 0$ and arbitrarily large $\nu > 0$, where C_ν is a constant independent of \hbar .

Remark 6.20. The result of integration by part is not very simple. But we have only to consider the order w.r.t. the parameter \hbar , since we allow the constant C_ν to depend on the derivatives of g . Hence it is not too difficult to do. Just note that θ satisfies the condition (6.1).

1785 This estimate for sufficiently large ν and (5.28) yields the required estimate. The second claim is proved in the parallel manner. \square

As we noted in the beginning of this section, the operator $\mathcal{L}_g \circ \mathcal{M}(\chi_\hbar)$ may not extend to a bounded operator from $\mathcal{H}_\hbar^r(\mathbb{R}^{2d})$ to itself, even though $g \in \mathcal{G}_\hbar$ is very close to the identity map. The next proposition (and hyperbolicity of f) will compensate this inconvenience.

1790 **Proposition 6.21.** *For any $g \in \mathcal{G}_\hbar$, we have*

$$\|\mathcal{L}_g \circ \mathcal{M}(\chi_\hbar)\|_{\mathcal{H}_\hbar^{r,+}(\mathbb{R}^{2d}) \rightarrow \mathcal{H}_\hbar^r(\mathbb{R}^{2d})} \leq C_0 \quad \text{and} \quad \|\mathcal{L}_g \circ \mathcal{M}(\chi_\hbar)\|_{\mathcal{H}_\hbar^r(\mathbb{R}^{2d}) \rightarrow \mathcal{H}_\hbar^{r,-}(\mathbb{R}^{2d})} \leq C_0 \quad (6.24)$$

for sufficiently small $\hbar > 0$, where $C_0 > 1$ is a constant that depends only on n, r, d, θ and the choice of the escape functions W and W^\pm in subsection 4.3. (In particular, C_0 is independent of the choice of the family \mathcal{G}_\hbar .)

The conclusion of this proposition is quite natural in view of the facts that

$$\mathcal{W}_\hbar^r \circ G(x, \xi) \cdot \chi_\hbar(x) \leq \mathcal{W}_\hbar^{r,+}(x, \xi), \quad \mathcal{W}_\hbar^{r,-} \circ G(x, \xi) \cdot \chi_\hbar(x) \leq \mathcal{W}_\hbar^r(x, \xi) \quad (6.25)$$

1795 for the canonical map

$$G : \mathbb{R}^{2d} \oplus \mathbb{R}^{2d} \rightarrow \mathbb{R}^{2d} \oplus \mathbb{R}^{2d}, \quad G(x, \xi) = (g(x), {}^t(Dg(x))^{-1}(\xi))$$

1800 associated to the operator \mathcal{L}_g . (Recall the argument in Subsection 2.1.) And it can be proved in essentially same ways as the argument given in the papers [6] and [19], where Littlewood-Paley theory and the theory of pseudodifferential operator is used respectively. Below we give a proof below by interpreting the argument in [6] in terms of the Bargmann transform. (But the reader may skip it because this is not a very essential part of our argument and may be proved in various ways.)

Proof. We decompose the operator $\mathcal{B}_x \circ \mathcal{L}_g \circ \mathcal{M}(\chi_{\hbar}) \circ \mathcal{B}_x^*$ as

$$\begin{aligned} \mathcal{B}_x \circ \mathcal{L}_g \circ \mathcal{M}(\chi_{\hbar}) \circ \mathcal{B}_x^* &= \mathcal{M}(1 - Y_{\hbar}) \circ \mathcal{B}_x \circ \mathcal{L}_g \circ \mathcal{M}(\chi_{\hbar}) \circ \mathcal{B}_x^* \\ &\quad + \mathcal{M}(Y_{\hbar}) \circ \mathcal{B}_x \circ (\mathcal{L}_g - \text{Id}) \circ \mathcal{M}(\chi_{\hbar}) \circ \mathcal{B}_x^* \\ &\quad + \mathcal{M}(1 - Y_{\hbar}) \circ \mathcal{B}_x \circ \mathcal{M}(\chi_{\hbar}) \circ \mathcal{B}_x^*. \end{aligned}$$

If we apply Lemma 6.19 (or more precisely (6.22) in the proof) to the second term and Lemma 6.15 and corollary 6.3 to the third term, we see that these two operators are bounded operators on $\mathcal{H}_{\hbar}^r(\mathbb{R}^{2d})$ and the operator norms are bounded by an absolute constant. Hence, in order to prove the former claim of the theorem, it suffices to show that the operator norm of

$$\mathcal{M}(1 - Y_{\hbar}) \circ \mathcal{B}_x \circ \mathcal{L}_g \circ \mathcal{M}(\chi_{\hbar}) \circ \mathcal{B}_x^* : L^2(\mathbb{R}^{2d}, (\mathcal{W}_{\hbar}^{r,+})^2) \rightarrow L^2(\mathbb{R}^{2d}, (\mathcal{W}_{\hbar}^r)^2) \quad (6.26)$$

is bounded by a constant C_0 with the same property as stated in the proposition. (The latter claim is proved in the parallel manner.) Below we give a proof¹⁶.

We take and fix $1/3 < a^+ < b^+ < a < b < 1/2$. Then we introduce a C^∞ partition of unity $\{\psi_n\}_{n \in \mathbb{Z}}$ (resp. $\{\psi_n^+\}_{n \in \mathbb{Z}}$) on \mathbb{R}^{2d} with the following properties:

- The function ψ_n (resp. ψ_n^+) is supported on the disk $|\xi| \leq 1$ if $n = 0$ and on the annulus $2^{|n|-1} \leq |\xi| \leq 2^{|n|+1}$ otherwise.
- The function ψ_n is supported on the cone $\mathbf{C}_+(b)$ if $n > 0$ and on the cone $\mathbf{C}_-(1/a) = \mathbb{R}^{2d} \setminus \mathbf{C}_+(a)$ if $n < 0$. Respectively, the function ψ_n^+ is supported on the cone $\mathbf{C}_+(b^+)$ if $n > 0$ and on the cone $\mathbf{C}_-(1/a^+) = \mathbb{R}^{2d} \setminus \mathbf{C}_+(a^+)$ if $n < 0$.
- The normalized functions $\xi \mapsto \psi_n(2^n \xi)$ (resp. $\xi \mapsto \psi_n^+(2^n \xi)$) are uniformly bounded in C^∞ norm.

For each $\hbar > 0$, we define functions ψ_n and ψ_n^+ on $T^*\mathbb{R}^{2d} = \mathbb{R}^{2d} \oplus \mathbb{R}^{2d}$ for $n \in \mathbb{Z}$ by

$$\psi_{n,\hbar}(x, \xi) = \psi_n(\hbar^{-1/2}\zeta) \quad \text{resp.} \quad \psi_{n,\hbar}^+(x, \xi) = \psi_n^+(\hbar^{-1/2}\zeta)$$

where $\zeta = (\zeta_p, \zeta_q)$ is the coordinates introduced in Proposition 2.15. Then, from the definition of the partition of unities above, we have

$$\|\mathcal{W}_{\hbar}^r \cdot u\|_{L^2}^2 \leq C_0 \sum_{n=-\infty}^{\infty} 2^{2rn} \|\psi_{n,\hbar} \cdot u\|_{L^2}^2$$

and also

$$\sum_{n=-\infty}^{\infty} 2^{2rn} \|\psi_{n,\hbar}^+ \cdot u\|_{L^2}^2 \leq C_0 \|\mathcal{W}_{\hbar}^{r,+} \cdot u\|_{L^2}^2 \quad (6.27)$$

¹⁶The argument in the following part is a little sketchy. For the details, we refer the argument in [6], though it will not be very necessary.

for any function $u(x, \xi) \in C^\infty(\mathbb{R}^{2d} \oplus \mathbb{R}^{2d})$. From the first inequality above and the definition of the function Y_{\hbar} , we have

$$\begin{aligned} & \|\mathcal{W}_{\hbar}^r \cdot (1 - Y_{\hbar}) \cdot \mathcal{B}_x \circ \mathcal{L}_g \circ \mathcal{M}(\chi_{\hbar}) \circ \mathcal{B}_x^* u\|_{L^2} \\ & \leq C_0 \sum_{|n| \geq \hbar^{-\theta}} \sum_{n' = -\infty}^{\infty} 2^{2rn} \|\psi_{n, \hbar} \cdot \mathcal{B}_x \circ \mathcal{L}_g \circ \mathcal{M}(\chi_{\hbar}) \circ \mathcal{B}_x^*(\psi_{n', \hbar}^+ \cdot u)\|_{L^2}^2. \end{aligned}$$

For the summands on the right hand side, we observe that

- From Lemma 4.2, the L^2 -operator norm of $\mathcal{B}_x \circ \mathcal{L}_g \circ \mathcal{M}(\chi_{\hbar}) \circ \mathcal{B}_x^*$ is bounded by 1, so

$$\|\psi_{n, \hbar} \cdot \mathcal{B}_x \circ \mathcal{L}_g \circ \mathcal{M}(\chi_{\hbar}) \circ \mathcal{B}_x^*(\psi_{n', \hbar}^+ \cdot u)\|_{L^2}^2 \leq \|\psi_{n', \hbar}^+ \cdot u\|_{L^2}^2.$$

- If either $|n| - |n'| \geq 3$ or $n < 0 < n'$, we have

$$\text{dist}(Dg_x^t(\text{supp } \psi_{n, \hbar}), \text{supp } \psi_{n', \hbar}) > C_0 \cdot \hbar^{1/2} 2^{\max\{n, n'\}} \quad \text{for } x \in \text{supp } \chi_{\hbar}$$

1825

and, by crude estimate using integration by parts, we get the estimate

$$\|\psi_{n, \hbar} \cdot \mathcal{B}_x \circ \mathcal{L}_g \circ \mathcal{M}(\chi_{\hbar}) \circ \mathcal{B}_x^*(\psi_{n', \hbar}^+ \cdot u)\|_{L^2}^2 \leq C_{\nu}(g) \cdot 2^{-\nu \cdot \max\{n, n'\}} \|\psi_{n', \hbar}^+ \cdot u\|_{L^2}^2$$

where the constant $C_{\nu}(g)$ may depend on g and ν but not on \hbar . Otherwise we have $n < n' + 3$ and $2^{rn} \leq C_0 2^{rn'}$.

From these observations and (6.27), we can conclude the required estimate:

$$\|\mathcal{W}_{\hbar}^r \cdot (1 - Y_{\hbar}) \cdot \mathcal{B}_x \circ \mathcal{L}_g \circ \mathcal{M}(\chi_{\hbar}) \circ \mathcal{B}_x^* u\|_{L^2}^2 \leq C_0 \sum_{n' = -\infty}^{\infty} 2^{2rn'} \|\psi_{n', \hbar}^+ \cdot u\|_{L^2}^2 \leq C_0 \|\mathcal{W}_{\hbar} u\|_{L^2}^2$$

for sufficiently small $\hbar > 0$. □

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The next lemma will be used in the key step in the proof of Theorem 7.1. This lemma implies that we can ignore the action of nonlinear diffeomorphisms in \mathcal{G}_{\hbar} when we restrict it to the image of the projectors $t_{\hbar}^{(k)}$.

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Lemma 6.22. *There exist constants $\epsilon > 0$ and $C > 0$ independent of \hbar such that the following holds: Let $\psi \in \mathcal{X}_{\hbar}$ be supported on the disk $\mathbb{D}(2\hbar^{1/2-\theta})$ and let $g \in \mathcal{G}_{\hbar}$, $0 \leq k \leq n$, then it holds*

$$\left\| (\mathcal{L}_g - \text{Id}) \circ \mathcal{M}(\psi) \circ t_{\hbar}^{(k)} \right\|_{\mathcal{H}_{\hbar}^r(\mathbb{R}^{2d})} \leq C \hbar^{\epsilon}$$

and

$$\left\| t_{\hbar}^{(k)} \circ (\mathcal{L}_g - \text{Id}) \circ \mathcal{M}(\psi) \right\|_{\mathcal{H}_{\hbar}^r(\mathbb{R}^{2d})} \leq C \hbar^{\epsilon}.$$

Proof. We decompose the operator $(\mathcal{L}_g - \text{Id}) \circ \mathcal{M}(\psi) \circ t_h^{(k)}$ into

$$(\mathcal{L}_g \circ \mathcal{M}(\chi_h) - \text{Id}) \circ \mathcal{Y}_h \circ \mathcal{M}(\psi) \circ t_h^{(k)} \quad \text{and} \quad (\mathcal{L}_g \circ \mathcal{M}(\chi_h) - \text{Id}) \circ (\text{Id} - \mathcal{Y}_h) \circ \mathcal{M}(\psi) \circ t_h^{(k)}. \quad (6.28)$$

(Note that we have $\chi_h \cdot \psi = \psi$ from the assumption.) The operator norm of the latter is also bounded by $C\hbar^\epsilon$ from Lemma 6.16 and Proposition 6.21. We further write the former part as

$$(\mathcal{L}_g - \text{Id}) \circ \mathcal{M}(\psi) \circ \mathcal{Y}_h \circ t_h^{(k)} + (\mathcal{L}_g \circ \mathcal{M}(\chi_h) - \text{Id}) \circ [\mathcal{Y}_h, \mathcal{M}(\psi)] \circ t_h^{(k)}$$

Then we see that the operator norm of the former part is bounded by $C\hbar^\epsilon$, from Lemma 6.19, Corollary 5.13, Lemma 6.15 and Proposition 6.21. The latter part is also bounded by $C\hbar^\epsilon$ from Lemma 6.6. Thus we obtain the former claim. The latter claim can be proved in the parallel manner. \square

Corollary 6.23. *There exist constants $\epsilon > 0$ and $C > 0$ independent of \hbar such that, for any $\psi \in \mathcal{X}_\hbar$ and $g \in \mathcal{G}_\hbar$ it holds*

$$\left\| [\mathcal{L}_g \circ \mathcal{M}(\psi), t_h^{(k)}] \right\|_{\mathcal{H}_\hbar^r(\mathbb{R}^{2d})} \leq C\hbar^\epsilon \quad \text{for } 0 \leq k \leq n$$

and also

$$\left\| [\mathcal{L}_g \circ \mathcal{M}(\psi), \tilde{t}_h] \right\|_{\mathcal{H}_\hbar^r(\mathbb{R}^{2d})} \leq C\hbar^\epsilon.$$

Proof. The former claim is an immediate consequence of the last lemma and Lemma 6.7. The latter claim then follows from the relation $\tilde{t}_h = \text{Id} - \sum_{k=0}^n t_h^{(k)}$. \square

Lemma 6.24. *There exist constants $\epsilon > 0$ and $C > 0$, independent of \hbar , such that the following holds true: Let $\psi \in \mathcal{X}_\hbar$ be supported on the disk $\mathbb{D}(2\hbar^{1/2-\theta})$ and let $g \in \mathcal{G}_\hbar$, $0 \leq k \leq n$, then it holds*

$$\left\| (\mathcal{L}_g - \text{Id}) \circ \mathcal{M}(\psi) \circ t_h^{(k)} : \mathcal{H}_\hbar^r(\mathbb{R}^{2d}) \rightarrow \mathcal{H}_\hbar^r(\mathbb{R}^{2d}) \right\|_{tr} \leq C\hbar^{-\theta d + \epsilon}$$

$$\left\| t_h^{(k)} \circ (\mathcal{L}_g - \text{Id}) \circ \mathcal{M}(\psi) : \mathcal{H}_\hbar^r(\mathbb{R}^{2d}) \rightarrow \mathcal{H}_\hbar^r(\mathbb{R}^{2d}) \right\|_{tr} \leq C\hbar^{-\theta d + \epsilon}$$

and

$$\left\| [\mathcal{L}_g \circ \mathcal{M}(\psi), t_h^{(k)}] : \mathcal{H}_\hbar^r(\mathbb{R}^{2d}) \rightarrow \mathcal{H}_\hbar^r(\mathbb{R}^{2d}) \right\|_{tr} \leq C\hbar^{-\theta d + \epsilon}$$

Proof. We write the operators $(\mathcal{L}_g - \text{Id}) \circ \mathcal{M}(\psi) \circ t_h^{(k)}$ as

$$(\mathcal{L}_g - \text{Id}) \circ \mathcal{M}(\psi) \circ t_h^{(k)} = \left((\mathcal{L}_g - \text{Id}) \circ \mathcal{M}(\psi) \circ t_h^{(k)} \right) \circ \left(t_h^{(k)} \circ \mathcal{M}(\chi_h) \right) + \left((\mathcal{L}_g - \text{Id}) \circ \mathcal{M}(\psi) \right) \circ [\mathcal{M}(\chi_h), t_h^{(k)}].$$

Then we obtain the first inequality by Lemma 6.22, Lemma 6.10, Proposition 6.21 and Corollary 6.14. We obtain the second inequality in the same manner. Then the third inequality follows from Corollary 6.14. \square

7 Proof of the Theorem 1.17 for the band spectrum of the prequantum transfer operator

In Section 7.1 we show how to deduce Theorem 3.5 from Theorem 7.1. Then the subsequent subsections will be devoted to the proof of Theorem 7.1.

7.1 Proof of Theorem 1.17

Recall that the damping function $D(x) = V(x) - V_0(x)$ has been defined in (1.25).

Theorem 7.1. *Let $n \geq 0$ and take sufficiently large r accordingly. There exist a constant C_0 , which is independent of V and f , and a constant $N_0 > 0$ such that for every $|N| > N_0$ one has a decomposition of the Hilbert space $\mathcal{H}_N^r(P)$ independent on V :*

$$\mathcal{H}_N^r(P) = \mathcal{H}_0 \oplus \mathcal{H}_1 \dots \oplus \mathcal{H}_n \oplus \mathcal{H}_{n+1} \tag{7.1}$$

such that, writing $\tau^{(k)}$ for the projection onto the component \mathcal{H}_k along the other components,

(1) For some constant $\epsilon > 0$ and $C > 0$ independent of \hbar , we have

$$|\dim \mathcal{H}_k - r(k, d) \cdot N^d \cdot Vol_\omega(M)| \leq CN^{d-\epsilon} \quad \text{for } 0 \leq k \leq n$$

where $r(k, d) = \binom{d+k-1}{d-1}$, while $\dim \mathcal{H}_{n+1} = \infty$,

(2) $\|\tau^{(k)}\| < C_0$ for $0 \leq k \leq n + 1$,

(3) For some constant $\epsilon > 0$ and $C > 0$ independent of \hbar , we have, if $k \neq l$, that

$$\|\tau^{(k)} \circ \hat{F}_N \circ \tau^{(l)}\| \leq CN^{-\epsilon},$$

(4) For every $0 \leq k \leq n + 1$,

$$\|\tau^{(k)} \circ \hat{F}_N \circ \tau^{(k)}\|_{\mathcal{H}_N^r(P)} \leq C_0 \sup_{x \in M} \left(e^{D(x)} \|Df_x|_{E_u}\|_{\min}^{-k} \right), \tag{7.2}$$

(5) For every $0 \leq k \leq n$ and $u \in \mathcal{H}_k$,

$$\left\| \left(\tau^{(k)} \circ \hat{F}_N \right) u \right\|_{\mathcal{H}_N^r(P)} \geq C_0^{-1} \inf_{x \in M} \left(e^{D(x)} \|Df_x|_{E_u}\|_{\max}^{-k} \right) \|u\|_{\mathcal{H}_N^r(P)}, \tag{7.3}$$

Now we show how to deduce Theorem 1.17 from Theorem 7.1. Let $m \geq 1$ and apply

Theorem 7.1 to f^m and \hat{F}_N^m . On the right hand side of (7.2) and (7.3) we have

$$r_{k,m}^+ := C_0 \sup_{x \in M} \left(e^{D_m(x)} \left\| (Df_{/E_u}^m(x))^{-1} \right\|^{+k} \right)$$

and

$$r_{k,m}^- := C_0^{-1} \cdot \inf_{x \in M} \left(e^{D_m(x)} \left\| Df_{/E_u}^m(x) \right\|^{-k} \right),$$

with $D_m(x) := \sum_{j=0}^{m-1} D(f^{-j}(x))$. From Eq.(1.28) one has $\lim_{m \rightarrow \infty} (r_{k,m}^\pm)^{1/m} = r_k^\pm$. So let $\varepsilon > 0$ and take m large enough so that

$$\forall k, \quad (r_{k,m}^+)^{1/m} < r_k^+ + \varepsilon \quad \text{and} \quad (r_{k,m}^-)^{1/m} > r_k^- - \varepsilon. \quad (7.4)$$

The following arguments using Neumann series for resolvents are very standard. For $0 \leq k \leq n+1$, let

$$A_{k,m} := \tau^{(k)} \circ \hat{F}_N^m \circ \tau^{(k)} \quad : \mathcal{H}_k \rightarrow \mathcal{H}_k,$$

From Theorem 7.1 we have $\|A_{k,m}\| \leq r_{k,m}^+$ and for $k \leq n$ we have also $\|A_{k,m}^{-1}\| \leq (r_{k,m}^-)^{-1}$ (recall that $A_{k,m}$ is invertible and finite rank for $k \leq n$). For convenience, if $k = n+1$, we define $r_{k,m}^- = 0$.

Lemma 7.2. *Let $0 \leq k \leq n+1$, and $z \in \mathbb{C}$ with $|z| < r_{k,m}^-$ or $|z| > r_{k,m}^+$. Then $(z - A_{k,m})$ is invertible and its inverse $R_{A_{k,m}}(z) := (z - A_{k,m})^{-1}$, the resolvent operator, satisfies*

$$\|R_{A_{k,m}}(z)\| \leq \frac{1}{\text{dist}(|z|, [r_{k,m}^-, r_{k,m}^+])}$$

Proof. If $|z| > r_{k,m}^+ \geq \|A_{k,m}\|$, then $\left\| \frac{A_{k,m}}{z} \right\| < 1$ and we can write a convergent Neuman series for

$$R_{A_{k,m}}(z) = (z - A_{k,m})^{-1} = \frac{1}{z} \left(1 - \frac{A_{k,m}}{z} \right)^{-1}$$

giving

$$\|R_{A_{k,m}}(z)\| \leq \frac{1}{|z|} \left(1 - \frac{\|A_{k,m}\|}{|z|} \right)^{-1} = \frac{1}{(|z| - \|A_{k,m}\|)} \leq \frac{1}{\text{dist}(|z|, [r_{k,m}^-, r_{k,m}^+])}$$

Similarly if $|z| < r_{k,m}^- \leq \|A_{k,m}^{-1}\|^{-1}$ then we have $\|zA_{k,m}^{-1}\| < 1$ and a convergent Neuman series for

$$R_{A_{k,m}}(z) = (z - A_{k,m})^{-1} = -A_{k,m}^{-1} (1 - zA_{k,m}^{-1})^{-1}$$

giving

$$\|R_{A_{k,m}}(z)\| \leq \|A_{k,m}^{-1}\| (1 - |z| \|A_{k,m}^{-1}\|)^{-1} = \frac{1}{(\|A_{k,m}^{-1}\|^{-1} - |z|)} \leq \frac{1}{\text{dist}(|z|, [r_{k,m}^-, r_{k,m}^+])}$$

□

Thus the operator

$$A_m := A_{0,m} \oplus A_{1,m} \oplus \dots \oplus A_{n+1,m} : \mathcal{H}_h^r(P) \rightarrow \mathcal{H}_h^r(P) \tag{7.5}$$

has a resolvent $R_{A_m}(z)$ which satisfies

$$\|R_{A_m}(z)\| \leq \sum_{k=0}^{n+1} \frac{1}{\text{dist}(|z|, [r_{k,m}^-, r_{k,m}^+])} \tag{7.6}$$

From Theorem 7.1(3), the operator \hat{F}_N^m can be written $\hat{F}_N^m = A_m + B_m$ with $\|B_m\| \leq C\hbar^\epsilon$ with C and ϵ independent on \hbar (but depends on m). We use a standard perturbation argument [48, p.311] to show that if $z \in \mathbb{C}$ is such that $\|R_{A_m}(z)\| \|B_m\| < 1$ then it is not in the spectrum of \hat{F}_N^m : $z \notin \sigma(\hat{F}_N^m)$. For this we write

$$\begin{aligned} z - \hat{F}_N^m &= z - A_m - B_m \\ &= (z - A_m) (1 - (z - A_m)^{-1} B_m) \end{aligned}$$

hence

$$(z - \hat{F}_N^m)^{-1} = (1 - (z - A_m)^{-1} B_m)^{-1} (z - A_m)^{-1}$$

and using Neumann series we deduce:

$$\|R_{\hat{F}_N^m}(z)\| \leq (1 - \|R_{A_m}(z)\| \|B_m\|)^{-1} \|R_{A_m}(z)\|$$

hence z is in the resolvent set of \hat{F}_N^m , i.e. $z \notin \sigma(\hat{F}_N^m)$.

From Eq.(7.6) and $\|B_m\| \leq C\hbar^\epsilon$, we see that the condition $\|R_{A_m}(z)\| \|B_m\| < 1$ is satisfied if for every $k \in \{0, n + 1\}$

$$\text{dist}(|z|, [r_{k,m}^-, r_{k,m}^+]) > (n + 1) C\hbar^\epsilon.$$

In that case we have $\|R_{\hat{F}_N^m}(z)\| < C$ with C independent on N . In other terms, replacing z by z^m and taking the power $1/m$ of the previous estimates, we have that if $z \in \mathbb{C}$ and

$$(r_{k+1,m}^+ + (n + 1) C\hbar^\epsilon)^{1/m} < |z| < (r_{k,m}^- + (n + 1) C\hbar^\epsilon)^{1/m} \tag{7.7}$$

for every $0 \leq k \leq n + 1$ then $\|R_{\hat{F}_N^m}(z^m)\| < C$ and $z \notin \sigma(\hat{F}_N^m)$.

Considering (7.4), take N_ϵ large enough such that for every $|N| = \frac{1}{2\pi\hbar} > N_\epsilon$ we have $(r_{k+1,m}^+ + (n + 1) C\hbar^\epsilon)^{1/m} < r_{k+1}^+ + \epsilon$ and $(r_{k,m}^- + (n + 1) C\hbar^\epsilon)^{1/m} > r_k^- - \epsilon$. Then if $z \in \mathbb{C}$ is such that $r_{k+1}^+ + \epsilon < |z| < r_k^- - \epsilon$, we have (7.7) and therefore

$$\left\| (z^m - \hat{F}_N^m)^{-1} \right\| < C \tag{7.8}$$

and $z \notin \sigma(\hat{F}_N)$ from above. We have obtained the results presented in Theorem 1.17 except for the bound on the resolvent (1.29) that we derive now from (7.8).

From Theorem 7.1 we have that $\|\hat{F}_N\| \leq C$ is bounded independent on N . For $z \in \mathbb{C}$
 1900 in the resolvent set we have the relation¹⁷

$$(z - \hat{F}_N)^{-1} = \left(\sum_{r=0}^{m-1} z^{m-1-r} \hat{F}_N^r \right) (z^m - \hat{F}_N^m)^{-1}$$

hence we deduce that

$$\begin{aligned} \left\| (z - \hat{F}_N)^{-1} \right\| &\leq \left(\sum_{r=0}^{m-1} |z|^{m-1-r} C^r \right) \left\| (z^m - \hat{F}_N^m)^{-1} \right\| \\ &\stackrel{(7.8)}{\leq} C' \end{aligned}$$

with some constant C' independent on N . We have obtained (1.29).

Suppose that $r_1^+ < r_0^-$. We have defined in (1.32) by Π_h the spectral projector on the external band of \hat{F}_N . The projector $\tau^{(0)}$ introduced in Theorem 7.1 is the projector on
 1905 the component $A_{0,m}$ of the decomposition (7.5) of the operator A_m . We aim to finish this subsection by showing that

$$\|\Pi_h - \tau^{(0)}\| \leq C\hbar^\epsilon \quad (7.9)$$

with some constant $C > 0, \epsilon > 0$ independent on $\hbar = 1/(2\pi N)$. Let m as above and let γ be a clockwise path a circle of radius $(r_1^+)^m < |z| < (r_0^-)^m$. By Cauchy formula we have

$$\begin{aligned} \Pi_h &= \frac{1}{2\pi i} \oint_{\gamma} R_{\hat{F}_N^m}(z) dz, & R_{\hat{F}_N^m}(z) &:= (z - \hat{F}_N^m)^{-1}, \\ \tau^{(0)} &= \frac{1}{2\pi i} \oint_{\gamma} R_{A_m}(z) dz, & R_{A_m}(z) &:= (z - A_m)^{-1}. \end{aligned}$$

1910 We have written above that $\hat{F}_N^m = A_m + B_m$ with $\|B_m\| \leq C\hbar^\epsilon$. For any $z \in \gamma$ we have shown that the resolvent are bounded: $\|R_{\hat{F}_N^m}(z)\| \leq C, \|R_{A_m}(z)\| \leq C$. From the general relation

$$R_{\hat{F}_N^m}(z) - R_{A_m}(z) = \frac{1}{2} \left(R_{\hat{F}_N^m}(z) B_m R_{A_m}(z) + R_{A_m}(z) B_m R_{\hat{F}_N^m}(z) \right)$$

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Proof. For complex numbers $a, F \in \mathbb{C}$ we have

$$(z - F) \left(\sum_{r=0}^{m-1} z^{m-1-r} F^r \right) = \sum_{r=0}^{m-1} z^{m-r} F^r - \sum_{r=0}^{m-1} z^{m-1-r} F^{r+1} = \sum_{r=0}^{m-1} z^{m-r} F^r - \sum_{r=1}^m z^{m-r} F^r = z^m - F^m$$

□

we deduce that $\left\| R_{\hat{F}_N^m}(z) - R_{A_m}(z) \right\| \leq C' \hbar^\epsilon$ and then from Cauchy formula above that $\left\| \Pi_{\hbar} - \tau^{(0)} \right\| \leq C \hbar^\epsilon$.

1915 7.2 Local charts on M and local trivialization of the bundle $P \rightarrow M$

In this section and the next ones we give the proof of the Theorem 3.5. We henceforth consider the setting assumed in Section 1. In particular $\lambda > 1$ is the constant in the condition (1.2) in the definition (Definition 1.1) that f is an Anosov diffeomorphism. Note that, by replacing f by its iterate if necessary, we may and do suppose that λ is a large
1920 number. Below we write C_0 for positive constants independent of f , V and \hbar and write C for those independent of \hbar but may (or may not) dependent on f and V . Also we assume (6.6) for the choice of r .

As in (6.1), we fix a constant $0 < \theta < \beta/8$ with $0 < \beta < 1$ being the Hölder exponent of the stable and unstable sub-bundle given in (1.3). Below we take an atlas on M depending
1925 on the semiclassical parameter $\hbar = \frac{1}{2\pi N} > 0$ so that it consists of charts of diameter of order $\hbar^{\frac{1}{2}-\theta}$. We consider \mathbb{R}^{2d} as a linear symplectic space with coordinates $x = (q, p) = (q^1, \dots, q^d, p^1, \dots, p^d)$ and symplectic form $\omega = \sum_{i=1}^d dq^i \wedge dp^i$. The open ball of radius $c > 0$ is denoted by $\mathbb{D}(c) := \{x \in \mathbb{R}^{2d} \mid |x| < c\}$. The following Proposition is illustrated in Figure 7.1.

1930 **Proposition 7.3. “Local chart and trivialization”.** *For each $\hbar = \frac{1}{2\pi N} > 0$, there exist a set of distinct points*

$$\mathcal{P}_{\hbar} = \{m_i \in M \mid 1 \leq i \leq I_{\hbar}\}$$

and a coordinate map associated to each point $m_i \in \mathcal{P}_{\hbar}$,

$$\kappa_i = \kappa_{i,\hbar} : \mathbb{D}(c) \subset \mathbb{R}^{2d} \rightarrow M, \quad 1 \leq i \leq I_{\hbar}$$

with $c > 0$ a constant independent of \hbar , so that the following conditions hold:

- (1) $\kappa_i(0) = m_i$.
- 1935 (2) *The differential of κ_i at the origin 0 maps the subspaces $\mathbb{R}^d \oplus \{0\}$ and $\{0\} \oplus \mathbb{R}^d$ (or, the q - and p - axis) isometrically onto the unstable and stable subspace respectively:*

$$(D\kappa_i)_0(\mathbb{R}^d \oplus \{0\}) = E_u(m_i), \quad (D\kappa_i)_0(\{0\} \oplus \mathbb{R}^d) = E_s(m_i).$$

Further, the map $\kappa_i \circ (D\kappa_i)_0^{-1}$ is not far from the exponential map in the sense that

$$\left\| \exp_{m_i}^{-1} \circ \kappa_i \circ (D\kappa_i)_0^{-1} : \mathbb{D}(m_i, c) \rightarrow T_{m_i}M \right\|_{C^k} \leq C_k$$

with C_k a constant independent of \hbar nor $1 \leq i \leq I_{\hbar}$, where $\mathbb{D}(m_i, c)$ denotes the disk in $T_{m_i}M$ with radius c and center at the origin.

1940 (3) The open subsets $U_i := \kappa_i \left(\mathbb{D} \left(3\hbar^{\frac{1}{2}-\theta} \right) \right) \subset M$ for $1 \leq i \leq I_\hbar$ cover the manifold M . The cardinality I_\hbar of the set \mathcal{P}_\hbar is bounded by $C_0 \cdot \hbar^{-d(1-2\theta)}$ and we have

$$\max_{1 \leq i \leq I_\hbar} \#\{1 \leq j \leq I_\hbar \mid U_i \cap U_j \neq \emptyset\} \leq C_0 \quad (7.10)$$

with C_0 a constant independent of \hbar .

(4) For every $1 \leq i \leq I_\hbar$, $\kappa_i^*(\omega) = \sum_i dq^i \wedge dp^i$ on U_i and with an appropriate choice of a section $\tau_i : U_i \rightarrow P$, the statement of Proposition 2.15 holds true.

1945 (5) If $U_i \cap U_j \neq \emptyset$, we denote the coordinate change transformation by $\kappa_{j,i} := \kappa_j^{-1} \circ \kappa_i : \mathbb{D}(c) \rightarrow \mathbb{R}^{2d}$. Then there exists symplectic and isometric affine map $A_{j,i} : \mathbb{R}^{2d} \rightarrow \mathbb{R}^{2d}$ that belongs to \mathcal{A} (see Definition 5.14) such that $g_{j,i} := A_{j,i} \circ \kappa_{j,i}$ satisfies

$$g_{j,i}(0) = 0, \quad \|Dg_{j,i}(0) - \text{Id}\|_{C^1} \leq C_1 \cdot \hbar^{\beta(\frac{1}{2}-\theta)} \quad \text{and} \quad \|g_{j,i}\|_{C^s} < C_s \quad \text{for } s \geq 2$$

where C_s for $s \geq 1$ are constant independent of \hbar and $1 \leq i, j \leq I_\hbar$.

1950 (6) There exists a family of C^∞ functions $\{\psi_i : \mathbb{R}^{2d} \rightarrow [0, 1]\}_{i=1}^{I_\hbar}$ which is supported on the disk $\mathbb{D}(\hbar^{1/2-\theta})$ and gives a partition of unity on M :

$$\sum_{i=1}^{I_\hbar} \psi_i \circ \kappa_i^{-1} \equiv 1 \text{ on } M. \quad (7.11)$$

The set of functions ψ_i satisfies the conditions (C1) and (C2) in Subsection 6.1.

Remark 7.4. Since the unstable and stable vector sub-bundles, E_u and E_s may be non-trivial in general, we need to put the affine isometries $A_{j,i} \in \mathcal{A}$ in the condition (5) above.

1955 *Proof.* For each point $m \in M$, we first define κ_m as the composition of the exponential mapping (in Riemannian geometry) $\exp_m : T_m M \rightarrow M$ and a linear map $\mathbb{R}^{2d} \rightarrow T_m M$ so that the condition 1 (with $\kappa_i = \kappa_m$) holds true. Then, using Darboux theorem, we can deform such κ_m into a symplectic map to ensure condition 4 with keeping the condition 1. (See Lemma 3.14 in [37, p.94] and its proof). We may then take a section τ as in the proof of Proposition 2.15 so that the condition 4 (with $\kappa_i = \kappa_m$) holds. It is then clear that, if we take the points in \mathcal{P}_\hbar appropriately, the conditions 1 to 3 hold true with setting 1960 $\kappa_i := \kappa_{m_i}$. The condition 5 and 6 are also obvious from this construction. \square

In the following subsections, we fix the set \mathcal{P}_\hbar , the coordinate maps κ_i , the isometric affine maps $A_{j,i} \in \mathcal{A}$ and the functions ψ_i taken in Proposition 7.3 above.

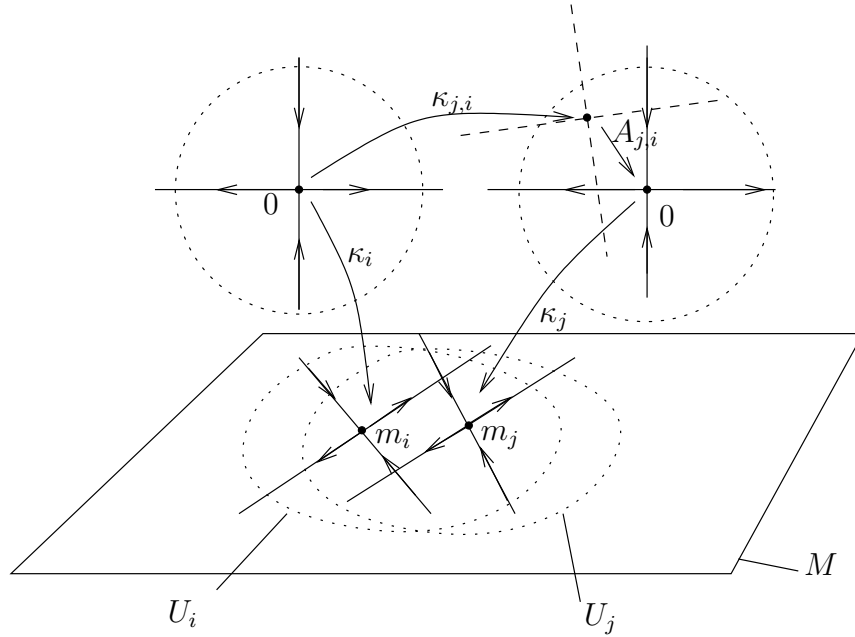


Figure 7.1: Illustration of the local charts U_i and maps κ_i defined in Proposition 7.3.

7.3 The prequantum transfer operator decomposed on local charts

1965 To proceed, we express the transfer operator \hat{F}_\hbar as the totality of operators between local charts. First we discuss about an expression of an equivariant section $u \in C_N^\infty(P)$ as a set of functions on local charts.

Definition 7.5. Let

$$\mathcal{E}_\hbar := \bigoplus_{i=1}^{I_\hbar} C_0^\infty(\mathbb{D}(\hbar^{1/2-\theta})).$$

1970 Let $\mathbf{I}_\hbar : C_N^\infty(P) \rightarrow \mathcal{E}$ be the operator that associates to each equivariant function $u \in C_N^\infty(P)$ a set of functions $\mathbf{I}_\hbar(u) = (u_i)_{i=1}^{I_\hbar} \in \mathcal{E}_\hbar$ on local charts:

$$\mathbf{I}_\hbar : \begin{cases} C_N^\infty(P) & \rightarrow \mathcal{E}_\hbar \\ u & \rightarrow u_i(x) = \psi_i(x) \cdot u(\tau_i(\kappa_i(x))) \end{cases} \quad \text{for } 1 \leq i \leq I_\hbar. \quad (7.12)$$

The inverse operation is given as follows¹⁸.

Proposition 7.6. Let $\mathbf{I}_\hbar^* : \bigoplus_{i=1}^{I_\hbar} \mathcal{S}(\mathbb{R}^{2d}) \rightarrow C_N^\infty(P)$ be the operator defined by

$$(\mathbf{I}_\hbar^*((u_i)_{i=1}^{I_\hbar})) (p) = \sum_{i=1}^{I_\hbar} e^{iN \cdot \alpha_i(p)} \cdot \chi_\hbar(x) \cdot u_i(x) \quad (7.13)$$

¹⁸Beware that \mathbf{I}_\hbar^* is not the L^2 adjoint of \mathbf{I}_\hbar here.

where χ_{\hbar} is the function defined in (6.21), and $x = \kappa_i^{-1}(\pi(p))$ and $\alpha_i(p)$ is the real number such that $p = e^{i\alpha_i(p)} \cdot \tau_i(\pi(p))$. This operator reconstructs $u \in C_N^\infty(P)$ from its local data $u_i = (\mathbf{I}_{\hbar}(u))_i$:

$$\mathbf{I}_{\hbar}^* \circ \mathbf{I}_{\hbar} = \text{Id}_{C_N^\infty(P)}. \quad (7.14)$$

Consequently, $\mathbf{I}_{\hbar} \circ \mathbf{I}_{\hbar}^* : \mathcal{E}_{\hbar} \rightarrow \mathcal{E}_{\hbar}$ is a projection onto the image of \mathbf{I}_{\hbar} .

Proof. Notice that

$$\chi_{\hbar} \cdot \psi_i = \psi_i \quad (7.15)$$

Let $w := (\mathbf{I}_{\hbar}^* \circ \mathbf{I}_{\hbar})(v)$. From the expressions of \mathbf{I}_{\hbar} and \mathbf{I}_{\hbar}^* and equivariance of v , we compute

$$w(p) = \sum_{i=1}^{I_{\hbar}} e^{iN\alpha_i(p)} (\chi_{\hbar}(x) \cdot \psi_i(x) \cdot v(\tau_i(\kappa_i(x)))) = \sum_{i=1}^{I_{\hbar}} \psi_i(x) v(p) = v(p).$$

Finally $\mathbf{I}_{\hbar} \circ \mathbf{I}_{\hbar}^*$ is a projector since $(\mathbf{I}_{\hbar} \circ \mathbf{I}_{\hbar}^*)^2 = \mathbf{I}_{\hbar} \circ (\mathbf{I}_{\hbar}^* \circ \mathbf{I}_{\hbar}) \circ \mathbf{I}_{\hbar}^* = \mathbf{I}_{\hbar} \circ \mathbf{I}_{\hbar}^*$. \square

Definition 7.7. We define the *lift of the prequantum transfer operator* \hat{F}_{\hbar} with respect to \mathbf{I}_{\hbar} as

$$\mathbf{F}_{\hbar} := \mathbf{I}_{\hbar} \circ \hat{F}_N \circ \mathbf{I}_{\hbar}^* : \bigoplus_{i=1}^{I_{\hbar}} \mathcal{S}(\mathbb{R}^{2d}) \rightarrow \mathcal{E}_{\hbar} \subset \bigoplus_{i=1}^{I_{\hbar}} \mathcal{S}(\mathbb{R}^{2d}). \quad (7.16)$$

The operator \mathbf{F}_{\hbar} is nothing but the prequantum transfer operator $\hat{F}_N : C_N^\infty(P) \rightarrow C_N^\infty(P)$ viewed through the local charts and local trivialization that we have chosen. This is a matrix of operators that describe transition between local data that \hat{F}_{\hbar} induces. The next proposition gives it in a concrete form.

Definition 7.8. We write $i \rightarrow j$ for $0 \leq i, j \leq I_{\hbar}$ if and only if $f(U_i) \cap U_j \neq \emptyset$.

Clearly we have

$$\max_{1 \leq i \leq I_{\hbar}} \#\{1 \leq j \leq I_{\hbar} \mid i \rightarrow j\} \leq C(f) \quad (7.17)$$

for some constant $C(f)$ which may depend on f but not on \hbar .

