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Prequantum transfer operator for Anosov diffeomorphism (Preliminary version)

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Abstract

This is a preliminary version and some other results will appear in the next version. We define the prequantization of a symplectic Anosov diffeomorphism, which is a $\mathbf{U}(1)$ extension of the diffeomorphism preserving an associated specific connection. We study the spectrum of the associated transfer operator, called *prequantum transfer operator*, restricted to the N -th Fourier mode with respect to the $\mathbf{U}(1)$ action on P . We investigate the spectral property in the limit $N \rightarrow \infty$, regarding the transfer operator as a Fourier integral operator and using semi-classical analysis. In the main result, we show a “*band structure*” of the spectrum, that is, the spectrum is contained in a few separated annuli and a disk concentric at the origin.

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1 Introduction and results

1.1 Introduction

We consider a smooth symplectic Anosov diffeomorphism $f : M \rightarrow M$ on a $2d$ -dimensional compact symplectic manifold (M, ω) as a standard model of "chaotic" dynamical system. Following the geometric quantization procedure introduced by Kostant, Souriau and Kirillov in 1970s', we set up the prequantum bundle $P \rightarrow M$. This is the $\mathbf{U}(1)$ -principal bundle over M equipped with a connection whose curvature is $-2\pi i \cdot \omega$. Then we introduce the prequantum map $\tilde{f} : P \rightarrow P$ as the $\mathbf{U}(1)$ -equivariant lift of the map f preserving the connection. The prequantum map \tilde{f} thus defined is known to be exponentially mixing¹, that is, any smooth probability density which evolves under the iteration of \tilde{f} converges weakly towards the uniform equilibrium distribution on P and the speed of convergence is exponentially fast if it is measured by a smooth observable. We study the fluctuations in this convergence to the equilibrium by investigating spectral properties of the transfer operator \hat{F} associated to the prequantum map \tilde{f} . Following the approach taken by David Ruelle in his study of expanding dynamical systems[32], we first show that the transfer operator displays discrete spectrum, which is sometimes called *Ruelle-Pollicott resonances*. Precisely we consider the restriction \hat{F}_N of the transfer operator \hat{F} to the N -th Fourier mode with respect to the $\mathbf{U}(1)$ action on P and show that its natural extension to appropriate generalized Sobolev spaces of distributions has discrete spectrum. This result concerning discrete spectrum is already known in the preceding works [33],[9, 20],[5, theorem 1.1],[17, theorem 1] and will be recalled in Theorem 1.14. The new result of this paper is in Theorem 1.16, where we show that the spectrum of \hat{F}_N has a particular "band" structure: for every N large enough, there is an annulus that contains finitely many (but increasing to infinity as $N \rightarrow \infty$) eigenvalues, separated from the rest of the internal spectrum by a gap. This means that the convergence to the equilibrium mentioned above, restricted to the N -th Fourier mode, is described by a finite rank operator F_N , up to relatively small exponentially decaying errors. The finite rank operator F_N is the spectral restriction of the prequantum transfer operator \hat{F}_N on the external annulus. We show, in Theorem 1.19, that the rank of \hat{F}_N is proportional to N^d asymptotically as² $N \rightarrow \infty$. These results are generalizations of the results in [16] for the linear Arnold cat map to the case of general non-linear symplectic Anosov diffeomorphisms.

Motivations of the study From the construction above, the prequantum map $\tilde{f} : P \rightarrow P$ is partially hyperbolic, that is, hyperbolic in the directions transverse to the fibers but is neutral (because of equivariance) in the direction of the fibers. Also note that \tilde{f} preserves the connection one form on the prequantum bundle P which is a contact form on P . (See Remark 1.7) These properties of the prequantum map are very similar to those of

¹Exponential mixing of the map \tilde{f} is already known [12] but is also a direct consequence of results presented in this paper.

²The precise value of $\text{rank}(\hat{F}_N)$ is given by an index formula of Atiyah-Singer.

the time- t -map of the geodesic flow $\phi_1 : T_1^*N \rightarrow T_1^*N$ on a closed Riemannian manifold N with negative curvature, acting on the unit cotangent bundle T_1^*N : In the latter case the time- t -map of the geodesic flow is partially hyperbolic and preserves the canonical Liouville contact one form ξdx on T_1^*N . (See [26, 39, 40, 18].) With this point of view, the prequantum transfer operator can be considered as a model of the transfer operators for the geodesic flows on negatively curved manifolds. One of our objective behind the present work is to show some band structure of the spectrum for the case of geodesic flow. In the special case of manifolds with constant curvature, such a band structure is readily observed from the classical theorem of Selberg on zeta functions [35].

Another motivation already discussed in [16] is the following observation: The finite rank operator \mathcal{F}_N which describes the long time classical correlation functions of the map \tilde{f} has the properties of a "quantum map" i.e. a "quantization of f " with the Planck constant $\hbar = 1/(2\pi N)$. It satisfies the Gutzwiller Trace formula with an error term which decreases exponentially fast in large time, an exact Egorov theorem, etc. Surprising is that this "quantization" or quantum behavior, appears here dynamically (after long time) in the "classical" map \tilde{f} . A movie of this phenomenon can be seen on the web page [15]. This aspect will be discussed and developed in more detail in the next version of this paper.

Semiclassical approach The general method that we use to obtain the main results is *semiclassical analysis*. We regard the prequantum transfer operator as a Fourier Integral Operator (FIO), which means that we consider its action on wave packets in the high frequency limit $N \rightarrow \infty$. From the general idea in semiclassical analysis, this action is effectively described by the associated canonical map $(Df^*)^{-1}$ on the cotangent space T^*M equipped with the symplectic structure $\Omega = dx \wedge d\zeta + \pi^*\omega$ (where $dx \wedge d\zeta$ stands for the canonical symplectic structure on T^*M). For the action of the canonical map $(Df^*)^{-1}$, the non-wandering set is the zero section $K \subset T^*M$ and is called the *trapped set*. The additional term $\pi^*\omega$ in Ω makes K a symplectic submanifold. We will see that these facts are the core of our argument and give the band structure of the spectrum in the main theorem.

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1.2 Definitions

1.2.1 Symplectic Anosov map

Let M be a C^∞ closed connected symplectic manifold of dimension $2d$ with symplectic two form ω . Let $f : M \rightarrow M$ be a C^∞ symplectic Anosov diffeomorphism, i.e. a C^∞ Anosov

diffeomorphism such that $f^*\omega = \omega$. We recall the definition of an Anosov diffeomorphism:

Definition 1.1. [24, p.263] A diffeomorphism $f : M \rightarrow M$ is said to be Anosov if there exists a C^∞ Riemannian metric g on M , an f -invariant continuous decomposition of TM ,

$$T_x M = E_u(x) \oplus E_s(x), \quad \forall x \in M \quad (1.1)$$

and a constant $\lambda > 1$, such that, for any $x \in M$, hold

$$\begin{aligned} |D_x f(v_s)|_g &\leq \frac{1}{\lambda} |v_s|_g \quad \forall v_s \in E_s(x), \quad \text{and} \\ |D_x f^{-1}(v_u)|_g &\leq \frac{1}{\lambda} |v_u|_g \quad \forall v_u \in E_u(x). \end{aligned} \quad (1.2)$$

This subspaces E_s (resp. E_u) in which f is uniformly contracting (resp. expanding) is called the stable (resp. unstable) sub-bundle. See figure 1.1.

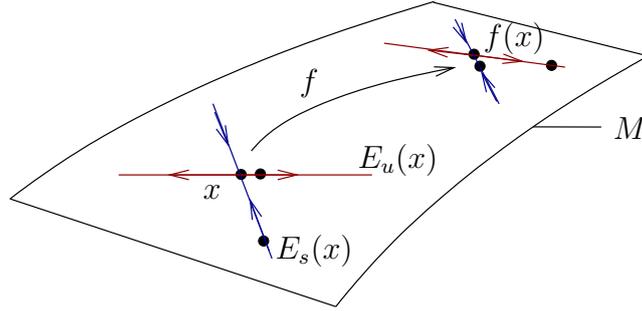


Figure 1.1: A symplectic Anosov map f

Remark 1.2. (1) The subspaces $E_u(x)$ and $E_s(x)$ do not depend smoothly on the point x in general. However it is known that they are Hölder continuous in x with some Hölder exponent [31]. In what follows, we assume that the Hölder exponent is

$$0 < \beta < 1. \quad (1.3)$$

The subspaces $E_u(x)$ and $E_s(x)$ are Lagrangian linear subspace of $T_x M$ and have both dimension d .

(2) The *Arnold cat map*[2] is a simple example of a symplectic Anosov diffeomorphism on the torus $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$,

$$f_0 \begin{pmatrix} q \\ p \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} q \\ p \end{pmatrix} \quad \text{mod } \mathbb{Z}^2. \quad (1.4)$$

It preserves the symplectic form $\omega = dq \wedge dp$. If $h : M \rightarrow M$ is a diffeomorphism close enough to identity in the C^1 norm and preserves the symplectic form ω , the *perturbed cat map*

$$f(x) := h(f_0(x)) \quad (1.5)$$

is also a (probably non-linear) symplectic Anosov diffeomorphism [24, p.266]. Similarly, we get examples of symplectic Anosov diffeomorphisms on \mathbb{T}^{2d} from any linear map $f_0 \in Sp_{2d}(\mathbb{Z})$ with no eigenvalues on the unit circle.

1.2.2 The prequantum bundle and the lift map \tilde{f}

A prequantum bundle is a $\mathbf{U}(1)$ -principal bundle P equipped with a specific connection. In a few paragraphs below, we recall the definition of a $\mathbf{U}(1)$ -principal bundle and that of a connection on it. (For the detailed account, we refer [42].) The one-dimensional unitary group $\mathbf{U}(1)$ is the multiplicative group of complex numbers of the form $e^{i\theta}$, $\theta \in \mathbb{R}$. A $\mathbf{U}(1)$ -principal bundle P over M is a manifold with a free action of $\mathbf{U}(1)$, written

$$p \in P \rightarrow (e^{i\theta}p) \in P, \quad (1.6)$$

such that the quotient space is $M = P/\mathbf{U}(1)$. We write $\pi : P \rightarrow M$ for the projection map. Then the space P has a local product structure over M as follows: There exist a finite cover of M by simply connected open subsets $U_\alpha \subset M$, $\alpha \in I$, and smooth sections $\tau_\alpha : U_\alpha \rightarrow P$ on each of U_α , called a *local smooth section*; A *local trivialization* of P over U_α is defined as the diffeomorphism

$$T_\alpha : U \times \mathbf{U}(1) \rightarrow \pi^{-1}(U_\alpha), \quad T_\alpha(x, e^{i\theta}) \rightarrow e^{i\theta}\tau_\alpha(x). \quad (1.7)$$

A *connection* on P is a differential one form $A \in C^\infty(P, \Lambda^1 \otimes (i\mathbb{R}))$ on P with values in the Lie algebra $\mathfrak{u}(1) = i\mathbb{R}$ which is invariant by the action of $\mathbf{U}(1)$ and normalized so that

$$A\left(\frac{\partial}{\partial\theta}\right) = i \quad (1.8)$$

where $\frac{\partial}{\partial\theta}$ denotes the vector field on P generating the action of $\mathbf{U}(1)$. Consequently the pull-back of the connection A on P by the trivialization map (1.7) is written as

$$T_\alpha^*A = id\theta - i2\pi\eta_\alpha \quad (1.9)$$

where $\eta_\alpha \in C^\infty(U_\alpha, \Lambda^1)$ is a one-form on U_α which depends on the choice of the local section τ_α . A different local section $\tau_\beta : U_\beta \rightarrow P$ with $U_\alpha \cap U_\beta \neq \emptyset$ is written as $\tau_\beta = e^{i\chi}\tau_\alpha$ with using a function $\chi : U_\alpha \cap U_\beta \rightarrow \mathbb{R}$ and hence the connection A pulled-back by the corresponding trivialization T_β is written as

$$\eta_\beta = \eta_\alpha - \frac{1}{2\pi}d\chi \quad \text{on } U_\alpha \cap U_\beta. \quad (1.10)$$

The *curvature* of the connection A is the two form $\Theta = dA$ on P . In the local trivialization (1.7), we have $T_\alpha^* \Theta = -i2\pi (d\eta_\alpha)$ and (1.10) implies that $d\eta_\alpha = d\eta_\beta$. Therefore the curvature two form is written as

$$\Theta = -i(2\pi)(\pi^* \tilde{\omega})$$

where $\tilde{\omega} = d\eta_\alpha$ is a closed two form on M independent of the trivialization.

Since there is a given symplectic two form ω on M in our setting, we naturally require below in (1.11) that the two form $\tilde{\omega}$ coincides with ω .

For the construction of the prequantum bundle and prequantum transfer operator, we will need the following two assumptions:

Assumption 1 : The cohomology class $[\omega] \in H^2(M, \mathbb{R})$ represented by the symplectic form ω is integral, that is, $[\omega] \in H^2(M, \mathbb{Z})$.

Assumption 2 : The integral homology group $H_1(M, \mathbb{Z})$ has no torsion part and that 1 is not an eigenvalue of the linear map $f_* : H_1(M, \mathbb{R}) \rightarrow H_1(M, \mathbb{R})$ induced by $f : M \rightarrow M$.

Remark 1.3. The second assumption above is not restrictive and may not be necessary. In fact this hypothesis is conjectured to be true in general. For the case $M = \mathbb{T}^{2d}$, this is always satisfied.

Theorem 1.4. *Under Assumption 1 above, there exists a $\mathbf{U}(1)$ -principal bundle $\pi : P \rightarrow M$ and a connection $A \in C^\infty(P, \Lambda^1 \otimes (i\mathbb{R}))$ on P such that the curvature two form $\Theta = dA$ satisfies*

$$\Theta = -i(2\pi)(\pi^* \omega). \quad (1.11)$$

If we put Assumption 2 in addition, we can choose the connection A as above so that there exists an equivariant lift $\tilde{f} : P \rightarrow P$ of the map $f : M \rightarrow M$ preserving the connection A :

$$(\pi \circ \tilde{f})(p) = (f \circ \pi)(p), \quad \forall p \in P \quad : \tilde{f} \text{ is a lift of } f. \quad (1.12)$$

$$\tilde{f}(e^{i\theta} p) = e^{i\theta} \tilde{f}(p), \quad \forall p \in P, \forall \theta \in \mathbb{R} \quad : \tilde{f} \text{ is equivariant w.r.t. the } \mathbf{U}(1) \text{ action.} \quad (1.13)$$

$$\tilde{f}^* A = A \quad : \tilde{f} \text{ preserves the connection } A. \quad (1.14)$$

(See figure 1.2.)

The proof of Theorem 1.4 is given in Section 2.1.

Definition 1.5. The $\mathbf{U}(1)$ -principal bundle $\pi : P \rightarrow M$ equipped with the connection $A \in C^\infty(P, \Lambda^1 \otimes (i\mathbb{R}))$ satisfying (1.11) is called *prequantum bundle* over the symplectic manifold (M, ω) . The map $\tilde{f} : P \rightarrow P$ satisfying the conditions (1.12), (1.13) and (1.14) is called *prequantum map*.

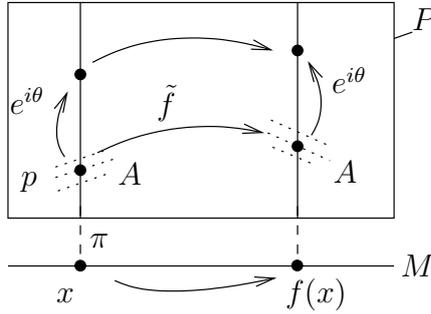


Figure 1.2: A picture of the prequantum map \tilde{f} . The dash lines at point p represent the horizontal space $H_p P = \text{Ker}(A_p)$ which is preserved by \tilde{f} .

Remark 1.6. (Uniqueness of the prequantum bundle and the prequantum map) The prequantum bundle P is unique (as a smooth manifold) if it exists, because it is determined by its first Chern class $c_1(P) = [\omega] \in H^2(M, \mathbb{Z}^2)$. However the connection A on the prequantum bundle P is not unique. In the proof of the theorem above, we will explicitly show that there may be finitely many connections A which satisfy the condition (1.11) and they differ from each other by a flat connection. Once the prequantum bundle P and the connection A on it is given, the lifted map \tilde{f} is unique up to a global phase $e^{i\theta_0} \in \mathbf{U}(1)$, i.e. another map \tilde{g} satisfies the conditions in (2) of Theorem 1.4 if and only if $\tilde{g} = e^{i\theta_0} \tilde{f}$ for some $e^{i\theta_0} \in \mathbf{U}(1)$.

Remark 1.7. Let $\alpha := \frac{i}{2\pi} A$. Then the differential $(2d + 1)$ -form

$$\mu_P := \frac{1}{d!} \alpha \wedge (d\alpha)^d \quad (1.15)$$

is a non-degenerate volume form on P . This means that α is a *contact one form* on P preserved by \tilde{f} .

1.2.3 The prequantum transfer operator \hat{F} and the reduced operator \hat{F}_N

As usual in dynamical system theory, we consider the transfer operator associated to the prequantum map \tilde{f} :

Definition 1.8. Let $V \in C^\infty(M)$ be a real-valued smooth function, called *potential*. The *prequantum transfer operator* is defined as

$$\hat{F} : C^\infty(P) \rightarrow C^\infty(P), \quad \hat{F}(u) = e^{V \circ \pi} (u \circ \tilde{f}^{-1}) \quad (1.16)$$

where $V \circ \pi \in C^\infty(P)$ is the function V lifted on P .

Remark 1.9. The fact that \tilde{f}^{-1} appears instead of \tilde{f} in (1.16) is a matter of choice. In our choice, \tilde{f} maps the support of u to that of $\hat{F}u$.

From the equivariance property (1.13), the prequantum transfer operator commutes with the action of $U(1)$ on functions on P and therefore is naturally decomposed into each Fourier mode with respect to the $U(1)$ action.

Definition 1.10. For a given $N \in \mathbb{Z}$, we consider the space of functions in the N -th Fourier mode

$$C_N^\infty(P) := \{u \in C^\infty(P) \mid \forall p \in P, \forall \theta \in \mathbb{R}, \quad u(e^{i\theta}p) = e^{iN\theta}u(p)\}. \quad (1.17)$$

The prequantum transfer operator \hat{F} restricted to $C_N^\infty(P)$ is denoted by:

$$\hat{F}_N := \hat{F}|_{C_N^\infty(P)} : C_N^\infty(P) \rightarrow C_N^\infty(P). \quad (1.18)$$

Remark 1.11. The complex conjugation maps $C_N^\infty(P)$ to $C_{-N}^\infty(P)$ and commutes with \hat{F} . It is therefore enough to study \hat{F}_N with $N \geq 0$.

Remark 1.12. The space of equivariant functions $C_N^\infty(P)$ defined in (1.17) can be identified with the space of smooth sections of an associated Hermitian complex line bundle $L^{\otimes N}$ over M (i.e. the N tensor power of a line bundle $L \rightarrow M$) with covariant derivative D , called the *prequantum line bundle* i.e. we have

$$C_N^\infty(P) \cong C^\infty(M, L^{\otimes N}).$$

See [38, p.502, eq.(6.1)]. In order to simplify the presentation we will not use this identification in this paper although it will be present implicitly. Notice however that most of references about geometric quantization use the “line bundle terminology”.

In this paper the main object of study is the resonance spectrum of the operator \hat{F}_N , (1.18), in the limit $N \rightarrow \infty$. For $N > 0$, we set

$$\hbar = \frac{1}{2\pi N}. \quad (1.19)$$

This new variable \hbar is in one-to-one correspondence to N , and $\hbar \rightarrow +0$ as $N \rightarrow \infty$. We introduce it for convenience in referring some argument in semi-classical analysis where \hbar is regarded as the Plank constant and the limit $\hbar \rightarrow +0$ is considered.

Remark 1.13. In the following, we will confuse the parameters N and \hbar in the notation. For instance, the operator \hat{F}_N will be written \hat{F}_\hbar sometimes.

1.3 Results on the spectrum of the prequantum operator \hat{F}_N

The following theorem has been obtained essentially in the works of Rugh [33], Liverani et al. [9, 20], Baladi et al.[5, theorem 1.1], Faure et al. [17, theorem 1]. The method employed in the present paper is close to the semiclassical approach given in [17, theorem 1]. Before giving the Theorem, let us mention that the transfer operator \hat{F}_N has been defined on the space of smooth functions $C_N^\infty(P)$ and can be extended by duality to the distributions space $\mathcal{D}'_N(P)$. We will introduce a family of Hilbert spaces $\mathcal{H}_N^r(P)$ for arbitrarily large $r > 0$, called anisotropic Sobolev space such that $C_N^\infty(P) \subset \mathcal{H}_N^r(P) \subset \mathcal{D}'_N(P)$ and show

Theorem 1.14. *“Discrete spectrum of prequantum transfer operators”. For any $N \in \mathbb{Z}$, the operator \hat{F}_N extends to a bounded operator*

$$\hat{F}_N : \mathcal{H}_N^r(P) \rightarrow \mathcal{H}_N^r(P),$$

and its essential spectral radius $r_{ess}(\hat{F}_N)$ is bounded by $\varepsilon_r := \frac{1}{\lambda^r} \max e^V$, which shrinks to zero if $r \rightarrow +\infty$. The discrete eigenvalues of \hat{F}_N on the domain $|z| \geq \varepsilon_r$ (and their associated eigenspaces) are independent on the choice of r and are therefore intrinsic to the Anosov map f . These discrete eigenvalues $\text{Res}(\hat{F}_N) := \{\lambda_i\}_i \subset \mathbb{C}^$ are called Ruelle-Pollicott resonances.*

Remark 1.15. See [6],[17, cor. 1.3] for a general argument about this independence of $\text{Res}(\hat{F}_N)$ on the choice of r .

The new result of this paper is the following. It is illustrated in Figure 1.3.

Theorem 1.16. *“The spectrum of \hat{F}_N has band structure”. In the semiclassical limit $N \rightarrow \infty$, the Ruelle-Pollicott resonances of \hat{F}_N is contained in a small neighborhood of the union of annuli*

$$\mathcal{A}_k := \{z \in \mathbb{C}, |z| \in [r_k^-, r_k^+]\} \quad \text{for } k \geq 0 \quad (1.20)$$

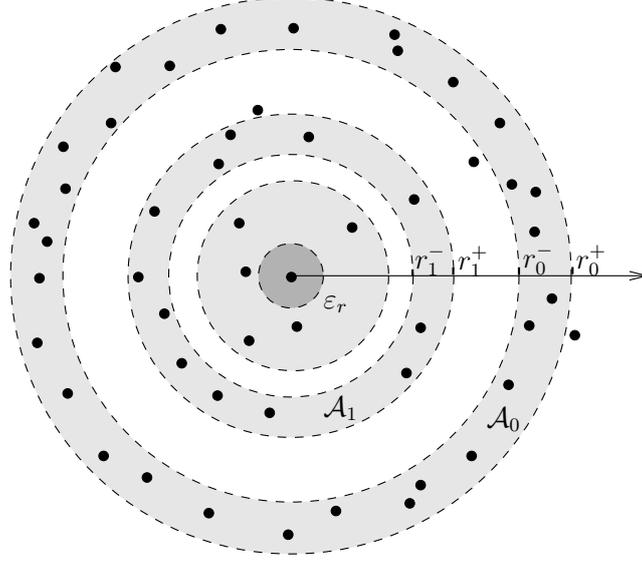


Figure 1.3: A “band structure” of the spectrum of \hat{F}_N .

where r_k^+ and r_k^- are defined by

$$r_k^- := \lim_{n \rightarrow \infty} \left| \inf_{x \in M} \left(e^{V_n(x)} \|Df_x^n|_{E_u}\|_{\max}^{-k} |\det Df_x^n|_{E_u}|^{-1/2} \right) \right|^{1/n}, \quad (1.21)$$

$$r_k^+ := \lim_{n \rightarrow \infty} \left| \sup_{x \in M} \left(e^{V_n(x)} \|Df_x^n|_{E_u}\|_{\min}^{-k} |\det Df_x^n|_{E_u}|^{-1/2} \right) \right|^{1/n}$$

with setting

$$V_n(x) := \sum_{j=1}^n V(f^j(x))$$

and

$$\|L\|_{\max} := \|L\|, \quad \|L\|_{\min} := \|L^{-1}\|^{-1} \quad (1.22)$$

for a linear map L . More precisely, we have

$$\text{Res}(\hat{F}_N) \subset \mathcal{N}_{\delta(N)}(\mathbf{A}) := \{z \in \mathbb{C}, \text{dist}(z, \mathbf{A}) \leq \delta(N)\}, \quad \mathbf{A} := \left\{ \bigcup_{k \geq 0} \mathcal{A}_k \right\} \quad (1.23)$$

where $\delta(N) = CN^{-\varepsilon} \xrightarrow{N \rightarrow \infty} 0$ with some $\varepsilon > 0$ and some constant C independent of f , V and N . If the outmost annulus \mathcal{A}_0 is isolated from other annuli, that is if $\mathcal{A}_0 \cap \mathcal{A}_l = \emptyset$ for any $l > 0$, the number of resonances in its neighborhood satisfies the estimate

$$C^{-1}N^d < \# \left\{ \text{Res}(\hat{F}_N) \cap \mathcal{N}_{\delta(N)}(\mathcal{A}_0) \right\} < CN^d$$

for some constant C .

Theorem 1.16 is deduced from Theorem 1.18 below and the proof is given in Section A.1.

Remark 1.17. (1) Since $\|Df_x^n|_{E_u}\|_{\max} \geq \|Df_x^n|_{E_u}\|_{\min} > 1$ we have obviously $r_{k+1}^- < r_k^-$ and $r_{k+1}^+ < r_k^+$ for every $k \geq 0$. However we don't always have $r_{k+1}^+ < r_k^-$ therefore the annuli may intersect each other.

(2) Let $\tilde{E}_u \subset TM$ be a smooth approximation of the unstable sub-bundle $E_u \subset TM$ in C^0 norm. If one chooses the potential:

$$V(x) := \frac{1}{2} \log |\det Df_x|_{\tilde{E}_u}| \quad (1.24)$$

one can have r_0^-, r_0^+ (arbitrarily) closed to one and the annulus \mathcal{A}_0 of the external band get isolated from the other ones ($\mathcal{A}_0 \cap \mathcal{A}_k = \emptyset$ for $k \neq 0$).

(3) In the simple case of a linear hyperbolic map on the torus \mathbb{T}^2 , i.e. example (1.4) with $V(x) = 0$, then $r_k^+ = r_k^- = \lambda^{-k-\frac{1}{2}}$, with $\lambda = Df_0|_{E_u} = \frac{3+\sqrt{5}}{2} \simeq 2.6$ (constant), i.e. each annulus \mathcal{A}_k is a circle. In this case Theorem 1.16 has been obtained in [16] and is depicted on Figure (1-b) in [16]. If one chooses $V(x) = \frac{1}{2} \log |\det Df_x|_{E_u}| = \frac{1}{2} \log \lambda$ the external band \mathcal{A}_0 is the unit circle and it is shown in [16] that the Ruelle-Pollicott resonances on the external band coincide with the spectrum of the quantized map called the ‘‘quantum cat map’’.

The following Theorem gives a more precise description of the prequantum transfer operator that leads to the spectral property stated in Theorem 1.16, though it looks more technical. The proof of Theorem 1.18 is given in Section 6.5.

Theorem 1.18. *‘‘Detailed description of the band structure’’. Let $n \geq 0$ and take sufficiently large r accordingly. Then there exists a small constant $\varepsilon > 0$, a constant C_0 , which is independent of V , f and N , and a decomposition of the Hilbert space $\mathcal{H}_N^r(P)$:*

$$\mathcal{H}_N^r(P) = \mathcal{H}_0 \oplus \mathcal{H}_1 \dots \oplus \mathcal{H}_n \oplus \mathcal{H}_{n+1} \quad (1.25)$$

such that, writing $\tau^{(k)}$ for the projection onto the component \mathcal{H}_k along other components,

$$(1) \frac{1}{C_0} \hbar^{-d} \leq \dim \mathcal{H}_k \leq C_0 \hbar^{-d} \text{ for } 0 \leq k \leq n \text{ while } \dim \mathcal{H}_{n+1} = \infty,$$

$$(2) \|\tau^{(k)}\| < C_0 \text{ for } 0 \leq k \leq n+1,$$

$$(3) \|\tau^{(k)} \circ \hat{F}_N \circ \tau^{(l)}\| \leq C_0 \hbar^\varepsilon \text{ if } k \neq l,$$

(4) for $0 \leq k \leq n+1$, it holds

$$\|\tau^{(k)} \circ \hat{F}_N \circ \tau^{(k)}\|_{\mathcal{H}_N^r(P)} \leq C_0 \sup_{x \in M} \left(e^{V(x)} \|Df_x|_{E_u}\|_{\min}^{-k} |\det Df_x|_{E_u}|^{-1/2} \right), \quad (1.26)$$

(5) for $0 \leq k \leq n$ and $u \in \mathcal{H}_k$ it holds

$$\left\| \left(\tau^{(k)} \circ \hat{F}_N \right) u \right\|_{\mathcal{H}_N^r(P)} \geq C_0^{-1} \inf_{x \in M} \left(e^{V(x)} \|Df_x|_{E_u}\|_{\max}^{-k} |\det Df_x|_{E_u}|^{-1/2} \right) \|u\|_{\mathcal{H}_N^r(P)}, \quad (1.27)$$

provided that $\hbar > 0$ is sufficiently small.

We suppose now that the external annulus \mathcal{A}_0 defined in (1.20) is isolated from other annuli $\bigcup_{k \geq 1} \mathcal{A}_k$. We have seen in (1.24), how to achieve such situation by a suitable choice of the potential $V(x)$.

Theorem 1.19. “Weyl formula and index formula for the number of resonances”. *If the external annulus \mathcal{A}_0 is isolated, i.e. $r_1^+ < r_0^-$, then the number of resonances in the external annulus \mathcal{A}_0 is given by the “Weyl formula”*

$$\# \left\{ \text{Res} \left(\hat{F}_N \right) \cap \mathcal{N}_{\delta(N)} \left(\mathcal{A}_0 \right) \right\} = N^d \text{Vol}_\omega(M) + \mathcal{O} \left(N^{d-1} \right) \quad (1.28)$$

with $\mathcal{N}_{\delta(N)} \left(\mathcal{A}_0 \right)$ defined in (1.23) and with the symplectic volume of M given by $\text{Vol}_\omega(M) := \int_M \omega^{\wedge d} / d!$.

More precisely, for N large enough, let Π_0 denotes the spectral projector associated to the isolated external band (i.e. on the spectral domain $|z| > r_0^- - \delta(N)$) then the rank of Π_0 is exactly given by the Atiyah-Singer index formula:

$$\text{rank}(\Pi_0) = \int_M \left[e^{N\omega} \text{Todd}(TM) \right]_{2d} \quad (1.29)$$

where

$$e^{N\omega} = 1 + N\omega + \dots + \frac{N^d \omega^d}{d!}$$

is the Chern character,

$$\text{Todd}(TM) = \det \left(\frac{\Omega(TM)}{1 - \exp(-\Omega(TM))} \right) = 1 + \frac{\Omega(TM)}{2} + \dots \in H_{DR}^\bullet(M)$$

is the Todd class of the tangent bundle defined from the Riemannian curvature $\Omega(TM)$ and $[\cdot]_{2d}$ denotes the restriction to volume $2d$ -forms.

Theorem 1.19 above follows from two theorems that will be stated in Section 2.4 where we will introduce a differential operator $\Delta = D^*D$ acting in $C_N^\infty(P)$ called the *rough*

Laplacian. In Theorem 2.21, we will show that its low energy spectrum has band spectrum and that the cardinality of the eigenvalues in the first (or the lowest) band equals the quantity on the right hand side of the formula (1.29). The latter is actually a consequence of a theorem in geometry. Then, in Theorem 2.21, we will show that the rank of the projector Π_0 coincides with the rank of the spectral projector for eigenvalues in the first band. We thus obtain the formula (1.29). Then the Weyl formula (1.28) is a direct consequence. Indeed we have $[e^{N\omega}\text{Todd}(TM)]_{2d} = \frac{N^d \omega^d}{d!} + O(N^{d-1})$ and hence

$$\int_M [e^{N\omega}\text{Todd}(TM)]_{2d} = N^d \text{Vol}_\omega(M) + O(N^{d-1}).$$

Remark 1.20. In the case of $M = \mathbb{T}^2$ which correspond to example (1.4) and treated in [16], the projector Π_0 has exactly $\text{rank}(\Pi_0) = N$. Indeed, for Riemann surfaces M of genus g , we have $\text{Todd}(TM) = 1 + \frac{c_1(TM)}{2}$ with first Chern number $\int_M c_1(TM) = 2 - 2g$ (the Gauss-Bonnet integral formula). Hence $\text{rank}(\Pi_0) = \int_M (N\omega) + \int_M c_1(TM) = N$ for $M = \mathbb{T}^2$.

2 Preliminary results and sketch of the proofs

In this Section we begin with establishing some preliminary results which will be used in the rest of the paper. Then we sketch the proofs of the main theorems presented in Section 1.

2.1 Proof of Theorem 1.4

This Section 2.1 may be skipped for a first reading.

Under Assumption 1 on page 8, existence of a $\mathbf{U}(1)$ -principal bundle $\pi : P \rightarrow M$ with a connection A satisfying the condition (1.11) is standard in differential geometry. See [25][42, prop 8.3.1]. Notice that the connection one form A satisfying (1.11) is determined up to addition by a connection A_0 with $dA_0 = 0$ i.e. a flat connection. Below we choose a connection appropriately so that the second claim in Theorem 1.4 holds true. We first prove the following lemma.

Lemma 2.1. *Let $\pi : P \rightarrow M$ be a prequantum bundle over a closed symplectic manifold (M, ω) with a connection 1-form A such that $dA = -i(2\pi)(\pi^*\omega)$. Let $f : M \rightarrow M$ be a diffeomorphism. The following conditions are equivalent*

- (1) *There exists an equivariant lift $\tilde{f} : P \rightarrow P$ preserving the connection.*
- (2) *For any closed path $\gamma \subset M$, we have*

$$h_A(f(\gamma)) = h_A(\gamma) \tag{2.1}$$

where $h_A(\gamma) \in \mathbf{U}(1)$ denotes the holonomy along γ (with respect to the connection A).

(3) f preserves the curvature two-form on M :

$$f^*\omega = \omega, \quad \omega := \frac{i}{2\pi} (dA) \quad (2.2)$$

and the homomorphism

$$r_A : H_1(M, \mathbb{Z}) \rightarrow \mathbf{U}(1), \quad r_A([\gamma]) = \frac{h_A(f(\gamma))}{h_A(\gamma)} \quad (2.3)$$

(which is well-defined if (2.2) holds true) is trivial:

$$r_A \equiv 1 \quad (2.4)$$

The equivariant lift \tilde{f} as above is unique up to a global phase (if it exists): \tilde{g} is another equivariant lift if and only if there exists $e^{i\theta} \in \mathbf{U}(1)$ such that $\tilde{g} = e^{i\theta} \tilde{f}$.

Proof. The proof of Lemma 2.1 can be found in [43, Prop 2.2 p.632]. We give it here since some details of the proof will be useful later on. The idea of the proof is illustrated in Figure 2.1.

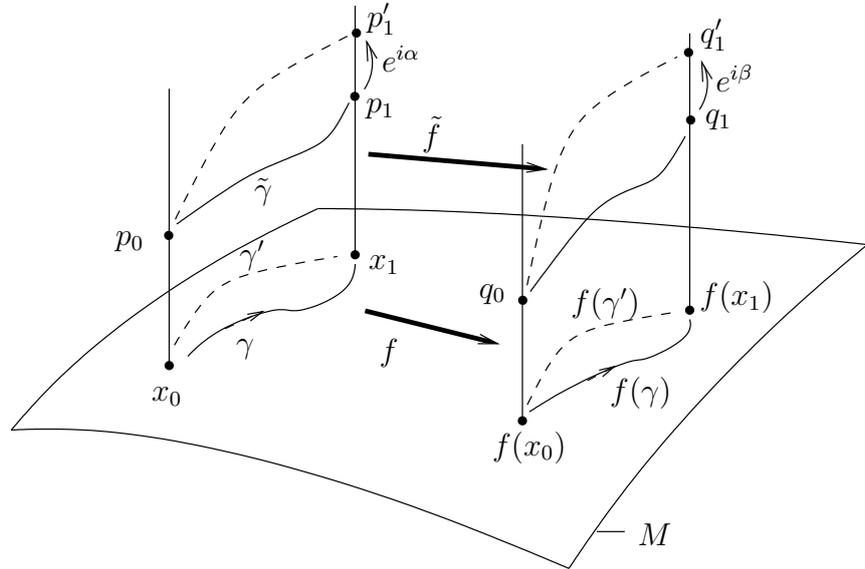


Figure 2.1: Picture of \tilde{f} construction.

The assertion (1) \Rightarrow (2) is obvious because holonomy is defined from the connection

preserved by \tilde{f} hence holonomy of closed paths is preserved by f .

To prove the assertion (2) \Rightarrow (1) we construct \tilde{f} explicitly. Let $p_0 \in P$ and $x_0 = \pi(p_0) \in M$ be some given points of reference. We choose $q_0 \in P_{f(x_0)}$ an arbitrary point in the fiber $P_{f(x_0)}$ and set $\tilde{f}(p_0) = q_0$. By equivariance, this defines \tilde{f} on the fiber P_{x_0} : for any $e^{i\theta} \in \mathbf{U}(1)$, we have to set $\tilde{f}(e^{i\theta}p_0) = e^{i\theta}q_0$. Let $x_1 \in M$ be any point. We want to define \tilde{f} on the fiber P_{x_1} . We choose a path $\gamma : [0, 1] \rightarrow M$ which joins $\gamma(0) = x_0$ to $\gamma(1) = x_1$ and then take the unique horizontal lift³ $\tilde{\gamma} : [0, 1] \rightarrow P$ of γ such that $\tilde{\gamma}(0) = p_0$. Put $p_1 := \tilde{\gamma}(1) \in P_{x_1}$. Next let $\widetilde{f(\gamma)}$ be the unique horizontal lift of $f(\gamma)$ such that $\widetilde{f(\gamma)}(0) = q_0$. Since \tilde{f} preserves the connection, it send $\tilde{\gamma}$ to a horizontal lift of $f(\gamma)$. We define $\tilde{f}(p_1) = q_1 := \widetilde{f(\gamma)}(1) \in P_{f(x_1)}$. For equivariance, we define \tilde{f} on the fiber P_{x_1} so that $\tilde{f}(e^{i\theta}p_1) = e^{i\theta}q_1$ for any $e^{i\theta} \in \mathbf{U}(1)$.

The definition of \tilde{f} described above depends a priori on the choice of the path γ . Below we check that the condition (2) guarantees the well definiteness (or independence of the choice of the path γ) of this definition. Suppose that γ' is another path such that $\gamma'(0) = x_0$ and $\gamma'(1) = x_1$ and that we define $p'_1 \in P_{x_1}$ and $q'_1 \in P_{f(x_1)}$ in the similar manner as above using $\tilde{\gamma}$ in the place of γ . Then we have $p'_1 = e^{i\alpha}p_1$ for some $e^{i\alpha} \in \mathbf{U}(1)$ and $q'_1 = e^{i\beta}q_1$ for some $e^{i\beta} \in \mathbf{U}(1)$. From the definition above, we have $\tilde{f}(p'_1) = \tilde{f}(e^{i\alpha}p_1) = e^{i\alpha}\tilde{f}(p_1) = e^{i\alpha}q_1$. For well definiteness, we have to check that $q'_1 = \tilde{f}(p'_1)$ or, equivalently, that $e^{i\alpha} = e^{i\beta}$. Note that $\Gamma := \gamma' \circ \gamma^{-1}$ is a closed path with holonomy⁴ $h_A(\Gamma) = e^{i\alpha}$ and $f(\Gamma) = f(\gamma') \circ f(\gamma)^{-1}$ is that with holonomy $h_A(f(\Gamma)) = e^{i\beta}$. Therefore the required condition $e^{i\alpha} = e^{i\beta}$ is equivalent to the condition (2). By construction \tilde{f} preserves the horizontal bundle hence the connection A . We have obtained (1).

Let us show that (2) and (3) are equivalent. Let $\gamma = \partial\sigma$ be a closed path which borders a surface $\sigma \subset M$ i.e. $[\gamma] = 0$ in $H_1(M, \mathbb{Z})$. The curvature formula [11] gives the holonomy as

$$h_A(\gamma) = \exp\left(-i2\pi \int_{\sigma} \omega\right).$$

Also

$$h_A(f(\gamma)) = \exp\left(-i2\pi \int_{f(\sigma)} \omega\right) = \exp\left(-i2\pi \int_{\sigma} f^*\omega\right).$$

The condition $h_A(f(\gamma)) = h_A(\gamma)$ for any closed path $\gamma = \partial\sigma$ as above is therefore equivalent to the local condition $f^*\omega = \omega$. In that case, for any closed paths γ and γ' such that $[\gamma] = [\gamma'] \in H_1(M, \mathbb{Z})$, we have $h_A(f(\gamma) \circ f(\gamma')^{-1}) = h_A(\gamma \circ (\gamma')^{-1})$, and hence

$$\frac{h_A(f(\gamma'))}{h_A(\gamma')} = \frac{h_A(f(\gamma))}{h_A(\gamma)}.$$

³By definition, $\tilde{\gamma} \in P$ is a *horizontal lift* of the path $\gamma(t) \in M$ if $\pi(\tilde{\gamma}(t)) = \gamma(t)$ and if the tangent vector is horizontal at every point: $A\left(\frac{d\tilde{\gamma}}{dt}\right) = 0$. It does not depend on the parametrization of γ .

⁴By definition, the *holonomy* of a closed path $\Gamma(t) \in M$, $\Gamma(1) = \Gamma(0)$ is $h(\Gamma) \in \mathbf{U}(1)$ computed as follows. We construct $\tilde{\Gamma}(t) \in P$, a horizontal lift of Γ and write $\tilde{\Gamma}(1) = e^{ih(\Gamma)}\tilde{\Gamma}(0) \in \pi^{-1}(\Gamma(0))$.

Therefore the map (2.3) is well defined. Now the equivalence of the conditions (2) and (3) is obvious. \square

The next lemma give the choice of the connection in the latter statement of Theorem 1.4.

Lemma 2.2. *Let $\pi : P \rightarrow M$ be a prequantum bundle over a closed symplectic manifold (M, ω) with connection 1-form A such that $dA = -i(2\pi)(\pi^*\omega)$. Let $f : M \rightarrow M$ a symplectic diffeomorphism and $f_* : H_1(M, \mathbb{R}) \rightarrow H_1(M, \mathbb{R})$ the linear map induced in the homology group. If Assumption 2 on page 8 holds, there exists a flat connection A_0 such that (2.4) holds for the modified connection $A + A_0$.*

Proof. If A_0 is a flat connection (i.e. $dA_0 = 0$) let $A' = A + A_0$ be a modified connection. For a closed path γ the modified holonomy is

$$h_{A'}(\gamma) = h_A(\gamma) \cdot h_{A_0}(\gamma).$$

We have a well-defined homomorphism $P_{A_0} : H_1(M, \mathbb{Z}) \rightarrow \mathbb{R}/\mathbb{Z}$, called the *period map*, such that

$$h_{A_0}(\gamma) = e^{i2\pi P_{A_0}(\gamma)}. \quad (2.5)$$

Suppose that f is symplectic i.e. $f^*\omega = \omega$. For the connections A and $A' = A + A_0$, we have the relation:

$$\frac{h_{A'}(f(\gamma))}{h_{A'}(\gamma)} = \frac{h_A(f(\gamma))}{h_A(\gamma)} \frac{h_{A_0}(f(\gamma))}{h_{A_0}(\gamma)}$$

or, in terms of the maps (2.3) and (2.5), we have

$$r_{A'} = r_A \exp(i2\pi(P_{A_0}(f_* - I))).$$

Hence if we choose the flat connection A_0 so that

$$\exp(i2\pi(P_{A_0}(f_* - I))) = r_A,$$

then the condition (2.4) is realized for the modified connection $A' = A + A_0$. From Assumption 2, this is possible. Indeed, if we can write $r_A = e^{i2\pi R_A}$ with $R_A : H_1(M, \mathbb{Z}) \rightarrow \mathbb{R}$ and choose a flat connection A_0 so that $P_{A_0} = -R_A(f_* - I)^{-1}$. \square

From Lemma 2.1, the conclusion of the corollary above implies that there is an equivariant lifted map $\tilde{f} : P \rightarrow P$, which is unique up to a global phase. This proves Theorem 1.4.

2.2 Semiclassical description of the prequantum operator \hat{F}_N

We give a local expression of the transfer operator \hat{F}_N defined in (1.18) with respect to local charts and local trivialization of the bundle P . These local expressions will be useful in the sequel of the paper.

As in Section 1.2.2, let $(U_\alpha)_{\alpha \in I}$ be a finite collection of simply connected open subsets which cover M and, for every open set $U_\alpha \subset M$, let

$$\tau_\alpha : U_\alpha \rightarrow P$$

be a local section of the bundle. For a given $N \in \mathbb{Z}$, an equivariant function $u \in C_N^\infty(P)$ defines the set of *associated functions*⁵ $u_\alpha : U_\alpha \rightarrow \mathbb{C}$, $\alpha \in I$, defined by

$$u_\alpha(x) := u(\tau_\alpha(x)) \quad (2.6)$$

for $x \in U_\alpha$. Conversely one reconstructs u from $(u_\alpha)_{\alpha \in I}$ by the relation

$$u(p) = u(e^{i\theta}\tau_\alpha(x)) = e^{iN\theta}u(\tau_\alpha(x)) = e^{iN\theta}u_\alpha(x) \quad \text{for } p = e^{i\theta}\tau_\alpha(x) \text{ and } x \in U_\alpha.$$

Recall the one forms η_α in the local expression (1.9) of the connection A .

Proposition 2.3. “Local expression of \hat{F}_N ”. *Let $u \in C_N^\infty(P)$ and $u' := \hat{F}_N u \in C_N^\infty(P)$ with respective associated functions $u_\alpha(x) = u(\tau_\alpha(x))$ and $u'_\alpha(x) = u'(\tau_\alpha(x))$, with $x \in U_\beta$, $f^{-1}(x) \in U_\alpha$. Then*

$$u'_\beta = e^V \cdot \mathcal{L}_f u_\alpha$$

with

$$(\mathcal{L}_f u_\alpha)(x) := e^{i2\pi N \mathcal{A}_{\beta,\alpha}(x)} u_\alpha(f^{-1}(x)) \quad (2.7)$$

where

$$\begin{aligned} \mathcal{A}_{\beta,\alpha}(x) &= \int_{f^{-1}(\gamma)} \eta_\alpha - \int_\gamma \eta_\beta + c(x_0) \\ &= \int_\gamma ((f^{-1})^*(\eta_\alpha) - \eta_\beta) + c(x_0). \end{aligned} \quad (2.8)$$

In the last integral, $x_0 \in U_\beta$ is any point of reference, $\gamma \subset U_\beta$ is a path from x_0 to x and $c(x_0)$ does not depend on x . In other words, the prequantum operator \hat{F}_N is expressed locally by the transfer operator $e^V \mathcal{L}_f$ defined in (2.7).

