HAL open science

Generalized Riesz Products on the Bohr compactification of

El Houcein El Abdalaoui

To cite this version:

El Houcein El Abdalaoui. Generalized Riesz Products on the Bohr compactification of . 2012. hal-00703137v4

HAL Id: hal-00703137

https://hal.science/hal-00703137v4

Preprint submitted on 5 Jun 2012

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

GENERALIZED RIESZ PRODUCTS ON THE BOHR COMPACTIFICATION OF $\mathbb{R}^{(*)}$

E. H. EL ABDALAOUI

Abstract. We study a class of generalized Riesz products connected to the spectral type of some class of rank one flows on $\mathbb{R}$. Applying a Central Limit Theorem of Kac, we exhibit a large class of singular generalized Riesz products on the Bohr compactification of $\mathbb{R}$. Moreover, we discuss the problem of the flat polynomials in this setting.

*Dedicated to Professors Jean-Paul Thouvenot and Bernard Host.

AMS Subject Classifications (2010) Primary: 42A05, 42A55; Secondary: 11L03, 42A61.

Key words and phrases: Generalized Riesz products, almost periodic functions, Bohr compactification, mean value, Besicovitch space, Kac Central Limit Theorem, Kakutani criterion .

1. Introduction

We stress that the main purpose of this paper is to extend and complete the study of the notion of generalized Riesz product associated to the rank one flows on $\mathbb{R}$ formulated in the same manner as Peyriére in [3]. The authors in [3] mentioned that Peyriére extended the notion of Riesz product to the real line using a class of kernel functions. Furthermore, it is noted by Peyriére in his pioneer paper [41] that an alternative extension of the classical Riesz products can be done using the Bohr compactification of $\mathbb{R}$.

Indeed, it is usual that the extension of some notions from the periodic setting to the almost periodic deals with the Bohr compactification $\mathbf{b}\mathbb{R}$ of $\mathbb{R}$ (or more generally, the Bohr compactification of Local Abelian Groups). The Bohr compactification plays in the almost periodic case the same role played by the torus $\mathbb{T} \stackrel{\text{def}}{=} \{t \in \mathbb{C}, |t| = 1\}$ in the periodic case as the domain of the fast scale variables. As opposed to the torus, the Bohr compactification is often a non-separable compact topological space and this lack of separability is a source of difficulties in trying to adapt the arguments from the periodic context to the almost periodic one. Peyriére [41] mentioned this difficulty and introduced the Riesz products on $\mathbb{R}$ associated to some class of kernels.

Notice that here we use the notion of separability used in Hirotada-Kakutani paper [26], That is, the topological group (or space) is called separable if it satisfies the second countability axiom of Hausdorff which means that it has a countable basis. In that paper, Hirotada and Kakutani established that the Bohr compactification of a given locally compact Abelian group has a countable basis if and only if the union of the spectrum of all almost periodic functions is countable.

Our analysis here is also motivated by the recent growing interest in the problem of the flat polynomials suggested by A.A. Prikhod'ko [42] in the context of $\mathbb{R}$. It turns out that the main idea developed in [42] does not seem well adapted to the context of our extension of generalized Riesz products to the Bohr compactification of $\mathbb{R}$. This is due to the fact that the sequence of trigonometric polynomials constructed by A.A. Prikhod'ko is only locally L^1-flat and not L^1-flat in the usual sense (see, for instance, Remark 6.11).

The paper is organized as follows. In section 2 we review some standard facts on the almost periodic functions including the ergodicity of the action of $\mathbb{R}$ by translations on its Bohr compactification. In section 3 we define the notion of the generalized Riesz products on the Bohr compactification of $\mathbb{R}$. In section 4 we summarize and extend the relevant material on the Kakutani criterion and the Bourgain criterion on the singularity of the generalized Riesz products introduced in section 3. In section 5 we state and prove the

Central Limit Theorem due to M. Kac. In section 6 we apply the Central Limit Theorem of Kac to prove our main result concerning the singularity of a large class of generalized Riesz products on $\mathbf{b}\mathbb{R}$. Finally, in the appendix, we consider the problem of the flat polynomials on the Bohr compactification of $\mathbb{R}$ and we add a short note based on Hirodata-Kakutani paper [26] on the Bohr compactification compared with the Stone-Čech compactification.