Proposition 7.9. *The operator \mathbf{F}_{\hbar} is written as*

$$\mathbf{F}_{\hbar}((v_i)_{i \in I_{\hbar}}) = \left(\sum_{i=1}^{I_{\hbar}} \mathbf{F}_{j,i}(v_i) \right)_{j \in I_{\hbar}}$$

where the component

$$\mathbf{F}_{j,i} : \mathcal{S}(\mathbb{R}^{2d}) \rightarrow C_0^\infty(\mathbb{D}(\hbar^{1/2-\theta}))$$

is defined by $\mathbf{F}_{j,i} \equiv 0$ if $i \not\rightarrow j$ and, otherwise, by

$$\mathbf{F}_{j,i}(v_i) = \mathcal{L}_{f_{j,i}}(e^{V \circ f \circ \kappa_i} \cdot \psi_{j,i} \cdot \chi_{\hbar} \cdot v_i)$$

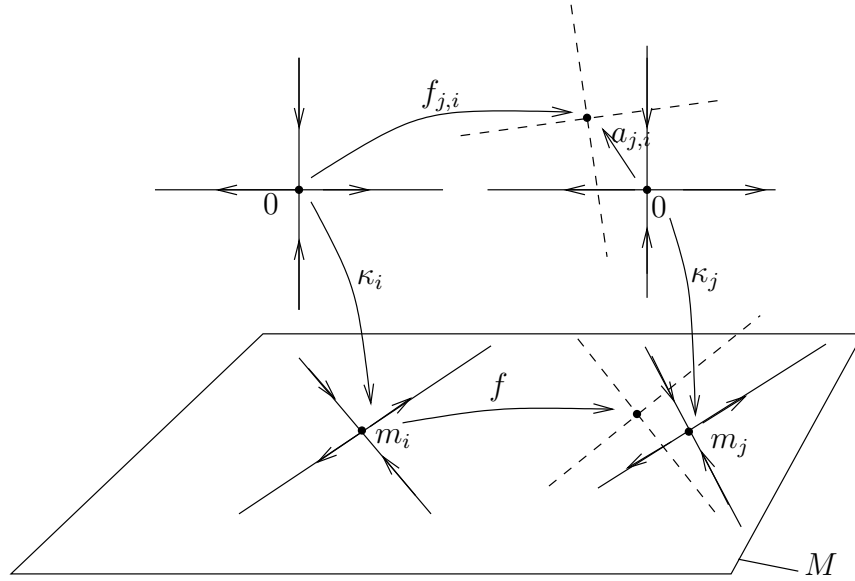


Figure 7.2: Illustration of the local map $f_{j,i}$ defined in (7.18) and $a_{j,i}$ defined in (7.22).

where we set

$$f_{j,i} := \kappa_j^{-1} \circ f \circ \kappa_i, \quad (7.18)$$

$$\psi_{j,i} := \psi_j \circ f_{j,i} \quad (7.19)$$

and $\mathcal{L}_{f_{j,i}}$ is the Euclidean prequantum transfer operator defined in (6.19) with $g = f_{j,i}$.

Remark 7.10. χ_{\hbar} is actually not necessary.

The maps $f_{j,i}$ is illustrated on Figure 7.2.

1995 *Proof.* The expression of the operator \hat{F}_N in local coordinates has been given in Proposition 2.4. Taking the multiplication by functions ψ_i , $1 \leq i \leq I_{\hbar}$, in the definitions of the operators \mathbf{I}_{\hbar} and \mathbf{I}_{\hbar}^* into account, we obtain the expression of $\mathbf{F}_{j,i}$ as above. \square

We define

$$V_j = \max\{V(m) \mid m \in U_j\} \quad \text{for } 1 \leq j \leq I_{\hbar}.$$

Since the function V is almost constant on each U_j , we have

2000 **Lemma 7.11.** *If we set*

$$\mathcal{X}_{\hbar} = \{\psi_{j,i} \cdot \chi_{\hbar} \mid 1 \leq i, j \leq I_{\hbar}, i \rightarrow j\} \quad (\text{resp. } \mathcal{X}_{\hbar} = \{e^{V \circ f \circ \kappa_i} \cdot \psi_{j,i} \cdot \chi_{\hbar} \mid 1 \leq i, j \leq I_{\hbar}, i \rightarrow j\}),$$

it satisfies the conditions (C1) and (C2) in Subsection 6.1. (The constants C and C_{α} will depend on f and V though not on \hbar .) For $1 \leq i, j \leq I_{\hbar}$ such that $i \rightarrow j$, we have

$$\left\| \mathcal{M}(e^{V \circ f \circ \kappa_i} \cdot \psi_{j,i} \cdot \chi_{\hbar}) - e^{V_j} \cdot \mathcal{M}(\psi_{j,i} \cdot \chi_{\hbar}) \right\|_{\mathcal{H}_{\hbar}^r(\mathbb{R}^{2d})} \leq C(f, V) \cdot \hbar^{\theta}$$

for some constant $C(f, V)$ independent of \hbar .

2005 *Proof.* The former claim should be obvious from the choice of the coordinates κ_i and the functions ψ_i for $i \in I_{\hbar}$. We can get the latter claim if we apply Corollary 6.3 to the multiplication operators by $e^{V \circ f \circ \kappa_i} \cdot \psi_{j,i} \cdot \chi_{\hbar} - e^{V_j} \cdot \psi_{j,i} \cdot \chi_{\hbar} = (e^{V \circ f \circ \kappa_i} - e^{V_j}) \cdot \psi_{j,i} \cdot \chi_{\hbar}$. \square

7.4 The anisotropic Sobolev spaces

Definition 7.12. The *Anisotropic Sobolev space* $\mathcal{H}_{\hbar}^r(P)$ is defined as the completion of $C_N^{\infty}(P)$ with respect to the norm

$$\|u\|_{\mathcal{H}_{\hbar}^r(P)} := \left(\sum_{i=1}^{I_{\hbar}} \|u_i\|_{\mathcal{H}_{\hbar}^r(\mathbb{R}^{2d})}^2 \right)^{1/2} \quad \text{for } u \in C_N^{\infty}(P),$$

2010 where $u_i = (\mathbf{I}_{\hbar}(u))_i \in C_0^{\infty}(\mathbb{D}(\hbar^{1/2-\theta}))$ are the local data defined in (7.12) and $\|u_i\|_{\mathcal{H}_{\hbar}^r(\mathbb{R}^{2d})}^2$ is the anisotropic Sobolev norm on $C_0^{\infty}(\mathbb{R}^{2d})$ in Definition 5.9. We define the Hilbert spaces $\mathcal{H}_{\hbar}^{r,\pm}(P)$ in the parallel manner, replacing $\|u_i\|_{\mathcal{H}_{\hbar}^r(\mathbb{R}^{2d})}^2$ by the norms $\|u_i\|_{\mathcal{H}_{\hbar}^{r,\pm}(\mathbb{R}^{2d})}^2$ respectively.

Remark 7.13. (1) By definition, the operation \mathbf{I}_{\hbar} extends uniquely to an isometric injection

$$\mathbf{I}_{\hbar} : \mathcal{H}_{\hbar}^r(P) \rightarrow \bigoplus_{i=1}^{I_{\hbar}} \mathcal{H}_{\hbar}^r(\mathbb{D}(\hbar^{1/2-\theta})) \subset \bigoplus_{i=1}^{I_{\hbar}} \mathcal{H}_{\hbar}^r(\mathbb{R}^{2d})$$

2015 where $\mathcal{H}_{\hbar}^r(\mathbb{D}(\hbar^{1/2-\theta}))$ denotes the subspace that consists of elements supported on the disk $\mathbb{D}(\hbar^{1/2-\theta})$.

(2) From (7.14), we have $\mathbf{I}_{\hbar}^* \circ \mathbf{I}_{\hbar} = \text{Id}$ on $\mathcal{H}_{\hbar}^r(P)$ and also on $\mathcal{H}_{\hbar}^{r,\pm}(P)$.

Lemma 7.14. *The projector $\mathbf{I}_{\hbar} \circ \mathbf{I}_{\hbar}^* : \mathcal{E}_{\hbar} \rightarrow \mathcal{E}_{\hbar}$ extends to bounded operators*

$$\mathbf{I}_{\hbar} \circ \mathbf{I}_{\hbar}^* : \bigoplus_{i=1}^{I_{\hbar}} \mathcal{H}_{\hbar}^{r,+}(\mathbb{R}^{2d}) \rightarrow \bigoplus_{i=1}^{I_{\hbar}} \mathcal{H}_{\hbar}^r(\mathbb{D}(\hbar^{1/2-\theta}))$$

and

$$\mathbf{I}_{\hbar} \circ \mathbf{I}_{\hbar}^* : \bigoplus_{i=1}^{I_{\hbar}} \mathcal{H}_{\hbar}^r(\mathbb{R}^{2d}) \rightarrow \bigoplus_{i=1}^{I_{\hbar}} \mathcal{H}_{\hbar}^{r,-}(\mathbb{D}(\hbar^{1/2-\theta})).$$

2020 *Further the operator norms of these projectors are bounded by a constant independent of \hbar .*

Remark 7.15. The operator $\mathbf{I}_{\hbar} \circ \mathbf{I}_{\hbar}^*$ will not be a bounded operator from $\bigoplus_{i=1}^{I_{\hbar}} \mathcal{H}_{\hbar}^r(\mathbb{R}^{2d})$ to itself.

Proof. To prove the claim, it is enough to apply Proposition 6.21 and Corollary 6.3 to each component of $\mathbf{I}_{\hbar} \circ \mathbf{I}_{\hbar}^*$ with setting

$$\mathcal{G}_{\hbar} = \{A_{j,i} \circ \kappa_{j,i} \mid 1 \leq i, j \leq I_{\hbar}, U_i \cap U_j \neq \emptyset\} \quad (7.20)$$

2025 and

$$\mathcal{X}_\hbar = \{\psi_j \circ \kappa_{j,i} \cdot \chi_\hbar \mid 1 \leq i, j \leq I_\hbar, U_i \cap U_j \neq \emptyset\},$$

and use (7.10). (See also the remark below.) \square

Remark 7.16. The affine transformation $A_{j,i}$ in (7.20) is that appeared in the choice of local coordinates in Proposition 7.3. Note that the prequantum transfer operator $\mathcal{L}_{A_{j,i}}$ is a unitary operator on $\mathcal{H}_\hbar^r(\mathbb{R}^{2d})$ (and on $\mathcal{H}_\hbar^{r,\pm}(\mathbb{R}^{2d})$), by Lemma 5.15, and hence we may neglect the post- or pre-composition of $\mathcal{L}_{A_{j,i}}$ when we consider the operator norm on $\mathcal{H}_\hbar^r(\mathbb{R}^{2d})$ (and on $\mathcal{H}_\hbar^{r,\pm}(\mathbb{R}^{2d})$). For the later argument, we also note that, from Lemma 5.15, the prequantum transfer operator $\mathcal{L}_{A_{j,i}}$ commutes with the projection operators $t_\hbar^{(k)}$ defined in (5.26).

For the operator \hat{F}_\hbar on the Hilbert space $\mathcal{H}_\hbar^r(P)$, we confirm the following fact at this point, though we will give a more detailed description later.

Lemma 7.17. *The operator \mathbf{F}_\hbar defined in (7.16) extends uniquely to the bounded operator*

$$\mathbf{F}_\hbar : \bigoplus_{i=1}^{I_\hbar} \mathcal{H}_\hbar^r(\mathbb{R}^{2d}) \rightarrow \bigoplus_{i=1}^{I_\hbar} \mathcal{H}_\hbar^r(\mathbb{D}(\hbar^{1/2-\theta})) \quad (7.21)$$

and the operator norm is bounded by a constant independent of \hbar . Consequently the same result holds for the prequantum transfer operator $\hat{F}_\hbar : \mathcal{H}_\hbar^r(P) \rightarrow \mathcal{H}_\hbar^r(P)$.

Proof. From (7.17), it is enough to prove that the operators $\mathbf{F}_{j,i}$ for $1 \leq i, j \leq I_\hbar$ with $i \rightarrow j$ are bounded operators on $\mathcal{H}_\hbar^r(\mathbb{R}^{2d})$ and that the operator norms are bounded by a constant independent of \hbar . To see this, we express the diffeomorphism $f_{j,i}$ in (7.18) as a composition

$$f_{j,i} = a_{j,i} \circ g_{j,i} \circ B_{j,i} \quad (7.22)$$

where

- $a_{j,i} : \mathbb{R}^{2d} \rightarrow \mathbb{R}^{2d}$ is a translation on \mathbb{R}^{2d} ,
- $B_{j,i} : \mathbb{R}^{2d} \rightarrow \mathbb{R}^{2d}$ is a linear map of the form (5.18), i.e. $B_{j,i} = \begin{pmatrix} A & 0 \\ 0 & {}_t A^{-1} \end{pmatrix}$, with A an expanding map such that $\|A^{-1}\| \leq 1/\lambda$,
- $g_{j,i}$ is a diffeomorphism such that $\mathcal{G}_\hbar = \{g_{j,i}\}_{1 \leq i, j \leq I_\hbar}$ satisfies the condition (G1), (G2) and (G3) in Subsection 6.4.

This is possible because, if we let $B_{j,i}$ be the linearization of $f_{j,i}$ at the origin and let $a_{j,i} \in \mathcal{A}$ be the translation such that $a_{j,i}(f_{j,i}(0)) = 0$, then $a_{j,i}$, $B_{j,i}$ and $g_{j,i} := a_{j,i}^{-1} \circ f_{j,i} \circ B_{j,i}^{-1}$ satisfies the required conditions.

Remark 7.18. This decomposition of the diffeomorphism $f_{j,i}$ will be used later in the proof of Proposition 7.20 where we study more detailed properties of $f_{j,i}$.

From the expression (7.22) of $f_{j,i}$ above, the operator $\mathbf{F}_{j,i}$ is expressed as the composition

$$\mathbf{F}_{j,i} = \mathcal{L}^{(0)} \circ \mathcal{L}^{(1)} \circ \mathcal{L}^{(2)} \quad (7.23)$$

2055 where $\mathcal{L}^{(0)} := \mathcal{L}_{a_{j,i}}$ and $\mathcal{L}^{(2)} := \mathcal{L}_{B_{j,i}}$ are the Euclidean prequantum transfer operators (6.19) for the diffeomorphism a_{ij} and B_{ij} respectively, while $\mathcal{L}^{(1)}$ is the operator of the form

$$\mathcal{L}^{(1)}u = \mathcal{L}_{g_{j,i}} \left(((e^{V \circ f \circ \kappa_i} \cdot \psi_{j,i} \cdot \chi_{\hbar}) \circ B_{j,i}^{-1}) \cdot u \right)$$

with $\psi_{j,i}$ the function defined in (7.19). Note that the functions

$$(e^{V \circ f \circ \kappa_i} \cdot \psi_{j,i} \cdot \chi_{\hbar}) \circ B_{j,i}^{-1} = (e^{V \circ f \circ \kappa_i} \cdot \chi_{\hbar}) \circ B_{j,i}^{-1} \cdot (\psi_j \circ a_{j,i} \circ g_{j,i})$$

is supported on the disk $\mathbb{D}(2\hbar^{1/2-\theta})$, provided that \hbar is sufficiently small. Hence we may
2060 write the operator $\mathcal{L}^{(1)}$ as

$$\mathcal{L}^{(1)}u = \mathcal{L}_{g_{j,i}} \circ \mathcal{M}(\chi_{\hbar}) \left(((e^{V \circ f \circ \kappa_i} \cdot \psi_{j,i} \cdot \chi_{\hbar}) \circ B_{j,i}^{-1}) \cdot u \right).$$

From Lemma 5.15, $\mathcal{L}_{a_{j,i}} : \mathcal{H}_{\hbar}^r(\mathbb{R}^{2d}) \rightarrow \mathcal{H}_{\hbar}^r(\mathbb{R}^{2d})$ is a unitary operator. From Lemma 5.12, the operator $\mathcal{L}_{B_{j,i}} : \mathcal{H}_{\hbar}^r(\mathbb{R}^{2d}) \rightarrow \mathcal{H}_{\hbar}^{r,+}(\mathbb{R}^{2d})$ is bounded and the operator norm is bounded by a constant independent of \hbar . From Lemma 6.21 and Corollary 6.2, so is the operator $\mathcal{L}^{(1)} : \mathcal{H}_{\hbar}^{r,+}(\mathbb{R}^{2d}) \rightarrow \mathcal{H}_{\hbar}^r(\mathbb{R}^{2d})$, because

$$\mathcal{X}_{\hbar} = \{(e^{V \circ f \circ \kappa_i} \cdot \psi_{j,i} \cdot \chi_{\hbar}) \circ B_{j,i}^{-1}\}_{1 \leq i, j \leq I_{\hbar}}, \quad \mathcal{G}_{\hbar} = \{g_{j,i}\}_{1 \leq i, j \leq I_{\hbar}} \quad (7.24)$$

2065 satisfy respectively the conditions (C1), (C2) in Section 6.1 and (G1), (G2), (G3) in Section 6.4. \square

7.5 The main propositions

In this subsection, we give two key propositions which will give Theorem 7.1 as a consequence. To state the propositions, we introduce the projection operators

$$\mathbf{t}_{\hbar}^{(k)} : \bigoplus_{i=1}^{I_{\hbar}} \mathcal{H}_{\hbar}^r(\mathbb{R}^{2d}) \rightarrow \bigoplus_{i=1}^{I_{\hbar}} \mathcal{H}_{\hbar}^r(\mathbb{R}^{2d}), \quad \mathbf{t}_{\hbar}^{(k)}((u_i)_{i=1}^{I_{\hbar}}) = (t_{\hbar}^{(k)}(u_i))_{i=1}^{I_{\hbar}}, \quad (7.25)$$

2070 for $0 \leq k \leq n$ and

$$\tilde{\mathbf{t}}_{\hbar} : \bigoplus_{i=1}^{I_{\hbar}} \mathcal{H}_{\hbar}^r(\mathbb{R}^{2d}) \rightarrow \bigoplus_{i=1}^{I_{\hbar}} \mathcal{H}_{\hbar}^r(\mathbb{R}^{2d}), \quad \tilde{\mathbf{t}}_{\hbar}((u_i)_{i=1}^{I_{\hbar}}) = (\tilde{t}_{\hbar}(u_i))_{i=1}^{I_{\hbar}}, \quad (7.26)$$

which are just applications of the projection operators $t_{\hbar}^{(k)}$ and \tilde{t}_{\hbar} introduced in (5.26) and (5.27) to each component. For brevity of notation, we set

$$\mathbf{t}_{\hbar}^{(n+1)} = \tilde{\mathbf{t}}_{\hbar}. \quad (7.27)$$

Then the set of operators $\{\mathbf{t}_h^{(k)}\}_{k=0}^{n+1}$ are complete sets of mutually commuting projection operators .

2075 The following Proposition shows that the projectors $\mathbf{t}_h^{(k)}$ almost commute with the projector $(\mathbf{I}_h \circ \mathbf{I}_h^*)$.

Proposition 7.19. *There are constants $\epsilon > 0$ and $C > 0$, independent of \hbar , such that the following holds: We have that*

$$\begin{aligned} \left\| \mathbf{t}_h^{(k)} \circ (\mathbf{I}_h \circ \mathbf{I}_h^*) \right\|_{\bigoplus_{i=1}^{I_h} \mathcal{H}_h^{r,-}(\mathbb{R}^{2d}) \rightarrow \bigoplus_{i=1}^{I_h} \mathcal{H}_h^{r,+}(\mathbb{R}^{2d})} &< C, \quad \text{and} \\ \left\| (\mathbf{I}_h \circ \mathbf{I}_h^*) \circ \mathbf{t}_h^{(k)} \right\|_{\bigoplus_{i=1}^{I_h} \mathcal{H}_h^{r,-}(\mathbb{R}^{2d}) \rightarrow \bigoplus_{i=1}^{I_h} \mathcal{H}_h^{r,+}(\mathbb{R}^{2d})} &< C \end{aligned}$$

2080 for $0 \leq k \leq n$. (Hence the same statement holds as operators on $\bigoplus_{i=1}^{I_h} \mathcal{H}_h^r(\mathbb{R}^{2d})$.) Also we have, for the norm of the commutators, that

$$\left\| \left[\mathbf{t}_h^{(k)}, (\mathbf{I}_h \circ \mathbf{I}_h^*) \right] \right\|_{\bigoplus_{i=1}^{I_h} \mathcal{H}_h^{r,-}(\mathbb{R}^{2d}) \rightarrow \bigoplus_{i=1}^{I_h} \mathcal{H}_h^{r,+}(\mathbb{R}^{2d})} \leq C\hbar^\epsilon \quad (7.28)$$

for $0 \leq k \leq n$.

Proof. From Lemma 7.14, $\mathbf{I}_h \circ \mathbf{I}_h^*$ are bounded as operators from $\bigoplus_{i=1}^{I_h} \mathcal{H}_h^{r,+}(\mathbb{R}^{2d})$ to $\bigoplus_{i=1}^{I_h} \mathcal{H}_h^r(\mathbb{R}^{2d})$ (resp. from $\bigoplus_{i=1}^{I_h} \mathcal{H}_h^r(\mathbb{R}^{2d})$ to $\bigoplus_{i=1}^{I_h} \mathcal{H}_h^{r,-}(\mathbb{R}^{2d})$) and the operator norm is bounded by a constant independent of \hbar . From Lemma 5.13, so are the operators $\mathbf{t}_h^{(k)}$ as operators from $\bigoplus_{i=1}^{I_h} \mathcal{H}_h^{r,-}(\mathbb{R}^{2d})$ to $\bigoplus_{i=1}^{I_h} \mathcal{H}_h^r(\mathbb{R}^{2d})$ (resp. from $\bigoplus_{i=1}^{I_h} \mathcal{H}_h^r(\mathbb{R}^{2d})$ to $\bigoplus_{i=1}^{I_h} \mathcal{H}_h^{r,+}(\mathbb{R}^{2d})$). Hence we obtain the first two inequalities. To prove (7.28), we take $\mathbf{u} = (u_i) \in \bigoplus_{i=1}^{I_h} \mathcal{H}_h^{r,-}(\mathbb{R}^{2d})$ arbitrarily. From the definition, we have

$$\mathbf{t}_h^{(k)} \circ (\mathbf{I}_h \circ \mathbf{I}_h^*)(\mathbf{u}) = \left(\sum_{i:U_i \cap U_j \neq \emptyset} \mathbf{t}_h^{(k)} \circ \mathcal{M}(\psi_j) \circ \mathcal{L}_{\kappa_{j,i}} \circ \mathcal{M}(\chi_h)(u_i) \right)_{j=1}^{I_h}$$

and

$$(\mathbf{I}_h \circ \mathbf{I}_h^*) \circ \mathbf{t}_h^{(k)}(\mathbf{u}) = \left(\sum_{i:U_i \cap U_j \neq \emptyset} \mathcal{M}(\chi_j) \circ \mathcal{L}_{\kappa_{j,i}} \circ \mathcal{M}(\chi_h) \circ \mathbf{t}_h^{(k)}(u_i) \right)_{j=1}^{I_h}.$$

Applying Corollary 6.23 and Corollary 6.7 to each components with the setting (7.20) and recalling Remark 7.16 and (7.10), we obtain (7.28). \square

2085 The next Proposition stated for \mathbf{F}_h is now very close to Theorem 7.1.

Proposition 7.20. *There are constants $\epsilon > 0$ and $C > 0$ independent of \hbar such that*

$$\left\| \left[\mathbf{F}_h, \mathbf{t}_h^{(k)} \right] \right\|_{\bigoplus_{i=1}^{I_h} \mathcal{H}_h^r(\mathbb{R}^{2d}) \rightarrow \bigoplus_{i=1}^{I_h} \mathcal{H}_h^r(\mathbb{R}^{2d})} \leq C\hbar^\epsilon \quad \text{for } 1 \leq k \leq n+1. \quad (7.29)$$

Further there exists a constant $C_0 > 0$, which is independent of f , V and \hbar , such that

(1) For $0 \leq k \leq n+1$, it holds

$$\left\| \mathbf{t}_h^{(k)} \circ \mathbf{F}_h \circ \mathbf{t}_h^{(k)} \right\|_{\bigoplus_{i=1}^{I_h} \mathcal{H}_h^r(\mathbb{R}^{2d}) \rightarrow \bigoplus_{i=1}^{I_h} \mathcal{H}_h^r(\mathbb{R}^{2d})} \leq C_0 \sup (|e^V| \|Df|_{E_u}\|_{\min}^{-k} |\det Df|_{E_u}|^{-1/2})$$

(2) If $\mathbf{u} \in \bigoplus_{i=1}^{I_h} \mathcal{H}_h^r(\mathbb{R}^{2d})$ satisfies $\mathbf{I}_h \circ \mathbf{I}_h^*(\mathbf{u}) = \mathbf{u}$ and

$$\|\mathbf{u} - (\mathbf{I}_h \circ \mathbf{I}_h^*) \circ \mathbf{t}_h^{(k)}(\mathbf{u})\|_{\mathcal{H}_h^r} < \|\mathbf{u}\|_{\mathcal{H}_h^r}/2 \quad \text{for some } 0 \leq k \leq n,$$

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then we have

$$\|\mathbf{t}_h^{(k)} \circ \mathbf{F}_h \circ \mathbf{t}_h^{(k)}(\mathbf{u})\|_{\mathcal{H}_h^r} \geq C_0^{-1} \cdot \inf (|e^V| \|Df|_{E_u}\|_{\max}^{-k} |\det Df|_{E_u}|^{-1/2}) \cdot \|\mathbf{u}\|_{\mathcal{H}_h^r}.$$

Proof. We recall the argument in the proof of Lemma 7.17, in particular, the expression (7.23) of the operator \mathbf{F}_{ij} . Then we observe that, for each i, j such that $i \rightarrow j$,

(i) From Proposition 5.11 and Lemma 5.15, the projection operators $t_h^{(k)}$ for $0 \leq k \leq n$ and \tilde{t}_h commute with the operator $\mathcal{L}^{(0)}$ and $\mathcal{L}^{(2)}$ (defined in (7.23)).

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(ii) From Lemma 5.15, the operator $\mathcal{L}^{(0)}$ is a unitary operator on $\mathcal{H}_h^r(\mathbb{R}^{2d})$ and also on $\mathcal{H}_h^{r,\pm}(\mathbb{R}^{2d})$.

(iii) From Proposition 6.19, the operator $\mathcal{L}^{(1)}$ extends to a bounded operator from $\mathcal{H}_h^{r,+}(\mathbb{R}^{2d})$ to $\mathcal{H}_h^r(\mathbb{R}^{2d})$ (resp. from $\mathcal{H}_h^r(\mathbb{R}^{2d})$ to $\mathcal{H}_h^{r,-}(\mathbb{R}^{2d})$) and the operator norm is bounded by a constant C_0 , provided that \tilde{h} is sufficiently small.

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(iv) Applying Proposition 5.11 to $\mathcal{L}^{(2)}$, we see that the operator $\mathcal{L}^{(2)}$ is a bounded operator on $\mathcal{H}_h^r(\mathbb{R}^{2d})$ and that

$$C_0^{-1} \|B_{j,i}|_{E^+}\|^{-k} \cdot |\det B_{j,i}|_{E^+}|^{-1/2} \leq \frac{\|\mathcal{L}^{(2)}u\|_{\mathcal{H}_h^r(\mathbb{R}^{2d})}}{\|u\|_{\mathcal{H}_h^r(\mathbb{R}^{2d})}} \leq C_0 \|B_{j,i}^{-1}|_{E^+}\|^k \cdot |\det B_{j,i}|_{E^+}|^{-1/2}$$

for $0 \neq u \in H'_k := \text{Im } t_h^{(k)}$ and for $0 \leq k \leq n$, where $E^+ = \mathbb{R}^{2d} \oplus \{0\}$. Further we have

$$\|\mathcal{L}^{(2)}u\|_{\mathcal{H}_h^r(\mathbb{R}^{2d})} \leq C_0 \|B_{j,i}^{-1}|_{E^+}\|^{n+1} |\det B_{j,i}|_{E^+}|^{-1/2} \|u\|_{\mathcal{H}_h^r(\mathbb{R}^{2d})}$$

for $u \in \tilde{H}' := \text{Im } \tilde{t}_h$.

(v) By simple comparison, we have

$$\begin{aligned} C_0^{-1} \cdot \inf (|e^V| \|Df|_{E_u}\|_{\max}^{-k} |\det Df|_{E_u}|^{-1/2}) \\ < e^{V_j} \cdot \|B_{j,i}^{-1}|_{E^+}\|^k \cdot |\det B_{j,i}|_{E^+}|^{-1/2} \\ < C_0 \sup (|e^V| \|Df|_{E_u}\|_{\min}^{-k} |\det Df|_{E_u}|^{-1/2}). \end{aligned}$$

2105 (vi) Applying Lemma 6.22 to the setting (7.24) and Lemma 7.11, we have

$$\|\mathcal{L}^{(1)} \circ t_h^{(k)} - e^{V_j} \cdot \mathcal{M}((\psi_{j,i} \cdot \chi_h) \circ B_{j,i}^{-1}) \circ t_h^{(k)}\|_{\mathcal{H}_h^r(\mathbb{R}^{2d})} \leq C\hbar^\epsilon \quad \text{for } 0 \leq k \leq n \quad (7.30)$$

with some positive constants C and ϵ independent of \hbar .

(vii) Applying Corollary 6.23 to the setting (7.24), we have that

$$\|[\mathcal{L}^{(1)}, t_h^{(k)}]\|_{\mathcal{H}_h^r(\mathbb{R}^{2d})} \leq C\hbar^\epsilon \quad \text{for } 0 \leq k \leq n+1$$

with setting $t_h^{(n+1)} = \tilde{t}_h$ for the case $k = n+1$. This is true with $\mathcal{H}_h^r(\mathbb{R}^{2d})$ replaced by $\mathcal{H}_h^{r,\pm}(\mathbb{R}^{2d})$.

2110 From the observations (i), (ii), (iv) and (vii) above, it follows

$$\|[\mathbf{F}_{j,i}, t_h^{(k)}]\|_{\mathcal{H}_h^r(\mathbb{R}^{2d}) \rightarrow \mathcal{H}_h^r(\mathbb{R}^{2d})} \leq C\hbar^\epsilon \quad \text{for } 0 \leq k \leq n+1.$$

This, together with (7.17), implies (7.29).

We prove Claim (1). Take $\mathbf{u} = (u_i)_{i=1}^{I_h} \in \bigoplus_{i=1}^{I_h} \mathcal{H}_h^r(\mathbb{R}^{2d})$ arbitrarily. Let $0 \leq k \leq n+1$ and set

$$v_{j,i} = t_h^{(k)} \circ \mathbf{F}_{j,i} \circ t_h^{(k)}(u_i), \quad u_{j,i} = \psi_{j,i} \cdot \chi_h \cdot u_i = (\psi_j \circ f_{j,i}) \cdot \chi_h \cdot u_i \quad (7.31)$$

for $1 \leq i, j \leq I_h$ such that $i \rightarrow j$. Suppose that $0 \leq k \leq n$. Then, using the expression (7.23) of $\mathbf{F}_{j,i}$, we obtain, by (vi) and Corollary 6.7,

$$\begin{aligned} \|v_{j,i}\|_{\mathcal{H}_h^r(\mathbb{R}^{2d})} &= \|t_h^{(k)} \circ \mathcal{L}^{(1)} \circ \mathcal{L}^{(2)} \circ t_h^{(k)} u_i\|_{\mathcal{H}_h^r(\mathbb{R}^{2d})} \\ &= e^{V_j} \cdot \|\mathcal{L}^{(2)} \circ \mathcal{M}(\psi_{j,i} \cdot \chi_h) \circ t_h^{(k)}(u_i)\|_{\mathcal{H}_h^r(\mathbb{R}^{2d})} + \mathcal{O}(\hbar^\epsilon \|u_i\|_{\mathcal{H}_h^r(\mathbb{R}^{2d})}) \\ &= e^{V_j} \cdot \|\mathcal{L}^{(2)} \circ t_h^{(k)}(u_{j,i})\|_{\mathcal{H}_h^r(\mathbb{R}^{2d})} + \mathcal{O}(\hbar^\epsilon \cdot \|u_i\|_{\mathcal{H}_h^r(\mathbb{R}^{2d})}) \end{aligned}$$

where $\mathcal{O}(\hbar^\epsilon \cdot \|u_i\|_{\mathcal{H}_h^r(\mathbb{R}^{2d})})$ denotes positive terms that are bounded by $C\hbar^\epsilon \cdot \|u_i\|_{\mathcal{H}_h^r(\mathbb{R}^{2d})}$.

2115 Hence, from (iv), we get the estimates

$$\|v_{j,i}\|_{\mathcal{H}_h^r} \leq C_0 \cdot e^{V_j} \cdot \|B_{j,i}|_{E^+}\|_{\min}^{-k} \cdot |\det B_{j,i}|_{E^+}|^{-1/2} \cdot \|t_h^{(k)}(u_{j,i})\|_{\mathcal{H}_h^r(\mathbb{R}^{2d})} + \mathcal{O}(\hbar^\epsilon \cdot \|u_i\|_{\mathcal{H}_h^r(\mathbb{R}^{2d})}) \quad (7.32)$$

and

$$\|v_{j,i}\|_{\mathcal{H}_h^r} \geq C_0^{-1} \cdot e^{V_j} \cdot \|B_{j,i}|_{E^+}\|_{\max}^{-k} \cdot |\det B_{j,i}|_{E^+}|^{-1/2} \cdot \|t_h^{(k)}(u_{j,i})\|_{\mathcal{H}_h^r(\mathbb{R}^{2d})} - \mathcal{O}(\hbar^\epsilon \cdot \|u_i\|_{\mathcal{H}_h^r(\mathbb{R}^{2d})}) \quad (7.33)$$

for $0 \leq k \leq n$.

Actually the upper estimate (7.32) can be strengthened by modifying the argument above, so that it also holds for $k = n+1$. (Note that the argument above is not true for $k = n+1$, because (7.30) does not hold in that case.) Indeed we can show that

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$$\|v_{j,i}\|_{\mathcal{H}_h^{r,+}(\mathbb{R}^{2d})} \leq C_0 e^{V_j} \|B_{j,i}|_{E^+}\|_{\min}^{-k} |\det B_{j,i}|_{E^+}|^{-1/2} \cdot \|t_h^{(k)}(u_{j,i})\|_{\mathcal{H}_h^r(\mathbb{R}^{2d})} \quad (7.34)$$

for all $0 \leq k \leq n + 1$. Let $B_0 : \mathbb{R}^{2d} \rightarrow \mathbb{R}^{2d}$ be the linear map defined by

$$B_0(x_+, x_-) = (\lambda_0 \cdot x_+, \lambda_0^{-1} \cdot x_-) \quad \text{for } (x_+, x_-) \in \mathbb{R}^{2d} = \mathbb{R}^d \oplus \mathbb{R}^d$$

where λ_0 is an absolute constant greater than 9. (Say $\lambda_0 = 10$.) Then we write the operator \mathbf{F}_{ij} as

$$\mathbf{F}_{ij} = \mathcal{L}^{(0)} \circ \mathcal{L}_{B_0} \circ \tilde{\mathcal{L}}^{(1)} \circ \tilde{\mathcal{L}}^{(2)} \quad \text{with setting } \tilde{\mathcal{L}}^{(1)} = \mathcal{L}_{B_0^{-1}} \circ \mathcal{L}^{(1)} \circ \mathcal{L}_{B_0}, \quad \tilde{\mathcal{L}}^{(2)} = \mathcal{L}_{B_0^{-1} \circ B_{j,i}}.$$

The operator \mathcal{L}_{B_0} is a bounded operator from $\mathcal{H}_h^{r,-}(\mathbb{R}^{2d})$ to $\mathcal{H}_h^{r,+}(\mathbb{R}^{2d})$, from Lemma 5.12. The operator $\tilde{\mathcal{L}}^{(1)}$ is a bounded operator from $\mathcal{H}_h^r(\mathbb{R}^{2d})$ to $\mathcal{H}_h^{r,-}(\mathbb{R}^{2d})$ and the operator norm is bounded by $C_0 e^{V_j}$, from Proposition 6.21. And the observation (iv) holds true with $\mathcal{L}^{(2)}$ replaced by $\tilde{\mathcal{L}}^{(2)}$. Hence we obtain (7.34).

From (7.10) in the choice of the coordinate system $\{\kappa_i\}_{i=1}^{I_h}$ (see Proposition 7.3) and from Lemma 5.16, we have

$$\|\mathbf{F} \circ \mathbf{t}^{(k)}(\mathbf{u})\|_{\mathcal{H}_h^r}^2 = \left\| \left(\sum_{i:i \rightarrow j} v_{j,i} \right)_j \right\|_{\mathcal{H}_h^r}^2 \leq C_0 \sum_{i,j:i \rightarrow j} \|v_{j,i}\|_{\mathcal{H}_h^r(\mathbb{R}^{2d})}^2 + \mathcal{O}(\hbar^\epsilon \cdot \|\mathbf{u}\|_{\mathcal{H}_h^r}^2)$$

and

$$\sum_{i,j:i \rightarrow j} \|u_{j,i}\|_{\mathcal{H}_h^r(\mathbb{R}^{2d})}^2 \leq C_0 \|\mathbf{u}\|_{\mathcal{H}_h^r}^2,$$

provided \hbar is sufficiently small. Hence we obtain Claim 1 as a consequence of (7.34) and the observation (v).

Remark 7.21. Because of the inconvenient property of the inner product $(\cdot, \cdot)_{\mathcal{H}_h^r}$, noted in the paragraph just before Lemma 5.16, the two inequalities above are not an immediate consequence of the estimate (7.10) on the intersection multiplicities of the supports of $v_{j,i}$ and $u_{j,i}$. We have to use Lemma 5.16.

We prove Claim (2). We continue the argument in the proof of Claim (1). Note that we already have the estimate (7.33) for each $v_{j,i}$. Below we show that the functions $v_{j,i}$ do not cancel out too much when we sum up them with respect to i such that $i \rightarrow j$. More precisely, we prove the estimate

$$\sum_{i,i',j:i \rightarrow j, i' \rightarrow j, i \neq i'} \operatorname{Re}(v_{j,i}, v_{j,i'})_{\mathcal{H}_h^r(\mathbb{R}^{2d})} \geq -C\hbar^\epsilon \cdot \|\mathbf{u}\|_{\mathcal{H}_h^r}^2 \quad (7.35)$$

where $\sum_{i,i',j:i \rightarrow j, i' \rightarrow j, i \neq i'}$ denotes the sum over $1 \leq i, i', j \leq I_h$ that satisfies $i \rightarrow j$, $i' \rightarrow j$ and $i \neq i'$. For $1 \leq j \leq I_h$, let $I(j)$ be the set of integers $1 \leq \ell \leq I_h$ such that there exists $1 \leq \ell', \ell'' \leq I_h$ satisfying $U_\ell \cap U_{\ell'} \neq \emptyset$, $U_{\ell'} \cap U_{\ell''} \neq \emptyset$ and $\ell' \rightarrow j$. Note that we have

$$\max_{1 \leq j \leq I_h} \#I(j) \leq C_0. \quad (7.36)$$

Consider $1 \leq i, i', j \leq I_h$ that satisfies $i \rightarrow j$, $i \rightarrow j'$ and $i \neq i'$. We express $v_{j,i}$ as

$$v_{j,i} = t_h^{(k)} \circ \mathcal{M}(e^{V \circ \kappa_j} \cdot \psi_j) \circ \mathcal{L}_{f_{j,i}} \circ t_h^{(k)} \left(\psi_i \cdot \sum_{\ell \in I(j)} \mathcal{L}_{\kappa_{i,\ell}}(u_\ell) \right).$$

2145 We can of course write $v_{j,i'}$ in the same form with i replaced by i' , but we rewrite it as

$$v_{j,i'} = t_h^{(k)} \circ \mathcal{M}(e^{V \circ \kappa_j} \cdot \psi_j) \circ \mathcal{L}_{f_{j,i}} \circ \mathcal{L}_{\kappa_{i,i'}} \circ t_h^{(k)} \left(\psi_{i'} \cdot \sum_{\ell \in I(j)} \mathcal{L}_{\kappa_{i',\ell}}(u_\ell) \right).$$

We change the order of operators on the right hand sides above, estimating the commutators by Corollary 6.7 and Corollary 6.23 and noting the relation $\kappa_{i,i'} \circ \kappa_{i',\ell} = \kappa_{i,\ell}$. Then we get

$$\left\| v_{j,i} - \mathcal{M}(e^{V \circ \kappa_{i'}} \cdot \psi_j \cdot \psi_i \circ f_{j,i}^{-1}) \circ \mathcal{L}_{f_{j,i}} \circ t_h^{(k)} \left(\sum_{\ell \in I(j)} \mathcal{L}_{\kappa_{i,\ell}}(u_\ell) \right) \right\|_{\mathcal{H}_h^r(\mathbb{R}^{2d})} \leq C\hbar^\epsilon \sum_{\ell \in I(j)} \|u_\ell\|_{\mathcal{H}_h^r(\mathbb{R}^{2d})}$$

and

$$\left\| v_{j,i'} - \mathcal{M}(e^{V \circ \kappa_{i'}} \cdot \psi_j \cdot \psi_{i'} \circ f_{j,i'}^{-1}) \circ \mathcal{L}_{f_{j,i}} \circ t_h^{(k)} \left(\sum_{\ell \in I(j)} \mathcal{L}_{\kappa_{i,\ell}}(u_\ell) \right) \right\|_{\mathcal{H}_h^r(\mathbb{R}^{2d})} \leq C\hbar^\epsilon \sum_{\ell \in I(j)} \|u_\ell\|_{\mathcal{H}_h^r(\mathbb{R}^{2d})}.$$

Therefore, by Corollary 6.4, we get

$$\begin{aligned} \operatorname{Re}(v_{j,i}, v_{j,i'}) &\geq \left\| \mathcal{M} \left(e^{V \circ \kappa_{i'}} \psi_j \cdot \sqrt{\psi_i \circ f_{j,i}^{-1} \cdot \psi_{i'} \circ f_{j,i'}^{-1}} \right) \circ \mathcal{L}_{f_{j,i}} \circ t_h^{(k)} \left(\sum_{\ell \in I(j)} \mathcal{L}_{\kappa_{i,\ell}}(u_\ell) \right) \right\|_{\mathcal{H}_h^r(\mathbb{R}^{2d})}^2 \\ &\quad - C\hbar^\epsilon \cdot \sum_{\ell \in I(j)} \|u_\ell\|_{\mathcal{H}_h^r(\mathbb{R}^{2d})}^2 \\ &\geq -C\hbar^\epsilon \cdot \sum_{\ell \in I(j)} \|u_\ell\|_{\mathcal{H}_h^r(\mathbb{R}^{2d})}^2. \end{aligned}$$

2150 Summing up the both sides of the inequality above for all j, i, i' with $i \rightarrow j$, $i' \rightarrow j$ and $i \neq i'$ and using (7.36), we obtain (7.35).