⁵In the language of associate line bundle, these functions ψ_α are sections of $L^{\otimes N}$ expressed with respect to the trivializations.

Remark 2.4. Notice that the integral (2.8) does not depend on the path γ from x_0 to x because the one form $((f^{-1})^*(\eta_\alpha) - \eta_\beta)$ is closed. Indeed $d((f^{-1})^*(\eta_\alpha) - \eta_\beta) = (f^{-1})^*\omega - \omega = 0$ since f^{-1} is symplectic.

Proof. From definition (1.16) of the transfer operator

$$u'_\beta(x) = u'(\tau_\beta(x)) = (\hat{F}u)(\tau_\beta(x)) e^{V(x)} u(\tilde{f}^{-1}(\tau_\beta(x))).$$

To prove the proposition, we have to show $\tilde{f}^{-1}(\tau_\beta(x)) = \mathcal{L}_f u_\alpha$. Let $\gamma \subset U_\beta$ be a path from x_0 to x . Let $\tilde{\gamma} : t \rightarrow \tilde{\gamma}(t)$ be the lifted path parallel transported above γ starting from $\tau_\beta(x_0)$ and ending at point p . (See Figure 2.2.) Since the connection one form vanishes along the path $\tilde{\gamma}$, we have

$$0 = (T_\beta^* A) \left(\frac{d\tilde{\gamma}_\beta}{dt} \right) = (id\theta - i2\pi\eta_\beta) \left(\frac{d\tilde{\gamma}_\beta}{dt} \right)$$

with $\tilde{\gamma}_\beta = T_\beta^{-1}(\tilde{\gamma})$. From the construction of the lifted map \tilde{f} in the proof of Lemma 2.1, we have

$$p = e^{ia(x)} \tau_\beta(x) \tag{2.9}$$

with

$$a(x) = \int_{\tilde{\gamma}} d\theta = 2\pi \int_{\gamma} \eta_\beta.$$

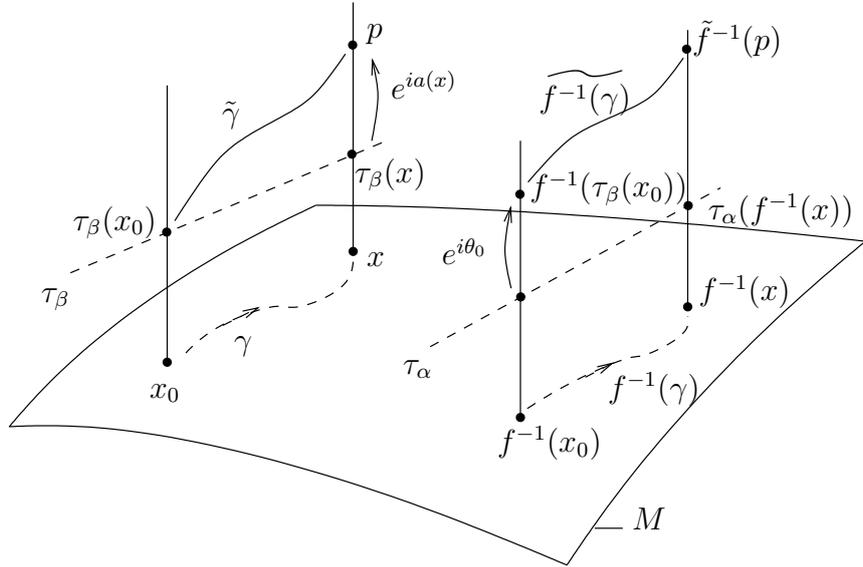


Figure 2.2:

Let θ_0 given by $\tilde{f}^{-1}(\tau_\beta(x_0)) = e^{i\theta_0} \tau_\alpha(f^{-1}(x_0))$. Similarly we have

$$\tilde{f}^{-1}(p) = e^{i\theta_0} e^{ia'(x)} \tau_\alpha(f^{-1}(x))$$

with

$$a'(x) = 2\pi \int_{f^{-1}(\gamma)} \eta_\alpha.$$

From equivariance of \tilde{f} and (2.9), we have $\tilde{f}^{-1}(p) = e^{ia(x)} \tilde{f}^{-1}(\tau_\beta(x))$. Therefore

$$\tilde{f}^{-1}(\tau_\beta(x)) = e^{-ia(x)} \tilde{f}^{-1}(p) = e^{-ia(x)} e^{i\theta_0} e^{ia'(x)} \tau_\alpha(f^{-1}(x)) = e^{i2\pi \mathcal{A}_{\beta,\alpha}(x)} \tau_\alpha(f^{-1}(x))$$

with

$$\mathcal{A}_{\beta,\alpha}(x) = \int_{f^{-1}(\gamma)} \eta_\alpha - \int_\gamma \eta_\beta + \frac{\theta_0}{2\pi}.$$

We conclude

$$\begin{aligned} u'_\beta(x) &= e^{V(x)} u(\tilde{f}^{-1}(\tau_\beta(x))) = e^{V(x)} u(e^{i2\pi \mathcal{A}_{\beta,\alpha}(x)} \tau_\alpha(f^{-1}(x))) \\ &= e^{V(x)} e^{i2\pi N \mathcal{A}_{\beta,\alpha}(x)} u(\tau_\alpha(f^{-1}(x))) \\ &= e^{V(x)} e^{i2\pi N \mathcal{A}_{\beta,\alpha}(x)} u_\alpha(f^{-1}(x)) \end{aligned}$$

as required. □

The following composition formula should be obvious but will be useful.

Lemma 2.5. “Composition formula”. *If $f_1, f_2 : M \rightarrow M$ are symplectic diffeomorphism with corresponding prequantum transfer operators $\hat{F}_{f_1} u := u \circ \tilde{f}_1^{-1}$, $\hat{F}_{f_2} u := u \circ \tilde{f}_2^{-1}$ then*

$$\hat{F}_{f_2} \circ \hat{F}_{f_1} = \hat{F}_{f_2 \circ f_1}$$

(equality up to some global rotation). Consequently in local charts their local expression (2.7) satisfy

$$\mathcal{L}_{f_2} \circ \mathcal{L}_{f_1} = \mathcal{L}_{f_2 \circ f_1} \tag{2.10}$$

with appropriate setting of local trivializations.

The next few results will concern some semiclassical aspects of the transfer operator. The first easy result but of fundamental importance is the following Proposition. (It is not used in the proofs in this paper and the explanation below may be a bit sloppy. But it gives the background of the following argument.)

Remark 2.6. For the general definition of the Fourier Integral Operator, we refer to Martinez [28], Evans-Zworski [14] or Duistermaat [13]. If the reader is not familiar with Fourier Integral Operator, it is enough for the reading of this paper to understand the rough idea of a Fourier integral operator \hat{F} , which is quite simple as explained in [10]: if a function ψ is localized at point $x \in M$ and its Fourier transform is localized at point $\xi \in T_x^*M$ (which means that these functions decay fast outside these points and we say that φ is

micro-localized at $(x, \xi) \in T^*M$) then the operator \hat{F} transforms this function ψ to a function ψ' micro-localized in another point $(x', \xi') \in T^*M$. The map $F : (x, \xi) \rightarrow (x', \xi')$ is called the associated canonical map. Note that we are interested in the situation $|\xi| \gtrsim N = (2\pi\hbar)^{-1} \gg 1$ and, in the limit $N \rightarrow \infty$ (or $\hbar \rightarrow +0$), we will normalize ξ by multiplying \hbar .

Proposition 2.7. *The prequantum transfer operator \hat{F}_N is a Fourier Integral Operator if we view it in the local trivializations as in (2.7). The associated canonical map on the cotangent space is given by*

$$F_{\alpha,\beta} : T^*U_\alpha \rightarrow T^*U_\beta, \quad F_{\alpha,\beta} \left(\begin{array}{c} x \\ \xi \end{array} \right) = \left(\begin{array}{c} f(x) \\ {}^t(Df_{x'}^{-1})(\xi + \eta_\alpha(x)) - \eta_\beta(x') \end{array} \right) \quad (2.11)$$

where $x \in U_\alpha$, $f(x) \in U_\beta$ and $\xi \in T_x^*U_\alpha$. The map $F_{\alpha,\beta}$ preserves the canonical symplectic structure

$$\Omega = \sum_{j=1}^{2d} dx_j \wedge d\xi_j$$

on the cotangent space on T^*U_α .

Proof. The transfer operator is given in local chart by (2.7). This expression shows that it is the composition of two operators $\hat{F}_2 \circ \hat{F}_1$ where \hat{F}_1 is the pull-back by the diffeomorphism f^{-1} and \hat{F}_2 is the multiplication by a phase function. Both operators are basic examples of F.I.O as explained in [28, chap.5] and the Proposition 2.7 follows. See the explanation below. \square

We give here a more detailed explanation of the associated canonical map (2.11). From (2.7), the transfer operator \hat{F}_N can be decomposed as elementary operators. Let

$$\hat{F}_1 : u(x) \rightarrow u(f^{-1}(x)).$$

This is a F.I.O. with canonical map

$$F_1 \left(\begin{array}{c} x \\ \xi \end{array} \right) = \left(\begin{array}{c} x' \\ \xi' \end{array} \right) = \left(\begin{array}{c} f(x) \\ {}^t(Df_{x'}^{-1})\xi \end{array} \right) \quad (2.12)$$

Indeed it is clear that $\text{supp}(\psi)$ is transported to $f(\text{supp}(\psi))$ hence $x' = f(x)$. Also an oscillating function $u(x) = e^{\frac{i}{\hbar}\xi \cdot x}$ is transformed to $u'(y) = e^{\frac{i}{\hbar}\xi \cdot f^{-1}(y)}$ and, for $y = f(x) + y'$ with $|y'| \ll 1$, we have $f^{-1}(y) = x + Df_{x'}^{-1} \cdot y' + o(|y'|)$, hence

$$u'(y) \simeq e^{\frac{i}{\hbar}\xi \cdot (x + Df_{x'}^{-1} \cdot y')} = C e^{\frac{i}{\hbar} {}^t(Df_{x'}^{-1})\xi \cdot y} = C e^{\frac{i}{\hbar}\xi' \cdot y}$$

with $\xi' = {}^t(Df_{x'}^{-1})\xi$ and $C = e^{\frac{i}{\hbar}\xi \cdot (x - Df_{x'}^{-1} \cdot f(x))}$. We deduce (2.12).

Next we consider a multiplication operator by a “fast oscillating phase” (recall $\hbar \ll 1$):

$$\hat{F}_2 : \psi(x) \rightarrow e^{iS(x)/\hbar} \psi(x)$$

For the same reasons, it is a F.I.O. and its canonical map is

$$F_2 \left(\begin{array}{c} x \\ \xi \end{array} \right) = \left(\begin{array}{c} x \\ \xi + dS(x) \end{array} \right)$$

Indeed $u(x) = e^{\frac{i}{\hbar}\xi \cdot x}$ is transformed to $u'(y) = e^{\frac{i}{\hbar}(\xi \cdot y + S(y))}$ and, for $y = x + y'$ with $|y'| \ll 1$, we have

$$u'(x') \simeq e^{\frac{i}{\hbar}(\xi \cdot y + dS \cdot y)}$$

with $\xi' = \xi + dS$. From these two previous examples and (2.7), we can deduce (2.11). Notice that the multiplication operator by e^V does not appear in the canonical map because it is not a “fast oscillating function” (in the limit $\hbar \rightarrow 0$).

Notice that the canonical maps $F_{\alpha,\beta}$ in the last proposition is not consistent for different choice of local trivializations if we simply regard T^*U_α and T^*U_β as a subset of T^*M . This is because of the curvature of the bundle P . The following proposition gives a global and geometric description of the canonical map (2.11).

Proposition 2.8. *Consider the following change of variable on T^*U_α for every $\alpha \in I$:*

$$\left(\begin{array}{c} x \\ \xi \end{array} \right) \in T^*U_\alpha \rightarrow \left(\begin{array}{c} x \\ \zeta := \xi + \eta_\alpha(x) \end{array} \right) \in T^*M \quad (2.13)$$

*Then the **canonical map** (2.11) get the simpler and global expression (independent on the set U_α) on the phase space T^*M*

$$F : T^*M \rightarrow T^*M, \quad F \left(\begin{array}{c} x \\ \zeta \end{array} \right) = (DF^*)^{-1} \left(\begin{array}{c} x \\ \zeta \end{array} \right) = \left(\begin{array}{c} f(x) \\ {}^t(Df_{x'}^{-1})\zeta \end{array} \right). \quad (2.14)$$

The symplectic form preserved by F is:

$$\Omega = \sum_{j=1}^{2d} (dx_j \wedge d\zeta_j) + \pi^*(\omega). \quad (2.15)$$

Remark 2.9. The change of variables (2.13) to get the global geometric description (2.14) is standard in the semiclassical problems involving large magnetic fields on manifolds. In a more geometric (but more abstract) approach one can show that the cotangent space (T^*M, Ω) is interpreted as an “affine cotangent space” and can be geometrically defined as the space of connections on the principal bundle $P \rightarrow M$, see [8, appendix].

Proof. The relation (2.14) is obvious from (2.11) and (2.13). To prove (2.15), we write in coordinates

$$\eta_\alpha = \sum_{j=1}^{2d} \eta_j dx_j$$

and the fact

$$\omega = d\eta_\alpha = \sum_{i,j} \left(\frac{\partial \eta_j}{\partial x_i} \right) (dx_i \wedge dx_j).$$

Then

$$\begin{aligned} \Omega &= \sum_{j=1}^{2d} dx_j \wedge d\xi_j = \sum_{j=1}^{2d} dx_j \wedge (d\zeta_j - d\eta_j) = \sum_{j=1}^{2d} dx_j \wedge \left(d\zeta_j - \sum_{i=1}^{2d} \left(\frac{\partial \eta_j}{\partial x_i} \right) dx_i \right) \\ &= \sum_{j=1}^{2d} (dx_j \wedge d\zeta_j) + \omega \end{aligned}$$

□

In relation to the global expression (2.14) of the canonical map associated to the pre-quantum transfer operator \hat{F}_N , it should be natural to introduce the following definition.

Definition 2.10. The *trapped set* $K \Subset T^*M$ is the set of points $(x, \xi) \in T^*M$ which do not escape to infinity in the past neither in the future $n \rightarrow \pm\infty$ under the dynamics of the canonical map F :

$$K := \{(x, \xi), \exists C > 0, |\xi(n)| \leq C, \forall n \in \mathbb{Z}, \text{ with } (x(n), \xi(n)) := F^n(x, \xi)\}.$$

In terms of the theory of dynamical systems, the trapped set K is the *non-wandering set* for the dynamical system generated by F .

As one expects naturally, we will show that the trapped set $K \subset T^*M$ and the dynamics of the canonical map F on its vicinity are the main geometrical objects which explain all the results of this paper about the resonances. It should be noted that, in the context of quantum scattering, this idea has been investigated extensively (but, for different types of evolution equations than the prequantum transfer operator), initiated by Combes and Baslev in the 70' and Helffer-Sjöstrand in the 80'.

Proposition 2.11. “*Description of the trapped set K* ”. The trapped set $K \subset T^*M$ is the zero section:

$$K = \{(x, \zeta) \in T^*M \mid x \in M, \zeta = 0\}. \quad (2.16)$$

This is a symplectic submanifold of (T^*M, Ω) isomorphic to (M, ω) . For every point $\rho \in K$ the tangent space is decomposed as an Ω -orthogonal sum of symplectic linear subspaces

$$T_\rho(T^*M) = T_\rho K \oplus^\perp (T_\rho K)^\perp \quad (2.17)$$

Moreover each part is decomposed into isotropic unstable/stable linear spaces

$$T_\rho K = \mathcal{E}_u^{(1)}(\rho) \oplus \mathcal{E}_s^{(1)}(\rho), \quad (T_\rho K)^\perp = \mathcal{E}_u^{(2)}(\rho) \oplus \mathcal{E}_s^{(2)}(\rho) \quad (2.18)$$

where the subspaces $\mathcal{E}_\sigma^{(i)}(\rho)$ for $i = 1, 2$ and $\sigma = s, u$ are d -dimensional subspaces defined by

$$\mathcal{E}_\sigma^{(1)}(\rho) = T_\rho K \cap E_\sigma^*(\rho), \quad \mathcal{E}_\sigma^{(2)}(\rho) = (T_\rho K)^\perp \cap E_\sigma^*(\rho) \quad \text{for } \sigma = s, u.$$

Here the decomposition

$$T_\rho(T^*M) = E_u^*(\rho) \oplus E_s^*(\rho)$$

is that into the unstable and stable subspaces with respect to the action of DF , (which is dual to the decomposition (1.1)):

$$E_u^*(\rho) := \{v \in T_\rho(T^*M) \mid |DF_\rho^{-n}(v)| \rightarrow 0 \text{ as } n \rightarrow \infty\}$$

and

$$E_s^*(\rho) := \{v \in T_\rho(T^*M) \mid |DF_\rho^n(v)| \rightarrow 0 \text{ as } n \rightarrow \infty\}.$$

All the decompositions above are invariant by the map F . See figure 2.3.

Remark 2.12. There is another F -invariant decomposition:

$$\underbrace{T_\rho(T^*M)}_{4d} = \underbrace{T_\rho K}_{2d} \oplus \underbrace{T_x^*M}_{2d}$$

with $x = \tilde{\pi}(\rho)$ and T_x^*M the fiber of the cotangent space. However this sum is not Ω -orthogonal and moreover T_x^*M is Ω -Lagrangian.

Proof. First it is clear that the trapped set is the zero section $\{(x, \zeta), \zeta = 0\}$ since it is invariant and, if $\zeta \neq 0$, we have $|F^n(x, \zeta)| \rightarrow \infty$ at least either as $n \rightarrow \infty$ or $n \rightarrow -\infty$. From (2.15) $\Omega_{/K} = \omega$ therefore $\tilde{\pi} : (K, \omega) \rightarrow (M, \omega)$ is a symplectomorphism. The symplectic maps $f : M \rightarrow M$ and $F : K \rightarrow K$ are conjugated. For every point $\rho \in K$, $T_\rho K$ is a linear symplectic subspace of the symplectic linear space $T_\rho(T^*M)$ and therefore admits a unique symplectic orthogonal $(T_\rho K)^\perp$. The decomposition (2.17) is invariant

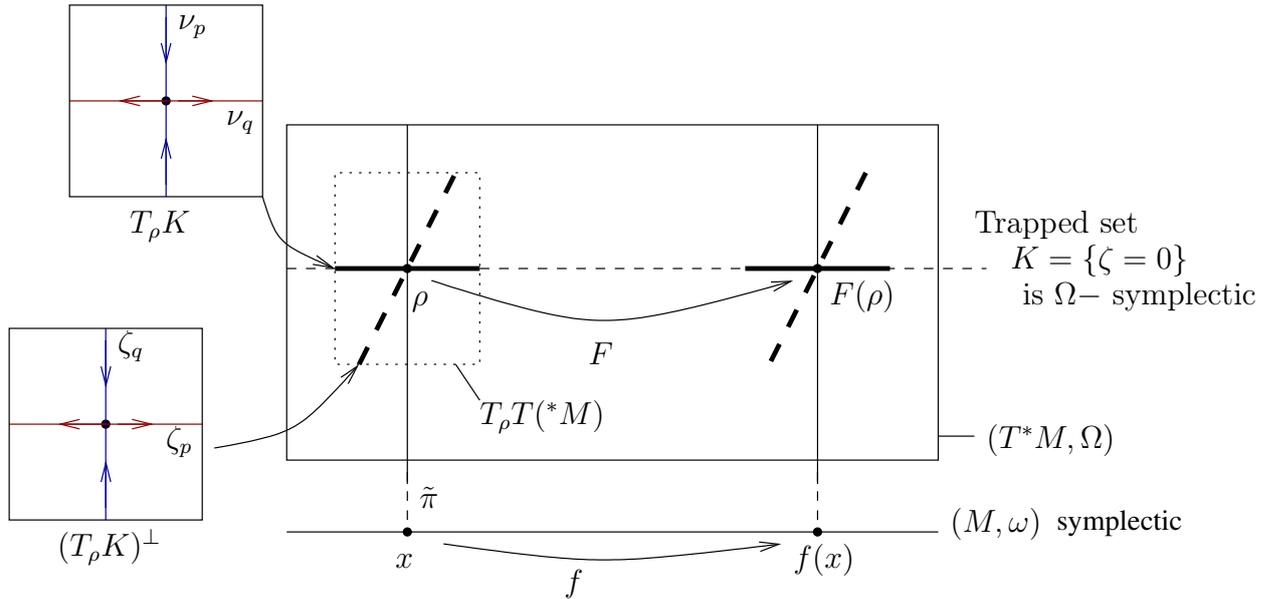


Figure 2.3: The decompositions of the tangent space $T_\rho(T^*M)$.

under the map F because the trapped set K is invariant and because F preserves the symplectic form Ω . \square

In the next proposition, we introduce convenient local coordinates, called normal or Darboux coordinates. We will use them later in the proof.

Proposition 2.13. “Normal coordinates”. *On a sufficiently small neighborhood U of every point $x \in M$, there exist coordinates*

$$x = (q, p) = (q^1, \dots, q^d, p^1, \dots, p^d)$$

and a trivialization of P such that the connection one-form in (1.9) is given by

$$\eta = \sum_{i=1}^d \left(\frac{1}{2} q^i dp^i - \frac{1}{2} p^i dq^i \right) \quad (2.19)$$

and consequently the symplectic form ω is given by

$$\omega = d\eta = \sum_{i=1}^d dq^i \wedge dp^i. \quad (2.20)$$

On the cotangent bundle T^*U , there is a change of coordinates $\Phi : (x, \xi) \rightarrow (\nu, \zeta)$ where

the variables $\zeta = (\zeta_q^j, \zeta_p^j)_{j=1, \dots, d}$ are already defined in (2.13) as

$$\begin{aligned}\zeta_p^i &: = \xi_p^i + \eta_p^i = \xi_p^i + \frac{1}{2}q^i \\ \zeta_q^i &: = \xi_q^i + \eta_q^i = \xi_q^i - \frac{1}{2}p^i\end{aligned}\tag{2.21}$$

while $\nu = (\nu_q^j, \nu_p^j)_{j=1, \dots, d}$ are given by

$$\begin{aligned}\nu_q^j &= q^j - \zeta_p^j = \frac{1}{2}q^j - \xi_p^j \\ \nu_p^j &= p^j + \zeta_q^j = \frac{1}{2}p^j + \xi_q^j\end{aligned}\tag{2.22}$$

This change of variables transforms the symplectic form Ω in (2.15) to the normal form:

$$\Omega = \sum_{j=1}^d (d\nu_q^j \wedge d\nu_p^j) + (d\zeta_p^j \wedge d\zeta_q^j).\tag{2.23}$$

Remark 2.14. Recall from (2.16) that $T_\rho K = \{(\nu, \zeta), \zeta = 0\}$. Then (2.23) implies that its symplectic orthogonal is given by $(T_\rho K)^\perp = \{(\nu, \zeta), \nu = 0\}$. In other words, (ν, ζ) are symplectic coordinates related to the decomposition (2.17). These coordinates were introduced in the paper [16] treating the linear case, under the different names $(Q_1, P_1) \equiv (\nu_q, \nu_p)$ and $(Q_2, P_2) \equiv (\zeta_p, \zeta_q)$.

Proof. Darboux theorem on symplectic structure (see [1]) tells that, if we take sufficiently small neighborhood U of x , there exist coordinates $x = (q, p) = (q^1, \dots, q^d, p^1, \dots, p^d)$ such that the symplectic form is expressed in the normal form $\omega = dq \wedge dp = \sum_{i=1}^d dq^i \wedge dp^i$. Take any local smooth section $\tau' : U \rightarrow P$ and let η' be the local connection one form (see (1.9)) with respect to the corresponding local trivialization of P . Since we have $d(\eta' - \eta) = \omega - \omega = 0$ for η in (2.19), there is a smooth function $\chi : U \rightarrow \mathbb{R}$ such that $\eta' - \eta = \frac{1}{2\pi}d\chi$. Setting $\tau = e^{i\chi}\tau'$ and recalling the formula (1.10), we see that the former statement of the proposition holds for the coordinates $x = (q, p) = (q^1, \dots, q^d, p^1, \dots, p^d)$ and the trivialization of P associated to the local smooth section τ thus taken.

We prove the latter statement. (2.13) and (2.19) imply (2.21). Clearly (2.21) and (2.22) are coordinates on U as we can give the inverse explicitly. Starting from (2.15) we

get

$$\begin{aligned}
\Omega &= \sum_{j=1}^d (dq^j \wedge d\zeta_q^j + dp^j \wedge d\zeta_p^j + dq^j \wedge dp^j) \\
&= \sum_{j=1}^d ((d\nu_q^j + d\zeta_p^j) \wedge d\zeta_q^j + (d\nu_p^j - d\zeta_q^j) \wedge d\zeta_p^j + d(\nu_q^j + \zeta_p^j) \wedge (d\nu_p^j - d\zeta_q^j)) \\
&= \sum_{j=1}^d d\nu_q^j \wedge d\nu_p^j - d\zeta_q^j \wedge d\zeta_p^j = \sum_{j=1}^d d\nu_q^j \wedge d\nu_p^j + d\zeta_p^j \wedge d\zeta_q^j.
\end{aligned}$$

This completes the proof. \square

Remark 2.15. Since the symplectic 2-form $\omega = \sum_j dq^j \wedge dp^j$ on $T_x^*M \equiv \mathbb{R}^{2d}$ is non degenerate, it defines an isomorphism, called *flat operator*,

$$b : \mathbb{R}^{2d} \rightarrow (\mathbb{R}^{2d})^*, \quad v^b = \omega(v, \cdot).$$

Its inverse is called the *sharp operator*.

$$\sharp = b^{-1} : (\mathbb{R}^{2d})^* \rightarrow \mathbb{R}^{2d}.$$

For a one-form $\alpha \in (\mathbb{R}^{2d})^*$, α^\sharp is defined by the relation $\alpha = \omega(\alpha^\sharp, \cdot)$. In coordinates, for

$$v = \sum_{j=1}^n v_q^j \frac{\partial}{\partial q^j} + v_p^j \frac{\partial}{\partial p^j} \equiv (v_q, v_p) \in \mathbb{R}^{2d} \quad \text{and} \quad \alpha = \sum_{j=1}^n \alpha_q^j dq^j + \alpha_p^j dp^j \equiv (\alpha_q, \alpha_p) \in (\mathbb{R}^{2d})^*$$

we have

$$v^b = \sum_{j=1}^n -v_p^j dq^j + v_q^j dp^j \equiv (-v_p, v_q) \in (\mathbb{R}^{2d})^* \quad (2.24)$$

$$\alpha^\sharp = \sum_{j=1}^n \alpha_p^j \frac{\partial}{\partial q^j} - \alpha_q^j \frac{\partial}{\partial p^j} \equiv (\alpha_p, -\alpha_q) \in \mathbb{R}^{2d} \quad (2.25)$$

We also have

$$\alpha(v) = -v^b(\alpha^\sharp) \in \mathbb{R}, \quad b^t = -b, \quad \sharp^t = -\sharp. \quad (2.26)$$

where b^t, \sharp^t denote the transposed maps. Using these notation, the relation (2.22) can be written in a more intrinsic manner:

$$\nu := x - \zeta^\sharp. \quad (2.27)$$

Proposition 2.16. *The normal coordinates $(\nu_q, \nu_p, \zeta_p, \zeta_q)$ introduced in the last proposition can be chosen so that the differential of the coordinate map $\Phi : \pi^{-1}(U) \rightarrow \mathbb{R}^{4d}$ at any point $\rho \in K$ such that $\tilde{\pi}(\rho) = x \in U$ carries the subspaces $\mathcal{E}_u^{(1)}(\rho)$, $\mathcal{E}_s^{(1)}(\rho)$, $\mathcal{E}_u^{(2)}(\rho)$, $\mathcal{E}_s^{(2)}(\rho)$ in (2.18) to the subspaces*

$$\begin{aligned} \mathbb{R}_{\nu_q}^d &:= \{(\nu_q, 0, 0, 0) \mid \nu_q \in \mathbb{R}^d\}, & \mathbb{R}_{\nu_p}^d &:= \{(0, \nu_p, 0, 0) \mid \nu_p \in \mathbb{R}^d\}, \\ \mathbb{R}_{\zeta_p}^d &:= \{(0, 0, \zeta_p, 0) \mid \zeta_p \in \mathbb{R}^d\}, & \mathbb{R}_{\zeta_q}^d &:= \{(0, 0, 0, \zeta_q) \mid \zeta_q \in \mathbb{R}^d\} \end{aligned}$$

respectively. That is to say,

$$\begin{array}{ccc} T_\rho(T^*M) & = & \underbrace{\mathcal{E}_u^{(1)}(\rho) \oplus \mathcal{E}_s^{(1)}(\rho)}_{T_\rho K} \overset{\perp}{\oplus} \underbrace{\mathcal{E}_u^{(2)}(\rho) \oplus \mathcal{E}_s^{(2)}(\rho)}_{(T_\rho K)^\perp} \\ D\Phi \downarrow & & \downarrow \qquad \qquad \downarrow \\ T^*\mathbb{R}_{(q,p)}^{2d} & = & \underbrace{\left(\mathbb{R}_{\nu_q}^d \oplus \mathbb{R}_{\nu_p}^d\right)}_{T^*\mathbb{R}_{\nu_q}^d} \overset{\perp}{\oplus} \underbrace{\left(\mathbb{R}_{\zeta_p}^d \oplus \mathbb{R}_{\zeta_q}^d\right)}_{T^*\mathbb{R}_{\zeta_p}^d} \end{array}$$

With respect to these coordinates the differential of the canonical map $DF_\rho : T_\rho(T^*M) \rightarrow T_{F(\rho)}(T^*M)$ is expressed as

$$\Phi \circ DF_\rho \circ \Phi^{-1} = F^{(1)} \oplus F^{(2)}, \quad F^{(1)} \equiv \begin{pmatrix} A_x & 0 \\ 0 & {}^t A_x^{-1} \end{pmatrix}, \quad F^{(2)} \equiv \begin{pmatrix} A_x & 0 \\ 0 & {}^t A_x^{-1} \end{pmatrix} \quad (2.28)$$

where $A_x : \mathbb{R}^d \rightarrow \mathbb{R}^d$ is an expanding linear map.

Proof. We can take the coordinates $x = (q, p) = (q^1, \dots, q^d, p^1, \dots, p^d)$ in the beginning of the proof of the last proposition so that the stable and unstable subspaces, $E_s(x)$ and $E_u(x)$, correspond to the subspaces given by the equations $p = 0$ and $q = 0$ respectively, because they are Lagrangian subspaces. Then the coordinates and the trivialization constructed in the proof have the required property. The differential of the Anosov map $Df_x : T_x M \rightarrow T_{f(x)} M$ splits according to the invariant decomposition $T_x M = E_u(x) \oplus E_s(x)$ as $Df_x = (A_x, B_x)$ with

$$A_x := Df|_{E_u(x)} : E_u(x) \rightarrow E_u(f(x))$$

and

$$B_x := Df|_{E_s(x)} : E_s(x) \rightarrow E_s(f(x))$$

But since $E_u(x)$ and $E_s(x)$ are Lagrangian subspaces, ω provides an isomorphism $\flat : E_s(x) \rightarrow E_u(x)^*$ by $\flat(S) : U \in E_u(x) \rightarrow \omega(S, U) \in \mathbb{R}$ for $S \in E_s(x)$. Because Df_x is

symplectic, i.e. preserves ω , then $B_x = \flat \circ {}^t A_x^{-1} \circ \flat^{-1}$ is isomorphic to ${}^t A_x^{-1}$ and

$$Df_x \equiv \begin{pmatrix} A_x & 0 \\ 0 & {}^t A_x^{-1} \end{pmatrix} \quad (2.29)$$

□

2.3 The covariant derivative D

The connection one form A on P induces a differential operator D called the covariant derivative. We recall its general definition and give its expression in local coordinates. We will use it for the definition of the rough Laplacian operator Δ in the Section 2.4 and also to treat the affine models on \mathbb{R}^{2d} in Section 4.2.

Recall that the exterior derivative du of a function $u \in C^\infty(P)$ (i.e. the differential of u) is defined at point $p \in P$ by

$$(du)_p(V) = V(u)(p)$$

where $V \in T_p P$ is any tangent vector. The connection one form $A \in C^\infty(P, \Lambda^1 \otimes (i\mathbb{R}))$ on the principal bundle $\pi : P \rightarrow M$ defines a splitting of the tangent space at every point $p \in P$:

$$T_p P = V_p P \oplus H_p P$$

with $H_p P = \text{Ker}(A(p))$ and $V_p P = \text{Ker}((D\pi)(p))$. The subspaces $V_p P$ and $H_p P$ called respectively vertical subspace and horizontal subspace. We will denote

$$H : T_p P \rightarrow H_p P \quad (2.30)$$

the projection onto the horizontal space with $\text{Ker}(H) = V_p P$. Explicitly if $V \in T_p P$ then from (1.8) its horizontal component is

$$HV = V + iA(V) \frac{\partial}{\partial \theta}. \quad (2.31)$$

This can be checked easily from the requirements that $A(HV) = 0$ and $V - HV \in V_p P$.

Definition 2.17. If $u \in C^\infty(P)$ is a smooth function, its *exterior covariant derivative* $Du \in C^\infty(P; \Lambda^1)$ is a one form on P defined by

$$(Du)_p(V) = ((HV)(u))(p) \quad \text{for } p \in P, V \in T_p P \quad (2.32)$$

The operator D is equivariant with respect to (or commutes with) the $\mathbf{U}(1)$ action (1.6) in P and therefore restricts naturally to the operator

$$D : C_N^\infty(P) \rightarrow C_N^\infty(P, \Lambda^1)$$

for every $N \in \mathbb{Z}$.

2.3.1 Expression of D in local charts

Proposition 2.18. *With respect to the local trivialization (1.7) of the bundle P over open sets $U_\alpha \subset M$, if $u \in C_N^\infty(P)$ then Du is expressed as the first order differential operator $D_\alpha : C^\infty(U_\alpha) \rightarrow C^\infty(U_\alpha, \Lambda^1)$ given by:*

$$D_\alpha u_\alpha = du_\alpha + \frac{i}{\hbar} u_\alpha \eta_\alpha \quad (2.33)$$

with $u_\alpha := (u \circ \tau_\alpha) \in C^\infty(U_\alpha)$ and $D_\alpha u_\alpha := (Du) \circ \tau_\alpha \in C^\infty(U_\alpha, \Lambda^1)$. More specifically, in the normal coordinates $x = (q, p) = (q^1, \dots, q^d, p^1, \dots, p^d)$ and the local trivialization on U_α in Proposition 2.13, it is expressed as

$$D_\alpha u_\alpha = \frac{i}{\hbar} \sum_{j=1}^d \left(\widehat{\zeta}_q^j u_\alpha \right) dq^j + \left(\widehat{\zeta}_p^j u_\alpha \right) dp^j \quad (2.34)$$

with the basis (dq, dp) of Λ_x^1 , where $\widehat{\zeta}_q^j$ and $\widehat{\zeta}_p^j$ are the differential operators on $C^\infty(\mathbb{R}^{2d})$ defined respectively by

$$\begin{aligned} \widehat{\zeta}_q^j &:= \widehat{\xi}_q^j - \frac{1}{2} p^j & \text{with } \widehat{\xi}_q^j &:= -i\hbar \frac{\partial}{\partial q^j} & \text{and by} & & (2.35) \\ \widehat{\zeta}_p^j &:= \widehat{\xi}_p^j + \frac{1}{2} q^j & \text{with } \widehat{\xi}_p^j &:= -i\hbar \frac{\partial}{\partial p^j}. \end{aligned}$$

Remark 2.19. Notice that the canonical variables (ζ_q, ζ_p) defined in (2.21) are the symbol of the operators (2.35) as a pseudodifferential operator. In more geometrical terms, the symbol of the covariant derivative $-i\hbar D$ is the one form $\sigma(-i\hbar D) = \zeta dx = \sum_j \zeta_q^j dq^j + \zeta_p^j dp^j$ on T^*U_α . This can be understood as a generalization of the simpler case of the exterior derivative $d : C^\infty(M, \Lambda^p) \rightarrow C^\infty(M, \Lambda^{p+1})$ (by taking $\eta = 0$, i.e. a connection with zero curvature), for which the principal symbol is known to be $\sigma(d) = \frac{i}{\hbar} (\xi dx) \wedge \cdot$, [37, (10.12) on p.162].

Proof. Consider local coordinates $x = (x^1, \dots, x^{2d}) \in U_\alpha \subset M$ and let $V = V^x \frac{\partial}{\partial x} + V^\theta \frac{\partial}{\partial \theta}$ be a vector field on P . From (2.31) and (1.9) we have

$$\begin{aligned} HV &= V + iA(V) \frac{\partial}{\partial \theta} = V + (-d\theta(V) + 2\pi\eta_\alpha(V)) \frac{\partial}{\partial \theta} \\ &= V^x \frac{\partial}{\partial x} + 2\pi\eta_\alpha(V^x) \frac{\partial}{\partial \theta}. \end{aligned}$$

Then, from the definition (2.32),

$$(Du)(V) = (HV)(u) = V^x \frac{\partial u}{\partial x} + 2\pi\eta_\alpha(V^x) \frac{\partial u}{\partial \theta}.$$

Suppose now that $u \in C_N^\infty(P)$ and write $p = e^{i\theta}\tau_\alpha(x) \in P$. Then

$$u(e^{i\theta}\tau_\alpha(x)) = e^{iN\theta}u(\tau_\alpha(x)) = e^{iN\theta}u_\alpha(x)$$

and hence

$$\begin{aligned} (Du)_p(V) &= e^{iN\theta} \left(V^x \frac{\partial u_\alpha}{\partial x} + iN2\pi\eta_\alpha(V^x)u_\alpha \right) \\ &= e^{iN\theta} (du_\alpha(V) + iN2\pi\eta_\alpha(V)u_\alpha) \\ &= e^{iN\theta} \left(du_\alpha + \frac{i}{\hbar}u_\alpha\eta_\alpha \right) (V). \end{aligned}$$

Hence

$$D_\alpha u_\alpha = (Du)_p \circ \tau_\alpha = du_\alpha + \frac{i}{\hbar}u_\alpha\eta_\alpha$$

We obtain the rest of the claims by simple calculation. \square

2.4 The rough Laplacian Δ

In order to define the adjoint operator D^* and the Laplacian $\Delta = D^*D$ which is used in *geometric quantization* we need an additional structure on the manifold M , namely a metric g compatible with ω . References are [41, p.400],[7, p.168],[38, p.504].

2.4.1 Compatible metrics and Laplacian

We recall [34, p.72] that on a symplectic manifold (M, ω) , there exists a Riemannian metric g compatible with ω in the sense that there exists an *almost complex structure* J on M such that

$$\omega(Ju, Jv) = \omega(u, v) \text{ and } g(u, v) = \omega(u, Jv) \text{ for all } x \in M \text{ and } u, v \in T_x M. \quad (2.36)$$

In general J is not integrable, i.e. it is not a complex structure. In the rest of this Section we suppose given such a metric g and an almost complex structure J on M .

The metric g on M induces an equivariant metric g_P on P by declaring that [38, ex.1,ex.2 p.508]:

- (1) for every point $p \in P$, $V_p P \perp H_p P$ are orthogonal,
- (2) on the horizontal space $H_p P$, g_P is the pull back of g by π : $(g_P)_{/H_p P} = \pi^*(g)$
- (3) on the vertical space $V_p P$, g_P is the canonical (Killing) metric on $\mathfrak{u}(1)$ i.e. $\left\| \frac{\partial}{\partial \theta} \right\|_{g_P} = 1$.

This metric g_P induces a L^2 scalar product $\langle \alpha | \beta \rangle_{\Lambda^1(p)}$ in the space of one forms $\Lambda^1(p)$. Using the volume form μ_P on P in (1.15), we define a L^2 scalar product in the space of differential one forms $C^\infty(P, \Lambda^1)$ by

$$\langle \alpha | \beta \rangle_{L^2(P, \Lambda^1)} := \int \langle \alpha(p) | \beta(p) \rangle_{\Lambda^1(p)} d\mu_P(p) \text{ for } \alpha, \beta \in C^\infty(P, \Lambda^1).$$

The L^2 product of functions is of course define by

$$\langle \alpha | \beta \rangle_{L^2(P)} := \int \overline{\alpha(p)} \cdot \beta(p) d\mu_P(p) \quad \text{for } \alpha, \beta \in C^\infty(P, \Lambda^1).$$

Then the operators D^* and Δ are defined as follows.

Definition 2.20. The **adjoint covariant derivative** $D^* : C^\infty(P, \Lambda^1) \rightarrow C^\infty(P)$ is defined by the relation

$$\langle u | D^* \alpha \rangle_{L^2(P, \Lambda^1)} = \langle Du | \alpha \rangle_{L^2(P)} \quad \text{for all } u \in C^\infty(P) \quad \text{and } \alpha \in C^\infty(P, \Lambda^1).$$

The *rough Laplacian* $\Delta : C^\infty(P) \rightarrow C^\infty(P)$ is defined as the composition

$$\Delta = D^* D.$$

The operators introduced above is equivariant, i.e. $D^* R_\theta = R_\theta D^*$ and $\Delta R_\theta = R_\theta \Delta$, because so is the metric g_P . Hence D^* and Δ restrict naturally to

$$D^* : C_N^\infty(P, \Lambda^1) \rightarrow C_N^\infty(P) \quad \text{and} \quad \Delta_N : C_N^\infty(P) \rightarrow C_N^\infty(P)$$

for each $N \in \mathbb{Z}$. We have denoted Δ_N for the restriction of Δ to $C_N^\infty(P)$.

It is known that, for every $N \in \mathbb{Z}$, the operator Δ_N is an essentially self-adjoint positive operator with compact resolvent. Hence its spectrum is discrete and consists of real positive eigenvalues. The next theorem shows that these eigenvalues form some “clusters” (also called “bands”) in the lower part of this spectrum. Precisely the eigenvalues of $\frac{1}{2\pi N} \Delta_N$ concentrate around the specific half-integer values $\frac{1}{2}d + k$ with $d = \frac{1}{2} \dim M$, $k \in \mathbb{N}$. (These half-integer values correspond essentially to the eigenvalues of a harmonic oscillator model as we will see later.) See Figure 2.4. These clusters of eigenvalues are called *Landau levels* or *Landau bands* in physics. The existence of the first band is given in various papers, see [27, cor 1.2] and reference therein.⁶

Theorem 2.21. “*The bottom spectrum of Δ_N has band structure*”. For any $\alpha > 0$, the spectral set of the rough Laplacian $\frac{1}{2\pi N} \Delta_N$ in the interval $[0, \alpha]$ is contained in the $N^{-\epsilon}$ -neighborhood of the subset $\{\frac{d}{2} + k, k \in \mathbb{N} \cup \{0\}\}$ for sufficiently large N , with some $\epsilon > 0$. The number of eigenvalues in the $N^{-\epsilon}$ -neighborhood of $\frac{d}{2} + k$ (or in

⁶ The result for all the bands seems to be known to specialists although it does not appear explicitly in the literature to the best of our knowledge.

the k -th band) is proportional to N^d , that is, if we write \mathfrak{P}_k for the spectral projector for the eigenvalues on the k -th band, we have

$$C^{-1}N^d < \text{rank } \mathfrak{P}_k < CN^d$$

for some constant C independent of N . In particular, for the spectral projector \mathfrak{P}_0 for the first band, we have

$$\text{rank } (\mathfrak{P}_0) = \int_M [e^{N\omega} \text{Todd}(TM)]_{2d}. \quad (2.37)$$

Further, for the relation to the prequantum transfer operator \hat{F}_N , we have

$$\text{rank } \mathfrak{P}_k = \text{rank } \tau^{(k)} = \dim \mathcal{H}_k \quad \text{for } 0 \leq k \leq n$$

for sufficiently large N , where n , $\tau^{(k)}$ and \mathcal{H}_k are those in Theorem 1.18

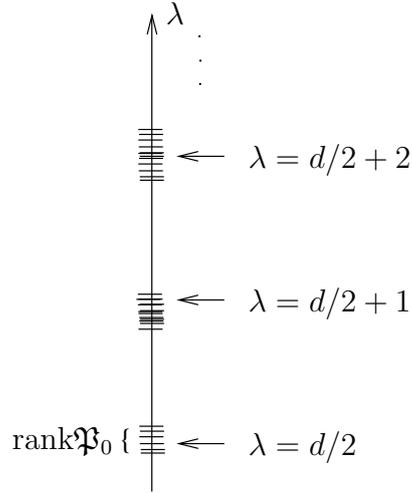


Figure 2.4: The *Landau levels* of the spectrum of the rough Laplacian $\frac{1}{2\pi N}\Delta : L_N^2(P) \rightarrow L_N^2(P)$ for $N \gg 1$.

A proof of this theorem will be given in Section 6.6. Note that the proof of the index formula (2.37) is given in [21, th. 2], see also [27, cor. 1.2] and references therein.

2.4.2 Expression of D^* and Δ in local charts

Let us see the expression of differential operators D^* and Δ introduced above in the local trivialization. Consider a local trivialization (1.7) of the bundle P over an open set $U_\alpha \subset M$.

The operator $D : C_N^\infty(P) \rightarrow C_N^\infty(P, \Lambda^1)$ in such local trivialization is represented by the operator

$$D_\alpha : C^\infty(U_\alpha) \rightarrow C^\infty(U_\alpha, \Lambda^1), \quad D_\alpha u_\alpha = (Du) \circ \tau_\alpha$$

where $u \in C_N^\infty(P)$ and $u_\alpha := (u \circ \tau_\alpha) \in C^\infty(U_\alpha)$. Similarly $D^* : C_N^\infty(P, \Lambda^1) \rightarrow C_N^\infty(P)$ and $\Delta_N = D^*D$ are represented by operators

$$D_\alpha^* : C^\infty(U_\alpha, \Lambda^1) \rightarrow C^\infty(U_\alpha) \quad \text{and} \quad \Delta_\alpha : C^\infty(U_\alpha) \rightarrow C^\infty(U_\alpha).$$

The next proposition gives the explicit expression of the operators D_α , D_α^* and Δ_α using a local coordinates (x^1, \dots, x^{2d}) on U_α , though we have already obtain such expression for D_α in Proposition 2.18. Note that the operators D_α^* and Δ_α depend on the Riemann metric g on M and also on N (or \hbar) though it is not explicit in the notation. We write $g = \sum_{j,k} g_{j,k} dx^j \otimes dx^k$ for the metric tensor and $g^{j,k} = (g_{j,k})_{j,k}^{-1}$ for the entries of the inverse matrix.