2. The Bohr compactification of $\mathbb{R}$

The Bohr compactification of $\mathbb{R}$ is based on the theory of almost periodic functions initiated by H. Bohr [7] in connection with the celebrated ζ-function of Riemann. In this section we are going to recall the basic ingredients of this theory. For the basic facts about almost periodic functions and generalizations of this concept the reader is referred to the classical presentation of Bohr [7] and Besicovitch [6].

We point out that the theory of almost periodic functions can be extended to more general setting with applications in many context including the non-linear differential equations [15].

Definition 2.1. Let $f : \mathbb{R} \longrightarrow \mathbb{C}$ be a bounded continuous function and $\varepsilon > 0$; we say that $\tau \in \mathbb{R}$ is an ε-almost period for f if

$$\sup_{x\in\mathbb{R}} |f(x+\tau) - f(x)| \stackrel{\text{def}}{=} ||f(.+\tau) - f(.)||_\infty < \varepsilon.$$

The mapping f is said to be almost periodic if for any $\varepsilon > 0$ the set of ε-almost periods of f is relatively dense, i.e., there is $l = l(\varepsilon) > 0$ such that any interval with length l contains at least one ε-almost period.

The space of all almost periodic functions is denoted by $\mathcal{AP}(\mathbb{R})$. From the above definition we easily deduce that $\mathcal{AP}(\mathbb{R})$ is a subspace of the space of bounded continuous functions on $\mathbb{R}$. An important characterization of almost periodic functions is due to Bochner and it can be stated as follows

Theorem 2.2 (Bochner's characterization of $\mathcal{AP}(\mathbb{R})$)**.** A bounded function f is almost periodic function if, and only if, the family of translates $\{f(.+t)\}_{t\in\mathbb{R}}$ is relatively compact in the space bounded continuous functions on $\mathbb{R}$ endowed with the sup-norm topology.

The proof of Theorem 2.2 can be found in [7], [6] or [18]. Furthermore, we have the following fundamental theorem (see for instance [7] or [6]).

Theorem 2.3 (Bohr)**.** A bounded continuous function f is almost periodic function if, and only if, f is uniformly approximated by finite linear combinations of functions in the set $\{\cos(tx), \sin(tx)\}_{t\in\mathbb{R}}$.

The space of all continuous functions on $\mathbf{b}\mathbb{R}$ is denoted by $C(\mathbf{b}\mathbb{R})$. $C(\mathbf{b}\mathbb{R})$ is a commutative $C^\star$-algebra under pointwise multiplication and addition. Below, we give an important topological characterization of the Bohr compactification of $\mathbb{R}$ due to Gelfand, Raikov and Chilov. They obtain this

characterization as an application of their theory of commutative Banach algebras.

Theorem 2.4 (Gelfand-Raikov-Chilov [21])**.** The group $\mathbb{R}$, equipped with the usual addition operation, may be embedded as a dense subgroup of a compact Abelian group $\mathbf{b}\mathbb{R}$ in such way as to make $\mathcal{AP}(\mathbb{R})$ the family of all restrictions functions $f_{|\mathbb{R}}$ to $\mathbb{R}$ of functions $f \in C(\mathbf{b}\mathbb{R})$. The operator $f \longmapsto f_{|\mathbb{R}}$ is an isometric $\star$-isomorphism of $C(\mathbf{b}\mathbb{R})$ onto $\mathcal{AP}(\mathbb{R})$. Moreover, the addition operation $+ \ : \ \mathbb{R}\times\mathbb{R} \longrightarrow \mathbb{R}$ extends uniquely to the continuous group operation of $\mathbf{b}\mathbb{R}$, $+ \ : \mathbf{b}\mathbb{R} \times \mathbf{b}\mathbb{R} \longrightarrow \mathbf{b}\mathbb{R}$. The group $\mathbf{b}\mathbb{R}$ is called the Bohr compactification of $\mathbf{b}\mathbb{R}$.

For simplicity of notation, for any f in $\mathcal{AP}(\mathbb{R})$, we use the same letter f for its canonical extension to $\mathbf{b}\mathbb{R}$. As a consequence of Theorem 2.4 combined with the Riesz representation Theorem we have

Theorem 2.5 ([18])**.** The dual of the space $\mathcal{AP}(\mathbb{R})$ is isometrically isomorphic to the space $\mathcal{M}(\mathbf{b}\mathbb{R})$ of all Radon measures on the Bohr compactification of $\mathbb{R}$. The isomorphism $x^* \longmapsto \mu_{x^*}$ is given by the formula

$$x^*(f) = \int_{\mathbf{b}\mathbb{R}} f(t) d\mu_{x^*}(t).$$

We recall in the following the definition of the characters. The characters play a important role in the Abelian group and, by Koopmann observation, in the spectral analysis of dynamical systems.