From (7.35), (7.33) and the observation (iv), we get

$$\begin{aligned} \|\mathbf{F}_h \circ \mathbf{t}^{(k)}(\mathbf{u})\|_{\mathcal{H}_h^r}^2 &= \sum_j \sum_{i,i'} \Re(v_{j,i}, v_{j,i'})_{\mathcal{H}_h^r(\mathbb{R}^{2d})} \\ &\geq \sum_{i,j:i \rightarrow j} \|v_{j,i}\|_{\mathcal{H}_h^r(\mathbb{R}^{2d})}^2 - C\hbar^\epsilon \|\mathbf{u}\|_{\mathcal{H}_h^r}^2 \\ &\geq \sum_{i,j:i \rightarrow j} C_0^{-1} e^{V_j} \cdot \|B_{j,i}|_{E^+}\|_{\max}^{-k} |\det B_{j,i}|_{E^+}^{-1/2} \|t_h^{(k)}(u_{j,i})\|_{\mathcal{H}_h^r(\mathbb{R}^{2d})}^2 - C\hbar^\epsilon \|\mathbf{u}\|_{\mathcal{H}_h^r}^2 \\ &\geq C_0^{-1} \inf(|e^V| \|Df|_{E_u}\|_{\max}^{-k} |\det Df|_{E_u}|^{-1/2}) \sum_{i,j:i \rightarrow j} \|t_h^{(k)}(u_{j,i})\|_{\mathcal{H}_h^r(\mathbb{R}^{2d})}^2 - C\hbar^\epsilon \|\mathbf{u}\|_{\mathcal{H}_h^r}^2. \quad (7.37) \end{aligned}$$

To finish the proof, we compare $\sum_{i,j:i \rightarrow j} \|t_h^{(k)}(u_{j,i})\|_{\mathcal{H}_h^r(\mathbb{R}^{2d})}^2$ and $\|\mathbf{u}\|_{\mathcal{H}_h^r}^2$. To this end, we use the assumptions in Claim (2), of course. From the assumption and (7.28) in Proposition 7.19, we have

$$\begin{aligned} \|\mathbf{u}\|_{\mathcal{H}_h^r} &\leq 2 \cdot \|(\mathbf{I}_h \circ \mathbf{I}_h^*) \circ \mathbf{t}_h^{(k)}(\mathbf{u})\|_{\mathcal{H}_h^r} \leq 2\|\mathbf{t}_h^{(k)} \circ (\mathbf{I}_h \circ \mathbf{I}_h^*)(\mathbf{u})\|_{\mathcal{H}_h^r} + C\hbar^\epsilon \|\mathbf{u}\|_{\mathcal{H}_h^r} \\ &= 2\|\mathbf{t}_h^{(k)}(\mathbf{u})\|_{\mathcal{H}_h^r} + C\hbar^\epsilon \|\mathbf{u}\|_{\mathcal{H}_h^r}. \end{aligned} \quad (7.38)$$

We also have, from Corollary 6.7, Lemma 5.16 and (7.10), that

$$\begin{aligned} \|\mathbf{t}_h^{(k)}(\mathbf{u})\|_{\mathcal{H}_h^r}^2 &= \sum_i \|t_h^{(k)}(u_i)\|^2 = \left\| \sum_i t_h^{(k)} \left(\sum_{j:i \rightarrow j} u_{j,i} \right) \right\|_{\mathcal{H}_h^r(\mathbb{R}^{2d})}^2 \\ &= \sum_i \left\| t_h^{(k)} \left(\sum_{j:i \rightarrow j} \mathcal{M}(\psi_{j,i} \cdot \chi_h) u_i \right) \right\|_{\mathcal{H}_h^r(\mathbb{R}^{2d})}^2 \\ &\leq \sum_i \left\| \sum_{j:i \rightarrow j} \mathcal{M}(\psi_{j,i} \cdot \chi_h) \circ t_h^{(k)}(u_i) \right\|_{\mathcal{H}_h^r(\mathbb{R}^{2d})}^2 + C\hbar^\epsilon \|\mathbf{u}\|_{\mathcal{H}_h^r}^2 \\ &\leq C_0 \sum_{i,j:i \rightarrow j} \|\mathcal{M}(\psi_{j,i} \cdot \chi_h) \circ t_h^{(k)}(u_j)\|_{\mathcal{H}_h^r(\mathbb{R}^{2d})}^2 + C\hbar^\epsilon \|\mathbf{u}\|_{\mathcal{H}_h^r}^2 \\ &\leq C_0 \sum_{i,j:i \rightarrow j} \|t_h^{(k)}(u_{j,i})\|_{\mathcal{H}_h^r(\mathbb{R}^{2d})}^2 + C\hbar^\epsilon \|\mathbf{u}\|_{\mathcal{H}_h^r}^2. \end{aligned}$$

We conclude Claim (2) from these inequalities and (7.37). \square

7.6 Proof of Theorem 7.1

We finish the proof of Theorem 7.1. Actually we have almost done with the essential part of the proof. Below we give a formal argument to complete it. Let us begin with introducing the operators

$$\check{\tau}_h^{(k)} = \mathbf{I}_h^* \circ \mathbf{t}_h^{(k)} \circ \mathbf{I}_h : \mathcal{H}_h^r(P) \rightarrow \mathcal{H}_h^r(P) \quad (7.39)$$

for $0 \leq k \leq n+1$. From Proposition 7.19, these are bounded operators with operator norms bounded by a constant C independent of \hbar , and satisfy

$$\check{\tau}_h^{(1)} + \check{\tau}_h^{(2)} + \cdots + \check{\tau}_h^{(n)} + \check{\tau}_h^{(n+1)} = \text{Id}, \quad (7.40)$$

$$\|\check{\tau}_h^{(k)} \circ \check{\tau}_h^{(k)} - \check{\tau}_h^{(k)}\|_{\mathcal{H}_h^r(P)} \leq C\hbar^\epsilon \quad \text{for } 0 \leq k \leq n+1, \quad \text{and} \quad (7.41)$$

$$\|\check{\tau}_h^{(k)} \circ \check{\tau}_h^{(k')}\|_{\mathcal{H}_h^r(P)} \leq C\hbar^\epsilon \quad \text{for } 0 \leq k, k' \leq n+1 \text{ with } k \neq k' \quad (7.42)$$

for some constants $\epsilon > 0$ and $C > 0$.

Lemma 7.22. *The operators $\check{\tau}_h^{(k)}$, $0 \leq k \leq n$, are trace class operators. There exist constants $\epsilon > 0$ and $C > 0$, independent of \hbar , such that*

$$\|\check{\tau}_h^{(k)} : \mathcal{H}_h^r(P) \rightarrow \mathcal{H}_h^r(P)\|_{tr} \leq C\hbar^{-d}$$

2160 and that

$$\|\check{\tau}_h^{(k)} - \check{\tau}_h^{(k)} \circ \check{\tau}_h^{(k)} : \mathcal{H}_h^r(P) \rightarrow \mathcal{H}_h^r(P)\|_{tr} \leq C\hbar^{-d+\epsilon}$$

Proof. It is enough to prove the corresponding statement for

$$\mathbf{I}_h \circ \check{\tau}_h^{(k)} \circ \mathbf{I}_h^* = (\mathbf{I}_h \circ \mathbf{I}_h^*) \circ \mathbf{t}_h^{(k)} \circ (\mathbf{I}_h \circ \mathbf{I}_h^*) : \bigoplus_{i=1}^{I_h} \mathcal{H}_h^r(\mathbb{R}^{2d}) \rightarrow \bigoplus_{i=1}^{I_h} \mathcal{H}_h^r(\mathbb{R}^{2d}).$$

2165 Applying Corollary 6.13 and Proposition 6.21 (to deal with the non-linearity of the coordinate change transformations), we see that each component of this operator is a trace class operator and hence so is itself. Recalling Proposition 7.3(3), we obtain the first claim by summing up the trace norm of the components. To prove the second claim, we compare the operator above with

$$(\mathbf{I}_h \circ \check{\tau}_h^{(k)} \circ \mathbf{I}_h^*) \circ (\mathbf{I}_h \circ \check{\tau}_h^{(k)} \circ \mathbf{I}_h^*) = (\mathbf{I}_h \circ \mathbf{I}_h^*) \circ \mathbf{t}_h^{(k)} \circ (\mathbf{I}_h \circ \mathbf{I}_h^*) \circ \mathbf{t}_h^{(k)} \circ (\mathbf{I}_h \circ \mathbf{I}_h^*) : \bigoplus_{i=1}^{I_h} \mathcal{H}_h^r(\mathbb{R}^{2d}) \rightarrow \bigoplus_{i=1}^{I_h} \mathcal{H}_h^r(\mathbb{R}^{2d}).$$

We need to prove that the trace norm of the commutator

$$\left[(\mathbf{I}_h \circ \mathbf{I}_h^*), \mathbf{t}_h^{(k)} \right] : \bigoplus_{i=1}^{I_h} \mathcal{H}_h^r(\mathbb{R}^{2d}) \rightarrow \bigoplus_{i=1}^{I_h} \mathcal{H}_h^r(\mathbb{R}^{2d})$$

is bounded by $C\hbar^{-d+\epsilon}$. It is easy to obtain such estimate by using Corollary 6.14 and Lemma 6.24 to exchange the order of operators and also using Proposition 7.3(3). \square

2170 Now we will modify the operators $\check{\tau}_h^{(k)}$, $0 \leq k \leq n+1$, to get the projection operators $\tau_h^{(k)}$, $0 \leq k \leq n$, and $\tilde{\tau}_h = \tau_h^{(n+1)}$ in the statement of the theorem. The estimate (7.41) implies that the spectral set of the operator $\check{\tau}_h^{(k)} \circ \check{\tau}_h^{(k)} - \check{\tau}_h^{(k)}$ is contained in the disk $|z| \leq C\hbar^\epsilon$. By the spectral mapping theorem [16, Part I, VII.3.11], the spectral set of the operators $\check{\tau}_h^{(k)}$ is contained in the union of two small disks around 0 and 1:

$$\mathbb{D}(0, C\hbar^\epsilon) \cup \mathbb{D}(1, C\hbar^\epsilon) \quad \text{where } \mathbb{D}(z, r) := \{w \in \mathbb{C} \mid |w - z| < r\}. \quad (7.43)$$

2175 For $0 \leq k \leq n+1$, let $\hat{\tau}_h^{(k)}$ be the spectral projector of $\check{\tau}_h^{(k)}$ for the part of its spectral set contained in $\mathbb{D}(1, C\hbar^\epsilon)$. This is of finite rank because of compactness of $\check{\tau}_h^{(k)}$. The next lemma should be easy to prove. (We provide a proof in the appendix for completeness.)

Lemma 7.23. *There is a constant $C > 0$ independent of \hbar such that*

$$\|\check{\tau}_h^{(k)} - \hat{\tau}_h^{(k)}\|_{\mathcal{H}_h^r(P)} \leq C\hbar^\epsilon \quad \text{and} \quad \|\check{\tau}_h^{(k)} - \hat{\tau}_h^{(k)} : \mathcal{H}_h^r(P) \rightarrow \mathcal{H}_h^r(P)\|_{tr} \leq C\hbar^{-d+\epsilon}$$

for some $C > 0$ independent of \hbar .

2180 Thus we get the set of projection operators $\hat{\tau}_h^{(k)}$ for $0 \leq k \leq n+1$, which approximate $\check{\tau}_h^{(k)}$. As consequences of (7.40) and (7.42), we have

$$\|\text{Id} - (\hat{\tau}_h^{(0)} + \hat{\tau}_h^{(1)} + \cdots + \hat{\tau}_h^{(n)} + \hat{\tau}_h^{(n+1)})\|_{\mathcal{H}_h^r(P)} \leq C\hbar^\epsilon \quad (7.44)$$

and

$$\|\hat{\tau}_h^{(k)} \circ \hat{\tau}_h^{(k')}\|_{\mathcal{H}_h^r(P)} \leq C\hbar^\epsilon \quad \text{if } k \neq k'. \quad (7.45)$$

We set $\mathcal{H}_k := \text{Im } \hat{\tau}_h^{(k)}$ for $0 \leq k \leq n+1$ and put $\tilde{\mathcal{H}} = \mathcal{H}_{n+1}$. We have

Lemma 7.24. *The Hilbert space $\mathcal{H}_h^r(P)$ is decomposed into the direct sum:*

$$\mathcal{H}_h^r(P) = \mathcal{H}_0 \oplus \mathcal{H}_1 \oplus \mathcal{H}_2 \oplus \cdots \oplus \mathcal{H}_n \oplus \tilde{\mathcal{H}}.$$

2185 *Proof.* Since the sum $\hat{\tau}_h^{(0)} + \hat{\tau}_h^{(1)} + \cdots + \hat{\tau}_h^{(n)} + \hat{\tau}_h^{(n+1)}$ is invertible from (7.44), we can set

$$\tau_h^{(k)} := \hat{\tau}_h^{(k)} (\hat{\tau}_h^{(0)} + \hat{\tau}_h^{(1)} + \cdots + \hat{\tau}_h^{(n)} + \hat{\tau}_h^{(n+1)})^{-1} \quad \text{for } 0 \leq k \leq n+1. \quad (7.46)$$

We can express any $v \in \mathcal{H}_h^r(P)$ as

$$v = \sum_{k=1}^{n+1} v_k \quad \text{with } v_k := \tau_h^{(k)}(v) \in \mathcal{H}_k.$$

Thus the subspaces \mathcal{H}_k for $0 \leq k \leq n+1$ span the whole space $\mathcal{H}_h^r(P)$. Uniqueness of such expression follows from (7.45). \square

2190 From the argument in the proof above, the operator $\tau_h^{(k)} : \mathcal{H}_h^r(P) \rightarrow \mathcal{H}_k(P)$ for $0 \leq k \leq n+1$ in (7.46) are the projections to the subspace \mathcal{H}_k along other subspaces. Clearly we have

$$\|\tau_h^{(k)} - \hat{\tau}_h^{(k)}\|_{\mathcal{H}_h^r(P)} \leq C\hbar^\epsilon \quad \text{and hence} \quad \|\tau_h^{(k)} - \check{\tau}_h^{(k)}\|_{\mathcal{H}_h^r(P)} \leq C\hbar^\epsilon. \quad (7.47)$$

and also

$$\|\tau_h^{(k)} - \hat{\tau}_h^{(k)} : \mathcal{H}_h^r(P) \rightarrow \mathcal{H}_h^r(P)\|_{tr} \leq C\hbar^{-d+\epsilon} \quad \text{and hence} \quad \|\tau_h^{(k)} - \check{\tau}_h^{(k)} : \mathcal{H}_h^r(P) \rightarrow \mathcal{H}_h^r(P)\|_{tr} \leq C\hbar^{-d+\epsilon}.$$

7.6.1 Proof of Claim (1)

2195 For the external band $k=0$, Claim 1 is written in Corollary 1.64. But the same argument works for every $0 \leq k \leq n$, so we give it here. Since we have

$$\text{rank } \tau_h^{(k)} = \text{Tr } \tau_h^{(k)} = \text{Tr } \check{\tau}_h^{(k)} + \mathcal{O}(\hbar^{-d+\epsilon}),$$

it is enough to prove the claim that

$$\left| \text{Tr } \check{\tau}_h^{(k)} - \frac{r(k, d) \cdot \text{Vol}_\omega(M)}{(2\pi\hbar)^d} \right| < C\hbar^{-d+\epsilon}.$$

From the definition, we have that

$$\mathrm{Tr} \check{\tau}_h^{(k)} = \mathrm{Tr}(\mathbf{I}_h \circ \mathbf{t}_h^{(k)} \circ \mathbf{I}_h^*) := \sum_{i=1}^{I_h} \mathrm{Tr}(\mathbf{I}_h \circ \mathbf{t}_h^{(k)} \circ \mathbf{I}_h^*)_{i,i}.$$

From Lemma 6.10 and Corollary 6.14, we see that

$$\left| \mathrm{Tr}(\mathbf{I}_h \circ \mathbf{t}_h^{(k)} \circ \mathbf{I}_h^*)_{i,i} - \frac{r(k, d)}{(2\pi\hbar)^d} \int \psi_i dx \right| \leq C\hbar^{-\theta d + \epsilon}$$

Since $I_h < C\hbar^{-(1-\theta)d}$ and $\sum_i \psi_i \circ \kappa_i^{-1} \equiv 1$, we obtain

$$\left| \mathrm{Tr} \check{\tau}_h^{(k)} - \frac{r(k, d) \cdot \mathrm{Vol}_\omega(M)}{(2\pi\hbar)^d} \right| < C\hbar^{-(1-\theta)d} \cdot \hbar^{-\theta d + \epsilon} = C\hbar^{-d + \epsilon}.$$

2200 7.6.2 Proof of Claim (2)

For the proof of Claim (2)-(5), it is enough to prove the statements with $\tau_h^{(k)}$ replaced by $\check{\tau}_h^{(k)}$ because we have (7.47). The operator norm of $\check{\tau}_h^{(k)} : \mathcal{H}_h^r(P) \rightarrow \mathcal{H}_h^r(P)$ is bounded by a constant independent of \hbar from Proposition 7.19, as we noted in the beginning of this subsection.

2205 7.6.3 Proof of Claim (3) and (4)

From the definition of the operators \mathbf{F}_h and $\check{\tau}_h^{(k)}$, the following diagram commutes:

$$\begin{array}{ccc} \bigoplus_{i=1}^{I_h} \mathcal{H}_h^r(\mathbb{R}^{2d}) & \xrightarrow{\mathbf{I}_h \circ \mathbf{I}_h^* \circ \mathbf{t}_h^{(k)} \circ \mathbf{F}_h \circ \mathbf{t}_h^{(k')}} & \bigoplus_{i=1}^{I_h} \mathcal{H}_h^r(\mathbb{R}^{2d}) \\ \uparrow \mathbf{I}_h & & \uparrow \mathbf{I}_h \\ \mathcal{H}_h^r(P) & \xrightarrow{\check{\tau}_h^{(k)} \circ \hat{F}_h \circ \check{\tau}_h^{(k')}} & \mathcal{H}_h^r(P) \end{array}$$

Since the operator \mathbf{I}_h in the vertical direction is an isometric embedding, we have

$$\|\check{\tau}_h^{(k)} \circ \hat{F}_h \circ \check{\tau}_h^{(k')}\|_{\mathcal{H}_h^r(P)} \leq \|\mathbf{I}_h \circ \mathbf{I}_h^* \circ \mathbf{t}_h^{(k)} \circ \mathbf{F}_h \circ \mathbf{t}_h^{(k')}\|_{\mathcal{H}_h^r} = \|(\mathbf{I}_h \circ \mathbf{I}_h^* \circ \mathbf{t}_h^{(k)}) \circ (\mathbf{t}_h^{(k')} \circ \mathbf{F}_h \circ \mathbf{t}_h^{(k')})\|_{\mathcal{H}_h^r}.$$

From Proposition 7.19, we have

$$\left\| \mathbf{I}_h \circ \mathbf{I}_h^* \circ \mathbf{t}_h^{(k)} \right\|_{\mathcal{H}_h^r} \leq C_0 \quad \text{for } 0 \leq k \leq n$$

where C_0 is a constant independent of \hbar , f and V . These estimates give also

$$\left\| \mathbf{I}_h \circ \mathbf{I}_h^* \circ \mathbf{t}_h^{(n+1)} \right\|_{\bigoplus_{i=1}^{I_h} \mathcal{H}_h^{r,+}(\mathbb{R}^{2d}) \rightarrow \bigoplus_{i=1}^{I_h} \mathcal{H}_h^r(\mathbb{R}^{2d})} \leq C_0,$$

2210 because of the relation $\mathbf{t}_h^{(n+1)} = \tilde{\mathbf{t}}_h = \mathrm{Id} - \sum_{k=0}^n \mathbf{t}_h^{(k)}$ and Lemma 7.14. Now Claim (4) is an immediate consequence of Proposition 7.20 (1). From Proposition 7.20 and Lemma 7.17, we have

$$\|\mathbf{t}_h^{(k)} \circ \mathbf{F}_h \circ \mathbf{t}_h^{(k')}\|_{\mathcal{H}_h^r} \leq \|\mathbf{t}_h^{(k)} \circ \mathbf{t}_h^{(k')} \circ \mathbf{F}_h\|_{\mathcal{H}_h^r} + C\hbar^\epsilon = C\hbar^\epsilon$$

if $k \neq k'$ and hence Claim (3) follows.

7.6.4 Proof of Claim (5)

Take $0 \neq u \in \mathcal{H}_k = \text{Im } \tau_h^{(k)}$ for $0 \leq k \leq n$ arbitrarily and set $\mathbf{u} = \mathbf{I}_h(u)$. Then we have, from (7.47), that

$$\begin{aligned} \|\mathbf{u} - (\mathbf{I}_h \circ \mathbf{I}_h^*) \circ \mathbf{t}_h^{(k)}(\mathbf{u})\|_{\mathcal{H}_h^r} &= \|\mathbf{I}_h(u) - \mathbf{I}_h \circ \check{\tau}_h^{(k)}(u)\|_{\mathcal{H}_h^r} = \|u - \check{\tau}_h^{(k)}(u)\|_{\mathcal{H}_h^r(P)} \\ &= \|(\tau_h^{(k)} - \check{\tau}_h^{(k)})u\|_{\mathcal{H}_h^r(P)} \leq C\hbar^\epsilon \|u\|_{\mathcal{H}_h^r(P)} = C\hbar^\epsilon \|\mathbf{u}\|_{\mathcal{H}_h^r} \end{aligned}$$

Hence we can apply the second claim in Proposition 7.20 to \mathbf{u} and obtain Claim (5), noting that $\|\mathbf{F}_h \mathbf{u}\|_{\mathcal{H}_h^r} = \|F_h u\|_{\mathcal{H}_h^r(P)}$ by definition.

7.7 Proof of Theorem 1.55

We will derive Theorem 1.55 as a consequence of Lemma 6.10 on local charts, gluing together the expression given in (6.15). Let $\varphi \in \mathcal{S}_d$ be a symbol, i.e. a \hbar -family of smooth functions $(\varphi_h)_h$ on M with regularity estimates as in Definition 1.52. Below we write φ for φ_h .

Recall the definition of the operators \mathbf{I}_h and \mathbf{I}_h^* in (7.12) and (7.13). We define

$$\mathbf{I}_{i,h} : C_N^\infty(P) \rightarrow C_0^\infty(\mathbb{D}(\hbar^{1/2-\theta})), \quad u \mapsto u_i(x) = \psi_i(x) \cdot u(\tau_i(\kappa_i(x)))$$

and

$$\mathbf{I}_{i,h}^* : C_0^\infty(\mathbb{D}(\hbar^{1/2-\theta})) \rightarrow C_N^\infty(P), \quad u \mapsto e^{N\alpha_i(p)} \cdot \chi_h \cdot u$$

for $1 \leq i \leq I_h$. Then we have

$$\begin{aligned} \mathcal{M}(\varphi) \circ \check{\tau}_h^{(0)} &\stackrel{(7.39)}{=} \mathcal{M}(\varphi) \circ \mathbf{I}_h^* \circ \mathbf{t}_h^{(0)} \circ \mathbf{I}_h \\ &\stackrel{(7.12),(7.13)}{=} \mathcal{M}(\varphi) \circ \sum_{i=1}^{I_h} \mathbf{I}_{i,h}^* \circ \mathbf{t}_h^{(0)} \circ \mathcal{M}(\psi_i) \circ \mathbf{I}_{i,h} \end{aligned} \quad (7.48)$$

$$= \sum_{i=1}^{I_h} \mathbf{I}_{i,h}^* \circ \mathcal{M}(\varphi \circ \kappa_i) \circ \mathbf{t}_h^{(0)} \circ \mathcal{M}(\psi_i) \circ \mathbf{I}_{i,h} \quad (7.49)$$

In the last line above, we have used the fact that $\mathcal{M}(\varphi) \circ \mathbf{I}_{i,h}^* = \mathbf{I}_{i,h}^* \circ \mathcal{M}(\varphi \circ \kappa_i)$. From Definition 1.52, the family of functions $\mathcal{X}_h := \{\chi_h \cdot \varphi \circ \kappa_i; 1 \leq i \leq I_h\}$ satisfies the conditions in Setting I in Subsection 6.1. Let $\hat{\pi}_0(\nu)$ be the rank one projection operator $\hat{\pi}_\alpha(\nu)$ defined in (6.11) for $\alpha = 0 \in \mathbb{N}^0$. We continue (7.48) and get (we will justify the second line below)

$$\begin{aligned} \mathcal{M}(\varphi) \circ \check{\tau}_h^{(0)} &= \sum_{i=1}^{I_h} \mathbf{I}_{i,h}^* \circ \left(\mathcal{M}(\varphi \circ \kappa_i) \circ \mathbf{t}_h^{(0)} \circ \mathcal{M}(\psi_i) \right) \circ \mathbf{I}_{i,h} \\ &\stackrel{(6.14)}{=} \sum_{i=1}^{I_h} \mathbf{I}_{i,h}^* \circ \left(\int_{\mathbb{R}^{2d}} \varphi \circ \kappa_i(\nu) \hat{\pi}_0(\nu) \frac{d\nu}{(2\pi\hbar)^d} \right) \circ \mathcal{M}(\psi_i) \circ \mathbf{I}_{i,h} + O(\hbar^\theta) \\ &= \frac{1}{(2\pi\hbar)^d} \int_M \sum_{i=1}^{I_h} \varphi(x) \cdot \chi_h(\kappa_i^{-1}(x)) \cdot \hat{\pi}_i^{(1)}(x) dx + O(\hbar^\theta) \end{aligned} \quad (7.50)$$

where we have put

$$\hat{\pi}_i^{(1)}(x) := \mathbf{I}_{i,\hbar}^* \circ \hat{\pi}_0(\kappa_i^{-1}(x)) \circ \mathcal{M}(\psi_i) \circ \mathbf{I}_{i,\hbar}, \quad i \in \{1, \dots, I_\hbar\}, x \in M. \quad (7.51)$$

and $O(\hbar^\theta)$ denotes the error term whose operator norm is bounded by $C\hbar^\epsilon$ with C a constant independent of \hbar . In order to justify this small error term, for every $i \in \{1, \dots, I_\hbar\}$, let

$$T_i := \mathbf{I}_{i,\hbar}^* \circ \left(\mathcal{M}(\varphi) \circ t_\hbar^{(0)} - \left(\int_{\mathbb{R}^{2d}} \varphi_i(\nu) \hat{\pi}_0(\nu) \frac{d\nu}{(2\pi\hbar)^d} \right) \right) \circ \mathcal{M}(\psi_i) \circ \mathbf{I}_{i,\hbar}.$$

For every $i \in \{1, \dots, I_\hbar\}$, we have from (6.14) that $\|T_i\| = O(\hbar^\theta)$. Due to truncations functions, for every $0 \leq i \leq I_\hbar$, we have that $T_i \circ T_k^* = 0$ and $T_i^* \circ T_k = 0$ except for a finite number (bounded uniformly in \hbar) of k . Hence from the discrete version of the Cotlar-Stein Lemma¹⁹ we deduce that $\left\| \sum_j T_j \right\| = O(\hbar^\theta)$. This justifies the second line of (7.50).

The operator $\hat{\pi}_i^{(1)}(x)$ is of rank one because $\hat{\pi}_0(\nu)$ is a rank one projector from Lemma 6.10. Its trace is

$$\begin{aligned} \text{Tr} \left(\hat{\pi}_j^{(1)}(x) \right) &= (\chi_\hbar \circ \kappa_i^{-1})(x) \cdot \text{Tr} \left(\hat{\pi}_0(\kappa_i^{-1}(x)) \mathcal{M}(\psi_i) \right) \\ &= \psi_i(\kappa_i^{-1}(x)) \cdot \text{Tr} \left(\hat{\pi}_0(\kappa_i^{-1}(x)) \right) + O(\hbar^\theta) \quad \text{by Lemma 6.9} \\ &= \psi_i(\kappa_i^{-1}(x)) + O(\hbar^\theta) \end{aligned} \quad (7.52)$$

Hence we have

$$\left(\hat{\pi}_i^{(1)}(x) \right)^2 = \psi_i(\kappa_i^{-1}(x)) \cdot \hat{\pi}_i^{(1)}(x) + O(\hbar^\theta) \quad (7.53)$$

We define the operator $\hat{\pi}^{(2)}(x)$ by summing the operators $\hat{\pi}_j^{(1)}(x)$ over the different charts:

$$\hat{\pi}^{(2)}(x) := \sum_{1 \leq i \leq I_\hbar} \hat{\pi}_i^{(1)}(x)$$

Notice that $\hat{\pi}_i^{(1)}(x) = 0$ if $(\chi_\hbar \circ \kappa_i^{-1})(x) = 0$. Hence in the previous equation, the sum over j contains only a finite number of terms and this number is bounded uniformly with respect to \hbar .

In order to proceed with this definition, we need the following lemma.

Lemma 7.26. (1) *There exist constants $\epsilon > 0$ and $C > 0$ such that we have that*

$$\left\| \psi_j(\kappa_j^{-1}(x)) \cdot \hat{\pi}_k^{(1)}(x) - \psi_k(\kappa_k^{-1}(x)) \cdot \hat{\pi}_j^{(1)}(x) \right\| < C\hbar^\epsilon \quad \text{for every } 1 \leq j, k \leq I_\hbar.$$

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Lemma 7.25. “Discrete version of the Cotlar-Stein Lemma”: If $(T_j)_j$ is a family of bounded operators, if $A := \sup_j \sum_k \|T_j T_k^*\|^{1/2} < \infty$ and $B := \sup_j \sum_k \|T_j^* T_k\|^{1/2} < \infty$ then $\sum_j T_j$ converges in the strong operator topology and $\left\| \sum_j T_j \right\| \leq \sqrt{AB}$.

(2) For any $m > 0$, there exists a constant $C_m > 0$ such that

$$\left\| \hat{\pi}_k^{(1)}(x) \circ \hat{\pi}_j^{(1)}(y)^* \right\| \leq C_m \left\langle \frac{|x-y|}{\sqrt{\hbar}} \right\rangle^{-m} \quad \text{and} \quad \left\| \hat{\pi}_k^{(1)}(x)^* \circ \hat{\pi}_j^{(1)}(y) \right\| \leq C_m \left\langle \frac{|x-y|}{\sqrt{\hbar}} \right\rangle^{-m} \quad \text{for } x, y \in M$$

Consequently we have (for possibly different constant C_m) that

$$\left\| \hat{\pi}_k^{(2)}(x) \circ \hat{\pi}_j^{(2)}(y)^* \right\| \leq C_m \left\langle \frac{|x-y|}{\sqrt{\hbar}} \right\rangle^{-m} \quad \text{and} \quad \left\| \hat{\pi}_k^{(2)}(x)^* \circ \hat{\pi}_j^{(2)}(y) \right\| \leq C_m \left\langle \frac{|x-y|}{\sqrt{\hbar}} \right\rangle^{-m} \quad \text{for } x, y \in M$$

The claim of the lemma above may look rather obvious. But note that the relation between the operators $\hat{\pi}_j^{(1)}(x)$ (or its adjoint) for different indices j involves the coordinate change transformation $\kappa_{j,i}$, which is close to the identity but non-linear. So we have to go through an argument similar to that in Section 6. We give a proof of Lemma 7.26 at the end.

Note that the operator $\hat{\pi}^{(2)}(x)$ is not rank one in general. We are going to construct a rank one projection operator $\hat{\pi}(x)$ from $\hat{\pi}^{(2)}(x)$ as its spectral projector. First we check

$$\text{Tr}(\hat{\pi}^{(2)}(x)) \stackrel{(7.52)}{=} \sum_j \psi_j(\kappa_j^{-1}(x)) + O(\hbar^\theta) = 1 + O(\hbar^\theta) \quad (7.54)$$

This is because the term $\psi_j(\kappa_j^{-1}(x)) \cdot \hat{\pi}_l^{(1)}(x)$ is not zero only if j and l are indices for intersecting local charts. Hence, using the first claim of Lemma 7.26, we see

$$\begin{aligned} \psi_j(\kappa_j^{-1}(x)) \cdot \hat{\pi}^{(2)}(x) &= \sum_l \psi_j(\kappa_j^{-1}(x)) \hat{\pi}_l^{(1)}(x) = \sum_l \psi_l(\kappa_l^{-1}(x)) \cdot \hat{\pi}_j^{(1)}(x) + O(\hbar^\epsilon) \\ &= \hat{\pi}_j^{(1)}(x) \sum_l \psi_l(\kappa_l^{-1}(x)) + O(\hbar^\theta) = \hat{\pi}_j^{(1)}(x) + O(\hbar^\epsilon) \end{aligned} \quad (7.55)$$

and

$$\begin{aligned} (\psi_j(\kappa_j^{-1}(x)) \cdot \hat{\pi}^{(2)}(x))^2 &\stackrel{(7.55)}{=} \left(\hat{\pi}_j^{(1)}(x) \right)^2 + O(\hbar^\theta) \stackrel{(7.53)}{=} \psi_j(\kappa_j^{-1}(x)) \cdot \hat{\pi}_j^{(1)}(x) + O(\hbar^\epsilon) \\ &= (\psi_j(\kappa_j^{-1}(x)))^2 \cdot \hat{\pi}^{(2)}(x) + O(\hbar^\epsilon) \end{aligned}$$

So choosing j such that $\psi_j(\kappa_j^{-1}(x)) > 1/C$ we get

$$(\hat{\pi}^{(2)}(x))^2 = \hat{\pi}^{(2)}(x) + O(\hbar^\epsilon) \quad (7.56)$$

Hence from (7.54) and (7.56), the operator $\hat{\pi}^{(2)}(x)$ has an isolated simple eigenvalue λ_1 at a distance $O(\hbar^\epsilon)$ to 1 and the rest of its spectrum is at distance $O(\hbar^\epsilon)$ of the origin. We consider a fixed path γ in \mathbb{C} with center 1 of radius $C\hbar^\epsilon$ (with large C) so that by a Dunford integral, we get the spectral projector of the operator $\hat{\pi}^{(2)}(x)$ for its simple eigenvalue λ_1 and we denote it:

$$\hat{\pi}(x) := \frac{1}{2\pi i} \oint_\gamma (z - \hat{\pi}^{(2)}(x))^{-1} dz.$$

Note that we may rewrite this definition as

$$\hat{\pi}(x) = \left(\frac{1}{2\pi i} \oint_{\gamma} \frac{1}{z} (z - \hat{\pi}^{(2)}(x))^{-1} dz \right) \circ \hat{\pi}^{(2)}(x) = \hat{\pi}^{(2)}(x) \circ \left(\frac{1}{2\pi i} \oint_{\gamma} \frac{1}{z} (z - \hat{\pi}^{(2)}(x))^{-1} dz \right) \quad (7.57)$$

2265 The operators $\hat{\pi}(x)$ are rank one projection operators depending smoothly on $x \in M$. There exists a constant $C > 0$ independent of \hbar such that $\|\hat{\pi}(x)\| \leq C$ and that $\|\hat{\pi}^{(2)}(x) - \hat{\pi}(x)\| < C\hbar^\epsilon$. (For these estimates, we refer the proof of Lemma 7.23 in Appendix A.2 where the argument is in parallel.) Hence

$$\psi_j(\kappa_j^{-1}(x)) \cdot \hat{\pi}(x) \stackrel{(7.55)}{=} \hat{\pi}_j^{(1)}(x) + O(\hbar^\epsilon). \quad (7.58)$$

Continuing (7.50), we deduce that

$$\begin{aligned} \mathcal{M}(\varphi) \tilde{\tau}_\hbar^{(0)} &= \frac{1}{(2\pi\hbar)^d} \int_M \sum_{j=1}^{I_\hbar} \varphi(x) \hat{\pi}_j^{(1)}(x) dx + O(\hbar^\theta) \\ &\stackrel{(7.58)}{=} \frac{1}{(2\pi\hbar)^d} \int_M \sum_{j=1}^{I_\hbar} \varphi(x) \psi_j(\kappa_j^{-1}(x)) \cdot \hat{\pi}(x) dx + O(\hbar^\epsilon) \\ &= \frac{1}{(2\pi\hbar)^d} \int_M \varphi(x) \cdot \hat{\pi}(x) dx + O(\hbar^\epsilon) \end{aligned} \quad (7.59)$$

2270 In the second line above, we have to use again the integral version of the Cotlar-Stein Lemma 6.11. We do this in two steps, let us consider the family of operators

$$T(x) := T_1(x) + T_2(x) = \sum_{j=1}^{I_\hbar} \varphi(x) \left(\psi_j(\kappa_j^{-1}(x)) \cdot \hat{\pi}(x) - \hat{\pi}_j^{(1)}(x) \right)$$

with

$$\begin{aligned} T_1(x) &:= \sum_{j=1}^{I_\hbar} \varphi(x) \left(\psi_j(\kappa_j^{-1}(x)) \cdot \hat{\pi}^{(2)}(x) - \hat{\pi}_j^{(1)}(x) \right) \\ T_2(x) &:= \sum_{j=1}^{I_\hbar} \varphi(x) \cdot \psi_j(\kappa_j^{-1}(x)) \cdot (\hat{\pi}(x) - \hat{\pi}^{(2)}(x)) = \varphi(x) (\hat{\pi}(x) - \hat{\pi}^{(2)}(x)). \end{aligned}$$

From Lemma 7.26 and the expression (7.57) of $\hat{\pi}(x)$, we see that, for $k = 1, 2$,

$$\|T_k(x) T_k(y)^*\|_{\mathcal{H}_\hbar^r(\mathbb{R}^{2d})} \leq C\hbar^{2\epsilon} \quad \text{and} \quad \|T_k(x)^* T_k(y)\|_{\mathcal{H}_\hbar^r(\mathbb{R}^{2d})} \leq C\hbar^{2\epsilon}$$

2275 and also

$$\|T_k(x) T_k(y)^*\|_{\mathcal{H}_\hbar^r(\mathbb{R}^{2d})} \leq C_m \left\langle \frac{|x-y|}{\sqrt{\hbar}} \right\rangle^{-m} \quad \text{and} \quad \|T_k(x)^* T_k(y)\|_{\mathcal{H}_\hbar^r(\mathbb{R}^{2d})} \leq C_m \left\langle \frac{|x-y|}{\sqrt{\hbar}} \right\rangle^{-m}$$

for any $m > 0$ with a constant C_m independent of \hbar . It follows

$$\|T_k(x) T_k(y)^*\|_{\mathcal{H}_\hbar^r(\mathbb{R}^{2d})} \leq C_m \hbar^\epsilon \cdot \left\langle \frac{|x-y|}{\sqrt{\hbar}} \right\rangle^{-m} \quad \text{and} \quad \|T_k(x)^* T_k(y)\|_{\mathcal{H}_\hbar^r(\mathbb{R}^{2d})} \leq C_m \hbar^\epsilon \cdot \left\langle \frac{|x-y|}{\sqrt{\hbar}} \right\rangle^{-m}.$$

Then Cotlar-Stein Lemma 6.11 implies that $\frac{1}{(2\pi\hbar)^d} \left\| \int_M T_k(x) dx \right\|_{\mathcal{H}_\hbar^r(\mathbb{R}^{2d})} = O(\hbar^\epsilon)$ and therefore $\frac{1}{(2\pi\hbar)^d} \left\| \int_M T(x) dx \right\|_{\mathcal{H}_\hbar^r(\mathbb{R}^{2d})} = O(\hbar^\epsilon)$, giving (7.59) above. We have finally obtained that in norm operator:

$$\left\| \mathcal{M}(\varphi) \tilde{\tau}_\hbar^{(0)} - \frac{1}{(2\pi\hbar)^d} \int_M \varphi(x) \cdot \hat{\pi}(x) dx \right\|_{\mathcal{H}_\hbar^r(\mathbb{R}^{2d})} \leq C\hbar^\epsilon$$

2280 With Lemma 7.23 and (7.47), we deduce that

$$\left\| \mathcal{M}(\varphi) \tau_\hbar^{(0)} - \frac{1}{(2\pi\hbar)^d} \int_M \varphi(x) \cdot \hat{\pi}(x) dx \right\|_{\mathcal{H}_\hbar^r(\mathbb{R}^{2d})} \leq C\hbar^\epsilon$$

We can argue just in parallel manner to give

$$\left\| \tau_\hbar^{(0)} \mathcal{M}(\varphi) - \frac{1}{(2\pi\hbar)^d} \int_M \varphi(x) \cdot \hat{\pi}(x) dx \right\|_{\mathcal{H}_\hbar^r(\mathbb{R}^{2d})} \leq C\hbar^\epsilon \quad (7.60)$$

Therefore

$$\left\| \left[\tau_\hbar^{(0)}, \mathcal{M}(\varphi) \right] \right\|_{\mathcal{H}_\hbar^r(\mathbb{R}^{2d})} \leq C\hbar^\epsilon.$$

Suppose that $r_1^+ < r_0^-$. We have defined in (1.32) by Π_\hbar the finite rank spectral projector on the external band of \hat{F}_N . We define

$$\pi_x := \Pi_\hbar \circ \hat{\pi}(x) \circ \Pi_\hbar.$$

2285 We have shown in (7.9) that

$$\left\| \tau_\hbar^{(0)} - \Pi_\hbar \right\| < C\hbar^\epsilon \quad (7.61)$$

Therefore we deduce from above that

$$\left\| \Pi_\hbar \mathcal{M}(\varphi) \Pi_\hbar - \frac{1}{(2\pi\hbar)^d} \int_M \varphi(x) \cdot \pi_x dx \right\| \leq C\hbar^\epsilon$$

and

$$\left\| \left[\Pi_\hbar, \mathcal{M}(\varphi) \right] \right\| \leq C\hbar^\epsilon.$$

This finishes the proof of Theorem 1.55. Finally we prove Lemma 7.26.

2290 *Proof of Lemma 7.26* . Recall the operator \mathcal{Y}_\hbar which truncate the functions in the phase space. We decompose the operators $\hat{\pi}_j^{(1)}(\nu)$ into

$$\hat{\pi}_{j,1}^{(1)}(x) := \mathbf{I}_{i,\hbar}^* \circ \mathcal{Y}_\hbar \circ \hat{\pi}_0(\kappa_j^{-1}(x)) \circ \mathcal{M}(\psi_j) \circ \mathbf{I}_{i,\hbar}$$

and

$$\hat{\pi}_{j,2}^{(1)}(x) := \mathbf{I}_{i,\hbar}^* \circ (1 - \mathcal{Y}_\hbar) \circ \hat{\pi}_0(\kappa_j^{-1}(x)) \circ \mathcal{M}(\psi_j) \circ \mathbf{I}_{i,\hbar}.$$

For the former part $\hat{\pi}_{j,1}^{(1)}(x)$, we apply Proposition 6.19 and the estimate (6.23) on the kernel in the proof to see that the non-linear coordinate change transformation hardly affect this part. Thus we have both of the claims when we replace the operator $\hat{\pi}_j^{(1)}(\nu)$ by $\hat{\pi}_{j,1}^{(1)}(\nu)$. For the latter part, we have

$$\|\hat{\pi}_{j,2}^{(1)}(x)\|_{\mathcal{H}_\hbar^r(\mathbb{R}^{2d})} < C\hbar^\theta$$

from Lemma 6.9. This completes the proof of the first claim (1). By inspecting the kernels of the lifted operators, we also see

$$\|(\hat{\pi}_k^{(1)}(x))^* \circ \hat{\pi}_{j,2}^{(1)}(x)\|_{\mathcal{H}_\hbar^r(\mathbb{R}^{2d})} = \mathcal{O}(\hbar^\infty), \quad \|\hat{\pi}_k^{(1)}(x) \circ (\hat{\pi}_{j,2}^{(1)}(x))^*\|_{\mathcal{H}_\hbar^r(\mathbb{R}^{2d})} = \mathcal{O}(\hbar^\infty).$$

Since we have only to consider points $x, y \in M$ with $d(x, y) \leq C\hbar^{1/2-\theta}$, this gives the second claim (2). \square

2300 8 Proof of Theorem 3.5 for the spectrum of the rough Laplacian.

8.1 The harmonic oscillator

In this subsection we present the harmonic oscillator in the setting of Bargmann transform. (We refer [23],[57] for a more detailed treatment.) We will need it in dealing with the (Euclidean) rough Laplacian Δ_\hbar in Section 8.2. Associated to the standard coordinates

$$(x, \xi) = (x_1, x_2, \dots, x_D, \xi_1, \xi_2, \dots, \xi_D)$$

on $T^*\mathbb{R}^D = \mathbb{R}^{2D}$, we consider the operators

$$\hat{x}_i : \begin{cases} \mathcal{S}(\mathbb{R}^{2D}) & \rightarrow \mathcal{S}(\mathbb{R}^{2D}) \\ u & \rightarrow (\mathcal{P}_\hbar \circ \mathcal{M}(x_i) \circ \mathcal{P}_\hbar) u \end{cases} \quad \text{and} \quad \hat{\xi}_i : \begin{cases} \mathcal{S}(\mathbb{R}^{2D}) & \rightarrow \mathcal{S}(\mathbb{R}^{2D}) \\ u & \rightarrow (\mathcal{P}_\hbar \circ \mathcal{M}(\xi_i) \circ \mathcal{P}_\hbar) u \end{cases}$$

where $\mathcal{M}(x_i)$ and $\mathcal{M}(\xi_i)$ on the right hand sides denote the multiplication by the corresponding functions and \mathcal{P}_\hbar is the Bargmann projector (4.6). These operators are usually called Toeplitz quantization of the functions x_i and ξ_i . Then we set

$$\hat{P} := \frac{1}{2\hbar} (\hat{x}^2 + \hat{\xi}^2) := \frac{1}{2\hbar} \sum_{i=1}^D (\hat{x}_i \circ \hat{x}_i + \hat{\xi}_i \circ \hat{\xi}_i), \tag{8.1}$$

$$\mathcal{H} := \mathcal{B}_\hbar^* \hat{P} \mathcal{B}_\hbar, \quad L^2(\mathbb{R}^D) \rightarrow L^2(\mathbb{R}^D)$$

which is usually called the *harmonic oscillator operator*. (the operators $\mathcal{B}_\hbar^*, \mathcal{B}_\hbar$ are defined in Section 4.1).