Proposition 2.22. *With respect to the local trivialization and coordinate system described above, we have the following expressions for D , D^* and $\Delta = D^*D$:*

$$D_\alpha u_\alpha = \frac{i}{\hbar} \sum_{j=1}^{2d} (\widehat{\zeta}^j u_\alpha) dx^j, \quad \text{with } \widehat{\zeta}^j = -i\hbar \frac{\partial}{\partial x^j} + \eta_j, \quad (2.38)$$

$$D_\alpha^* \left(\sum_{j=1}^{2d} v_j dx^j \right) = -\frac{i}{\hbar} \sum_{j,k=1}^{2d} \left(g^{j,k} \widehat{\zeta}^j - i\hbar (\partial_j g^{j,k}) \right) v_k,$$

$$\Delta_\alpha u_\alpha = \frac{1}{\hbar^2} \sum_{j,k=1}^{2d} \left(g^{j,k} \widehat{\zeta}^j \widehat{\zeta}^k - i\hbar (\partial_j g^{j,k}) \widehat{\zeta}^k \right) u_\alpha. \quad (2.39)$$

Proof. The expression (2.34) gives (2.38). Let $v = \sum_{j=1}^{2d} v_j dx^j \in C^\infty(U_\alpha, \Lambda^1)$. Using integration by parts, we have

$$\begin{aligned} (u, D_\alpha^* v)_{L^2(P)} &= (D_\alpha u, v)_{L^2(P, \Lambda)} = \int \langle du + \frac{i}{\hbar} \eta u | v \rangle dx \\ &= \sum_{j,k} \int \overline{\left(\partial_j u + \frac{i}{\hbar} \eta_j u \right)} g^{j,k} v_k dx = - \sum_{j,k} \int \left(\bar{u} \partial_j (g^{j,k} v_k) + \frac{i}{\hbar} \eta_j \bar{u} g^{j,k} v_k \right) dx \\ &= - \int \bar{u} \sum_{j,k} \left((\partial_j g^{j,k}) v_k + g^{j,k} \left(\partial_j v_k + \frac{i}{\hbar} \eta_j v_k \right) \right) dx. \end{aligned}$$

Hence

$$D_\alpha^* v = -\frac{i}{\hbar} \sum_{j,k} \left(g^{j,k} \widehat{\zeta}^j - i\hbar (\partial_j g^{j,k}) \right) v_k.$$

We deduce (2.39) from the computation

$$\begin{aligned}\Delta_\alpha u_\alpha &= D_\alpha^* D_\alpha u_\alpha = -\frac{i}{\hbar} \sum_{j,k} \left(g^{jk} \widehat{\zeta}^j - i\hbar (\partial_j g^{jk}) \right) \frac{i}{\hbar} \left(\widehat{\zeta}^k u_\alpha \right) \\ &= \frac{1}{\hbar^2} \sum_{j,k} \left(g^{jk} \widehat{\zeta}^j \widehat{\zeta}^k - i\hbar (\partial_j g^{jk}) \widehat{\zeta}^k \right) u_\alpha\end{aligned}$$

□

Corollary 2.23. *In local Darboux coordinates $x = (q, p) = (q^1, \dots, q^d, p^1, \dots, p^d)$ on U_α and in the special case of the Euclidean metric $g = \frac{1}{2} \sum_{j=1}^d dq^j \otimes dq^j + dp^j \otimes dp^j$, we have*

$$\Delta_\alpha = \frac{1}{2\hbar^2} \left(\sum_{j=1}^d \widehat{\zeta}_p^j{}^2 + \widehat{\zeta}_q^j{}^2 \right) \quad (2.40)$$

with $\widehat{\zeta}_q^j, \widehat{\zeta}_p^j$ given in (2.35).

The operator (2.40) is called the *Euclidean rough Laplacian*. In Section 4.6, we will deduce the cluster structure (or the Landau levels) of the spectrum of the Euclidean rough Laplacian by identifying it with the *harmonic oscillator*.

2.5 Sketch of the proof of the main theorems

So far, we have presented preliminary results which contain some aspects essential in the proof of the main theorems of this paper. The main idea is somehow depicted in Figure 2.3 page 26: It give a schematic picture of the action of the canonical map F associated to the prequantum transfer operator \widehat{F}_N ; the trapped set K is a compact symplectic submanifold of the phase space T^*M ; and the dynamics is hyperbolic in the transverse directions.

In order to focus on the action of the canonical map on the vicinity of the trapped set and relate it to the spectral properties of the prequantum transfer operator \widehat{F}_N , we use an *escape function* (or a *weight function*) $W_h(x, \xi)$ on the phase space, which decreases strictly along the flow outside a vicinity of the trapped set K , and use it to define some associate norm and associated *anisotropic Sobolev spaces*⁷. From the fact that the trapped set is compact in phase space and from the property of the escape function mentioned above, we deduce that the spectrum of the prequantum transfer operator \widehat{F}_N is discrete

⁷This idea of defining a generalized Sobolev space using an escape (or weight) function on the phase space has been used several times before and not new in this paper. For instance, in the context of semiclassical scattering theory in the phase space, such an idea was developed by B. Helffer and J. Sjöstrand in [22]. In the context of transfer operators for hyperbolic dynamical systems, it was developed in [4, 5] more recently.

in these anisotropic Sobolev spaces. The eigenvalues are called “**resonances**” (from the physical meaning in scattering theory).

The fundamental process of using an escape function $W_{\hbar}(x, \xi)$ in phase space will be explain below. Here are the main stages for the proof (though those will be presented in different order):

- (1) In Section 6.1 we will consider a system of local charts on the manifold M depending on \hbar (or on N), which is of small size $\hbar^{1/2-\theta} \ll 1$ in the semiclassical limit $\hbar \rightarrow 0$, and then consider the local trivializations of the prequantum bundle $P \rightarrow M$ on each chart, as in Proposition 2.13 and 2.16. On each of such charts, the map f is approximated at first order by its linear approximation, namely its differential Df which is a linear hyperbolic map as given in (2.29). In Section 6.2, we show how to decompose the global prequantum operator into “prequantum operators on charts” and how to recompose it, i.e. passing from global to local and *vice versa*.
- (2) In view of the decomposition above, we study first the spectrum of resonances of a prequantum operator associated to a linear hyperbolic map. This is done in Section 4. The prequantum operator for a linear hyperbolic map turns out to be the tensor product of two operators, according to the decomposition (2.28): one operator is a prequantum operator associated to the linear map tangent to the trapped set K and the other is that associated to the linear map in its (symplectic) orthogonal. The first part is a unitary operator, while the second part is treated by using the property of the escape function $W_{\hbar}(x, \xi)$ as described above and shown to display discrete spectrum in Proposition 3.20. For rigorous argument, we will present some technical tools first:
 - (a) the Bargmann transform \mathcal{B}_{\hbar} which represents functions and operators in phase space, is explained in Section 3.1.
 - (b) definition of the escape function in phase space and the associated anisotropic Sobolev space, for the hyperbolic dynamics orthogonal to the trapped set, in Section 3.3.
- (3) In Section 5 we develop some results that will be used in Section 6 in order to show that non-linearity of the map f can be neglected in the reconstruction of the global prequantum operator from its local parts.
- (4) Finally in Section 6, we assemble all these previous results and obtain the proofs of the main theorems.

In parallel to the treatment of the prequantum operator, we will also consider the rough Laplacian $\Delta = D^*D$ in every sections. In Section 6.6 the Laplacian is decomposed into local charts and approximated by an Euclidean Laplacian. In Section 5.4 we establish some lemmas in order to show that non-linearity can be neglected. In Section 4.6, we describe

the spectrum of the rough Euclidean Laplacian. For this, we use the Harmonic oscillator described in Section 3.5.

Finally let us mention that Sections 3 and 4, where we study the resonances of the linear model, are the core of our argument because they reveal the main mechanism responsible for the band structure of the spectrum described in Theorem 1.16. This mechanism was discovered in the paper [16] originally in the study of the prequantum linear cat map.

3 Resonances of linear expanding maps

This Section is self-contained. The main result of this Section is Proposition 3.20.

3.1 The Bargmann transform

3.1.1 Definitions

In this section, we recall some basic facts related to the so-called Bargmann transform. For more detailed account about the Bargmann transform, we refer the books [28, chap.3] [19, p.39] [3] [30, p.19].

Let D be a positive integer. Let $\hbar > 0$. We consider the Euclidean space \mathbb{R}^D with its canonical Euclidean norm written $|\cdot|$. For each point $(x, \xi) \in T^*\mathbb{R}^D = \mathbb{R}^D \oplus \mathbb{R}^D$, we assign the complex-valued smooth function $\phi_{x,\xi} \in \mathcal{S}(\mathbb{R}^D)$ defined by

$$\phi_{x,\xi}(y) = a_D \exp\left(\frac{i}{\hbar}\xi \cdot (y - \frac{x}{2}) - \frac{1}{2\hbar}|y - x|^2\right) \quad (3.1)$$

with

$$a_D = (\pi\hbar)^{-D/4}. \quad (3.2)$$

We will henceforth consider the measure dx on \mathbb{R}^D defining the Hilbert spaces $L^2(\mathbb{R}^D)$ and the measure $(2\pi\hbar)^{-D} dx d\xi$ on $T^*\mathbb{R}^D = \mathbb{R}^{2D}$ defining $L^2(\mathbb{R}^{2D})$. The constant a_D is taken so that $\|\phi_{x,\xi}\|_{L^2(\mathbb{R}^D)} = 1$.

Definition 3.1. The *Bargmann transform* is the continuous operator

$$\mathcal{B}_\hbar : \mathcal{S}(\mathbb{R}^D) \rightarrow \mathcal{S}(\mathbb{R}^{2D}), \quad (\mathcal{B}_\hbar u)(x, \xi) = \int \overline{\phi_{x,\xi}(y)} \cdot u(y) dy \quad (3.3)$$

on the Schwartz space $\mathcal{S}(\mathbb{R}^D)$. The (formal) adjoint of \mathcal{B}_\hbar is

$$\mathcal{B}_\hbar^* : \mathcal{S}(\mathbb{R}^{2D}) \rightarrow \mathcal{S}(\mathbb{R}^D), \quad (\mathcal{B}_\hbar^* v)(y) = \int \phi_{x,\xi}(y) \cdot v(x, \xi) \frac{dx d\xi}{(2\pi\hbar)^D}.$$

Lemma 3.2. [28, p70, Proposition 3.1.1] *We have that*

- (1) \mathcal{B}_\hbar extends uniquely to an isometric embedding $\mathcal{B}_\hbar : L^2(\mathbb{R}^D) \rightarrow L^2(\mathbb{R}^{2D})$.

(2) \mathcal{B}_\hbar^* extends uniquely to a bounded operator $\mathcal{B}_\hbar^* : L^2(\mathbb{R}^{2D}) \rightarrow L^2(\mathbb{R}^D)$.

(3) $\mathcal{B}_\hbar^* \circ \mathcal{B}_\hbar = \text{Id}$ on $L^2(\mathbb{R}^D)$.

Proof. For any $u \in \mathcal{S}(\mathbb{R}^D)$, we have

$$\begin{aligned} \|\mathcal{B}_\hbar u\|_{L^2(\mathbb{R}^{2D})} &= \frac{a_D^2}{(2\pi\hbar)^D} \int \phi_{x,\xi}(y') \overline{u(y')} \cdot \overline{\phi_{x,\xi}(y) u(y)} dx d\xi dy dy' \\ &= \frac{a_D^2}{(2\pi\hbar)^D} \int u(y) \overline{u(y')} \exp\left(\frac{i}{\hbar} \xi(y' - y) - \frac{1}{2\hbar} (|x - y|^2 + |x - y'|^2)\right) dx d\xi dy dy' \\ &= (\pi\hbar)^{-D/2} \int |u(y)|^2 \exp(-|x - y|^2/\hbar) dx dy \\ &= \int |u(y)|^2 dy = \|u\|_{L^2(\mathbb{R}^D)}. \end{aligned}$$

This gives the claims of the lemma by the usual continuity argument. \square

This lemma implies that the space $L^2(\mathbb{R}^{2D})$ is orthogonally decomposed as

$$L^2(\mathbb{R}^{2D}) = \text{Im} \mathcal{B}_\hbar \oplus \ker \mathcal{B}_\hbar^*. \quad (3.4)$$

3.1.2 Bargmann projector

Proposition 3.3. *The Bargmann projector \mathcal{P}_\hbar is defined by*

$$\mathcal{P}_\hbar := \mathcal{B}_\hbar \circ \mathcal{B}_\hbar^* : L^2(\mathbb{R}^{2D}) \rightarrow L^2(\mathbb{R}^{2D}). \quad (3.5)$$

This is the orthogonal projection onto $\text{Im} \mathcal{B}_\hbar \subset L^2(\mathbb{R}^{2D})$. It can be expressed as an integral operator $\mathcal{P}_\hbar v(z) = \int K_{\mathcal{P},\hbar}(z, z') v(z') dz'$ with the kernel:

$$K_{\mathcal{P},\hbar}(z, z') = \exp\left(\frac{i}{2\hbar} \omega(z, z') - \frac{1}{4\hbar} |z - z'|^2\right) \quad (3.6)$$

with $z = (x, \xi)$, $z' = (x', \xi') \in \mathbb{R}^{2D}$, the measure $dz' = dx' d\xi' / (2\pi\hbar)^D$, the Euclidean norm $|z|^2 := |x|^2 + |\xi|^2$ and the canonical symplectic form on $T^\mathbb{R}^D$, $\omega(z, z') = x \cdot \xi' - \xi \cdot x'$*

Proof. \mathcal{P}_\hbar is an orthogonal projection because $\mathcal{P}_\hbar^* = (\mathcal{B}_\hbar \circ \mathcal{B}_\hbar^*)^* = \mathcal{P}_\hbar$ and $\mathcal{P}_\hbar^2 = \mathcal{B}_\hbar \circ \mathcal{B}_\hbar^* \circ \mathcal{B}_\hbar \circ \mathcal{B}_\hbar^* = \mathcal{P}_\hbar$ from Lemma 3.2. From Definition 3.1, the kernel of $\mathcal{P}_\hbar = \mathcal{B}_\hbar \circ \mathcal{B}_\hbar^*$ is

$$\begin{aligned} K_{\mathcal{P},\hbar}(z, z') &= \int dy \overline{\phi_z(y)} \phi_{z'}(y) \\ &= a_D^2 \int_{\mathbb{R}^D} dy \exp\left(-i \frac{\xi}{\hbar} (y - \frac{x}{2}) + i \frac{\xi'}{\hbar} (y - \frac{x'}{2}) - \frac{1}{2\hbar} (|y - x|^2 + |y - x'|^2)\right) \\ &= a_D^2 \exp\left(i \frac{1}{2\hbar} (\xi x - \xi' x') - \frac{1}{2\hbar} (|x|^2 + |x'|^2)\right) \\ &\quad \times \int dy \exp\left(\frac{1}{\hbar} \langle i(\xi' - \xi) + (x + x') | y \rangle - \frac{1}{\hbar} |y|^2\right). \end{aligned}$$

We use the following formula for Gaussian integral in \mathbb{R}^D :

$$\int_{\mathbb{R}^D} e^{-\frac{1}{2}\langle y|Ay\rangle + b \cdot y} dy = \sqrt{\frac{(2\pi)^D}{\det A}} \exp\left(\frac{1}{2}\langle b|A^{-1}b\rangle\right), \quad b \in \mathbb{C}^D, A \in \mathcal{L}(\mathbb{R}^D) \quad (3.7)$$

with $A = (2/\hbar) \cdot \text{Id}$, $b = \frac{i}{\hbar}(\xi' - \xi) + \frac{1}{\hbar}(x + x')$ and get

$$\begin{aligned} K_{\mathcal{P},\hbar}(z, z') &= \frac{1}{(\pi\hbar)^{D/2}} \frac{(2\pi)^{D/2}}{(2/\hbar)^{D/2}} \exp\left(\frac{1}{4\hbar}(i(\xi' - \xi) + (x + x'))^2\right) \\ &\quad \times \exp\left(i\frac{1}{2\hbar}(\xi x - \xi' x') - \frac{1}{2\hbar}(|x|^2 + |x'|^2)\right) \\ &= \exp\left(\frac{1}{4\hbar}(-|\xi' - \xi|^2 - |x - x'|^2) + \frac{i}{2\hbar}(\xi x - \xi' x' + (\xi' - \xi)(x + x'))\right) \\ &= \exp\left(\frac{i}{2\hbar}\omega(z, z') - \frac{1}{4\hbar}|z - z'|^2\right). \end{aligned}$$

□

3.1.3 The Bargmann transform in more general setting

We have seen that the Bargmann transform gives a phase-space representation, i.e. a unitary isomorphism: $\mathcal{B}_\hbar : L^2(\mathbb{R}^D) \rightarrow \text{Im}(\mathcal{B}_\hbar) \subset L^2(T^*\mathbb{R}^D)$. The next proposition gives the Bargmann transform in a slightly more general setting. We start from a symplectic linear space (E, ω) of dimension $2D$ and a Lagrangian subspace $L \subset E$. Let $g(\cdot, \cdot)$ be a scalar product on E that is compatible with the symplectic form ω on E in the sense that there is a linear map $J : E \rightarrow E$ such that $J \circ J = -\text{Id}$ and holds

$$g(z, z') = \omega(z, Jz') \quad \text{for all } z, z' \in E.$$

This is nothing but the point-wise version of the condition (2.36). Let L^{\perp_g} be the orthogonal complement of L with respect to the inner product g , so that a point $z \in E$ can be decomposed uniquely as $z = x + \xi$, $x \in L$, $\xi \in L^{\perp_g}$.

For each point $z = x + \xi$, $x \in L$, $\xi \in L^{\perp_g}$, we define the wave packet $\phi_z(y) \in \mathcal{S}(L)$ by

$$\phi_z(y) = a_D \exp\left(\frac{i}{\hbar}\omega(\xi, y - \frac{x}{2}) - \frac{1}{2\hbar}|y - \frac{x}{2}|_g^2\right) \quad \text{for } y \in L.$$

We define the Bargmann transform $\mathcal{B}_\hbar : \mathcal{S}(L) \rightarrow \mathcal{S}(E)$ and its adjoint $\mathcal{B}_\hbar^* : \mathcal{S}(E) \rightarrow \mathcal{S}(L)$ as in Definition 3.1. Then the statement corresponding to Lemma 3.2 and Proposition 3.3 holds. Namely

Proposition 3.4. *For the Bargmann transform \mathcal{B}_\hbar and its adjoint defined as above,*

- (1) \mathcal{B}_\hbar extends uniquely to an isometric embedding $\mathcal{B}_\hbar : L^2(L) \rightarrow L^2(E)$.

(2) \mathcal{B}_h^* extends uniquely to a bounded operator $\mathcal{B}_h^* : L^2(E) \rightarrow L^2(L)$.

(3) $\mathcal{B}_h^* \circ \mathcal{B}_h = \text{Id}$ on $L^2(L)$.

(4) The Bargmann projector $\mathcal{P}_h : L^2(E) \rightarrow L^2(E)$, defined by $\mathcal{P}_h := \mathcal{B}_h \circ \mathcal{B}_h^*$, is the orthogonal projection onto $\text{Im } \mathcal{B}_h \subset L^2(E)$. It is expressed as an integral operator with kernel (3.6) with $|\cdot|$ replaced by $|\cdot|_g$.

Remark 3.5. Notice that the Bargmann projector $\mathcal{P}_h : L^2(E) \rightarrow L^2(E)$ can be defined directly from its kernel (3.6) and is independent on the choice of the Lagrangian subspace L .

Proof. There are linear isomorphisms $\psi : \mathbb{R}^D \rightarrow L$ and $\Psi : \mathbb{R}^{2D} \rightarrow E$ such that the following diagram commutes:

$$\begin{array}{ccc} \mathbb{R}^{2D} & \xrightarrow{\Psi} & E \\ p \downarrow & & p' \downarrow \\ \mathbb{R}^D & \xrightarrow{\psi} & L \end{array}$$

where $p : \mathbb{R}^{2D} = \mathbb{R}^D \oplus \mathbb{R}^D \rightarrow \mathbb{R}^D$ is the projection to the first D components: $p(x, \xi) = x$, and $p' : E \rightarrow L$ is the orthogonal projection to L with respect to g , and moreover that the pull-back of the symplectic form ω and the inner product g by Ψ coincides with the standard Euclidean inner product $g_0(z, z') = z \cdot z'$ and the standard symplectic form:

$$\omega_0(z, z') = x \cdot \xi' - \xi \cdot x' \quad \text{for } z=(x, \xi), \quad z'=(x', \xi') \in T^*\mathbb{R}^D.$$

Through such correspondence by Ψ and ψ , the definition of the Bargmann transform and its adjoint above coincides with those that we made in the last subsection. Therefore the claims are just restatement of Lemma 3.2 and Proposition 3.3. \square

The next proposition will be useful later on in Lemma 4.6. Note that the Bargmann transform and its adjoint are defined for each combination (E, ω, g, L) of a linear space E , symplectic form ω , a compatible Euclidean metric g on E and a Lagrangian subspace $L \subset E$. Let $(E_i, \omega_i, g_i, L_i)$ for $i = 1, 2$, be two such combination and suppose that (E, ω, g, L) is the direct product of them in the sense that

$$E = E_1 \oplus E_2, \quad \omega = \omega_1 \oplus \omega_2, \quad g = g_1 \oplus g_2, \quad L = L_1 \oplus L_2.$$

Then the Bargmann transform \mathcal{B}_h , its adjoint \mathcal{B}_h^* and the Bargmann projection \mathcal{P}_h defined for (E, ω, g, L) are the tensor product of those, denoted by $\mathcal{B}_{i,h}$, $\mathcal{B}_{i,h}^*$ and $\mathcal{P}_{i,h}$, defined for $(E_i, \omega_i, g_i, L_i)$, $i = 1, 2$. More precisely

Proposition 3.6. *The following diagram commutes:*

$$\begin{array}{ccccccc} L^2(E) & \xrightarrow{\mathcal{B}_h^*} & L^2(L) & \xrightarrow{\mathcal{B}_h} & L^2(E) \\ \downarrow & & \downarrow & & \downarrow \\ L^2(E_1) \otimes L^2(E_2) & \xrightarrow{\mathcal{B}_{1,h}^* \otimes \mathcal{B}_{2,h}^*} & L^2(L_1) \otimes L^2(L_2) & \xrightarrow{\mathcal{B}_{1,h} \otimes \mathcal{B}_{2,h}} & L^2(E_1) \otimes L^2(E_2) \end{array}$$

where the vertical arrows denotes the natural identification. Consequently the Bargmann projector $\mathcal{P}_h = \mathcal{B}_h \circ \mathcal{B}_h^*$ defined for (E, ω, g, L) is identified with the tensor product $\mathcal{P}_{1,h} \otimes \mathcal{P}_{2,h} = (\mathcal{B}_{1,h}^* \circ \mathcal{B}_{1,h}) \otimes (\mathcal{B}_{2,h}^* \circ \mathcal{B}_{2,h})$.

3.1.4 Scaling

The operators \mathcal{B}_h and \mathcal{B}_h^* are related to \mathcal{B}_1 and \mathcal{B}_1^* (i.e. with $\hbar = 1$) by the simple scaling $x \mapsto \hbar^{1/2}x$. Though this fact should be obvious, we give the relations explicitly for the later use. Let us introduce the unitary operators

$$s_h : L^2(\mathbb{R}^D) \rightarrow L^2(\mathbb{R}^D), \quad s_h u(x) = \hbar^{-D/4} u(\hbar^{-1/2}x) \quad (3.8)$$

and

$$S_h : L^2(\mathbb{R}^{2D}) \rightarrow L^2(\mathbb{R}^{2D}), \quad S_h u(x, \xi) = u(\hbar^{-1/2}x, \hbar^{-1/2}\xi) \quad (3.9)$$

Then we have⁸

Lemma 3.7. *The following diagram commutes:*

$$\begin{array}{ccc} L^2(\mathbb{R}^{2D}) & \xrightarrow{S_h} & L^2(\mathbb{R}^{2D}) \\ \mathcal{B}_h \uparrow & & \mathcal{B}_1 \uparrow \\ L^2(\mathbb{R}^D) & \xrightarrow{s_h} & L^2(\mathbb{R}^D) \end{array}$$

3.2 Action of linear transforms

Definition 3.8. The lift of a bounded operator $L : L^2(\mathbb{R}^D) \rightarrow L^2(\mathbb{R}^D)$ with respect to the Bargmann transform \mathcal{B}_h is defined as the operator

$$L^{\text{lift}} := \mathcal{B}_h \circ L \circ \mathcal{B}_h^* : L^2(\mathbb{R}^{2D}) \rightarrow L^2(\mathbb{R}^{2D}) \quad (3.10)$$

By definition, it makes the following diagram commutes:

$$\begin{array}{ccc} L^2(\mathbb{R}^{2D}) & \xrightarrow{L^{\text{lift}}} & L^2(\mathbb{R}^{2D}) \\ \mathcal{B}_h \uparrow & & \mathcal{B}_h \uparrow \\ L^2(\mathbb{R}^D) & \xrightarrow{L} & L^2(\mathbb{R}^D). \end{array} \quad (3.11)$$

Since $\mathcal{P}_h \circ L^{\text{lift}} \circ \mathcal{P}_h = L^{\text{lift}}$, the lift L^{lift} is always trivial on the second factor with respect to the decomposition $L^2(\mathbb{R}^{2D}) = \text{Im}\mathcal{B}_h \oplus \ker \mathcal{B}_h^* = \text{Im}\mathcal{P}_h \oplus \ker \mathcal{P}_h$ in (3.4), that is,

$$L^{\text{lift}} = (\mathcal{B}_h \circ L \circ \mathcal{B}_h^*)_{\text{Im}\mathcal{P}_h} \oplus (\mathbf{0})_{\text{Ker}\mathcal{P}_h}. \quad (3.12)$$

⁸Recall the convention on the norm on $L^2(\mathbb{R}^D)$ and $L^2(\mathbb{R}^{2D})$ made in the beginning of this section.

Lemma 3.9. For a invertible linear transformation $A : \mathbb{R}^D \rightarrow \mathbb{R}^D$, we associate a unitary transfer operator defined by

$$L_A : L^2(\mathbb{R}^D) \rightarrow L^2(\mathbb{R}^D), \quad L_A u = \frac{1}{\sqrt{\det A}} \cdot u \circ A^{-1} \quad (3.13)$$

Then we have

$$L_A = d(A) \cdot \mathcal{B}_h^* \circ L_{A \oplus {}^t A^{-1}} \circ \mathcal{B}_h, \quad (3.14)$$

where $L_{A \oplus {}^t A^{-1}} : L^2(\mathbb{R}^{2D}) \rightarrow L^2(\mathbb{R}^{2D})$ is the unitary transfer operator given by

$$(L_{A \oplus {}^t A^{-1}} u)(x, \xi) := u(A^{-1}x, {}^t A \xi) \quad (3.15)$$

and we set

$$d(A) = \det \left(\frac{1}{2} (A + {}^t A^{-1}) \right)^{1/2}.$$

Consequently the lift of L_A ,

$$L_A^{\text{lift}} := \mathcal{B}_h \circ L_A \circ \mathcal{B}_h^* \quad (3.16)$$

is an isometry on $\text{Im } \mathcal{B}_h = \text{Im } \mathcal{P}_h \subset L^2(\mathbb{R}^{2D})$ (hence, a bounded operator from (3.12)) and expressed as

$$L_A^{\text{lift}} = d(A) \cdot \mathcal{P}_h \circ L_{A \oplus {}^t A^{-1}} \circ \mathcal{P}_h. \quad (3.17)$$

Remark 3.10. (1) The expression (3.14) shows that L_A can be expressed as an operator on the phase space defined in terms of the Bargmann projector and the transfer operator $L_{A \oplus {}^t A^{-1}}$, but with an additional factor $d(A)$, sometimes called the *metaplectic correction*. This may be regarded as a realization of the idea explained in the last section: L_A can be seen as a Fourier integral operator and canonical map is $A \oplus {}^t A^{-1}$ on $T^*\mathbb{R}^D$. But notice that the correction term $d(A)$ will be crucially important for our argument.

(2) For an orthogonal transform $A \in SO(D)$, we have $d(A) = 1$.

Proof. To prove (3.14), we write the operator $\mathcal{B}_h^* \circ L_{A \oplus {}^t A^{-1}} \circ \mathcal{B}_h$ as an integral operator

$$\mathcal{B}_h^* \circ L_{A \oplus {}^t A^{-1}} \circ \mathcal{B}_h u(y) = \int K(y, y') u(y') dy'$$

with the kernel

$$K(y, y') = \int \phi_{x, \xi}(y) \cdot \overline{\phi_{A^{-1}x, {}^t A \xi}(y')} \frac{dx d\xi}{(2\pi\hbar)^D}.$$

Using the formula (3.7) for the Gaussian integral and change of variables, we can calculate the integral on the right hand side as

$$\begin{aligned}
K(y, y') &= \int \phi_{Ax', tA^{-1}\xi'}(y) \cdot \overline{\phi_{x', \xi'}(y')} \frac{dx' d\xi'}{(2\pi\hbar)^D} \quad (x' = A^{-1}x, \xi' = {}^tA\xi) \\
&= a_D^2 \cdot \int e^{\frac{i}{\hbar}(\xi', y' - A^{-1}y) - |y' - x'|^2/(2\hbar) - |y - Ax'|^2/(2\hbar)} \frac{dx' d\xi'}{(2\pi\hbar)^D} \\
&= (\pi\hbar)^{-D/2} \cdot \delta(y' - A^{-1}y) \cdot \int e^{-|A^{-1}y - x'|^2/(2\hbar) - |y - Ax'|^2/(2\hbar)} dx' \\
&= \pi^{-D/2} \cdot \delta(y' - A^{-1}y) \cdot \int e^{-|t|^2/2 - |At|^2/2} dt \quad (t = (x' - A^{-1}y)/\hbar) \\
&= \det((I + {}^tAA)/2)^{-1/2} \cdot \delta(y' - A^{-1}y).
\end{aligned}$$

Therefore we have

$$(\mathcal{B}_\hbar^* \circ L_{A \oplus {}^tA^{-1}} \circ \mathcal{B}_\hbar)u(y) = \det((I + {}^tAA)/2)^{-1/2} \cdot u(A^{-1}y) = d(A)^{-1} \cdot (L_A u)(y)$$

and hence the claim (3.14) follows. This implies

$$L_A^{\text{lift}} = \mathcal{B}_\hbar \circ L_A \circ \mathcal{B}_\hbar^* = d(A) \cdot \mathcal{P}_\hbar \circ L_{A \oplus {}^tA^{-1}} \circ \mathcal{P}_\hbar.$$

The other claims follow immediately. \square

Lemma 3.11. For $(x_0, \xi_0) \in T^*\mathbb{R}^D = \mathbb{R}^{2D}$, we associate a unitary operator defined by

$$T_{(x_0, \xi_0)} : L^2(\mathbb{R}^D) \rightarrow L^2(\mathbb{R}^D), \quad (T_{(x_0, \xi_0)}v)(y) = e^{\frac{i}{\hbar}\xi_0 \cdot (y + \frac{x_0}{2})} v(y - x_0) \quad (3.18)$$

Then we have

$$T_{(x_0, \xi_0)} = \mathcal{B}_\hbar^* \circ \mathcal{T}_{(x_0, \xi_0)} \circ \mathcal{B}_\hbar, \quad (3.19)$$

where $\mathcal{T}_{(x_0, \xi_0)} : L^2(\mathbb{R}^{2D}) \rightarrow L^2(\mathbb{R}^{2D})$ is the unitary transfer operator given by

$$\mathcal{T}_{(x_0, \xi_0)}u(x, \xi) := e^{\frac{i}{2\hbar}(\xi_0 \cdot x - x_0 \cdot \xi)} u(x - x_0, \xi - \xi_0) \quad (3.20)$$

Consequently the lift of $T_{(x_0, \xi_0)}$ is

$$T_{(x_0, \xi_0)}^{\text{lift}} := \mathcal{B}_\hbar \circ T_{(x_0, \xi_0)} \circ \mathcal{B}_\hbar^* = \mathcal{P}_\hbar \circ \mathcal{T}_{(x_0, \xi_0)} \circ \mathcal{P}_\hbar. \quad (3.21)$$

Proof. The Schwartz kernel of $\mathcal{B}_\hbar^* \circ \mathcal{T}_{(x_0, \xi_0)} \circ \mathcal{B}_\hbar$ is

$$\begin{aligned}
K(y, y') &= \int_{\mathbb{R}^{2D}} e^{\frac{i}{2\hbar}(\xi_0 \cdot x - x_0 \cdot \xi)} \phi_{x, \xi}(y) \cdot \overline{\phi_{x - x_0, \xi - \xi_0}(y')} \frac{dx d\xi}{(2\pi\hbar)^D} \\
&= a_D^2 \cdot \int \frac{dx d\xi}{(2\pi\hbar)^D} e^{\frac{i}{2\hbar}(\xi_0 \cdot x - x_0 \cdot \xi)} e^{\frac{i}{\hbar}(\xi \cdot (y - \frac{x}{2}) - (\xi - \xi_0) \cdot (y' - \frac{x - x_0}{2}))} e^{-\frac{1}{2\hbar}(|y' - (x - x_0)|^2 + |y - x|^2)} \\
&= (\pi\hbar)^{-D/2} \delta(y' - y + x_0) e^{\frac{i}{\hbar}\xi_0 \cdot (y + \frac{x_0}{2})} \int e^{-\frac{1}{\hbar}|y - x|^2} dx \\
&= \delta(y' - y + x_0) \cdot e^{\frac{i}{\hbar}\xi_0 \cdot (y + \frac{x_0}{2})}.
\end{aligned}$$

This is the kernel of the operator $T_{(x_0, \xi_0)}$. \square

Corollary 3.12. *The lift of the operator $T_{(x_0, \xi_0)}$ is expressed as*

$$T_{(x_0, \xi_0)}^{\text{lift}} := \mathcal{B}_h \circ T_{(x_0, \xi_0)} \circ \mathcal{B}_h^* = \mathcal{T}_{(x_0, \xi_0)} \circ \mathcal{P}_h = \mathcal{P}_h \circ \mathcal{T}_{(x_0, \xi_0)}. \quad (3.22)$$

Proof. We can check the equality $\mathcal{T}_{(x_0, \xi_0)} \circ \mathcal{P}_h = \mathcal{P}_h \circ \mathcal{T}_{(x_0, \xi_0)}$ on the left by showing that the Schwartz kernels of $\mathcal{T}_{(x_0, \xi_0)} \circ \mathcal{P}_h$ and $\mathcal{P}_h \circ \mathcal{T}_{(x_0, \xi_0)}$ are equal. This is an easy computation using the expressions (3.6) and (3.20). The rest of the claim follows from Lemma 3.11. \square

3.3 The weighted L^2 spaces : $L^2(\mathbb{R}^{2D}, (W_h^r)^2)$

For each $t > 0$, we define the cones

$$\mathbf{C}_+(t) = \{(x, \xi) \in \mathbb{R}^{2D} \mid |\xi| \leq t \cdot |x|\}, \quad \mathbf{C}_-(t) = \{(x, \xi) \in \mathbb{R}^{2D} \mid |x| \leq t \cdot |\xi|\}.$$

Take and fix a C^∞ function $m : \mathbb{P}(\mathbb{R}^{2D}) \rightarrow [-r, r]$, called *order function*, on the projective space $\mathbb{P}(\mathbb{R}^{2D})$ so that

$$m([x, \xi]) = \begin{cases} -r, & \text{if } (x, \xi) \in \mathbf{C}_+(1/2); \\ +r, & \text{if } (x, \xi) \in \mathbf{C}_-(1/2). \end{cases} \quad (3.23)$$

We then define the *escape function* (or *the weight function*) by

$$W^r : \mathbb{R}^{2D} \rightarrow \mathbb{R}_+, \quad W^r(x, \xi) = \langle |(x, \xi)| \rangle^{m([x, \xi])}$$

where $\langle s \rangle := (1 + s^2)^{1/2}$ for $s \in \mathbb{R}$ and $|(x, \xi)|^2 := |x|^2 + |\xi|^2$. From this definition we have

$$W^r(x, \xi) \sim \langle |(x, \xi)| \rangle^{-r} \quad \text{if } |x| \geq 2|\xi| \text{ and } |(x, \xi)| \gg 1$$

and

$$W^r(x, \xi) \sim \langle |(x, \xi)| \rangle^r \quad \text{if } |x| \leq |\xi|/2 \text{ and } |(x, \xi)| \gg 1.$$

For convenience in the later argument, we also take and fix C^∞ functions

$$m^+, m^- : \mathbb{P}(\mathbb{R}^{2D}) \rightarrow [-r, r]$$

such that

$$m^+([x, \xi]) = \begin{cases} -r, & \text{if } (x, \xi) \in \mathbf{C}_+(1/9); \\ +r, & \text{if } (x, \xi) \in \mathbf{C}_-(3), \end{cases}$$

and

$$m^-([x, \xi]) = \begin{cases} -r, & \text{if } (x, \xi) \in \mathbf{C}_+(3); \\ +r, & \text{if } (x, \xi) \in \mathbf{C}_-(1/9), \end{cases}$$

and define the functions $W^{r,\pm} : \mathbb{R}^{2D} \rightarrow \mathbb{R}_+$ by

$$W^{r,\pm}(x, \xi) := \langle |(x, \xi)| \rangle^{m^\pm(|(x, \xi)|)}. \quad (3.24)$$

Obviously we have

$$W^{r,-}(x, \xi) \leq W^r(x, \xi) \leq W^{r,+}(x, \xi). \quad (3.25)$$

These functions, W^r and $W^{r,\pm}$, satisfy the following preferable condition that we will make use of later on: For any $\epsilon > 0$ and multi-index α , there exists a constant $C_{\alpha,\epsilon} > 0$ such that

$$|\partial_{x,\xi}^\alpha W^r(x, \xi)| \leq C_{\alpha,\epsilon} \langle |(x, \xi)| \rangle^{-(1-\epsilon)|\alpha|} \cdot W^r(x, \xi) \quad \text{for all } (x, \xi) \in \mathbb{R}^{2D} \quad (3.26)$$

and the same inequalities for $W^{r,\pm}(\cdot)$ hold.

Definition 3.13. For $\hbar > 0$, let

$$W_h^r : \mathbb{R}^{2D} \rightarrow \mathbb{R}_+, \quad W_h^r(x, \xi) := W^r(\hbar^{-1/2}x, \hbar^{-1/2}\xi) = W^r \circ S_h(x, \xi) \quad (3.27)$$

where S_h is the operator defined in (3.9). We consider the weighted L^2 space defined as

$$L^2(\mathbb{R}^{2D}, (W_h^r)^2) = \{v \in L^2_{\text{loc}}(\mathbb{R}^{2D}) \mid \|W_h^r \cdot v\|_{L^2(\mathbb{R}^{2D})} < \infty\}. \quad (3.28)$$

Likewise, we define the functions $W_h^{r,\pm}$ and the weighted L^2 spaces $L^2(\mathbb{R}^{2D}, (W_h^{r,\pm})^2)$ in the parallel manner, replacing W^r by $W^{r,\pm}$.

Note that the function W^r (and $W^{r,\pm}$) satisfies the condition

$$W^r(x, \xi) \leq C \cdot W^r(y, \eta) \cdot \langle |(x, \xi) - (y, \eta)| \rangle^{2r} \quad \text{for any } x, y \in \mathbb{R}^d \quad (3.29)$$

for some constant $C > 0$. Consequently the function W_h^r (and $W_h^{r,\pm}$) satisfies

$$W_h^r(x, \xi) \leq C \cdot W_h^r(y, \eta) \cdot \langle \hbar^{-1/2} |(x, \xi) - (y, \eta)| \rangle^{2r} \quad \text{for any } x, y \in \mathbb{R}^d. \quad (3.30)$$

The next Lemma characterizes a class of bounded integral operators in $L^2(\mathbb{R}^{2D}, (W_h^r)^2)$ in terms of its kernel.

Lemma 3.14. *If $R : \mathcal{S}(\mathbb{R}^{2D}) \rightarrow \mathcal{S}(\mathbb{R}^{2D})$ is an integral operator of the form*

$$Ru(x, \xi) = \int K_R(x, \xi; x', \xi') u(x', \xi') dx' d\xi'$$

and if the kernel $K_R(\cdot)$ is a continuous function satisfying

$$|K_R(x, \xi; x', \xi')| \leq \langle \hbar^{-1/2} |(x, \xi) - (x', \xi')| \rangle^{-\nu}$$

for some $\nu > 2r + 2D$, then it extends to a bounded operator on $L^2(\mathbb{R}^{2D}, (W_h^r)^2)$ and

$$\|R : L^2(\mathbb{R}^{2D}, (W_h^r)^2) \rightarrow L^2(\mathbb{R}^{2D}, (W_h^r)^2)\| \leq C_\nu$$

where C_ν is a constant which depends only on ν .

Proof. From (3.30), we have

$$\left| \frac{W_h^r(x, \xi)}{W_h^r(x', \xi')} \cdot K_R(x, \xi; x', \xi') \right| \leq C \langle \hbar^{-1/2} |(x, \xi) - (x', \xi')| \rangle^{2r-\nu}.$$

Hence, by Young inequality, the operator norm of $u \mapsto W_h^r \cdot R((W_h^r)^{-1} \cdot u)$ with respect to the L^2 norm is bounded by a constant C_ν . This implies the claim of the lemma. \square

From expression (3.6), the Bargmann projector \mathcal{P}_h satisfies the assumption of the last lemma. Thus we have

Corollary 3.15. *The Bargmann projector \mathcal{P}_h is a bounded operator on $L^2(\mathbb{R}^{2D}, (W_h^r)^2)$.*

3.4 Spectrum of transfer operator for linear expanding map

Below we consider the action of a linear expanding map.

Lemma 3.16. *If $A : \mathbb{R}^D \rightarrow \mathbb{R}^D$ is an expanding linear map satisfying*

$$\|A^{-1}\| \leq \frac{1}{\lambda} \quad \text{for some } \lambda > 1. \quad (3.31)$$

Then the lift L_A^{lift} of L_A , defined in (3.16), extends to a bounded operator

$$L_A^{\text{lift}} : L^2(\mathbb{R}^{2D}, (W_h^r)^2) \rightarrow L^2(\mathbb{R}^{2D}, (W_h^r)^2). \quad (3.32)$$

Further, if $\lambda > 1$ is sufficiently large (say $\lambda > 9$), L_A^{lift} extends to a bounded operator

$$L_A^{\text{lift}} : L^2(\mathbb{R}^{2D}, (W_h^{r,-})^2) \rightarrow L^2(\mathbb{R}^{2D}, (W_h^{r,+})^2). \quad (3.33)$$

Proof. From (3.17) in Lemma 3.9 and Corollary 3.15, we have only to check boundedness of $L_{A \oplus A^{-1}}$ as an operator on $L^2(\mathbb{R}^{2D}, (W^r)^2)$ (resp. from $L^2(\mathbb{R}^{2D}, (W^{r,-})^2)$ to $L^2(\mathbb{R}^{2D}, (W^{r,+})^2)$). But this is clear from the definitions of W^r and $W^{r,\pm}$. \square

To look into more detailed structure of the operator L_A and L_A^{lift} , we introduce some definitions. For $k \in \mathbb{N} \cup \{0\}$, let $\text{Polynom}^{(k)}$ be the space of homogeneous polynomial on \mathbb{R}^D of order k . Then we consider the operator

$$T^{(k)} : C^\infty(\mathbb{R}^D) \rightarrow \text{Polynom}^{(k)}, \quad (T^{(k)}u)(x) = \sum_{|\alpha|=k} \frac{\partial^\alpha u(0)}{\alpha!} \cdot x^\alpha. \quad (3.34)$$

This is a projector which extracts the terms of order k in the Taylor expansion. Clearly the operator $T^{(k)}$ is of finite rank and satisfies the following relations

$$T^{(k)} \circ T^{(k')} = \begin{cases} T^{(k)}, & \text{if } k = k'; \\ 0 & \text{otherwise,} \end{cases} \quad (3.35)$$

and

$$T^{(k)} \circ L_A = L_A \circ T^{(k)}. \quad (3.36)$$

As in (3.10) we define the lift of the operator $T^{(k)}$ by

$$\mathcal{T}_h^{(k)} := \mathcal{B}_h \circ T^{(k)} \circ \mathcal{B}_h^*. \quad (3.37)$$

Lemma 3.17. *Let $n \in \mathbb{N}$ and $r > 0$ such that*

$$r > n + 2 + \frac{D}{2}. \quad (3.38)$$

Then for $0 \leq k \leq n$ the operator $\mathcal{T}_h^{(k)}$ extends naturally to bounded operators

$$\mathcal{T}_h^{(k)} : L^2(\mathbb{R}^{2D}, (W_h^{r,-})^2) \rightarrow L^2(\mathbb{R}^{2D}, (W_h^{r,+})^2) \quad (3.39)$$

and

$$\mathcal{T}_h^{(k)} : L^2(\mathbb{R}^{2D}, (W_h^r)^2) \rightarrow L^2(\mathbb{R}^{2D}, (W_h^r)^2). \quad (3.40)$$

Further if we write the operator $\mathcal{T}_h^{(k)}$ as an integral operator

$$\left(\mathcal{T}_h^{(k)} u \right) (x, \xi) = \int K(x, \xi; x', \xi') u(x', \xi') dx' d\xi',$$

the kernel $K(\cdot)$ satisfies the estimate

$$\left| \frac{W_h^{r,+}(x, \xi)}{W_h^{r,-}(x', \xi')} \cdot K(x, \xi; x', \xi') \right| \leq C \langle \hbar^{-1/2} |(x, \xi)| \rangle^{k-r} \cdot \langle \hbar^{-1/2} |(x', \xi')| \rangle^{k-r} \quad (3.41)$$

$$\leq C' \langle \hbar^{-1/2} |(x, \xi) - (x', \xi')| \rangle^{k-r} \quad (3.42)$$

for some constants $C, C' > 0$ that do not depend on $\hbar > 0$.