Definition 2.6. Let G be an Abelian group and e its identity element, then a character of G is a complex valued function χ defined on G such that $\chi(e) = 1$ and $\chi(st) = \chi(s)\chi(t)$ for all $s, t \in G$.

Let us recall the following basic fact on compact Abelian groups due to Peter and Weyl.

Theorem 2.7 (Peter-Weyl)**.** Let G be a compact Abelian group, with $\mathcal{B}(G)$ its Borel field and h its Haar measure. Then the set of continuous characters is fundamental both in $C(G)$ and in $L^2(G, \mathcal{B}(G), h)$.

The proof of Peter-Weyl Theorem can be found in [50] and for the thorough treatment we refer the reader to [50].

It is obvious that the continuous characters of $\mathbf{b}\mathbb{R}$ are the functions $e^{i\omega} \ : \ \mathbf{b}\mathbb{R} \longrightarrow \mathbb{T}$. In addition the orthogonality of two distinct characters can be checked directly. Indeed,

$$\int_{\mathbf{b}\mathbb{R}} e^{i\omega t}.e^{i\omega' t} dh(t) = \lim_{T \longrightarrow +\infty} \frac{1}{2T} \int_{-T}^{T} e^{i(\omega-\omega')t} dt = 0, \text{ whenever } \omega \neq \omega'.$$

For $f \in \mathcal{A}P(\mathbb{R})$ we denote by $\displaystyle\fint_{\mathbb{R}} f(t)dt$ the asymptotic mean value of f, given by

$$\fint_{\mathbb{R}} f(t)dt = \lim_{T\longrightarrow+\infty} \frac{1}{2T}\int_{-T}^{T} f(t)dt.$$

As a consequence of the averaging properties of almost periodic functions we have the following

Lemma 1. For any $f \in \mathcal{A}P(\mathbb{R})$ we have

$$\int_{\mathbf{b}\mathbb{R}} f dh(t) = \fint_{\mathbb{R}} f(t)dt,$$

where $dh(t)$ is the Haar measure in $\mathbf{b}\mathbb{R}$, normalized to be a probability measure, and dt is the usual Lebesgue measure in $\mathbb{R}$. Moreover we have

$$\int_{\mathbf{b}\mathbb{R}} f(t)dh(t) = \lim_{T\longrightarrow+\infty} \frac{1}{T|K|}\int_{T.K} f(t)dt,$$

where K is any bounded subset of $\mathbb{R}$ with $|K| \neq 0$, $|K|$ is the Lebesgue measure of K.

Following [31], for any $f \in \mathcal{A}P(\mathbb{R})$, we introduce the notation

$$\widehat{f}\big(\{\lambda\}\big) = \fint_{\mathbb{R}} f(t)e^{-i\lambda t}dt.$$

That is, $\big(\widehat{f}\big(\{\lambda\}\big)\big)_{\lambda\in\mathbb{R}}$ are the Fourier coefficients of f relative to orthonormal family $\{e^{i\lambda t}\}_{\lambda\in\mathbb{R}}$; the inner product is defined by

$$< f, g >=< f, g >= \int_{\mathbf{b}\mathbb{R}} f(\kappa)\overline{g}(\kappa)dh(\kappa).$$

2.1. **On Besicovitch space.** Since the functions $f \in \mathcal{A}P(\mathbb{R})$ correspond to restrictions, $f = \widetilde{f}_{|\mathbb{R}}$, of continuous functions f on $\mathbf{b}\mathbb{R}$, a natural question is whether it is possible to define a class of functions f which correspond to "restrictions" $f = \widetilde{f}_{|\mathbb{R}}$, of functions $\widetilde{f} \in L^1(\mathbf{b}\mathbb{R})$. This motivates the following definition.

Definition 2.8. Given $p \in [1, +\infty)$ the space $\mathrm{BAP}^p(\mathbb{R})$, of Besicovitch's generalized almost periodic functions on $\mathbb{R}$, consists of those functions $f \in L^p_{\mathrm{loc}}(\mathbb{R})$ for which there exists a sequence $f_n \in \mathcal{AP}(\mathbb{R})$ satisfying

$$\lim_{n \longrightarrow +\infty} \limsup_{T \longrightarrow +\infty} \frac{1}{2T} \int_{-T}^{T} \left| f_n(x) - f(x) \right|^p dx = 0 \tag{1}$$

We denote $\mathrm{BAP}^p(\mathbb{R})$ simply by $\mathcal{B}^p(\mathbb{R})$.