Lemma 8.1. "Spectrum of the harmonic oscillator". *The operator \mathcal{H} in (8.1) is a closed self-adjoint operator on $L^2(\mathbb{R}^D)$ and its spectral set consists of eigenvalues*

$$\frac{D}{2} + k, \quad k \in \mathbb{N}.$$

2315 *For every $k \in \mathbb{N}$, the spectral projector $Q_\hbar^{(k)}$ for the eigenvalue $\frac{D}{2} + k$ is an orthogonal projection operator of rank $\binom{D+k-1}{D-1}$. We have*

$$\bigoplus_{i=0}^k \text{Im} Q_\hbar^{(i)} = \{\varphi_0 \cdot p \mid p \text{ is a polynomial of degree } \leq k\} \tag{8.2}$$

where

$$\varphi_0(x) = e^{-|x|^2/(2\hbar)}.$$

In particular, we have the following orthogonal decomposition of $L^2(\mathbb{R}^D)$:

$$L^2(\mathbb{R}^D) = \overline{\bigoplus_{k=0}^{\infty} \text{Im} Q_\hbar^{(k)}}.$$

Proof. Since \hat{x}_i and $\hat{\xi}_i$ are the lift of the operators

$$u \mapsto x_i \cdot u \quad \text{and} \quad u \mapsto -i\hbar \cdot \partial_{x_i} u$$

2320 respectively, the operator $\hat{P} = \frac{1}{2\hbar} (\hat{x}^2 + \hat{\xi}^2)$ is the lift of

$$\mathcal{H} : L^2(\mathbb{R}^D) \rightarrow L^2(\mathbb{R}^D), \quad (\mathcal{H}u)(x) = \frac{1}{\hbar} \left(-\frac{\hbar^2}{2} \left(\sum_{i=1}^D \frac{\partial^2 u}{\partial x_i^2} \right) (x) + \frac{1}{2} |x|^2 \cdot u(x) \right) \tag{8.3}$$

Therefore the conclusion follows from the argument on quantization of the harmonic oscillator \mathcal{H} [51, p.105]. □

2325 *Remark 8.2.* It is possible to compute directly that the eigenfunctions of \hat{P} using the "creation operator" $a_j = \hat{x}_j + i\hat{\xi}_j$ and showing that it is multiplication by $z_j = x_j + i\xi_j$ in phase space (see [23, 27]). Then we can identify $Q_\hbar^{(i)} = \mathcal{B}_\hbar^* Q_\hbar^{(i)} \mathcal{B}_\hbar$ as the projection onto homogeneous polynomials of degree i in $(z_j)_j$.

Recall that operators $Q_h^{(k)}$ and $T^{(k)}$ (Section 4.4) have the same rank $\binom{D+k-1}{D-1}$. The next lemma gives a more precise relation between them.

Lemma 8.3. *For $0 \leq k \leq n$, the operator $Q_h^{(k)} = \mathcal{B}_h^* Q_h^{(k)} \mathcal{B}_h$ extends to a continuous operator*

$$Q_h^{(k)} : \mathcal{S}'(\mathbb{R}^{2D}) \rightarrow \mathcal{S}(\mathbb{R}^{2D}).$$

The restrictions:

$$\left(\bigoplus_{i=0}^k Q_h^{(i)} \right) : \bigoplus_{i=1}^k \text{Im} \mathcal{T}_h^{(i)} \rightarrow \bigoplus_{i=0}^k \text{Im} Q_h^{(i)} \quad (8.4)$$

and

$$\left(\bigoplus_{i=0}^k \mathcal{T}_h^{(i)} \right) : \bigoplus_{i=0}^k \text{Im} Q_h^{(i)} \rightarrow \bigoplus_{i=0}^k \text{Im} \mathcal{T}_h^{(i)} \quad (8.5)$$

are well-defined and bijective. The operator norms of (8.4), (8.5) and their inverses are bounded by a constant independent of \hbar .

Proof. Recall that the operator $Q_h^{(k)}$ is the orthogonal projection to its image (8.2), which is finite dimensional and contained in $\mathcal{S}(\mathbb{R}^{2D})$. Hence we have the first claim and well definiteness of the operators (8.4) and (8.5) is an immediate consequence.

To prove that (8.4) and (8.5) are bijective, we have only to show that they are injective, because the subspaces in the source and target have the same finite dimension. We prove injectivity of (8.4). Let $u \in \left(\bigoplus_{i=0}^k \text{Im} \mathcal{T}_h^{(i)} \right)$. Such u can be expressed as $u = \mathcal{B}_h p$ with p a polynomial of degree at most k on \mathbb{R}^D . Suppose that $\left(\bigoplus_{i=0}^k Q_h^{(i)} \right) u = 0$. Since $\left(\bigoplus_{i=0}^k Q_h^{(i)} \right)$ is an orthogonal projection operator, u is orthogonal to $\bigoplus_{i=0}^k \text{Im} Q_h^{(i)}$ and hence we have

$$(p, q \cdot \varphi_0)_{L^2(\mathbb{R}^D)} = (\mathcal{B}_h p, \mathcal{B}_h(q \cdot \varphi_0))_{L^2(\mathbb{R}^{2D})} = 0$$

for any polynomial q of order at most k on \mathbb{R}^D . Setting $q = p$, we see that $(p, p \cdot \varphi_0)_{L^2(\mathbb{R}^D)} = (|p|^2, \varphi_0)_{L^2(\mathbb{R}^D)} = 0$, showing $p = 0$ and $u = 0$. We have shown that (8.4) is injective.

To prove injectivity of (8.5), let $u = q \varphi_0 \in \left(\bigoplus_{i=0}^k \text{Im} Q_h^{(i)} \right)$ with q a polynomial of degree at most k on \mathbb{R}^D . Let $p = \left(\bigoplus_{i=0}^k \mathcal{T}_h^{(i)} \right) u$ be the Taylor expansion of u at 0 up to order k . Suppose that $p = 0$. Since $\varphi_0(0) \neq 0$, φ_0 is invertible as a formal power series, we deduce that $q = 0$. Hence (8.5) is injective.

The operators (8.4) (resp. (8.5)) for different $\hbar > 0$ are related by the scaling (4.10) and hence we get the last claim. \square

8.2 The rough Laplacian on \mathbb{R}^{2d}

As in Subsection 5.1, we consider \mathbb{R}^{2d} as a symplectic linear space with $\omega = \sum_{i=1}^d dq^i \wedge dp^i$ and with the additional compatible Euclidean metric

$$g = \sum_{i=1}^d dq^i \otimes dq^i + dp^i \otimes dp^i$$

We have seen in (3.12) that these data define the Euclidean rough Laplacian as the operator

$$\Delta_{\hbar} = D^*D : C^\infty(\mathbb{R}^{2d}) \rightarrow C^\infty(\mathbb{R}^{2d}).$$

2355 This operator Δ_{\hbar} is a closed self-adjoint operator on $L^2(\mathbb{R}^{2d})$ and its domain of definition is

$$\mathcal{D}(\Delta_{\hbar}) = \{u \in L^2(\mathbb{R}^{2d}) \mid \|\Delta_{\hbar}u\|_{L^2} < \infty\}.$$

Note that $\mathcal{D}(\Delta_{\hbar})$ becomes a Hilbert space if we consider the norm

$$\|u\|_{\Delta_{\hbar}} = ((u, u)_{\Delta_{\hbar}})^{1/2} \quad (8.6)$$

induced by the inner product

$$(u, v)_{\Delta_{\hbar}} = (u, v)_{L^2} + (\Delta_{\hbar}u, \Delta_{\hbar}v)_{L^2}.$$

Obviously, Δ_{\hbar} gives a bounded operator

$$\Delta_{\hbar} : (\mathcal{D}(\Delta_{\hbar}), \|\cdot\|_{\Delta_{\hbar}}) \rightarrow L^2(\mathbb{R}^{2d}).$$

2360 An important property of the operator Δ_{\hbar} that follows from the definition is that it is invariant with respect to the action of prequantum transfer operators for symplectic isometries:

Lemma 8.4. *Suppose that $f : \mathbb{R}^{2d} \rightarrow \mathbb{R}^{2d}$ is an isometric affine map preserving the symplectic form ω , then we have $\Delta_{\hbar} \circ \mathcal{L}_f = \mathcal{L}_f \circ \Delta_{\hbar}$ for the associated prequantum transfer operator \mathcal{L}_f given in (5.7).*

2365

From the expression of the Euclidean rough Laplacian obtained in (3.12), we have

$$\hbar\Delta = \mathcal{U} \circ \left(\text{Id} \otimes \frac{1}{\hbar} \left(\widehat{\zeta}_q^2 + \widehat{\zeta}_p^2 \right) \right) \circ \mathcal{U}^{-1}$$

where \mathcal{U} has been defined in (5.15).

By considering a commutative diagram corresponding to (5.23), we obtain the following commutative diagram similar to (5.24):

$$\begin{array}{ccc} L^2(\mathbb{R}_x^{2d}) & \xrightarrow{\hbar\Delta_{\hbar}} & L^2(\mathbb{R}_x^{2d}) \\ \uparrow \mathcal{U} & & \uparrow \mathcal{U} \\ L^2(\mathbb{R}_{\nu_q}^2) \otimes L^2(\mathbb{R}_{\zeta_p}^d) & \xrightarrow{\text{Id} \otimes \mathcal{H}} & L^2(\mathbb{R}_{\nu_q}^2) \otimes L^2(\mathbb{R}_{\zeta_p}^d) \end{array}$$

2370 where \mathcal{H} is the harmonic oscillator operator defined in (8.3) with setting $D = d$.

Thus we may invoke the argument in Subsection 8.1, especially Lemma 8.1 and 8.3, to derive the next proposition on the spectral structure of the Euclidean rough Laplacian Δ_{\hbar} . For $k \geq 0$, let us consider the spectral projection operator

$$q_{\hbar}^{(k)} := \mathcal{U} \circ \left(\text{Id} \otimes Q_{\hbar}^{(k)} \right) \circ \mathcal{U}^{-1} : L^2(\mathbb{R}_x^{2d}) \rightarrow L^2(\mathbb{R}_x^{2d}) \quad (8.7)$$

where $Q_{\hbar}^{(k)}$ is the projection operator on level k of the harmonic oscillator \mathcal{H} . Note that it restricts to a bounded operator

$$q_{\hbar}^{(k)} : L^2(\mathbb{R}^{2d}) \rightarrow (\mathcal{D}(\Delta_{\hbar}), \|\cdot\|_{\Delta_{\hbar}}) \subset L^2(\mathbb{R}^{2d}) \quad (8.8)$$

whose operator norm is bounded by a constant independent of \hbar .

Proposition 8.5. *The rough Laplacian $\Delta_{\hbar} = D^*D$ on the Euclidean space \mathbb{R}^{2d} is a closed self-adjoint operator on $L^2(\mathbb{R}^{2d})$ and its spectrum consists of integer eigenvalues $d + 2k$ with $k \in \mathbb{N}$. The spectral projector corresponding to the eigenvalue $d + 2k$ is the operator $q_{\hbar}^{(k)}$ given in (8.7) and together they form a complete set of mutually commuting orthogonal projections in $L^2(\mathbb{R}^{2d})$. Consequently we have*

$$L^2(\mathbb{R}^{2d}) = \overline{\bigoplus_{k=0}^{\infty} H_k''}, \quad \text{with } H_k'' := \text{Im } q_{\hbar}^{(k)}.$$

The next proposition is an immediate consequence of Lemma 8.3.

Proposition 8.6. *The operator $\bigoplus_{i=0}^k q_{\hbar}^{(i)}$ and $\bigoplus_{i=0}^k t_{\hbar}^{(i)}$ restricts to the bijections*

$$\bigoplus_{i=0}^k q_{\hbar}^{(i)} : \bigoplus_{i=0}^k \text{Im } t_{\hbar}^{(i)} \rightarrow \bigoplus_{i=0}^k \text{Im } q_{\hbar}^{(i)}$$

and

$$\bigoplus_{i=0}^k t_{\hbar}^{(i)} : \bigoplus_{i=0}^k \text{Im } q_{\hbar}^{(i)} \rightarrow \bigoplus_{i=0}^k \text{Im } t_{\hbar}^{(i)}$$

respectively. The operator norms of these operators and their inverses are bounded by some constant independent of \hbar .

8.3 The multiplication operators and the rough Laplacian on \mathbb{R}^{2d}

Recall the space of functions \mathcal{X}_{\hbar} defined in Setting I, page 6.1.

Lemma 8.7. *For any $\psi \in \mathcal{X}_{\hbar}$, we have*

$$\|[\mathcal{M}(\psi), \Delta_{\hbar}]\|_{(\mathcal{D}(\Delta_{\hbar}), \|\cdot\|_{\Delta_{\hbar}}) \rightarrow L^2(\mathbb{R}^{2d})} \leq C\hbar^{\theta}$$

and

$$\left\| \left[\mathcal{M}(\psi), q_{\hbar}^{(k)} \right] \right\|_{L^2(\mathbb{R}^{2d})} \leq C\hbar^{\theta}$$

where C is a constant independent of $\psi \in \mathcal{X}_{\hbar}$ and \hbar .

Proof. The first claim can be checked easily from the expression of Δ_{\hbar} given in Proposition 3.6. For the second claim, we can just follow the argument in the proof of Lemma 6.6, replacing $t_{\hbar}^{(k)}$ by $q_{\hbar}^{(k)}$. (The proof is simpler actually.) \square

2395 **8.4 Proof of Theorem 3.5**

In this subsection, we give a proof of Theorem 3.5 on the rough Laplacian. In former part of the proof, we consider a rough Laplacian $\tilde{\Delta}_\hbar$ constructed from local data instead of the geometric rough Laplacian Δ_\hbar , and prove the claims of Theorem 3.5 for $\tilde{\Delta}_\hbar$. In the latter part, we show that we can deform the rough Laplacian $\tilde{\Delta}_\hbar$ continuously to Δ_\hbar keeping
 2400 the “band structure” of the eigenvalues. This will imply that the cardinality of eigenvalues in the first (or lowest) band coincides for $\tilde{\Delta}_\hbar$ and Δ_\hbar . We note at this moment that, for the argument on rough Laplacian below, we do not need Condition (2) in Proposition 7.3 (i.e. orthogonality of stable and unstable subspaces) in the choice of the coordinate charts $\{\kappa_i\}_{i=1}^{I_\hbar}$ in Proposition 7.3, that is, our argument below holds true for any choice of
 2405 coordinate charts $\{\kappa_i\}_{i=1}^{I_\hbar}$ satisfying the conditions other than that condition. Also since our proof about the Laplacian is independent on the dynamics of f , in our choices the value of $0 < \beta < 1$ can be taken close to 1.

We introduce a rough Laplacian $\tilde{\Delta}_\hbar$ acting on the space $C_N^\infty(P)$ of equivariant functions. We start from the operators on local data. Let

$$\Delta_\hbar : \bigoplus_{i=1}^{I_\hbar} \mathcal{D}(\Delta_\hbar) \rightarrow \bigoplus_{i=1}^{I_\hbar} L^2(\mathbb{R}^{2d}), \quad \Delta((u_i)_{i=1}^{I_\hbar}) = (\Delta_\hbar u_i)_{i=1}^{I_\hbar}$$

2410 where Δ_\hbar denotes the Euclidean rough Laplacian on \mathbb{R}^{2d} defined in Subsection 8.2. The next proposition is an immediate consequence of Lemma 8.7. (So we omit the proof.)

Proposition 8.8. *There exist constants $C > 0$ and $\epsilon > 0$, independent of \hbar , such that*

$$\|[\Delta_\hbar, (\mathbf{I}_\hbar \circ \mathbf{I}_\hbar^*)]\|_{\bigoplus_{i=1}^{I_\hbar} (\mathcal{D}(\Delta_\hbar), \|\cdot\|_{\Delta_\hbar}) \rightarrow \bigoplus_{i=1}^{I_\hbar} L^2(\mathbb{R}^{2d})} \leq C\hbar^\epsilon. \tag{8.9}$$

We define a rough Laplacian $\tilde{\Delta}_\hbar$ acting on $C_N^\infty(P)$ by

$$\tilde{\Delta}_\hbar := \mathbf{I}_\hbar^* \circ \Delta_\hbar \circ \mathbf{I}_\hbar \quad : C_N^\infty(P) \rightarrow C_N^\infty(P). \tag{8.10}$$

Remark 8.9. Notice that this rough Laplacian operator $\tilde{\Delta}_\hbar$ is defined by gluing Euclidean
 2415 rough Laplacian on local charts and does *not* coincide with the geometric rough Laplacian $\Delta_\hbar = D^*D$ with respect to a global metric on M defined in Subsection 3.2.

The operator $\tilde{\Delta}_\hbar$ defined above is a closed densely defined operator on $C_N^\infty(P)$. Its domain of definition is by definition

$$\mathcal{D}(\tilde{\Delta}_\hbar) = \{u \in L_N^2(P) \mid \|\tilde{\Delta}_\hbar u\|_{L^2} < \infty\},$$

which becomes a Hilbert space if we equip it with the inner product

$$(u, v)_{\tilde{\Delta}_\hbar} = (u, v)_{L^2} + (\tilde{\Delta}_\hbar u, \tilde{\Delta}_\hbar v)_{L^2}.$$

2420 We will write $\|\cdot\|_{\tilde{\Delta}_\hbar}$ for the corresponding norm. It is easy to see the following Lemma. (So we omit the proof.)

Lemma 8.10. *The norm $\|\cdot\|_{\tilde{\Delta}_h}$ above is equivalent to the norm*

$$\|u\|'_{\tilde{\Delta}_h} = \left(\sum_{i=1}^{I_h} \|u_i\|_{\Delta_h}^2 \right)^{1/2}$$

defined in terms of local data, where $\|\cdot\|_{\Delta_h}$ on the right hand side denotes the norm defined in (8.6). Consequently we have

$$\mathcal{D}(\tilde{\Delta}_h) = \left\{ u \in L_N^2(P) \mid \sum_{i=1}^{I_h} \|u_i\|_{\Delta_h}^2 < \infty \right\}.$$

2425

Below we are going to construct the spectral projectors for $\tilde{\Delta}_h$, corresponding to the “bands of eigenvalues”. Again we start from local data: We consider the operators

$$\mathbf{q}_h^{(k)} : \bigoplus_{i=1}^{I_h} L^2(\mathbb{R}^{2d}) \rightarrow \bigoplus_{i=1}^{I_h} (\mathcal{D}(\Delta_h), \|\cdot\|_{\Delta_h}) \subset \bigoplus_{i=1}^{I_h} L^2(\mathbb{R}^{2d}), \quad \mathbf{q}_h^{(k)}((u_i)_{i=1}^{I_h}) = (q_h^{(k)}(u_i))_{i=1}^{I_h},$$

for $0 \leq k \leq n$, where $q_h^{(k)}$ is the operator defined in (8.7). Recall (8.8) for boundedness of these projection operators. The remainder is denoted as

$$\tilde{\mathbf{q}}_h = \mathbf{q}_h^{(n+1)} : \bigoplus_{i=1}^{I_h} L^2(\mathbb{R}^{2d}) \rightarrow \bigoplus_{i=1}^{I_h} L^2(\mathbb{R}^{2d}), \quad \tilde{\mathbf{q}}_h = \mathbf{q}_h^{(n+1)} := \text{Id} - (\mathbf{q}_h^{(1)} + \mathbf{q}_h^{(2)} + \cdots + \mathbf{q}_h^{(n)}).$$

The last operator restricts to a bounded operator

$$\tilde{\mathbf{q}}_h = \mathbf{q}_h^{(n+1)} : \bigoplus_{i=1}^{I_h} (\mathcal{D}(\Delta_h), \|\cdot\|_{\Delta_h}) \rightarrow \bigoplus_{i=1}^{I_h} (\mathcal{D}(\Delta_h), \|\cdot\|_{\Delta_h}).$$

2430

We next introduce the operators

$$\check{\lambda}_h^{(k)} := \mathbf{I}_h^* \circ \mathbf{q}_h^{(k)} \circ \mathbf{I}_h : L_N^2(P) \rightarrow (\mathcal{D}(\tilde{\Delta}_h), \|\cdot\|_{\tilde{\Delta}_h})$$

for $0 \leq k \leq n$. These are bounded operators and the operator norms are bounded by a constant independent of \hbar . For $k = n + 1$, we set

$$\check{\lambda}_h^{(n+1)} := \mathbf{I}_h^* \circ \mathbf{q}_h^{(n+1)} \circ \mathbf{I}_h = \text{Id} - (\check{\lambda}_h^{(0)} + \check{\lambda}_h^{(1)} + \cdots + \check{\lambda}_h^{(n)}). \quad (8.11)$$

Further we can prove the estimates

$$\left\| \check{\lambda}_h^{(k)} \circ \check{\lambda}_h^{(k)} - \check{\lambda}_h^{(k)} \right\|_{L_N^2(P) \rightarrow (\mathcal{D}(\tilde{\Delta}_h), \|\cdot\|_{\tilde{\Delta}_h})} \leq C\hbar^\epsilon \quad \text{for } 0 \leq k \leq n+1 \quad (8.12)$$

and

$$\left\| \check{\lambda}_h^{(k)} \circ \check{\lambda}_h^{(k')} \right\|_{L_N^2(P) \rightarrow (\mathcal{D}(\tilde{\Delta}_h), \|\cdot\|_{\tilde{\Delta}_h})} \leq C\hbar^\epsilon \quad \text{for } 0 \leq k, k' \leq n+1 \text{ with } k \neq k' \quad (8.13)$$

for some constants $\epsilon > 0$ and $C > 0$. (For the case where either of k or k' equals $n + 1$, use
 2435 the definition (8.11) to check (8.12) and (8.13).)

Now we proceed in parallel to the argument in Subsection 7.6 and obtain the following lemma.

Lemma 8.11. *There exist a decomposition of the Hilbert space $(\mathcal{D}(\tilde{\Delta}_\hbar), \|\cdot\|_{\tilde{\Delta}_\hbar})$*

$$\mathcal{D}(\tilde{\Delta}_\hbar) = \mathfrak{H}_0 \oplus \mathfrak{H}_1 \oplus \mathfrak{H}_2 \oplus \cdots \oplus \mathfrak{H}_n \oplus \tilde{\mathfrak{H}}$$

and that of $L_N^2(P)$

$$L_N^2(P) = \mathfrak{H}_0 \oplus \mathfrak{H}_1 \oplus \mathfrak{H}_2 \oplus \cdots \oplus \mathfrak{H}_n \oplus \overline{\mathfrak{H}}$$

2440 where $\overline{\mathfrak{H}}$ is the closure of $\tilde{\mathfrak{H}}$ in $L_N^2(P)$. The subspaces \mathfrak{H}_k for $0 \leq k \leq n$ are of finite dimension. If we write $\lambda_\hbar^{(k)}$ for projection operators to \mathfrak{H}_k (resp. to $\overline{\mathfrak{H}}$ in the case $k=n+1$) along other subspaces, then we have

$$(1) \left\| \lambda_\hbar^{(k)} - \check{\lambda}_\hbar^{(k)} \right\|_{(\mathcal{D}(\tilde{\Delta}_\hbar), \|\cdot\|_{\tilde{\Delta}_\hbar})} \leq C\hbar^\epsilon \text{ for } 0 \leq k \leq n,$$

(2) if $k \neq k'$,

$$\left\| \lambda_\hbar^{(k)} \circ \hbar \tilde{\Delta}_\hbar \circ \lambda_\hbar^{(k')} \right\|_{(\mathcal{D}(\Delta_\hbar), \|\cdot\|_{\Delta_\hbar}) \rightarrow L_N^2(P)} \leq C\hbar^\epsilon,$$

2445 (3) if $0 \leq k \leq n$,

$$\left\| \lambda_\hbar^{(k)} \circ \hbar \tilde{\Delta}_\hbar \circ \lambda_\hbar^{(k)} - (d + 2k) \cdot \lambda_\hbar^{(k)} \right\|_{(\mathcal{D}(\Delta_\hbar), \|\cdot\|_{\Delta_\hbar}) \rightarrow L_N^2(P)} \leq C\hbar^\epsilon,$$

(4) for $k = n + 1$, we have

$$\left\| \lambda_\hbar^{(n+1)} \circ \hbar \tilde{\Delta}_\hbar(u) \right\|_{L^2} \geq (d + 2k + 1 - C\hbar^\epsilon) \|u\|_{L^2} \quad \text{for } u \in \tilde{\mathfrak{H}} = \mathfrak{H}^{(n+1)}.$$

Therefore, by the general theorem on perturbation of closed linear operators [29, chap.IV, th. 1.16], we obtain an analogue of Theorem 3.5 for the rough Laplacian $\tilde{\Delta}_\hbar$.

Theorem 8.12. *There exists a small constant $\epsilon > 0$ such that, for any $\alpha > 0$, we have*

$$\text{dist} \left(\text{Spec} \left(\hbar \tilde{\Delta}_\hbar \right) \cap \{|z| < \alpha\}, \{d + 2k, k \in \mathbb{N}\} \right) \leq \hbar^\epsilon$$

2450 when \hbar is sufficiently small.

Further we have

Lemma 8.13. *There exists $C_0 > 0$ s.t. for sufficiently small $\hbar > 0$, we have*

$$C_0^{-1} \hbar^{-d} \leq \dim \mathcal{H}_k = \dim \mathfrak{H}_k \leq C_0 \hbar^{-d} \quad \text{for } 0 \leq k \leq n.$$

Proof. To prove the equality $\dim \mathcal{H}_k = \dim \mathfrak{H}_\epsilon$, it is enough to check that

$$\text{rank } \tau_{\hbar}^{(k)} = \text{rank } \lambda_{\hbar}^{(k)} \quad \text{for } 0 \leq k \leq n,$$

or equivalently

$$\text{rank } \bigoplus_{i=0}^k \tau_{\hbar}^{(i)} = \text{rank } \bigoplus_{i=0}^k \lambda_{\hbar}^{(i)} \quad \text{for } 0 \leq k \leq n.$$

2455 The latter relation would follow if we show

$$c \left\| \left(\bigoplus_{i=0}^k \lambda_{\hbar}^{(i)} \right) u \right\|_{L^2} \leq \left\| \left(\bigoplus_{i=0}^k \lambda_{\hbar}^{(i)} \right) \circ \left(\bigoplus_{i=0}^k \tau_{\hbar}^{(i)} \right) \circ \left(\bigoplus_{i=0}^k \lambda_{\hbar}^{(i)} \right) u \right\|_{L^2} \leq C \left\| \left(\bigoplus_{i=0}^k \lambda_{\hbar}^{(i)} \right) u \right\|_{L^2}$$

and

$$c \left\| \left(\bigoplus_{i=0}^k \tau_{\hbar}^{(i)} \right) u \right\|_{\mathcal{H}_{\hbar}^r} \leq \left\| \left(\bigoplus_{i=0}^k \tau_{\hbar}^{(i)} \right) \circ \left(\bigoplus_{i=0}^k \lambda_{\hbar}^{(i)} \right) \circ \left(\bigoplus_{i=0}^k \tau_{\hbar}^{(i)} \right) u \right\|_{\mathcal{H}_{\hbar}^r} \leq C \left\| \left(\bigoplus_{i=0}^k \tau_{\hbar}^{(i)} \right) u \right\|_{\mathcal{H}_{\hbar}^r}$$

for a constant $0 < c < C$, independent of \hbar . But these are immediate consequences of Proposition 8.6 (and the construction of the projection operators $\tau_{\hbar}^{(k)}$ and $\lambda_{\hbar}^{(k)}$).

It remains to show

$$\text{rank } \lambda_{\hbar}^{(k)} \asymp \hbar^{-d} \quad \text{for } 0 \leq k \leq n. \quad (8.14)$$

2460 For each point $x \in M$, we associate a smooth function

$$\varphi_x := \check{\lambda}_{\hbar}^{(k)}(\delta_x)$$

where δ_x is the Dirac measure at the point x . (The right hand side is well-defined and give a smooth function that concentrates around x .) Consider positive constants $0 < c < C$ and take a finite subset of points Q_{\hbar} on M so that the mutual distance between two points in Q_{\hbar} is in between $c \cdot \hbar^{1/2}$ and $C \cdot \hbar^{1/2}$. Let

$$\mathcal{Q}_{\hbar} = \{ \lambda_{\hbar}^{(k)}(\varphi_x) \mid x \in Q_{\hbar} \} \subset \text{Im } \lambda_{\hbar}^{(k)}.$$

2465 Note that $\lambda_{\hbar}^{(k)}(\varphi_x)$ is close to φ_x from Claim 1 in Proposition 8.11. It is not difficult to check that

- (1) if we let the constants c, C large, the subset \mathcal{Q}_{\hbar} is linearly independent, and
- (2) if we let the constants c, C small, the subset \mathcal{Q}_{\hbar} span the whole space $\text{Im } \lambda_{\hbar}^{(k)}$.

Indeed, to prove (1), we have only to observe that, if the constants c, C are sufficiently large, the L^2 -scalar product between different elements $\varphi_x, \varphi_{x'}$ in \mathcal{Q}_{\hbar} decay rapidly with respect the distance between the corresponding points x, x' (relative to the size $\hbar^{1/2}$). To prove (2), we see that, if the constants c, C are sufficiently small, any element of $\text{Im } \lambda_{\hbar}^{(k)}$ is well approximated by the linear combinations of the element in \mathcal{Q}_{\hbar} and then, by successive approximation, it is really contained in the subspace spanned by \mathcal{Q}_{\hbar} . Clearly the claims

2475 (1) and (2) imply (8.14). \square

Finally we complete the proof of Theorem 3.5 and Claim 1 of Theorem 7.1. We show that the operator $\tilde{\Delta}_h$ is continuously deformed to the geometric Laplacian Δ_h , keeping the band structure described in Theorem 8.12. For this purpose, we take a continuous one-parameter family of splitting of the tangent bundle

$$TM = E_+^{(t)} \oplus E_-^{(t)}$$

2480 with $t \in [0, 1]$ the parameter, such that

- $E_+^{(0)} = E_u$ and $E_-^{(0)} = E_s$, that is, the splitting above coincides with the hyperbolic splitting associated to f when $t = 0$.
- the sub-bundles $E_{\pm}^{(1)}$ for $t = 1$ are C^∞ and orthogonal with respect to the Riemann metric g on M .

2485 Then we consider a continuous deformation $\{\kappa_{i,t}\}_{i=1}^{I_h}$ of the atlas $\{\kappa_i\}_{i=1}^{I_h}$ and, correspondingly, the deformation $\{\psi_{i,t}; 1 \leq i \leq I_h\}$ of the family $\{\psi_i; 1 \leq i \leq I_h\}$ of functions so that all the conditions in Proposition 7.3 hold uniformly for $t \in [0, 1]$, but with the sub-bundles E^u and E^s in the condition (2) replaced by E_+^t and E_-^t .

2490 We consider the rough Laplacian $\tilde{\Delta}_{h,t}$ defined, similarly to $\tilde{\Delta}_h$ in (8.10), from the Euclidean rough Laplacian on local charts $\{\kappa_{i,t}\}_{i=1}^{I_h}$ and the family of functions $\{\psi_{i,t}; 1 \leq i \leq I_h\}$. The argument in the former part of this subsection holds true uniformly for $t \in [0, 1]$, that is, we can consider the spectral projection operators $\lambda_{h,t}^{(k)}$, $0 \leq k \leq n+1$, for $\tilde{\Delta}_{h,t}$, which corresponds to $\lambda_h^{(k)}$ for $\tilde{\Delta}_h$. Since the deformation is continuous, we see by homotopy argument that $\text{rank } \lambda_{h,\rho}^{(k)} = \dim \mathfrak{H}_{k,\rho}$ is constant for $t \in [0, 1]$. In particular we
2495 have $\text{rank } \lambda_{h,1}^{(k)} = \text{rank } \lambda_{h,0}^{(k)}$.

Note that the operator $\tilde{\Delta}_{h,1}$ is close to the geometric rough Laplacian Δ_h . In fact, since the derivative $(D\kappa_{i,1})_0$ at the origin is an isometry with respect to the Euclidean metric on \mathbb{R}^{2d} and the Riemann metric g on M , we can check by using the local expression of the geometric rough Laplacian in Proposition 3.11 that we have

$$\|\hbar\tilde{\Delta}_{h,1} - \hbar\Delta_h : (\mathcal{D}(\Delta_h), \|\cdot\|_{\Delta_h}) \rightarrow L_N^2(P)\| \leq C\hbar^\epsilon.$$

2500 Therefore, by the perturbation theorem [29, chap.IV, th. 1.16] for closed operators, we obtain the “band structure” stated in the former part of Theorem 3.5. We can also see that the number of eigenvalues of the geometric rough Laplacian $\Delta_h = D^*D$ in the k -th band is same as that for $\tilde{\Delta}_h$. Hence, from Lemma 8.13, we obtain the rough upper and lower bound on the rank of spectral projectors \mathfrak{P}_k in Theorem 3.5 and also Claim (1) of
2505 Theorem 7.1.

9 Proof of Th. 1.34, 1.35. Extension of the transfer operator to the Grassmanian bundle.

In this section, we explain how we obtain the results stated in Subsection 1.4, namely Theorem 1.34, Theorem 1.35, Theorem 1.36 and Theorem 1.37 for the Grassmanian extensions.

2510 The proof are obtained by modifying the argument in the previous sections, and, for the
most part, the extensions are rather formal and easy. The most essential difference is in
Proposition 9.22, where we take systems of local coordinate charts depending on N (or \hbar).
Unlike the corresponding statement, Proposition 7.3, the local coordinate charts will be
(metrically) singular in the fiber directions and the singularity will increase as N tends to
2515 infinity. With this choice of local coordinate charts, the non-smooth section E_u will look
“flat” in the local coordinates. A problem that may happen with this choice of singular
local coordinates is that the coordinate change transformations and the flow viewed in
such local coordinate charts may be also singular. In the proof of Proposition 9.22, we
show that this problem actually does not occur. We will give a detailed argument on this
2520 point. Once we establish the local coordinates, we can follow the argument in the previous
sections almost literally. So we will just give the corresponding statements to clarify the
correspondence and skip most the proofs referring those of the corresponding statements
in the previous sections.

9.1 Discussion about the linear model

2525 We first discuss about the extension of the argument in Section 4 and 5 about the pre-
quantum transfer operators for linear hyperbolic maps. Instead of a hyperbolic symplectic
linear map B in (5.18), we consider a linear map $\tilde{B} : \mathbb{R}^{2d+d'} \rightarrow \mathbb{R}^{2d+d'}$ of the form

$$\tilde{B}(q, p, s) = (Aq, {}^t A^{-1}p, \hat{A}s) \quad (9.1)$$

where (q, p, s) with $q, p \in \mathbb{R}^d$ and $s \in \mathbb{R}^{d'}$ denote the coordinates on $\mathbb{R}^{2d+d'} = \mathbb{R}^d \oplus \mathbb{R}^d \oplus \mathbb{R}^{d'}$,
 $A : \mathbb{R}^d \rightarrow \mathbb{R}^d$ is an expanding linear map satisfying $\|A^{-1}\| \leq 1/\lambda$ for some $\lambda > 1$ and
2530 $\hat{A} : \mathbb{R}^{d'} \rightarrow \mathbb{R}^{d'}$ is a contracting linear map satisfying $\|\hat{A}\| \leq 1/\lambda$. This is a hyperbolic linear
map with stable subspace $\{0\} \oplus \mathbb{R}^d \oplus \mathbb{R}^{d'}$ and unstable subspace $\mathbb{R}^d \oplus \{0\} \oplus \{0\}$. The
 L^2 -normalized transfer operator associated to \tilde{B} is

$$\tilde{\mathcal{L}} : L^2(\mathbb{R}^{2d+d'}) \rightarrow L^2(\mathbb{R}^{2d+d'}), \quad \tilde{\mathcal{L}}u = \frac{1}{\sqrt{|\det \hat{A}|}} \cdot u \circ A^{-1}.$$

Before we proceed, we put a remark.

Remark 9.1. A simple idea to treat the transfer operator $\tilde{\mathcal{L}}$ as above is to regard it as
2535 the tensor product of two transfer operators, one associated to the hyperbolic linear map
 $A \oplus {}^t A^{-1}$ and the other associated to the contracting linear map \hat{A} . We may then apply
the results in Section 5 to the former factor and that in 4 to (the adjoint of) the latter,
and show a band structure of the spectrum of the transfer operator $\tilde{\mathcal{L}}$ on a Hilbert space.
However the Hilbert space that appears in such an argument has singular properties with
2540 respect to the action of non-linear diffeomorphisms that break the product structure. For
this reason, we take a similar but different way.

Let

$$\mathcal{B}_{(x,s)} : L^2(\mathbb{R}_{(x,s)}^{2d+d'}) \rightarrow L^2(\mathbb{R}_{(x,s)}^{2d+d'} \oplus \mathbb{R}_{(\xi_x, \xi_s)}^{2d+d'})$$

be the Bargmann transform defined by

$$\mathcal{B}_{(x,s)} := \mathcal{B}_x \otimes \mathcal{B}_s \quad (9.2)$$

where $\mathcal{B}_x : L^2(\mathbb{R}_x^{2d}) \rightarrow L^2(\mathbb{R}_{(x,\xi)}^{4d})$ is the slight modification of the Bargmann transform given in (5.11) and $\mathcal{B}_s : L^2(\mathbb{R}_s^{d'}) \rightarrow L^2(\mathbb{R}_{(s,\xi_s)}^{2d'})$ is the standard Bargmann transform given in (4.3) with setting $D = d'$. Let

$$\mathcal{B}_{(x,s)}^* := \mathcal{B}_x^* \times \mathcal{B}_s^* : L^2(\mathbb{R}_{(x,s)}^{2d+d'} \oplus \mathbb{R}_{(\xi_x,\xi_s)}^{2d+d'}) \rightarrow L^2(\mathbb{R}_{(x,s)}^{2d+d'})$$

be the L^2 adjoint of $\mathcal{B}_{(x,s)}$. The lift of the operator $\tilde{\mathcal{L}}$ with respect to the Bargmann transform $\mathcal{B}_{(x,s)}$ is defined as before:

$$\tilde{\mathcal{L}}^{\text{lift}} := \mathcal{B}_{(x,s)} \circ \tilde{\mathcal{L}} \circ \mathcal{B}_{(x,s)}^* : L^2(\mathbb{R}_{(x,s)}^{2d+d'} \oplus \mathbb{R}_{(\xi_x,\xi_s)}^{2d+d'}) \rightarrow L^2(\mathbb{R}_{(x,s)}^{2d+d'} \oplus \mathbb{R}_{(\xi_x,\xi_s)}^{2d+d'}).$$

Here the space $\mathbb{R}_{(x,s)}^{2d+d'} \oplus \mathbb{R}_{(\xi_x,\xi_s)}^{2d+d'}$ is identified with the cotangent bundle of $\mathbb{R}_{(x,s)}^{2d+d'}$ equipped with the coordinates

$$(q, p, s, \xi_q, \xi_p, \xi_s) = (x, s, \xi_x, \xi_s)$$

where $x = (q, p) \in \mathbb{R}^d \oplus \mathbb{R}^d$ and $s \in \mathbb{R}^{d'}$ is the coordinates on $\mathbb{R}^{2d+d'} = \mathbb{R}^{2d} \oplus \mathbb{R}^{d'}$ and $\xi_x = (\xi_q, \xi_p) \in \mathbb{R}^d \oplus \mathbb{R}^d$ and $\xi_s \in \mathbb{R}^{d'}$ are their respective dual coordinates. Imitating the argument in Section 5, we introduce a different coordinate system

$$(\nu_q, \nu_p, \zeta_q, \zeta_p, s, \xi_s) = (\nu, \zeta, s, \xi_s) \quad (9.3)$$

on $\mathbb{R}^{2d+d'} \oplus \mathbb{R}^{2d+d'}$, where $(\nu_q, \nu_p, \zeta_q, \zeta_p) = (\nu, \zeta) \in \mathbb{R}^{4d} = \mathbb{R}^{2d} \oplus \mathbb{R}^{2d}$ is the coordinates introduced in Proposition 2.15 while we do not change the coordinates (s, ξ_s) . The corresponding coordinate change transformation is written

$$\tilde{\Phi} : \mathbb{R}_{(x,s)}^{2d+d'} \oplus \mathbb{R}_{(\xi_x,\xi_s)}^{2d+d'} \rightarrow \mathbb{R}_\nu^{2d} \oplus \mathbb{R}_\zeta^{2d} \oplus \mathbb{R}_s^{d'} \oplus \mathbb{R}_{\xi_s}^{d'}$$

$$\tilde{\Phi}(q, p, s, \xi_q, \xi_p, \xi_s) = (\nu_q, \nu_p, \zeta_p, \zeta_q, s, \xi_s).$$

By this transformation, the standard symplectic form $\tilde{\Omega}_0 = dx \wedge d\xi_x + ds \wedge d\xi_s$ is transferred to

$$(D\tilde{\Phi}^*)^{-1}(\tilde{\Omega}_0) = d\nu_q \wedge d\nu_p + d\zeta_p \wedge d\zeta_q + ds \wedge d\xi_s$$

and the metric $\tilde{g}_0 = \frac{1}{2}dx^2 + 2d\xi^2 + ds^2 + d\xi_s^2$ is transferred to the standard Euclidean metric

$$(D\tilde{\Phi}^*)^{-1}(g_0) = d\nu^2 + d\zeta^2 + ds^2 + d\xi_s^2.$$

The unitary operator associated to the coordinate change $\tilde{\Phi}$ is defined as

$$\tilde{\Phi}^* : L^2\left(\mathbb{R}_\nu^{2d} \oplus \mathbb{R}_\zeta^{2d} \oplus \mathbb{R}_s^{d'} \oplus \mathbb{R}_{\xi_s}^{d'}\right) \rightarrow L^2\left(\mathbb{R}_{(x,s)}^{2d+d'} \oplus \mathbb{R}_{(\xi_x,\xi_s)}^{2d+d'}\right), \quad \tilde{\Phi}^* u := u \circ \tilde{\Phi}.$$

Under these settings, we can follow the arguments in Section 5 and obtain the next proposition, which corresponds to Proposition 5.8.

Proposition 9.2. *The following diagram commutes:*

$$\begin{array}{ccc}
 L^2 \left(\mathbb{R}_{(x,s)}^{2d+d'} \right) & \xrightarrow{\tilde{\mathcal{L}}} & L^2 \left(\mathbb{R}_{(x,s)}^{2d+d'} \right) \\
 \uparrow \tilde{\mathcal{U}} & & \uparrow \tilde{\mathcal{U}} \\
 L^2 \left(\mathbb{R}_{\nu_q}^d \right) \otimes L^2 \left(\mathbb{R}_{(\zeta_p, \xi_s)}^{d+d'} \right) & \xrightarrow{L_A \otimes L_{A \oplus t \hat{A}^{-1}}} & L^2 \left(\mathbb{R}_{\nu_q}^d \right) \otimes L^2 \left(\mathbb{R}_{(\zeta_p, \xi_s)}^{d+d'} \right)
 \end{array} \tag{9.4}$$

2565 *with the unitary operator $\tilde{\mathcal{U}}$ defined by*

$$\tilde{\mathcal{U}} = \mathcal{B}_{(x,s)}^* \circ \tilde{\Phi}^* \circ (\mathcal{B}_{\nu_q} \otimes \mathcal{B}_{(\zeta_p, \xi_s)}).$$

Equivalently, for the lifted operators, the following diagram commutes:

$$\begin{array}{ccc}
 L^2 \left(\mathbb{R}_{(x,s)}^{2d+d'} \oplus \mathbb{R}_{(\xi_x, \xi_s)}^{2d+d'} \right) & \xrightarrow{\tilde{\mathcal{L}}^{\text{lift}}} & L^2 \left(\mathbb{R}_{(x,s)}^{2d+d'} \oplus \mathbb{R}_{(\xi_x, \xi_s)}^{2d+d'} \right) \\
 \uparrow \tilde{\Phi}^* & & \uparrow \tilde{\Phi}^* \\
 L^2 \left(\mathbb{R}_{\nu}^{2d} \right) \otimes L^2 \left(\mathbb{R}_{\zeta}^{2d} \oplus (\mathbb{R}_s^{d'} \oplus \mathbb{R}_{\xi_s}^{d'}) \right) & \xrightarrow{L_A^{\text{lift}} \otimes L_{A \oplus t \hat{A}^{-1}}^{\text{lift}}} & L^2 \left(\mathbb{R}_{\nu}^{2d} \right) \otimes L^2 \left(\mathbb{R}_{\zeta}^{2d} \oplus (\mathbb{R}_s^{d'} \oplus \mathbb{R}_{\xi_s}^{d'}) \right)
 \end{array} \tag{9.5}$$

We next introduce the anisotropic Sobolev space for the extended situation.