Proof. For each multi-index $\alpha \in (\mathbb{Z}_+)^D$ with $|\alpha| = k$, we set

$$T^{(\alpha)} : \mathcal{S}(\mathbb{R}^D) \rightarrow \mathcal{S}(\mathbb{R}^D)', \quad T^{(\alpha)} u(x) = \frac{\partial^\alpha u(0)}{\alpha!} \cdot x^\alpha$$

and hence to

$$\mathcal{T}_h^{(\alpha)} = \mathcal{B}_h \circ T^{(\alpha)} \circ \mathcal{B}_h^* : \mathcal{S}(\mathbb{R}^{2D}) \rightarrow \mathcal{S}(\mathbb{R}^{2D})'.$$

Since $\mathcal{T}_h^{(k)} = \sum_{\alpha: |\alpha|=k} \mathcal{T}_h^{(\alpha)}$, the claims of the lemma follows if one prove the corresponding claim for $\mathcal{T}_h^{(\alpha)}$. The kernel of the operator $\mathcal{T}_h^{(\alpha)}$ is written as

$$K(x, \xi; x', \xi') = \frac{1}{\alpha!} \cdot k_+(x, \xi) \cdot k_-(x', \xi')$$

with

$$k_+(x, \xi) = \int \overline{\phi_{x, \xi}(y)} \cdot (\hbar^{-1/2}y)^\alpha dy, \quad k_-(x', \xi') = \hbar^{k/2} \cdot \partial^\alpha \phi_{x', \xi'}(0).$$

Applying integration by parts to the integral in the definition of $k_+(\cdot)$ above, we see

$$|k_+(x, \xi)| \leq C_\nu \cdot \langle \hbar^{-1/2}|x| \rangle^k \cdot \langle \hbar^{-1/2}|\xi| \rangle^{-\nu}$$

for arbitrarily large integer ν , where C_ν is a constant depending only on ν . Also a straight-forward computation gives the similar estimate for $k_-(\cdot)$:

$$|k_-(x', \xi')| \leq C_\nu \cdot \langle \hbar^{-1/2}|\xi'| \rangle^k \cdot \langle \hbar^{-1/2}|x'| \rangle^{-\nu}.$$

These estimates for sufficiently large ν imply that

$$W_h^r(x, \xi) \cdot |k_+(x, \xi)| \leq C \langle \hbar^{-1/2}|(x, \xi)| \rangle^{k-r}$$

and

$$\frac{1}{W_h^r(x', \xi')} \cdot |k_-(x', \xi')| \leq C \langle \hbar^{-1/2}|(x', \xi')| \rangle^{k-r}$$

for some constant $C > 0$ independent of $\hbar > 0$. Thus we have obtained the last claim on the kernel $K(\cdot)$. Since $r - k \geq r - n > 2D$ from the assumption (3.38) on the choice of r , the former claim on boundedness of the operators follows from this claim and Young inequality. \square

The following is a direct consequence of the relation (3.36).

Corollary 3.18. *For $0 \leq k, k' \leq n$, we have*

$$\mathcal{T}_h^{(k)} \circ \mathcal{T}_h^{(k')} = \begin{cases} \mathcal{T}_h^{(k)}, & \text{if } k = k'; \\ 0, & \text{otherwise,} \end{cases}$$

and

$$L_A^{\text{lift}} \circ \mathcal{T}_h^{(k)} = \mathcal{T}_h^{(k)} \circ L_A^{\text{lift}} = \mathcal{B}_h \circ L_A \circ T^{(k)} \circ \mathcal{B}_h^*.$$

Let us set

$$\tilde{\mathcal{T}}_h = \text{Id} - \sum_{k=0}^n \mathcal{T}_h^{(k)} : L^2(\mathbb{R}^{2D}, (W_h^r)^2) \rightarrow L^2(\mathbb{R}^{2D}, (W_h^r)^2). \quad (3.43)$$

Then the set of operators $\mathcal{T}_h^{(k)}$, $0 \leq k \leq n$, and $\tilde{\mathcal{T}}_h$ form a complete set of mutually commuting projection operators on $L^2(\mathbb{R}^{2D}, (W_h^r)^2)$ such that

$$\text{rank } \mathcal{T}_h^{(k)} = \dim \text{Polynom}^{(k)} = \binom{D+k-1}{D-1} = \frac{(D+k-1)!}{(D-1)!k!}, \quad \text{rank } \tilde{\mathcal{T}}_h = \infty.$$

Let

$$H_k := \text{Im } \mathcal{T}_h^{(k)} \quad \text{and} \quad \tilde{H} = \text{Im } \tilde{\mathcal{T}}_h.$$

Then the Hilbert space $L^2(\mathbb{R}^{2D}, (W_h^r)^2)$ is decomposed as

$$L^2(\mathbb{R}^{2D}, (W_h^r)^2) = H_0 \oplus H_1 \oplus H_2 \oplus \cdots \oplus H_n \oplus \tilde{H} \quad (3.44)$$

Since the operator L_A^{lift} commutes with the projections $\mathcal{T}_h^{(k)}$ and $\tilde{\mathcal{T}}_h$, it preserves this decomposition and therefore the operator L_A^{lift} acting on $L(\mathbb{R}^{2D}, W_h^r)$ is identified with the direct sum of the operators

$$L_A^{\text{lift}} : H_k \rightarrow H_k \quad \text{for } 0 \leq k \leq n, \quad \text{and} \quad L_A^{\text{lift}} : \tilde{H} \rightarrow \tilde{H}.$$

The former is identified with the action of L_A on $\text{Polynom}^{(k)}$, because the diagram

$$\begin{array}{ccc} H_k & \xrightarrow{L_A^{\text{lift}}} & H_k \\ \mathcal{B}_h \uparrow & & \mathcal{B}_h \uparrow \\ \text{Polynom}^{(k)} & \xrightarrow{L_A} & \text{Polynom}^{(k)} \end{array} \quad (3.45)$$

commutes and the operator $\mathcal{B}_h : \text{Polynom}^{(k)} \rightarrow H_k$ in the vertical direction is an isomorphism between finite dimensional linear spaces.

To state the next proposition which is the main result of this Section, we introduce the following definition:

Definition 3.19. The Hilbert space $H_h^r(\mathbb{R}^D) \subset \mathcal{S}'(\mathbb{R}^D)$ of distributions is the completion of $\mathcal{S}(\mathbb{R}^D)$ with respect to the norm induced by the scalar product

$$(u, v)_{H_h^r(\mathbb{R}^D)} := (\mathcal{B}_h u, \mathcal{B}_h v)_{L^2(\mathbb{R}^{2D}, (W_h^r)^2)} = \int (W_h^r)^2 \cdot \overline{\mathcal{B}_h u} \cdot \mathcal{B}_h v \frac{dx d\xi}{(2\pi\hbar)^D} \quad \text{for } u, v \in \mathcal{S}(\mathbb{R}^D).$$

The induced norm on $H_h^r(\mathbb{R}^D)$ will be written as $\|u\|_{H_h^r(\mathbb{R}^D)} := \|\mathcal{B}_h u\|_{L^2(\mathbb{R}^{2D}, (W_h^r)^2)}$.

Proposition 3.20. "Discrete spectrum of the linear expanding map". Let $A : \mathbb{R}^D \rightarrow \mathbb{R}^D$ be a linear expanding map satisfying $\|A^{-1}\| \leq 1/\lambda$ for some $\lambda > 1$. Let L_A be the unitary transfer operator defined in (3.13): $L_A u = \frac{1}{\sqrt{\det A}} u \circ A^{-1}$. Let $n > 0$ and $r > n + 2 + \frac{D}{2}$. Then the Hilbert space $H_h^r(\mathbb{R}^D)$ defined above is decomposed into subspaces of homogeneous polynomial of degree k for $0 \leq k \leq n$ and the remainder:

$$H_h^r(\mathbb{R}^D) = \left(\bigoplus_{k=0}^n \text{Polynom}^{(k)} \right) \oplus \tilde{\mathcal{H}}_h$$

where $\tilde{\mathcal{H}}_h := \tilde{T}(H_h^r(\mathbb{R}^D))$ with setting

$$\tilde{T} := \text{Id} - \sum_{k=0}^n T^{(k)}. \quad (3.46)$$

(The operators $T^{(k)}$ are defined in (3.34).) This decomposition is preserved by L_A . There exists a constant $C_0 > 0$ independent of A and \hbar such that

(1) For $0 \leq k \leq n$ and $0 \neq u \in \text{Polynom}^{(k)}$, we have

$$C_0^{-1} \|A\|_{\max}^{-k} \cdot |\det A|^{-1/2} \leq \frac{\|L_A u\|_{H_h^r(\mathbb{R}^D)}}{\|u\|_{H_h^r(\mathbb{R}^D)}} \leq C_0 \|A\|_{\min}^{-k} \cdot |\det A|^{-1/2} \quad (3.47)$$

(Recall (1.22) for the definition of $\|\cdot\|_{\max}$ and $\|\cdot\|_{\min}$.)

(2) The operator norm of the restriction of L_A to $\tilde{\mathcal{H}}_h$ is bounded by

$$C_0 \max\{\|A\|_{\min}^{-(n+1)} \cdot |\det A|^{-1/2}, \|A\|_{\min}^{-r} \cdot |\det A|^{1/2}\}. \quad (3.48)$$

The following equivalent statements holds for the lifted operator:

$$L_A^{\text{lift}} : L^2(\mathbb{R}^{2D}, (W_h^r)^2) \rightarrow L^2(\mathbb{R}^{2D}, (W_h^r)^2).$$

The operator L_A^{lift} preserves the decomposition of $L^2(\mathbb{R}^{2D}, (W_h^r)^2)$ in (3.44) and

(1) For $0 \leq k \leq n$ and for $0 \neq u \in H_k$, we have

$$C_0^{-1} \|A\|_{\max}^{-k} \cdot |\det A|^{-1/2} \leq \frac{\|L_A^{\text{lift}} u\|_{L^2(\mathbb{R}^{2D}, (W_h^r)^2)}}{\|u\|_{L^2(\mathbb{R}^{2D}, (W_h^r)^2)}} \leq C_0 \|A\|_{\max}^{-k} \cdot |\det A|^{-1/2}. \quad (3.49)$$

(2) The operator norm of the restriction of L_A^{lift} to \tilde{H} is bounded by (3.48).

Remark 3.21. Proposition 3.20 shows that the spectrum of the transfer operator L_A in the Hilbert space $H_h^r(\mathbb{R}^D)$ is discrete outside the radius given by (3.48). The eigenvalues outside this radius are given by the action of L_A in the finite dimensional space $\text{Polynom}^{(k)}$. These eigenvalues can be compute explicitly from the Jordan block decomposition of A . In particular if A is diagonal then the monomials are obviously eigenvectors.

Proof. For the proof of (3.47) and (3.49), we use the fact that the space $\text{Polynom}^{(k)}$ is identical to the space $\text{Sym}^k(\mathbb{R}^D)$ of totally symmetric tensors of rank k . For the linear operator $(A^{-1})^{\otimes k}$ acting on $(\mathbb{R}^D)^{\otimes k}$, we have a commutative diagram:

$$\begin{array}{ccc} (\mathbb{R}^D)^{\otimes k} & \xrightarrow{(A^{-1})^{\otimes k}} & (\mathbb{R}^D)^{\otimes k} \\ \text{Sym} \downarrow & & \text{Sym} \downarrow \\ \text{Sym}^k(\mathbb{R}^D) & \xrightarrow{|\det A|^{1/2} L_A} & \text{Sym}^k(\mathbb{R}^D) \end{array}$$

where Sym denotes the symmetrization operation. For every $0 \neq \tilde{u} \in (\mathbb{R}^D)^{\otimes k}$ we have

$$\|A\|_{\max}^{-k} \leq \frac{\|(A^{-1})^{\otimes k} \tilde{u}\|}{\|\tilde{u}\|} \leq \|A\|_{\min}^{-k}.$$

Since the spaces are finite dimensional (and hence all norms are equivalent), we deduce (3.47) for some constant $C_0 > 0$ independent of A , and also independent on \hbar because of the scaling invariance (3.27). The proof of the Claim (2) is postponed to Subsection 3.6, as it requires more detailed argument. \square

3.5 The harmonic oscillator

In this subsection we present the harmonic oscillator in the setting of Bargmann transform. (We refer [19],[14] for more detailed treatment.) We will need it in dealing with the (Euclidean) rough Laplacian Δ_{\hbar} later on. Associated to the standard coordinates

$$(x, \xi) = (x_1, x_2, \dots, x_D, \xi_1, \xi_2, \dots, \xi_D)$$

on $T^*\mathbb{R}^D = \mathbb{R}^{2D}$, we consider the operators

$$\hat{x}_i : \begin{cases} \mathcal{S}(\mathbb{R}^{2D}) & \rightarrow \mathcal{S}(\mathbb{R}^{2D}) \\ u & \rightarrow (\mathcal{P}_{\hbar} \circ \mathcal{M}(x_i) \circ \mathcal{P}_{\hbar}) u \end{cases} \quad \text{and} \quad \hat{\xi}_i : \begin{cases} \mathcal{S}(\mathbb{R}^{2D}) & \rightarrow \mathcal{S}(\mathbb{R}^{2D}) \\ u & \rightarrow (\mathcal{P}_{\hbar} \circ \mathcal{M}(\xi_i) \circ \mathcal{P}_{\hbar}) u \end{cases}$$

where $\mathcal{M}(x_i)$ and $\mathcal{M}(\xi_i)$ on the right hand sides denote the multiplication by the corresponding functions. These operators are usually called Toeplitz quantization of x_i and ξ_i . Then we set

$$\hat{P} := \frac{1}{2\hbar} (\hat{x}^2 + \hat{\xi}^2) := \frac{1}{2\hbar} \sum_{i=1}^D (\hat{x}_i \circ \hat{x}_i + \hat{\xi}_i \circ \hat{\xi}_i), \quad (3.50)$$

which is usually called the *harmonic oscillator* operator.

Lemma 3.22. *"Spectrum of the harmonic oscillator". The operator \hat{P} in (3.50) is a closed self-adjoint operator on $L^2(\mathbb{R}^{2D})$ and its spectral set consists of eigenvalues*

$$\frac{D}{2} + k, \quad k \in \mathbb{N} \cup \{0\}.$$

The spectral projector $\mathcal{Q}_{\hbar}^{(k)}$ for the eigenvalue $\frac{D}{2} + k$ is an orthogonal projection operator of rank $\binom{D+k-1}{D-1}$. We have

$$\text{Im} \mathcal{Q}_{\hbar}^{(k)} = \{ \mathcal{B}_{\hbar}(p \cdot \varphi_0) \mid p \in \text{Polynom}^{(k)} \} \quad (3.51)$$

where

$$\varphi_0(x) = e^{-|x|^2/(2\hbar)}.$$

In particular, we have the following orthogonal decomposition of $L^2(\mathbb{R}^{2D})$:

$$L^2(\mathbb{R}^{2D}) = \overline{\bigoplus_{k=0}^{\infty} \text{Im} \mathcal{Q}_h^{(k)}}.$$

Proof. Since \hat{x}_i and $\hat{\xi}_i$ are the lift of the operators

$$u \mapsto x_i \cdot u \quad \text{and} \quad u \mapsto -i\hbar \cdot \partial_{x_i} u$$

respectively, the operator $\hat{P} = \frac{1}{2\hbar} (\hat{x}^2 + \hat{\xi}^2)$ is the lift of

$$\mathcal{H} : L^2(\mathbb{R}^D) \rightarrow L^2(\mathbb{R}^D), \quad (\mathcal{H}u)(x) = \frac{1}{\hbar} \left(-\frac{\hbar^2}{2} \left(\sum_{i=1}^D \frac{\partial^2 u}{\partial x_i^2} \right) (x) + \frac{1}{2}|x|^2 \cdot u(x) \right) \quad (3.52)$$

Therefore the conclusion follows from the argument on quantization of the harmonic oscillator \mathcal{H} [38, p.105]. \square

Remark 3.23. It is possible to compute directly that the eigenfunctions of \hat{P} using the “creation operator” $a_j = \frac{1}{\sqrt{2}} (\hat{x}_j + i\hat{\xi}_j)$ and showing that it is multiplication par z_j in phase space. (See [19, 3]) Then we can identify $\mathcal{Q}_h^{(i)}$ as projection onto homogeneous polynomials of degree i in $z_i = x_i + i\xi_i$.

Recall that operators $\mathcal{Q}_h^{(k)}$ and $\mathcal{T}_h^{(k)}$ have the same rank $\binom{D+k-1}{D-1}$. The next lemma gives a more precise relation between them.

Lemma 3.24. *For $0 \leq k \leq n$, the operator $\mathcal{Q}_h^{(k)}$ extends to a continuous operator*

$$\mathcal{Q}_h^{(k)} : \mathcal{S}'(\mathbb{R}^{2D}) \rightarrow \mathcal{S}(\mathbb{R}^{2D}).$$

The restrictions:

$$\left(\bigoplus_{i=0}^k \mathcal{Q}_h^{(i)} \right) : \bigoplus_{i=1}^k \text{Im} \mathcal{T}_h^{(i)} \rightarrow \bigoplus_{i=0}^k \text{Im} \mathcal{Q}_h^{(i)} \quad (3.53)$$

and

$$\left(\bigoplus_{i=0}^k \mathcal{T}_h^{(i)} \right) : \bigoplus_{i=0}^k \text{Im} \mathcal{Q}_h^{(i)} \rightarrow \bigoplus_{i=0}^k \text{Im} \mathcal{T}_h^{(i)} \quad (3.54)$$

are well-defined and bijective. The operator norms of (3.53), (3.54) and their inverses are bounded by a constant independent of \hbar .

Proof. Recall that the operator $\mathcal{Q}_h^{(k)}$ is the orthogonal projection to its image (3.51), which is finite dimensional and contained in $\mathcal{S}(\mathbb{R}^{2D})$. Hence we have the first claim and well definiteness of the operators (3.53) and (3.54) is an immediate consequence.

To prove that (3.53) and (3.54) are bijective, we have only to show that they are injective, because the subspaces in the source and target have the same finite dimension. We prove injectivity of (3.53). Let $u \in \left(\bigoplus_{i=0}^k \text{Im} \mathcal{T}_h^{(i)} \right)$. Such u can be expressed as $u = \mathcal{B}_h p$

with p a polynomial of degree at most k on \mathbb{R}^D . Suppose that $\left(\oplus_{i=0}^k \mathcal{Q}_h^{(i)}\right)u = 0$. Since $\left(\oplus_{i=0}^k \mathcal{Q}_h^{(i)}\right)$ is an orthogonal projection operator, u is orthogonal to $\oplus_{i=0}^k \text{Im} \mathcal{Q}_h^{(i)}$ and hence we have

$$(p, q \cdot \varphi_0)_{L^2(\mathbb{R}^D)} = (\mathcal{B}_h p, \mathcal{B}_h(q \cdot \varphi_0))_{L^2(\mathbb{R}^{2D})} = 0$$

for any polynomial q of order at most k on \mathbb{R}^D . Setting $q = p$, we see that $(p, p \cdot \varphi_0)_{L^2(\mathbb{R}^D)} = (|p|^2, \varphi_0)_{L^2(\mathbb{R}^D)} = 0$, showing $p = 0$ and $u = 0$. We have shown that (3.53) is injective.

To prove injectivity of (3.54), let $u = q\varphi_0 \in \left(\oplus_{i=0}^k \text{Im} \mathcal{Q}_h^{(i)}\right)$ with q a polynomial of degree at most k on \mathbb{R}^D . Let $p = \left(\oplus_{i=0}^k \mathcal{T}_h^{(i)}\right)u$ be the Taylor expansion of u at 0 up to order k . Suppose that $p = 0$. Since $\varphi_0(0) \neq 0$, φ_0 is invertible as a formal power series, we deduce that $q = 0$. Hence (3.54) is injective.

The operators (3.53) (resp. (3.54)) for different $\hbar > 0$ are related by the scaling (3.9) and hence we get the last claim. \square

3.6 Proof of Claim (2) in Proposition 3.20

We prove Claim (2) on the lifted operator L_A^{lift} in the latter part of the statement, which is equivalent to Claim (2) in the former part. In the proof below, we may and do assume $\hbar = 1$, because the Bargmann transforms for different parameter \hbar are related by the scaling (3.9), as we noted in Subsection 3.1. Accordingly we will drop the subscript \hbar from the notation. Let $\chi : \mathbb{R}^D \rightarrow [0, 1]$ be a smooth function such that

$$\chi(x) = \begin{cases} 1 & \text{if } |x| \leq 1 \\ 0 & \text{if } |x| \geq 2. \end{cases} \quad (3.55)$$

Below we use C_0 as a generic symbol for the constants which do not depend on A (but may depend on r, n and D). Letting λ smaller if necessary, we suppose

$$\lambda = \|A^{-1}\|^{-1} > 1$$

for simplicity. We write $\mathcal{M}(\varphi)$ for the multiplication operator by φ .

To prove the claim, it is enough to show

$$\|L_A^{\text{lift}} \circ \tilde{\mathcal{T}}\|_{L^2(\mathbb{R}^{2D}, (W^r)^2)} \leq C_0 \cdot \max\{\lambda^{-n-1} |\det A|^{-1/2}, \lambda^{-r} |\det A|^{1/2}\}$$

where $\tilde{\mathcal{T}}$ is the operator defined in (3.43) with $\hbar = 1$ and $\|\cdot\|_{L^2(\mathbb{R}^{2D}, (W^r)^2)}$ denotes the operator norm on $L^2(\mathbb{R}^{2D}, (W^r)^2)$.

Let us consider the operators

$$X = \mathcal{B} \circ \mathcal{M}(\chi) \circ \mathcal{B}^* : L^2(\mathbb{R}^{2D}, W^r) \rightarrow L^2(\mathbb{R}^{2D}, W^r)$$

and

$$\Xi : L^2(\mathbb{R}^{2D}, (W^r)^2) \rightarrow L^2(\mathbb{R}^{2D}, (W^r)^2), \quad (\Xi v)(x, \xi) = \chi\left(\frac{|\xi|}{\lambda}\right) \cdot v(x, \xi).$$

The next lemma is the main ingredient of the proof.

Lemma 3.25. $\|L_A^{\text{lift}} \circ X \circ \tilde{\mathcal{T}} \circ \Xi\|_{L^2(\mathbb{R}^{2D}, (W^r)^2)} \leq C_0 \cdot \lambda^{-(n+1)} |\det A|^{-1/2}$.

Proof. Let $\mathbf{1}_{\mathbf{C}_-(2)}$ be the indicator function of the cone

$$\mathbf{C}_-(2) := \{(x, \xi) \mid |x| \leq 2|\xi|\} = \overline{\mathbb{R}^{2D} \setminus \mathbf{C}_+(1/2)}$$

and set

$$W_+^r(x, \xi) := \mathbf{1}_{\mathbf{C}_-(2)}(x, \xi) \cdot \langle |\xi| \rangle^r, \quad W_-^r(x, \xi) := \langle |x| \rangle^{-r}.$$

Then the weight function $W^r(x, \xi)$ satisfies

$$W^r(x, \xi) \leq C_0 \cdot W_+^r(x, \xi) + C_0 \cdot W_-^r(x, \xi)$$

for a constant $C_0 > 0$. Hence, to prove the lemma, it is enough to show the claim

$$\|W_\sigma^r \cdot \mathcal{B} \circ L_A \circ \mathcal{M}(\chi) \circ \tilde{\mathcal{T}} \circ \mathcal{B}^* \circ \Xi u\|_{L^2} \leq C_0 \cdot \lambda^{-(n+1)} |\det A|^{-1/2} \|u\| \quad \text{for any } u \in L^2(\mathbb{R}^{2D}, W^r) \quad (3.56)$$

in the cases $\sigma = \pm$, where $\tilde{\mathcal{T}}$ is the operator defined in (3.46). Before proceeding to the proof of (3.56), we prepare a few estimates. Take $u \in L^2(\mathbb{R}^{2D}, W^r)$ arbitrarily and set

$$v(y) := (\mathcal{B}^* \circ \Xi u)(y) = \int \phi_{x, \xi}(y) \chi\left(\frac{|\xi|}{\lambda}\right) u(x, \xi) dx d\xi.$$

Then, for any multi-index $\alpha \in \mathbb{N}^D$ and arbitrarily large ν , we have

$$|\partial_y^\alpha v(y)| \leq C_{\alpha, \nu} \int_{|\xi| \leq 2\lambda} \langle |x - y| \rangle^{-\nu} \cdot \langle |\xi| \rangle^{|\alpha|} \cdot |u(x, \xi)| dx d\xi \quad \text{for any } y \in \mathbb{R}^D. \quad (3.57)$$

Note that we have

$$\langle |x - y| \rangle^{-r} \cdot \langle |\xi| \rangle^r \leq C_0 \cdot W^r(x, \xi) \quad \text{for any } x, y, \xi \in \mathbb{R}^D \text{ with } |y| \leq 2.$$

Hence

$$\langle |x - y| \rangle^{-\nu} \cdot \langle |\xi| \rangle^{|\alpha|} \leq C_0 (\langle |x - y| \rangle^{-\nu+r} \cdot \langle |\xi| \rangle^{-D/2+1}) \cdot \langle |\xi| \rangle^{|\alpha|+D/2+1-r} \cdot W^r(x, \xi).$$

Putting this estimate in (3.57) with $\nu \geq D/2 + 1 + r$, we obtain, by Cauchy-Schwarz inequality,

$$|\partial_y^\alpha v(y)| \leq C_\alpha \lambda^{\max\{|\alpha|+D/2+1-r, 0\}} \|u\|_{L^2(\mathbb{R}^{2D}, (W^r)^2)} \quad \text{for any } y \in \mathbb{R}^D \text{ with } |y| \leq 2. \quad (3.58)$$

Notice that, if $|\alpha| \leq n + 1$, we have $|\alpha| + D/2 + 1 - r \leq 0$ from the assumption (3.38) and hence the last inequality implies

$$|\partial_y^\alpha v(y)| \leq C_\alpha \|u\|_{L^2(\mathbb{R}^{2D}, (W^r)^2)} \quad \text{for any } y \in \mathbb{R}^D \text{ with } |y| \leq 2. \quad (3.59)$$

Next we consider the function

$$w := \mathcal{M}(\chi) \circ \tilde{\mathcal{T}} v = \mathcal{M}(\chi) \circ \tilde{\mathcal{T}} \circ \mathcal{B}^* \circ \Xi u.$$

Note that the support of w is contained in that of χ . It follows from (3.58) that, for each multi-index α ,

$$|\partial^\alpha w(y)| \leq C_\alpha \lambda^{\max\{|\alpha|+D/2+1-r, 0\}} \|u\|_{L^2(\mathbb{R}^{2D}, (W^r)^2)} \quad \text{for all } y \in \mathbb{R}^D. \quad (3.60)$$

Further, if $|\alpha| \leq n+1$, it follows from (3.59) and the definition of \tilde{T} that

$$|\partial_y^\alpha w(y)| \leq C_\alpha |y|^{n+1-|\alpha|} \cdot \max_{|y| \leq 2} |\partial_y^{n+1} v| \leq C_\alpha |y|^{n+1-|\alpha|} \|u\|_{L^2(\mathbb{R}^{2D}, (W^r)^2)} \quad \text{for all } y \in \mathbb{R}^D. \quad (3.61)$$

Now we prove the claim (3.56) in the case $\sigma = +$. Let u, v, w be as above. We are going to estimate the quantity

$$\xi^\alpha \cdot (\mathcal{B} \circ L_A \circ \mathcal{M}(\chi) \circ \tilde{T} \circ \mathcal{B}^* \circ \Xi u)(x, \xi) = |\det A|^{-1/2} \cdot \xi^\alpha \int \overline{\phi_{x, \xi}(y)} \cdot w(A^{-1}y) dy.$$

By integration by parts, we see that this is bounded in absolute value by

$$C_{\alpha, \nu} |\det A|^{-1/2} \sum_{\alpha' \leq \alpha} \int \langle |x-y| \rangle^{-\nu} \cdot \lambda^{-|\alpha'|} \cdot |\partial^{\alpha'} w(A^{-1}y)| dy$$

for each $\nu > 0$, where $C_{\alpha, \nu}$ is a constant depending only on α and ν . If ν is sufficiently large, we have from (3.60), (3.61) and then from (3.38) that

$$\begin{aligned} & \sum_{\alpha' \leq \alpha} \int \langle |x-y| \rangle^{-\nu} \cdot \lambda^{-|\alpha'|} \cdot |\partial^{\alpha'} w(A^{-1}y)| dy \\ & \leq C_\alpha \lambda^{-|\alpha'|} \left(\sum_{\alpha': |\alpha'| \leq n+1} \left(\frac{\langle |x| \rangle}{\lambda} \right)^{n+1-|\alpha'|} + \sum_{\alpha': n+2 \leq |\alpha'| \leq |\alpha|} \lambda^{\max\{|\alpha'|+D/2+1-r, 0\}} \right) \|u\|_{L^2(\mathbb{R}^{2D}, (W^r)^2)} \\ & \leq C_\alpha \cdot \lambda^{-(n+1)} \langle |x| \rangle^{n+1} \cdot \|u\|_{L^2(\mathbb{R}^{2D}, (W^r)^2)}. \end{aligned}$$

Therefore we obtain

$$\langle |\xi| \rangle^\nu \cdot |\mathcal{B} \circ L_A \circ \mathcal{M}(\chi) \circ \tilde{T} \circ \mathcal{B}^* \circ \Xi u(x, \xi)| \leq C_\nu \lambda^{-(n+1)} |\det A|^{-1/2} \|u\|_{L^2(\mathbb{R}^{2D}, (W^r)^2)} \cdot \langle |x| \rangle^{n+1}$$

for arbitrarily large ν . For (x, ξ) on $\text{supp } W_+^r = \mathbf{C}_-(2)$, we have $\langle |x| \rangle \leq 2\langle |\xi| \rangle$ and hence

$$W_+^r(x, \xi) \leq \langle |\xi| \rangle^r \leq C_0 \langle |\xi| \rangle^{r+D/1+1+(n+1)} \cdot \langle |x| \rangle^{-D/2-1-(n+1)}.$$

Using this in the last inequality, we get

$$\begin{aligned} & W_+^r(x, \xi) |(\mathcal{B} \circ L_A \circ \mathcal{M}(\chi) \circ (\text{Id} - \mathbf{T}_n) \circ \mathcal{B}^* \circ \Xi u)(x, \xi)| \\ & \leq C_\nu \lambda^{-(n+1)} |\det A|^{-1/2} \|u\|_{L^2(\mathbb{R}^{2D}, (W^r)^2)} \langle |x| \rangle^{-D/2-1} \langle |\xi| \rangle^{-\nu+r+D/1+1+(n+1)}. \end{aligned}$$

This estimate for sufficiently large ν implies the claim (3.56) in the case $\sigma = +$, by Cauchy-Schwarz inequality.

We prove the claim (3.56) for $\sigma = -$. The proof is easier than the previous case actually. Note that we have

$$\|W_-^r \cdot \mathcal{B}\varphi\|_{L^2} \leq C_0 \cdot \|\langle \cdot \rangle^{-r} \cdot \varphi(\cdot)\|_{L^2(\mathbb{R}^D)}.$$

Hence, from (3.61) with $\alpha = \emptyset$, we get

$$\begin{aligned} \|W_-^r \cdot \mathcal{B} \circ L_A \circ \mathcal{M}(\chi) \circ \tilde{T} \circ \mathcal{B}^* \circ \Xi u\|_{L^2} &= \|W_-^r \cdot \mathcal{B} \circ L_A \circ w\|_{L^2} \\ &\leq C_0 \cdot |\det A|^{-1/2} \cdot \left| \int \langle x \rangle^{-2r} \langle x/\lambda \rangle^{2(n+1)} dx \right|^{1/2} \cdot \|u\|_{L^2(\mathbb{R}^{2D}, (W^r)^2)} \\ &\leq C_0 \cdot |\det A|^{-1/2} \cdot \lambda^{-n-1} \cdot \|u\|_{L^2(\mathbb{R}^{2D}, (W^r)^2)}. \end{aligned}$$

Clearly this implies (3.56) for $\sigma = -$. □

To finish, it is enough to show

$$\|L_A^{\text{lift}} \circ \tilde{\mathcal{T}} - L_A^{\text{lift}} \circ X \circ \tilde{\mathcal{T}} \circ \Xi\|_{L^2(\mathbb{R}^{2D}, (W^r)^2)} \leq C_0 \lambda^{-r} |\det A|^{1/2}. \quad (3.62)$$

Note the relations

$$L_A^{\text{lift}} \circ \tilde{\mathcal{T}} = \mathcal{B} \circ L_A \circ \tilde{T} \circ \mathcal{B} = \mathcal{B} \circ \tilde{T} \circ L_A \circ \mathcal{B} = \tilde{\mathcal{T}} \circ L_A^{\text{lift}}$$

and

$$\begin{aligned} L_A^{\text{lift}} \circ (\text{Id} - X) &= \mathcal{B} \circ L_A \circ \mathcal{M}(1 - \chi) \circ \mathcal{B}^* \\ &= \mathcal{B} \circ \mathcal{M}(1 - \chi_A) \circ L_A \circ \mathcal{B}^* = (\text{Id} - X_A) \circ L_A^{\text{lift}} \end{aligned}$$

where $\chi_A = \chi \circ A^{-1}$ and $X_A = \mathcal{B} \circ \chi_A \circ \mathcal{B}^*$. Below we will prove the claims

$$\|L_A^{\text{lift}} \circ (\text{Id} - X)\|_{L^2(\mathbb{R}^{2D}, (W^r)^2)} \leq C_0 \lambda^{-r} |\det A|^{1/2} \quad (3.63)$$

and

$$\|L_A^{\text{lift}} \circ (\text{Id} - \Xi)\|_{L^2(\mathbb{R}^{2D}, (W^r)^2)} \leq C_0 \lambda^{-r} |\det A|^{1/2}. \quad (3.64)$$

Since $\tilde{\mathcal{T}} = \text{Id} - \sum_{k=0}^n \mathcal{T}^{(k)}$ is a bounded operator on $L^2(\mathbb{R}^{2D}, W^r)$, these claims would imply

$$\|L_A^{\text{lift}} \circ (\text{Id} - X) \circ \tilde{\mathcal{T}}\|_{L^2(\mathbb{R}^{2D}, (W^r)^2)} \leq C_0 \cdot \lambda^{-r} |\det A|^{1/2}$$

and

$$\|L_A^{\text{lift}} \circ X \circ \tilde{\mathcal{T}} \circ (\text{Id} - \Xi)\|_{L^2(\mathbb{R}^{2D}, (W^r)^2)} = \|X_A \circ \tilde{\mathcal{T}} \circ L_A^{\text{lift}} \circ (\text{Id} - \Xi)\|_{L^2(\mathbb{R}^{2D}, (W^r)^2)} \leq C_0 \lambda^{-r} |\det A|^{1/2}$$

and therefore the conclusion (3.62) would follow.

We can prove (3.63) and (3.64) by straightforward estimate. Writing the kernel of the operator X_A explicitly and applying integration by parts to it, we get the estimate

$$|(\text{Id} - X_A)v(x, \xi)| \leq C_\nu \int K_1^{(\nu)}(x, \xi; x', \xi') \cdot v(x', \xi') dx' d\xi'$$

for arbitrarily large ν , where

$$K_1^{(\nu)}(x, \xi; x', \xi') = \int_{\text{supp}(1-\chi_A)} \langle |x-y| \rangle^{-\nu} \langle |y-x'| \rangle^{-\nu} \langle |\xi-\xi'| \rangle^{-\nu} dy.$$

From the expression (3.17) of the operator L_A^{lift} in Lemma 3.9, we also have

$$|L_A^{\text{lift}} v(x', \xi')| \leq C_\nu |\det A|^{1/2} \int K_2^{(\nu)}(x', \xi'; x'', \xi'') v(x'', \xi'') dx'' d\xi''$$

for arbitrarily large ν , where

$$K_2^{(\nu)}(x', \xi'; x'', \xi'') = \int \langle |x' - x_\dagger| \rangle^{-\nu} \langle |\xi' - \xi_\dagger| \rangle^{-\nu} \langle |A^{-1}x_\dagger - x''| \rangle^{-\nu} \langle |{}^t A \xi_\dagger - \xi''| \rangle^{-\nu} dx_\dagger d\xi_\dagger.$$

(We used $d(A) \leq \det A$ also.) From the definition of the function W^r and the expanding property (3.31) of A , we have

$$\frac{W^r(x_\dagger, \xi_\dagger)}{W^r(A^{-1}x_\dagger, {}^t A \xi_\dagger)} \cdot \langle |x-y| \rangle^{-2r} \leq C_0 \lambda^{-r} \quad \text{if } y \in \text{supp}(1-\chi_A).$$

Also note that, from the property (3.30) of W^r , we have

$$W^r(x, \xi) \cdot \langle |x-y| \rangle^{-2r} \langle |y-x'| \rangle^{-2r} \langle |x'-x_\dagger| \rangle^{-2r} \leq C_0 \cdot W^r(x_\dagger, \xi_\dagger)$$

and

$$\frac{1}{W^r(x'', \xi'')} \cdot \langle |A^{-1}x_\dagger - x''| \rangle^{-2r} \langle |{}^t A \xi_\dagger - \xi''| \rangle^{-2r} \leq C_0 \cdot \frac{1}{W^r(A^{-1}x_\dagger, {}^t A \xi_\dagger)}.$$

Summarizing these estimates, we obtain

$$\begin{aligned} & \frac{W^r(x, \xi)}{W^r(x'', \xi'')} \cdot \int K_1^{(\nu)}(x, \xi; x', \xi'; y) \cdot K_2^{(\nu)}(x', \xi'; x'', \xi'') dx' d\xi' \\ & \leq C_0 \lambda^{-r} \cdot \int \int K_1^{(\nu-4r)}(x, \xi; x', \xi'; y) \cdot K_2^{(\nu-2r)}(x', \xi'; x'', \xi'') dx' d\xi'. \end{aligned}$$

By Young inequality, the integral operators with the kernels $K_1^{(\nu-4r)}(\cdot)$ and $K_2^{(\nu-2r)}(\cdot)$ are bounded operators on $L^2(\mathbb{R}^{2D})$ and the operator norms are bounded by a constant that does not depend on A , provided ν is sufficiently large. Therefore the last estimate implies (3.63). We can prove the claim (3.64) in the same manner.

4 Resonance of hyperbolic linear prequantum maps

The main result of this section, Proposition 4.9, concerns the spectrum of the prequantum transfer operator for hyperbolic symplectic affine maps. In the last subsection, we will also consider the spectrum of the Euclidean rough Laplacian $\Delta = D^*D$ on \mathbb{R}^{2d} .

4.1 Prequantum transfer operator and rough Laplacian on \mathbb{R}^{2d}

In this section, we study prequantum transfer operators and rough Laplacian in a special and easy case: The Riemann manifold M is the linear space \mathbb{R}^{2d} with the coordinates

$$x \equiv (q, p) = (q^1, \dots, q^d, p^1, \dots, p^d) \in \mathbb{R}^{2d}. \quad (4.1)$$

We regard it as a symplectic manifold equipped with the symplectic two form

$$\omega = dq \wedge dp := \sum_{i=1}^d dq^i \wedge dp^i. \quad (4.2)$$

The prequantum bundle P is the trivial $U(1)$ -bundle $\pi : P = \mathbb{R}^{2d} \times U(1) \rightarrow \mathbb{R}^{2d}$ over \mathbb{R}^{2d} equipped with the connection one form $A = id\theta - i(2\pi)\eta$ where

$$\eta = \sum_{i=1}^d \left(\frac{1}{2} q^i dp^i - \frac{1}{2} p^i dq^i \right). \quad (4.3)$$

The corresponding curvature two form is then

$$\Theta = -i(2\pi)(\pi^*\omega)$$

because $\omega = d\eta$. Under these settings, we may rephrase the construction of the prequantum transfer operator for a symplectic diffeomorphism on \mathbb{R}^{2d} and also that of the rough Laplacian.

Let $f : U \rightarrow U'$ be a symplectic diffeomorphism between two domains U and U' in \mathbb{R}^{2d} with respect to the symplectic two form ω . Let $\tilde{f} : U \times \mathbf{U}(1) \rightarrow U' \times \mathbf{U}(1)$ be the equivariant lift of f preserving the connection A , that is, the map satisfying the conditions (1.12), (1.13) and (1.14) in Proposition 1.4. Then we define

$$\hat{F} : C^\infty(U \times \mathbf{U}(1)) \rightarrow C^\infty(U' \times \mathbf{U}(1)), \quad \hat{F}u(x) = u \circ \tilde{f}^{-1}(x)$$

and let

$$\hat{F}_N : C_N^\infty(U \times \mathbf{U}(1)) \rightarrow C_N^\infty(U' \times \mathbf{U}(1))$$

be its restriction to the space of functions in the N -th Fourier mode. This construction is exactly parallel to that we have given in Subsection 1.2, but for the set $V \equiv 0$. Let

$$\mathcal{L}_f : C^\infty(\mathbb{R}^{2d}) \rightarrow C^\infty(\mathbb{R}^{2d})$$

be the expression of the prequantum transfer operator \hat{F}_N with respect to the trivialization using the trivial section $\tau_0 : \mathbb{R}^{2d} \rightarrow P = \mathbb{R}^{2d} \times U(1)$ defined by $\tau_0(x) = (x, 1)$. This operator \mathcal{L}_f is called the *prequantum transfer operator* for a diffeomorphism $f : U \rightarrow U'$. (Note that \mathcal{L}_f depends on the integer $N \in \mathbb{Z}$ and hence on \hbar .) We recall its concrete form from Proposition 2.3.

Proposition 4.1. *The operator \mathcal{L}_f as above is written*

$$\mathcal{L}_f u(x) := e^{\frac{i}{\hbar} \mathcal{A}_f(x)} u(f^{-1}(x)) \quad (4.4)$$

with the (action) function

$$\mathcal{A}_f(x) = \int_{\gamma} (f^{-1})^* \eta - \eta \quad (4.5)$$

where γ is a path from a fixed point $x_0 \in U'$ to x .

Let us assume now an Euclidean (and ω -compatible) metric

$$g = \frac{1}{2} \left(\sum_{i=1}^d dq^i \otimes dq^i + dp^i \otimes dp^i \right)$$

on \mathbb{R}^{2d} . We may consider the rough Laplacian in the setting above as the operator

$$\Delta_{\hbar} = D^* D : C^\infty(\mathbb{R}^{2d}) \rightarrow C^\infty(\mathbb{R}^{2d}).$$

This operator is called the *Euclidean rough Laplacian* and given in Proposition 2.40.

4.2 Prequantum operator for a symplectic affine map on \mathbb{R}^{2d}

Let $f : \mathbb{R}^{2d} \rightarrow \mathbb{R}^{2d}$ be an affine map preserving the symplectic form ω , written:

$$f : \mathbb{R}^{2d} \rightarrow \mathbb{R}^{2d}, \quad f(x) = Bx + b \quad (4.6)$$

where $B : \mathbb{R}^{2d} \rightarrow \mathbb{R}^{2d}$ a linear symplectic map and $b \in \mathbb{R}^{2d}$ a constant vector.

Proposition 4.2. *The prequantum transfer operator \mathcal{L}_f for an affine map f as above is written as*

$$\mathcal{L}_f u(x) := e^{\frac{i}{\hbar} \mathcal{A}_f(x)} u(f^{-1}(x)) \quad (4.7)$$

with the (action) function

$$\mathcal{A}_f(x) = -\frac{1}{2} \omega(b, x). \quad (4.8)$$

Remark 4.3. Notice that the function \mathcal{A}_f in (4.8) does not depend on the linear map B which enters in (4.6).

Proof. For simpler notations we can write the one form η in (4.3) as $\eta = \frac{1}{2} x \wedge dx$. Then, for any $x \in \mathbb{R}^{2d}$, we have

$$\int_0^x \eta = \int_0^x \frac{1}{2} y \wedge dy = \int_0^1 \frac{1}{2} (tx \wedge x) dt = 0 \quad \text{with setting } y(t) = tx.$$

Therefore for a linear symplectic map $f_2(x) = Bx$, the action defined in (2.8) vanishes:

$$\mathcal{A}_{f_2}(x) = \int_{f_2^{-1}(0)}^{f_2^{-1}(x)} \eta - \int_0^x \eta = 0.$$

For a translation map $f_1(x) = x + b$, the action is

$$\mathcal{A}_{f_1}(x) = \int_{-b}^{x-b} \eta - \int_0^x \eta = \int_{-b}^{x-b} \frac{1}{2} y \wedge dy = \int_0^1 \frac{1}{2} ((tx - b) \wedge x) dt = -\frac{1}{2} b \wedge x$$

with setting $y(t) = tx - b$. Finally for the affine map $f(x) = Bx + b = (f_1 \circ f_2)(x)$, the action is

$$\mathcal{A}_f = \mathcal{A}_{f_1} + \mathcal{A}_{f_2} \circ f_2^{-1} = \mathcal{A}_{f_1} = -\frac{1}{2} b \wedge x = -\frac{1}{2} \omega(b, x).$$

□

We next consider the lift of the operator \mathcal{L}_f above with respect to the Bargmann transform \mathcal{B}_h . Following the idea explained in Subsection 2.2, we express it with respect to the coordinates (ν, ζ) introduced in Proposition 2.13. Then, in the next proposition, we obtain an expression of \mathcal{L}_f as a tensor product of two operators: each of the two operators is associated to the dynamics of the canonical map $F = {}^t Df^{-1} : T^*\mathbb{R}^{2d} \rightarrow T^*\mathbb{R}^{2d}$ of \mathcal{L}_f in the directions along and orthogonal to the trapped set K , which is

$$K = \{(x, \xi) \in \mathbb{R}^{2d} \mid \zeta = 0\}$$

in the simple setting we are considering. (See Proposition 2.13.)

Let us write the change of variable given in Proposition 2.13 as

$$\Phi : \left(\underbrace{q, p}_x, \underbrace{\xi_q, \xi_p}_\xi \right) \in \mathbb{R}^{2d} \oplus \mathbb{R}^{2d} \rightarrow \left(\underbrace{\nu_q, \nu_p}_\nu, \underbrace{\zeta_p, \zeta_q}_\zeta \right) \in \mathbb{R}^{2d} \oplus \mathbb{R}^{2d}. \quad (4.9)$$

It transfers the standard symplectic form $\Omega_0 = dx \wedge d\xi$ on $\mathbb{R}^{2d} \oplus \mathbb{R}^{2d}$ to

$$(D\Phi^*)^{-1}(\Omega_0) = d\nu_q \wedge d\nu_p + d\zeta_p \wedge d\zeta_q$$

and the metric $g_0 = \frac{1}{2} dx^2 + 2d\xi^2$ on $\mathbb{R}^{2d} \oplus \mathbb{R}^{2d}$ (g_0 is the metric induced by g on $T^*\mathbb{R}^{2d}$ as explained in Section 2.4.1) to the standard Euclidean metric on $\mathbb{R}^{2d} \oplus \mathbb{R}^{2d}$:

$$(D\Phi^*)^{-1}(g_0) = d\nu^2 + d\zeta^2.$$

The unitary operator associated to the coordinate change Φ is defined as

$$\Phi^* : L^2(\mathbb{R}_\nu^{2d} \oplus \mathbb{R}_\zeta^{2d}) \rightarrow L^2(\mathbb{R}_x^{2d} \oplus \mathbb{R}_\xi^{2d}), \quad (\Phi^* u) := u \circ \Phi.$$

Here (and henceforth) we make the convention that the subscript in the notation such as \mathbb{R}_ν^{2d} indicates the name of the coordinates on the space.

We define the operators

$$\mathcal{B}_{\nu_q} : L^2\left(\mathbb{R}_{\nu_q}^d\right) \rightarrow L^2\left(\mathbb{R}_{(\nu_q, \nu_p)}^{2d}\right) \quad \text{and} \quad \mathcal{B}_{\nu_q}^* : L^2\left(\mathbb{R}_{(\nu_q, \nu_p)}^{2d}\right) \rightarrow L^2\left(\mathbb{R}_{\nu_q}^d\right)$$

as the Bargmann transform \mathcal{B}_h and its adjoint \mathcal{B}_h^* in Subsection 3.1 for the case $D = d$. We define

$$\mathcal{B}_{\zeta_p} : L^2\left(\mathbb{R}_{\zeta_p}^d\right) \rightarrow L^2\left(\mathbb{R}_{(\zeta_p, \zeta_q)}^{2d}\right) \quad \text{and} \quad \mathcal{B}_{\zeta_p}^* : L^2\left(\mathbb{R}_{(\zeta_p, \zeta_q)}^{2d}\right) \rightarrow L^2\left(\mathbb{R}_{\zeta_p}^d\right)$$

similarly. Suppose that \mathcal{P}_{ν_q} and \mathcal{P}_{ζ_p} are defined correspondingly: That is to say, with setting $D = d$, we define

$$\mathcal{B}_{\nu_q} = \mathcal{B}_{\zeta_p} = \mathcal{B}_h, \quad \mathcal{B}_{\nu_q}^* = \mathcal{B}_{\zeta_p}^* = \mathcal{B}_h^* \quad \text{and} \quad \mathcal{P}_{\nu_q} = \mathcal{P}_{\zeta_p} = \mathcal{P}_h.$$

Next we define the operators

$$\mathcal{B}_x : L^2\left(\mathbb{R}_x^{2d}\right) \rightarrow L^2\left(\mathbb{R}_{(x, \xi)}^{4d}\right) \quad \text{and} \quad \mathcal{B}_x^* : L^2\left(\mathbb{R}_{(x, \xi)}^{4d}\right) \rightarrow L^2\left(\mathbb{R}_x^{2d}\right)$$

by

$$\mathcal{B}_x := \tilde{\sigma}^{-1} \circ \mathcal{B}_h \circ \sigma \quad \text{and} \quad \mathcal{B}_x^* := \sigma^{-1} \circ \mathcal{B}_h^* \circ \tilde{\sigma} \quad (4.10)$$

where \mathcal{B}_h and \mathcal{B}_h^* are now those in the case $D = 2d$, and

$$\sigma : L^2(\mathbb{R}_x^{2d}) \rightarrow L^2(\mathbb{R}_x^{2d}) \quad \text{and} \quad \tilde{\sigma} : L^2(\mathbb{R}_{(x, \xi)}^{4d}) \rightarrow L^2(\mathbb{R}_{(x, \xi)}^{4d})$$

are simple unitary operators defined by

$$\sigma u(x) = 2^{-d} u(2^{-1/2}x) \quad \text{and} \quad \tilde{\sigma} v(x, \xi) = v(2^{-1/2}x, 2^{1/2}\xi).$$

Correspondingly we set

$$\mathcal{P}_x = \mathcal{B}_x \circ \mathcal{B}_x^* = \tilde{\sigma}^{-1} \circ \mathcal{P}_h \circ \tilde{\sigma}. \quad (4.11)$$

Remark 4.4. (1) In terms of the (generalized) Bargmann transforms considered in Subsection 3.1.3, the operators \mathcal{B}_x and \mathcal{B}_x^* are the Bargmann transform and its adjoint for the combination of the Euclidean space $E = \mathbb{R}^{4d}$, the standard symplectic form $\Omega_0 = dx \wedge d\xi$, the metric $g_0 = \frac{1}{2}dx^2 + 2d\xi^2$ and the Lagrangian subspace \mathbb{R}_x^{2d} . Since the metric g_0 corresponds to the standard Euclidean metric through Φ , the definition of the operators \mathcal{B}_x , \mathcal{B}_x^* and \mathcal{P}_x above is more convenient and natural for our argument than those without σ and $\tilde{\sigma}$. Indeed, from Proposition 3.6, we have

$$\mathcal{P}_x^* \circ \Phi^* = \Phi^* \circ (\mathcal{P}_{\nu_q} \otimes \mathcal{P}_{\zeta_p})^*. \quad (4.12)$$

- (2) In the notation introduced above, the subscripts indicate the related coordinates. Though this may deviate from the standard usage, it is convenient for our argument. Notice that the operators introduced above, such as \mathcal{B}_x , depend on the parameter \hbar (and hence on N).

Lemma 4.5. *Let \mathcal{L}_f be the associated prequantum transfer operator (4.7) for a symplectic affine map in (4.6). Then the following diagram commutes:*

$$\begin{array}{ccc} L^2(\mathbb{R}_x^{2d}) & \xrightarrow{\mathcal{L}_f} & L^2(\mathbb{R}_x^{2d}) \\ \uparrow \mathcal{U} & & \uparrow \mathcal{U} \\ L^2(\mathbb{R}_{\nu_q}^d) \otimes L^2(\mathbb{R}_{\zeta_p}^d) & \xrightarrow{M_\nu(f) \otimes M_\zeta(B)} & L^2(\mathbb{R}_{\nu_q}^d) \otimes L^2(\mathbb{R}_{\zeta_p}^d) \end{array}$$

where \mathcal{U} , $M_\nu(f)$ and $M_\zeta(B)$ are the unitary operators defined respectively by

$$\mathcal{U} : L^2(\mathbb{R}_{\nu_q}^d) \otimes L^2(\mathbb{R}_{\zeta_p}^d) \rightarrow L^2(\mathbb{R}_x^{2d}), \quad \mathcal{U} = \mathcal{B}_x^* \circ \Phi^* \circ (\mathcal{B}_{\nu_q} \otimes \mathcal{B}_{\zeta_p}), \quad (4.13)$$

$$M_\nu(f) : L(\mathbb{R}_{\nu_q}^d) \rightarrow L(\mathbb{R}_{\nu_q}^d), \quad M_\nu(f) = \sqrt{d(B)} \cdot \mathcal{B}_{\nu_q}^* \circ (e^{\frac{i}{2\hbar}\omega(\nu,b)} \cdot L_f) \circ \mathcal{B}_{\nu_q},$$

$$M_\zeta(B) : L^2(\mathbb{R}_{\zeta_p}^d) \rightarrow L^2(\mathbb{R}_{\zeta_p}^d), \quad M_\zeta(B) = \sqrt{d(B)} \cdot \mathcal{B}_{\zeta_p}^* \circ L_B \circ \mathcal{B}_{\zeta_p}$$

with $d(B) = \det((B + {}^t B^{-1})/2)^{1/2}$, $e^{\frac{i}{2\hbar}\omega(\nu,b)}(L_f u)(\nu) = e^{\frac{i}{2\hbar}\omega(\nu,b)}(u \circ f^{-1})(\nu)$ and $L_B u = u \circ B^{-1}$ as before. Equivalently, in terms of lifted operators, we have the following commuting diagram:

$$\begin{array}{ccc} L^2(\mathbb{R}_x^{2d} \oplus \mathbb{R}_\xi^{2d}) & \xrightarrow{\mathcal{L}_f^{\text{lift}}} & L^2(\mathbb{R}_x^{2d} \oplus \mathbb{R}_\xi^{2d}) \\ \uparrow \Phi^* & & \uparrow \Phi^* \\ L^2(\mathbb{R}_\nu^{2d}) \otimes L^2(\mathbb{R}_\zeta^{2d}) & \xrightarrow{M_\nu^{\text{lift}}(f) \otimes M_\zeta^{\text{lift}}(B)} & L^2(\mathbb{R}_\nu^{2d}) \otimes L^2(\mathbb{R}_\zeta^{2d}) \end{array} \quad (4.14)$$

Proof. Recall the operators $\flat : \mathbb{R}^{2d} \rightarrow (\mathbb{R}^{2d})^* = \mathbb{R}^{2d}$ and $\sharp : (\mathbb{R}^{2d})^* = \mathbb{R}^{2d} \rightarrow \mathbb{R}^{2d}$ introduced in Remark 2.15. The expression (4.7) shows that \mathcal{L}_f can be written $\mathcal{L}_f = T_{(b, -\frac{1}{2}b^\flat)} \circ L_B$ where the unitary transfer operator $T_{(b, -\frac{1}{2}b^\flat)}$ is defined in (3.18). We apply Lemma 3.11, Lemma 3.9 to obtain

$$\begin{aligned} \mathcal{L}_f &= \mathcal{B}_x^* \circ \mathcal{T}_{(b, -\frac{1}{2}b^\flat)} \circ \mathcal{B}_x \circ (d(B) \cdot \mathcal{B}_x^*) \circ L_{B \oplus {}^t B^{-1}} \circ \mathcal{B}_x \\ &= d(B) \cdot \mathcal{B}_x^* \circ \mathcal{T}_{(b, -\frac{1}{2}b^\flat)} \circ \mathcal{B}_x \circ \mathcal{B}_x^* \circ L_{B \oplus {}^t B^{-1}} \circ \mathcal{B}_x \\ &= d(B) \cdot \mathcal{B}_x^* \circ (e^{\frac{i}{2\hbar}\varphi} \cdot L_F) \circ \mathcal{B}_x \end{aligned} \quad (4.15)$$

with $F(x, \xi) = (Bx + b, {}^t B^{-1}\xi - \frac{1}{2}b^\flat)$ and $\varphi(x, \xi) = -\frac{1}{2}b^\flat \cdot x - b \cdot \xi$. Since we have ${}^t B^{-1} = \flat \circ B \circ \flat^{-1}$ for B symplectic, we get the following expression of F in the new coordinates (ν, ζ) :

$$(\Phi \circ F \circ \Phi^{-1})(\nu, \zeta) = (B\nu + b, {}^t B^{-1}\zeta) = (f(\nu), {}^t Df^{-1}\zeta).$$

This implies

$$L_F = \Phi^* \circ (L_f \otimes L_B) \circ (\Phi^*)^{-1}.$$

From (4.12), we have

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$$\mathcal{B}_x^* \circ \Phi^* = \mathcal{B}_x^* \circ \mathcal{P}_x \circ \Phi^* = \mathcal{B}_x^* \circ \Phi^* \circ (\mathcal{P}_{\nu_q} \otimes \mathcal{P}_{\zeta_p}) = \mathcal{U} \circ (\mathcal{B}_{\nu_q} \otimes \mathcal{B}_{\zeta_p})^*$$

and

$$(\Phi^*)^{-1} \circ \mathcal{B}_x = (\Phi^*)^{-1} \circ \mathcal{P}_x \circ \mathcal{B}_x = (\mathcal{P}_{\nu_q} \otimes \mathcal{P}_{\zeta_p}) \circ (\Phi^*)^{-1} \circ \mathcal{B}_x = (\mathcal{B}_{\nu_q} \otimes \mathcal{B}_{\zeta_p}) \circ \mathcal{U}^{-1}.$$

Using these relations to continue (4.15) and noting that $\varphi = \omega(\nu, b)$, we conclude

$$\begin{aligned} \mathcal{L}_f &= d(B) \cdot \mathcal{B}_x^* \circ e^{\frac{i}{2\hbar}\varphi} L_F \circ \mathcal{B}_x \\ &= d(B) \cdot \mathcal{B}_x^* \circ \Phi^* \circ \left(e^{\frac{i}{2\hbar}\varphi} L_f \otimes L_B \right) \circ (\Phi^*)^{-1} \circ \mathcal{B}_x \\ &= d(B) \cdot \mathcal{U} \circ \left(\left(\mathcal{B}_{\nu_q}^* \circ (e^{\frac{i}{2\hbar}\varphi} \cdot L_f) \circ \mathcal{B}_{\nu_q} \right) \otimes \left(\mathcal{B}_{\zeta_p}^* \circ L_B \circ \mathcal{B}_{\zeta_p}^* \right) \right) \circ \mathcal{U}^{-1} \\ &= \mathcal{U} \circ (M_\nu(f) \otimes M_\zeta(B)) \circ \mathcal{U}^{-1}. \end{aligned}$$

□

Remarks:

- In the expression (??) for \mathcal{L}_f , it is remarkable that we get a product of two operators $M_\nu(f) \otimes M_\zeta(B)$ (this is due of course to the fact that Df is constant). Each of these operators is usually called a metaplectic operator, we refer to [19]. Here we have used a derivation in phase using the Bargmann transform.

4.3 The prequantum transfer operator for a linear hyperbolic map

In this subsection, we restrict the argument in the last subsection to the case where f in (4.6) is hyperbolic in the sense that f is expanding in $\mathbb{R}^d \oplus \{0\}$ while contracting in $\{0\} \oplus \mathbb{R}^d$. Since f preserves the symplectic form ω , we may express it as (see last remark in the proof of Proposition 2.16)

$$f(q, p) = B(q, p) = (Aq, {}^t A^{-1}p) \quad \text{where } B = \begin{pmatrix} A & 0 \\ 0 & {}^t A^{-1} \end{pmatrix} \quad (4.16)$$

with $A : \mathbb{R}^d \rightarrow \mathbb{R}^d$ an expanding linear map satisfying $\|A^{-1}\| \leq 1/\lambda$ for some $\lambda > 1$. Notice that, since $b = 0$, the action \mathcal{A} vanishes in (4.7) and the prequantum transfer operator gets the simpler expression:

$$(\mathcal{L}_f u)(x) = u(B^{-1}x) = L_B u(x). \quad (4.17)$$

The next proposition is a special case of Proposition 4.5.

Proposition 4.6. *The following diagram commutes:*

$$\begin{array}{ccc}
L^2(\mathbb{R}_x^{2d}) & \xrightarrow{\mathcal{L}_f} & L^2(\mathbb{R}_x^{2d}) \\
\uparrow \mathcal{U} & & \uparrow \mathcal{U} \\
L^2(\mathbb{R}_{\nu_q}^d) \otimes L^2(\mathbb{R}_{\zeta_p}^d) & \xrightarrow{L_A \otimes L_A} & L^2(\mathbb{R}_{\nu_q}^d) \otimes L^2(\mathbb{R}_{\zeta_p}^d)
\end{array} \tag{4.18}$$

with the unitary operator \mathcal{U} defined in (4.13). Equivalently using lifted operators, expressed in Lemma 3.9, we have the following commuting diagram:

$$\begin{array}{ccc}
L^2(\mathbb{R}_x^{2d} \oplus \mathbb{R}_\xi^{2d}) & \xrightarrow{\mathcal{L}_f^{\text{lift}}} & L^2(\mathbb{R}_x^{2d} \oplus \mathbb{R}_\xi^{2d}) \\
\uparrow \Phi^* & & \uparrow \Phi^* \\
L^2(\mathbb{R}_\nu^{2d}) \otimes L^2(\mathbb{R}_\zeta^{2d}) & \xrightarrow{L_A^{\text{lift}} \otimes L_A^{\text{lift}}} & L^2(\mathbb{R}_\nu^{2d}) \otimes L^2(\mathbb{R}_\zeta^{2d})
\end{array} \tag{4.19}$$

Proof. We have

$$d(B) = \det((B + {}^t B^{-1})/2)^{1/2} = (d(A))^2$$

and $L_f = L_B = L_{A \oplus {}^t A^{-1}}$. Hence, by the expression (3.14), we get

$$M_\nu(f) := \sqrt{d(B)} \cdot \mathcal{B}_{\nu_q}^* \circ (e^{\frac{i}{2\hbar}\omega(\nu,b)} \cdot L_f) \circ \mathcal{B}_{\nu_q} = d(A) \cdot \mathcal{B}_{\nu_q}^* \circ L_{A \oplus {}^t A^{-1}} \circ \mathcal{B}_{\nu_q} = L_A$$

and

$$M_\zeta(B) := \sqrt{d(B)} \cdot \mathcal{B}_{\zeta_p}^* \circ L_B \circ \mathcal{B}_{\zeta_p} = d(A) \cdot \mathcal{B}_{\zeta_p}^* \circ L_{A \oplus {}^t A^{-1}} \circ \mathcal{B}_{\zeta_p} = L_A.$$

Putting these in Proposition 4.5, we obtain the conclusion. \square

4.4 Anisotropic Sobolev space

In order to observe discrete spectrum of resonances, we have to consider the action of the prequantum transfer operator \mathcal{L}_f on an appropriate spaces of functions. As we explained in Section 2.5, we define such space of function, called anisotropic Sobolev space, by changing the norm in the directions transverse to the trapped set K (that is, in the directions of the variables ζ). Below is the precise definition.

Definition 4.7. We define the *escape function* or *weight function*

$$\mathcal{W}_h^r : \mathbb{R}_x^{2d} \oplus \mathbb{R}_\xi^{2d} \rightarrow \mathbb{R}_+ \quad \text{and} \quad \mathcal{W}_h^{r,\pm} : \mathbb{R}_x^{2d} \oplus \mathbb{R}_\xi^{2d} \rightarrow \mathbb{R}_+$$

by

$$\mathcal{W}_h^r(x, \xi) := W_h^r(\zeta_p, \zeta_q) \quad \text{and} \quad \mathcal{W}_h^{r,\pm}(x, \xi) := W_h^r(\zeta_p, \zeta_q) \tag{4.20}$$

where the functions W_h^r and $W_h^{r,\pm}$ are defined in Definition 3.13, and (ζ_p, ζ_q) is part of the coordinates introduced in (4.9). The *anisotropic Sobolev space* $\mathcal{H}_h^r(\mathbb{R}^{2d})$ is the Hilbert space obtained as the completion of the Schwartz space $\mathcal{S}(\mathbb{R}^{2d})$ with respect to the norm

$$\|u\|_{\mathcal{H}_h^r} := \|\mathcal{W}_h^r \cdot \mathcal{B}_x u\|_{L^2}$$

where \mathcal{B}_x is the operator defined in (4.10). Similarly, let $\mathcal{H}_h^{r,\pm}(\mathbb{R}^{2d})$ be the Hilbert space defined in the parallel manner by replacing $\mathcal{W}_h^r(\cdot)$ by $\mathcal{W}_h^{r,\pm}(\cdot)$.

By definition, the operator \mathcal{B}_x extends to an isometric embedding

$$\mathcal{B}_x : \mathcal{H}_h^r(\mathbb{R}^{2d}) \rightarrow L^2(\mathbb{R}_x^{2d} \oplus \mathbb{R}_\xi^{2d}, (\mathcal{W}_h^r)^2).$$

Since the weight function $\mathcal{W}^r(\cdot)$ can be expressed as

$$\mathcal{W}_h^r = (1 \otimes W_h^r) \circ \Phi,$$

where Φ is given in (4.9), we see that the following diagram commutes:

$$\begin{array}{ccc} \mathcal{H}_h^r(\mathbb{R}_x^{2d}) & \xrightarrow{\mathcal{L}_f} & \mathcal{H}_h^r(\mathbb{R}_x^{2d}) \\ \downarrow \mathcal{B}_x & & \downarrow \mathcal{B}_x \\ L^2(\mathbb{R}_x^{2d} \oplus \mathbb{R}_\xi^{2d}, (\mathcal{W}_h^r)^2) & \xrightarrow{\mathcal{L}^{\text{lift}}} & L^2(\mathbb{R}_x^{2d} \oplus \mathbb{R}_\xi^{2d}, (\mathcal{W}_h^r)^2) \\ \uparrow \Phi^* & & \uparrow \Phi^* \\ L^2(\mathbb{R}_\nu^{2d}) \otimes L^2(\mathbb{R}_\zeta^{2d}, (W_h^r)^2) & \xrightarrow{L_A^{\text{lift}} \otimes L_A^{\text{lift}}} & L^2(\mathbb{R}_\nu^{2d}) \otimes L^2(\mathbb{R}_\zeta^{2d}, (W_h^r)^2) \\ \uparrow \mathcal{B}_{\nu_q} \otimes \mathcal{B}_{\zeta_p} & & \uparrow \mathcal{B}_{\nu_q} \otimes \mathcal{B}_{\zeta_p} \\ L^2(\mathbb{R}_{\nu_q}^{2d}) \otimes H_h^r(\mathbb{R}_{\zeta_p}^d) & \xrightarrow{L_A \otimes L_A} & L^2(\mathbb{R}_{\nu_q}^{2d}) \otimes H_h^r(\mathbb{R}_{\zeta_p}^d) \end{array} \quad (4.21)$$

where Φ^* is an isomorphism between Hilbert spaces, and $H_h^r(\mathbb{R}_{\zeta_p}^d)$ in the last line is defined in Definition 3.19. Hence, skipping the lines in the middle, we get:

$$\begin{array}{ccc} \mathcal{H}_h^r(\mathbb{R}_x^{2d}) & \xrightarrow{\mathcal{L}_f} & \mathcal{H}_h^r(\mathbb{R}_x^{2d}) \\ \uparrow \mathcal{U} & & \uparrow \mathcal{U} \\ L^2(\mathbb{R}_{\nu_q}^{2d}) \otimes H_h^r(\mathbb{R}_{\zeta_p}^d) & \xrightarrow{L_A \otimes L_A} & L^2(\mathbb{R}_{\nu_q}^{2d}) \otimes H_h^r(\mathbb{R}_{\zeta_p}^d) \end{array} \quad (4.22)$$

with \mathcal{U} the unitary operator defined in (4.13). For the operator $L_A \otimes L_A$ on the bottom line, we know that the operator $L_A : L^2(\mathbb{R}_{\nu_q}^{2d}) \rightarrow L^2(\mathbb{R}_{\nu_q}^{2d})$ is unitary and Proposition 3.20 gives a description on the spectral structure of the operator $L_A : H_h^r(\mathbb{R}_{\zeta_p}^d) \rightarrow H_h^r(\mathbb{R}_{\zeta_p}^d)$.

Therefore we obtain the next proposition as a consequence. We fix some integer $n \geq 0$ and assume

$$r > n + 2 + \frac{d}{2}. \quad (4.23)$$

This assumption on r corresponds to (3.38) in the last section.

Definition 4.8. For $0 \leq k \leq n$, we consider the projection operators

$$t_h^{(k)} := \mathcal{U} \circ (\text{Id} \otimes T^{(k)}) \circ \mathcal{U}^{-1} : \mathcal{H}_h^r(\mathbb{R}^{2d}) \rightarrow \mathcal{H}_h^r(\mathbb{R}^{2d}) \quad (4.24)$$

and

$$\tilde{t}_h := \text{Id} - \sum_{k=0}^n t_h^{(k)} = \mathcal{U} \circ (\text{Id} \otimes \tilde{T}) \circ \mathcal{U}^{-1} : \mathcal{H}_h^r(\mathbb{R}^{2d}) \rightarrow \mathcal{H}_h^r(\mathbb{R}^{2d}) \quad (4.25)$$

where $T^{(k)}$ and \tilde{T} are the projection operators introduced in (3.34) and (3.46) respectively.

Proposition 4.9. *The operators $t_h^{(k)}$, $0 \leq k \leq n$, and \tilde{t}_h form a complete set of mutually commutative projection operators on $\mathcal{H}_h^r(\mathbb{R}^{2d})$. These operators also commute with the prequantum transfer operator \mathcal{L}_f defined in (4.17). Consequently the space $\mathcal{H}_h^r(\mathbb{R}^{2d})$ has a decomposition invariant under the action of \mathcal{L}_f :*

$$\mathcal{H}_h^r(\mathbb{R}^{2d}) = H'_0 \oplus H'_1 \oplus \cdots \oplus H'_n \oplus \tilde{H}' \quad \text{where } H'_k = \text{Im } t_h^{(k)} \text{ and } \tilde{H}' = \text{Im } \tilde{t}_h$$

For this decomposition, we have

- (1) For every $0 \leq k \leq n$, the operator $\mathcal{L}_f : H'_k \rightarrow H'_k$ is conjugated by the isometric bijection \mathcal{U} to the tensor product of the unitary operator $L_A : L^2(\mathbb{R}_{\nu_q}^2) \rightarrow L^2(\mathbb{R}_{\nu_q}^2)$ and the finite rank operator $L_A : \text{Polynom}^{(k)} \rightarrow \text{Polynom}^{(k)}$. In particular, we have for every $u \in H'_k$,

$$C_0^{-1} \|A\|^{-k} \cdot |\det(A)|^{-1/2} \cdot \|u\|_{\mathcal{H}_h^r} \leq \|\mathcal{L}_f u\|_{\mathcal{H}_h^r} \leq C_0 \|A^{-1}\|^k \cdot |\det(A)|^{-1/2} \cdot \|u\|_{\mathcal{H}_h^r}.$$

- (2) The operator norm of $\mathcal{L}_f : \tilde{H}' \rightarrow \tilde{H}'$ is bounded by

$$C_0 \cdot \max\{\|A^{-1}\|^{n+1} \cdot |\det A|^{-1/2}, \|A^{-1}\|^r \cdot |\det A|^{1/2}\}.$$

The constant C_0 is independent of A and \hbar .

4.5 A few technical lemmas

In this subsection, we collect a few miscellaneous technical lemmas related to the anisotropic Sobolev spaces $\mathcal{H}_h^r(\mathbb{R}^{2d})$ and $\mathcal{H}_h^{r,\pm}(\mathbb{R}^{2d})$, which we will use in the later sections. The following are immediate consequences of Lemma 3.16 and 3.17 respectively.

Lemma 4.10. *Suppose that f is a hyperbolic linear transformation (4.16) defined for an expanding linear map A satisfying (3.31) for some large λ (say $\lambda > 9$). Then the operator \mathcal{L}_f extends to a bounded operator $\mathcal{L}_f : \mathcal{H}_\hbar^{r,-}(\mathbb{R}^{2d}) \rightarrow \mathcal{H}_\hbar^{r,+}(\mathbb{R}^{2d})$ and the operator norm is bounded by a constant independent of $\hbar > 0$ and f .*

Corollary 4.11. *The operator $t_\hbar^{(k)}$ for $0 \leq k \leq n$, defined in (4.24), extends to a bounded operator*

$$t_\hbar^{(k)} : \mathcal{H}_\hbar^{r,-}(\mathbb{R}^{2d}) \rightarrow \mathcal{H}_\hbar^{r,+}(\mathbb{R}^{2d})$$

whose operator norm is bounded by a constant independent of \hbar .

For convenience in the later argument, let us put the following definition:

Definition 4.12. Let \mathcal{A} be the group of affine transformation on \mathbb{R}^{2d} that preserves the symplectic form ω , the Euclidean norm and the splitting $\mathbb{R}^{2d} = \mathbb{R}^d \oplus \mathbb{R}^d$ simultaneously.

Note that the function \mathcal{W}_\hbar^r is invariant with respect to the transformation $(a, {}^t D a^{-1})$ on $T^*\mathbb{R}^{2d} = \mathbb{R}^{2d} \times \mathbb{R}^{2d}$ when $a \in \mathcal{A}$. This fact, together with Lemma 3.9, yields

Lemma 4.13. *If $a \in \mathcal{A}$, then the prequantum operator \mathcal{L}_a , defined in (4.7) extends to an isometry on $\mathcal{H}_\hbar^r(\mathbb{R}^{2d})$.*

The norm $\|\cdot\|_{\mathcal{H}_\hbar^r}$ on the Hilbert space $\mathcal{H}_\hbar^r(\mathbb{R}^{2d})$ is induced by a (unique) inner product $(\cdot, \cdot)_{\mathcal{H}_\hbar^r(\mathbb{R}^{2d})}$. Notice that even if two functions u and v in $\mathcal{H}_\hbar^r(\mathbb{R}^{2d})$ have mutually disjoint supports, the inner product $(u, v)_{\mathcal{H}_\hbar^r(\mathbb{R}^{2d})}$ may not vanish. This is somewhat inconvenient. But we have the following “pseudo-local” property. We omit the proof because it can be given by a straightforward estimate.

Lemma 4.14. *Let $\epsilon > 0$. If $d(\text{supp } u, \text{supp } v) \geq \hbar^{(1-\epsilon)/2}$ for $u, v \in \mathcal{H}_\hbar^r(\mathbb{R}^{2d})$, we have*

$$|(u, v)_{\mathcal{H}_\hbar^r(\mathbb{R}^{2d})}| \leq C_{\nu, \epsilon} \cdot \hbar^\nu \cdot \|u\|_{\mathcal{H}_\hbar^r(\mathbb{R}^{2d})} \|v\|_{\mathcal{H}_\hbar^r(\mathbb{R}^{2d})} \quad \text{for } u, v \in \mathcal{H}_\hbar^r(\mathbb{R}^{2d})$$

for arbitrarily large ν , with $C_{\nu, \epsilon} > 0$ a constant depending on ϵ and ν .

From the definition of the function $\mathcal{W}_\hbar^r(\cdot)$ and (3.29), we have

$$\mathcal{W}_\hbar^r(x, \xi) \leq C \cdot \mathcal{W}_\hbar^r(y, \eta) \cdot \langle \hbar^{-1/2} |(x, \xi) - (y, \eta)| \rangle^{2r}. \quad (4.26)$$

The next lemma and corollary are direct consequences of this estimate. The proof is completely parallel to that of Lemma 3.14.

Lemma 4.15. *If $R_\hbar : \mathcal{S}(\mathbb{R}^{2d} \oplus \mathbb{R}^{2d}) \rightarrow \mathcal{S}(\mathbb{R}^{2d} \oplus \mathbb{R}^{2d})$ is an integral operator of the form*

$$(R_\hbar u)(x, \xi) = \int K_\hbar(x, \xi; x', \xi') u(x', \xi') dx' d\xi'$$

depending on \hbar and if the kernel $K_\hbar(\cdot; \cdot)$ is a continuous function satisfying

$$|K_\hbar(x, \xi; x', \xi')| \leq \langle \hbar^{-1/2} \cdot |(x, \xi) - (x', \xi')| \rangle^{-\nu}$$

for some $\nu > 2r + 4d$, then the operator R_{\hbar} extends uniquely to a bounded operator on $L^2(\mathbb{R}^{2d} \oplus \mathbb{R}^{2d}, (\mathcal{W}_{\hbar}^r)^2)$ and

$$\|R_{\hbar}\|_{L^2(\mathbb{R}^{2d} \oplus \mathbb{R}^{2d}, (\mathcal{W}_{\hbar}^r)^2)} \leq C_{\nu}$$

where C_{ν} is a constant independent of \hbar . The same holds true with \mathcal{W}_{\hbar}^r replaced by $\mathcal{W}_{\hbar}^{r,\pm}$ simultaneously.

Corollary 4.16. *The Bargmann projector \mathcal{P}_{\hbar} extends uniquely to a bounded operator on $L^2(\mathbb{R}^{2d} \oplus \mathbb{R}^{2d}, (\mathcal{W}_{\hbar}^r)^2)$ and its operator norm is bounded by a constant that does not depend on $\hbar > 0$. The same holds true with \mathcal{W}_{\hbar}^r replaced by $\mathcal{W}_{\hbar}^{r,\pm}$ simultaneously.*

4.6 The rough Laplacian on \mathbb{R}^{2d}

In this subsection, we consider the spectrum of the Euclidean rough Laplacian on \mathbb{R}^{2d} defined in Subsection 4.1. This operator Δ_{\hbar} is a closed self-adjoint operator on $L^2(\mathbb{R}^{2d})$ and its domain of definition is

$$\mathcal{D}(\Delta_{\hbar}) = \{u \in L^2(\mathbb{R}^{2d}) \mid \|\Delta_{\hbar}u\|_{L^2} < \infty\}.$$

Note that $\mathcal{D}(\Delta_{\hbar})$ becomes a Hilbert space if we consider the norm

$$\|u\|_{\Delta_{\hbar}} = ((u, u)_{\Delta_{\hbar}})^{1/2} \quad (4.27)$$

induced by the inner product

$$(u, v)_{\Delta_{\hbar}} = (u, v)_{L^2} + (\Delta_{\hbar}u, \Delta_{\hbar}v)_{L^2}.$$

Obviously, Δ_{\hbar} gives a bounded operator

$$\Delta_{\hbar} : (\mathcal{D}(\Delta_{\hbar}), \|\cdot\|_{\Delta_{\hbar}}) \rightarrow L^2(\mathbb{R}^{2d}).$$

An important property of the operator Δ_{\hbar} that follows from the definition is that it is invariant with respect to the action of prequantum transfer operators for symplectic isometries:

Lemma 4.17. *Suppose that $f : \mathbb{R}^{2d} \rightarrow \mathbb{R}^{2d}$ is an isometric affine map preserving the symplectic form ω , then we have $\Delta_{\hbar} \circ \mathcal{L}_f = \mathcal{L}_f \circ \Delta_{\hbar}$ for the associated prequantum transfer operator \mathcal{L}_f given in (4.7).*

By considering a commutative diagram corresponding to (4.21), we obtain the following commutative diagram similar to (4.22):

$$\begin{array}{ccc} L^2(\mathbb{R}_x^{2d}) & \xrightarrow{\hbar\Delta_{\hbar}} & L^2(\mathbb{R}_x^{2d}) \\ \uparrow \mathcal{U} & & \uparrow \mathcal{U} \\ L^2(\mathbb{R}_{\nu_q}^2) \otimes L^2(\mathbb{R}_{\zeta_p}^d) & \xrightarrow{\text{Id} \otimes \mathcal{K}} & L^2(\mathbb{R}_{\nu_q}^2) \otimes L^2(\mathbb{R}_{\zeta_p}^d) \end{array}$$

where \mathcal{H} is the harmonic oscillator operator defined in (3.52) with setting $D = d$.

From the expression of the Euclidean rough Laplacian obtained in (2.40), we have

$$\hbar\Delta = \mathcal{U} \circ \left(\text{Id} \otimes \frac{1}{2\hbar} \left(\widehat{\zeta}_q^2 + \widehat{\zeta}_p^2 \right) \right) \circ \mathcal{U}^{-1}.$$

Thus we may invoke the argument in Subsection 3.5, especially Lemma 3.22 and 3.24, to derive the next proposition on the spectral structure of the rough Laplacian Δ_\hbar . For $k \geq 0$, let us consider the spectral projection operator

$$q_\hbar^{(k)} := \mathcal{U} \circ \left(\text{Id} \otimes Q_\hbar^{(k)} \right) \circ \mathcal{U}^{-1} \quad : L^2(\mathbb{R}_x^{2d}) \rightarrow L^2(\mathbb{R}_x^{2d}) \quad (4.28)$$

where $Q_\hbar^{(k)}$ is the projection operator on level k of the harmonic oscillator \mathcal{H} . Note that it restricts to a bounded operator

$$q_\hbar^{(k)} : L^2(\mathbb{R}^{2d}) \rightarrow (\mathcal{D}(\Delta_\hbar), \|\cdot\|_{\Delta_\hbar}) \subset L^2(\mathbb{R}^{2d}) \quad (4.29)$$

whose operator norm is bounded by a constant independent of \hbar .

Proposition 4.18. *The rough Laplacian $\Delta_\hbar = D^*D$ on the Euclidean space \mathbb{R}^{2d} is a closed self-adjoint operator on $L^2(\mathbb{R}^{2d})$ and its spectrum consists of integer eigenvalues $\frac{d}{2} + k$ with $k \in \mathbb{N} \cup \{0\}$. The spectral projector corresponding to the eigenvalue $\frac{d}{2} + k$ is the operator $q_\hbar^{(k)}$ given in (4.28) and together they form a complete set of mutually commuting orthogonal projections in $L^2(\mathbb{R}^{2d})$. Consequently we have*

$$L^2(\mathbb{R}^{2d}) = \overline{\bigoplus_{k=0}^{\infty} H_k''}, \quad \text{with } H_k'' := \text{Im } q_\hbar^{(k)}.$$

The next proposition is an immediate consequence of Lemma 3.24.

Proposition 4.19. *The operator $\bigoplus_{i=0}^k q_\hbar^{(i)}$ and $\bigoplus_{i=0}^k t_\hbar^{(i)}$ restricts to the bijections*

$$\bigoplus_{i=0}^k q_\hbar^{(i)} : \bigoplus_{i=0}^k \text{Im } t_\hbar^{(i)} \rightarrow \bigoplus_{i=0}^k \text{Im } q_\hbar^{(i)}$$

and

$$\bigoplus_{i=0}^k t_\hbar^{(i)} : \bigoplus_{i=0}^k \text{Im } q_\hbar^{(i)} \rightarrow \bigoplus_{i=0}^k \text{Im } t_\hbar^{(i)}$$

respectively. The operator norms of these operators and their inverses are bounded by some constant independent of \hbar .

5 Nonlinear prequantum maps on \mathbb{R}^{2d}

In this section, we prepare some basic estimates on the effect of non-linearity of the Anosov diffeomorphism f on the anisotropic Sobolev space $\mathcal{H}_\hbar^r(\mathbb{R}^{2d})$. Most of the results in this section may be rather obvious at least for those readers who are familiar with Fourier analysis. However we have to be attentive to the following particular situations in our argument:

- The escape function \mathcal{W}_\hbar^r in the definition of the anisotropic Sobolev space $\mathcal{H}_\hbar^r(\mathbb{R}^{2d})$ has variable growth order $m(\cdot)$ depending on the directions. This leads to the fact that the (prequantum) transfer operators associated to a non-linear map may be unbounded even if the map is very close to identity.
- The spectral projection operators $t_\hbar^{(k)}$ in Proposition 4.9 is also very anisotropic and rather singular. It is not well-defined (or bounded) on any usual (isotropic) Sobolev spaces of positive or negative order.
- The escape function \mathcal{W}_\hbar^r is not very smooth, viewed in the scale ($\sim \hbar^{1/2}$) of the smallest-possible wave packets in the phase space. In terms of the theory of pseudodifferential operators, this implies that the escape function \mathcal{W}_\hbar^r belongs only to the symbol class of “critical order”. We have to avoid carefully the difficulties caused by this fact.

For these reasons, we are going to give the argument to some detail. The main result in this section is Proposition 5.12, which concerns the third item above.

Recall that the stable and unstable subspaces $E_s(x)$ and $E_u(x)$ for the Anosov diffeomorphism f depend on the point $x \in M$ not smoothly but only Hölder continuously. We let $0 < \beta < 1$ be the Hölder exponent. (See Remark 1.2(1).) In what follows, we fix a small positive constant θ such that

$$0 < \theta < \beta/8. \quad (5.1)$$

The open ball of radius $c > 0$ on \mathbb{R}^{2d} is denoted by

$$\mathbb{D}(c) = \{x \in \mathbb{R}^{2d} \mid |x| < c\}.$$

5.1 Truncation operations in the real space

We first consider the operation of truncating functions in the (real) space by multiplying smooth functions with small supports. Below we consider the following setting:

Setting I: For each $\hbar > 0$, there is a given set \mathcal{X}_\hbar of C^∞ functions on \mathbb{R}^{2d} such that, for all $\psi \in \mathcal{X}_\hbar$ and $\hbar > 0$,

(C1) the support of ψ is contained in the disk $\mathbb{D}(C_*\hbar^{1/2-\theta})$ and

(C2) $|\partial_x^\alpha \psi(x)| < C_\alpha \hbar^{(\frac{1}{2}-\theta)|\alpha|}$ for each multi-index $\alpha \in \mathbb{N}^{2d}$,

where $C_* > 0$ and $C_\alpha > 0$ are constants independent of $\psi \in \mathcal{X}_\hbar$ and $\hbar > 0$.

In the next section, we will consider a few specific sets of functions as \mathcal{X}_\hbar and apply the argument in this section to them.

Remark 5.1. The condition above on \mathcal{X}_\hbar is equivalent to the condition that the normalized family

$$\widetilde{\mathcal{X}}_\hbar = \{ \varphi(x) = \psi(\hbar^{1/2-\theta}x) \in C^\infty(\mathbb{R}^{2d}) \mid \psi \in \mathcal{X}_\hbar \} \quad \text{for } \hbar > 0 \quad (5.2)$$

are uniformly bounded in the (uniform) C^∞ topology and supported in a fixed bounded subset of \mathbb{R}^{2d} .

Recall the transformations

$$\mathcal{B}_x : L^2(\mathbb{R}^{2d}) \rightarrow L^2(\mathbb{R}^{2d} \oplus \mathbb{R}^{2d}), \quad \mathcal{B}_x^* : L^2(\mathbb{R}^{2d} \oplus \mathbb{R}^{2d}) \rightarrow L^2(\mathbb{R}^{2d})$$

and

$$\mathcal{P}_x := \mathcal{B}_x^* \circ \mathcal{B}_x : L^2(\mathbb{R}^{2d} \oplus \mathbb{R}^{2d}) \rightarrow L^2(\mathbb{R}^{2d} \oplus \mathbb{R}^{2d}),$$

which are defined in (4.10) and (4.11) as slight modifications of the Bargmann transform \mathcal{B}_\hbar , its adjoint \mathcal{B}_\hbar^* and the Bargmann projector \mathcal{P}_\hbar in the case $D = 2d$. Notice that the operators \mathcal{B}_x , \mathcal{B}_x^* and \mathcal{P}_x depend on the parameter \hbar (and hence on N).

Below we write $\mathcal{M}(\varphi)$ for the multiplication operator by a function φ . Since $\hbar^{1/2-\theta} \gg \hbar^{1/2}$ for small \hbar , the functions in \mathcal{X}_\hbar are very smooth (or flat) viewed in the scale of the wave packet $\phi_{x,\xi}(\cdot)$ used in the Bargmann transform \mathcal{B}_\hbar . This observation naturally leads to the following few statements.

Lemma 5.2. *For each $\psi \in \mathcal{X}_\hbar$, let $\mathcal{M}^{\text{lift}}(\psi) = \mathcal{B}_x \circ \mathcal{M}(\psi) \circ \mathcal{B}_x^*$ be the lift of the multiplication operator $\mathcal{M}(\psi)$ with respect to the (modified) Bargmann transform \mathcal{B}_x . Then it is approximated by the multiplication by the function $\psi \circ \pi$. Precisely, there exists a constant $C > 0$ such that, for any $\hbar > 0$ and $\psi \in \mathcal{X}_\hbar$, we have*

$$\| \mathcal{M}^{\text{lift}}(\psi) - \mathcal{M}(\psi \circ \pi) \circ \mathcal{P}_x \|_{L^2(\mathbb{R}^{2d} \oplus \mathbb{R}^{2d}, (\mathcal{W}_\hbar^r)^2)} < C\hbar^\theta \quad (5.3)$$

and

$$\| \mathcal{M}^{\text{lift}}(\psi) - \mathcal{P}_x \circ \mathcal{M}(\psi \circ \pi) \|_{L^2(\mathbb{R}^{2d} \oplus \mathbb{R}^{2d}, (\mathcal{W}_\hbar^r)^2)} < C\hbar^\theta$$

Consequently we have

$$\| [\mathcal{P}_x, \mathcal{M}(\psi \circ \pi)] \|_{L^2(\mathbb{R}^{2d} \oplus \mathbb{R}^{2d}, (\mathcal{W}_\hbar^r)^2)} < C\hbar^\theta \quad (5.4)$$

where $[A, B]$ denotes the commutator of two operators: $[A, B] = A \circ B - B \circ A$. The same statement holds true with \mathcal{W}_\hbar^r replaced by $\mathcal{W}_\hbar^{r,\pm}$.

Proof. The kernel of the operator

$$\mathcal{M}^{\text{lift}}(\psi) - \mathcal{M}(\psi \circ \pi) \circ \mathcal{P}_x = \mathcal{B}_x \circ \mathcal{M}(\psi) \circ \mathcal{B}_x^* - \mathcal{M}(\psi \circ \pi) \circ \mathcal{B}_x \circ \mathcal{B}_x^*$$

is written

$$K(x, \xi; x', \xi') = (2\pi\hbar)^{-d} \int e^{(i/\hbar)(\xi(y-x) - \xi'(x'-y))} \cdot e^{-|y-x|^2/4\hbar - |y-x'|^2/4\hbar} (\psi(y) - \psi(x)) dy.$$

We apply integration by parts, using the differential operator

$$L = \frac{1 - i(\xi - \xi')\partial_y}{1 + \hbar^{-1}(\xi - \xi')^2},$$

which satisfies $L(e^{(i/\hbar)(\xi(y-x) - \xi'(x'-y))}) = e^{(i/\hbar)(\xi(y-x) - \xi'(x'-y))}$ for ν times. Then we get

$$K(x, \xi; x', \xi') = (2\pi\hbar)^d \int e^{(i/\hbar)(\xi(y-x) - \xi'(x'-y))} \cdot ({}^tL)^\nu \left(e^{-|y-x|^2/4\hbar - |y-x'|^2/4\hbar} (\psi(y) - \psi(x)) \right) dy$$

where ${}^tL = (1 - i(\xi - \xi')\partial_y)/(1 + \hbar^{-1}(\xi - \xi')^2)$ is the transpose of L . Using the conditions (C1) and (C2) on the family \mathcal{X}_\hbar and, in particular, the estimate

$$|\psi(x) - \psi(y)| \cdot \langle \hbar^{-1/2}|x - y| \rangle^{-1} < C\hbar^\theta \quad (5.5)$$

that follows from the condition (C2), we see that the integrand is bounded in absolute value by

$$C\hbar^\theta \cdot \langle \hbar^{-1/2}|\xi - \xi'| \rangle^{-\nu} \cdot \langle \hbar^{-1/2}|x - y| \rangle^{-\nu} \cdot \langle \hbar^{-1/2}|x' - y| \rangle^{-\nu}.$$

Hence, letting ν large, we obtain

$$|K(x, \xi; x', \xi')| \leq C\hbar^\theta \cdot \langle \hbar^{-1/2}|\xi - \xi'| \rangle^{-\nu} \cdot \langle \hbar^{-1/2}|x - x'| \rangle^{-\nu}.$$

This estimate for sufficiently large ν and Lemma 4.15 give the first inequality (5.3). We can get the second inequality in the same manner. \square

Corollary 5.3. *The multiplication operator $\mathcal{M}(\psi)$ by $\psi \in \mathcal{X}_\hbar$ extends to a bounded operator on $\mathcal{H}_\hbar^r(\mathbb{R}^{2d})$ and, for the operator norm, we have $\|\mathcal{M}(\psi)\|_{\mathcal{H}_\hbar^r(\mathbb{R}^{2d})} < \|\psi\|_\infty + C\hbar^\theta$ for all $\psi \in \mathcal{X}_\hbar$, with a constant $C > 0$ independent of \hbar .*

Proof. From the commutative diagram (3.11), the operator norm of $\mathcal{M}(\psi) : \mathcal{H}_\hbar^r(\mathbb{R}^{2d}) \rightarrow \mathcal{H}_\hbar^r(\mathbb{R}^{2d})$ coincides with that of the operator

$$\mathcal{M}^{\text{lift}}(\psi) : L^2(\mathbb{R}^{2d} \oplus \mathbb{R}^{2d}, (\mathcal{W}_\hbar^r)^2) \rightarrow L^2(\mathbb{R}^{2d} \oplus \mathbb{R}^{2d}, (\mathcal{W}_\hbar^r)^2)$$

restricted to the image of $\mathcal{B}_x : \mathcal{H}_\hbar^r(\mathbb{R}^{2d}) \rightarrow L^2(\mathbb{R}^{2d} \oplus \mathbb{R}^{2d}, (\mathcal{W}_\hbar^r)^2)$. Hence the claim follows from the last lemma. \square

Corollary 5.4. *For $u, v \in \mathcal{H}_\hbar^r(\mathbb{R}^{2d})$ and $\psi \in \mathcal{X}_\hbar$ we have*

$$(u, \psi \cdot v)_{\mathcal{H}_\hbar^r(\mathbb{R}^{2d})} = (\bar{\psi} \cdot u, v)_{\mathcal{H}_\hbar^r} + \mathcal{O}(\hbar^\theta) \cdot \|u\|_{\mathcal{H}_\hbar^r(\mathbb{R}^{2d})} \cdot \|v\|_{\mathcal{H}_\hbar^r(\mathbb{R}^{2d})}$$

where $\mathcal{O}(\hbar^\theta)$ denotes a term whose absolute value is bounded by $C\hbar^\theta$ with C a constant independent of \hbar .