Definition 9.3. We define the *escape function* (or the *weight function*)

$$\tilde{\mathcal{W}}_h^r : \mathbb{R}_{(x,s)}^{2d+d'} \oplus \mathbb{R}_{(\xi_x, \xi_s)}^{2d+d'} \rightarrow \mathbb{R}_+ \quad \text{and} \quad \tilde{\mathcal{W}}_h^{r,\pm} : \mathbb{R}_{(x,s)}^{2d+d'} \oplus \mathbb{R}_{(\xi_x, \xi_s)}^{2d+d'} \rightarrow \mathbb{R}_+$$

by

$$\tilde{\mathcal{W}}_h^r(x, \xi) := W_h^r((\zeta_p, \xi_s), (\zeta_q, s)) \quad \text{and} \quad \tilde{\mathcal{W}}_h^{r,\pm}(x, \xi) := W_h^{r,\pm}((\zeta_p, \xi_s), (\zeta_q, s)) \tag{9.6}$$

2570 where the functions W_h^r and $W_h^{r,\pm}$ are those defined in Definition 4.12 with $D = d + d'$, and $\zeta_p, \zeta_q, s, \xi_s$ are those in the coordinates (9.3). The *anisotropic Sobolev space* $\tilde{\mathcal{H}}_h^r \left(\mathbb{R}_{(x,s)}^{2d+d'} \right)$ is the completion of the Schwartz space $\mathcal{S} \left(\mathbb{R}_{(x,s)}^{2d+d'} \right)$ with respect to the norm

$$\|u\|_{\tilde{\mathcal{H}}_h^r} := \left\| \tilde{\mathcal{W}}_h^r \cdot \mathcal{B}_{(x,s)} u \right\|_{L^2}.$$

Let $\tilde{\mathcal{H}}_h^{r,\pm} \left(\mathbb{R}^{2d+d'} \right)$ be the Hilbert space defined in the parallel manner with $\tilde{\mathcal{W}}_h^r(\cdot)$ replaced by $\tilde{\mathcal{W}}_h^{r,\pm}(\cdot)$.

2575 Below we fix an integer $n \geq 0$ and assume that the parameter r in the definition of the anisotropic Sobolev space $\tilde{\mathcal{H}}_h^r \left(\mathbb{R}_{(x,s)}^{2d+d'} \right)$ satisfies the condition

$$r > n + 2 + 2(2d + d') \tag{9.7}$$

which corresponds to (6.6) in Subsection 6.1. The next definition of projection operators correspond to Definition 5.26. Note that we will use the same symbol for the new projection operators as the corresponding projection operators in Definition 5.26. Since these two families of projection operators act on different Hilbert spaces, this will not introduce confusion.

Definition 9.4. For $0 \leq k \leq n$, we consider the projection operators

$$t_h^{(k)} := \tilde{\mathcal{U}} \circ (\text{Id} \otimes T^{(k)}) \circ \tilde{\mathcal{U}}^{-1} : \tilde{\mathcal{H}}_h^r(\mathbb{R}_{(x,s)}^{2d+d'}) \rightarrow \tilde{\mathcal{H}}_h^r(\mathbb{R}_{(x,s)}^{2d+d'}) \quad (9.8)$$

and

$$\tilde{t}_h := \text{Id} - \sum_{k=0}^n t_h^{(k)} = \tilde{\mathcal{U}} \circ (\text{Id} \otimes \tilde{T}) \circ \tilde{\mathcal{U}}^{-1} : \tilde{\mathcal{H}}_h^r(\mathbb{R}_{(x,s)}^{2d+d'}) \rightarrow \tilde{\mathcal{H}}_h^r(\mathbb{R}_{(x,s)}^{2d+d'}) \quad (9.9)$$

where $T^{(k)}$ and \tilde{T} are the projection operators introduced in (4.36) and (4.49) page 74 respectively for the setting $D = d + d'$.

We can translate the argument in Section 5 (especially that in Subsection 5.4) to get the next result, which corresponds to Proposition 5.11.

Proposition 9.5. *The projection operators $t_h^{(k)}$, $0 \leq k \leq n$, and \tilde{t}_h , defined in (9.8) and (9.9), form a complete set of mutually commutative projection operators on $\tilde{\mathcal{H}}_h^r(\mathbb{R}_{(x,s)}^{2d+d'})$. These operators also commute with the operator $\tilde{\mathcal{L}}$. Consequently the space $\tilde{\mathcal{H}}_h^r(\mathbb{R}_{(x,s)}^{2d+d'})$ has a decomposition invariant under the action of $\tilde{\mathcal{L}}$:*

$$\tilde{\mathcal{H}}_h^r(\mathbb{R}^{2d+d'}) = E'_0 \oplus E'_1 \oplus \dots \oplus E'_n \oplus \tilde{E}' \quad \text{where } E'_k = \text{Im } t_h^{(k)} \text{ and } \tilde{E}' = \text{Im } \tilde{t}_h.$$

For this decomposition, we have the following estimates:

(1) For every $0 \leq k \leq n$ and for every $u \in E'_k$, we have

$$C_0^{-1} \frac{|\det(A \oplus ({}^t\hat{A}^{-1}))|^{-1/2}}{\|A \oplus ({}^t\hat{A}^{-1})\|_{\max}^k} \cdot \|u\|_{\tilde{\mathcal{H}}_h^r} \leq \|\tilde{\mathcal{L}}u\|_{\tilde{\mathcal{H}}_h^r} \leq C_0 \frac{|\det(A \oplus ({}^t\hat{A}^{-1}))|^{-1/2}}{\|A \oplus ({}^t\hat{A}^{-1})\|_{\min}^k} \cdot \|u\|_{\tilde{\mathcal{H}}_h^r}.$$

(2) The operator norm of $\tilde{\mathcal{L}} : \tilde{E}' \rightarrow \tilde{E}'$ is bounded by

$$C_0 \cdot \max \left\{ \frac{|\det(A \oplus ({}^t\hat{A}^{-1}))|^{-1/2}}{\|A \oplus ({}^t\hat{A}^{-1})\|_{\min}^{n+1}}, \frac{|\det(A \oplus ({}^t\hat{A}^{-1}))|^{1/2}}{\|A \oplus ({}^t\hat{A}^{-1})\|_{\min}^r} \right\}.$$

The constant C_0 is independent of A and \hbar .

9.2 Treatment of non-linearity

We next explain how we modify the argument in Section 6 on the action of non-linear diffeomorphisms. Below we give definitions and related statements with some remarks on the correspondence to the argument in Section 6. We omit the proofs because we can obtain them by translating the corresponding ones in Section 6. Recall again that $0 < \beta < 1$ is the Hölder exponent of the hyperbolic splitting for the Anosov map $f : M \rightarrow M$. We take and fix a constant $0 < \theta < 1$ so small that

$$0 < \theta < \beta/20. \quad (9.10)$$

Let $\mathbb{D}^{(d)}(c)$ be the open ball of radius $c > 0$ on \mathbb{R}^d with center at the origin. Instead of the “Setting I” given in Subsection 6.1, we consider the following setting:

Setting I^{ext}: For each $\hbar > 0$, there is a given set $\widetilde{\mathcal{X}}_{\hbar}$ of C^∞ functions on $\mathbb{R}^{2d+d'}$ such that, for all $\psi \in \widetilde{\mathcal{X}}_{\hbar}$ and $\hbar > 0$,

(C1) the support of ψ is contained in the disk $\mathbb{D}^{(2d+d')}(C_*\hbar^{1/2-\theta}) \subset \mathbb{R}^{2d+d'}$ and

(C2) $|\partial_x^\alpha \psi(x)| < C_\alpha \hbar^{-(\frac{1}{2}-\theta)|\alpha|}$ for each multi-index $\alpha \in \mathbb{N}^{2d+d'}$,

where $C_* > 0$ and $C_\alpha > 0$ are constants independent of $\psi \in \widetilde{\mathcal{X}}_{\hbar}$ and $\hbar > 0$.

The Bargmann projection operator in the extended setting is

$$\mathcal{P}_{(x,s)} := \mathcal{B}_{(x,s)} \circ \mathcal{B}_{(x,s)}^* : L^2(\mathbb{R}_{(x,s)}^{2d+d'} \oplus \mathbb{R}_{(\xi_x, \xi_s)}^{2d+d'}) \rightarrow L^2(\mathbb{R}_{(x,s)}^{2d+d'} \oplus \mathbb{R}_{(\xi_x, \xi_s)}^{2d+d'}).$$

The following statements correspond to Lemma 6.2, Corollary 6.3 and 6.4 respectively.

For each $\psi \in \widetilde{\mathcal{X}}_{\hbar}$, let

$$\mathcal{M}^{\text{lift}}(\psi) = \mathcal{B}_{(x,s)} \circ \mathcal{M}(\psi) \circ \mathcal{B}_{(x,s)}^*$$

be the lift of the multiplication operator $\mathcal{M}(\psi)$.

Lemma 9.6. There exists a constant $C > 0$ such that, for any $\hbar > 0$ and $\psi \in \widetilde{\mathcal{X}}_{\hbar}$, we have

$$\|\mathcal{M}^{\text{lift}}(\psi) - \mathcal{M}(\psi \circ \pi) \circ \mathcal{P}_{(x,s)}\|_{L^2(\mathbb{R}^{2d+d'} \oplus \mathbb{R}^{2d+d'}, (\widetilde{\mathcal{W}}_{\hbar}^r)^2)} < C\hbar^\theta \quad (9.11)$$

and

$$\|\mathcal{M}^{\text{lift}}(\psi) - \mathcal{P}_{(x,s)} \circ \mathcal{M}(\psi \circ \pi)\|_{L^2(\mathbb{R}^{2d+d'} \oplus \mathbb{R}^{2d+d'}, (\widetilde{\mathcal{W}}_{\hbar}^r)^2)} < C\hbar^\theta.$$

where $\pi : \mathbb{R}_{(x,s,\xi_x,\xi_s)}^{4d+2d'} \rightarrow \mathbb{R}_{(x,s)}^{2d+d'}$ is the natural projection defined by $\pi(x, s, \xi_x, \xi_s) = (x, s)$. Consequently we have

$$\|[\mathcal{P}_{(x,s)}, \mathcal{M}(\psi \circ \pi)]\|_{L^2(\mathbb{R}^{2d+d'} \oplus \mathbb{R}^{2d+d'}, (\widetilde{\mathcal{W}}_{\hbar}^r)^2)} < C\hbar^\theta. \quad (9.12)$$

The same statement holds true with $\widetilde{\mathcal{W}}_{\hbar}^r$ replaced by $\widetilde{\mathcal{W}}_{\hbar}^{r,\pm}$.

Corollary 9.7. *The multiplication operator $\mathcal{M}(\psi)$ by $\psi \in \widetilde{\mathcal{X}}_{\hbar}$ extends to a bounded operator on $\widetilde{\mathcal{H}}_{\hbar}^r(\mathbb{R}^{2d+d'})$ and, for the operator norm, we have $\|\mathcal{M}(\psi)\|_{\widetilde{\mathcal{H}}_{\hbar}^r(\mathbb{R}^{2d+d'})} < \|\psi\|_{\infty} + C\hbar^{\theta}$ for all $\psi \in \widetilde{\mathcal{X}}_{\hbar}$, with a constant $C > 0$ independent of \hbar and ψ .*

2620 **Corollary 9.8.** *For $u, v \in \widetilde{\mathcal{H}}_{\hbar}^r(\mathbb{R}^{2d+d'})$ and $\psi \in \widetilde{\mathcal{X}}_{\hbar}$ we have*

$$\left| (u, \psi \cdot v)_{\widetilde{\mathcal{H}}_{\hbar}^r(\mathbb{R}^{2d+d'})} - (\overline{\psi} \cdot u, v)_{\widetilde{\mathcal{H}}_{\hbar}^r(\mathbb{R}^{2d+d'})} \right| < C\hbar^{\theta} \cdot \|u\|_{\widetilde{\mathcal{H}}_{\hbar}^r(\mathbb{R}^{2d+d'})} \cdot \|v\|_{\widetilde{\mathcal{H}}_{\hbar}^r(\mathbb{R}^{2d+d'})}$$

where C is a constant independent of u, v, ψ and \hbar .

The next lemma corresponds to Lemma 6.6.

Lemma 9.9. *There exists a constant $C > 0$ such that, for any $\hbar > 0$, $\psi \in \widetilde{\mathcal{X}}_{\hbar}$ and $0 \leq k \leq n$, we have*

$$\left\| \left[\mathcal{M}(\psi), t_{\hbar}^{(k)} \right] \right\|_{\widetilde{\mathcal{H}}_{\hbar}^{r,-}(\mathbb{R}^{2d+d'}) \rightarrow \widetilde{\mathcal{H}}_{\hbar}^{r,+}(\mathbb{R}^{2d+d'})} < C\hbar^{\theta}.$$

The following corresponds to Lemma 6.10, Corollary 6.13 and Corollary 6.14

Lemma 9.10. *For $\psi \in \widetilde{\mathcal{X}}_{\hbar}$ and $0 \leq k \leq n$, the operator*

$$\mathcal{M}(\psi) \circ t_{\hbar}^{(k)} : \widetilde{\mathcal{H}}_{\hbar}^r(\mathbb{R}^{2d+d'}) \rightarrow \widetilde{\mathcal{H}}_{\hbar}^r(\mathbb{R}^{2d+d'})$$

2625 *is a trace class operator. There exists a constant $C > 0$, independent of $\psi \in \mathcal{X}_{\hbar}$, $\hbar > 0$ and $0 \leq k \leq n$, such that*

$$\|\mathcal{M}(\psi) \circ t_{\hbar}^{(k)} : \mathcal{H}_{\hbar}^r(\mathbb{R}^{2d}) \rightarrow \mathcal{H}_{\hbar}^r(\mathbb{R}^{2d})\|_{tr} \leq \frac{r(k, d)}{(2\pi\hbar)^d} \int |\psi(x, 0)| dx + C\hbar^{-\theta d + \theta}$$

and

$$\left| \text{Tr} \left(\mathcal{M}(\psi) \circ t_{\hbar}^{(k)} : \mathcal{H}_{\hbar}^r(\mathbb{R}^{2d}) \rightarrow \mathcal{H}_{\hbar}^r(\mathbb{R}^{2d}) \right) - \frac{r(k, d)}{(2\pi\hbar)^d} \int \psi(x, 0) dx \right| \leq C\hbar^{-\theta d + \theta}$$

where $\|\cdot\|_{tr}$ denotes the trace norm of an operator and

$$r(k, d) = \binom{d+k-1}{d-1} = \text{rank } T^{(k)}.$$

The same statement holds true for $t_{\hbar}^{(k)} \circ \mathcal{M}(\psi)$.

2630 **Corollary 9.11.** *There exists a constant $C > 0$, such that, for $0 \leq k \leq n$ and $\psi \in \widetilde{\mathcal{X}}_{\hbar}$,*

$$\|\mathcal{M}(\psi) \circ t_{\hbar}^{(k)} : \widetilde{\mathcal{H}}_{\hbar}^{r,-}(\mathbb{R}^{2d+d'}) \rightarrow \widetilde{\mathcal{H}}_{\hbar}^{r,+}(\mathbb{R}^{2d+d'})\|_{Tr} \leq \frac{C \cdot r(k, d)}{(2\pi\hbar)^d} \int |\psi(x, 0)| dx + C\hbar^{-\theta d + \theta}$$

Corollary 9.12. *There exists a constant $C > 0$, such that*

$$\left\| \left[\mathcal{M}(\psi), t_h^{(k)} \right] : \widetilde{\mathcal{H}}_h^{r,-}(\mathbb{R}^{2d+d'}) \rightarrow \widetilde{\mathcal{H}}_h^{r,+}(\mathbb{R}^{2d+d'}) \right\|_{Tr} < C \hbar^{-\theta d + \theta} \quad \text{for } 0 \leq k \leq n$$

for any $\psi \in \mathcal{X}_h$.

Next let us consider the function

$$\widetilde{Y}_h : T^*\mathbb{R}^{2d+d'} = \mathbb{R}_{(x,s)}^{2d+d'} \oplus \mathbb{R}_{(\xi_x, \xi_s)}^{2d+d'} \rightarrow [0, 1], \quad Y_h(x, \xi) = \chi(\hbar^{2\theta-1/2}|((x, s), (\xi_x, \xi_s))|). \quad (9.13)$$

and the operator

$$\widetilde{\mathcal{Y}}_h : L^2(\mathbb{R}_{(x,s)}^{2d+d'}) \rightarrow L^2(\mathbb{R}_{(x,s)}^{2d+d'}), \quad \widetilde{\mathcal{Y}}_h = \mathcal{B}_{(x,s)}^* \circ \mathcal{M}(\widetilde{Y}_h) \circ \mathcal{B}_{(x,s)} \quad (9.14)$$

2635 with χ is as in (4.53). The following two lemmas correspond to Lemma 6.15 and 6.16 in Subsection 6.3.

Lemma 9.13. *The operator $\widetilde{\mathcal{Y}}_h$ extends naturally to a bounded operator on $\widetilde{\mathcal{H}}_h^r(\mathbb{R}^{2d+d'})$ and we have*

$$\|\widetilde{\mathcal{Y}}_h\|_{\widetilde{\mathcal{H}}_h^r(\mathbb{R}^{2d+d'})} < 1 + C \hbar^\theta$$

and

$$\|[\widetilde{\mathcal{Y}}_h, \mathcal{M}(\psi)]\|_{\widetilde{\mathcal{H}}_h^r(\mathbb{R}^{2d+d'})} < C \hbar^\theta \quad \text{for any } \psi \in \widetilde{\mathcal{X}}_h$$

2640 with some positive constants C independent of \hbar and ψ .

Lemma 9.14. *For $0 \leq k \leq n$ and $\psi \in \widetilde{\mathcal{X}}_h$, we have*

$$\|(\text{Id} - \widetilde{\mathcal{Y}}_h) \circ \mathcal{M}(\psi) \circ t_h^{(k)}\|_{\widetilde{\mathcal{H}}_h^{r,-}(\mathbb{R}^{2d+d'}) \rightarrow \widetilde{\mathcal{H}}_h^{r,+}(\mathbb{R}^{2d+d'})} < C \hbar^\theta$$

and

$$\|t_h^{(k)} \circ (\text{Id} - \widetilde{\mathcal{Y}}_h) \circ \mathcal{M}(\psi)\|_{\widetilde{\mathcal{H}}_h^{r,-}(\mathbb{R}^{2d+d'}) \rightarrow \widetilde{\mathcal{H}}_h^{r,+}(\mathbb{R}^{2d+d'})} < C \hbar^\theta$$

with some constant $C > 0$ independent of \hbar and ψ .

Below we give a few statements corresponding to those in Subsection 6.4. We now consider the following setting, in addition to Setting I^{ext}. This corresponds to “Setting II”
2645 in Subsection 6.4.

Setting II^{ext}: For every $\hbar > 0$, there is a given set $\tilde{\mathcal{G}}_\hbar$ of C^∞ diffeomorphisms

$$g : \mathbb{D}^{(2d+d')}(\hbar^{1/2-2\theta}) \rightarrow g(\mathbb{D}^{(2d+d')}(\hbar^{1/2-2\theta})) \subset \mathbb{R}^{2d+d'}$$

such that every $g \in \tilde{\mathcal{G}}_\hbar$ satisfies

(G0) g has a skew product structure with respect to the projection $p : \mathbb{R}_{(x,s)}^{2d+d'} \rightarrow \mathbb{R}_x^{2d}$, that is, we may write g as

$$g(x, s) = (\check{g}(x), \hat{g}(x, s)) \in \mathbb{R}^{2d} \oplus \mathbb{R}^{d'}$$

for $(x, s) \in \mathbb{D}^{(2d+d')}(\hbar^{1/2-2\theta}) \subset \mathbb{R}^{2d} \oplus \mathbb{R}^{d'}$,

(G1) \check{g} is symplectic with respect to the symplectic form ω on \mathbb{R}^{2d} in (5.2),

(G2) $\check{g}(0) = 0$, $|\hat{g}(0, 0)| < C\hbar^{1/2+\theta}$ and $\|Dg(0) - \text{Id}\| < C \max\{\hbar^{\beta(1/2-\theta)}, \hbar^{(1-\beta)(1/2-\theta)+2\theta}\}$, and

(G3) $|\partial^\alpha g| < C_\alpha \cdot \hbar^{-((1-\beta)(1/2-\theta)+2\theta)(|\alpha|-1)}$ for any multi-index α with $|\alpha| \geq 2$,

where C and C_α are positive constants that do not depend on \hbar nor $g \in \tilde{\mathcal{G}}_\hbar$.

Remark 9.15. Condition (G2) and (G3) in the setting above is weaker than the literal translation of those in ‘‘Setting II’’ in Subsection 6.4. Still we can get the proofs of the propositions below by translating those of the corresponding propositions, though we have to check that these weaker conditions are sufficient to get the conclusions. The point is that the diffeomorphisms g get closer to the identity in the C^∞ sense as $\hbar \rightarrow +0$, provided we look them in the scale $\hbar^{1/2}$. See Remark 9.17 below also.

For $g \in \tilde{\mathcal{G}}_\hbar$, we consider the Euclidean prequantum transfer operator

$$\tilde{\mathcal{L}}_g : C_0^\infty(\mathbb{D}^{(2d+d')}(\hbar^{1/2-2\theta})) \rightarrow C_0^\infty(g(\mathbb{D}^{(2d+d')}(\hbar^{1/2-2\theta}))) \quad (9.15)$$

defined by

$$\tilde{\mathcal{L}}_g u(x, s) = \frac{1}{|\det(Dg|_{\ker Dp})|} \cdot e^{-(i2\pi/\hbar) \cdot \mathcal{A}_g(p(g^{-1}(x,s)))} \cdot u(g^{-1}(x, s))$$

with \mathcal{A}_g be the function defined by (6.20) replacing g by \check{g} . Let

$$\tilde{\chi}_\hbar : \mathbb{R}_{(x,s)}^{2d+d'} \rightarrow [0, 1], \quad \tilde{\chi}_\hbar(x, s) = \chi(\hbar^{-1/2+\theta}|(x, s)|/2) \quad (9.16)$$

where χ is a C^∞ function satisfying (4.53). The next two propositions correspond to Proposition 6.19 and 6.21 respectively.

Proposition 9.16. *There exist constants $C > 0$ and $\epsilon > 0$ such that, for any $\hbar > 0$ and $g \in \tilde{\mathcal{G}}_\hbar$, we have*

$$\|\tilde{\mathcal{Y}}_\hbar \circ (\tilde{\mathcal{L}}_g - \text{Id}) \circ \mathcal{M}(\tilde{\chi}_\hbar)\|_{\tilde{\mathcal{H}}_\hbar^r(\mathbb{R}^{2d+d'})} < Ch^\epsilon \quad \text{and} \quad \|(\tilde{\mathcal{L}}_g - \text{Id}) \circ \tilde{\mathcal{Y}}_\hbar \circ \mathcal{M}(\tilde{\chi}_\hbar)\|_{\tilde{\mathcal{H}}_\hbar^r(\mathbb{R}^{2d+d'})} < Ch^\epsilon.$$

2660 *Remark 9.17.* Concerning Remark 9.15, we have to check that the argument in the proof of Proposition 6.19 works under the weaker assumption in the setting II^{ext} . This is not difficult if we take the last comment in Remark 9.15 into account and noting that, from (9.10), we have

$$-(1 - \beta)(1/2 - \theta) - 2\theta > -1/2 + 2\theta$$

for the exponent that appeared in the condition (G3) in the setting II^{ext} .

2665 **Proposition 9.18.** *For any $g \in \tilde{\mathcal{G}}_{\hbar}$, we have*

$$\left\| \tilde{\mathcal{L}}_g \circ \mathcal{M}(\tilde{\chi}_{\hbar}) \right\|_{\tilde{\mathcal{H}}_{\hbar}^{r,+}(\mathbb{R}^{2d+d'}) \rightarrow \tilde{\mathcal{H}}_{\hbar}^r(\mathbb{R}^{2d+d'})} \leq C_0 \quad \text{and} \quad \left\| \tilde{\mathcal{L}}_g \circ \mathcal{M}(\tilde{\chi}_{\hbar}) \right\|_{\tilde{\mathcal{H}}_{\hbar}^r(\mathbb{R}^{2d+d'}) \rightarrow \tilde{\mathcal{H}}_{\hbar}^{r,-}(\mathbb{R}^{2d+d'})} \leq C_0 \quad (9.17)$$

for sufficiently small $\hbar > 0$, where $C_0 > 1$ is a constant that depends only on n, r, d, θ and the choice of the escape functions W and W^{\pm} in subsection 4.3.

The following correspond to Lemma 6.22, Corollary 6.23 and Lemma 6.24.

2670 **Lemma 9.19.** *There exist constants $C > 0$ and $\epsilon > 0$ independent of \hbar such that the following holds: Let $\psi \in \tilde{\mathcal{X}}_{\hbar}$ be supported on the disk $\mathbb{D}^{(2d+d')}(2\hbar^{1/2-\theta})$ and let $g \in \tilde{\mathcal{G}}_{\hbar}$, $0 \leq k \leq n$, then it holds*

$$\left\| (\tilde{\mathcal{L}}_g - \text{Id}) \circ \mathcal{M}(\psi) \circ t_{\hbar}^{(k)} \right\|_{\tilde{\mathcal{H}}_{\hbar}^r(\mathbb{R}^{2d+d'})} \leq C\hbar^{\epsilon}$$

and

$$\left\| t_{\hbar}^{(k)} \circ (\tilde{\mathcal{L}}_g - \text{Id}) \circ \mathcal{M}(\psi) \right\|_{\tilde{\mathcal{H}}_{\hbar}^r(\mathbb{R}^{2d+d'})} \leq C\hbar^{\epsilon}.$$

Corollary 9.20. *There exist constants $C > 0$ and $\epsilon > 0$ independent of \hbar such that, for any $\psi \in \tilde{\mathcal{X}}_{\hbar}$ and $g \in \tilde{\mathcal{G}}_{\hbar}$, it holds*

$$\left\| [\tilde{\mathcal{L}}_g \circ \mathcal{M}(\psi), t_{\hbar}^{(k)}] \right\|_{\tilde{\mathcal{H}}_{\hbar}^r(\mathbb{R}^{2d+d'})} \leq C\hbar^{\epsilon} \quad \text{for } 0 \leq k \leq n$$

2675 and also

$$\left\| [\tilde{\mathcal{L}}_g \circ \mathcal{M}(\psi), \tilde{t}_{\hbar}] \right\|_{\tilde{\mathcal{H}}_{\hbar}^r(\mathbb{R}^{2d+d'})} \leq C\hbar^{\epsilon}.$$

Lemma 9.21. *There exist constants $\epsilon > 0$ and $C > 0$, independent of \hbar , such that the following holds true: Let $\psi \in \tilde{\mathcal{X}}_{\hbar}$ and let $g \in \tilde{\mathcal{G}}_{\hbar}$, $0 \leq k \leq n$, then it holds*

$$\left\| (\tilde{\mathcal{L}}_g - \text{Id}) \circ \mathcal{M}(\psi) \circ t_{\hbar}^{(k)} : \tilde{\mathcal{H}}_{\hbar}^r(\mathbb{R}^{2d}) \rightarrow \tilde{\mathcal{H}}_{\hbar}^r(\mathbb{R}^{2d}) \right\|_{T_r} \leq C\hbar^{-\theta d + \epsilon}$$

$$\left\| t_{\hbar}^{(k)} \circ (\tilde{\mathcal{L}}_g - \text{Id}) \circ \mathcal{M}(\psi) : \tilde{\mathcal{H}}_{\hbar}^r(\mathbb{R}^{2d}) \rightarrow \tilde{\mathcal{H}}_{\hbar}^r(\mathbb{R}^{2d}) \right\|_{T_r} \leq C\hbar^{-\theta d + \epsilon}$$

and

$$\left\| [\tilde{\mathcal{L}}_g \circ \mathcal{M}(\psi), t_{\hbar}^{(k)}] : \tilde{\mathcal{H}}_{\hbar}^r(\mathbb{R}^{2d}) \rightarrow \tilde{\mathcal{H}}_{\hbar}^r(\mathbb{R}^{2d}) \right\|_{T_r} \leq C\hbar^{-\theta d + \epsilon}$$

9.3 Proof of the main theorems in the setting of Grassmanian extension.

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Now we give the proofs of the main theorems, namely Theorem 1.34, Theorem 1.35 and Theorem 1.37 in the extended situation. (Theorem 1.36 will be proved in the next subsection.) The point in the following argument is in the choice of the local coordinate charts and a few basic estimates on the extended map f_G viewed in them. The modifications in the rest part will be rather obvious.

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Recall the set of points

$$\mathcal{P}_h = \{m_i \in M \mid 1 \leq i \leq I_h\},$$

local coordinate charts

$$\kappa_i = \kappa_{i,h} : \mathbb{D}(c) \subset \mathbb{R}^{2d} \rightarrow M, \quad 1 \leq i \leq I_h$$

and the sections $\tau_i : U_i \rightarrow P$ taken in Proposition 7.3. Also recall that we write E_u for the image of the section $E_u : M \rightarrow G$, which assigns the unstable subspace to each point and is Hölder continuous with exponent β . This is an attracting subset for f_G . (See (1.47).) Let $\tilde{m}_i := E_u(m_i) \in K_0$ and set

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$$\tilde{\mathcal{P}}_h = \{\tilde{m}_i \in G \mid 1 \leq i \leq I_h\}.$$

In the next proposition, we take local coordinate charts $\tilde{\kappa}_i = \tilde{\kappa}_{i,h}$ on a small neighborhood of the point \tilde{m}_i as an extension of $\kappa_i = \kappa_{i,h}$. Notice that the local coordinate charts $\tilde{\kappa}_i = \tilde{\kappa}_{i,h}$ are far from being conformal as the parameter h gets smaller. That is, $D\tilde{\kappa}_i|_{\mathbb{R}^{2d} \oplus \{0\}}$ is nearly isometry while $D\tilde{\kappa}_i|_{\{0\} \oplus \mathbb{R}^{d'}}$ is a conformal expansion by the rate $h^{-(1-\beta)(1/2-\theta)-2\theta}$. (See Condition (3) in the proposition below and remember Condition (2) in Proposition 7.3.) This is necessary when we deal with the problems caused by non-smoothness of the attracting set E_u .

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Proposition 9.22. *For each $h = \frac{1}{2\pi N} > 0$, there exist a system of local coordinate charts*

$$\tilde{\kappa}_i = \tilde{\kappa}_{i,h} : \mathbb{D}^{(2d)}(c) \times \mathbb{D}^{(d')}(c \cdot h^{(1-\beta)(1/2-\theta)+2\theta}) \subset \mathbb{R}^{2d+d'} \rightarrow G, \quad 1 \leq i \leq I_h \quad (9.18)$$

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on a neighborhood of E_u with $c > 0$ a constant independent of h , so that the following conditions hold for sufficiently small h :

(1) $\tilde{\kappa}_i(0) = \tilde{m}_i$ and the following diagram commutes

$$\begin{array}{ccc} \mathbb{D}^{(2d)}(c) \times \mathbb{D}^{(d')}(c \cdot h^{(1-\beta)(1/2-\theta)+2\theta}) & \xrightarrow{\tilde{\kappa}_i} & G \\ p \downarrow & & p \downarrow \\ \mathbb{D}^{(2d)}(c) & \xrightarrow{\kappa_i} & M \end{array}$$

- (2) The derivative $(D\tilde{\kappa}_i)_0 : T_0\mathbb{R}^{2d+d'} = \mathbb{R}^{2d+d'} \rightarrow T_{\tilde{m}_i}G$ maps the subspaces $\{0\} \oplus \mathbb{R}^{d'}$ and $\mathbb{R}^{2d} \oplus \{0\}$ respectively to the subspace $\ker(Dp)_{\tilde{m}_i} \subset T_{\tilde{m}_i}G$ and its orthogonal complement respectively. The restriction of $(D\tilde{\kappa}_i)_0$ to the subspace $\{0\} \oplus \mathbb{R}^{d'}$ is a conformal expansion by the rate $\hbar^{-(1-\beta)(1/2-\theta)-2\theta}$ with respect to the Euclidean metric in the source and the Riemann metric on G in the target. Further, the map $\tilde{\kappa}_i \circ (D\tilde{\kappa}_i)_0^{-1}$ is not far from the exponential map in the sense that

$$\|\exp_{\tilde{m}_i}^{-1} \circ \tilde{\kappa}_i \circ (D\tilde{\kappa}_i)_0^{-1} : \mathbb{D}(\tilde{m}_i, c) \rightarrow T_{\tilde{m}_i}G\|_{C^k} \leq C_k$$

with C_k a constant independent of \hbar nor $1 \leq i \leq I_h$, where $\mathbb{D}(\tilde{m}_i, c)$ denotes the disk in $T_{\tilde{m}_i}G$ with radius c and center at the origin.

- (3) The union of the images $\tilde{U}_i := \tilde{\kappa}_i(\mathbb{D}^{(2d+d')}(h^{1/2-\theta}))$, $1 \leq i \leq I_h$, cover the $\hbar^{\beta(1/2-\theta)-2\theta}/2$ -neighborhood of the section E_u and contained in the $2\hbar^{\beta(1/2-\theta)-2\theta}$ -neighborhood of the section E_u .
- (4) For the coordinate change transformation $\tilde{\kappa}_{j,i} := \tilde{\kappa}_j^{-1} \circ \tilde{\kappa}_i$, defined for $1 \leq i, j \leq I_h$ with $\tilde{U}_i \cap \tilde{U}_j \neq \emptyset$, there exists an isometric affine map $\tilde{A}_{j,i} : \mathbb{R}^{2d+d'} \rightarrow \mathbb{R}^{2d+d'}$ of the form $\tilde{A}_{j,i} = A_{j,i} \oplus \hat{A}_{j,i}$ with $A_{j,i}$ given in Proposition 7.3 and $\hat{A}_{j,i} : \mathbb{R}^{d'} \rightarrow \mathbb{R}^{d'}$ an isometric linear map, such that, if we set $\mathcal{G}_h = \{\tilde{A}_{j,i} \circ \tilde{\kappa}_{j,i}\}$, it satisfies the conditions (G0)-(G3) in the Setting II^{ext} in Subsection 9.2.
- (5) There exists a family of C^∞ functions $\{\tilde{\psi}_i : \mathbb{R}^{2d+d'} \rightarrow [0, 1]\}_{i=1}^{I_h}$ which is supported on the disk $\mathbb{D}^{(2d+d')}(h^{1/2-\theta})$, such that

$$\sum_{i=1}^{I_h} \tilde{\psi}_i \circ \tilde{\kappa}_i^{-1} \equiv 1 \text{ on the } (\hbar^{\beta(1/2-\theta)-2\theta}/4)\text{-neighborhood of the section } E_u. \quad (9.19)$$

The set of functions $\mathcal{X}_h = \{\tilde{\psi}_i\}$ satisfies the conditions, (C1) and (C2), in the Setting I^{ext} in Subsection 9.2.

Proof. We continue the argument in the proof of Proposition 7.3, in which we defined the local chart $\kappa_m : \mathbb{D}(c) \rightarrow M$ for each point $m \in M$ as a composition of a linear map from \mathbb{R}^{2d} to T_mM and a modification of the exponential mapping. By a parallel construction, we can and do take a local chart $\hat{\kappa}_m : \mathbb{D}(c) \rightarrow G$ for each point $m \in M$ so that the following conditions hold:

- $\hat{\kappa}_m(0) = \tilde{m} := E_u(m) \in K_0$ and we have $p \circ \hat{\kappa}_m = \kappa_m \circ p$,
- The derivative $(D\hat{\kappa}_m)_0 : T_0\mathbb{R}^{2d+d'} = \mathbb{R}^{2d+d'} \rightarrow T_{\tilde{m}}G$ maps the subspaces $\{0\} \oplus \mathbb{R}^{d'}$ and $\mathbb{R}^{2d} \oplus \{0\}$ to the subspace $\ker(Dp)_{\tilde{m}} \subset T_{\tilde{m}}G$ and its orthogonal complement respectively. The restriction of $(D\hat{\kappa}_m)_0$ to the subspace $\{0\} \oplus \mathbb{R}^{d'}$ is an isometric

linear map with respect to the Euclidean metric in the source and the Riemann metric on G in the target. Further, the map $\hat{\kappa}_{\tilde{m}} \circ (D\hat{\kappa}_{\tilde{m}})_0^{-1}$ is not far from the exponential map in the sense that

$$\| \exp_{\tilde{m}}^{-1} \circ \hat{\kappa}_{\tilde{m}} \circ (D\hat{\kappa}_{\tilde{m}})_0^{-1} : \mathbb{D}(\tilde{m}, c) \rightarrow T_{\tilde{m}}G \|_{C^k} \leq C_k$$

with C_k a constant independent of \hbar nor $1 \leq i \leq I_h$.

We then define the local charts $\tilde{\kappa}_i$ in the statement of the proposition by

$$\tilde{\kappa}_i(x, s) := \tilde{\kappa}_{\tilde{m}_i}(x, \hbar^{-(1-\beta)(1/2-\theta)-2\theta} s).$$

It is not difficult to check that the coordinate maps $\tilde{\kappa}_i$ thus defined satisfies the required conditions. The conditions (1), (2) and (3) should be obvious. Also so should be (5) once we check the conditions (1)-(4). Thus we check the condition (4). The condition (G0) and (G1) are direct consequences of the construction. To check the condition (G2), we first observe that the diffeomorphism $\hat{\kappa}_j^{-1} \circ \hat{\kappa}_i$ is written in the form

$$\hat{\kappa}_j^{-1} \circ \hat{\kappa}_i \begin{pmatrix} q \\ p \\ s \end{pmatrix} = \begin{pmatrix} q_0 \\ p_0 \\ s_0 \end{pmatrix} + \begin{pmatrix} A_{1,1} & A_{1,2} & 0 \\ A_{2,1} & A_{2,2} & 0 \\ A_{3,1} & A_{3,2} & A_{3,3} \end{pmatrix} \begin{pmatrix} q \\ p \\ s \end{pmatrix} + K(q, p, s) \tag{9.20}$$

where $K(q, p, s)$ satisfies $K(0, 0, 0) = 0$ and $DK(0, 0, 0) = 0$. From the assumption $U_i \cap U_j \neq \emptyset$ and the choice of β , we have

- $|q_0| < C\hbar^{1/2-\theta}$, $|p_0| < C\hbar^{1/2-\theta}$, $|s_0| < C\hbar^{\beta(1/2-\theta)}$,
- $\|A_{1,2}\| < C\hbar^{\beta(1/2-\theta)}$, $\|A_{2,1}\| < C\hbar^{\beta(1/2-\theta)}$, and
- $\|A_{1,1} - I\| < C\hbar^{\beta(1/2-\theta)}$, $\|A_{2,2} - I\| < C\hbar^{\beta(1/2-\theta)}$, $\|A_{3,3} - I'\| < C\hbar^{(1/2-\theta)}$ for some orthogonal transformations $I, I' : \mathbb{R}^d \rightarrow \mathbb{R}^d$.

From the definition, the diffeomorphism $\tilde{\kappa}_{j,i} := \tilde{\kappa}_j^{-1} \circ \tilde{\kappa}_i$ is then written

$$\begin{aligned} \tilde{\kappa}_{j,i} \begin{pmatrix} q \\ p \\ s \end{pmatrix} &= \begin{pmatrix} q_0 \\ p_0 \\ \hbar^{(1-\beta)(1/2-\theta)+2\theta} s_0 \end{pmatrix} \\ &+ \begin{pmatrix} A_{1,1} & A_{1,2} & 0 \\ A_{2,1} & A_{2,2} & 0 \\ \hbar^{(1-\beta)(1/2-\theta)+2\theta} A_{3,1} & \hbar^{(1-\beta)(1/2-\theta)+2\theta} A_{3,2} & A_{3,3} \end{pmatrix} \begin{pmatrix} q \\ p \\ s \end{pmatrix} + K'(q, p, s) \end{aligned} \tag{9.21}$$

where $K'(q, p, s)$ satisfies $K'(0, 0, 0) = 0$ and $DK'(0, 0, 0) = 0$. From this expression and the estimates above, we find the affine map $\tilde{A}_{j,i}$ as claimed in (4) so that the condition (G2) holds true. To prove the condition (G3), we express the term $K(q, p, s)$ in (9.20) as

$$K \begin{pmatrix} x \\ s \end{pmatrix} = \begin{pmatrix} K_1(x) \\ K_2(x, s) \end{pmatrix}$$

and the differentials of $K_1(x)$ and $K_2(x, s)$ are uniformly bounded with respect to $1 \leq i, j \leq I_h$ and \hbar . Then $K'(q, p, s)$ in (9.21) is written

$$K' \begin{pmatrix} x \\ s \end{pmatrix} = \begin{pmatrix} K_1(x) \\ \hbar^{(1-\beta)(1/2-\theta)+2\theta} \cdot K_2(x, \hbar^{-(1-\beta)(1/2-\theta)-2\theta} s) \end{pmatrix}.$$

The condition (G3) follows immediately from this expression. \square

2755 We define a C^∞ section $\tilde{\tau}_i : \tilde{U}_i \rightarrow P_G$ of the $\mathbf{U}(1)$ -bundle P_G as the pull-back of τ_i by the projection p :

$$\tilde{\tau}_i(z) = \tau_i \circ p(z).$$

Once we have set up the local coordinates and local sections as above, we can follow the argument in Section 7 by confirming the correspondence between the objects and statements. The following few paragraphs we follow the argument in Subsection 7.3 with
2760 obvious modifications.

Definition 9.23. Set

$$\tilde{\mathcal{E}}_h := \bigoplus_{i=1}^{I_h} C_0^\infty(\mathbb{D}^{2d+d'}(\hbar^{1/2-\theta})).$$

and let $\tilde{\mathbf{I}}_h : C_N^\infty(P_G) \rightarrow \tilde{\mathcal{E}}_h$ be the operator that assign to each equivariant function $u \in C_N^\infty(P_G)$ a set of functions $\tilde{\mathbf{I}}_h(u) = (u_i)_{i=1}^{I_h} \in \tilde{\mathcal{E}}_h$ on local charts defined by the relation

$$u_i(x, s) = \tilde{\psi}_i(x, s) \cdot u(\tilde{\tau}_i(\tilde{\kappa}_i(x, s))) \quad \text{for } 1 \leq i \leq I_h. \quad (9.22)$$

Let $\tilde{\mathbf{I}}_h^* : \bigoplus_{i=1}^{I_h} \mathcal{S}(\mathbb{R}^{2d+d'}) \rightarrow C_N^\infty(P_G)$ be the operator defined by

$$\left(\tilde{\mathbf{I}}_h^* \left((u_i)_{i=1}^{I_h} \right) \right) (p) = \sum_{i=1}^{I_h} e^{i2\pi N \cdot \alpha_i(p)} \cdot \tilde{\chi}_h(x, s) \cdot u_i(x, s) \quad (9.23)$$

2765 where $\tilde{\chi}_h$ is the function defined in (9.16), $(x, s) = \kappa_i^{-1}(\pi(p))$ and $\alpha_i(p)$ is the real number such that $p = e^{i2\pi \alpha_i(p)} \cdot \tilde{\tau}_i(\pi(p))$. Then we have

$$\tilde{\mathbf{I}}_h^* \circ \tilde{\mathbf{I}}_h u = u \quad \text{for } u \in C_N^\infty(P_G) \quad (9.24)$$

and $\tilde{\mathbf{I}}_h \circ \tilde{\mathbf{I}}_h^* : \tilde{\mathcal{E}}_h \rightarrow \tilde{\mathcal{E}}_h$ is a projection onto the image of $\tilde{\mathbf{I}}_h$. We define the *lift of the prequantum transfer operator* \hat{F}_h with respect to $\tilde{\mathbf{I}}_h$ by

$$\tilde{\mathbf{F}}_h := \tilde{\mathbf{I}}_h \circ \hat{F}_N \circ \tilde{\mathbf{I}}_h^* : \bigoplus_{i=1}^{I_h} \mathcal{S}(\mathbb{R}^{2d+d'}) \rightarrow \tilde{\mathcal{E}}_h \subset \bigoplus_{i=1}^{I_h} \mathcal{S}(\mathbb{R}^{2d+d'}). \quad (9.25)$$

Proposition 9.24. *The operator $\tilde{\mathbf{F}}_{\hbar}$ can be written as*

$$\tilde{\mathbf{F}}_{\hbar}((v_i)_{i \in I_{\hbar}}) = \left(\sum_{i=1}^{I_{\hbar}} \tilde{\mathbf{F}}_{j,i}(v_i) \right)_{j \in I_{\hbar}}$$

2770 *where the component*

$$\tilde{\mathbf{F}}_{j,i} : \mathcal{S}(\mathbb{R}^{2d+d'}) \rightarrow C_0^{\infty}(\mathbb{D}^{(2d+d')})(\hbar^{1/2-\theta})$$

is defined by $\tilde{\mathbf{F}}_{j,i} \equiv 0$ if $i \not\rightarrow j$ and, otherwise, by

$$\tilde{\mathbf{F}}_{j,i}(v_i) = \tilde{\mathcal{L}}_{\tilde{f}_{j,i}} \left(e^{\tilde{V} \circ f_G \circ \tilde{\kappa}_i} \cdot \tilde{\psi}_{j,i} \cdot \tilde{\chi}_{\hbar} \cdot v_i \right)$$

where we set

$$\tilde{f}_{j,i} := \tilde{\kappa}_j^{-1} \circ f_G \circ \tilde{\kappa}_i, \tag{9.26}$$

$$\tilde{\psi}_{j,i} := \tilde{\psi}_j \circ \tilde{f}_{j,i} \tag{9.27}$$

and $\tilde{\mathcal{L}}_{\tilde{f}_{j,i}}$ is the Euclidean prequantum transfer operator defined in (9.15) for $g = \tilde{f}_{j,i}$.

We define

$$\tilde{V}_j = \max\{\tilde{V}(m) \mid m \in \tilde{U}_j\} \quad \text{for } 1 \leq j \leq I_{\hbar}.$$

The next lemma corresponds to Lemma 7.11.

2775 **Lemma 9.25.** *If we set*

$$\tilde{\mathcal{X}}_{\hbar} = \{\tilde{\psi}_{j,i} \cdot \tilde{\chi}_{\hbar} \mid 1 \leq i, j \leq I_{\hbar}, i \rightarrow j\},$$

it satisfies the conditions (C1) and (C2) in Setting I^{ext} in Subsection 9.2. (The constants C and C_{α} will depend on f_G and \tilde{V} though not on \hbar .) For $1 \leq i, j \leq I_{\hbar}$ such that $i \rightarrow j$, we have

$$\left\| \mathcal{M}(e^{\tilde{V} \circ f_G \circ \tilde{\kappa}_i} \cdot \tilde{\psi}_{j,i} \cdot \tilde{\chi}_{\hbar}) - e^{\tilde{V}_j} \cdot \mathcal{M}(\tilde{\psi}_{j,i} \cdot \tilde{\chi}_{\hbar}) \right\|_{\tilde{\mathcal{H}}_{\hbar}^r(\mathbb{R}^{2d+d'})} \leq C(f_G, \tilde{V}) \cdot \hbar^{\theta}$$

2780 *for some constant $C(f_G, \tilde{V})$ independent of \hbar . The same statement holds for the set of functions*

$$\tilde{\mathcal{X}}_{\hbar} = \{e^{\tilde{V} \circ f_G \circ \tilde{\kappa}_i} \cdot \tilde{\psi}_{j,i} \cdot \tilde{\chi}_{\hbar} \mid 1 \leq i, j \leq I_{\hbar}, i \rightarrow j\}.$$

Proof. We obtain the proof by following the argument in that of Lemma 7.11 with obvious correspondence, using Lemma 9.7 instead of Lemma 6.3. \square

We next proceed to the argument corresponding to that in Subsection 7.4. The anisotropic Sobolev space in the extended setting is defined as follows.

2785 **Definition 9.26.** Let $C_N^\infty(P_G, E_u)$ be the set of functions $u \in C_N^\infty(P_G)$ that is supported on the inverse image (with respect to the projection $\pi_G : P_G \rightarrow G$) of the $(\hbar^{\beta(1/2-\theta)}/4)$ -neighborhood of the section E_u . The *anisotropic Sobolev space* $\tilde{\mathcal{H}}_h^r(P_G, E_u)$ is defined as the completion of the space $C_N^\infty(P_G, E_u)$ with respect to the norm

$$\|u\|_{\tilde{\mathcal{H}}_h^r} := \left(\sum_{i=1}^{I_h} \|u_i\|_{\tilde{\mathcal{H}}_h^r(\mathbb{R}^{2d+d'})}^2 \right)^{1/2} \quad \text{for } u \in C_N^\infty(P_G, E_u),$$

where $u_i = (\mathbf{I}_h(u))_i \in C_0^\infty(\mathbb{D}^{(2d+d')}(\hbar^{1/2-\theta}))$ are the local data defined in (9.22) and $\|u_i\|_{\tilde{\mathcal{H}}_h^r(\mathbb{R}^{2d+d'})}^2$ is the anisotropic Sobolev norm on $C_0^\infty(\mathbb{R}^{2d+d'})$ in Definition 9.3. We define the Hilbert spaces $\tilde{\mathcal{H}}_h^{r,\pm}(P_G, E_u)$ in the parallel manner, replacing $\|u_i\|_{\tilde{\mathcal{H}}_h^r(\mathbb{R}^{2d+d'})}^2$ by the norms $\|u_i\|_{\tilde{\mathcal{H}}_h^{r,\pm}(\mathbb{R}^{2d+d'})}^2$ respectively.

The next lemma corresponds to Lemma 7.14.

Lemma 9.27. *The projector $\tilde{\mathbf{I}}_h \circ \tilde{\mathbf{I}}_h^* : \tilde{\mathcal{E}}_h \rightarrow \tilde{\mathcal{E}}_h$ extends to bounded operators*

$$\tilde{\mathbf{I}}_h \circ \tilde{\mathbf{I}}_h^* : \bigoplus_{i=1}^{I_h} \tilde{\mathcal{H}}_h^{r,+}(\mathbb{R}^{2d+d'}) \rightarrow \bigoplus_{i=1}^{I_h} \tilde{\mathcal{H}}_h^r(\mathbb{D}^{(2d+d')}(\hbar^{1/2-\theta}))$$

2795 and

$$\tilde{\mathbf{I}}_h \circ \tilde{\mathbf{I}}_h^* : \bigoplus_{i=1}^{I_h} \tilde{\mathcal{H}}_h^r(\mathbb{R}^{2d+d'}) \rightarrow \bigoplus_{i=1}^{I_h} \tilde{\mathcal{H}}_h^{r,-}(\mathbb{D}^{(2d+d')}(\hbar^{1/2-\theta})).$$

Further the operator norms of these projection operators are bounded by a constant independent of \hbar .

Proof. The proof is obtained by following that of Lemma 7.14, setting

$$\tilde{\mathcal{G}}_h = \{\tilde{A}_{j,i} \circ \tilde{\kappa}_{j,i} \mid 1 \leq i, j \leq I_h, U_i \cap U_j \neq \emptyset\} \quad (9.28)$$

and

$$\tilde{\mathcal{X}}_h = \{\tilde{\psi}_j \circ \tilde{\kappa}_{j,i} \cdot \tilde{\chi}_h \mid 1 \leq i, j \leq I_h, U_i \cap U_j \neq \emptyset\},$$

2800 and by using Proposition 9.18 and Corollary 9.7 instead of Proposition 6.21 and Corollary 6.3. \square

The next lemma corresponds to Lemma 7.17. We suppose that the hyperbolicity exponent $\lambda > 1$ of the flow f_G is sufficiently large. (Say $\lambda > 9$.)

Lemma 9.28. *The operator $\tilde{\mathbf{F}}_h$ defined in (9.25) extends uniquely to the bounded operator*

$$\tilde{\mathbf{F}}_h : \bigoplus_{i=1}^{I_h} \tilde{\mathcal{H}}_h^r(\mathbb{R}^{2d+d'}) \rightarrow \bigoplus_{i=1}^{I_h} \tilde{\mathcal{H}}_h^r(\mathbb{D}^{(2d+d')}(\hbar^{1/2-\theta})) \quad (9.29)$$

2805 and the operator norm is bounded by a constant independent of \hbar . Consequently the same result holds for the prequantum transfer operator $\tilde{F}_h : \tilde{\mathcal{H}}_h^r(P_G, E_u) \rightarrow \tilde{\mathcal{H}}_h^r(P_G, E_u)$.

Proof. We follow the argument in the proof of Lemma 7.17 with slight modification. We express the diffeomorphism $\tilde{f}_{j,i}$ in (9.26) as a composition

$$\tilde{f}_{j,i} = \tilde{a}_{j,i} \circ \tilde{g}_{j,i} \circ \tilde{B}_{j,i} \quad (9.30)$$

where

- $\tilde{a}_{j,i} : \mathbb{R}^{2d+d'} \rightarrow \mathbb{R}^{2d+d'}$ is a translation in a direction in $\mathbb{R}^{2d} \oplus \{0\}$,
- $\tilde{B}_{j,i} : \mathbb{R}^{2d} \rightarrow \mathbb{R}^{2d}$ is a linear map of the form (9.1) considered in Subsection 9.1.
- $\tilde{g}_{j,i}$ is a diffeomorphism such that $\tilde{\mathcal{G}}_h = \{\tilde{g}_{j,i}\}_{1 \leq i, j \leq I_h}$ satisfies the condition (G0),(G1),(G2) and (G3) in Setting II^{ext} in Subsection 9.2.

This corresponds to the expression (7.22) in the proof of Lemma 7.17.

Remark 9.29. (1) The prequantum transfer operator $\tilde{L}_{\tilde{a}_{j,i}}$ associated to a translation on $\mathbb{R}^{2d+d'}$ acts on the anisotropic Sobolev space $\tilde{\mathcal{H}}_h^r(\mathbb{R}^{2d+d'})$ as an isometry if the direction of translation belongs to the subspace $\mathbb{R}^{2d} \oplus \{0\}$. But this is not true for translations in the other directions.

(2) The proof of the claim on $\tilde{g}_{j,i}$ need some argument. But this is essentially parallel to that in the proof of Proposition 9.22.

From the expression (9.30), the operator $\mathbf{F}_{j,i}$ is expressed as the composition

$$\mathbf{F}_{j,i} = \tilde{\mathcal{L}}^{(0)} \circ \tilde{\mathcal{L}}^{(1)} \circ \tilde{\mathcal{L}}^{(2)} \quad (9.31)$$

where $\tilde{\mathcal{L}}^{(0)} := \mathcal{L}_{\tilde{a}_{j,i}}$ and $\tilde{\mathcal{L}}^{(2)} := \tilde{\mathcal{L}}_{\tilde{B}_{j,i}}$ are the Euclidean prequantum transfer operators (9.15) associated to the diffeomorphisms $g = a_{ij}$ and $g = B_{ij}$ respectively, while $\tilde{\mathcal{L}}^{(1)}$ is the operator of the form

$$\tilde{\mathcal{L}}^{(1)}u = \tilde{\mathcal{L}}_{\tilde{g}_{j,i}} \left(\left((e^{\tilde{V} \circ \tilde{f} \circ \tilde{\kappa}_i} \cdot \tilde{\psi}_{j,i} \cdot \tilde{\chi}_h) \circ \tilde{B}_{j,i}^{-1} \right) \cdot u \right)$$

with $\tilde{\psi}_{j,i}$ the function defined in (9.27). Then we follow the argument in the latter part of the proof of Lemma 7.17, replacing some proposition by those prepared in the last subsection. \square

Next we introduce the projection operators

$$\mathbf{t}_h^{(k)} : \bigoplus_{i=1}^{I_h} \tilde{\mathcal{H}}_h^r(\mathbb{R}^{2d+d'}) \rightarrow \bigoplus_{i=1}^{I_h} \tilde{\mathcal{H}}_h^r(\mathbb{R}^{2d+d'}), \quad \mathbf{t}_h^{(k)}((u_i)_{i=1}^{I_h}) = (t_h^{(k)}(u_i))_{i=1}^{I_h}, \quad (9.32)$$

for $0 \leq k \leq n$ and

$$\tilde{\mathbf{t}}_h : \bigoplus_{i=1}^{I_h} \tilde{\mathcal{H}}_h^r(\mathbb{R}^{2d+d'}) \rightarrow \bigoplus_{i=1}^{I_h} \tilde{\mathcal{H}}_h^r(\mathbb{R}^{2d+d'}), \quad \tilde{\mathbf{t}}_h((u_i)_{i=1}^{I_h}) = (\tilde{t}_h(u_i))_{i=1}^{I_h},$$

2830 where $t_h^{(k)}$ and \tilde{t}_h are the projection operators introduced in (9.8) and (9.9). As before, we set $\mathbf{t}_h^{(n+1)} = \tilde{t}_h$, so that the set of operators $\{t_h^{(k)}\}_{k=0}^{n+1}$ are complete sets of mutually commuting projection operators. (Notice that we are using the same notation $\mathbf{t}_h^{(k)}$ and \tilde{t}_h in this extended setting as (7.25) and (7.26) in Subsection 7.5.)

2835 The following two propositions corresponds to Proposition 7.19 and 7.20. For the proofs, we have only to follow those of Proposition 7.19 and 7.20 respectively, checking the correspondence in the notation and replacing the propositions by the corresponding ones.

Proposition 9.30. *There are constants $\epsilon > 0$ and $C_0 > 0$, independent of \hbar , such that the following holds: We have that*

$$\left\| \mathbf{t}_h^{(k)} \circ (\tilde{\mathbf{I}}_h \circ \tilde{\mathbf{I}}_h^*) \right\|_{\bigoplus_{i=1}^{I_h} \tilde{\mathcal{H}}_h^r(\mathbb{R}^{2d+d'}) \rightarrow \bigoplus_{i=1}^{I_h} \tilde{\mathcal{H}}_h^r(\mathbb{R}^{2d+d'})} < C_0, \quad \text{and}$$

$$\left\| (\tilde{\mathbf{I}}_h \circ \tilde{\mathbf{I}}_h^*) \circ \mathbf{t}_h^{(k)} \right\|_{\bigoplus_{i=1}^{I_h} \tilde{\mathcal{H}}_h^r(\mathbb{R}^{2d+d'}) \rightarrow \bigoplus_{i=1}^{I_h} \tilde{\mathcal{H}}_h^r(\mathbb{R}^{2d+d'})} < C_0$$

2840 for $0 \leq k \leq n$. Also we have, for the norm of the commutators, that

$$\left\| \left[\mathbf{t}_h^{(k)}, (\tilde{\mathbf{I}}_h \circ \tilde{\mathbf{I}}_h^*) \right] \right\|_{\bigoplus_{i=1}^{I_h} \tilde{\mathcal{H}}_h^r(\mathbb{R}^{2d+d'}) \rightarrow \bigoplus_{i=1}^{I_h} \tilde{\mathcal{H}}_h^r(\mathbb{R}^{2d+d'})} \leq C_0 \hbar^\epsilon \quad (9.33)$$

for $0 \leq k \leq n$.

Proposition 9.31. *There are constants $\epsilon > 0$ and $C > 0$, independent of \hbar , such that*

$$\left\| \left[\tilde{\mathbf{F}}_h, \mathbf{t}_h^{(k)} \right] \right\|_{\bigoplus_{i=1}^{I_h} \tilde{\mathcal{H}}_h^r(\mathbb{R}^{2d+d'}) \rightarrow \bigoplus_{i=1}^{I_h} \tilde{\mathcal{H}}_h^r(\mathbb{R}^{2d+d'})} \leq C \hbar^\epsilon \quad \text{for } 1 \leq k \leq n+1. \quad (9.34)$$

Further there exists a constant $C_0 > 0$, which is independent of f_G , \tilde{V} and \hbar , such that

2845 (1) For $0 \leq k \leq n+1$, it holds

$$\left\| \mathbf{t}_h^{(k)} \circ \tilde{\mathbf{F}}_h \circ \mathbf{t}_h^{(k)} \right\|_{\bigoplus_{i=1}^{I_h} \tilde{\mathcal{H}}_h^r(\mathbb{R}^{2d+d'}) \rightarrow \bigoplus_{i=1}^{I_h} \tilde{\mathcal{H}}_h^r(\mathbb{R}^{2d+d'})} \leq C_0 \sup \left(|e^{\tilde{V}}| \|Df|_{E_u}\|_{\min}^{-k} |\det Df|_{E_u}|^{-1/2} \right)$$

(2) If $\mathbf{u} \in \bigoplus_{i=1}^{I_h} \tilde{\mathcal{H}}_h^r(\mathbb{R}^{2d+d'})$ satisfies $\tilde{\mathbf{I}}_h \circ \tilde{\mathbf{I}}_h^*(\mathbf{u}) = \mathbf{u}$ and

$$\|\mathbf{u} - (\tilde{\mathbf{I}}_h \circ \tilde{\mathbf{I}}_h^*) \circ \mathbf{t}_h^{(k)}(\mathbf{u})\|_{\tilde{\mathcal{H}}_h^r} < \|\mathbf{u}\|_{\tilde{\mathcal{H}}_h^r} / 2 \quad \text{for some } 0 \leq k \leq n,$$

then we have

$$\|\mathbf{t}_h^{(k)} \circ \tilde{\mathbf{F}}_h \circ \mathbf{t}_h^{(k)}(\mathbf{u})\|_{\tilde{\mathcal{H}}_h^r} \geq C_0^{-1} \cdot \inf \left(|e^{\tilde{V}}| \|Df|_{E_u}\|_{\max}^{-k} |\det Df|_{E_u}|^{-1/2} \right) \cdot \|\mathbf{u}\|_{\tilde{\mathcal{H}}_h^r}.$$

Once we have obtained the propositions above, we can prove the next theorem, just in the same manner as we deduced Theorem 7.1 in Subsection 7.6.

2850 **Theorem 9.32.** *Let $n \geq 0$ and take sufficiently large r accordingly. Then there exists a small constant $\varepsilon > 0$, a constant C_0 , which is independent of V , f and N , and a decomposition of the Hilbert space $\tilde{\mathcal{H}}_N^r(P_G, E_u)$ independent of V :*

$$\tilde{\mathcal{H}}_N^r(P_G, E_u) = \mathcal{H}'_0 \oplus \mathcal{H}'_1 \dots \oplus \mathcal{H}'_n \oplus \mathcal{H}'_{n+1} \quad (9.35)$$

such that, writing $\tau^{(k)}$ for the projection onto the component \mathcal{H}'_k along other components,

(1) For some constant $\epsilon > 0$ and $C > 0$ independent of \hbar , we have

$$|\dim \mathcal{H}'_k - N^d \cdot \text{Vol}_\omega(M)| \leq CN^{d-\epsilon} \quad \text{for } 0 \leq k \leq n.$$

2855 while $\dim \mathcal{H}'_{n+1} = \infty$,

(2) $\|\tau^{(k)}\| < C_0$ for $0 \leq k \leq n+1$,

(3) $\|\tau^{(k)} \circ \tilde{F}_N \circ \tau^{(l)}\| \leq C/N^\varepsilon$ if $k \neq l$, with C independent on N (but may depend on f_G).

(4) for $0 \leq k \leq n+1$, it holds

$$\|\tau^{(k)} \circ \tilde{F}_N \circ \tau^{(k)}\|_{\tilde{\mathcal{H}}_N^r(P_G, E_u)} \leq C_0 \sup_{x \in M} \left(e^{D(x)} \|Df_x|_{E_u}\|_{\min}^{-k} \right), \quad (9.36)$$

2860 (5) for $0 \leq k \leq n$ and $u \in \mathcal{H}'_k$ it holds

$$\left\| \left(\tau^{(k)} \circ \tilde{F}_N \right) u \right\|_{\tilde{\mathcal{H}}_N^r(P_G, E_u)} \geq C_0^{-1} \inf_{x \in M} \left(e^{D(x)} \|Df_x|_{E_u}\|_{\max}^{-k} \right) \|u\|_{\tilde{\mathcal{H}}_N^r(P_G, E_u)}, \quad (9.37)$$

provided that N is sufficiently large.

Now we can deduce Theorem 1.34 and Theorem 1.35 from the theorem above by the argument parallel to that in Subsection 7.1. (But see the remark below.) The former part of the statements in Theorem 1.37 is an immediate sequence of Theorem 1.34 and Theorem 2865 1.35. The proof of the angular equidistribution law in Theorem 1.37 is parallel to that of Theorem 1.23, which we will present in Subsection 10.3.

Remark. The Hilbert space $\tilde{\mathcal{H}}_N^r(P_G, E_u)$ consists of distributions supported on a small neighborhood of the attractor E_u depending on $\hbar > 0$. To get the Hilbert space $\mathcal{H}_\hbar^r(P_G)$ in the statement of Theorem 1.34 and Theorem 1.35, we need a little formal argument 2870 to construct $\mathcal{H}_\hbar^r(P_G)$ from $\tilde{\mathcal{H}}_N^r(P_G, E_u)$ so that $\mathcal{H}_\hbar^r(P_G)$ contains $C_N^\infty(P_G)$ and that the operator $\tilde{F}_N \circ \hat{\chi}$ on $\mathcal{H}_\hbar^r(P_G)$ has the same spectral property as that on $\tilde{\mathcal{H}}_N^r(P_G, E_u)$. But, recalling the argument in Subsection 1.4.3 on the absorbing neighborhoods of the attractor E_u and noting the precomposition of the operator $\hat{\chi}$ in $\tilde{F}_N \circ \hat{\chi}$, such an argument can be provided easily in various ways.

2875 **9.4 A remark about the relation between the transfer operators**
 \hat{F}_N and \tilde{F}_N (and a proof of Theorem 1.36).

The prequantum transfer operator \tilde{F}_N on G is an extension of the prequantum transfer operator \hat{F}_N . But the relation between the spectrum of \tilde{F}_N and that of \hat{F}_N is not clear in general. In this subsection, we relate the spectra of the transfer operators \tilde{F}_N and \hat{F}_N in the respective “outermost” bands. This will reduce Theorem 1.36 (and the corresponding part of Theorem 1.37) to Theorem 1.21.

Below we suppose that the potential function V and \tilde{V} in their definitions are related as

$$\tilde{V} = V \circ p \tag{9.38}$$

and that they are both smooth²⁰. We also assume $r_1^+ < r_0^-$ so that the outermost annuli in Theorem 1.17 and Theorem 1.34 are separated from the inner annuli. Let us consider the pull-back operator

$$p^* : C_N^\infty(P) \rightarrow C_N^\infty(P_G), \quad p^*u(z) = u(p(x))$$

by the projection $p : G \rightarrow M$, and its dual

$$p_* : (C_N^\infty(P_G))' \rightarrow (C_N^\infty(P))', \quad \langle p_*v, u \rangle = \langle v, p_*u \rangle$$

Let $\tilde{\mathcal{H}}_h^r(P_G, E_u)_0$ be the closed subspace of the Hilbert space $\tilde{\mathcal{H}}_h^r(P_G, E_u)$ that consists of elements supported on (the image of) the section E_u .

2890 **Proposition 9.33.** *The operator p_* above restricts to a bounded operator*

$$p_* : \tilde{\mathcal{H}}_h^r(P_G, E_u)_0 \rightarrow \mathcal{H}_h^r(P) \tag{9.39}$$

and the following diagram commutes:

$$\begin{array}{ccc} \tilde{\mathcal{H}}_h^r(P_G, E_u)_0 & \xrightarrow{\tilde{F}_N} & \tilde{\mathcal{H}}_h^r(P_G, E_u)_0 \\ p_* \downarrow & & p_* \downarrow \\ \mathcal{H}_h^r(P) & \xrightarrow{\hat{F}_N} & \mathcal{H}_h^r(P) \end{array}$$

Further we have that

- (1) The generalized eigenvectors of $\tilde{F}_N : \tilde{\mathcal{H}}_h^r(P_G, E_u) \rightarrow \tilde{\mathcal{H}}_h^r(P_G, E_u)$ for the eigenvalues in the outmost band $\{r_0^- - \epsilon < |z| < r_0^+ + \epsilon\}$ is contained in $\tilde{\mathcal{H}}_h^r(P_G, E_u)_0$ and its image by p_* does not vanish.
- (2) The image of p_* in (9.39) contains $C_N^\infty(P) \subset \mathcal{H}_h^r(P)$.

²⁰Good part of the argument below holds without the assumption on smoothness of the potential V .

Before proving this proposition, we give the following consequence.

Corollary 9.34. *Under the same assumptions as in 9.33, the spectrum of the operators $\hat{F}_N : \mathcal{H}_h^r(P) \rightarrow \mathcal{H}_h^r(P)$ and $\tilde{F}_N : \tilde{\mathcal{H}}_h^r(P_G, E_u) \rightarrow \tilde{\mathcal{H}}_h^r(P_G, E_u)$ in the respective outermost annulus $\{r_0^- - \epsilon < |z| < r_0^+ + \epsilon\}$ with sufficiently small $\epsilon > 0$ coincides up to multiplicity (provided \hbar is sufficiently small according to ϵ).*

Proof. Proposition 9.33 tells that, for a generalized eigenvector v of $\tilde{F}_N : \tilde{\mathcal{H}}_h^r(P_G, E_u) \rightarrow \tilde{\mathcal{H}}_h^r(P_G, E_u)$ for an eigenvalue in the outermost band, its image $p_*(v)$ is a generalized eigenvector of $\hat{F}_N : \mathcal{H}_h^r(P) \rightarrow \mathcal{H}_h^r(P)$ for the same eigenvalue. Thus the eigenvalues of $\tilde{F}_N : \tilde{\mathcal{H}}_h^r(P_G, E_u) \rightarrow \tilde{\mathcal{H}}_h^r(P_G, E_u)$ in the outmost band is contained in those of $\hat{F}_N : \mathcal{H}_h^r(P) \rightarrow \mathcal{H}_h^r(P)$ up to multiplicity²¹.

We prove the converse. The image Imp_* of p_* in (9.39) can be identified with the quotient space $\tilde{\mathcal{H}}_h^r(P_G, E_u)/\ker p_*$. By this mean, we regard Imp_* as a Hilbert space, which contains $C_N^\infty(P) \subset \mathcal{H}_h^r(P)$ from Proposition 9.33(2). Thus we have two Hilbert spaces $\mathcal{H}_h^r(P)$ and Imp_* which contains $C_N^\infty(P)$ in common as dense subsets and the operator \hat{F}_N act on both of the Hilbert space boundedly as natural extensions of its action on $C_N^\infty(P)$. Then, by a general argument (see Appendix of [7] for instance), the discrete spectra of the operators $\hat{F}_N : \mathcal{H}_h^r(P) \rightarrow \mathcal{H}_h^r(P)$ and $\hat{F}_N : \text{Imp}_* \rightarrow \text{Imp}_*$ coincide up to multiplicity on the outside of their essential spectral radius. This implies that the eigenvalues of $\hat{F}_N : \mathcal{H}_h^r(P) \rightarrow \mathcal{H}_h^r(P)$ in the outermost band is contained in those of $\tilde{F}_N : \tilde{\mathcal{H}}_h^r(P_G, E_u) \rightarrow \tilde{\mathcal{H}}_h^r(P_G, E_u)$ up to multiplicity. \square

With this corollary, we finish the proof of Theorem 1.36.

Proof of Theorem 1.36. Theorem 1.36 follows from Corollary 9.34 and Theorem 1.21, if the potential function \tilde{V} satisfies (9.38) for some smooth function V close to V_0 . But the rank of the spectral projection operator for the outermost band does not depend on the potential function \tilde{V} (provided that the outermost band is isolated) because it coincides with the number of the eigenvalues of $\mathbf{t}_h^{(0)}$ in a small neighborhood of 1. So we get Theorem 1.36. \square

For the proof of Proposition 9.33, we first prove the following simpler version in the linearized setting.

Lemma 9.35. *The operator $p_* : \mathcal{S}(\mathbb{R}^{2d+d'}) \rightarrow \mathcal{S}(\mathbb{R}^{2d})$ defined by*

$$p_*u(x) = \int u(x, s)ds$$

extends to a bounded operator $p_ : \tilde{\mathcal{H}}_h^r(\mathbb{R}^{2d+d'}) \rightarrow \mathcal{H}_h^r(\mathbb{R}^{2d})$ and the operator norm is bounded by a constant independent of \hbar . It restricts to a bijection between the subspaces H'_0 in*

²¹That is, the multiplicity of the eigenvalues of the former is not greater than the latter.

2930 *Proposition 5.11 and E'_0 in Proposition 9.5. Further there exists a constant K independent of \hbar such that*

$$\|p_*(u)\|_{\widetilde{\mathcal{H}}_{\hbar}^r(\mathbb{R}^{2d+d'})} = K \cdot \|u\|_{\mathcal{H}_{\hbar}^r(\mathbb{R}^{2d})}$$

for all $u \in H'_0 = \text{Im } t_{\hbar}^{(0)}$.

Proof. We first prove boundedness of the operator $p_* : \widetilde{\mathcal{H}}_{\hbar}^r(\mathbb{R}^{2d+d'}) \rightarrow \mathcal{H}_{\hbar}^r(\mathbb{R}^{2d})$. Let us consider the operator $P_* : \mathcal{S}(\mathbb{R}^{4d+2d'}) \rightarrow \mathcal{S}(\mathbb{R}^{4d})$ defined by

$$P_*v(x, \xi_x) = a_{d'}^{-1} \cdot (2\pi\hbar)^{-d'/2} \cdot \int v(x, s, \xi_x, \xi_s) \exp(-\hbar^{-1}|\xi_s|^2/2) \frac{ds d\xi_s}{(2\pi\hbar)^{d'}}.$$

It makes the following diagram commute:

$$\begin{array}{ccc} \mathcal{S}\left(\mathbb{R}_{(x,s)}^{2d+d'}\right) & \xrightarrow{\mathcal{B}_{(x,s)}} & \mathcal{S}\left(\mathbb{R}_{(x,s,\xi_x,\xi_s)}^{4d+2d'}\right) \\ \downarrow p_* & & \downarrow P_* \\ \mathcal{S}\left(\mathbb{R}_x^{2d}\right) & \xrightarrow{\mathcal{B}_x} & \mathcal{S}\left(\mathbb{R}_{(x,\xi_x)}^{2d}\right) \end{array}$$

2935 Since we have, from the definition of $\mathcal{W}_{\hbar}^r(x, \xi_x)$ in (5.22) and that of $\widetilde{\mathcal{W}}_{\hbar}^r(x, s, \xi_x, \xi_s)$ in (9.6), that

$$\mathcal{W}_{\hbar}^r(x, \xi_x)^2 \cdot \exp(-\hbar^{-1}|\xi_s|^2) \leq C \cdot \widetilde{\mathcal{W}}_{\hbar}^r(x, s, \xi_x, \xi_s)^2$$

for some constant $C > 0$ independent of \hbar , we obtain, by Schwartz inequality,

$$\begin{aligned} \mathcal{W}_{\hbar}^r(x, \xi_x)^2 |P_*v(x, \xi_x)|^2 &= a_{d'}^{-2} \cdot (2\pi\hbar)^{-d'/2} \cdot \left| \int \mathcal{W}_{\hbar}^r(x, \xi_x) v(x, s, \xi_x, \xi_s) \exp(-\hbar^{-1}|\xi_s|^2) ds d\xi_s \right|^2 \\ &\leq C \int \widetilde{\mathcal{W}}_{\hbar}^r(x, s, \xi_x, \xi_s)^2 |v(x, s, \xi_x, \xi_s)|^2 ds d\xi_s \end{aligned}$$

with $C > 0$ a constant independent of \hbar . Integrating the both sides with respect to the variables x and ξ_x , we get

$$\|\mathcal{W}_{\hbar}^r \cdot P_*v\|_{L^2} \leq C \|\widetilde{\mathcal{W}}_{\hbar}^r \cdot v\|_{L^2}.$$

This implies that $p_* : \widetilde{\mathcal{H}}_{\hbar}^r(\mathbb{R}^{2d+d'}) \rightarrow \mathcal{H}_{\hbar}^r(\mathbb{R}^{2d})$ is bounded.

2940 We next prove the remaining claims. Recall that the operator $T^{(0)}$ defined in (4.36) is the rank one projection operator that assigns a function the constant term of its Taylor expansion at the origin 0. For the operator $p'_* : \mathcal{S}\left(\mathbb{R}_{(\zeta_p, \xi_s)}^{d+d'}\right) \rightarrow \mathcal{S}\left(\mathbb{R}_{\zeta_p}^d\right)$ defined by $p'_*v(\zeta_p) = v(\zeta_p, 0)$, we have the commutative diagram:

$$\begin{array}{ccc} \mathcal{S}\left(\mathbb{R}_{(\zeta_p, \xi_s)}^{d+d'}\right) & \xrightarrow{T^{(0)}} & \mathcal{S}\left(\mathbb{R}_{(\zeta_p, \xi_s)}^{d+d'}\right) \\ \downarrow p'_* & & \downarrow p'_* \\ \mathcal{S}\left(\mathbb{R}_{\zeta_p}^d\right) & \xrightarrow{T^{(0)}} & \mathcal{S}\left(\mathbb{R}_{\zeta_p}^d\right) \end{array}$$

2945 The images of the operators $T^{(0)}$ on the upper and lower rows are one-dimensional subspaces that consists of the constant functions. The operator p'_* restricts to an isomorphisms between them. The commutative diagram above extends to

$$\begin{array}{ccc} H_h^r \left(\mathbb{R}_{(\zeta_p, \xi_s)}^{d+d'} \right) & \xrightarrow{T^{(0)}} & H_h^r \left(\mathbb{R}_{(\zeta_p, \xi_s)}^{d+d'} \right) \\ \downarrow p'_* & & \downarrow p'_* \\ H_h^r \left(\mathbb{R}_{\zeta_p}^d \right) & \xrightarrow{T^{(0)}} & H_h^r \left(\mathbb{R}_{\zeta_p}^d \right) \end{array}$$

and hence trivially to

$$\begin{array}{ccc} L^2 \left(\mathbb{R}_{\nu_q}^d \right) \otimes H_h^r \left(\mathbb{R}_{(\zeta_p, \xi_s)}^{d+d'} \right) & \xrightarrow{\text{Id} \otimes T^{(0)}} & L^2 \left(\mathbb{R}_{\nu_q}^d \right) \otimes H_h^r \left(\mathbb{R}_{(\zeta_p, \xi_s)}^{d+d'} \right) \\ \downarrow \text{Id} \otimes p'_* & & \downarrow \text{Id} \otimes p'_* \\ L^2 \left(\mathbb{R}_{\nu_q}^d \right) \otimes H_h^r \left(\mathbb{R}_{\zeta_p}^d \right) & \xrightarrow{\text{Id} \otimes T^{(0)}} & L^2 \left(\mathbb{R}_{\nu_q}^d \right) \otimes H_h^r \left(\mathbb{R}_{\zeta_p}^d \right) \end{array} \quad (9.40)$$

This commutative diagram viewed through the isomorphisms

$$\mathcal{U} : L^2 \left(\mathbb{R}_{\nu_q}^d \right) \otimes H_h^r \left(\mathbb{R}_{\zeta_p}^d \right) \otimes \mathcal{H}_h^r \left(\mathbb{R}_x^{2d} \right) \quad \text{and} \quad \tilde{\mathcal{U}} : L^2 \left(\mathbb{R}_{\nu_q}^d \right) \otimes H_h^r \left(\mathbb{R}_{(\zeta_p, \xi_s)}^{d+d'} \right) \rightarrow \mathcal{H} \left(\mathbb{R}_{(x,s)}^{2d+d'} \right)$$

is just

$$\begin{array}{ccc} \mathcal{H}_h^r \left(\mathbb{R}_{(x,s)}^{2d+d'} \right) & \xrightarrow{t_h^{(0)}} & \mathcal{H}_h^r \left(\mathbb{R}_{(x,s)}^{2d+d'} \right) \\ \downarrow p_* & & \downarrow p_* \\ \mathcal{H}_h^r \left(\mathbb{R}_x^{2d} \right) & \xrightarrow{t_h^{(0)}} & \mathcal{H}_h^r \left(\mathbb{R}_x^{2d} \right) \end{array}$$

2950 Therefore for the proofs of the remaining claims of the proposition, it is enough to prove the corresponding claims in the diagram (9.40). But they are now obvious from the fact that $T^{(0)}$ is a rank one projection operator. \square

2955 *Proof of Proposition 9.33 .* The proofs of the claims other than Claim 2 are obtained by applying Lemma 9.35 to the local data and showing that the effect of non-linearity of the coordinate transformations is negligible. We omit the detail of the argument because it should be clear if we recall the argument in Section 7 and the preceding subsections in this section.

2960 We prove Claim 2. Recall that $E_u : M \rightarrow G$ is the section that assigns the unstable subspace to each point. Let $\mu_u = (E_u)_*(\text{Vol}_\omega)$ where Vol_ω is the symplectic volume on M . We consider the operator

$$\iota : C_N^\infty(P) \rightarrow C_N^\infty(P_G)' \quad \iota(u) = (u \circ p) \cdot \mu_u$$

which satisfies $p_* \circ \iota = \text{Id}$. To show that $\widetilde{\mathcal{H}}_h^r(P)$ contains the space $C_N^\infty(P)$, it is enough to prove that the image $\iota(C_N^\infty(P))$ of this operator is contained in $\widetilde{\mathcal{H}}_h^r(P_G, E_u)$. We define a Hilbert space of distributions $\widehat{\mathcal{H}}_h^r(G, E_u) \subset C^\infty(G)'$ on G as the completion of the space $C_h^\infty(G, E_u)$ with respect to the norm

$$\|u\|_{\widehat{\mathcal{H}}_h^r} := \left(\sum_{i=1}^{I_h} \|W_h^r \cdot \mathcal{B}_x u_i\|_{L^2}^2 \right)^{1/2}$$

where $u_i(x) := \psi_i(x) \cdot u(\kappa_i(x))$ and $W_h^r : \mathbb{R}^{2d+d'} \oplus \mathbb{R}^{2d+d'} \rightarrow \mathbb{R}$ is defined by

$$W_h^r(p, q, s, \xi_p, \xi_q, \xi_s) = W^r(\hbar^{-1/2}\xi_p, (\hbar^{-1/2}\xi_q, \hbar^{-1/2}\xi_s)).$$

Then, from the results in [7] and [19], the Perron-Frobenius operator

$$Q : C_h^\infty(G, E_u) \rightarrow C_h^\infty(G, E_u), \quad Qu(x) = \frac{1}{|\det(Df_G|_{\ker p})|} \cdot u \circ f_G^{-1}$$

extends to a bounded operator on $\widehat{\mathcal{H}}_h^r(G, E_u)$. The spectral radius of $Q : \widehat{\mathcal{H}}_h^r(G, E_u) \rightarrow \widehat{\mathcal{H}}_h^r(G, E_u)$ is 1 and the essential spectral radius is smaller than 1. Further 1 is the unique eigenvalue on the unit circle, which is simple, and the corresponding eigenvector is μ_u . In particular, the measure μ_u belongs to $\widehat{\mathcal{H}}_h^r(G, E_u)$. To finish, we consider the operator

$$j : C_h^\infty(G, E_u) \times C_N^\infty(P_G) \rightarrow C_N^\infty(P_G), \quad j(u, v) = u \cdot v$$

and check that it extends to a continuous operator²²

$$j : \widehat{\mathcal{H}}_h^r(G, E_u) \times C_N^\infty(P_G) \rightarrow \widetilde{\mathcal{H}}_h^r(P_G, E_u).$$

Since $\iota(u) = j(\mu_u, u)$, this implies that $\iota(C_N^\infty(P))$ is contained in $\widetilde{\mathcal{H}}_h^r(P_G, E_u)$. \square

10 Proof of Th. 1.23. Concentration of most of the external eigenvalues on a circle.

10.1 Time average and Birkhoff's ergodic theorem

Let us write \widehat{F}_V and $\widehat{F}_{V,N}$ for the prequantum transfer operators \widehat{F} and \widehat{F}_N defined respectively in (1.19) and (1.21), specifying dependence on the potential function V . In this subsection, we prove the first part of Theorem 1.23, namely that almost all eigenvalues of $\widehat{F}_{V,N}$ of the external band concentrate at the value $e^{(V-V_0)}$ as $N \rightarrow \infty$, where V_0 is the potential of reference defined in (1.31) and whose importance has been shown in Theorem

²²Note that this claim is for each fixed \hbar and we do not claim any uniformly in \hbar . So the proof is easy if we consider in the local charts.

1.37. Take $\epsilon > 0$ arbitrarily and let V_ϵ be a smooth approximation of the function V_0 such that

$$|V_0(x) - V_\epsilon(x)| < \epsilon \quad \text{for all } x \in M.$$

We introduce the “(approximate) damping function” as the difference

$$D := V - V_\epsilon$$

2985 The next lemma shows that the transfer operator \hat{F}_V is conjugate (and hence has the same spectrum) to an operator \mathcal{L}_{D_n} which is the operator of reference \hat{F}_{V_ϵ} with an additional potential D_n obtained by the time average of the damping function D . This presentation of the operator \hat{F}_V will be very convenient for our purpose.

Lemma 10.1. *For any $n \geq 1$,*

$$\hat{F}_V = e^{G_n} \circ \mathcal{L}_{D_n} \circ e^{-G_n} \tag{10.1}$$

2990 where

$$\mathcal{L}_{D_n} := e^{D_n} F_{V_\epsilon}$$

and

$$D_n := \frac{1}{n} \sum_{k=1}^n D \circ \tilde{f}^{-k}$$

is the time averaged of the “damping function” D and G_n is the multiplication operator by the function (with same name):

$$G_n = \frac{1}{n} \sum_{k=1}^n k D \circ \tilde{f}^{-n+k}$$

Proof. From (1.19), we have that

$$e^{G_n} \circ \left(e^{D_n} \hat{F}_{V_\epsilon} \right) \circ e^{-G_n} = e^{G_n + D_n - G_n \circ \tilde{f}^{-1}} \hat{F}_{V_\epsilon}.$$

We compute that

$$\begin{aligned} G_n + D_n - G_n \circ \tilde{f}^{-1} &= \frac{1}{n} \sum_{k=1}^n \left(k D \circ \tilde{f}^{-n+k} + D \circ \tilde{f}^{-n+k-1} \right) - \frac{1}{n} \sum_{k=0}^{n-1} (k+1) D \circ \tilde{f}^{-n+k} \\ &= D = V - V_\epsilon \end{aligned}$$

2995 Hence

$$e^{G_n} \circ \left(e^{D_n} \hat{F}_{V_\epsilon} \right) \circ e^{-G_n} = e^{V - V_\epsilon} \hat{F}_{V_\epsilon} = \hat{F}_V.$$

□

Due to ergodicity of the map $f : M \rightarrow M$, the time averaged function D_n converges for $n \rightarrow \infty$ almost everywhere to its spatial average:

$$\langle D \rangle := \frac{1}{\text{Vol}_\omega(M)} \int_M D dx$$

In particular we will use later on that

$$\int_M (e^{D_n} - e^{\langle D \rangle}) dx \xrightarrow{n \rightarrow \infty} 0. \quad (10.2)$$

3000 Let

$$\mathcal{L}_{\langle D \rangle} := e^{\langle D \rangle} F_{V_\epsilon}$$

Let $\mathcal{L}_{\langle D \rangle, N}$ be the restriction of $\mathcal{L}_{\langle D \rangle}$ to $\mathcal{H}_N^r(P)$. Its spectrum is that of $\hat{F}_{V_\epsilon, N}$ multiplied by the constant factor $e^{\langle D \rangle}$, in particular its external part concentrates (as $N \rightarrow \infty$) to the annulus $e^{\langle D \rangle - \epsilon} \leq |z| \leq e^{\langle D \rangle + \epsilon}$ and the disk $|z| \leq (1/\lambda)e^{\langle D \rangle + \epsilon}$ from Theorem 1.17. Let $\tau_N^{(0)}$ be the (finite rank) approximate projection operator on the external band of $\hat{F}_{V_\epsilon, N}$ introduced in Theorem 7.1. (We put the subscript N now to make the dependence on N explicit.) Recall that this approximate spectral projector $\tau_N^{(0)}$ does not depend on the choice of the potential V , hence it is also suitable for the transfer operators \mathcal{L}_{D_n} and $\mathcal{L}_{\langle D \rangle}$. We want now to study the “quantity of spectrum” of the external band of $\mathcal{L}_{D_n, N}$ which does not concentrate on the annulus $e^{\langle D \rangle - \epsilon} \leq |z| \leq e^{\langle D \rangle + \epsilon}$. To this end, we introduce the operator:

$$S_{n, N} := \tau_N^{(0)} (\mathcal{L}_{D_n, N} - \mathcal{L}_{\langle D \rangle, N}) \tau_N^{(0)} = \tau_N^{(0)} \left((e^{D_n} - e^{\langle D \rangle}) \cdot \hat{F}_{V_\epsilon, N} \right) \tau_N^{(0)}$$

and $T_{n, N} := \mathcal{L}_{D_n, N} - S_{n, N}$ giving the following decomposition:

$$\mathcal{L}_{D_n, N} = T_{n, N} + S_{n, N}$$

Notice that $T_{n, N}$ is in some sense $\mathcal{L}_{D_n, N}$ but with the the external band replaced by the spectrum of $\mathcal{L}_{\langle D \rangle, N}$. As a consequence, from Theorem 1.17, for N large enough, the operator $T_{n, N}$ has no spectrum in the spectral domain²³

$$W_\epsilon := \left\{ z \in \mathbb{C}, \quad \frac{1}{\lambda} e^{\sup D_n + \epsilon} \leq |z| \leq e^{\langle D \rangle - 2\epsilon} \text{ or } e^{\langle D \rangle + 2\epsilon} \leq |z| \right\}$$

3015 and, moreover, on W_ϵ we have a bound on the norm of the resolvent of $T_{n, N}$: there exists $C_\epsilon > 0$, and N_ϵ such that for any $N \geq N_\epsilon$,

$$\|(z - T_{n, N})^{-1}\| \leq C_\epsilon \quad \text{uniformly for } z \in W_\epsilon \quad (10.3)$$

and

$$\|T_{n, N}\| \leq e^{\langle D \rangle + 2\epsilon}. \quad (10.4)$$

²³Here we used the fact that V_ϵ is an ϵ -approximation of V_0 .