Proof. This is a consequence of the equality

$$\begin{aligned} (u, \psi \cdot v)_{\mathcal{H}_\hbar^r(\mathbb{R}^{2d})} &= (\mathcal{B}_x u, \mathcal{M}^{\text{lift}}(\psi) \circ \mathcal{B}_x v)_{L^2(\mathbb{R}^{2d} \oplus \mathbb{R}^{2d}, (\mathcal{W}_\hbar^r)^2)} \\ &= (\mathcal{B}_x u, \mathcal{M}(\psi \circ \pi) \circ \mathcal{B}_x v)_{L^2(\mathbb{R}^{2d} \oplus \mathbb{R}^{2d}, (\mathcal{W}_\hbar^r)^2)} + \mathcal{O}(\hbar^\theta) \cdot \|u\|_{\mathcal{H}_\hbar^r(\mathbb{R}^{2d})} \cdot \|v\|_{\mathcal{H}_\hbar^r(\mathbb{R}^{2d})} \end{aligned}$$

and the parallel estimate for $(\bar{\psi} \cdot u, v)_{\mathcal{H}_\hbar^r(\mathbb{R}^{2d})}$, which follow from Lemma 5.2. \square

Remark 5.5. The statements of Corollary 5.3 and Corollary 5.4 above hold true with $\mathcal{H}_h^r(\mathbb{R}^{2d})$ replaced by $\mathcal{H}_h^{r,\pm}(\mathbb{R}^{2d})$ and the proofs are completely parallel. This is the case for a few statements (Lemma 5.8, Proposition 5.12, Lemma 5.16 and Corollary 5.17, precisely) in this section.

Next we recall the projection operators $t_h^{(k)}$ for $0 \leq k \leq n$ in (4.24) and \tilde{t}_h in (4.25). We henceforth assume

$$r > n + 2 + 4d \quad (5.6)$$

for the choice of r . (This is a little more restrictive than (4.23).)

Lemma 5.6. *There exists a constant $C > 0$ such that*

$$\left\| \left[\mathcal{M}(\psi), t_h^{(k)} \right] \right\|_{\mathcal{H}_h^{r,-}(\mathbb{R}^{2d}) \rightarrow \mathcal{H}_h^{r,+}(\mathbb{R}^{2d})} < C\hbar^\theta$$

for any $\hbar > 0$, $\psi \in \mathcal{X}_h$ and $0 \leq k \leq n$.

Proof. From (4.12) and the definition of the operator $t_h^{(k)}$, we have

$$\begin{aligned} \mathcal{M}(\psi) \circ t_h^{(k)} &= \mathcal{B}_x^* \circ \mathcal{M}^{\text{lift}}(\psi) \circ \mathcal{B}_x \circ \mathcal{U} \circ (\text{Id} \otimes T^{(k)}) \circ \mathcal{U}^{-1} \\ &= \mathcal{B}_x^* \circ \mathcal{M}^{\text{lift}}(\psi) \circ \Phi^* \circ (\mathcal{B}_{\nu_q} \otimes \mathcal{B}_{\zeta_p}) \circ (\text{Id} \otimes T^{(k)}) \circ (\mathcal{B}_{\nu_q}^* \otimes \mathcal{B}_{\zeta_p}^*) \circ (\Phi^*)^{-1} \circ \mathcal{B}_x \\ &= \mathcal{B}_x^* \circ \mathcal{M}^{\text{lift}}(\psi) \circ \Phi^* \circ (\mathcal{P}_{\nu_q} \otimes \mathcal{T}_h^{(k)}) \circ (\Phi^*)^{-1} \circ \mathcal{B}_x \end{aligned}$$

and, similarly

$$t_h^{(k)} \circ \mathcal{M}(\psi) = \mathcal{B}_x^* \circ \Phi^* \circ (\mathcal{P}_{\nu_q} \otimes \mathcal{T}_h^{(k)}) \circ (\Phi^*)^{-1} \circ \mathcal{M}^{\text{lift}}(\psi) \circ \mathcal{B}_x.$$

Thus, from 5.2, it is enough to show that

$$\left\| \left[\mathcal{M}(\psi \circ \pi), \Phi^* \circ (\mathcal{P}_{\nu_q} \otimes \mathcal{T}_h^{(k)}) \circ (\Phi^*)^{-1} \right] \right\|_{L^2(\mathbb{R}^{2d} \oplus \mathbb{R}^{2d}, (\mathcal{W}_h^{r,-})^2) \rightarrow L^2(\mathbb{R}^{2d} \oplus \mathbb{R}^{2d}, (\mathcal{W}_h^{r,+})^2)} < C\hbar^\theta. \quad (5.7)$$

From Proposition 3.3 and Lemma 3.17, if we write $K(x, \xi; x', \xi')$ for the kernel of the operator $\Phi^* \circ (\mathcal{P}_{\nu_q} \otimes \mathcal{T}_h^{(k)}) \circ (\Phi^*)^{-1}$, it satisfies

$$\frac{\mathcal{W}_h^{r,+}(x, \xi)}{\mathcal{W}_h^{r,-}(x', \xi')} |K(x, \xi; x', \xi')| \quad (5.8)$$

$$\leq C_\nu \langle \hbar^{-1/2} |\nu_q - \nu'_q| \rangle^{-\nu} \langle \hbar^{-1/2} |\nu_p - \nu'_p| \rangle^{-\nu} \langle \hbar^{-1/2} |(\zeta_p, \zeta_q)| \rangle^{-(r-k)} \langle \hbar^{-1/2} |(\zeta'_p, \zeta'_q)| \rangle^{-(r-k)} \quad (5.9)$$

$$\leq C'_\nu \langle \hbar^{-1/2} |(\nu_q, \nu_p) - (\nu'_q, \nu'_p)| \rangle^{-\nu} \langle \hbar^{-1/2} |(\zeta_p, \zeta_q) - (\zeta'_p, \zeta'_q)| \rangle^{-(r-k)} \quad (5.10)$$

for arbitrarily large $\nu > 0$, where $C_\nu, C'_\nu, C''_\nu > 0$ are constants independent of \hbar . The variables $\nu_q, \nu_p, \zeta_q, \zeta_p$ (resp. $\nu'_q, \nu'_p, \zeta'_q, \zeta'_p$) are the coordinates for (x, ξ) (resp. (x', ξ')) introduced in (4.9) and $|\cdot|$ denotes the Euclidean norms. The kernel $\tilde{K}(x, \xi; x', \xi')$ of the commutator in (5.7) is then

$$\tilde{K}(x, \xi; x', \xi') = (\psi(x) - \psi(x')) \cdot K(x, \xi; x', \xi').$$

By (5.10) with sufficiently large ν and (5.5), we get

$$\frac{\mathcal{W}_\hbar^{r,+}(x, \xi)}{\mathcal{W}_\hbar^{r,-}(x', \xi')} |\tilde{K}(x, \xi; x', \xi')| \leq C\hbar^\theta \cdot \langle \hbar^{-1/2} |(x, \xi) - (y, \eta)| \rangle^{-(r-k-1)}.$$

Hence we obtain the required estimate by Young inequality, noting that $r - k - 1 \geq r - n - 1 > 4d$ from the assumption (5.6). \square

Corollary 5.7. *There exists a constant $C > 0$, such that*

$$\left\| \left[\mathcal{M}(\psi), t_\hbar^{(k)} \right] \right\|_{\mathcal{H}_\hbar^r(\mathbb{R}^{2d})} < C\hbar^\theta \quad \text{for } 0 \leq k \leq n$$

and

$$\left\| \left[\mathcal{M}(\psi), \tilde{t}_\hbar \right] \right\|_{\mathcal{H}_\hbar^r(\mathbb{R}^{2d})} < C\hbar^\theta$$

for any $\psi \in \mathcal{X}_\hbar$, with $C > 0$ a constant independent of \hbar .

Proof. The former claim is an immediate consequence of the last lemma. Since $\tilde{t}_\hbar = \text{Id} - \sum_{k=0}^n t_\hbar^{(k)}$ by definition, the latter claim follows. \square

5.2 Truncation operations in the phase space

In order to truncate functions in the phase space $T^*\mathbb{R}^{2d} = \mathbb{R}^{2d} \oplus \mathbb{R}^{2d}$, we consider the smooth function

$$Y_\hbar : T^*\mathbb{R}^{2d} \rightarrow [0, 1], \quad Y_\hbar(x, \xi) = \chi(\hbar^{2\theta-1/2} |(x, \xi)|) \quad (5.11)$$

with $\chi : \mathbb{R} \rightarrow [0, 1]$ a C^∞ function satisfying (3.55), and then introduce the operator

$$\mathcal{Y}_\hbar : L^2(\mathbb{R}^{2d}) \rightarrow L^2(\mathbb{R}^{2d}), \quad \mathcal{Y}_\hbar u = \mathcal{B}_x^* \circ \mathcal{M}(Y_\hbar) \circ \mathcal{B}_x. \quad (5.12)$$

Note that the size ($\sim \hbar^{1/2-2\theta}$) of the support of the function Y_\hbar is much larger than the size ($\sim \hbar^{1/2-\theta}$) of the region on which the Bargmann transform of the functions in \mathcal{X}_\hbar concentrates, when $\hbar > 0$ is small.

First of all, we show

Lemma 5.8. *The operator \mathcal{Y}_\hbar extends naturally to a bounded operator on $\mathcal{H}_\hbar^r(\mathbb{R}^{2d})$ and we have*

$$\|\mathcal{Y}_\hbar\|_{\mathcal{H}_\hbar^r(\mathbb{R}^{2d})} < 1 + C\hbar^\theta$$

and

$$\left\| \left[\mathcal{Y}_\hbar, \mathcal{M}(\psi) \right] \right\|_{\mathcal{H}_\hbar^r(\mathbb{R}^{2d})} < C\hbar^\theta \quad \text{for any } \psi \in \mathcal{X}_\hbar$$

with some positive constants C independent of \hbar .

Proof. It is enough to show

$$\|\mathcal{P}_x \circ \mathcal{M}(Y_h) : L^2(\mathbb{R}^{2d} \oplus \mathbb{R}^{2d}, (\mathcal{W}_h^r)^2) \rightarrow L^2(\mathbb{R}^{2d} \oplus \mathbb{R}^{2d}, (\mathcal{W}_h^r)^2)\| < 1 + C\hbar^\theta$$

and

$$\|[\mathcal{P}_x \circ \mathcal{M}(Y_h) \circ \mathcal{P}_x, \mathcal{M}^{\text{lift}}(\psi)] : L^2(\mathbb{R}^{2d} \oplus \mathbb{R}^{2d}, (\mathcal{W}_h^r)^2) \rightarrow L^2(\mathbb{R}^{2d} \oplus \mathbb{R}^{2d}, (\mathcal{W}_h^r)^2)\| < C\hbar^\theta.$$

Note that we have $\|[\mathcal{P}_x, \mathcal{M}(Y_h)]\|_{L^2(\mathbb{R}^{2d} \oplus \mathbb{R}^{2d}, (\mathcal{W}_h^r)^2)} \leq C\hbar^\theta$ by a simple estimate on the kernel. The first claim is a consequence of this estimate. For the second, we use Lemma 5.2. \square

The next lemma tells roughly that the truncation operator \mathcal{Y}_h hardly affect the projection operators $t_h^{(k)}$, $0 \leq k \leq n$, defined in Definition 4.24, if we view it in the anisotropic Sobolev spaces.

Lemma 5.9. *For $0 \leq k \leq n$ and $\psi \in \mathcal{X}_h$, we have*

$$\|(\text{Id} - \mathcal{Y}_h) \circ \mathcal{M}(\psi) \circ t_h^{(k)}\|_{\mathcal{H}_h^{r,-}(\mathbb{R}^{2d}) \rightarrow \mathcal{H}_h^{r,+}(\mathbb{R}^{2d})} < C\hbar^\theta$$

and

$$\|t_h^{(k)} \circ (\text{Id} - \mathcal{Y}_h) \circ \mathcal{M}(\psi)\|_{\mathcal{H}_h^{r,-}(\mathbb{R}^{2d}) \rightarrow \mathcal{H}_h^{r,+}(\mathbb{R}^{2d})} < C\hbar^\theta$$

with some constant $C > 0$ independent of \hbar .

Proof. For the proof of the first inequality, it suffices to show the estimate

$$\|A : L^2(\mathbb{R}^{2d} \oplus \mathbb{R}^{2d}, (\mathcal{W}_h^{r,-})^2) \rightarrow L^2(\mathbb{R}^{2d} \oplus \mathbb{R}^{2d}, (\mathcal{W}_h^{r,+})^2)\| < C\hbar^\theta$$

for the operator

$$A := \mathcal{P}_x \circ (\text{Id} - \mathcal{M}(Y_h)) \circ \mathcal{P}_x \circ \mathcal{M}^{\text{lift}}(\psi) \circ \Phi^* \circ (\mathcal{P}_{\nu_q} \otimes \mathcal{T}_h^{(k)}) \circ (\Phi^*)^{-1}.$$

Recall that we already have the estimates (3.6) and (5.9) respectively for the kernel of the operator \mathcal{P}_x and $\Phi^* \circ (\mathcal{P}_{\nu_q} \otimes \mathcal{T}_h^{(k)}) \circ (\Phi^*)^{-1}$. Using those estimates with the property (4.26) of the escape function $\mathcal{W}_h^{r,\pm}$ and noting that

$$|(x, \xi)| \geq \hbar^{1/2-2\theta} \quad \text{for } (x, \xi) \in \text{supp } Y_h \quad (\text{resp. } |x| \leq 2\hbar^{1/2-\theta} \quad \text{for } x \in \text{supp } \psi),$$

we can estimate the kernel of the operator $\mathcal{M}(\mathcal{W}_h^{r,+}) \circ A \circ \mathcal{M}(\mathcal{W}_h^{r,-})^{-1}$ in absolute value and obtain the required estimate. The second inequality can be proved in the parallel manner. We omit the tedious details. \square

5.3 Prequantum transfer operators for non-linear transformations close to the identity

In this subsection, we study the Euclidean prequantum transfer operators for diffeomorphisms defined on small open subsets on \mathbb{R}^{2d} and is close to the identity map. Roughly we show that the action of those prequantum operators are close to the identity as an operator on $\mathcal{H}_\hbar^r(\mathbb{R}^{2d})$, though this is *not* true in the literal sense.

In this subsection, we consider the following setting in addition to Setting I:

Setting II: For every $\hbar > 0$, there is a given set \mathcal{G}_\hbar of C^∞ diffeomorphisms

$$g : \mathbb{D}(\hbar^{1/2-2\theta}) \rightarrow g(\mathbb{D}(\hbar^{1/2-2\theta})) \subset \mathbb{R}^{2d}$$

such that every $g \in \mathcal{G}_\hbar$ satisfies

(G1) g is symplectic with respect to the symplectic form ω in (4.2),

(G2) $g(0) = 0$ and $\|Dg(0) - \text{Id}\| < C\hbar^{\beta(1/2-\theta)}$, and

(G3) $\|g\|_{C^s} < C_s$

where C and C_s are positive constants that does not depend on \hbar nor $g \in \mathcal{G}_\hbar$.

Remark 5.10. In the next section we will consider a few different sets of diffeomorphisms as \mathcal{G}_\hbar and apply the argument below. At this moment, the meaning of the bound $C\hbar^{\beta(1/2-\theta)}$ in the condition (G2) may not be clear. This is a consequence of the fact that the hyperbolic splitting (1.1) is β -Hölder continuous. The reason will become clear when we introduce a family of local coordinates on M in the beginning of the next section.

For $g \in \mathcal{G}_\hbar$, we consider the Euclidean prequantum transfer operator \mathcal{L}_g defined in Subsection 4.1. Recall from Proposition 4.1 that this operator is of the form

$$\mathcal{L}_g : C_0^\infty(\mathbb{D}(\hbar^{1/2-2\theta})) \rightarrow C_0^\infty(g(\mathbb{D}(\hbar^{1/2-2\theta}))), \quad \mathcal{L}_g u(x) = e^{(i/\hbar) \cdot \mathcal{A}_g(x)} \cdot u(g^{-1}(x)) \quad (5.13)$$

with

$$\mathcal{A}_g(x) = \int_\gamma (g^{-1})^* \eta - \eta$$

where γ is a path from the origin 0 to x . (For convenience, we take the origin 0 as a fixed point of reference.)

We first show the following lemma for the (action) function $\mathcal{A}_g(x)$.

Lemma 5.11. *If $g \in \mathcal{G}_\hbar$, we have*

$$|\partial^\alpha \mathcal{A}_g(x)| \leq C_\alpha \cdot \min\{|x|^{3-|\alpha|}, 1\}$$

for any multi-index α with $|\alpha| > 0$, where $C_\alpha > 0$ is a constant independent of \hbar .

Proof. From the definition, we have

$$|\partial^\alpha \mathcal{A}_g(x)| \leq C_\alpha$$

for any multi-index α . Hence the conclusion holds obviously in the case $|\alpha| \geq 3$. The first derivatives of \mathcal{A}_g at 0 vanishes from the assumption $g(0) = 0$ in the condition (G2). Actually the second derivatives of \mathcal{A}_g also vanishes. Indeed, from the condition (G1), we have $(g^{-1})^*\omega = \omega$. If we write the diffeomorphism g^{-1} as

$$g^{-1}(p, q) = (g_p(p, q), g_q(p, q))$$

in the coordinate $x = (p, q)$ in (4.1), we see that the condition $(g^{-1})^*\omega = \omega$ implies

$$\frac{\partial g_q}{\partial q_i} \cdot \frac{\partial g_p}{\partial q_j} - \frac{\partial g_q}{\partial q_j} \cdot \frac{\partial g_p}{\partial q_i} = 0, \quad \frac{\partial g_q}{\partial p_i} \cdot \frac{\partial g_p}{\partial p_j} - \frac{\partial g_q}{\partial p_j} \cdot \frac{\partial g_p}{\partial p_i} = 0$$

and

$$\frac{\partial g_q}{\partial q_j} \cdot \frac{\partial g_p}{\partial p_i} - \frac{\partial g_q}{\partial p_i} \cdot \frac{\partial g_p}{\partial q_j} = \begin{cases} 1 & (i = j); \\ 0 & (i \neq j). \end{cases}$$

If we write the one form $(g^{-1})^* \eta - \eta$ in the coordinates as above, we have

$$(g^{-1})^* \eta - \eta = \frac{1}{2} \sum_{i=1}^d \left(g_q \cdot \frac{\partial g_p}{\partial p_i} - g_p \cdot \frac{\partial g_q}{\partial p_i} - q_i \right) dp_i + \frac{1}{2} \sum_{i=1}^d \left(g_q \cdot \frac{\partial g_p}{\partial q_i} - g_p \cdot \frac{\partial g_q}{\partial q_i} + p_i \right) dq_i.$$

Then we can check that all of the first order partial derivatives of the coefficients of dp_i and dq_i vanish at the origin $0 \in \mathbb{R}^{2d}$ from the equalities above. (Note that we have $g_p(0, 0) = g_q(0, 0) = 0$ from (G2).) This implies that the second derivatives of \mathcal{A}_g vanishes at the origin. The claim of the lemma for the case $|\alpha| \leq 2$ then follows immediately. \square

In the next section, we consider the action of the operator \mathcal{L}_g mainly on functions supported on $\mathbb{D}(\hbar^{1/2-\theta})$ (or sometimes on $\mathbb{D}(2\hbar^{1/2-\theta})$). For this reason, we take the C^∞ function

$$\chi_\hbar : \mathbb{R}^{2d} \rightarrow [0, 1], \quad \chi_\hbar(x) = \chi(\hbar^{-1/2+\theta}x/2) \tag{5.14}$$

with letting χ be a C^∞ function satisfying (3.55) and consider the operator

$$\mathcal{L}_g \circ \mathcal{M}(\chi_\hbar) : C^\infty(\mathbb{R}^{2d}) \rightarrow C_0^\infty(\mathbb{R}^{2d})$$

instead of the operator \mathcal{L}_g itself. The next lemma is the main ingredient of this subsection, which tells roughly that the operator \mathcal{L}_g for $g \in \mathcal{G}_\hbar$ is close to the identity, under the effect of truncation by the operator \mathcal{Y}_\hbar .

Proposition 5.12. *There exist constants $C > 0$ and $\epsilon > 0$ such that, for any $\hbar > 0$ and $g \in \mathcal{G}_\hbar$, we have*

$$\|\mathcal{Y}_\hbar \circ (\mathcal{L}_g - \text{Id}) \circ \mathcal{M}(\chi_\hbar)\|_{\mathcal{H}_\hbar^r(\mathbb{R}^{2d})} < C\hbar^\epsilon \quad \text{and} \quad \|(\mathcal{L}_g - \text{Id}) \circ \mathcal{Y}_\hbar \circ \mathcal{M}(\chi_\hbar)\|_{\mathcal{H}_\hbar^r(\mathbb{R}^{2d})} < C\hbar^\epsilon.$$

Proof. The proof below is elementary but a little demanding. We will use the following estimate which follows from Lemma 5.11 and the conditions in Setting II on \mathcal{G}_\hbar : For any $x \in \mathbb{R}^{2d}$ with $|x| \leq \hbar^{1/2-\theta}$, it holds

$$\begin{aligned} |\mathcal{A}_g(x) - \mathcal{A}_g(0)| &\leq C|x|^3 < C\hbar^{3(1/2-\theta)}, & \|D\mathcal{A}_g(x)\| &\leq C|x|^2 < C\hbar^{2(1/2-\theta)}, \\ \|Dg(x) - \text{Id}\| &\leq C\hbar^{\beta(1/2-\theta)} \quad \text{and} & |g(x) - x| &\leq C|x|^{1+\beta} < C\hbar^{(1+\beta)(1/2-\theta)} \end{aligned}$$

with C a constant independent of $\hbar > 0$ and $g \in \mathcal{G}_\hbar$. Also we note that, if $(x, \xi) \in \text{supp } Y_\hbar$, we have $|(x, \xi)| \leq 2\hbar^{1/2-2\theta}$ and, in particular, $|\xi| \leq 2\hbar^{1/2-2\theta}$.

From Corollary 4.16, the first claim follows if we show

$$\|\mathcal{M}(Y_\hbar) \circ \mathcal{B}_x \circ (\mathcal{L}_g - \text{Id}) \circ \mathcal{M}(\chi_\hbar) \circ \mathcal{B}_x^*\|_{L^2(\mathbb{R}^{2d} \oplus \mathbb{R}^{2d}, (\mathcal{W}_\hbar^r)^2)} \leq C\hbar^\epsilon. \quad (5.15)$$

Recalling the definition of the operators \mathcal{B}_x and \mathcal{B}_x^* in (4.10), we write the operator $\mathcal{B}_x \circ (\mathcal{L}_g - \text{Id}) \circ \mathcal{M}(\chi_\hbar) \circ \mathcal{B}_x^*$ as an integral operator of the form

$$(\mathcal{B}_x \circ (\mathcal{L}_g - \text{Id}) \circ \mathcal{M}(\chi_\hbar) \circ \mathcal{B}_x^* u)(x, \xi) = \int K(2^{-1/2}x, 2^{1/2}\xi; 2^{-1/2}x', 2^{1/2}\xi') u(x', \xi') \frac{dx' d\xi'}{(2\pi\hbar)^{2d}},$$

where

$$K(x, \xi; x', \xi') = a_D^2 \int e^{(i/\hbar)\xi((x/2)-y) + (i/\hbar)\xi'(y-(x'/2))} \cdot e^{-|y-x|^2/(2\hbar) - |y-x'|^2/(2\hbar)} \cdot \chi_\hbar(y) \cdot k(x, \xi, x', \xi', y) dy$$

and

$$k(x, \xi, x', \xi', y) = e^{(i/\hbar)\mathcal{A}_g(g(y)) - (i/\hbar)\xi(g(y)-y) - (|g(y)-x|^2 - |y-x|^2)/(2\hbar)} - 1.$$

(The factor $2^{\pm 1}$ appears because of the change of variable $\tilde{\sigma}$ in the definition of \mathcal{B}_x and \mathcal{B}_x^* . But this is not important in any sense.) Applying integration by parts to the integral above for ν times, we see

$$\begin{aligned} &K(x, \xi; x', \xi') \\ &= a_D^2 \int L^\nu (e^{(i/\hbar)\xi((x/2)-y) + (i/\hbar)\xi'(y-(x'/2))}) \cdot e^{-|y-x|^2/(2\hbar) - |y-x'|^2/(2\hbar)} \chi_\hbar(y) k(x, \xi, x', \xi', y) dy \\ &= a_D^2 \int e^{(i/\hbar)\xi((x/2)-y) + (i/\hbar)\xi'(y-(x'/2))} \cdot ({}^t L)^\nu (e^{-|y-x|^2/(2\hbar) - |y-x'|^2/(2\hbar)} \chi_\hbar(y) k(x, \xi, x', \xi', y)) dy \end{aligned}$$

where L is the differential operators defined by

$$Lu = \frac{1}{1 + \hbar^{-1}|\xi - \xi'|^2} \cdot \left(1 + i \sum_{j=1}^{2d} (\xi_j - \xi'_j) \frac{\partial}{\partial y_j} \right) u$$

and tL is its transpose:

$${}^tLu = \left(1 - i \sum_{j=1}^{2d} (\xi_j - \xi'_j) \frac{\partial}{\partial y_j} \right) \left(\frac{1}{1 + \hbar^{-1} |\xi - \xi'|^2} \cdot u \right).$$

Using the estimates noted in the beginning in the resulting terms, we can get the estimate

$$|K(x, \xi; x', \xi')| \leq C_\nu \cdot \hbar^\epsilon \cdot \langle \hbar^{-1/2} |(x, \xi) - (x', \xi')| \rangle^{-\nu} \quad \text{for } (x, \xi) \in \text{supp } \mathcal{Y}_\hbar$$

for a small constant $\epsilon > 0$ and arbitrarily large $\nu > 0$, where C_ν is a constant independent of \hbar .

Remark 5.13. The result of integration by part is not very simple. But we have only to consider the order about the parameter \hbar , since we allow the constant C_ν to depend on the derivatives of g . Hence it is not too difficult to do. Just note that θ satisfies the condition (5.1).

This estimate for sufficiently large ν and (4.26) yields the required estimate. The second claim is proved in the parallel manner. \square

As we noted in the beginning of this section, the operator $\mathcal{L}_g \circ \mathcal{M}(\chi_\hbar)$ may not extend to a bounded operator from $\mathcal{H}_\hbar^r(\mathbb{R}^{2d})$ to itself, even though $g \in \mathcal{G}_\hbar$ is very close to the identity map. The next proposition (and hyperbolicity of f) will compensate this inconvenience.

Proposition 5.14. *For any $g \in \mathcal{G}_\hbar$, we have*

$$\|\mathcal{L}_g \circ \mathcal{M}(\chi_\hbar)\|_{\mathcal{H}_\hbar^{r,+}(\mathbb{R}^{2d}) \rightarrow \mathcal{H}_\hbar^r(\mathbb{R}^{2d})} \leq C_0 \quad \text{and} \quad \|\mathcal{L}_g \circ \mathcal{M}(\chi_\hbar)\|_{\mathcal{H}_\hbar^r(\mathbb{R}^{2d}) \rightarrow \mathcal{H}_\hbar^{r,-}(\mathbb{R}^{2d})} \leq C_0 \quad (5.16)$$

for sufficiently small $\hbar > 0$, where $C_0 > 1$ is a constant that depend only on n, r, d, θ and the choice of the escape functions W and W^\pm in subsection 3.3. (In particular, C_0 is independent of the choice of the family \mathcal{G}_\hbar .)

The conclusion of this proposition is quite natural in view of the facts that

$$\mathcal{W}_\hbar^r \circ G(x, \xi) \cdot \chi_\hbar(x) \leq \mathcal{W}_\hbar^{r,+}(x, \xi), \quad \mathcal{W}_\hbar^{r,-} \circ G(x, \xi) \cdot \chi_\hbar(x) \leq \mathcal{W}_\hbar^r(x, \xi) \quad (5.17)$$

for the canonical map

$$G : \mathbb{R}^{2d} \oplus \mathbb{R}^{2d} \rightarrow \mathbb{R}^{2d} \oplus \mathbb{R}^{2d}, \quad G(x, \xi) = (g(x), {}^t(Dg(x))^{-1}(\xi))$$

associated to the operator \mathcal{L}_g . (Recall the argument in Subsection 2.2.) And it can be proved in essentially same ways as the argument given in the papers [5] and [17], where Littlewood-Paley theory and the theory of pseudodifferential operator is used respectively. Below we give a proof below by interpreting the argument in [5] in terms of the Bargmann transform. (But the reader may skip it because this is not a very essential part of our argument and may be proved in various ways.)

Proof. We decompose the operator $\mathcal{B}_x \circ \mathcal{L}_g \circ \mathcal{M}(\chi_{\hbar}) \circ \mathcal{B}_x^*$ as

$$\begin{aligned} \mathcal{B}_x \circ \mathcal{L}_g \circ \mathcal{M}(\chi_{\hbar}) \circ \mathcal{B}_x^* &= \mathcal{M}(1 - Y_{\hbar}) \circ \mathcal{B}_x \circ \mathcal{L}_g \circ \mathcal{M}(\chi_{\hbar}) \circ \mathcal{B}_x^* \\ &\quad + \mathcal{M}(Y_{\hbar}) \circ \mathcal{B}_x \circ (\mathcal{L}_g - \text{Id}) \circ \mathcal{M}(\chi_{\hbar}) \circ \mathcal{B}_x^* \\ &\quad + \mathcal{M}(1 - Y_{\hbar}) \circ \mathcal{B}_x \circ \mathcal{M}(\chi_{\hbar}) \circ \mathcal{B}_x^*. \end{aligned}$$

If we apply Lemma 5.12 (or more precisely (5.15) in the proof) to the second term and Lemma 5.8 to the third term, we see that these two operators are bounded operators on $\mathcal{H}_{\hbar}^r(\mathbb{R}^{2d})$ and the operator norms are bounded by an absolute constant. Hence, in order to prove the former claim of the theorem, it suffices to show that the operator norm of

$$\mathcal{M}(1 - Y_{\hbar}) \circ \mathcal{B}_x \circ \mathcal{L}_g \circ \mathcal{M}(\chi_{\hbar}) \circ \mathcal{B}_x^* : L^2(\mathbb{R}^{2d}, (\mathcal{W}_{\hbar}^{r,+})^2) \rightarrow L^2(\mathbb{R}^{2d}, (\mathcal{W}_{\hbar}^r)^2) \quad (5.18)$$

is bounded by a constant C_0 with the same property as stated in the proposition. (The latter claim is proved in the parallel manner.) Below we give a proof⁹.

We take and fix $1/3 < a^+ < b^+ < a < b < 1/2$. Then we introduce a C^∞ partition of unity $\{\psi_n\}_{n \in \mathbb{Z}}$ (resp. $\{\psi_n^+\}_{n \in \mathbb{Z}}$) on \mathbb{R}^{2d} with the following properties:

- The function ψ_n (resp. ψ_n^+) is supported on the disk $|\xi| \leq 1$ if $n = 0$ and on the annulus $2^{|n|-1} \leq |\xi| \leq 2^{|n|+1}$ otherwise.
- The function ψ_n is supported on the cone $\mathbf{C}_+(b)$ if $n > 0$ and on the cone $\mathbf{C}_-(1/a) = \overline{\mathbb{R}^{2d}} \setminus \mathbf{C}_+(a)$ if $n < 0$. Respectively, the function ψ_n^+ is supported on the cone $\mathbf{C}_+(b^+)$ if $n > 0$ and on the cone $\mathbf{C}_-(1/a^+) = \overline{\mathbb{R}^{2d}} \setminus \mathbf{C}_+(a^+)$ if $n < 0$.
- The normalized functions $\xi \mapsto \psi_n(2^n \xi)$ (resp. $\xi \mapsto \psi_n^+(2^n \xi)$) are uniformly bounded in C^∞ norm.

For each $\hbar > 0$, we define functions ψ_n and ψ_n^+ on $T^*\mathbb{R}^{2d} = \mathbb{R}^{2d} \oplus \mathbb{R}^{2d}$ for $n \in \mathbb{Z}$ by

$$\psi_{n,\hbar}(x, \xi) = \psi_n(\hbar^{-1/2} \zeta) \quad \text{resp.} \quad \psi_{n,\hbar}^+(x, \xi) = \psi_n^+(\hbar^{-1/2} \zeta)$$

where $\zeta = (\zeta_p, \zeta_q)$ is the coordinates introduced in Proposition 2.13. Then, from the definition of the partition of unities above, we have

$$\|\mathcal{W}_{\hbar}^r \cdot u\|_{L^2}^2 \leq C_0 \sum_{n=-\infty}^{\infty} 2^{2rn} \|\psi_{n,\hbar} \cdot u\|_{L^2}^2$$

and also

$$\sum_{n=-\infty}^{\infty} 2^{2rn} \|\psi_{n,\hbar}^+ \cdot u\|_{L^2}^2 \leq C_0 \|\mathcal{W}_{\hbar}^{r,+} \cdot u\|_{L^2}^2 \quad (5.19)$$

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Remark 5.15. The argument in the following part is a little sketchy. For the details, we refer the argument in [6], though it will not be very necessary.

for any function $u(x, \xi) \in C^\infty(\mathbb{R}^{2d} \oplus \mathbb{R}^{2d})$. From the first inequality above and the definition of the function Y_{\hbar} , we have

$$\begin{aligned} & \|\mathcal{W}_{\hbar}^r \cdot (1 - Y_{\hbar}) \cdot \mathcal{B}_x \circ \mathcal{L}_g \circ \mathcal{M}(\chi_{\hbar}) \circ \mathcal{B}_x^* u\|_{L^2} \\ & \leq C_0 \sum_{|n| \geq \hbar^{-\theta}} \sum_{n' = -\infty}^{\infty} 2^{2rn} \|\psi_{n, \hbar} \cdot \mathcal{B}_x \circ \mathcal{L}_g \circ \mathcal{M}(\chi_{\hbar}) \circ \mathcal{B}_x^*(\psi_{n', \hbar}^+ \cdot u)\|_{L^2}^2. \end{aligned}$$

For the summands on the right hand side, we observe that

- From Lemma 3.2, the L^2 -operator norm of $\mathcal{B}_x \circ \mathcal{L}_g \circ \mathcal{M}(\chi_{\hbar}) \circ \mathcal{B}_x^*$ is bounded by 1, so

$$\|\psi_{n, \hbar} \cdot \mathcal{B}_x \circ \mathcal{L}_g \circ \mathcal{M}(\chi_{\hbar}) \circ \mathcal{B}_x^*(\psi_{n', \hbar}^+ \cdot u)\|_{L^2}^2 \leq \|\psi_{n', \hbar}^+ \cdot u\|_{L^2}^2.$$

- If either $|n| - |n'| \geq 3$ or $n < 0 < n'$, we have

$$\text{dist}(Dg_x^t(\text{supp } \psi_{n, \hbar}), \text{supp } \psi_{n', \hbar}) > C_0 \cdot \hbar^{1/2} 2^{\max\{n, n'\}} \quad \text{for } x \in \text{supp } \chi_{\hbar}$$

and, by crude estimate using integration by parts, we get the estimate

$$\|\psi_{n, \hbar} \cdot \mathcal{B}_x \circ \mathcal{L}_g \circ \mathcal{M}(\chi_{\hbar}) \circ \mathcal{B}_x^*(\psi_{n', \hbar}^+ \cdot u)\|_{L^2}^2 \leq C_{\nu}(g) \cdot 2^{-\nu \cdot \max\{n, n'\}} \|\psi_{n', \hbar}^+ \cdot u\|_{L^2}^2$$

where the constant $C_{\nu}(g)$ may depend on g and ν but not on \hbar . Otherwise we have $n < n + 3$ and $2^{rn} \leq C_0 2^{rn'}$.

From these observations and (5.19), we can conclude the required estimate:

$$\|\mathcal{W}_{\hbar}^r \cdot (1 - Y_{\hbar}) \cdot \mathcal{B}_x \circ \mathcal{L}_g \circ \mathcal{M}(\chi_{\hbar}) \circ \mathcal{B}_x^* u\|_{L^2}^2 \leq C_0 \sum_{n' = -\infty}^{\infty} 2^{2rn} \|\psi_{n', \hbar}^+ \cdot u\|_{L^2}^2 \leq C_0 \|\mathcal{W}_{\hbar} u\|_{L^2}^2$$

for sufficiently small $\hbar > 0$. □

The next lemma will be used in the key step in the proof of Theorem 1.18.

Lemma 5.16. *There exist constants $\epsilon > 0$ and $C > 0$ independent of \hbar such that the following holds: Let $\psi \in \mathcal{X}_{\hbar}$ be supported on the disk $\mathbb{D}(2\hbar^{1/2-\theta})$ and let $g \in \mathcal{G}_{\hbar}$, $0 \leq k \leq n$, then it holds*

$$\left\| (\mathcal{L}_g - \text{Id}) \circ \mathcal{M}(\psi) \circ t_{\hbar}^{(k)} \right\|_{\mathcal{H}_{\hbar}^r(\mathbb{R}^{2d})} \leq C \hbar^{\epsilon}$$

and

$$\left\| t_{\hbar}^{(k)} \circ (\mathcal{L}_g - \text{Id}) \circ \mathcal{M}(\psi) \right\|_{\mathcal{H}_{\hbar}^r(\mathbb{R}^{2d})} \leq C \hbar^{\epsilon}.$$

Proof. We decompose the operator $(\mathcal{L}_g - \text{Id}) \circ \mathcal{M}(\psi) \circ t_{\hbar}^{(k)}$ into

$$(\mathcal{L}_g \circ \mathcal{M}(\chi_{\hbar}) - \text{Id}) \circ \mathcal{Y}_{\hbar} \circ \mathcal{M}(\psi) \circ t_{\hbar}^{(k)} \quad \text{and} \quad (\mathcal{L}_g \circ \mathcal{M}(\chi_{\hbar}) - \text{Id}) \circ (\text{Id} - \mathcal{Y}_{\hbar}) \circ \mathcal{M}(\psi) \circ t_{\hbar}^{(k)}. \quad (5.20)$$

(Note that we have $\chi_{\hbar} \cdot \psi = \psi$ from the assumption.) The operator norm of the latter is also bounded by $C\hbar^\epsilon$ from Lemma 5.9 and Proposition 5.14. We further write the former part as

$$(\mathcal{L}_g - \text{Id}) \circ \mathcal{M}(\psi) \circ \mathcal{Y}_{\hbar} \circ t_{\hbar}^{(k)} + (\mathcal{L}_g \circ \mathcal{M}(\chi_{\hbar}) - \text{Id}) \circ [\mathcal{Y}_{\hbar}, \mathcal{M}(\psi)] \circ t_{\hbar}^{(k)}$$

Then we see that the operator norm of the former part is bounded by $C\hbar^\epsilon$, from Lemma 5.12, Lemma 4.11, Lemma 5.8 and Proposition 5.14. Thus we obtain the former claim. The latter claim can be proved in the parallel manner. \square

Corollary 5.17. *There exist constants $\epsilon > 0$ and $C > 0$ independent of \hbar such that, for any $\psi \in \mathcal{X}_{\hbar}$ and $g \in \mathcal{G}_{\hbar}$ it holds*

$$\left\| [\mathcal{L}_g \circ \mathcal{M}(\psi), t_{\hbar}^{(k)}] \right\|_{\mathcal{H}_{\hbar}^r(\mathbb{R}^{2d}) \rightarrow \mathcal{H}_{\hbar}^r(\mathbb{R}^{2d})} \leq C\hbar^\epsilon \quad \text{for } 0 \leq k \leq n$$

and also

$$\left\| [\mathcal{L}_g \circ \mathcal{M}(\psi), \tilde{t}_{\hbar}] \right\|_{\mathcal{H}_{\hbar}^r(\mathbb{R}^{2d})} \leq C\hbar^\epsilon.$$

Proof. The former claim is an immediate consequence of the last lemma and Lemma 5.7. The latter claim then follows from the relation $\tilde{t}_{\hbar} = \text{Id} - \sum_{k=0}^n t_{\hbar}^{(k)}$. \square

5.4 The multiplication operators and the rough Laplacian on \mathbb{R}^{2d}

We close this section by the following lemma on the Euclidean rough Laplacian Δ_{\hbar} considered in Subsection 4.6.

Lemma 5.18. *For any $\psi \in \mathcal{X}_{\hbar}$, we have*

$$\left\| [\mathcal{M}(\psi), \Delta_{\hbar}] \right\|_{(\mathcal{D}(\Delta_{\hbar}), \|\cdot\|_{\Delta_{\hbar}}) \rightarrow L^2(\mathbb{R}^{2d})} \leq C\hbar^\theta$$

and

$$\left\| [\mathcal{M}(\psi), q_{\hbar}^{(k)}] \right\|_{L^2(\mathbb{R}^{2d})} \leq C\hbar^\theta$$

where C is a constant independent of $\psi \in \mathcal{X}_{\hbar}$ and \hbar .

Proof. The first claim can be checked easily from the expression of Δ_{\hbar} given in Proposition 2.22. For the second claim, we can just follow the argument in the proof of Lemma 5.6, replacing $t_{\hbar}^{(k)}$ by $q_{\hbar}^{(k)}$. (The proof is simpler actually.) \square

6 Proofs of the main theorems

In this section, we give the proofs of the main theorems of this paper, Theorem 1.18 and Theorem 2.21. We henceforth consider the setting assumed in Section 1. In particular $\lambda > 1$ is the constant in the condition (1.2) in the definition (Definition 1.1) that f is an Anosov diffeomorphism. Note that, by replacing f by its iterate if necessary, we may and do suppose that λ is a large number. Below we write C_0 for positive constants independent of f , V and \hbar and write C for those independent of \hbar but may (or may not) dependent on f and V . Also we assume (5.6) for the choice of r .

6.1 Local charts on M and local trivialization of the bundle $P \rightarrow M$

As in the last section, we fix a constant $0 < \theta < \beta/8$ with $0 < \beta < 1$ being the Hölder exponent of the stable and unstable sub-bundle given in (1.3). Below we take an atlas on M depending on the semiclassical parameter $\hbar = \frac{1}{2\pi N} > 0$ so that it consists of charts of diameter of order $\hbar^{\frac{1}{2}-\theta}$. We consider \mathbb{R}^{2d} as a linear symplectic space with coordinates $x = (q, p) = (q^1, \dots, q^d, p^1, \dots, p^d)$ and symplectic form $\omega = \sum_{i=1}^d dq^i \wedge dp^i$. The open ball of radius $c > 0$ is denoted by $\mathbb{D}(c) := \{x \in \mathbb{R}^{2d} \mid |x| < c\}$.

Proposition 6.1. “Local chart and trivialization”. *For each $\hbar = \frac{1}{2\pi N} > 0$, there exist a set of distinct points*

$$\mathcal{P}_\hbar = \{m_i \in M \mid 1 \leq i \leq I_\hbar\}$$

and a coordinate map associated to each point $m_i \in \mathcal{P}_\hbar$,

$$\kappa_i = \kappa_{i,\hbar} : \mathbb{D}(c) \subset \mathbb{R}^{2d} \rightarrow M, \quad 1 \leq i \leq I_\hbar$$

with $c > 0$ a constant independent of \hbar , so that the following conditions hold:

- (1) $\kappa_i(0) = m_i$.
- (2) The differential of κ_i at the origin 0 maps the subspaces $\mathbb{R}^d \oplus \{0\}$ and $\{0\} \oplus \mathbb{R}^d$ (or, the q - and p - axis) isometrically onto the unstable and stable subspace respectively:

$$(D\kappa_i)_0(\mathbb{R}^d \oplus \{0\}) = E_u(m_i), \quad (D\kappa_i)_0(\{0\} \oplus \mathbb{R}^d) = E_s(m_i).$$

- (3) The open subsets $U_i := \kappa_i\left(\mathbb{D}\left(3\hbar^{\frac{1}{2}-\theta}\right)\right) \subset M$ for $1 \leq i \leq I_\hbar$ cover the manifold M . The cardinality I_\hbar of the set \mathcal{P}_\hbar is bounded by $C_0 \cdot \hbar^{-d(1-2\theta)}$ and we have

$$\max_{1 \leq i \leq I_\hbar} \#\{1 \leq j \leq I_\hbar \mid U \cap U_j \neq \emptyset\} \leq C_0 \tag{6.1}$$

with C_0 a constant independent of \hbar .

- (4) For every $1 \leq i \leq I_\hbar$, $\kappa_i^*(\omega) = \sum_i dq^i \wedge dp^i$ on U_i and with an appropriate choice of a section $\tau_i : U_i \rightarrow P$, the statement of Proposition 2.13 holds true.
- (5) If $U_i \cap U_j \neq \emptyset$, we denote the coordinate change transformation by $\kappa_{j,i} := \kappa_j^{-1} \circ \kappa_i : \mathbb{D}(c) \rightarrow \mathbb{R}^{2d}$. Then there exists symplectic and isometric affine map $A_{j,i} : \mathbb{R}^{2d} \rightarrow \mathbb{R}^{2d}$ that belongs to \mathcal{A} (see Definition 4.12) such that $g_{j,i} := A_{j,i} \circ \kappa_{j,i}$ satisfies

$$g_{j,i}(0) = 0, \quad \|Dg_{j,i}(0) - Id\|_{C^1} \leq C_1 \cdot \hbar^{\beta(\frac{1}{2}-\theta)} \quad \text{and} \quad \|g_{j,i}\|_{C^s} < C_s \quad \text{for } s \geq 2$$

where C_s for $s \geq 1$ are constant independent of \hbar and $1 \leq i, j \leq I_\hbar$.

(6) There exists a family of C^∞ functions $\{\psi_i : \mathbb{R}^{2d} \rightarrow [0, 1]\}_{i=1}^{I_h}$ which is supported on the disk $\mathbb{D}(\hbar^{1/2-\theta})$ and gives a partition of unity on M :

$$\sum_{i=1}^{I_h} \psi_i \circ \kappa_i^{-1} \equiv 1 \text{ on } M. \quad (6.2)$$

The set of functions ψ_i satisfies the conditions (C1) and (C2) in Subsection 5.1.

Remark 6.2. Since the vector unstable and stable sub-bundles, E_u and E_s may be non-trivial in general, we need to put the affine isometries $A_{j,i} \in \mathcal{A}$ in the condition (5) above.

Proof. For each point $m \in M$, we first define κ_m as the composition of the exponential mapping (in Riemannian geometry) $\exp_m : T_m M \rightarrow M$ and a linear map $\mathbb{R}^{2d} \rightarrow T_m M$ so that the condition 1 (with $\kappa_i = \kappa_m$) holds true. Then, using Darboux theorem, we can deform such κ_m into a symplectic map (condition 4) with keeping the condition 1. (See Lemma 3.14 in [29, p.94] and its proof). We may then take a section τ as in the proof of Proposition 2.13 so that the condition 4 (with $\kappa_i = \kappa_m$) holds. It is then clear that, if we take the points in \mathcal{P}_\hbar appropriately, the conditions 1 to 3 hold true with setting $\kappa_i := \kappa_{m_i}$. The condition 5 and 6 are also obvious from this construction. \square

In the following subsections, we fix the set \mathcal{P}_\hbar , the coordinate maps κ_i , the isometric affine maps $A_{j,i} \in \mathcal{A}$ and the functions ψ_i taken in Proposition 6.1 above.

6.2 The prequantum transfer operator decomposed on local charts

To proceed, we express the transfer operator \hat{F}_\hbar as the totality of operators between local charts. First we discuss about an expression of an equivariant section $u \in C_N^\infty(P)$ as a set of functions on local charts.

Definition 6.3. Let

$$\mathcal{E}_\hbar := \bigoplus_{i=1}^{I_h} C_0^\infty(\mathbb{D}(\hbar^{1/2-\theta})).$$

Let $\mathbf{I}_\hbar : C_N^\infty(P) \rightarrow \mathcal{E}$ be the operator that associates each equivariant function $u \in C_N^\infty(P)$ a set of functions $\mathbf{I}_\hbar(u) = (u_i)_{i=1}^{I_h} \in \mathcal{E}_\hbar$ on local charts defined by the relation

$$u_i(x) = \psi_i(x) \cdot u(\tau_i(\kappa_i(x))) \quad \text{for } 1 \leq i \leq I_h. \quad (6.3)$$

The inverse operation is given as follows.

Proposition 6.4. Let $\mathbf{I}_\hbar^* : \bigoplus_{i=1}^{I_h} \mathcal{S}(\mathbb{R}^{2d}) \rightarrow C_N^\infty(P)$ be the operator defined by

$$(\mathbf{I}_\hbar^*((u_i)_{i=1}^{I_h}))(p) = \sum_{i=1}^{I_h} e^{iN \cdot \alpha_i(p)} \cdot \chi_\hbar(x) \cdot u_i(x) \quad (6.4)$$

where χ_{\hbar} is the function defined in (5.14), and $x = \kappa_i^{-1}(\pi(p))$ and $\alpha_i(p)$ is the real number such that $p = e^{i\alpha_i(p)} \cdot \tau_i(\pi(p))$. This operator reconstructs $u \in C_N^\infty(P)$ from its local data $u_i = (\mathbf{I}_{\hbar}(u))_i$:

$$\mathbf{I}_{\hbar}^* \circ \mathbf{I}_{\hbar} = \text{Id}_{C_N^\infty(P)}. \quad (6.5)$$

Consequently, $\mathbf{I}_{\hbar} \circ \mathbf{I}_{\hbar}^* : \mathcal{E}_{\hbar} \rightarrow \mathcal{E}_{\hbar}$ is a projection onto the image of \mathbf{I}_{\hbar} .

Proof. Let $w := (\mathbf{I}_{\hbar}^* \circ \mathbf{I}_{\hbar})(v)$. From the expressions of \mathbf{I}_{\hbar} and \mathbf{I}_{\hbar}^* and equivariance of v , we compute

$$w(p) = \sum_{i=1}^{I_{\hbar}} e^{iN\alpha_i(p)} (\chi_{\hbar}(x) \cdot \psi_i(x) \cdot v(\tau_i(\kappa_i(x)))) = \sum_{i=1}^{I_{\hbar}} \psi_i(x) v(p) = v(p).$$

Finally $\mathbf{I}_{\hbar} \circ \mathbf{I}_{\hbar}^*$ is a projector since $(\mathbf{I}_{\hbar} \circ \mathbf{I}_{\hbar}^*)^2 = \mathbf{I}_{\hbar} \circ (\mathbf{I}_{\hbar}^* \circ \mathbf{I}_{\hbar}) \circ \mathbf{I}_{\hbar}^* = \mathbf{I}_{\hbar} \circ \mathbf{I}_{\hbar}^*$. \square

Definition 6.5. We define the *lift of the prequantum transfer operator* \hat{F}_{\hbar} with respect to \mathbf{I}_{\hbar} as

$$\mathbf{F}_{\hbar} := \mathbf{I}_{\hbar} \circ \hat{F}_N \circ \mathbf{I}_{\hbar}^* : \bigoplus_{i=1}^{I_{\hbar}} \mathcal{S}(\mathbb{R}^{2d}) \rightarrow \mathcal{E}_{\hbar} \subset \bigoplus_{i=1}^{I_{\hbar}} \mathcal{S}(\mathbb{R}^{2d}). \quad (6.6)$$

The operator \mathbf{F}_{\hbar} is nothing but the prequantum transfer operator $\hat{F}_N : C_N^\infty(P) \rightarrow C_N^\infty(P)$ viewed through the local charts and local trivialization that we have chosen. This is a matrix of operators that describe transition between local data that \hat{F}_{\hbar} induces. The next proposition gives it in a concrete form.

Definition 6.6. We write $i \rightarrow j$ for $0 \leq i, j \leq I_{\hbar}$ if and only if $f(U_i) \cap U_j \neq \emptyset$.

Clearly we have

$$\max_{1 \leq i \leq I_{\hbar}} \#\{1 \leq j \leq I_{\hbar} \mid i \rightarrow j\} \leq C(f) \quad (6.7)$$

for some constant $C(f)$ which may depend on f but not on \hbar .

Proposition 6.7. *The operator \mathbf{F}_{\hbar} is written as*

$$\mathbf{F}_{\hbar}((v_i)_{i \in I_{\hbar}}) = \left(\sum_{i=1}^{I_{\hbar}} \mathbf{F}_{j,i}(v_i) \right)_{j \in I_{\hbar}}$$

where the component

$$\mathbf{F}_{j,i} : \mathcal{S}(\mathbb{R}^{2d}) \rightarrow C_0^\infty(\mathbb{D}(\hbar^{1/2-\theta}))$$

is defined by $\mathbf{F}_{j,i} \equiv 0$ if $i \not\rightarrow j$ and, otherwise, by

$$\mathbf{F}_{j,i}(v_i) = \mathcal{L}_{f_{j,i}}(e^{V \circ f \circ \kappa_i} \cdot \psi_{j,i} \cdot \chi_{\hbar} \cdot v_i)$$

where we set

$$f_{j,i} := \kappa_j^{-1} \circ f \circ \kappa_i, \quad (6.8)$$

$$\psi_{j,i}(x) := \psi_j \circ f_{j,i} \quad (6.9)$$

and $\mathcal{L}_{f_{j,i}}$ is the Euclidean prequantum transfer operator defined in (2.7) with $g = f_{j,i}$.

Proof. The expression of the operator \hat{F}_N in local coordinates has been given in Proposition 2.3. Taking the multiplication by functions ψ_i , $1 \leq i \leq I_h$, in the definitions of the operators \mathbf{I}_h and \mathbf{I}_h^* into account, we obtain the expression of $\mathbf{F}_{j,i}$ as above. \square

We define

$$V_j = \max\{V(m) \mid m \in U_j\} \quad \text{for } 1 \leq j \leq I_h.$$

Since the function V is almost constant on each U_j , we have

Lemma 6.8. *If we set*

$$\mathcal{X}_h = \{\psi_{j,i} \cdot \chi_h \mid 1 \leq i, j \leq I_h, i \rightarrow j\} \quad (\text{resp. } \mathcal{X}_h = \{e^{V \circ f \circ \kappa_i} \cdot \psi_{j,i} \cdot \chi_h \mid 1 \leq i, j \leq I_h, i \rightarrow j\}),$$

it satisfies the conditions (C1) and (C2) in Subsection 5.1. (The constants C and C_α will depend on f and V though not on \hbar .) For $1 \leq i, j \leq I_h$ such that $i \rightarrow j$, we have

$$\left\| \mathcal{M}(e^{V \circ f \circ \kappa_i} \cdot \psi_{j,i} \cdot \chi_h) - e^{V_j} \cdot \mathcal{M}(\psi_{j,i} \cdot \chi_h) \right\|_{\mathcal{H}_h^r(\mathbb{R}^{2d})} \leq C(f, V) \cdot \hbar^\theta$$

for some constant $C(f, V)$ independent of \hbar .

Proof. The former claim should be obvious from the choice of the coordinates κ_i and the functions ψ_i for $i \in I_h$. We can get the latter claim if we apply Corollary 5.3 to the multiplication operators by $e^{V \circ f \circ \kappa_i} \cdot \psi_{j,i} \cdot \chi_h - e^{V_j} \cdot \psi_{j,i} \cdot \chi_h = (e^{V \circ f \circ \kappa_i} - e^{V_j}) \cdot \psi_{j,i} \cdot \chi_h$. \square

6.3 The anisotropic Sobolev spaces

Definition 6.9. The *Anisotropic Sobolev space* $\mathcal{H}_h^r(P)$ is defined as the completion of $C_N^\infty(P)$ with respect to the norm

$$\|u\|_{\mathcal{H}_h^r} := \left(\sum_{i=1}^{I_h} \|u_i\|_{\mathcal{H}_h^r(\mathbb{R}^{2d})}^2 \right)^{1/2} \quad \text{for } u \in C_N^\infty(P),$$

where $u_i = (\mathbf{I}_h(u))_i \in C_0^\infty(\mathbb{D}(\hbar^{1/2-\theta}))$ are the local data defined in (6.3) and $\|u_i\|_{\mathcal{H}_h^r(\mathbb{R}^{2d})}^2$ is the anisotropic Sobolev norm on $C_0^\infty(\mathbb{R}^{2d})$ in Definition 4.7. We define the Hilbert spaces $\mathcal{H}_h^{r,\pm}(P)$ in the parallel manner, replacing $\|u_i\|_{\mathcal{H}_h^r(\mathbb{R}^{2d})}^2$ by the norms $\|u_i\|_{\mathcal{H}_h^{r,\pm}(\mathbb{R}^{2d})}^2$ respectively.

Remark 6.10. (1) By definition, the operation \mathbf{I}_h extends uniquely to an isometric injection

$$\mathbf{I}_h : \mathcal{H}_h^r(P) \rightarrow \bigoplus_{i=1}^{I_h} \mathcal{H}_h^r(\mathbb{D}(\hbar^{1/2-\theta})) \subset \bigoplus_{i=1}^{I_h} \mathcal{H}_h^r(\mathbb{R}^{2d})$$

where $\mathcal{H}_h^r(\mathbb{D}(\hbar^{1/2-\theta}))$ denotes the subspace that consists of elements supported on the disk $\mathbb{D}(\hbar^{1/2-\theta})$.

(2) From (6.5), we have $\mathbf{I}_h^* \circ \mathbf{I}_h = \text{Id}$ on $\mathcal{H}_h^r(P)$ and also on $\mathcal{H}_h^{r,\pm}(P)$.

Lemma 6.11. *The projector $\mathbf{I}_\hbar \circ \mathbf{I}_\hbar^* : \mathcal{E}_\hbar \rightarrow \mathcal{E}_\hbar$ extends to bounded operators*

$$\mathbf{I}_\hbar \circ \mathbf{I}_\hbar^* : \bigoplus_{i=1}^{I_\hbar} \mathcal{H}_\hbar^{r,+}(\mathbb{R}^{2d}) \rightarrow \bigoplus_{i=1}^{I_\hbar} \mathcal{H}_\hbar^r(\mathbb{D}(\hbar^{1/2-\theta}))$$

and

$$\mathbf{I}_\hbar \circ \mathbf{I}_\hbar^* : \bigoplus_{i=1}^{I_\hbar} \mathcal{H}_\hbar^r(\mathbb{R}^{2d}) \rightarrow \bigoplus_{i=1}^{I_\hbar} \mathcal{H}_\hbar^{r,-}(\mathbb{D}(\hbar^{1/2-\theta})).$$

Further the operator norms of these projectors are bounded by a constant independent of \hbar .

Remark 6.12. The operator $\mathbf{I}_\hbar \circ \mathbf{I}_\hbar^*$ will not be a bounded operator from $\bigoplus_{i=1}^{I_\hbar} \mathcal{H}_\hbar^r(\mathbb{R}^{2d})$ to itself.

Proof. To prove the claim, it is enough to apply Proposition 5.14 and Corollary 5.3 to each component of $\mathbf{I}_\hbar \circ \mathbf{I}_\hbar^*$ with setting

$$\mathcal{G}_\hbar = \{A_{j,i} \circ \kappa_{j,i} \mid 1 \leq i, j \leq I_\hbar, U_i \cap U_j \neq \emptyset\} \quad (6.10)$$

and

$$\mathcal{X}_\hbar = \{\psi_j \circ \kappa_{j,i} \cdot \chi_\hbar \mid 1 \leq i, j \leq I_\hbar, U_i \cap U_j \neq \emptyset\},$$

and use (6.1). (See also the remark below.) \square

Remark 6.13. The affine transformation $A_{j,i}$ in (6.10) is that appeared in the choice of local coordinates in Proposition 6.1. Note that the prequantum transfer operator $\mathcal{L}_{A_{j,i}}$ is a unitary operator on $\mathcal{H}_\hbar^r(\mathbb{R}^{2d})$ (and on $\mathcal{H}_\hbar^{r,\pm}(\mathbb{R}^{2d})$), by Lemma 4.13, and hence we may neglect the post- or pre-composition of $\mathcal{L}_{A_{j,i}}$ when we consider the operator norm on $\mathcal{H}_\hbar^r(\mathbb{R}^{2d})$ (and on $\mathcal{H}_\hbar^{r,\pm}(\mathbb{R}^{2d})$). For the later argument, we also note that, from Lemma 4.13, the the prequantum transfer operator $\mathcal{L}_{A_{j,i}}$ commutes with the projection operators $t_\hbar^{(k)}$ defined in (4.24).

For the operator \hat{F}_\hbar on the Hilbert space $\mathcal{H}_\hbar^r(P)$, we confirm the following fact at this point, though we will give a more detailed description later.

Lemma 6.14. *The operator \mathbf{F}_\hbar defined in (6.6) extends uniquely to the bounded operator*

$$\mathbf{F}_\hbar : \bigoplus_{i=1}^{I_\hbar} \mathcal{H}_\hbar^r(\mathbb{R}^{2d}) \rightarrow \bigoplus_{i=1}^{I_\hbar} \mathcal{H}_\hbar^r(\mathbb{D}(\hbar^{1/2-\theta})) \quad (6.11)$$

and the operator norm is bounded by a constant independent of \hbar . Consequently the same result holds for the prequantum transfer operator $\hat{F}_\hbar : \mathcal{H}_\hbar^r(P) \rightarrow \mathcal{H}_\hbar^r(P)$.

Proof. From (6.7), it is enough to prove that the operators $\mathbf{F}_{j,i}$ for $1 \leq i, j \leq I_\hbar$ with $i \rightarrow j$ are bounded operators on $\mathcal{H}_\hbar^r(\mathbb{R}^{2d})$ and that the operator norms are bounded by a constant independent of \hbar . To see this, we express the diffeomorphism $f_{j,i}$ in (6.8) as a composition

$$f_{j,i} = a_{j,i} \circ g_{j,i} \circ B_{j,i} \quad (6.12)$$

where

- $a_{j,i} : \mathbb{R}^{2d} \rightarrow \mathbb{R}^{2d}$ is a translation on \mathbb{R}^{2d} ,
- $B_{j,i} : \mathbb{R}^{2d} \rightarrow \mathbb{R}^{2d}$ is a linear map of the form (4.16), i.e. $B_{j,i} = \begin{pmatrix} A & 0 \\ 0 & {}_tA^{-1} \end{pmatrix}$, with A an expanding map such that $\|A^{-1}\| \leq 1/\lambda$,
- $g_{j,i}$ is a diffeomorphism such that $\mathcal{G}_h = \{g_{j,i}\}_{1 \leq i, j \leq I_h}$ satisfies the condition (G1), (G2) and (G3) in Subsection 5.3.

This is possible because, if we let $B_{j,i}$ be the linearization of $f_{j,i}$ at the origin and let $a_{j,i} \in \mathcal{A}$ be the translation such that $a_{j,i}(f_{j,i}(0)) = 0$, then $a_{j,i}$, $B_{j,i}$ and $g_{j,i} := a_{j,i}^{-1} \circ f_{j,i} \circ B_{j,i}^{-1}$ satisfies the required conditions.

Remark 6.15. This decomposition of the diffeomorphism $f_{j,i}$ will be used later in the proof of Proposition 6.17 where we study more detailed properties of $f_{j,i}$.

From the expression (6.12) of $f_{j,i}$ above, the operator $\mathbf{F}_{j,i}$ is expressed as the composition

$$\mathbf{F}_{j,i} = \mathcal{L}^{(0)} \circ \mathcal{L}^{(1)} \circ \mathcal{L}^{(2)} \quad (6.13)$$

where $\mathcal{L}^{(0)} := \mathcal{L}_{a_{j,i}}$ and $\mathcal{L}^{(2)} := \mathcal{L}_{B_{j,i}}$ are the Euclidean prequantum transfer operators (2.7) for the diffeomorphism a_{ij} and B_{ij} respectively, while $\mathcal{L}^{(1)}$ is the operator of the form

$$\mathcal{L}^{(1)}u = \mathcal{L}_{g_{j,i}} \left(\left((e^{V \circ f \circ \kappa_i} \cdot \psi_{j,i} \cdot \chi_h) \circ B_{j,i}^{-1} \right) \cdot u \right)$$

with $\psi_{j,i}$ the function defined in (6.9). Note that the functions

$$(e^{V \circ f \circ \kappa_i} \cdot \psi_{j,i} \cdot \chi_h) \circ B_{j,i}^{-1} = (e^{V \circ f \circ \kappa_i} \cdot \chi_h) \circ B_{j,i}^{-1} \cdot (\psi_j \circ a_{j,i} \circ g_{j,i})$$

is supported on the disk $\mathbb{D}(2\hbar^{1/2-\theta})$, provided that \hbar is sufficiently small. Hence we may write the operator $\mathcal{L}^{(1)}$ as

$$\mathcal{L}^{(1)}u = \mathcal{L}_{g_{j,i}} \circ \mathcal{M}(\chi_h) \left(\left((e^{V \circ f \circ \kappa_i} \cdot \psi_{j,i} \cdot \chi_h) \circ B_{j,i}^{-1} \right) \cdot u \right).$$

From Lemma 4.13, $\mathcal{L}_{a_{j,i}} : \mathcal{H}_h^r(\mathbb{R}^{2d}) \rightarrow \mathcal{H}_h^r(\mathbb{R}^{2d})$ is a unitary operator. From Lemma 4.10, the operator $\mathcal{L}_{B_{j,i}} : \mathcal{H}_h^r(\mathbb{R}^{2d}) \rightarrow \mathcal{H}_h^{r,+}(\mathbb{R}^{2d})$ is bounded and the operator norm is bounded by a constant independent of \hbar . From Lemma 5.14 and Corollary 5.2, so is the operator $\mathcal{L}^{(1)} : \mathcal{H}_h^{r,+}(\mathbb{R}^{2d}) \rightarrow \mathcal{H}_h^r(\mathbb{R}^{2d})$, because

$$\mathcal{X}_h = \{(e^{V \circ f \circ \kappa_i} \cdot \psi_{j,i} \cdot \chi_h) \circ B_{j,i}^{-1}\}_{1 \leq i, j \leq I_h}, \quad \mathcal{G}_h = \{g_{j,i}\}_{1 \leq i, j \leq I_h} \quad (6.14)$$

satisfy respectively the conditions (C1), (C2) in Section 5.1 and (G1), (G2), (G3) in Section 5.3. \square

6.4 The main propositions

In this subsection, we give two key propositions which will give Theorem 1.18 as a consequence. To state the propositions, we introduce the projection operators

$$\mathbf{t}_h^{(k)} : \bigoplus_{i=1}^{I_h} \mathcal{H}_h^r(\mathbb{R}^{2d}) \rightarrow \bigoplus_{i=1}^{I_h} \mathcal{H}_h^r(\mathbb{R}^{2d}), \quad \mathbf{t}_h^{(k)}((u_i)_{i=1}^{I_h}) = (t_h^{(k)}(u_i))_{i=1}^{I_h},$$

for $0 \leq k \leq n$ and

$$\tilde{\mathbf{t}}_h : \bigoplus_{i=1}^{I_h} \mathcal{H}_h^r(\mathbb{R}^{2d}) \rightarrow \bigoplus_{i=1}^{I_h} \mathcal{H}_h^r(\mathbb{R}^{2d}), \quad \tilde{\mathbf{t}}_h((u_i)_{i=1}^{I_h}) = (\tilde{t}_h(u_i))_{i=1}^{I_h},$$

which are just applications of the projection operators $t_h^{(k)}$ and \tilde{t}_h introduced in (4.24) and (4.25) to each component. For brevity of notation, we set

$$\mathbf{t}_h^{(n+1)} = \tilde{\mathbf{t}}_h. \quad (6.15)$$

Then the set of operators $\{\mathbf{t}_h^{(k)}\}_{k=0}^{n+1}$ are complete sets of mutually commuting projection operators.

The following Proposition shows that the projectors $\mathbf{t}_h^{(k)}$ almost commute with the projector $(\mathbf{I}_h \circ \mathbf{I}_h^*)$.

Proposition 6.16. *There are constants $\epsilon > 0$ and $C > 0$, independent of \hbar , such that the following holds: We have that*

$$\left\| \mathbf{t}_h^{(k)} \circ (\mathbf{I}_h \circ \mathbf{I}_h^*) \right\|_{\bigoplus_{i=1}^{I_h} \mathcal{H}_h^{r,-}(\mathbb{R}^{2d}) \rightarrow \bigoplus_{i=1}^{I_h} \mathcal{H}_h^{r,+}(\mathbb{R}^{2d})} < C, \quad \text{and}$$

$$\left\| (\mathbf{I}_h \circ \mathbf{I}_h^*) \circ \mathbf{t}_h^{(k)} \right\|_{\bigoplus_{i=1}^{I_h} \mathcal{H}_h^{r,-}(\mathbb{R}^{2d}) \rightarrow \bigoplus_{i=1}^{I_h} \mathcal{H}_h^{r,+}(\mathbb{R}^{2d})} < C$$

for $0 \leq k \leq n$. (Hence the same statement holds as operators on $\bigoplus_{i=1}^{I_h} \mathcal{H}_h^r(\mathbb{R}^{2d})$.) Also we have, for the norm of the commutators, that

$$\left\| \left[\mathbf{t}_h^{(k)}, (\mathbf{I}_h \circ \mathbf{I}_h^*) \right] \right\|_{\bigoplus_{i=1}^{I_h} \mathcal{H}_h^{r,-}(\mathbb{R}^{2d}) \rightarrow \bigoplus_{i=1}^{I_h} \mathcal{H}_h^{r,+}(\mathbb{R}^{2d})} \leq C\hbar^\epsilon \quad (6.16)$$

for $0 \leq k \leq n$.

Proof. From Lemma 6.11, $\mathbf{I}_h \circ \mathbf{I}_h^*$ are bounded as operators from $\bigoplus_{i=1}^{I_h} \mathcal{H}_h^{r,+}(\mathbb{R}^{2d})$ to $\bigoplus_{i=1}^{I_h} \mathcal{H}_h^r(\mathbb{R}^{2d})$ (resp. from $\bigoplus_{i=1}^{I_h} \mathcal{H}_h^r(\mathbb{R}^{2d})$ to $\bigoplus_{i=1}^{I_h} \mathcal{H}_h^{r,-}(\mathbb{R}^{2d})$) and the operator norm is bounded by a constant independent of \hbar . From Lemma 4.11, so are the operators $\mathbf{t}_h^{(k)}$ as operators from $\bigoplus_{i=1}^{I_h} \mathcal{H}_h^{r,-}(\mathbb{R}^{2d})$ to $\bigoplus_{i=1}^{I_h} \mathcal{H}_h^r(\mathbb{R}^{2d})$ (resp. from $\bigoplus_{i=1}^{I_h} \mathcal{H}_h^r(\mathbb{R}^{2d})$ to $\bigoplus_{i=1}^{I_h} \mathcal{H}_h^{r,+}(\mathbb{R}^{2d})$). Hence

we obtain the first two inequalities. To prove (6.16), we take $\mathbf{u} = (u_i) \in \bigoplus_{i=1}^{I_h} \mathcal{H}_h^r(\mathbb{R}^{2d})$ arbitrarily. From the definition, we have

$$\mathbf{t}^{(k)} \circ (\mathbf{I}_h \circ \mathbf{I}_h^*)(\mathbf{u}) = \left(\sum_{i:U_i \cap U_j \neq \emptyset} t_h^{(k)} \circ \mathcal{M}(\psi_j) \circ \mathcal{L}_{\kappa_{j,i}} \circ \mathcal{M}(\chi_h)(u_i) \right)_{j=1}^{I_h}$$

and

$$(\mathbf{I}_h \circ \mathbf{I}_h^*) \circ \mathbf{t}^{(k)}(\mathbf{u}) = \left(\sum_{i:U_i \cap U_j \neq \emptyset} \mathcal{M}(\chi_j) \circ \mathcal{L}_{\kappa_{j,i}} \circ \mathcal{M}(\chi_h) \circ t_h^{(k)}(u_i) \right)_{j=1}^{I_h}.$$

Applying Corollary 5.17 and Corollary 5.7 to each components with the setting (6.10) and recalling Remark 6.13, we obtain (6.16). \square

The next Proposition stated for \mathbf{F}_h is now very close to Theorem 1.18.

Proposition 6.17. *There are constants $\epsilon > 0$ and $C > 0$ independent of \hbar such that*

$$\left\| \left[\mathbf{F}_h, \mathbf{t}_h^{(k)} \right] \right\|_{\bigoplus_{i=1}^{I_h} \mathcal{H}_h^{r,-}(\mathbb{R}^{2d}) \rightarrow \bigoplus_{i=1}^{I_h} \mathcal{H}_h^r(\mathbb{R}^{2d})} \leq C \hbar^\epsilon \quad \text{for } 1 \leq k \leq n+1. \quad (6.17)$$

Further there exists a constant $C_0 > 0$, which is independent of f , V and \hbar , such that

(1) For $0 \leq k \leq n+1$, it holds

$$\left\| \mathbf{t}_h^{(k)} \circ \mathbf{F}_h \circ \mathbf{t}_h^{(k)} \right\|_{\bigoplus_{i=1}^{I_h} \mathcal{H}_h^r(\mathbb{R}^{2d}) \rightarrow \bigoplus_{i=1}^{I_h} \mathcal{H}_h^{r,+}(\mathbb{R}^{2d})} \leq C_0 \sup (|e^V| \|Df|_{E_u}\|_{\min}^{-k} |\det Df|_{E_u}|^{-1/2})$$

(2) If $\mathbf{u} \in \bigoplus_{i=1}^{I_h} \mathcal{H}_h^r(\mathbb{R}^{2d})$ satisfies $\mathbf{I}_h \circ \mathbf{I}_h^*(\mathbf{u}) = \mathbf{u}$ and

$$\|\mathbf{u} - (\mathbf{I}_h \circ \mathbf{I}_h^*) \circ \mathbf{t}_h^{(k)}(\mathbf{u})\|_{\mathcal{H}_h^r} < \|\mathbf{u}\|_{\mathcal{H}_h^r} / 2 \quad \text{for some } 0 \leq k \leq n,$$

then we have

$$\|\mathbf{t}_h^{(k)} \circ \mathbf{F}_h \circ \mathbf{t}_h^{(k)}(\mathbf{u})\|_{\mathcal{H}_h^r} \geq C_0^{-1} \cdot \inf (|e^V| \|Df|_{E_u}\|_{\max}^{-k} |\det Df|_{E_u}|^{-1/2}) \cdot \|\mathbf{u}\|_{\mathcal{H}_h^r}.$$

Proof. We recall the argument in the proof of Lemma 6.14, in particular, the expression (6.13) of the operator \mathbf{F}_{ij} . Then we observe that, for each i, j such that $i \rightarrow j$,

- (i) From Proposition 4.9 and Lemma 4.13, the projection operators $t_h^{(k)}$ for $0 \leq k \leq n$ and \tilde{t}_h commute with the operator $\mathcal{L}^{(0)}$ and $\mathcal{L}^{(2)}$ (defined in (6.13)).
- (ii) From Lemma 4.13, the operator $\mathcal{L}^{(0)}$ is a unitary operator on $\mathcal{H}_h^r(\mathbb{R}^{2d})$ and also on $\mathcal{H}_h^{r,\pm}(\mathbb{R}^{2d})$.

(iii) From Proposition 5.12, the operator $\mathcal{L}^{(1)}$ extends to a bounded operator from $\mathcal{H}_h^{r,+}(\mathbb{R}^{2d})$ to $\mathcal{H}_h^r(\mathbb{R}^{2d})$ (resp. from $\mathcal{H}_h^r(\mathbb{R}^{2d})$ to $\mathcal{H}_h^{r,-}(\mathbb{R}^{2d})$) and the operator norm is bounded by a constant C_0 , provided that \hbar is sufficiently small.

(iv) Applying Proposition 4.9 to $\mathcal{L}^{(2)}$, we see that the operator $\mathcal{L}^{(2)}$ is a bounded operator on $\mathcal{H}_h^r(\mathbb{R}^{2d})$ and that

$$C_0^{-1} \|B_{j,i}|_{E^+}\|^{-k} \cdot |\det B_{j,i}|_{E^+}|^{-1/2} \leq \frac{\|\mathcal{L}^{(2)}u\|_{\mathcal{H}_h^r(\mathbb{R}^{2d})}}{\|u\|_{\mathcal{H}_h^r(\mathbb{R}^{2d})}} \leq C_0 \|B_{j,i}^{-1}|_{E^+}\|^k \cdot |\det B_{j,i}|_{E^+}|^{-1/2}$$

for $0 \neq u \in H'_k := \text{Im } t_h^{(k)}$ and for $0 \leq k \leq n$, where $E^+ = \mathbb{R}^{2d} \oplus \{0\}$. Further we have

$$\|\mathcal{L}^{(2)}u\|_{\mathcal{H}_h^r(\mathbb{R}^{2d})} \leq C_0 \|B_{j,i}^{-1}|_{E^+}\|^{n+1} |\det B_{j,i}|_{E^+}|^{-1/2} \|u\|_{\mathcal{H}_h^r(\mathbb{R}^{2d})}$$

for $u \in \tilde{H}' := \text{Im } \tilde{t}_h$. (We used (5.6) to see that the former term in (??) dominates the latter.)

(v) By simple comparison, we have

$$\begin{aligned} C_0^{-1} \cdot \inf (|e^V| \|Df|_{E_u}\|_{\max}^{-k} |\det Df|_{E_u}|^{-1/2}) \\ < e^{V_j} \cdot \|B_{j,i}^{-1}|_{E^+}\|^k \cdot |\det B_{j,i}|_{E^+}|^{-1/2} \\ < C_0 \sup (|e^V| \|Df|_{E_u}\|_{\min}^{-k} |\det Df|_{E_u}|^{-1/2}). \end{aligned}$$

Here (and henceforth) C_0 denotes positive constants independent of f , V and \hbar . For the operator $\mathcal{L}^{(1)}$, we further observe from the argument in Section 5 that

(vi) Applying Lemma 5.16 to the setting (6.14) and Lemma 6.8, we have

$$\|\mathcal{L}^{(1)} \circ t_h^{(k)} - e^{V_j} \cdot \mathcal{M}(\psi_{j,i} \cdot \chi_h) \circ B_{j,i}^{-1} \circ t_h^{(k)}\|_{\mathcal{H}_h^r(\mathbb{R}^{2d})} \leq C\hbar^\epsilon \quad \text{for } 0 \leq k \leq n \quad (6.18)$$

with some positive constants C and ϵ independent of \hbar .

(vii) Applying Corollary 5.17 to the setting (6.14), we have that

$$\|[\mathcal{L}^{(1)}, t_h^{(k)}]\|_{\mathcal{H}_h^r(\mathbb{R}^{2d})} \leq C\hbar^\epsilon \quad \text{for } 0 \leq k \leq n+1$$

with setting $t_h^{(n+1)} = \tilde{t}_h$ for the case $k = n+1$. This is true with $\mathcal{H}_h^r(\mathbb{R}^{2d})$ replaced by $\mathcal{H}_h^{r,\pm}(\mathbb{R}^{2d})$.

From the observations (i), (ii),(iv) and (vii) above, it follows

$$\|[\mathbf{F}_{j,i}, t_h^{(k)}]\|_{\mathcal{H}_h^{r,-}(\mathbb{R}^{2d}) \rightarrow \mathcal{H}_h^{r,+}(\mathbb{R}^{2d})} \leq C\hbar^\epsilon \quad \text{for } 0 \leq k \leq n+1.$$

This, together with (6.7), implies (6.17).

We prove Claim (1). Take $\mathbf{u} = (u_i)_{i=1}^{I_h} \in \bigoplus_{i=1}^{I_h} \mathcal{H}_h^r(\mathbb{R}^{2d})$ arbitrarily. Let $0 \leq k \leq n+1$ and set

$$v_{j,i} = t_h^{(k)} \circ \mathbf{F}_{j,i} \circ t_h^{(k)}(u_i), \quad u_{j,i} = \psi_{j,i} \cdot \chi_h \cdot u_i = (\psi_j \circ f_{j,i}) \cdot \chi_h \cdot u_i \quad (6.19)$$

for $1 \leq i, j \leq I_h$ such that $i \rightarrow j$. Suppose that $0 \leq k \leq n$. Then, using the expression (6.13) of $\mathbf{F}_{j,i}$, we obtain, by (vi), (vii) and Corollary 5.7,

$$\begin{aligned} \|v_{j,i}\|_{\mathcal{H}_h^r(\mathbb{R}^{2d})} &= \|t_h^{(k)} \circ \mathcal{L}^{(1)} \circ \mathcal{L}^{(2)} \circ t_h^{(k)} u_i\|_{\mathcal{H}_h^r(\mathbb{R}^{2d})} \\ &= e^{V_j} \cdot \|\mathcal{L}^{(2)} \circ \mathcal{M}(\psi_{j,i} \cdot \chi_h) \circ t_h^{(k)}(u_i)\|_{\mathcal{H}_h^r(\mathbb{R}^{2d})} + \mathcal{O}(\hbar^\epsilon \|u_i\|_{\mathcal{H}_h^r(\mathbb{R}^{2d})}) \\ &= e^{V_j} \cdot \|\mathcal{L}^{(2)} \circ t_h^{(k)}(u_{j,i})\|_{\mathcal{H}_h^r(\mathbb{R}^{2d})} + \mathcal{O}(\hbar^\epsilon \cdot \|u_i\|_{\mathcal{H}_h^r(\mathbb{R}^{2d})}) \end{aligned}$$

where $\mathcal{O}(\hbar^\epsilon \cdot \|u_i\|_{\mathcal{H}_h^r(\mathbb{R}^{2d})})$ denotes positive terms that are bounded by $C\hbar^\epsilon \cdot \|u_i\|_{\mathcal{H}_h^r(\mathbb{R}^{2d})}$. Hence, from (iv), we get the estimates

$$\|v_{j,i}\|_{\mathcal{H}_h^r} \leq C_0 \cdot e^{V_j} \cdot \|B_{j,i}|_{E^+}\|_{\min}^{-k} \cdot |\det B_{j,i}|_{E^+}|^{-1/2} \cdot \|t_h^{(k)}(u_{j,i})\|_{\mathcal{H}_h^r} + \mathcal{O}(\hbar^\epsilon \cdot \|u_i\|_{\mathcal{H}_h^r(\mathbb{R}^{2d})}) \quad (6.20)$$

and

$$\|v_{j,i}\|_{\mathcal{H}_h^r} \geq C_0^{-1} \cdot e^{V_j} \cdot \|B_{j,i}|_{E^+}\|_{\max}^{-k} \cdot |\det B_{j,i}|_{E^+}|^{-1/2} \cdot \|t_h^{(k)}(u_{j,i})\|_{\mathcal{H}_h^r} - \mathcal{O}(\hbar^\epsilon \cdot \|u_i\|_{\mathcal{H}_h^r(\mathbb{R}^{2d})}) \quad (6.21)$$

for $0 \leq k \leq n$.

Actually the upper estimate (6.21) can be strengthened by modifying the argument above, so that it also holds for $k = n+1$. (Note that the argument above is not true for $k = n+1$, because (6.18) does not hold in that case.) Indeed we can show that

$$\|v_{j,i}\|_{\mathcal{H}_h^{r,+}} \leq C_0 e^{V_j} \|B_{j,i}|_{E^+}\|_{\min}^{-k} |\det B_{j,i}|_{E^+}|^{-1/2} \cdot \|t_h^{(k)}(u_{j,i})\|_{\mathcal{H}_h^r} \quad (6.22)$$

for all $0 \leq k \leq n+1$. Let $B_0 : \mathbb{R}^{2d} \rightarrow \mathbb{R}^{2d}$ be the linear map defined by

$$B_0(x_+, x_-) = (\lambda_0 \cdot x_+, \lambda_0^{-1} \cdot x_-) \quad \text{for } (x_+, x_-) \in \mathbb{R}^{2d} = \mathbb{R}^d \oplus \mathbb{R}^d$$

where λ_0 is an absolute constant greater than 9. (Say $\lambda_0 = 10$.) Then we write the operator \mathbf{F}_{ij} as

$$\mathbf{F}_{ij} = \mathcal{L}^{(0)} \circ \mathcal{L}_{B_0} \circ \tilde{\mathcal{L}}^{(1)} \circ \tilde{\mathcal{L}}^{(2)} \quad \text{with setting } \tilde{\mathcal{L}}^{(1)} = \mathcal{L}_{B_0^{-1}} \circ \mathcal{L}^{(1)} \circ \mathcal{L}_{B_0}, \quad \tilde{\mathcal{L}}^{(2)} = \mathcal{L}_{B_0^{-1} \circ B_{j,i}}.$$

The operator \mathcal{L}_{B_0} is a bounded operator from $\mathcal{H}_h^{r,-}(\mathbb{R}^{2d})$ to $\mathcal{H}_h^{r,+}(\mathbb{R}^{2d})$, from Lemma 4.10. The operator $\tilde{\mathcal{L}}^{(1)}$ is a bounded operator from $\mathcal{H}_h^r(\mathbb{R}^{2d})$ to $\mathcal{H}_h^{r,-}(\mathbb{R}^{2d})$ and the operator norm is bounded by $C_0 e^{V_j}$, from Proposition 5.14. And the observation (iv) holds true with $\mathcal{L}^{(2)}$ replaced by $\tilde{\mathcal{L}}^{(2)}$. Hence we obtain (6.22).

From (6.1) in the choice of the coordinate system $\{\kappa_i\}_{i=1}^{I_h}$ (see Proposition 6.1) and from Lemma 4.14, we have

$$\|\mathbf{F} \circ \mathbf{t}^{(k)}(\mathbf{u})\|_{\mathcal{H}_h^r(\mathbb{R}^{2d})}^2 = \left\| \left(\sum_{i:i \rightarrow j} v_{j,i} \right)_j \right\|_{\mathcal{H}_h^r}^2 \leq C_0 \sum_{i,j:i \rightarrow j} \|v_{j,i}\|_{\mathcal{H}_h^r(\mathbb{R}^{2d})}^2 + \mathcal{O}(\hbar^\epsilon \cdot \|\mathbf{u}\|_{\mathcal{H}_h^r}^2)$$

and

$$\sum_{i,j:i \rightarrow j} \|u_{j,i}\|_{\mathcal{H}_h^r(\mathbb{R}^{2d})}^2 \leq C_0 \|\mathbf{u}\|_{\mathcal{H}_h^r}^2.$$

Hence we obtain Claim 1 as a consequence of (6.22) and the observation (v).

Remark 6.18. Because of the inconvenient property of the inner product $(\cdot, \cdot)_{\mathcal{H}_h^r}$, noted in the paragraph just before Lemma 4.14, the two inequalities above are not an immediate consequence of the estimate (6.1) on the intersection multiplicities of the supports of $v_{j,i}$ and $u_{j,i}$. We have to use Lemma 4.14.

We prove Claim (2). We continue the argument in the proof of Claim (1). Note that we already have the estimate (6.21) for each $v_{j,i}$. Below we show that the functions $v_{j,i}$ do not cancel out too much when we sum up them with respect to i such that $i \rightarrow j$. More precisely, we prove the estimate

$$\sum_{i,i',j:i \rightarrow j, i' \rightarrow j, i \neq i'} \operatorname{Re}(v_{j,i}, v_{j,i'})_{\mathcal{H}_h^r(\mathbb{R}^{2d})} \geq -C\hbar^\epsilon \cdot \|\mathbf{u}\|_{\mathcal{H}_h^r}^2 \quad (6.23)$$

where $\sum_{i,i',j:i \rightarrow j, i' \rightarrow j, i \neq i'}$ denotes the sum over $1 \leq i, i', j \leq I_h$ that satisfies $i \rightarrow j$, $i' \rightarrow j$ and $i \neq i'$. For $1 \leq j \leq I_h$, let $I(j)$ be the set of integers $1 \leq \ell \leq I_h$ such that there exists $1 \leq \ell', \ell'' \leq I_h$ satisfying $U_\ell \cap U_{\ell'} \neq \emptyset$, $U_{\ell'} \cap U_{\ell''} \neq \emptyset$ and $\ell' \rightarrow j$. Note that we have

$$\max_{1 \leq j \leq I_h} \#I(j) \leq C_0. \quad (6.24)$$

Consider $1 \leq i, i', j \leq I_h$ that satisfies $i \rightarrow j$, $i' \rightarrow j$ and $i \neq i'$. We express $v_{j,i}$ as

$$v_{j,i} = t_h^{(k)} \circ \mathcal{M}(e^{V \circ \kappa_j} \cdot \psi_j) \circ \mathcal{L}_{f_{j,i}} \circ t_h^{(k)} \left(\psi_i \cdot \sum_{\ell \in I(j)} \mathcal{L}_{\kappa_{i,\ell}}(u_\ell) \right).$$

We can of course write $v_{j,i'}$ in the same form with i replaced by i' , but we rewrite it as

$$v_{j,i'} = t_h^{(k)} \circ \mathcal{M}(e^{V \circ \kappa_j} \cdot \psi_j) \circ \mathcal{L}_{f_{j,i}} \circ \mathcal{L}_{\kappa_{i',\ell}} \circ t_h^{(k)} \left(\psi_{i'} \cdot \sum_{\ell \in I(j)} \mathcal{L}_{\kappa_{i',\ell}}(u_\ell) \right).$$

We change the order of operators on the right hand sides above, estimating the commutators by Corollary 5.7 and Corollary 5.17 and noting the relation $\kappa_{i,i'} \circ \kappa_{i',\ell} = \kappa_{i,\ell}$. Then we get

$$\left\| v_{j,i} - \mathcal{M}(e^{V \circ \kappa_{i'}} \cdot \psi_j \cdot \psi_i \circ f_{j,i}^{-1}) \circ \mathcal{L}_{f_{j,i}} \circ t_h^{(k)} \left(\sum_{\ell \in I(j)} \mathcal{L}_{\kappa_{i,\ell}}(u_\ell) \right) \right\|_{\mathcal{H}_h^r(\mathbb{R}^{2d})} \leq C\hbar^\epsilon \sum_{\ell \in I(j)} \|u_\ell\|_{\mathcal{H}_h^r(\mathbb{R}^{2d})}$$

and

$$\left\| v_{j,i'} - \mathcal{M}(e^{V \circ \kappa_{i'}} \cdot \psi_j \cdot \psi_{i'} \circ f_{j,i'}^{-1}) \circ \mathcal{L}_{f_{j,i}} \circ t_h^{(k)} \left(\sum_{\ell \in I(j)} \mathcal{L}_{\kappa_{i,\ell}}(u_\ell) \right) \right\|_{\mathcal{H}_h^r(\mathbb{R}^{2d})} \leq C\hbar^\epsilon \sum_{\ell \in I(j)} \|u_\ell\|_{\mathcal{H}_h^r(\mathbb{R}^{2d})}.$$

Therefore, by Corollary 5.4, we get

$$\begin{aligned}
\operatorname{Re}(v_{j,i}, v_{j,i'}) &\geq \left\| \mathcal{M} \left(e^{V \circ \kappa_{i'}} \psi_j \cdot \sqrt{\psi_i \circ f_{j,i}^{-1} \cdot \psi_{i'} \circ f_{j,i'}^{-1}} \right) \circ \mathcal{L}_{f_{j,i}} \circ t_h^{(k)} \left(\sum_{\ell \in I(j)} \mathcal{L}_{\kappa_{i,\ell}}(u_\ell) \right) \right\|_{\mathcal{H}_h^r(\mathbb{R}^{2d})}^2 \\
&\quad - C\hbar^\epsilon \cdot \sum_{\ell \in I(j)} \|u_\ell\|_{\mathcal{H}_h^r(\mathbb{R}^{2d})}^2 \\
&\geq -C\hbar^\epsilon \cdot \sum_{\ell \in I(j)} \|u_\ell\|_{\mathcal{H}_h^r(\mathbb{R}^{2d})}^2.
\end{aligned}$$

Summing up the both sides of the inequality above for all j, i, i' with $i \rightarrow j$, $i' \rightarrow j$ and $i \neq i'$ and using (6.24), we obtain (6.23).