Recall that the number of eigenvalues in the external band is $C_0 \cdot N^d + \mathcal{O}(N^{-\epsilon})$ with $C_0 = Vol_\omega(M)$ from Theorem 7.1. The next lemma concerns the trace norm of $S_{n,N}$ and is the key to show that the “perturbation” $S_{n,N}$ may add only a relatively negligible number of eigenvalues on W_ϵ .

Lemma 10.2. *For any $\epsilon' > 0$, there exists $n \geq 1$, $N_{\epsilon'} > 0$, such that, for any $N > N_{\epsilon'}$,*

$$\|S_{n,N}\|_{\text{Tr}} \leq \epsilon' N^d. \quad (10.5)$$

Proof. We have $S_{n,N} = \tau_N^{(0)}(e^{D_n} - e^{\langle D \rangle}) \hat{F}_{V_\epsilon, N} \tau_N^{(0)}$ so

$$\|S_{n,N}\|_{\text{Tr}} \leq \left\| \tau_N^{(0)}(e^{D_n} - e^{\langle D \rangle}) \right\|_{\text{Tr}} \left\| \hat{F}_{V_\epsilon, N} \tau_N^{(0)} \right\| \leq C \left\| \tau_N^{(0)}(e^{D_n} - e^{\langle D \rangle}) \right\|_{\text{Tr}}$$

Let $\epsilon' > 0$. From (7.60), we have

$$\left\| \tau_N^{(0)}(e^{D_n} - e^{\langle D \rangle}) \right\|_{\text{Tr}} \leq CN^d \left(\int_M |e^{D_n} - e^{\langle D \rangle}| dx \right)$$

From (10.2), there exists n such that $\left| \int_M (e^{D_n} - e^{\langle D \rangle}) \right| < \epsilon'/C^2$, giving $\|S_{n,N}\|_{\text{Tr}} \leq \epsilon' N^d$ for $N \geq N_{\epsilon'} := N_n$. \square

10.2 Proof of concentration of the moduli of the resonance to the circle $|z| = e^{\langle V - V_0 \rangle}$

Now, as a consequence of (10.3) and (10.5) concerning the decomposition $\mathcal{L}_{D_n, N} = T_{n, N} + S_{n, N}$, we prove

Lemma 10.3. *For any $\epsilon > 0$, there exists n and N_ϵ such that, for any $N > N_\epsilon$, the number of eigenvalues of $\mathcal{L}_{D_n, N}$ in W_ϵ counting multiplicities, is bounded by ϵN^d .*

Before giving the proof, remark that due to the conjugation (10.1), the same result holds for the operator $\tilde{F}_{\tilde{V}, N}$ and this finishes the proof for the claim on “radial concentration of eigenvalues” in Theorem 1.23. The proof that Lemma 10.3 follows from Lemma 10.2 should be well-known. Here we propose two different proofs, because they are both interesting in their own right. The first proof is based on Weyl inequality while the second is based on Jensen’s formula applied to the relative determinant.

10.2.1 Proof of Lemma 10.3 using Weyl inequality

We will just apply Lemma 10.5 stated below to the family of operators $A_N = e^{-\langle D \rangle} T_{n, N}$ and $B_N = e^{-\langle D \rangle} S_{n, N}$ with setting $F(N) = \epsilon' N^d e^{\langle D \rangle}$ given from (10.5).

Lemma 10.4. *Suppose that A_N, B_N is a family of bounded operators on an Hilbert space, depending on $N \in \mathbb{Z}$, such that for every N , we have $\|A_N\| \leq 1$, $\|B_N\| \leq C_B$ with some*

3045 $C_B > 0$ and, moreover, the operators B_N are trace class and $\|B_N\|_{\text{Tr}} \leq F(N)$ with some function $F : \mathbb{N} \rightarrow \mathbb{R}^+$. Then for any $\epsilon > 0$,

$$\#\{\sigma(A_N + B_N) \cap \{z \in \mathbb{C}, |z| \geq 1 + \epsilon\}\} \leq \frac{1}{\epsilon} C \cdot F(N)$$

with $C = \frac{(2+C_B)}{2\log(1+\epsilon)}$.

Proof. Let $C_N := A_N + B_N$ and $P_N := C_N^* C_N$. Let us write

$$P_N = \mathcal{A}_N + \mathcal{B}_N$$

3050 with $\mathcal{A}_N := A_N^* A_N$, $\mathcal{B}_N := A_N^* B_N + B_N^* A_N + B_N^* B_N$. We have that $\|\mathcal{A}_N\| \leq 1$, $\|\mathcal{B}_N\| \leq 2C_B + C_B^2$, \mathcal{B}_N is in the trace class and $\|\mathcal{B}_N\|_{\text{Tr}} \leq (2 + C_B) F(N)$. The operators C_N and P_N have discrete spectrum outside the circle of radius 1. For each N , let $(\lambda_j)_{j=1,2,\dots,M_N}$ denote the eigenvalues of C_N in the domain $\{z \in \mathbb{C}, |z| > 1 + \epsilon\}$, ordered in such a way that $|\lambda_1| \geq |\lambda_2| \dots \geq |\lambda_{M_N}|$. (For simplicity, we do not indicate the dependence on N). Observe that P_N is self-adjoint and positive. Let $p_1 \geq p_2 \geq \dots \geq p_{M_N}$ denote the eigenvalues of P_N counting multiplicity. The $(p_j^{1/2})_j$ are the singular values of C_N . Weyl inequalities give
3055 (see [24, p.50] for a proof):

$$\sum_{j=1}^{M_N} \log |\lambda_j| \leq \frac{1}{2} \sum_{j=1}^{M_N} \log p_j \tag{10.6}$$

Since $|\lambda_j| \geq 1 + \epsilon$, that is, $\log |\lambda_i| \geq \log(1 + \epsilon)$, this implies and $\log p_j \leq p_j - 1$.

$$M_N \cdot \log(1 + \epsilon) \leq \frac{1}{2} \sum_{j=1}^{M_N} \log p_j \leq \frac{1}{2} \sum_{j=1}^{M_N} (p_j - 1). \tag{10.7}$$

Let $(f_j)_{j \in \{1, \dots, M_N\}}$ denote the associated eigenvectors of P_N for eigenvalues $(p_j)_{j \in \{1, \dots, M_N\}}$. Put $\mathcal{E}_N := \text{Span}(f_j, j \in \{1, \dots, M_N\})$. Then we have

$$\begin{aligned} \sum_{j=1}^{M_N} p_j &= \text{Tr}(P_N |_{\mathcal{E}_N}) = \text{Tr}(\mathcal{A}_N |_{\mathcal{E}_N}) + \text{Tr}(\mathcal{B}_N |_{\mathcal{E}_N}) \\ &\leq \sum_{j=1}^{M_N} |(f_j, \mathcal{A}_N f_j)| + \sum_{j=1}^{M_N} |(f_j, \mathcal{B}_N f_j)| \\ &\leq \left(\sum_{j=1}^{M_N} \|\mathcal{A}_N\| \right) + \|\mathcal{B}_N\|_{\text{Tr}} \leq M_N + (2 + C_B) F(N) \end{aligned} \tag{10.8}$$

That is

$$\sum_{j=1}^{M_N} (p_j - 1) \leq (2 + C_B) F(N).$$

Putting this in (10.7), we obtain the conclusion. □

Note that the simple lemma above already gives the conclusion of Lemma 10.3 on
 3060 the outer connected component $\{|z| \geq e^{(D)+2\epsilon}\}$ of W_ϵ . In order to look into the inner
 component of W_ϵ , we need a little more argument.

Lemma 10.5. *Let $U \subset \mathbb{C}$ be an open disk and $z_0 \in U$. Let f be a Möbius transformation
 such that $f(U) = \{z \in \mathbb{C}, |z| > 1\}$ and $f(z_0) = \infty$. Suppose that A_N and B_N are family of
 3065 bounded operators on an Hilbert space, depending on $N \in \mathbb{N}$. Suppose also that, for some
 constant $C > 0$ independent of N , we have*

$$\|A_N\| \leq C, \quad \|B_N\| \leq C, \quad \|(z_0 - (A_N + B_N))^{-1}\| \leq C$$

and

$$\|(z - A_N)^{-1}\| \leq C \quad \text{for all } z \in U.$$

Moreover, suppose that the operators B_N are in the trace class and $\|B_N\|_{\text{Tr}} \leq F(N)$ with
 some function $F : \mathbb{N} \rightarrow \mathbb{R}^+$. Then there exists a constant $C' > 0$, which depends only on
 the constant C , such that, for every N , we have

$$\sharp\{\sigma(A_N + B_N) \cap U\} \leq C' \cdot F(N). \quad (10.9)$$

3070 *Proof.* Let $C_N := A_N + B_N$. Using holomorphic functional calculus, we define:

$$A'_N = f(A_N) := \frac{1}{2\pi i} \oint_{\gamma} f(z) R_{A_N}(z) dz, \quad \text{where } R_{A_N}(z) := (z - A_N)^{-1} \quad (10.10)$$

Here the path $\gamma = \gamma_1 \cup \gamma_2$ is taken as follows so that they enclose the spectrum of A_N for all
 N : the closed path γ_1 is taken so that $f(\gamma_1)$ is the clockwise circle of radius $1 - \epsilon_C$ with some
 $\epsilon_C > 0$ (this is possible because $\text{dist}(z, \sigma(A_N)) \geq \|R_{A_N}(z)\|^{-1} \geq C^{-1}$ for $z \in U$); the close
 path γ_2 is a counterclockwise circle of sufficiently large radius, say, $C + \epsilon_C \geq \|A_N\| + \epsilon_C$.
 3075 From (10.10), we see that A'_N is bounded and moreover that $\|A'^k_N\| < 1$ for some k which
 depends only on f and C . Equivalently we can modify the norm $\|\cdot\|$ (in a way which
 depends on f and C only) such that $\|A'_N\| < 1$. Similarly we define

$$C'_N = f(C_N) := \frac{1}{2\pi i} \oint_{\gamma'} f(z) R_{C_N}(z) dz, \quad \text{where } R_{C_N}(z) := (z - C_N)^{-1}$$

where $\gamma' = \gamma'_1 \cup \gamma_2$ with γ'_1 a small clockwise circle around the point z_0 so that it enclose
 the spectrum of C_N . From the assumption $\|(z_0 - (A_N + B_N))^{-1}\| \leq C$, the operator C'_N
 3080 thus defined is bounded uniformly with respect to N . Let

$$B'_N := C'_N - A'_N = \frac{1}{2\pi i} \oint_{\gamma'} f(z) (R_{C_N}(z) - R_{A_N}(z)) dz$$

From the relation

$$R_{C_N}(z) - R_{A_N}(z) = \frac{1}{2} (R_{C_N}(z) B_N R_{A_N}(z) + R_{A_N}(z) B_N R_{C_N}(z))$$

we deduce that $\|R_{C_N}(z) - R_{A_N}(z)\|_{\text{Tr}} \leq CF(N)$ and then that $\|B'_N\|_{\text{Tr}} \leq CF(N)$ with some $C > 0$ independent on N . We can apply Lemma 10.4 to the operators A'_N, B'_N and get that

$$\#\{\sigma(A'_N + B'_N) \cap \{z \in \mathbb{C}, |z| > 1\}\} \leq C' \cdot F(N)$$

3085 for some C' independent on N . By the spectral mapping theorem, we have

$$\#\{\sigma(A_N + B_N) \cap U\} = \#\{\sigma(A'_N + B'_N) \cap \{z \in \mathbb{C}, |z| > 1\}\}.$$

So we get (10.9). □

Now we apply this lemma to the setting mentioned in the beginning. We take finitely many disks $U_i \subset \mathbb{C}$ so that we may apply Lemma 10.5 to each U_i . (In particular we take U_i so that it intersects the region $e^{-\langle D \rangle}(r_1^+ + \varepsilon)^n < |z| < e^{-\langle D \rangle}(r_0^- - \varepsilon)^n$ for small $\varepsilon > 0$ 3090 where the resolvent $(z - A_N - B_N)^{-1}$ should be bounded uniformly in N and n , and where we take the pint z_0 in Lemma 10.5.) By covering the inner connected component of the region $e^{-\langle D \rangle}W_\varepsilon$ by such disks, we conclude Lemma 10.2.

10.2.2 Proof of Lemma 10.3 using Jensen formula

From the expression $\mathcal{L}_{D_n, N} = T_{n, N} + S_{n, N}$, we write $z - \mathcal{L}_{D_n, N}$ for $z \in W_\varepsilon$ as

$$z - \mathcal{L}_{D_n, N} = z - T_{n, N} - S_{n, N} = (z - T_{n, N})(1 - \mathcal{K}(z)) \quad (10.11)$$

3095 with setting

$$\mathcal{K}(z) := (z - T_{n, N})^{-1} S_{n, N}.$$

Since $z - T_{n, N}$ is invertible for $z \in W_\varepsilon$ from (10.3), we have

$$(z - T_{n, N})^{-1} (z - \mathcal{L}_{D_n, N}) = (1 - \mathcal{K}(z)) \quad \text{for } z \in W_\varepsilon.$$

Since $\mathcal{K}(x)$ is a finite rank operators (because so is $S_{n, N}$), we can define

$$\mathcal{D}(z) := \det(1 - \mathcal{K}(z))$$

and see that the eigenvalues of $\mathcal{L}_{D_n, N}$ in W_ε coincide with the zeroes of $\mathcal{D}(z)$ in W_ε up to multiplicity. From the formula $\log \mathcal{D}(z) = \text{Tr} \log(1 - \mathcal{K}(z))$ and the simple inequality $\log(1 - x) \leq x$ for $x > -1$, we have that, for $N > \max(N_\varepsilon, N_{\varepsilon'})$,

$$\log |\mathcal{D}(z)| \leq \|\mathcal{K}(z)\|_{\text{Tr}} \leq \|(z - T_{n, N})^{-1}\| \|S_{n, N}\|_{\text{Tr}} \leq C_\varepsilon \varepsilon' N^d \leq \varepsilon N^d \quad (10.12)$$

from (10.3) and (10.5). In the last inequality, we chose $\varepsilon' > 0$ such that $C_\varepsilon \varepsilon' < \varepsilon$. We next show that $\log |\mathcal{D}(z)|$ is not too small on some part of the region W_ε . More precisely, we let

$$W'_\varepsilon := \left\{ z \in \mathbb{C}, \frac{1}{\lambda} e^{\sup D_n + 2\varepsilon} \leq |z| \leq e^{\inf D_n - 3\varepsilon} \text{ or } e^{\sup D_n + 3\varepsilon} \leq |z| \right\} \subset W_\varepsilon$$

3100 and show that

$$\log |\mathcal{D}(z)| \geq -\epsilon N^d \quad \text{for } z \in W'_\epsilon. \quad (10.13)$$

From Theorem 1.17 applied to $\mathcal{L}_{D_n, N}$, we see that W'_ϵ is in a gap (or on the outside) of the bands given there and that

$$\|(z\text{Id} - \mathcal{L}_{D_n, N})^{-1}\| \leq C_\epsilon \quad \text{for all } z \in W'_\epsilon.$$

So, from (10.11), we see that $1 - \mathcal{K}(z)$ for $z \in W'_\epsilon$ is invertible and that

$$\begin{aligned} (1 - \mathcal{K}(z))^{-1} &= (z - \mathcal{L}_{D_n, N})^{-1} (z - T_{n, N}) = (z - \mathcal{L}_{D_n, N})^{-1} (z - \mathcal{L}_{D_n, N} + S_{n, N}) \\ &= 1 + (z - \mathcal{L}_{D_n, N})^{-1} S_{n, N}. \end{aligned}$$

Hence, similarly to (10.12), we get

$$-\log |\mathcal{D}(z)| \leq \|(z - \mathcal{L}_{D_n, N})^{-1} S_{n, N}\|_{\text{Tr}} \leq \|(z - \mathcal{L}_{D_n, N})^{-1}\| \|S_{n, N}\|_{\text{Tr}} \leq C_\epsilon \epsilon' N^d < \epsilon N^d$$

for $N > \max(N_\epsilon, N_{\epsilon'})$.

3105 Finally, we employ a theorem, Jensen's formula [38, p.236], in complex analysis to show that the inequalities (10.12) and (10.13) imply that, for arbitrarily small $\tilde{\epsilon} > 0$, we have

$$\#\{z \in W_\epsilon, \mathcal{D}(z) = 0\} < \tilde{\epsilon} N^d \quad (10.14)$$

for sufficiently large N . This finish the proof of Lemma 10.3.

3110 We cover the domain W_ϵ by finitely many open topological disks D_i , $1 \leq i \leq l$, so that $D_i \cap W'_\epsilon \neq \emptyset$ for every i . Let $\Phi_i : D_i \rightarrow \mathbb{D} := \{w \in \mathbb{C}, |w| < 1\}$ be a Riemann mapping such that $\Phi_i(z_i) = 0$ for some $z_i \in D_i \cap W'_\epsilon$. Take $\delta > 0$ so small that $\bigcup_i \Phi_i^{-1}(\{|w| < 1 - \delta\})$ also cover W_ϵ . Let

$$\mathcal{D}_i(w) := \mathcal{D}(\Phi_i(w))$$

which is a holomorphic function on \mathbb{D} with zeroes $w_j \in \mathbb{D}$, which correspond to the zeroes of $\mathcal{D}(z)$.

Jensen's formula writes

$$\sum_j -\log |w_j| = -\log |\mathcal{D}_i(0)| + \frac{1}{2\pi} \int_{|w|=1} \log |\mathcal{D}_i(w)| dw$$

3115 Inequality (10.13) gives $-\log |\mathcal{D}_i(0)| < \epsilon N^d$, and (10.12) gives $\frac{1}{2\pi} \int_{|w|=1} \log |\mathcal{D}_i(w)| dw < \epsilon N^d$. For the zero w_j in the disk $\{|w| < 1 - \delta\}$ we have $-\log |w_j| > -\log(1 - \delta) \geq \delta$ and, hence, we see that the number \mathcal{N}_i of such zeros is bounded:

$$\mathcal{N}_i \leq \sum_{j, |w_j| < 1 - \delta} \frac{-\log |w_j|}{\delta} \leq \frac{1}{\delta} \left(\sum_{j, w_j \in \mathbb{D}} -\log |w_j| \right) < \frac{\epsilon N^d}{\delta}.$$

Summing over the sets D_i and noting arbitrariness of $\epsilon > 0$, we conclude (10.14).

10.3 Proof of equidistribution of the arguments of the resonances

3120 In this subsection we prove the second part of Theorem 1.23, namely the equidistribution of the arguments. We write the eigenvalues of $\hat{\mathcal{F}}_{\hbar}$ as

$$\lambda_j = \rho_j e^{i\theta_j}, \quad \rho_j \geq 0, \quad \theta_j \in \mathbb{R}, \quad j = 1, 2, \dots, \mathcal{N}_{\hbar},$$

with $\mathcal{N}_{\hbar} := \dim \mathcal{H}_{\hbar}$. Let us consider the following distribution (for fixed \hbar) in θ on the circle S^1 :

$$s_{\hbar} := \frac{1}{\mathcal{N}_{\hbar}} \sum_{j=1}^{\mathcal{N}_{\hbar}} \delta(\theta - \theta_j)$$

3125 We want to show that s_{\hbar} converges (weakly) to the uniform probability measure on S^1 in the limit $\hbar \rightarrow 0$. This is equivalent to show that, for every fixed $n \in \mathbb{Z}$,

$$\langle e^{in\theta}, s_{\hbar} \rangle \xrightarrow{\hbar \rightarrow 0} \begin{cases} 1, & \text{if } n = 0; \\ 0, & \text{otherwise.} \end{cases} \quad (10.15)$$

Let us show now (10.15). If $n = 0$, we have simply $\langle 1, s_{\hbar} \rangle = \frac{1}{\mathcal{N}_{\hbar}} \mathcal{N}_{\hbar} = 1$. Suppose $n > 0$, since $\langle e^{-in\theta}, s_{\hbar} \rangle = \overline{\langle e^{in\theta}, s_{\hbar} \rangle}$. Let $r = e^{\langle V - V_0 \rangle}$. We write

$$\langle e^{in\theta}, s_{\hbar} \rangle = \frac{1}{\mathcal{N}_{\hbar}} \sum_{j=1}^{\mathcal{N}_{\hbar}} e^{in\theta_j} = \frac{1}{\mathcal{N}_{\hbar}} \sum_{j=1}^{\mathcal{N}_{\hbar}} \left(1 - \frac{\rho_j}{r}\right) e^{in\theta_j} + \frac{1}{\mathcal{N}_{\hbar}} \sum_{j=1}^{\mathcal{N}_{\hbar}} \frac{\rho_j}{r} e^{in\theta_j}$$

Since we have

$$\sum_{j=1}^{\mathcal{N}_{\hbar}} \rho_j e^{in\theta_j} = \text{Tr} \hat{\mathcal{F}}_{\hbar}^n,$$

we see

$$|\langle e^{in\theta}, s_{\hbar} \rangle| \leq \frac{1}{\mathcal{N}_{\hbar}} \sum_{j=1}^{\mathcal{N}_{\hbar}} \left|1 - \frac{\rho_j}{r}\right| + \frac{1}{r \mathcal{N}_{\hbar}} \left| \text{Tr} \hat{\mathcal{F}}_{\hbar}^n \right|.$$

3130 From the accumulation of the moduli of eigenvalues to r , proved in the last section, we have

$$\frac{1}{\mathcal{N}_{\hbar}} \sum_{j=1}^{\mathcal{N}_{\hbar}} \left|1 - \frac{\rho_j}{r}\right| \xrightarrow{\hbar \rightarrow 0} 0.$$

Therefore it is enough to show, for each fixed $n > 0$, that

$$\frac{1}{\mathcal{N}_{\hbar}} \left| \text{Tr} \hat{\mathcal{F}}_{\hbar}^n \right| \xrightarrow{\hbar \rightarrow 0} 0. \quad (10.16)$$

3135 From the results given in Subsection 1.7, we see that the trace of $\hat{\mathcal{F}}_{\hbar}^n = \Pi_{\hbar} \circ \hat{F}_{\hbar}^n \circ \Pi_{\hbar}$ is expressed as the sum of contributions from its restriction to neighborhoods of the finite number of fixed points $x = f^n(x)$ of f^n and obtain (10.16). We finished the proof of Theorem 1.23.

11 Proof of Th. 1.38. Gutzwiller trace formula.

11.1 The Atiyah-Bott trace formula

In this subsection, we recall the Atiyah-Bott trace formula in a general setting[4, cor.5.4,p.393]:

3140

Definition 11.1. “Flat Trace of a transfer operator”. Suppose that $f : M \rightarrow M$ is a smooth diffeomorphism on a manifold M whose periodic points are all hyperbolic, that $\pi : E \rightarrow M$ is a vector bundle and that $B : E \rightarrow E$ a vector bundle map projecting on f , i.e. such that the following diagram commutes:

$$\begin{array}{ccc} E & \xrightarrow{B} & E \\ \pi \downarrow & & \downarrow \pi \\ M & \xrightarrow{f} & M \end{array}$$

We can define the associated **transfer operator** acting on smooth sections of this vector bundle

$$\hat{F} : C^\infty(M, E) \rightarrow C^\infty(M, E), \quad (\hat{F}u)(x) := B_{f^{-1}(x)}(u(f^{-1}(x))) \quad (11.1)$$

For any $n \geq 1$, the **flat trace of the transfer operator** \hat{F}^n is defined by

$$\text{Tr}^b(\hat{F}^n) := \int_M \text{Tr}(K_n(x, x)) dx \quad (11.2)$$

where $K_n(x, y) dy \in \mathcal{L}(E_y \rightarrow E_x)$ denotes the Schwartz kernel of \hat{F}^n .

For $n \geq 1$, let $B_x^{(n)} := \prod_{k=0}^{n-1} B_{f^k(x)} : E_x \rightarrow E_{f^n(x)}$. For a periodic point $x = f^n(x)$ then $B_x^{(n)}$ is an endomorphism on E_x and $\text{Tr}(B_x^{(n)})$ is well defined and does not depend on x on the orbit.

3145

Lemma 11.2. For any $n \geq 1$, the **Atiyah-Bott trace formula** reads:

$$\text{Tr}^b(\hat{F}^n) = \sum_{x=f^n(x)} \frac{\text{Tr}(B_x^{(n)})}{|\det(I - Df_x^n)|} \quad (11.3)$$

This is a finite sum over periodic points of f with period n .

Proof. (As in [4, cor.5.4,p.393]). From (11.1) we have

$$\left(\left(\hat{F}^n u \right) (x) \right)_i = \left(B_{f^{-n}(x)}^{(n)} (u (f^{-n}(x))) \right)_i = \sum_j \int_M \delta_{f^{-n}(x)}(y) (B_y^{(n)})_{i,j} u_j(y) dy$$

where $i, j = 1 \dots \dim E$ are indices for components in the fibers E_y with respect to some local trivialization and δ_x is the Dirac distribution at x . So the Schwartz kernel of \hat{F}^n is

$$K_n(x, y) = B_y^{(n)} \delta_{f^{-n}(x)}(y)$$

3150 hence

$$\mathrm{Tr}(K_n(x, x)) = \delta_{f^{-n}(x)}(x) \mathrm{Tr} B_x^{(n)}.$$

From definition (11.2),

$$\begin{aligned} \mathrm{Tr}^b(\hat{F}^n) &= \int_M \delta(x - f^{-n}(x)) \mathrm{Tr} B_x^{(n)} dx \\ &= \sum_{x=f^n(x)} \mathrm{Tr} B_x^{(n)} \frac{1}{|\det(I - Df_x^{-n})|} = \sum_{x=f^n(x)} \frac{\mathrm{Tr} B_x^{(n)}}{|\det(I - Df_x^n)|} \end{aligned}$$

where in the second line we have used the change of variable $x \rightarrow y = x - f^{-n}(x)$ in the vicinity of $y = 0$. For the last equality we have used that $\det Df_x^n = 1$ since f preserves the volume form. \square

3155 *Remark 11.3.* A standard example of bundle map is the differential map $Df : TM \rightarrow TM$ or extension in tensor bundles as $f_{(k)} : \Lambda^k(T^*M) \rightarrow \Lambda^k(T^*M)$ (in antisymmetric tensor bundle of order k that we will use), etc.

11.2 The Gutzwiller Trace formula from the Atiyah-Bott trace formula

3160 In this subsection we show how the Gutzwiller trace formula can be expressed as a sum of flat traces of transfer operators acting on differential forms on the Grassmann extension G . We first extend the argument in the last section to the case of smooth diffeomorphisms on the Grassmann bundle G . This is rather trivial. We just replace the manifold M and the diffeomorphism f by the Grassman bundle G and the natural extension f_G of f .
3165 Lemma 11.2 remains true for such extension. Below we will always truncate the transfer operators on G to a small neighborhood K_0 of the attracting section E_u which represents the unstable subbundle. (Recall the argument in Subsection 1.4.3.) So, on the right hand side of (11.3), the sum will be over only the periodic points of f_G contained in K_0 . Those periodic points must be contained in E_u and in one-to-one correspondence to the periodic
3170 points for $f : M \rightarrow M$.

The next lemma shows how the amplitude $|\det(1 - Df_x^n)|^{-1/2}$ appear and the following corollary shows how the phases $e^{iS_{n,x}/\hbar}$ appear. For $l \in G$, we will denote

$$V_l G := (T_l G)_{\mathrm{vert}} = \ker Dp_l,$$

the vertical part of the tangent space. Then $VG = \cup_{l \in G} V_l G$ is a smooth subbundle of TG .

Lemma 11.4. For $0 \leq k \leq d$, $0 \leq m \leq d^2$, let $\Lambda^{(k,m)} \rightarrow G$ be the vector bundle with fiber $\Lambda^{(k,m)}(l) := \Lambda^k(T_l G) \otimes \Lambda^m(V_l G)$ over $l \in G$ and let $F_{k,m}$ be the transfer operator $F_{k,m} : C^\infty(G, \Lambda^{(k,m)}) \rightarrow C^\infty(G, \Lambda^{(k,m)})$ defined by

$$F_{k,m}u(l) = \chi \circ p(f_G^{-1}(l)) \cdot \frac{|\det(Df_x|l)|^{1/2}}{|\det(Df_G|_{V_l G}(l))|} \left(\Lambda^k(Df_{/l}^{-1}) \otimes \Lambda^m(Df|_{V_l G}) \right) u(f_G^{-1}(l)) \quad (11.4)$$

where $\chi : G \rightarrow [0, 1]$ is a smooth function such that $\chi(l) = 0$ for $l \notin K_0$ and $\chi(l) = 1$ for $l \in f_G(K_0)$ and $p : G \rightarrow M$ is the projection. Then the Atiyah-Bott trace formula gives:

$$\sum_{x=f^n(x)} \frac{1}{\sqrt{|\det(1 - Df_x^n)|}} = \sum_{k=0}^d \sum_{m=0}^{d^2} (-1)^{k+m} \text{Tr}^b(F_{k,m}^n) \quad (11.5)$$

Proof. We first derive an useful expression for the differential of the map $f_G : G \rightarrow G$. For $x \in M$, we have a continuous decomposition of the symplectic tangent space

$$T_x M = E_u(x) \oplus E_s(x),$$

where $E_u(x)$ and $E_s(x)$ are linear Lagrangian space. We will use the fact that the symplectic form ω provides an isomorphism between these complementary Lagrangian subspaces

$$E_u(x) \stackrel{\omega}{\cong} E_s^*(x) \quad (11.6)$$

by

$$U \in E_u(x) \longleftrightarrow \omega(U, \cdot) \in E_s^*(x)$$

The symplectic map $f : M \rightarrow M$ gives a symplectic hyperbolic linear map

$$Df_x : T_x M \rightarrow T_{f(x)} M$$

which decomposes accordingly into

$$Df_x = L_u(x) \oplus L_s(x) \quad (11.7)$$

where

$$L_u(x) := Df_x|_{E_u(x)} : E_u(x) \rightarrow E_u(f(x))$$

$$L_s(x) := Df_x|_{E_s(x)} : E_s(x) \rightarrow E_s(f(x))$$

The map $L_u(x)$ is expanding while $L_s(x)$ is contracting. Similarly in the cotangent bundle (with the usual convention here such as $E_u^*(E_s) = 0$), we have

$${}^t L_u^{-1}(x) := {}^t Df_x^{-1}|_{E_u^*(x)} : E_u^*(x) \rightarrow E_u^*(f(x))$$

$${}^t L_s^{-1}(x) \equiv {}^t Df_x^{-1} |_{E_s^*(x)}: E_s^*(x) \rightarrow E_s^*(f(x))$$

For $x \in M$, the graph of a linear map

$$\mathcal{L}: E_u(x) \rightarrow E_s(x)$$

3190 defines an element $l \in G_x$. In particular the linear map $\mathcal{L} \equiv 0$ is associated to the subspace $E_u(x) \in G_x$. In tensorial notations we have that

$$\mathcal{L} \in (E_u^*(x) \otimes E_s(x)) \quad (11.8)$$

Conversely, there is a neighborhood of $E_u(x)$ in G_x such that every element $l \in G_x$ in this neighborhood can be expressed as the graph of a such linear map \mathcal{L} . The map $f_G: G \rightarrow G$ defined in (1.38), is linear in the fiber and has derivative

$$Df_G |_{\ker p} \equiv \underbrace{\left({}^t L_u^{-1}(x) \otimes L_s(x) \right)}_{Df_G |_{V_i G}} \quad (11.9)$$

3195 Now let $x = f^n(x)$ be a periodic point. The isomorphism (11.6) gives

$$L(x) := L_u(x) \stackrel{\omega}{\equiv} {}^t L_s^{-1}(x) \quad (11.10)$$

We will note

$$L^{(n)}(x) := \prod_{k=0}^{n-1} L(f^k(x)), \quad L^{-(n)}(x) := (L^{(n)}(x))^{-1}.$$

We first express the ‘‘Gutzwiller amplitude’’ using (11.7) and (11.10) as follows:

$$\begin{aligned} \mathcal{B}_x &:= \frac{1}{\sqrt{|\det(1 - Df_x^n)|}} = \frac{1}{\left| \det(1 - L_u^{(n)}(x)) \right|^{1/2} \left| \det(1 - L_s^{(n)}(x)) \right|^{1/2}} \quad (11.11) \\ &= \frac{\left| \det(L_u^{(n)}(x)) \right|^{-1/2}}{\left| \det(1 - L_u^{-(n)}(x)) \right|^{1/2} \left| \det(1 - L_s^{(n)}(x)) \right|^{1/2}} \\ &= \left| \det(L^{-(n)}) \right|^{1/2} \cdot \frac{1}{\left| \det(1 - L^{-(n)}) \right|} \end{aligned}$$

Notice that the last expression is presented in such a way that the first factor is a multiplicative cocycle and the second one converges to 1 as $n \rightarrow +\infty$. Let us consider the map 3200 $f_G: G \rightarrow G$ and the transfer operator (11.4) with $(k, m) = (0, 0)$:

$$(F_{0,0}u) = |\det(Df_x(l))|^{1/2} \cdot |\det(Df_G |_{V_i G}(l))| \cdot u \circ f_G^{-1}$$

The Atiyah-Bott trace formula reads

$$\mathrm{Tr}^b(F_{0,0}^n) = \sum_{x=f^n(x)} \mathcal{A}_x \quad (11.12)$$

with the amplitude

$$\mathcal{A}_x := \left(\frac{|\det(Df_x^n(E_u(x)))|^{1/2}}{|\det(Df_G^n(E_u(x))|_{\ker p})|} \right) \frac{1}{\left| \det \left(1 - \left(D\tilde{f}_G^{-n} \right) (E_u(x)) \right) \right|}$$

From (11.9) and (11.10), we can express \mathcal{A}_x as

$$\begin{aligned} \mathcal{A}_x &= (|\det(1 - L_u^{-(n)})| |\det(1 - L_s^{-(n)})| |\det(1 - {}^t L_u^{(n)} \otimes L_s^{-(n)})|)^{-1} \left(\frac{(\det L_u^{(n)})^{1/2}}{|\det({}^t L_u^{-(n)} \otimes L_s^{(n)})|} \right) \\ &= (|\det(1 - L^{-(n)})| |\det(1 - L^{(n)})| |\det(1 - L^{(n)} \otimes L^{(n)})|)^{-1} \left(\frac{(\det L^{(n)})^{1/2}}{|\det(L^{-(n)} \otimes L^{-(n)})|} \right) \\ &= |\det(L^{-(n)})|^{1/2} |\det(1 - L^{-(n)})|^{-2} |\det(1 - L^{-(n)} \otimes L^{-(n)})|^{-1} \end{aligned}$$

3205 Comparing the expression of \mathcal{A}_x and \mathcal{B}_x we see that they have the same leading behavior for $n \rightarrow +\infty$. More precisely:

$$\mathcal{B}_x = \mathcal{A}_x |\det(1 - L^{-(n)})| |\det(1 - L^{-(n)} \otimes L^{-(n)})|$$

Recall that we aim to express \mathcal{B}_x as a sum of "amplitudes" related to some transfer operators. By construction, the amplitude \mathcal{A}_x already comes from a transfer operator. Therefore we would like to express the remaining factor $|\det(1 - L^{-(n)})| |\det(1 - L^{-(n)} \otimes L^{-(n)})|$ as a sum of "cocycles" related to some potentials as in (11.3). For this purpose we use
3210 some relations of linear algebra [42, p.396]: if $M : \mathbb{R}^m \rightarrow \mathbb{R}^m$ is a linear endomorphism and $\Lambda^k(M)$ denotes its natural action on the antisymmetric tensor algebra $\Lambda^k(\mathbb{R}^m)$ (with $k \in \{0, \dots, m\}$) then

$$\det(1 + M) = \sum_{k=0}^m \mathrm{Tr}(\Lambda^k(M)), \quad \Lambda^k(-M) = (-1)^k \Lambda^k(M), \quad \Lambda^k(M^n) = (\Lambda^k(M))^n.$$

Also for two linear endomorphisms M_1, M_2 one has $\mathrm{Tr}(M_1 \otimes M_2) = \mathrm{Tr}(M_1) \mathrm{Tr}(M_2)$ and $M_1^n \otimes M_2^n = (M_1 \otimes M_2)^n$. Using this, we get:

$$\begin{aligned} |\det(1 - L^{-(n)})| |\det(1 - L^{-(n)} \otimes L^{-(n)})| &= \left(\sum_{k=0}^d (-1)^k \mathrm{Tr}(\Lambda^k(L^{-(n)})) \right) \\ &\quad \left(\sum_{m=0}^{d^2} (-1)^m \mathrm{Tr}(\Lambda^m(L^{-(n)} \otimes L^{-(n)})) \right) \\ &= \sum_{k=0}^d \sum_{m=0}^{d^2} (-1)^{k+m} \mathrm{Tr}(\Lambda^k(L^{-(n)}) \otimes \Lambda^m(L^{-(n)} \otimes L^{-(n)})) \end{aligned}$$

And therefore

$$\sum_{x=f^n(x)} \mathcal{B}_x = \sum_{k=0}^d \sum_{m=0}^{d^2} (-1)^{k+m} \left(\sum_{x=f^n(x)} \mathcal{A}_x \text{Tr} (\Lambda^k (L^{-(n)}) \otimes \Lambda^m (L^{-(n)} \otimes L^{-(n)})) \right)$$

On the other hand, in the same way we derived (11.12), we check that the Atiyah-Bott trace formula of (11.4) reads:

$$\text{Tr}^b (F_{k,m}^n) = \sum_{x=f^n(x)} \mathcal{A}_x \text{Tr} (\Lambda^k (L^{-(n)}) \otimes \Lambda^m (L^{-(n)} \otimes L^{-(n)}))$$

We have obtained (11.5) and finished the proof of Lemma 11.4. □

3220 The next corollary extends Lemma 11.4 to the prequantum transfer operators with an arbitrary potential function \tilde{V} restricted to the N -th Fourier mode. But notice that, in the statement below, we define the Atiyah-Bott trace of the prequantum transfer operators (restricted to the N -th Fourier mode) by using their expressions in local charts, rather than applying Definition 11.1 naively. (See Remark 11.6 below.)

3225 **Corollary 11.5.** *Let $\tilde{V} \in C^\infty (G)$. Let $\Lambda^{(k,m)} \rightarrow P_G$ be the bundle $\Lambda^{(k,m)} \rightarrow G$ pulled back by $\pi_G : P_G \rightarrow G$. Let $\tilde{F}_{\tilde{V},k,m} : C^\infty (P_G, \Lambda^{(k,m)}) \rightarrow C^\infty (P_G, \Lambda^{(k,m)})$ be the (vector-valued) prequantum transfer operator defined by*

$$\tilde{F}_{\tilde{V},k,m} u(p) = \chi \circ p(f_G^{-1}(l)) \cdot \frac{e^{\tilde{V}(l)}}{|\det (D\tilde{f}_G |_{V_l G}(l))|} \left(\Lambda^k (Df_{\tilde{V}}^{-1}) \otimes \Lambda^m (D\tilde{f} |_{V_l G}) \right) u \left(\tilde{f}_G^{-1}(p) \right) \tag{11.13}$$

where we set $l = \pi_G(p)$. Let

$$\left(\tilde{F}_{\tilde{V},k,m} \right)_N : C_N^\infty (P_G, \Lambda^{(k,m)}) \rightarrow C_N^\infty (P_G, \Lambda^{(k,m)}) \tag{11.14}$$

be its restriction to N -equivariant functions. Then

$$\sum_{x=f^n(x)} \frac{e^{(V-V_0)_n(x)} e^{iS_{n,x}/\hbar}}{\sqrt{|\det (1 - Df_x^n)|}} = \sum_{k=0}^d \sum_{m=0}^{d^2} (-1)^{k+m} \text{Tr}^b \left(\tilde{F}_{\tilde{V},k,m} \right)_N^n \tag{11.15}$$

where $V_0(x) = \frac{1}{2} \log (\det Df_x |_{E_u(x)})$ is the potential of reference.

3230 *Remark 11.6.* As we noted above, we define the Atiyah-Bott trace of prequantum transfer operators by using its local expression. (If we adopt the “line bundle terminology” referred in Remark 1.13, this definition coincides with the Atiyah-Bott trace in Definition 11.1 with $E = L^{\otimes N}$.) For simplicity, let us consider the case $\tilde{F}_{\tilde{V},0,0}$ which acts on functions on P_G and let $\left(\tilde{F}_{\tilde{V},0,0} \right)_N$ be its restriction to the N -th Fourier mode. As we observed in

3235 Proposition 9.24 in Section 9, this operator $\left(\tilde{F}_{\tilde{V},0,0}\right)_N$ is lifted to a matrix of operators, denoted by $\tilde{\mathbf{F}}_{\hbar}$. Each component $\tilde{\mathbf{F}}_{j,i}$ of $\tilde{\mathbf{F}}_{\hbar}$ are simple transfer operators on Euclidean trace and the Atiyah-Bott trace is defined by Definition 11.1. We define the Atiyah-Bott trace of $\left(\tilde{F}_{\tilde{V},0,0}\right)_N$ as

$$\mathrm{Tr}^b\left(\tilde{F}_{\tilde{V},0,0}\right)_N := \sum_{i=1}^{I_{\hbar}} \mathrm{Tr}^b\left(\tilde{\mathbf{F}}_{i,i}\right)$$

3240 taking the sum of the flat traces of the diagonal elements. This definition actually does not depend on the choice of local trivializations. Also the extension to the case of vector-valued transfer operators such as $\left(\tilde{F}_{\tilde{V},k,m}\right)_N$ is straightforward.

3245 *Proof of Corollary 11.5.* Observe that we can go through the proof of Lemma 11.4 by considering the expressions of the transfer operators on local charts, as we noted in the remark above. Then, recalling Remark 1.8, we see that the action $e^{iS_{n,x}/\hbar}$ in the left hand side of (11.15) appears as a consequence of the rotation on the fiber that the prequantum map \tilde{f}_G^n induces. \square

11.3 Restriction to the external band

We will now relate the previous quite formal “flat trace formula” (11.15) with the spectrum of the transfer operators. This relation is given by the following result obtained in [7].

3250

Theorem 11.7. “Flat trace and spectrum”. *Let $f : M \rightarrow M$ be a smooth Anosov diffeomorphism. Suppose that the transfer operator \hat{F} is defined from the general setting (11.1). Let $(\lambda_j)_{j \in \mathbb{N}}$ denote its Ruelle discrete spectrum. Then, for any $\varepsilon > 0$, there exists $C_{\varepsilon} > 0$ such that for any $n > 0$,*

$$\left| \mathrm{Tr}^b \hat{F}^n - \sum_{j, |\lambda_j| \geq \varepsilon} \lambda_j^n \right| \leq C_{\varepsilon} \varepsilon^n. \quad (11.16)$$

3255 In our case, we have a family of operators $\left(\hat{F}_{\tilde{V},k,m}\right)_N$ depending on the semiclassical parameter $N = 1/(2\pi\hbar)$ and we want to get a result similar to (11.16) but with a control of the remainder uniformly with respect to N . This is the purpose of the next lemma which concerns the operator $\left(\hat{F}_{\tilde{V},k,m}\right)_N$ defined in (11.13) but with the particular value $(k, m) = (0, 0)$. It coincides with the transfer operator \tilde{F}_N defined in (1.44):

$$\left(\hat{F}_{\tilde{V},0,0}\right)_N = \tilde{F}_N$$

Recall that the quantum operator $\tilde{\mathcal{F}}_h : \mathcal{H}_h \rightarrow \mathcal{H}_h$ has been defined from \tilde{F}_N in (1.33) as its spectral restriction to the external band. $\tilde{\mathcal{F}}_h$ is finite rank so its trace $\text{Tr} \left(\tilde{\mathcal{F}}_h^n \right)$ below is well defined.