From (6.23), (6.21) and the observation (iv), we get

$$\begin{aligned}
\|\mathbf{F}_h \circ \mathbf{t}^{(k)}(\mathbf{u})\|_{\mathcal{H}_h^r}^2 &= \sum_j \sum_{i, i'} \Re(v_{j,i}, v_{j,i'})_{\mathcal{H}_h^r(\mathbb{R}^{2d})} \\
&\geq \sum_{i, j: i \rightarrow j} \|v_{j,i}\|_{\mathcal{H}_h^r(\mathbb{R}^{2d})}^2 - C\hbar^\epsilon \|\mathbf{u}\|_{\mathcal{H}_h^r}^2 \\
&\geq \sum_{i, j: i \rightarrow j} C_0^{-1} e^{V_j} \cdot \|B_{j,i}|_{E^+}\|_{\max}^{-k} |\det B_{j,i}|_{E^+}|^{-1/2} \|t_h^{(k)}(u_{j,i})\|_{\mathcal{H}_h^r(\mathbb{R}^{2d})}^2 - C\hbar^\epsilon \|\mathbf{u}\|_{\mathcal{H}_h^r}^2 \\
&\geq C_0^{-1} \inf(|e^V| \|Df|_{E_u}\|_{\max}^{-k} |\det Df|_{E_u}|^{-1/2}) \sum_{i, j: i \rightarrow j} \|t_h^{(k)}(u_{j,i})\|_{\mathcal{H}_h^r(\mathbb{R}^{2d})}^2 - C\hbar^\epsilon \|\mathbf{u}\|_{\mathcal{H}_h^r}^2. \quad (6.25)
\end{aligned}$$

To finish the proof, we compare $\sum_{i, j: i \rightarrow j} \|t_h^{(k)}(u_{j,i})\|_{\mathcal{H}_h^r(\mathbb{R}^{2d})}^2$ and $\|\mathbf{u}\|_{\mathcal{H}_h^r}^2$. To this end, we use the assumptions in Claim (2), of course. From the assumption and (6.16) in Proposition 6.16, we have

$$\begin{aligned}
\|\mathbf{u}\|_{\mathcal{H}_h^r} &\leq 2 \cdot \|(\mathbf{I}_h \circ \mathbf{I}_h^*) \circ \mathbf{t}_h^{(k)}(\mathbf{u})\|_{\mathcal{H}_h^r} \leq 2 \|\mathbf{t}_h^{(k)} \circ (\mathbf{I}_h \circ \mathbf{I}_h^*)(\mathbf{u})\|_{\mathcal{H}_h^r} + C\hbar^\epsilon \|\mathbf{u}\|_{\mathcal{H}_h^r} \\
&= 2 \|\mathbf{t}_h^{(k)}(\mathbf{u})\|_{\mathcal{H}_h^r} + C\hbar^\epsilon \|\mathbf{u}\|_{\mathcal{H}_h^r}. \quad (6.26)
\end{aligned}$$

We also have, from Corollary 5.7, Lemma 4.14 and (6.1), that

$$\begin{aligned}
\|\mathbf{t}_h^{(k)}(\mathbf{u})\|_{\mathcal{H}_h^r}^2 &= \sum_i \|t_h^{(k)}(u_i)\|^2 = \left\| \sum_i t_h^{(k)} \left(\sum_{j:i \rightarrow j} u_{j,i} \right) \right\|_{\mathcal{H}_h^r(\mathbb{R}^{2d})}^2 \\
&= \sum_i \left\| t_h^{(k)} \left(\sum_{j:i \rightarrow j} \mathcal{M}(\psi_{j,i} \cdot \chi_h) u_i \right) \right\|_{\mathcal{H}_h^r(\mathbb{R}^{2d})}^2 \\
&\leq \sum_i \left\| \sum_{j:i \rightarrow j} \mathcal{M}(\psi_{j,i} \cdot \chi_h) \circ t_h^{(k)}(u_i) \right\|_{\mathcal{H}_h^r(\mathbb{R}^{2d})}^2 + C\hbar^\epsilon \|\mathbf{u}\|_{\mathcal{H}_h^r} \\
&\leq C_0 \sum_{i,j:i \rightarrow j} \|\mathcal{M}(\psi_{j,i} \cdot \chi_h) \circ t_h^{(k)}(u_j)\|_{\mathcal{H}_h^r(\mathbb{R}^{2d})}^2 + C\hbar^\epsilon \|\mathbf{u}\|_{\mathcal{H}_h^r} \\
&C_0 \sum_{i,j:i \rightarrow j} \|t_h^{(k)}(u_{j,i})\|_{\mathcal{H}_h^r(\mathbb{R}^{2d})}^2 + C\hbar^\epsilon \|\mathbf{u}\|_{\mathcal{H}_h^r}.
\end{aligned}$$

We conclude Claim (2) from these inequalities and (6.25). \square

6.5 Proof of Theorem 1.18

We finish the proof of Theorem 1.18. Actually we have almost done with the essential part of the proof. Below we give a formal argument to complete it. Let us begin with introducing the operators

$$\check{\tau}_h^{(k)} = \mathbf{I}_h^* \circ \mathbf{t}_h^{(k)} \circ \mathbf{I}_h : \mathcal{H}_h^r(P) \rightarrow \mathcal{H}_h^r(P)$$

for $0 \leq k \leq n+1$. From Proposition 6.16, these are bounded operators with operator norms bounded by a constant C independent of \hbar , and satisfy

$$\check{\tau}_h^{(1)} + \check{\tau}_h^{(2)} + \cdots + \check{\tau}_h^{(n)} + \check{\tau}_h^{(n+1)} = \text{Id}, \quad (6.27)$$

$$\|\check{\tau}_h^{(k)} \circ \check{\tau}_h^{(k)} - \check{\tau}_h^{(k)}\|_{\mathcal{H}_h^r(P)} \leq C\hbar^\epsilon \quad \text{for } 0 \leq k \leq n+1, \quad \text{and} \quad (6.28)$$

$$\|\check{\tau}_h^{(k)} \circ \check{\tau}_h^{(k')}\|_{\mathcal{H}_h^r(P)} \leq C\hbar^\epsilon \quad \text{for } 0 \leq k, k' \leq n+1 \text{ with } k \neq k' \quad (6.29)$$

for some constants $\epsilon > 0$ and $C > 0$.

We modify the operators $\check{\tau}_h^{(k)}$, $0 \leq k \leq n+1$, to get the projection operators $\tau_h^{(k)}$, $0 \leq k \leq n$, and $\tilde{\tau}_h = \tau_h^{(n+1)}$ in the statement of the theorem. The estimate (6.28) implies that the spectral set of the operator $\check{\tau}_h^{(k)} \circ \check{\tau}_h^{(k)} - \check{\tau}_h^{(k)}$ is contained in the disk $|z| \leq C\hbar^\epsilon$. By the spectral mapping theorem [Dunford-Schwartz, Part I, VII.3.11], the spectral set of the operators $\check{\tau}_h^{(k)}$ is contained in the union of two small disks around 0 and 1:

$$\mathbb{D}(0, C\hbar^\epsilon) \cup \mathbb{D}(1, C\hbar^\epsilon) \quad \text{where } \mathbb{D}(z, r) := \{w \in \mathbb{C} \mid |w - z| < r\}. \quad (6.30)$$

For $0 \leq k \leq n+1$, let $\hat{\tau}_h^{(k)}$ be the spectral projector of $\check{\tau}_h^{(k)}$ for the part of its spectral set contained in $\mathbb{D}(1, C\hbar^\epsilon)$. The next lemma should be easy to prove. (We provide a proof in the appendix for completeness.)

Lemma 6.19. $\|\check{\tau}_\hbar^{(k)} - \hat{\tau}_\hbar^{(k)}\|_{\mathcal{H}_\hbar^r} \leq C\hbar^\epsilon$ for some $C > 0$ independent of \hbar .

Thus we get the set of projection operators $\hat{\tau}_\hbar^{(k)}$ for $0 \leq k \leq n+1$, which approximate $\check{\tau}_\hbar^{(k)}$. As consequences of (6.27) and (6.29), we have

$$\|\text{Id} - (\hat{\tau}_\hbar^{(0)} + \hat{\tau}_\hbar^{(1)} + \cdots + \hat{\tau}_\hbar^{(n)} + \hat{\tau}_\hbar^{(n+1)})\|_{\mathcal{H}_\hbar^r(P)} \leq C\hbar^\epsilon \quad (6.31)$$

and

$$\|\hat{\tau}_\hbar^{(k)} \circ \hat{\tau}_\hbar^{(k')}\|_{\mathcal{H}_\hbar^r(P)} \leq C\hbar^\epsilon \quad \text{if } k \neq k'. \quad (6.32)$$

We set $\mathcal{H}_k := \text{Im } \hat{\tau}_\hbar^{(k)}$ for $0 \leq k \leq n+1$ and put $\tilde{\mathcal{H}} = \mathcal{H}_{n+1}$. We have

Lemma 6.20. *The Hilbert space $\mathcal{H}_\hbar^r(P)$ is decomposed into the direct sum:*

$$\mathcal{H}_\hbar^r(P) = \mathcal{H}_0 \oplus \mathcal{H}_1 \oplus \mathcal{H}_2 \oplus \cdots \oplus \mathcal{H}_n \oplus \tilde{\mathcal{H}}.$$

Proof. Since the sum $\hat{\tau}_\hbar^{(0)} + \hat{\tau}_\hbar^{(1)} + \cdots + \hat{\tau}_\hbar^{(n)} + \hat{\tau}_\hbar^{(n+1)}$ is invertible from (6.31), we can set

$$\tau_\hbar^{(k)} := \hat{\tau}_\hbar^{(k)} (\hat{\tau}_\hbar^{(0)} + \hat{\tau}_\hbar^{(1)} + \cdots + \hat{\tau}_\hbar^{(n)} + \hat{\tau}_\hbar^{(n+1)})^{-1} \quad \text{for } 0 \leq k \leq n+1.$$

We can express any $v \in \mathcal{H}_\hbar^r(P)$ as

$$v = \sum_{k=1}^{n+1} v_k \quad \text{with } v_k := \tau_\hbar^{(k)}(v) \in \mathcal{H}_k.$$

Thus the subspaces \mathcal{H}_k for $0 \leq k \leq n+1$ span the whole space $\mathcal{H}_\hbar^r(P)$. Uniqueness of such expression follows from (6.32). \square

From the argument in the proof above, the operator $\tau_\hbar^{(k)} : \mathcal{H}_\hbar^r(P) \rightarrow \mathcal{H}_k(P)$ for $0 \leq k \leq n+1$ are the projections to the subspace \mathcal{H}_k along other subspaces. Clearly we have

$$\|\tau_\hbar^{(k)} - \hat{\tau}_\hbar^{(k)}\|_{\mathcal{H}_\hbar^r} \leq C\hbar^\epsilon \quad \text{and hence} \quad \|\tau_\hbar^{(k)} - \check{\tau}_\hbar^{(k)}\|_{\mathcal{H}_\hbar^r} \leq C\hbar^\epsilon. \quad (6.33)$$

6.5.1 Proof of Claim (1)

We actually postpone the proof of Claim 1 to the last part of the next subsection, Subsection 6.6, where we show that the rank of $\tau_\hbar^{(k)}$ for $0 \leq k \leq n$ is same as those of some spectral projection operator for the rough Laplacian Δ_\hbar .

6.5.2 Proof of Claim (2)

For the proof of Claim (2)-(5), it is enough to prove the statements with $\tau_\hbar^{(k)}$ replaced by $\check{\tau}_\hbar^{(k)}$ because we have (6.33). The operator norm of $\check{\tau}_\hbar^{(k)} : \mathcal{H}_\hbar^r(P) \rightarrow \mathcal{H}_\hbar^r(P)$ is bounded by a constant independent of \hbar from Proposition 6.16, as we noted in the beginning of this subsection.

6.5.3 Proof of Claim (3) and (4)

From the definition of the operators \mathbf{F}_h and $\check{\tau}_h^{(k)}$, the following diagram commutes:

$$\begin{array}{ccc} \bigoplus_{i=1}^{I_h} \mathcal{H}_h^r(\mathbb{R}^{2d}) & \xrightarrow{\mathbf{I}_h \circ \mathbf{I}_h^* \circ \mathbf{t}_h^{(k)} \circ \mathbf{F}_h \circ \mathbf{t}_h^{(k')}} & \bigoplus_{i=1}^{I_h} \mathcal{H}_h^r(\mathbb{R}^{2d}) \\ \uparrow \mathbf{I}_h & & \uparrow \mathbf{I}_h \\ \mathcal{H}_h^r(P) & \xrightarrow{\check{\tau}_h^{(k)} \circ \hat{F}_h \circ \check{\tau}_h^{(k')}} & \mathcal{H}_h^r(P) \end{array}$$

Since the operator \mathbf{I}_h in the vertical direction is an isometric embedding, we have

$$\|\check{\tau}_h^{(k)} \circ \hat{F}_h \circ \check{\tau}_h^{(k')}\|_{\mathcal{H}_h^r(P)} \leq \|\mathbf{I}_h \circ \mathbf{I}_h^* \circ \mathbf{t}_h^{(k)} \circ \mathbf{F}_h \circ \mathbf{t}_h^{(k')}\|_{\mathcal{H}_h^r} = \|(\mathbf{I}_h \circ \mathbf{I}_h^* \circ \mathbf{t}_h^{(k)}) \circ (\mathbf{t}_h^{(k')} \circ \mathbf{F}_h \circ \mathbf{t}_h^{(k')})\|_{\mathcal{H}_h^r}.$$

From Proposition 6.16, we have

$$\left\| \mathbf{I}_h \circ \mathbf{I}_h^* \circ \mathbf{t}_h^{(k)} \right\|_{\mathcal{H}_h^r} \leq C_0 \quad \text{for } 0 \leq k \leq n$$

where C is a constant independent of h , f and V . These estimates give also

$$\left\| \mathbf{I}_h \circ \mathbf{I}_h^* \circ \mathbf{t}_h^{(n+1)} \right\|_{\bigoplus_{i=1}^{I_h} \mathcal{H}_h^{r,+}(\mathbb{R}^{2d}) \rightarrow \bigoplus_{i=1}^{I_h} \mathcal{H}_h^r(\mathbb{R}^{2d})} \leq C_0,$$

because of the relation $\mathbf{t}_h^{(n+1)} = \tilde{\mathbf{t}}_h = \text{Id} - \sum_{k=0}^n \mathbf{t}_h^{(k)}$ and Lemma 6.11. Now Claim (4) is an immediate consequence of Proposition 6.17 (1). From Proposition 6.17 and Lemma 6.14, we have

$$\|\mathbf{t}_h^{(k)} \circ \mathbf{F}_h \circ \mathbf{t}_h^{(k')}\|_{\mathcal{H}_h^r} \leq \|\mathbf{t}_h^{(k)} \circ \mathbf{t}_h^{(k')} \circ \mathbf{F}_h\|_{\mathcal{H}_h^r} + C\hbar^\epsilon = C\hbar^\epsilon$$

if $k \neq k'$ and hence Claim (3) follows.

6.5.4 Proof of Claim (5)

Take $0 \neq u \in \mathcal{H}_k = \text{Im } \tau_h^{(k)}$ for $0 \leq k \leq n$ arbitrarily and set $\mathbf{u} = \mathbf{I}_h(u)$. Then we have, from (6.33), that

$$\begin{aligned} \|\mathbf{u} - (\mathbf{I}_h \circ \mathbf{I}_h^*) \circ \mathbf{t}_h^{(k)}(\mathbf{u})\|_{\mathcal{H}_h^r} &= \|\mathbf{I}_h(u) - \mathbf{I}_h \circ \check{\tau}_h^{(k)}(u)\|_{\mathcal{H}_h^r} = \|u - \check{\tau}_h^{(k)}(u)\|_{\mathcal{H}_h^r(P)} \\ &= \|(\tau_h^{(k)} - \check{\tau}_h^{(k)})u\|_{\mathcal{H}_h^r(P)} \leq C\hbar^\epsilon \|u\|_{\mathcal{H}_h^r(P)} = C\hbar^\epsilon \|\mathbf{u}\|_{\mathcal{H}_h^r} \end{aligned}$$

Hence we can apply the second claim in Proposition 6.17 to \mathbf{u} and obtain Claim (5), noting that $\|\mathbf{F}_h \mathbf{u}\|_{\mathcal{H}_h^r} = \|F_h u\|_{\mathcal{H}_h^r(P)}$ by definition.

6.6 Proof of Theorem 2.21

In this subsection, we give a proof of Theorem 2.21 on the rough Laplacian. In former part of the proof, we consider a rough Laplacian $\tilde{\Delta}_\hbar$ constructed from local data instead of the geometric rough Laplacian Δ_\hbar , and prove the claims of Theorem 2.21 for $\tilde{\Delta}_\hbar$. In the latter part, we show that we can deform the rough Laplacian $\tilde{\Delta}_\hbar$ continuously to Δ_\hbar keeping the “band structure” of the eigenvalues. This will imply that the cardinality of eigenvalues in the first (or lowest) band coincides for $\tilde{\Delta}_\hbar$ and Δ_\hbar . We note at this moment that, for the argument on rough Laplacian below, we do not need Condition (2) in Proposition 6.1 (i.e. orthogonality of stable and unstable subspaces) in the choice of the coordinate charts $\{\kappa_i\}_{i=1}^{I_\hbar}$ in Proposition 6.1, that is, our argument below holds true for any choice of coordinate charts $\{\kappa_i\}_{i=1}^{I_\hbar}$ satisfying the conditions other than that condition.

We introduce a rough Laplacian $\tilde{\Delta}_\hbar$ acting on the space $C_N^\infty(P)$ of equivariant functions. We start from the operators on local data. Let

$$\Delta_\hbar : \bigoplus_{i=1}^{I_\hbar} \mathcal{D}(\Delta_\hbar) \rightarrow \bigoplus_{i=1}^{I_\hbar} L^2(\mathbb{R}^{2d}), \quad \Delta((u_i)_{i=1}^{I_\hbar}) = (\Delta_\hbar u_i)_{i=1}^{I_\hbar}$$

where Δ_\hbar denotes the Euclidean rough Laplacian on \mathbb{R}^{2d} defined in Subsection 4.6. The next proposition is an immediate consequence of Lemma 5.18. (So we omit the proof.)

Proposition 6.21. *There exist constants $C > 0$ and $\epsilon > 0$, independent of \hbar , such that*

$$\|[\Delta_\hbar, (\mathbf{I}_\hbar \circ \mathbf{I}_\hbar^*)]\|_{\bigoplus_{i=1}^{I_\hbar} (\mathcal{D}(\Delta_\hbar), \|\cdot\|_{\Delta_\hbar}) \rightarrow \bigoplus_{i=1}^{I_\hbar} L^2(\mathbb{R}^{2d})} \leq C\hbar^\theta. \quad (6.34)$$

We define a rough Laplacian $\tilde{\Delta}_\hbar$ acting on $C_N^\infty(P)$ by

$$\tilde{\Delta}_\hbar := \mathbf{I}_\hbar^* \circ \Delta_\hbar \circ \mathbf{I}_\hbar \quad : C_N^\infty(P) \rightarrow C_N^\infty(P). \quad (6.35)$$

Remark 6.22. Notice that this rough Laplacian operator $\tilde{\Delta}_\hbar$ is defined by gluing Euclidean rough Laplacian on local charts and does *not* coincide with the geometric rough Laplacian $\Delta_\hbar = D^*D$ with respect to a global metric on M defined in Subsection 2.4.

The operator $\tilde{\Delta}_\hbar$ defined above is a closed densely defined operator on $C_N^\infty(P)$. Its domain of definition is by definition

$$\mathcal{D}(\tilde{\Delta}_\hbar) = \{u \in L_N^2(P) \mid \|\tilde{\Delta}_\hbar u\|_{L^2} < \infty\},$$

which becomes a Hilbert space if we equip it with the inner product

$$(u, v)_{\tilde{\Delta}_\hbar} = (u, v)_{L^2} + (\tilde{\Delta}_\hbar u, \tilde{\Delta}_\hbar v)_{L^2}.$$

We will write $\|\cdot\|_{\tilde{\Delta}_\hbar}$ for the corresponding norm. It is easy to see the following Lemma. (So we omit the proof.)

Lemma 6.23. *The norm $\|\cdot\|_{\tilde{\Delta}_h}$ above is equivalent to the norm*

$$\|u\|'_{\tilde{\Delta}_h} = \left(\sum_{i=1}^{I_h} \|u_i\|_{\Delta_h}^2 \right)^{1/2}$$

defined in terms of local data, where $\|\cdot\|_{\Delta_h}$ on the right hand side denotes the norm defined in (4.27). Consequently we have

$$\mathcal{D}(\tilde{\Delta}_h) = \left\{ u \in L_N^2(P) \mid \sum_{i=1}^{I_h} \|u_i\|_{\Delta_h}^2 < \infty \right\}.$$

Below we are going to construct the spectral projectors for $\tilde{\Delta}_h$, corresponding to the ‘‘bands of eigenvalues’’. Again we start from local data: We consider the operators

$$\mathbf{q}_h^{(k)} : \bigoplus_{i=1}^{I_h} L^2(\mathbb{R}^{2d}) \rightarrow \bigoplus_{i=1}^{I_h} (\mathcal{D}(\Delta_h), \|\cdot\|_{\Delta_h}) \subset \bigoplus_{i=1}^{I_h} L^2(\mathbb{R}^{2d}), \quad \mathbf{q}_h^{(k)}((u_i)_{i=1}^{I_h}) = (q_h^{(k)}(u_i))_{i=1}^{I_h},$$

for $0 \leq k \leq n$, where $q_h^{(k)}$ is the operator defined in (4.28). Recall (4.29) for boundedness of these projection operators. The remainder is denoted as

$$\tilde{\mathbf{q}}_h = \mathbf{q}_h^{(n+1)} : \bigoplus_{i=1}^{I_h} L^2(\mathbb{R}^{2d}) \rightarrow \bigoplus_{i=1}^{I_h} L^2(\mathbb{R}^{2d}), \quad \tilde{\mathbf{q}}_h = \mathbf{q}_h^{(n+1)} := \text{Id} - (\mathbf{q}_h^{(1)} + \mathbf{q}_h^{(2)} + \cdots + \mathbf{q}_h^{(n)}).$$

The last operator restricts to a bounded operator

$$\tilde{\mathbf{q}}_h = \mathbf{q}_h^{(n+1)} : \bigoplus_{i=1}^{I_h} (\mathcal{D}(\Delta_h), \|\cdot\|_{\Delta_h}) \rightarrow \bigoplus_{i=1}^{I_h} (\mathcal{D}(\Delta_h), \|\cdot\|_{\Delta_h}).$$

We next introduce the operators

$$\check{\lambda}_h^{(k)} := \mathbf{I}_h^* \circ \mathbf{q}_h^{(k)} \circ \mathbf{I}_h : L_N^2(P) \rightarrow (\mathcal{D}(\tilde{\Delta}_h), \|\cdot\|_{\tilde{\Delta}_h})$$

for $0 \leq k \leq n$. These are bounded operators and the operator norms are bounded by a constant independent of \tilde{h} . For $k = n + 1$, we set

$$\check{\lambda}_h^{(n+1)} := \mathbf{I}_h^* \circ \mathbf{q}_h^{(n+1)} \circ \mathbf{I}_h = \text{Id} - (\check{\lambda}_h^{(0)} + \check{\lambda}_h^{(1)} + \cdots + \check{\lambda}_h^{(n)}). \quad (6.36)$$

Further we can prove the estimates

$$\left\| \check{\lambda}_h^{(k)} \circ \check{\lambda}_h^{(k)} - \check{\lambda}_h^{(k)} \right\|_{L_N^2(P) \rightarrow (\mathcal{D}(\tilde{\Delta}_h), \|\cdot\|_{\tilde{\Delta}_h})} \leq C\tilde{h}^\epsilon \quad \text{for } 0 \leq k \leq n + 1 \quad (6.37)$$

and

$$\left\| \check{\lambda}_h^{(k)} \circ \check{\lambda}_h^{(k')} \right\|_{L_N^2(P) \rightarrow (\mathcal{D}(\tilde{\Delta}_h), \|\cdot\|_{\tilde{\Delta}_h})} \leq C\tilde{h}^\epsilon \quad \text{for } 0 \leq k, k' \leq n + 1 \text{ with } k \neq k' \quad (6.38)$$

for some constants $\epsilon > 0$ and $C > 0$. (For the case where either of k or k' equals $n + 1$, use the definition (6.36) to check (6.38).)

Now we proceed in parallel to the argument in Subsection 6.5 and obtain the following lemma.

Lemma 6.24. *There exist a decomposition of the Hilbert space $(\mathcal{D}(\tilde{\Delta}_h), \|\cdot\|_{\tilde{\Delta}_h})$*

$$\mathcal{D}(\tilde{\Delta}_h) = \mathfrak{H}_0 \oplus \mathfrak{H}_1 \oplus \mathfrak{H}_2 \oplus \cdots \oplus \mathfrak{H}_n \oplus \tilde{\mathfrak{H}}$$

and that of $L_N^2(P)$

$$L_N^2(P) = \mathfrak{H}_0 \oplus \mathfrak{H}_1 \oplus \mathfrak{H}_2 \oplus \cdots \oplus \mathfrak{H}_n \oplus \bar{\mathfrak{H}}$$

where $\bar{\mathfrak{H}}$ is the closure of $\tilde{\mathfrak{H}}$ in $L_N^2(P)$. The subspaces \mathfrak{H}_k for $0 \leq k \leq n$ are of finite dimension. If we write $\lambda_h^{(k)}$ for projection operators to \mathfrak{H}_k (resp. to $\bar{\mathfrak{H}}$ in the case $k=n+1$) along other subspaces, then we have

$$(1) \left\| \lambda_h^{(k)} - \tilde{\lambda}_h^{(k)} \right\|_{(\mathcal{D}(\tilde{\Delta}_h), \|\cdot\|_{\tilde{\Delta}_h})} \leq C\hbar^\epsilon \text{ for } 0 \leq k \leq n,$$

(2) if $k \neq k'$,

$$\left\| \lambda_h^{(k)} \circ \hbar \tilde{\Delta}_h \circ \lambda_h^{(k')} \right\|_{(\mathcal{D}(\Delta_h), \|\cdot\|_{\Delta_h}) \rightarrow L_N^2(P)} \leq C\hbar^\epsilon,$$

(3) if $0 \leq k \leq n$,

$$\left\| \lambda_h^{(k)} \circ \hbar \tilde{\Delta}_h \circ \lambda_h^{(k)} - \left(\frac{d}{2} + k \right) \cdot \lambda_h^{(k)} \right\|_{(\mathcal{D}(\Delta_h), \|\cdot\|_{\Delta_h}) \rightarrow L_N^2(P)} \leq C \cdot \hbar^\epsilon,$$

(4) for $k = n + 1$, we have

$$\left\| \lambda_h^{(n+1)} \circ \hbar \tilde{\Delta}_h(u) \right\|_{L^2} \geq \left(\frac{d}{2} + k + 1 - C\hbar^\epsilon \right) \|u\|_{L^2} \text{ for } u \in \tilde{\mathfrak{H}} = \mathfrak{H}^{(n+1)}.$$

Therefore, by the general theorem on perturbation of closed linear operators [23, chap.IV, th. 1.16], we obtain an analogue of Theorem 2.21 for the rough Laplacian $\tilde{\Delta}_h$.

Theorem 6.25. *There exists a small constant $\epsilon > 0$ such that, for any $\alpha > 0$, we have*

$$\text{dist} \left(\text{Spec} \left(\hbar \tilde{\Delta}_h \right) \cap [0, \alpha], \left\{ \frac{d}{2} + k, k \in \mathbb{N} \right\} \cap [0, \alpha] \right) \leq \hbar^\epsilon$$

when \hbar is sufficiently small.

Further we have

Lemma 6.26. *For sufficiently small $\hbar > 0$, we have*

$$\dim \mathcal{H}_k = \dim \mathfrak{H}_k \asymp \hbar^{-d} \asymp N^d \text{ for } 0 \leq k \leq n.$$

Proof. To prove the equality $\dim \mathcal{H}_k = \dim \mathfrak{H}_\mathfrak{k}$, it is enough to check that

$$\text{rank } \tau_{\hbar}^{(k)} = \text{rank } \lambda_{\hbar}^{(k)} \quad \text{for } 0 \leq k \leq n,$$

or equivalently

$$\text{rank } \bigoplus_{i=0}^k \tau_{\hbar}^{(i)} = \text{rank } \bigoplus_{i=0}^k \lambda_{\hbar}^{(i)} \quad \text{for } 0 \leq k \leq n.$$

The latter relation would follow if we show

$$c \left\| \left(\bigoplus_{i=0}^k \lambda_{\hbar}^{(i)} \right) u \right\|_{L^2} \leq \left\| \left(\bigoplus_{i=0}^k \lambda_{\hbar}^{(i)} \right) \circ \left(\bigoplus_{i=0}^k \tau_{\hbar}^{(i)} \right) \circ \left(\bigoplus_{i=0}^k \lambda_{\hbar}^{(i)} \right) u \right\|_{L^2} \leq C \left\| \left(\bigoplus_{i=0}^k \lambda_{\hbar}^{(i)} \right) u \right\|_{L^2}$$

and

$$c \left\| \left(\bigoplus_{i=0}^k \tau_{\hbar}^{(i)} \right) u \right\|_{\mathcal{H}_{\hbar}^r} \leq \left\| \left(\bigoplus_{i=0}^k \tau_{\hbar}^{(i)} \right) \circ \left(\bigoplus_{i=0}^k \lambda_{\hbar}^{(i)} \right) \circ \left(\bigoplus_{i=0}^k \tau_{\hbar}^{(i)} \right) u \right\|_{\mathcal{H}_{\hbar}^r} \leq C \left\| \left(\bigoplus_{i=0}^k \tau_{\hbar}^{(i)} \right) u \right\|_{\mathcal{H}_{\hbar}^r}$$

for a constant $0 < c < C$, independent of \hbar . But these are immediate consequences of Proposition 4.19 (and the construction of the projection operators $\tau_{\hbar}^{(k)}$ and $\lambda_{\hbar}^{(k)}$).

It remains to show

$$\text{rank } \lambda_{\hbar}^{(k)} \asymp \hbar^{-d} \quad \text{for } 0 \leq k \leq n. \quad (6.39)$$

For each point $x \in M$, we associate a smooth function

$$\varphi_x := \check{\lambda}_{\hbar}^{(k)}(\delta_x)$$

where δ_x is the Dirac measure at the point x . (The right hand side is well-defined and give a smooth function that concentrates around x .) Consider positive constants $0 < c < C$ and take a finite subset of points Q_{\hbar} on M so that the mutual distance between two points in Q_{\hbar} is in between $c \cdot \hbar^{1/2}$ and $C \cdot \hbar^{1/2}$. Let

$$\mathcal{Q}_{\hbar} = \{ \lambda_{\hbar}^{(k)}(\varphi_x) \mid x \in Q_{\hbar} \} \subset \text{Im } \lambda_{\hbar}^{(k)}.$$

Note that $\lambda_{\hbar}^{(k)}(\varphi_x)$ is close to φ_x from Claim 1 in Proposition 6.24. It is not difficult to check that

- (1) if we let the constants c, C large, the subset \mathcal{Q}_{\hbar} is linearly independent, and
- (2) if we let the constants c, C small, the subset \mathcal{Q}_{\hbar} span the whole space $\text{Im } \lambda_{\hbar}^{(k)}$.

Indeed, to prove (1), we have only to observe that, if the constants c, C are sufficiently large, the L^2 -scalar product between different elements $\varphi_x, \varphi_{x'}$ in \mathcal{Q}_{\hbar} decay rapidly with respect the distance between the corresponding points x, x' (relative to the size $\hbar^{1/2}$). To prove (2), we see that, if the constants c, C are sufficiently small, any element of $\text{Im } \lambda_{\hbar}^{(k)}$ is well approximated by the linear combinations of the element in \mathcal{Q}_{\hbar} and then, by successive approximation, it is really contained in the subspace spanned by \mathcal{Q}_{\hbar} . Clearly the claims (1) and (2) imply (6.39). \square

Finally we complete the proof of Theorem 2.21 and Claim 1 of Theorem 1.18. We show that the operator $\tilde{\Delta}_\hbar$ is continuously deformed to the geometric Laplacian Δ_\hbar , keeping the band structure described in Theorem 6.25. For this purpose, we take a continuous one-parameter family of splitting of the tangent bundle

$$TM = E_+^{(t)} \oplus E_-^{(t)}$$

with $t \in [0, 1]$ the parameter, such that

- $E_+^{(0)} = E_u$ and $E_-^{(0)} = E_s$, that is, the splitting above coincides with the hyperbolic splitting associated to f when $t = 0$.
- the sub-bundles $E_\pm^{(1)}$ for $t = 1$ are C^∞ and orthogonal with respect to the Riemann metric g on M .

Then we consider a continuous deformation $\{\kappa_{i,t}\}_{i=1}^{I_\hbar}$ of the atlas $\{\kappa_i\}_{i=1}^{I_\hbar}$ and, correspondingly, the deformation $\{\psi_{i,t}; 1 \leq i \leq I_\hbar\}$ of the family $\{\psi_i; 1 \leq i \leq I_\hbar\}$ of functions so that all the conditions in Proposition 6.1 hold uniformly for $t \in [0, 1]$, but with the sub-bundles E^u and E^s in the condition (2) replaced by E_+^t and E_-^t .

We consider the rough Laplacian $\tilde{\Delta}_{\hbar,t}$ defined, similarly to $\tilde{\Delta}_\hbar$ in (6.35), from the Euclidean rough Laplacian on local charts $\{\kappa_{i,t}\}_{i=1}^{I_\hbar}$ and the family of functions $\{\psi_{i,t}; 1 \leq i \leq I_\hbar\}$. The argument in the former part of this subsection holds true uniformly for $t \in [0, 1]$, that is, we can consider the spectral projection operators $\lambda_{\hbar,t}^{(k)}$, $0 \leq k \leq n+1$, for $\tilde{\Delta}_{\hbar,t}$, which corresponds to $\lambda_\hbar^{(k)}$ for $\tilde{\Delta}_\hbar$. Since the deformation is continuous, we see by homotopy argument that $\text{rank } \lambda_{\hbar,\rho}^{(k)} = \dim \mathfrak{H}_{k,\rho}$ is constant for $t \in [0, 1]$. In particular we have $\text{rank } \lambda_{\hbar,1}^{(k)} = \text{rank } \lambda_{\hbar,0}^{(k)}$.

Note that the operator $\tilde{\Delta}_{\hbar,1}$ is close to the geometric rough Laplacian Δ_\hbar . In fact, since the derivative $(D\kappa_{i,1})_0$ at the origin is an isometry with respect to the Euclidean metric on \mathbb{R}^{2d} and the Riemann metric g on M , we can check by using the local expression of the geometric rough Laplacian in Proposition 2.39 that we have

$$\|\hbar\tilde{\Delta}_{\hbar,1} - \hbar\Delta_\hbar : (\mathcal{D}(\Delta_\hbar), \|\cdot\|_{\Delta_\hbar}) \rightarrow L_N^2(P)\| \leq C\hbar^\epsilon.$$

Therefore, by the perturbation theorem [23, chap.IV, th. 1.16] for closed operators, we obtain the ‘‘band structure’’ stated in the former part of Theorem 2.21. We can also see that the number of eigenvalues of the geometric rough Laplacian $\Delta_\hbar = D^*D$ in the k -th band is same as that for $\tilde{\Delta}_\hbar$. Hence, from Lemma 6.26, we obtain the rough upper and lower bound on the rank of spectral projectors \mathfrak{P}_k in Theorem 2.21 and also Claim (1) of Theorem 1.18.

A Appendix

A.1 Proof of Theorem 1.16 from Theorem 1.18

Assume that Theorem 1.18 holds true. We will prove Theorem 1.16. Let $m \geq 1$ and apply Theorem 1.18 to f^m and \hat{F}_N^m . On the right hand side of (1.26) and (1.27) we have

$$r_{k,m}^+ := \sup_{x \in M} \left(e^{V_m(x)} \left\| (Df_{/E_u}^m(x))^{-1} \right\|^{+k} |\det Df_{/E_u}^m(x)|^{-1/2} \right)$$

and

$$r_{k,m}^- := C_0^{-1} \cdot \inf_{x \in M} \left(e^{V_m(x)} \left\| Df_{/E_u}^m(x) \right\|^{-k} |\det Df_{/E_u}^m(x)|^{-1/2} \right).$$

The following arguments using Neumann series for resolvents are very standard. For $0 \leq k \leq n+1$, let $A_k := \pi_k \circ \hat{F}_N \circ \pi_k : \mathcal{H}_k \rightarrow \mathcal{H}_k$. From Theorem 1.18 we have $\|A_k\| \leq r_{k,m}^+$ and for $k \leq n$ we have also $\|A_k^{-1}\| \leq (r_{k,m}^-)^{-1}$. For convenience, we define $r_{k,m}^- = 0$ if $k = n+1$.

Lemma A.1. *Let $0 \leq k \leq n+1$, and $z \in \mathbb{C}$ with $|z| < r_{k,m}^-$ or $|z| > r_{k,m}^+$. Then $(z - A_k)$ is invertible and its inverse $R_{A_k}(z)$, the resolvent operator, satisfies*

$$\|R_{A_k}(z)\| \leq \frac{1}{\text{dist}(|z|, [r_{k,m}^-, r_{k,m}^+])}$$

Proof. If $|z| > r_{k,m}^+ \geq \|A_k\|$, then $\|\frac{A_k}{z}\| < 1$ and we can write a convergent Neuman series for

$$R_{A_k}(z) = (z - A_k)^{-1} = \frac{1}{z} \left(1 - \frac{A_k}{z} \right)^{-1}$$

giving

$$\|R_{A_k}(z)\| \leq \frac{1}{|z|} \left(1 - \frac{\|A_k\|}{|z|} \right)^{-1} = \frac{1}{(|z| - \|A_k\|)} \leq \frac{1}{\text{dist}(|z|, [r_{k,m}^-, r_{k,m}^+])}$$

Similarly if $|z| < r_{k,m}^- \leq \|A_k^{-1}\|^{-1}$ then we have $\|zA_k^{-1}\| < 1$ and a convergent Neuman series for

$$R_{A_k}(z) = (z - A_k)^{-1} = -A_k^{-1} (1 - zA_k^{-1})^{-1}$$

giving

$$\|R_{A_k}(z)\| \leq \|A_k^{-1}\| (1 - |z| \|A_k^{-1}\|)^{-1} = \frac{1}{(\|A_k^{-1}\|^{-1} - |z|)} \leq \frac{1}{\text{dist}(|z|, [r_{k,m}^-, r_{k,m}^+])}$$

□

Thus $A := A_0 \oplus A_1 \oplus \dots \oplus A_{n+1}$ is an operator $A : \mathcal{H}_\hbar^r \rightarrow \mathcal{H}_\hbar^r$ and its resolvent satisfies

$$\|R_A(z)\| \leq \sum_{k=0}^{n+1} \frac{1}{\text{dist}(|z|, [r_{k,m}^-, r_{k,m}^+])} \quad (\text{A.1})$$

The operator \hat{F}_N^m can be written $\hat{F}_N^m = A + B$ with $\|B\| \leq C_0 \hbar^\varepsilon$. We use a standard ‘‘perturbation argument’’ [36, p.311] to show that if $z \in \mathbb{C}$ is such that $\|R_A(z)\| \|B\| < 1$ then it is not in the spectrum of \hat{F}_N^m : $z \notin \sigma(\hat{F}_N^m)$. For this we write

$$\begin{aligned} z - \hat{F}_N^m &= z - A - B \\ &= (z - A)(1 - (z - A)^{-1}B) \end{aligned}$$

hence

$$(z - \hat{F}_N^m)^{-1} = (1 - (z - A)^{-1}B)^{-1}(z - A)^{-1}$$

and using Neumann series we deduce:

$$\|R_{\hat{F}_N^m}(z)\| \leq (1 - \|R_A(z)\| \|B\|)^{-1} \|R_A(z)\|$$

hence z is in the resolvent set of \hat{F}_N^m , i.e. $z \notin \sigma(\hat{F}_N^m)$.

From (A.1) and $\|B\| \leq C_0 \hbar^\varepsilon$, we see that the condition $\|R_A(z)\| \|B\| < 1$ is satisfied if for every $k \in \{0, n+1\}$

$$\text{dist}(|z|, [r_{k,m}^-, r_{k,m}^+]) > (n+1) C_0 \hbar^\varepsilon.$$

and then $z \notin \sigma(\hat{F}_N^m)$. Taking the power $1/m$ and making $\hbar \rightarrow 0$ first and then $m \rightarrow \infty$ we get the result (1.23).

A.2 Proof of Lemma 6.19

It is enough to show that there exist constants $C_1 > 0$ and $C_2 > 0$ independent of \hbar such that

$$\|(\rho \cdot \text{Id} - \tilde{\tau}_\hbar^{(k)})^{-1}\|_{\mathcal{H}_\hbar^r(P)} \leq \frac{C_1}{\min\{|\rho|, |1 - \rho|\}} \quad (\text{A.2})$$

whenever $\rho \in \mathbb{C}$ satisfies

$$\min\{|\rho|, |1 - \rho|\} \geq C_2 \hbar^\varepsilon. \quad (\text{A.3})$$

In fact, the estimate (A.2) would imply that, for $r_0 = C_2 \hbar^\varepsilon$,

$$\begin{aligned} \|\tilde{\tau}_\hbar^{(k)} - \hat{\tau}_\hbar^{(k)}\|_{\mathcal{H}_\hbar^r} &= \left\| \int_{|\rho-1|=r_0, |\rho|=r_0} \rho(\rho \cdot \text{Id} - \tilde{\tau}_\hbar^{(k)})^{-1} d\rho - \int_{|\rho-1|=r_0} (\rho \cdot \text{Id} - \tilde{\tau}_\hbar^{(k)})^{-1} d\rho \right\|_{\mathcal{H}_\hbar^r(P)} \\ &\leq \left\| \int_{|\rho|=r_0} \rho(\rho \cdot \text{Id} - \tilde{\tau}_\hbar^{(k)})^{-1} d\rho \right\|_{\mathcal{H}_\hbar^r} + \left\| \int_{|\rho-1|=r_0} (\rho - 1)(\rho \cdot \text{Id} - \tilde{\tau}_\hbar^{(k)})^{-1} d\rho \right\|_{\mathcal{H}_\hbar^r(P)} \\ &\leq 2C_1 \cdot r_0 = 2C_1 \cdot C_2 \cdot \hbar^\varepsilon. \end{aligned}$$

To prove (A.2), we may and do assume $|\rho| \leq 2\|\tilde{\tau}_h^{(k)}\|_{\mathcal{H}_h^r(P)}$ because the claim is trivial otherwise. Take $u \in \mathcal{H}_h^r(P)$ arbitrarily. From the assumption made in the preceding sentence, we have

$$\begin{aligned} \|\rho^2 u - \tilde{\tau}_h^{(k)} \circ \tilde{\tau}_h^{(k)} u\|_{\mathcal{H}_h^r(P)} &\leq \|\rho^2 u - \rho \tilde{\tau}_h^{(k)} u\|_{\mathcal{H}_h^r(P)} + \|\tilde{\tau}_h^{(k)}(\rho u - \tilde{\tau}_h^{(k)} u)\|_{\mathcal{H}_h^r(P)} \\ &\leq (|\rho| + \|\tilde{\tau}_h^{(k)}\|_{\mathcal{H}_h^r(P)}) \cdot \|(\rho \cdot \text{Id} - \tilde{\tau}_h^{(k)})u\|_{\mathcal{H}_h^r(P)} \\ &\leq 3\|\tilde{\tau}_h^{(k)}\|_{\mathcal{H}_h^r(P)} \cdot \|(\rho \cdot \text{Id} - \tilde{\tau}_h^{(k)})u\|_{\mathcal{H}_h^r(P)}. \end{aligned}$$

On the other hand, from (6.28), we have

$$\begin{aligned} \|\rho^2 u - \tilde{\tau}_h^{(k)} \circ \tilde{\tau}_h^{(k)} u\|_{\mathcal{H}_h^r(P)} &\geq \|\rho^2 u - \tilde{\tau}_h^{(k)} u\|_{\mathcal{H}_h^r(P)} - C \cdot \hbar^\epsilon \|u\|_{\mathcal{H}_h^r(P)} \\ &\geq |\rho(\rho - 1)| \cdot \|u\|_{\mathcal{H}_h^r(P)} - \|(\rho \cdot \text{Id} - \tilde{\tau}_h^{(k)})u\|_{\mathcal{H}_h^r(P)} - C \cdot \hbar^\epsilon \|u\|_{\mathcal{H}_h^r(P)}. \end{aligned}$$

Hence we obtain the estimate

$$(|\rho(\rho - 1)| - C' \cdot \hbar^\epsilon) \cdot \|u\|_{\mathcal{H}_h^r} \leq (3\|\tilde{\tau}_h^{(k)}\|_{\mathcal{H}_h^r(P)} + 1) \cdot \|(\rho \cdot \text{Id} - \tilde{\tau}_h^{(k)})u\|_{\mathcal{H}_h^r(P)}$$

for some constant $C' > 0$. If we choose C_2 so large that $C_2 > 4C'$, the assumption (A.3) implies

$$|\rho(\rho - 1)| - C \cdot \hbar^\epsilon \geq \frac{1}{2} \min\{|\rho|, |1 - \rho|\} - C \cdot \hbar^\epsilon \geq \frac{1}{4} \min\{|\rho|, |1 - \rho|\}.$$

Therefore, with such choice of C_2 , the inequality (A.2) holds if we let

$$C_1 > 4 \cdot (3\|\tilde{\tau}_h^{(k)}\|_{\mathcal{H}_h^r(P)} + 1),$$

recalling that $\|\tilde{\tau}_h^{(k)}\|_{\mathcal{H}_h^r(P)}$ is bounded by a constant independent of \hbar .

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