3260 **Lemma 11.8.** *For any $\varepsilon > 0$, there exists C_ε and N_ε such that, for any $N > N_\varepsilon$ and any $n > 0$, we have*

$$\left| \text{Tr}^b \left(\tilde{F}_N^n \right) - \text{Tr} \left(\tilde{\mathcal{F}}_h^n \right) \right| \leq C_\varepsilon N^d (r_1^+ + \varepsilon)^n \quad (11.17)$$

Proof. We adapt the methods presented in [7] to our settings. Let $\Pi_0 : \mathcal{H}_N^r(P_G) \rightarrow \mathcal{H}_h$ denotes the finite rank spectral projector on the external band of \tilde{F}_N defined for N large enough. Note that it commutes with \tilde{F}_N by definition: $[\Pi_0, \tilde{F}_N] = 0$. Let

$$\tilde{F}_h := \Pi_0 \tilde{F}_N \Pi_0 \quad \text{and} \quad \tilde{\mathcal{R}}_h := (1 - \Pi_0) \tilde{F}_N (1 - \Pi_0).$$

3265 Then, for any $n \geq 1$, we have

$$\tilde{F}_N^n = \tilde{\mathcal{F}}_h^n + \tilde{\mathcal{R}}_h^n. \quad (11.18)$$

and hence

$$\text{Tr}^b \left(\tilde{F}_N^n \right) = \text{Tr} \left(\tilde{\mathcal{F}}_h^n \right) + \text{Tr}^b \left(\tilde{\mathcal{R}}_h^n \right). \quad (11.19)$$

We first recall some estimates related to the decomposition (11.18). From (1.30) we have

$$\left\| \tilde{\mathcal{F}}_h \right\|_{\text{Tr}} \leq CN^d \quad (11.20)$$

3270 with C independent on N . From (1.62), we have, for every $N > N_\varepsilon$ with some N_ε large enough:

$$\left\| \tilde{\mathcal{R}}_h^n \right\| \leq C_\varepsilon (r_1^+ + \varepsilon)^n \quad (11.21)$$

with C_ε independent on N and n . The following Lemma is central in the argument, whose proof is postpone for a moment.

Lemma 11.9. *There exists $C_\varepsilon > 0$ and $N_\varepsilon > 0$ such that for any $N > N_\varepsilon$ and any $n > 0$*

$$\left| \text{Tr}^b \left(\tilde{\mathcal{R}}_h^n \right) \right| \leq C_\varepsilon \cdot N^d (r_1^+ + \varepsilon)^n \quad (11.22)$$

From (11.22) and (11.19), we obtain (11.17), finishing the proof of Lemma 11.8. \square

3275

Proof of Lemma 11.9. The proof is obtained following the strategy presented in [7]. Here we explain how the uniform estimate on the remainder term with respect to the semi-classical parameter N (or \hbar) is obtained. In the argument below, we discuss about transfer operators on local charts and local trivializations. (Remind that we defined the Atiyah-Bott trace for prequantum transfer operators using local charts. See Remark 11.6.) But, 3280 for simplicity, we still write the operators on local charts by \tilde{F}_N abusively. (Moreover we

will confuse the objects on local coordinates and the corresponding global objects.) It is not difficult to put the following schematic argument into rigorous one. We refer the paper [7] for the detail.

3285 We consider the lifted operator on the phase space, $\tilde{F}_N^{\text{lift}} := \mathcal{B}_{(x,s)} \circ \tilde{F}_N \circ \mathcal{B}_{(x,s)}^*$, and decompose it into two parts:

$$\tilde{F}_N^{\text{lift}} = F_{\text{trace-free}} + F_{\text{trace}} \quad (11.23)$$

so that we get properties given in Lemma 11.10 below. Note that the decomposition (11.23) is not a spectral decomposition, like (11.18), but is obtained from a “phase space decomposition”. To define it, recall the weight (or escape) function $\mathcal{W}_h^r(x, s, \xi_x, \xi_s)$ defined in (5.22). For simplicity we write $z = (x, s, \xi_x, \xi_s) \in T^*M$. Let $K_F(z', z)$ be the Schwartz kernel of the lifted operator on the phase space $\tilde{F}_N^{\text{lift}} = \mathcal{B}_{(x,s)} \circ \tilde{F}_N \circ \mathcal{B}_{(x,s)}^*$, so that

$$\left(\tilde{F}_N^{\text{lift}} u\right)(z') = \int K_F(z', z) u(z) dz.$$

The main property of the escape function $\mathcal{W}_h^r(z)$ (which we have already made use of) is that there exists a compact neighborhood

$$U = \left\{ z = (x, \xi) \in T^*M, \quad |\zeta| \leq C\sqrt{h} \right\}$$

of the “trapped set” $\tilde{K} = \{z \in T^*G \mid \zeta = 0, s = 0\}$ such that

$$(z', z) \notin (U \times U) \Rightarrow \left| \frac{\mathcal{W}_h^r(z')}{\mathcal{W}_h^r(z)} K_F(z', z) \right| \leq \lambda^{-r}$$

3295 where ζ is the coordinate introduced in (9.3) (and Proposition 2.15).

Let $X \subset T^*M \times T^*M$ be the subset defined by

$$X := \left\{ (z', z) \in T^*M \times T^*M, \quad \mathcal{W}_h^r(z') \leq (\lambda/2)^{-r} \cdot \mathcal{W}_h^r(z) \right\} \quad (11.24)$$

and let $\mathbf{1}_X(z', z)$ be the characteristic function of the set X . We define the operator $F_{\text{trace-free}}$ by its integral expression:

$$(F_{\text{trace-free}} u)(z') := \int \mathbf{1}_X(z', z) K_F(z', z) u(z) dz$$

3300 In other words, we define $F_{\text{trace-free}}$ and F_{trace} as operators with the Schwartz kernel respectively $\mathbf{1}_X K_F$ and $(1 - \mathbf{1}_X) K_F$.

Lemma 11.10. *The operator $F_{\text{trace-free}}$ satisfies*

$$\text{Tr}^b(F_{\text{trace-free}}^n) = 0, \quad \text{and} \quad (11.25)$$

$$\|F_{\text{trace-free}}^n\| \leq C(\lambda/2)^{-rn} \cdot \left(\sup_{l \in K_0} \left(e^{\tilde{V}(l)} |\det(df_{p(l)}|_l)|^{1/2} \cdot |\det df_G^n|_{V_G(l)}|^{-1/2} \right) \right) \quad (11.26)$$

for any $n \geq 1$, with some $C > 0$ independent on n or N . The operator F_{trace} is of trace class and we have

$$\|F_{\text{trace}}\|_{\text{Tr}} \leq CN^d, \quad (11.27)$$

3305 with some $C > 0$ independent on N .

Proof. From the definition (11.24) of the subset X , it is clear that for any sequence of points $z_0, z_1, \dots, z_n \in T^*M$ with $z_0 = z_n$, we have

$$\mathbf{1}_X(z_n = z_0, z_{n-1}) \cdot \mathbf{1}_X(z_{n-1}, z_{n-2}) \cdots \mathbf{1}_X(z_1, z_0) = 0.$$

This implies that the Schwartz kernel of $F_{\text{trace-free}}^n$ vanishes on the diagonal and hence we have $\text{Tr}^b(F_{\text{trace-free}}^n) = 0$. For the second claim (11.26), we first observe that, if we
 3310 consider the action of this operator $F_{\text{trace-free}}^n$ with respect to the L^2 -norm (without the escape function $\mathcal{W}_h^r(\cdot)$), the operator norm is bounded by the right hand side of (11.26) without the term $C(\lambda/2)^{-rn}$. (This is because this is true for operator \tilde{F}_N and that $\mathcal{B}_{(x,s)}$ and $\mathcal{B}_{(x,s)}^*$ does not increase the L^2 norm.) Then, taking the escape function $\mathcal{W}_h^r(\cdot)$ into account and noting the definition of the subset X , we retain²⁴ the factor $C(\lambda/2)^{-rn}$ and
 3315 obtain (11.26). For (11.27), we observe that

$$\|F_{\text{trace}}\|_{\text{Tr}} \leq \int |(z - \mathbf{1}_X(z)) K_F(z, z)| dz$$

and then show that, for any $\nu > 0$, there exists a constant $C_\nu > 0$, which may depend on f_G but uniform for sufficiently large N , such that

$$|K_F(z, z)| \leq C_\nu \cdot \langle \hbar^{-1/2} \zeta \rangle^{-\nu}.$$

The last estimate is obtained by expressing the kernel $K_F(\cdot)$ as an (oscillatory) integral and then by applying integration by parts. (See the proofs of Proposition 6.19 and Proposition
 3320 6.21 for similar estimates.) Then we obtain (11.27). \square

We pursue the proof of Lemma 11.9. Using (11.18), we write

$$\text{Tr}^b(\tilde{F}_N^n) = \text{Tr}(\tilde{\mathcal{F}}_h^n) + \text{Tr}^b(\tilde{\mathcal{R}}_h^n)$$

Then (11.23) gives $\tilde{\mathcal{R}}_h^{\text{lift}} = F_{\text{trace-free}} + (F_{\text{trace}} - \tilde{\mathcal{F}}_h^{\text{lift}})$. We develop accordingly

$$\left(\tilde{\mathcal{R}}_h^{\text{lift}}\right)^n = F_{\text{trace-free}}^n + \sum_{k=0}^{n-1} F_{\text{trace-free}}^k \left(F_{\text{trace}} - \tilde{\mathcal{F}}_h^{\text{lift}}\right) \left(\tilde{\mathcal{R}}_h^{\text{lift}}\right)^{n-k-1}$$

²⁴For a rigorous proof of this, we can employ an argument similar to that in the proof of Proposition 6.21, using partition of unity on the phase space. For the detail we refer [7].

From (11.25), we know $\mathrm{Tr}^b(F_{\mathrm{trace-free}}^n) = 0$ for the first term. The operator $(F_{\mathrm{trace}} - \tilde{\mathcal{F}}_h^{\mathrm{lift}})$ is in the trace class and using the general fact $\|AB\|_{\mathrm{Tr}} \leq \|A\| \cdot \|B\|_{\mathrm{Tr}}$, we obtain

$$\left| \mathrm{Tr}^b(\tilde{\mathcal{R}}_h^n) \right| \leq \sum_{k=0}^{n-1} \|F_{\mathrm{trace-free}}^k\| \left\| F_{\mathrm{trace}} - \tilde{\mathcal{F}}_h^{\mathrm{lift}} \right\|_{\mathrm{Tr}} \left\| \tilde{\mathcal{R}}_h \right\|^{n-k-1}.$$

3325 From (11.27) and (11.20), we have

$$\left\| F_{\mathrm{trace}} - \tilde{\mathcal{F}}_h^{\mathrm{lift}} \right\|_{\mathrm{Tr}} \leq \|F_{\mathrm{trace}}\|_{\mathrm{Tr}} + \left\| \tilde{\mathcal{F}}_h \right\|_{\mathrm{Tr}} \leq CN^d.$$

By taking large r , we may and do assume that the right hand side of (11.26) is bounded by $C(r_1^+)^n$. Using these estimates and (11.21), we conclude

$$\left| \mathrm{Tr}^b(\tilde{\mathcal{R}}_h^n) \right| \leq CN^d \cdot n \cdot (r_1^+ + \varepsilon)^n.$$

This implies (11.22). We have finished the proof of Lemma 11.9. \square

3330 In order to finish the proof of Theorem 1.38, we have to consider the remaining terms on the right hand side of (11.15), that is,

$$(-1)^{k+m} \mathrm{Tr}^b \left(\tilde{F}_{\tilde{V},k,m} \right)_N^n \quad \text{for } (k, m) \neq (0, 0).$$

This is actually easier once we have done with the case $(k, m) = (0, 0)$. Recall the definition (11.13) of the operator $\tilde{F}_{\tilde{V},k,m}$ and observe that the extra term $\left(\Lambda^k \left(Df_{\mu}^{-1} \right) \otimes \Lambda^m \left(D\tilde{f}|_{V_iG} \right) \right)$ (compared with the case $(k, m) = (0, 0)$) is bounded in norm by $\|Df_{\mu}^{-1}\| < 1/\lambda < 1$. This observation gives the next lemma.

3335 **Lemma 11.11.** *Consider the transfer operator $\left(\hat{F}_{\tilde{V},k,m} \right)_N$ defined in (11.14). For any $\varepsilon > 0$, there exists a constant $C_\varepsilon > 0$ and $N_\varepsilon > 0$ such that*

$$\left| \mathrm{Tr}^b \left(\hat{F}_{\tilde{V},k,m} \right)_N^n \right| \leq C_\varepsilon N^d \cdot (r_1^+ + \varepsilon)^n \tag{11.28}$$

holds for any $N > N_\varepsilon$, $n \geq 1$ and any $(k, m) \neq (0, 0)$.

Proof. A trivial extension of Theorem 1.35 to the vector-valued case, gives the estimate

$$\left\| \left(\tilde{F}_{\tilde{V},k,m}^n \right)_N \right\| \leq C \sup_{x \in M} \left(e^{D_n(x)} \left\| \Lambda^k \left(Df_{\mu}^{-n} \right) \otimes \Lambda^m \left(D\tilde{f}^n|_{V_iG} \right) \right\| \right) \leq C_\varepsilon (r_1^+ + \varepsilon)^n$$

3340 where $D_n(x) = \sum_{j=1}^n D(f_G^j(x))$ is the Birkhoff sum of the damping factor $D(x) := V(x) - V_0(x)$. Then the argument in the proof of Lemma 11.9, using a “phase space decomposition” of $\left(\tilde{F}_{\tilde{V},k,m} \right)_N$ into the trace-free term and the trace class term similar to (11.23), enables us to obtain (11.28). \square

Finally, from (11.15), (11.28) and (11.17), we get

$$\left| \operatorname{Tr}(\mathcal{F}_h^n) - \sum_{x=f^n(x)} \frac{e^{D_n(x)} e^{iS_{n,x}/\hbar}}{\sqrt{|\det(1 - Df_x^n)|}} \right| \leq C_\varepsilon \cdot N^d \cdot (r_1^+ + \varepsilon)^n.$$

We have completed the proof of Th 1.38.

3345 12 Proof of Th. 1.69.

The argument of the proof proceed in two steps. First we prove the result for linear hyperbolic maps. We obtain an “exact result” in that case. Then using the same argument as for the main results of this paper we have that, using a partition of unity at small size, we can use the linear case as an approximation and get Theorem 1.69, where the error comes from estimates on non-linearities.

The function $\mathcal{M}(x)$ in Theorem 1.69 will be obtained in (12.2) below with the setting $A = Df|_{E_u(x)}$.

The quantum operator: We suppose that $f : \mathbb{R}^{2d} \rightarrow \mathbb{R}^{2d}$ is a linear symplectic hyperbolic map and that we have constructed the prequantum operator $\hat{F}_N = \mathcal{L}_f$ as in Section 5.3. By definition, the quantum operator is the restriction of \mathcal{L}_f to the outmost band:

$$\hat{\mathcal{F}}_h := \mathcal{L}_f|_{H'_0}$$

The quantum space is:

$$\mathcal{H}_h := H'_0$$

Proposition 4.9 and Eq. (4.22) give that

$$\hat{F}_N = \mathcal{U} \circ (L_A \otimes L_A) \circ \mathcal{U}^{-1}$$

and

$$\begin{aligned} \hat{\mathcal{F}}_h = \mathcal{L}_f|_{H'_0} &= \mathcal{U} \circ (L_A \otimes (T^{(0)} \circ L_A \circ T^{(0)})) \circ \mathcal{U}^{-1} \\ &= |\det(A)|^{-1/2} \cdot \mathcal{U} \circ (L_A \otimes T^{(0)}) \circ \mathcal{U}^{-1} \end{aligned}$$

In the last line we have used the fact that $T^{(0)}$ has rank one and that $(T^{(0)} L_A T^{(0)}) = |\det(A)|^{-1/2} T^{(0)}$.

The Laplacian operator: We suppose that g is a constant metric on \mathbb{R}^{2d} (but not necessary the canonical Euclidean metric). From (4.28) the spectral projector on the first Landau band of the rough Laplacian is

$$q_h^{(0)} = \mathcal{U} \circ (\operatorname{Id} \otimes Q_h^{(0)}) \circ \mathcal{U}^{-1}$$

where $Q_h^{(0)}$ has rank one. The Toeplitz space is

$$\mathcal{H}_T := H_0 = \text{Im} \left(q_h^{(0)} \right)$$

3365 **The Toeplitz operator:** By definition, the Toeplitz operator is

$$\begin{aligned} \mathcal{F}_T &:= q_h^{(0)} \hat{F}_N q_h^{(0)} |_{H_0} \\ &= \mathcal{U} \circ \left(L_A \otimes \left(Q_h^{(0)} L_A Q_h^{(0)} \right) \right) \circ \mathcal{U}^{-1} \end{aligned}$$

Since $Q_h^{(0)}$ is a rank one projector, we may write:

$$\left(Q_h^{(0)} L_A Q_h^{(0)} \right) = c(A) \cdot Q_h^{(0)}, \quad c(A) \in \mathbb{C} \quad (12.1)$$

i.e.

$$\begin{aligned} \mathcal{F}_T &= c(A) \cdot \mathcal{U} \circ \left(L_A \otimes Q_h^{(0)} \right) \circ \mathcal{U}^{-1} \\ &= c(A) \cdot \mathcal{U} \circ \left(L_A \otimes \text{Id} |_{\text{Im}(Q_h^{(0)})} \right) \circ \mathcal{U}^{-1} \end{aligned}$$

Isomorphism: We have seen in Lemma 3.23 that

$$\begin{aligned} Q^{(0)} &: \text{Im} \left(T^{(0)} \right) \rightarrow \text{Im} \left(Q^{(0)} \right) \\ T^{(0)} &: \text{Im} \left(Q^{(0)} \right) \rightarrow \text{Im} \left(T^{(0)} \right) \end{aligned}$$

3370 are bijective and invertible. Let

$$\Phi := \mathcal{U} \circ \left(\text{Id} \otimes Q_h^{(0)} \right) \circ \mathcal{U}^{-1} \quad : \mathcal{H}_h \rightarrow \mathcal{H}_T$$

which is invertible. (We may have used $T^{(0)}$ instead). Then from the previous expressions we deduce that

$$\Phi \mathcal{F}_h \Phi^{-1} = |\det(A)|^{-1/2} \cdot \mathcal{U} \circ \left(L_A \otimes \left(Q^{(0)} T^{(0)} (Q^{(0)})^{-1} \right) \right) \circ \mathcal{U}^{-1}$$

but $T^{(0)} |_{\text{Im}(T^{(0)})} = \text{Id}$ hence

$$\begin{aligned} \Phi \mathcal{F}_h \Phi^{-1} &= |\det(A)|^{-1/2} \cdot \mathcal{U} \circ \left(L_A \otimes \text{Id} |_{\text{Im}(Q^{(0)})} \right) \circ \mathcal{U}^{-1} \\ &= |\det(A)|^{-1/2} c(A)^{-1} \cdot \mathcal{F}_T \\ &= e^{\mathcal{M}} \mathcal{F}_T \end{aligned}$$

with

$$\mathcal{M} = -\frac{1}{2} \log |\det(A)| - \log c(A) \quad (12.2)$$

3375 In the next Section we compute the constant $c(A)$.

Computation of $Q_h^{(0)}$ We recall that we have coordinates $(\zeta_p, \zeta_q) \in \mathbb{R}^{2d}$ on $T^*\mathbb{R}_{\zeta_p}^d$.

The metric g compatible with ω is characterized²⁵ by a complex Lagrangian linear subspace $W \subset \left(T^*\mathbb{R}_{\zeta_p}^d\right)^\mathbb{C}$.

A Lagrangian linear subspace $W \subset \left(T^*\mathbb{R}_{\zeta_p}^d\right)^\mathbb{C}$ is uniquely characterized by its “generating function”, a quadratic function $S_W : \mathbb{R}_{\zeta_p}^d \rightarrow \mathbb{C}$,

$$S_W(\zeta_p) = \frac{1}{2} \langle \zeta_p | \mathbf{W} \zeta_p \rangle$$

where $\mathbf{W} \in \text{Sym}(\mathbb{C}^{d^2})$ is a symmetric $d \times d$ complex matrix. Precisely W is given by the graph of the differential dS_W :

$$\zeta_q = (dS_W)(\zeta_p) = \mathbf{W} \zeta_p$$

We associate a quadratic WKB function

$$\varphi_W(\zeta_p) := \exp\left(\frac{i}{\hbar} S_W(\zeta_p)\right) = \exp\left(\frac{i}{\hbar} \frac{1}{2} \langle \zeta_p | \mathbf{W} \zeta_p \rangle\right)$$

This function belongs to $\text{Im}\left(Q_h^{(0)}\right)$, hence

$$Q_h^{(0)} = \frac{|\varphi_W\rangle \langle \varphi_W|}{\langle \varphi_W | \varphi_W \rangle}$$

Example: if $d = 1$ and J is the standard complex structure $J \frac{\partial}{\partial z} = i \frac{\partial}{\partial z}$ with $z = \zeta_p + i\zeta_q$ then $\mathbf{W} = i$ and $\varphi_W(\zeta_p) = \exp\left(-\frac{1}{2\hbar} \zeta_p^2\right)$.

²⁵The natural construction of g and J is the following. At every point $x \in M$ the symplectic structure ω extends to a symplectic structure on the complexified tangent space $T_x M^\mathbb{C}$. Let us choose a non real Lagrangian subspace $W_x \subset T_x M^\mathbb{C}$ i.e. such that $\omega(W_x, W_x) = 0$, $\dim_{\mathbb{C}} W_x = d$ and $W_x \oplus \overline{W}_x = T_x M^\mathbb{C}$. Then W_x defines a complex structure J_x on $T_x M^\mathbb{C}$ by the requirement that if $u \in T_x M^\mathbb{C}$ decomposes as $u = u_W + u_{\overline{W}}$ with $u_W \in W_x$, $u_{\overline{W}} = \overline{u_W} \in \overline{W}_x$ then $J_x u = i u_W - i \overline{u_W}$. In other words, $T_x M^\mathbb{C} = W_x \oplus \overline{W}_x$ is the spectral decomposition of the operator J with respective eigenvalues $i, -i$. It is clear that $J_x^2 = -\text{Id}$ and that $\omega(Ju, Jv) = \omega(u, v)$ (it is enough to check this with $u \in W_x, v \in \overline{W}_x$). The space of such W_x is called the **Siegel generalized Upper half plane**. It is the homogeneous space $\mathcal{H}_d = \text{Sp}_{2d}(\mathbb{R}) / \text{U}_d$. Ref: [46, p.62][55, p.89,p.93]. In dimension $d = 1$, $\mathcal{H}_1 = \text{Sp}_2(\mathbb{R}) / \text{U}_1 = \text{SL}_2(\mathbb{R}) / \text{SO}_2$ is the Poincaré disk. In dimension $d = 1$ it clear that every complex structure J is compatible with ω because every one-dimensional subspace W is Lagrangian. Finally we require that the non degenerate symmetric form $g(u, v) := \omega(u, Jv)$ is positive definite. (It is easy to check that it is symmetric:

$$g(v, u) = \omega(v, Ju) = -\omega(Ju, v) = -\omega(J^2 u, Jv) = -\omega(-u, Jv) = g(u, v)$$

Computation of $c(A)$ Eq.(12.1) implies that

$$c(A) = \frac{\langle \varphi_W | L_A \varphi_W \rangle}{\langle \varphi_W | \varphi_W \rangle}$$

where

$$\begin{aligned} \langle \varphi_W | \varphi_W \rangle &= \int_{\mathbb{R}^d} \exp\left(-\frac{i}{\hbar} \frac{1}{2} \langle x, \overline{\mathbf{W}} x \rangle\right) \exp\left(\frac{i}{\hbar} \frac{1}{2} \langle x, \mathbf{W} x \rangle\right) dx \\ &= \int_{\mathbb{R}^d} \exp\left(-\frac{1}{\hbar} \langle x, \text{Im}(\mathbf{W}) x \rangle\right) dx \end{aligned}$$

Recall the operator L_A acting on $u \in \mathcal{S}(\mathbb{R}_{\zeta_p}^d)$ is given by

$$L_A u = |\det(A)|^{-1/2} \cdot u \circ A^{-1}$$

3390 Hence

$$c(A) = |\det(A)|^{-1/2} \frac{\langle \varphi_W | \varphi_W \circ A^{-1} \rangle}{\langle \varphi_W | \varphi_W \rangle}$$

We have that

$$\begin{aligned} \langle \varphi_W | \varphi_W \circ A^{-1} \rangle &= \int_{\mathbb{R}^d} \exp\left(-\frac{i}{\hbar} \frac{1}{2} \langle x, \overline{\mathbf{W}} x \rangle\right) \exp\left(\frac{i}{\hbar} \frac{1}{2} \langle A^{-1} x, \mathbf{W} A^{-1} x \rangle\right) dx \\ &= \int_{\mathbb{R}^d} \exp\left(-\frac{1}{\hbar} \langle x, \mathbf{W}' x \rangle\right) dx \end{aligned}$$

with

$$\mathbf{W}' = \frac{i}{2} (\overline{\mathbf{W}} - {}^t A^{-1} \mathbf{W} A^{-1})$$

We recall the Gaussian integral formula in \mathbb{R}^D :

$$\int_{\mathbb{R}^D} e^{-\frac{1}{2} \langle y | A y \rangle + b \cdot y} dy = \sqrt{\frac{(2\pi)^D}{\det A}} \exp\left(\frac{1}{2} \langle b | A^{-1} b \rangle\right), \quad b \in \mathbb{C}^D, A \in \mathcal{L}(\mathbb{R}^D) \quad (12.3)$$

giving

$$\langle \varphi_W | \varphi_W \rangle = \sqrt{\frac{(2\pi)^d}{\det \frac{2}{\hbar} \text{Im}(\mathbf{W})}} = \sqrt{\frac{(\pi \hbar)^d}{\det \text{Im}(\mathbf{W})}}$$

3395

$$\langle \varphi_W | \varphi_W \circ A^{-1} \rangle = \sqrt{\frac{(\pi \hbar)^d}{\det \text{Im}(\mathbf{W}')}}$$

hence

$$c(A) = \left(\frac{\det \text{Im}(\mathbf{W})}{|\det(A)| \det \text{Im}(\mathbf{W}')} \right)^{1/2}$$

A Appendix

A.1 Proof of Theorem 1.4

Under Assumption 1 on page 11, existence of a $\mathbf{U}(1)$ -principal bundle $\pi : P \rightarrow M$ with a connection A satisfying the condition (1.13) is standard in differential geometry. See [31][55, prop 8.3.1]. Notice that the connection one form A satisfying (1.13) is determined up to addition by a connection A_0 with $dA_0 = 0$ i.e. a flat connection. Below we choose a connection appropriately so that the second claim in Theorem 1.4 holds true. We first prove the following lemma.

Lemma A.1. *Let $\pi : P \rightarrow M$ be a prequantum bundle over a closed symplectic manifold (M, ω) with a connection 1-form A such that $dA = -i(2\pi)(\pi^*\omega)$. Let $f : M \rightarrow M$ be a diffeomorphism. The following conditions are equivalent*

- (1) *There exists an equivariant lift $\tilde{f} : P \rightarrow P$ preserving the connection.*
- (2) *For any closed path $\gamma \subset M$, we have*

$$h_A(f(\gamma)) = h_A(\gamma) \tag{A.1}$$

where $h_A(\gamma) \in \mathbf{U}(1)$ denotes the holonomy along γ (with respect to the connection A).

- (3) *f preserves ω (i.e. $f^*\omega = \omega$) and the homomorphism*

$$r_A : H_1(M, \mathbb{Z}) \rightarrow \mathbf{U}(1), \quad r_A([\gamma]) = \frac{h_A(f(\gamma))}{h_A(\gamma)} \tag{A.2}$$

(which is well-defined if $f^*\omega = \omega$ holds true) is trivial:

$$r_A \equiv 1 \tag{A.3}$$

The equivariant lift \tilde{f} as above is unique up to a global phase (if it exists): \tilde{g} is another equivariant lift if and only if there exists $e^{i\theta_0} \in \mathbf{U}(1)$ such that $\tilde{g} = e^{i\theta_0} \tilde{f}$.

Proof. The proof of Lemma A.1 can be found in [56, Prop 2.2 p.632]. We give it here since some details of the proof will be useful later on. The idea of the proof is illustrated in Figure A.1.

The assertion (1) \Rightarrow (2) is obvious because holonomy is defined from the connection preserved by \tilde{f} hence holonomy of closed paths is preserved by f .

To prove the assertion (2) \Rightarrow (1) we construct \tilde{f} explicitly. Let $p_0 \in P$ and $x_0 = \pi(p_0) \in M$ be some given points of reference. We choose $q_0 \in P_{f(x_0)}$ an arbitrary point in the

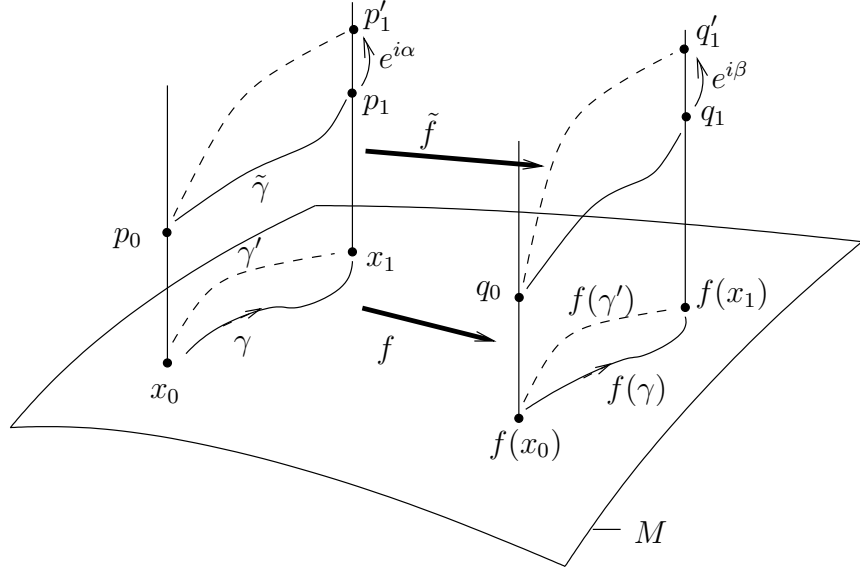


Figure A.1: Picture of \tilde{f} construction.

fiber $P_{f(x_0)}$ and set $\tilde{f}(p_0) = q_0$. By equivariance, this defines \tilde{f} on the fiber P_{x_0} : for any $e^{i\theta} \in \mathbf{U}(1)$, we have to set $\tilde{f}(e^{i\theta}p_0) = e^{i\theta}q_0$. Let $x_1 \in M$ be any point. We want to define \tilde{f} on the fiber P_{x_1} . We choose a path $\gamma : [0, 1] \rightarrow M$ which joins $\gamma(0) = x_0$ to $\gamma(1) = x_1$ and then take the unique horizontal lift²⁶ $\tilde{\gamma} : [0, 1] \rightarrow P$ of γ such that $\tilde{\gamma}(0) = p_0$. Put $p_1 := \tilde{\gamma}(1) \in P_{x_1}$. Next let $f(\gamma)$ be the unique horizontal lift of $f(\gamma)$ such that $f(\gamma)(0) = q_0$. Since \tilde{f} preserves the connection, it sends $\tilde{\gamma}$ to this horizontal lift $f(\gamma)$ of $f(\gamma)$. We define $\tilde{f}(p_1) = q_1 := f(\gamma)(1) \in P_{f(x_1)}$. For equivariance, we define \tilde{f} on the fiber P_{x_1} so that $\tilde{f}(e^{i\theta}p_1) = e^{i\theta}q_1$ for any $e^{i\theta} \in \mathbf{U}(1)$.

The definition of \tilde{f} described above depends a priori on the choice of the path γ . We check now that the condition (2) guarantees the well definiteness (or independence of the choice of the path γ) of this definition. Suppose that γ' is another path such that $\gamma'(0) = x_0$ and $\gamma'(1) = x_1$ and that we define $p'_1 \in P_{x_1}$ and $q'_1 \in P_{f(x_1)}$ in the similar manner as above using $\tilde{\gamma}'$ in the place of $\tilde{\gamma}$. Then we have $p'_1 = e^{i\alpha}p_1$ for some $e^{i\alpha} \in \mathbf{U}(1)$ and $q'_1 = e^{i\beta}q_1$ for some $e^{i\beta} \in \mathbf{U}(1)$. From the definition above, we have $\tilde{f}(p'_1) = \tilde{f}(e^{i\alpha}p_1) = e^{i\alpha}\tilde{f}(p_1) = e^{i\alpha}q_1$. For well definiteness, we have to check that $q'_1 = \tilde{f}(p'_1)$ or, equivalently, that $e^{i\alpha} = e^{i\beta}$. Note that $\Gamma := \gamma' \circ \gamma^{-1}$ is a closed path with holonomy²⁷ $h_A(\Gamma) = e^{i\alpha}$ and $f(\Gamma) = f(\gamma') \circ f(\gamma)^{-1}$ has holonomy $h_A(f(\Gamma)) = e^{i\beta}$. Therefore the required condition $e^{i\alpha} = e^{i\beta}$ is equivalent to the condition (2). By construction \tilde{f} preserves the horizontal bundle hence the connection A. We have obtained (1).

²⁶By definition, $\tilde{\gamma} \in P$ is a *horizontal lift* of the path $\gamma(t) \in M$ if $\pi(\tilde{\gamma}(t)) = \gamma(t)$ and if the tangent vector is horizontal at every point: $A\left(\frac{d\tilde{\gamma}}{dt}\right) = 0$. It does not depend on the parametrization of γ .

²⁷By definition, the *holonomy* of a closed path $\Gamma(t) \in M$, $\Gamma(1) = \Gamma(0)$ is $h(\Gamma) \in \mathbf{U}(1)$ computed as follows. We construct $\tilde{\Gamma}(t) \in P$, a horizontal lift of Γ and write $\tilde{\Gamma}(1) = e^{ih(\Gamma)}\tilde{\Gamma}(0) \in \pi^{-1}(\Gamma(0))$.

Let us show that (2) and (3) are equivalent. Let $\gamma = \partial\sigma$ be a closed path which borders a surface $\sigma \subset M$ i.e. $[\gamma] = 0$ in $H_1(M, \mathbb{Z})$. The curvature formula [13] gives the holonomy as

$$h_A(\gamma) = \exp\left(-i2\pi \int_{\sigma} \omega\right).$$

Also

$$h_A(f(\gamma)) = \exp\left(-i2\pi \int_{f(\sigma)} \omega\right) = \exp\left(-i2\pi \int_{\sigma} f^*\omega\right).$$

3435 The condition $h_A(f(\gamma)) = h_A(\gamma)$ for any closed path $\gamma = \partial\sigma$ as above is therefore equivalent to the local condition $f^*\omega = \omega$. In that case, for any closed paths γ and γ' such that $[\gamma] = [\gamma'] \in H_1(M, \mathbb{Z})$, we have $h_A(f(\gamma) \circ f(\gamma')^{-1}) = h_A(\gamma \circ (\gamma')^{-1})$, and hence

$$\frac{h_A(f(\gamma'))}{h_A(\gamma')} = \frac{h_A(f(\gamma))}{h_A(\gamma)}.$$

Therefore the map (A.2) is well defined. Now the equivalence of the conditions (2) and (3) is obvious. \square

3440 The next lemma gives the choice of the connection in the latter statement of Theorem 1.4.

Lemma A.2. *Let $\pi : P \rightarrow M$ be a prequantum bundle over a closed symplectic manifold (M, ω) with connection 1-form A such that $dA = -i(2\pi)(\pi^*\omega)$. Let $f : M \rightarrow M$ a symplectic diffeomorphism and $f_* : H_1(M, \mathbb{R}) \rightarrow H_1(M, \mathbb{R})$ the linear map induced in the homology group. If Assumption 2 on page 11 holds, there exists a flat connection A_0 such that (A.3) holds for the modified connection $A + A_0$.*

Proof. If A_0 is a flat connection (i.e. $dA_0 = 0$) let $A' = A + A_0$ be a modified connection (we assume $A_0(\frac{\partial}{\partial\theta})$ to ensure (1.8)). For a closed path γ the modified holonomy is

$$h_{A'}(\gamma) = h_A(\gamma) \cdot h_{A_0}(\gamma).$$

3445 We have a well-defined homomorphism $P_{A_0} : H_1(M, \mathbb{Z}) \rightarrow \mathbb{R}/\mathbb{Z}$, called the *period map*, such that

$$h_{A_0}(\gamma) = e^{i2\pi P_{A_0}(\gamma)}. \tag{A.4}$$

Suppose that f is symplectic i.e. $f^*\omega = \omega$. For the connections A and $A' = A + A_0$, we have the relation:

$$\frac{h_{A'}(f(\gamma))}{h_{A'}(\gamma)} = \frac{h_A(f(\gamma))}{h_A(\gamma)} \frac{h_{A_0}(f(\gamma))}{h_{A_0}(\gamma)}$$

or, in terms of the maps (A.2) and (A.4), it gives

$$r_{A'} = r_A \exp(i2\pi(P_{A_0}(f_* - I))).$$

Hence if we choose the flat connection A_0 so that

$$\exp(i2\pi(P_{A_0}(f_* - I))) = r_A^{-1},$$

3450 then the condition (A.3) is realized for the modified connection $A' = A + A_0$. From Assumption 2, this is possible. Indeed, if we can write $r_A = e^{i2\pi R_A}$ with $R_A : H_1(M, \mathbb{Z}) \rightarrow \mathbb{R}$ and choose a flat connection A_0 so that $P_{A_0} = -R_A(f_* - I)^{-1}$. \square

From Lemma A.1 and Lemma A.2 there exists an equivariant lifted map $\tilde{f} : P \rightarrow P$, which is unique up to a global phase. This proves Theorem 1.4.

3455 *Remark A.3.* The results of this Section may be expressed more clearly as follows.

(1) The symplectic form ω with the integral Assumption 1.12 page 11 defines a family of principal bundles $P_A \rightarrow M$ which are parametrized by a flat connection $A \in \mathcal{A}$. The space \mathcal{A} of flat connections is an affine space of finite dimension modeled on $H^1(M)$ (a torus).

3460 (2) The map $f : M \rightarrow M$ can be lifted without assumption on the family $P_A, A \in \mathcal{A}$ giving a map $\tilde{f} : P_A \rightarrow P_{f^\bullet(A)}$ between bundles with an induced map f^\bullet on \mathcal{A} . Corollary A.2 above was to find a flat connection A_0 which is a fixed point $A_0 = f^\bullet(A_0)$ so that we get a lifted map on a unique bundle $\tilde{f} : P_{A_0} \rightarrow P_{A_0}$. For this we need the additional assumption on f^\bullet .

3465 In the example of the Arnold cat map (1.4) of $M = \mathbb{T}^2$, then $\mathcal{A} = \mathbb{T}^2$ is also a torus (sometimes called Floquet parameters). For example in [11, eq.(2.1)] they use the notation $\kappa = (\kappa_1, \kappa_2) \in [0, 2\pi]^2 \equiv \mathcal{A}$, the map $f^\bullet : \mathcal{A} \rightarrow \mathcal{A}$ is given in [11, eq.(6.4)].

A.2 Proof of Lemma 7.23

It is enough to show that there exist constants $C_1 > 0$ and $C_2 > 0$ independent of \hbar such that

$$\|(\rho - \tilde{\tau}_\hbar^{(k)})^{-1}\|_{\mathcal{H}_\hbar^r(P)} \leq \frac{C_1}{\min\{|\rho|, |1 - \rho|\}} \quad (\text{A.5})$$

whenever $\rho \in \mathbb{C}$ satisfies

$$\min\{|\rho|, |1 - \rho|\} \geq C_2 \hbar^\epsilon. \quad (\text{A.6})$$

In fact, the estimate (A.5) would imply that, for $r_0 = C_2 \hbar^\epsilon$,

$$\begin{aligned} \|\tilde{\tau}_\hbar^{(k)} - \hat{\tau}_\hbar^{(k)}\|_{\mathcal{H}_\hbar^r} &= \left\| \int_{|\rho-1|=r_0, |\rho|=r_0} \rho(\rho - \tilde{\tau}_\hbar^{(k)})^{-1} d\rho - \int_{|\rho-1|=r_0} (\rho - \tilde{\tau}_\hbar^{(k)})^{-1} d\rho \right\|_{\mathcal{H}_\hbar^r(P)} \\ &\leq \left\| \int_{|\rho|=r_0} \rho(\rho - \tilde{\tau}_\hbar^{(k)})^{-1} d\rho \right\|_{\mathcal{H}_\hbar^r} + \left\| \int_{|\rho-1|=r_0} (\rho - 1)(\rho - \tilde{\tau}_\hbar^{(k)})^{-1} d\rho \right\|_{\mathcal{H}_\hbar^r(P)} \\ &\leq 2C_1 \cdot r_0 = 2C_1 \cdot C_2 \cdot \hbar^\epsilon. \end{aligned}$$

To prove (A.5), we may and do assume $|\rho| \leq 2\|\check{\tau}_h^{(k)}\|_{\mathcal{H}_h^r(P)}$ because the claim is trivial otherwise. Take $u \in \mathcal{H}_h^r(P)$ arbitrarily. From the assumption made in the preceding sentence, we have

$$\begin{aligned} \|\rho^2 u - \check{\tau}_h^{(k)} \circ \check{\tau}_h^{(k)} u\|_{\mathcal{H}_h^r(P)} &\leq \|\rho^2 u - \rho \check{\tau}_h^{(k)} u\|_{\mathcal{H}_h^r(P)} + \|\check{\tau}_h^{(k)}(\rho u - \check{\tau}_h^{(k)} u)\|_{\mathcal{H}_h^r(P)} \\ &\leq (|\rho| + \|\check{\tau}_h^{(k)}\|_{\mathcal{H}_h^r(P)}) \cdot \|(\rho - \check{\tau}_h^{(k)})u\|_{\mathcal{H}_h^r(P)} \\ &\leq 3\|\check{\tau}_h^{(k)}\|_{\mathcal{H}_h^r(P)} \cdot \|(\rho - \check{\tau}_h^{(k)})u\|_{\mathcal{H}_h^r(P)}. \end{aligned}$$

On the other hand, from (7.41), we have

$$\begin{aligned} \|\rho^2 u - \check{\tau}_h^{(k)} \circ \check{\tau}_h^{(k)} u\|_{\mathcal{H}_h^r(P)} &\geq \|\rho^2 u - \check{\tau}_h^{(k)} u\|_{\mathcal{H}_h^r(P)} - C \cdot \hbar^\epsilon \|u\|_{\mathcal{H}_h^r(P)} \\ &\geq |\rho(\rho - 1)| \cdot \|u\|_{\mathcal{H}_h^r(P)} - \|(\rho - \check{\tau}_h^{(k)})u\|_{\mathcal{H}_h^r(P)} - C \cdot \hbar^\epsilon \|u\|_{\mathcal{H}_h^r(P)}. \end{aligned}$$

Hence we obtain the estimate

$$(|\rho(\rho - 1)| - C' \cdot \hbar^\epsilon) \cdot \|u\|_{\mathcal{H}_h^r} \leq (3\|\check{\tau}_h^{(k)}\|_{\mathcal{H}_h^r(P)} + 1) \cdot \|(\rho - \check{\tau}_h^{(k)})u\|_{\mathcal{H}_h^r(P)}$$

for some constant $C' > 0$. If we choose C_2 so large that $C_2 > 4C'$, the assumption (A.6) implies

$$|\rho(\rho - 1)| - C \cdot \hbar^\epsilon \geq \frac{1}{2} \min\{|\rho|, |1 - \rho|\} - C \cdot \hbar^\epsilon \geq \frac{1}{4} \min\{|\rho|, |1 - \rho|\}.$$

3475 Therefore, with such choice of C_2 , the inequality (A.5) holds if we let

$$C_1 > 4 \cdot (3\|\check{\tau}_h^{(k)}\|_{\mathcal{H}_h^r(P)} + 1),$$

recalling that $\|\check{\tau}_h^{(k)}\|_{\mathcal{H}_h^r(P)}$ is bounded by a constant independent of \hbar .

To prove the second inequality on the trace norm, we observe, using Lemma 7.22,

$$\begin{aligned} \|\check{\tau}_h^{(k)} - \hat{\tau}_h^{(k)}\|_{tr} &\leq \|\check{\tau}_h^{(k)} \circ \check{\tau}_h^{(k)} - \hat{\tau}_h^{(k)}\|_{tr} + \|\check{\tau}_h^{(k)} \circ \check{\tau}_h^{(k)} - \check{\tau}_h^{(k)}\|_{tr} \\ &\leq \left\| \check{\tau}_h^{(k)} \circ \left(\int_{|\rho-1|=r_0} (\rho - (1/\rho))(\rho - \check{\tau}_h^{(k)})^{-1} d\rho \right) - \check{\tau}_h^{(k)} \circ \int_{|\rho|=r_0} \rho(\rho - \check{\tau}_h^{(k)})^{-1} d\rho \right\|_{tr} + C\hbar^{-d+\epsilon} \\ &\leq C\hbar^{-d+\epsilon}. \end{aligned}$$

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