The space of generalized almost periodic functions $\mathcal{B}^p(\mathbb{R})$ was introduced by Besicovitch, who also gave them a structural characterization. We refer to [6] for more details about functions in $\mathcal{B}^p(\mathbb{R})$. We immediately have $\mathcal{AP}(\mathbb{R}) \subset \mathcal{B}^p(\mathbb{R}) \subset \mathcal{B}^1(\mathbb{R})$ for any $p \geqslant 1$ and it is easy to see that any $f \in \mathcal{B}^1(\mathbb{R})$ has the mean value property, that is, for any bounded measurable $B \subset \mathbb{R}$, with $|B| \neq 0$, we have

$$M(f) = \lim_{L \longrightarrow +\infty} \frac{1}{L|B|} \int_{B_L} f dx,$$

where $B_L = \{x \in \mathbb{R} : \frac{x}{L} \in B\}$. The space corresponds to $L^p(\mathbf{b}\mathbb{R})$ in a way similar to the one in which the space $\mathcal{AP}(\mathbb{R})$ corresponds to $C(\mathbf{b}\mathbb{R})$. Indeed, notice first that the definition of $\mathcal{B}^p(\mathbb{R})$ immediately gives that the asymptotic mean value $\fint |f|^p dx$ of a function in $\mathcal{B}^p(\mathbb{R})$ is well defined; moreover, any approximating sequence $f_n \in \mathcal{AP}(\mathbb{R})$ satisfying (1) can be viewed as a Cauchy sequence in $L^p(\mathbf{b}\mathbb{R})$ and, hence, there exists $\widetilde{f} \in L^p(\mathbf{b}\mathbb{R})$ such that $\widetilde{f_n}$ converge to f in $L^p(\mathbf{b}\mathbb{R})$. Since $\widetilde{f}$ is easily seen to be independent of the approximating sequence, in this way we may associate with each $f \in \mathcal{B}^p(\mathbb{R})$ a well determined function $\widetilde{f} \in L^p(\mathbf{b}\mathbb{R})$ which we may view as an *"extension"* of f to $\mathbf{b}\mathbb{R}$. Notice that the map $f \mapsto \widetilde{f}$ is a linear map and that the approximation procedure together with Lemma 1 show that

$$\fint_{\mathbb{R}} \left| f \right|^p = \int_{\mathbf{b}\mathbb{R}} \left| \widetilde{f}(\kappa) \right|^p d\kappa, \forall f \in \mathcal{B}^p(\mathbb{R}). \tag{2}$$

As a consequence, the kernel of the map $f \longmapsto \widetilde{f}$ is made by the functions f such that the asymptotic mean value of $\left| f \right|^p$ is 0. The corresponding quotient space is denoted by $\mathcal{B}^p(\mathbb{R})/\sim$. For $f \in \mathcal{B}^p(\mathbb{R})$ we set

$$|f|_p \stackrel{\text{def}}{=} \left(\fint_{\mathbb{R}} \left| f \right|^p dx \right)^{\frac{1}{p}},$$

so that $\left|\ \right|_p$ is a semi-norm on $\mathcal{B}^p(\mathbb{R})$. It is well known that $\mathcal{B}^p(\mathbb{R})$ is complete with respect to the semi-norm $\left|\ \right|_p$ (see for instance [38]). The space $\mathcal{B}^2(\mathbb{R})$ is endowed with the scalar product

$$< f, g >= \fint_{\mathbb{R}} f(x)\overline{g(x)}dx = \int_{\mathbf{b}\mathbb{R}} \widetilde{f}(\kappa)\overline{\widetilde{g}(\kappa)}d\kappa, \tag{3}$$

The second equality follows by (2) with $p = 2$, implying that the scalar product is preserved under the map $f \mapsto \widetilde{f}$. Finally, we define

$$\mathcal{B}^\infty(\mathbb{R}) \stackrel{\text{def}}{=} \left\{ f \in \bigcap_{p\geqslant 1} \mathcal{B}^p(\mathbb{R}) \ : \ \sup_{p\geqslant 1} \left|f\right|_p < +\infty \right\},$$

Again, $\left|\ \right|_\infty$ is a semi-norm on $\mathcal{B}^\infty(\mathbb{R})$ and the corresponding quotient space is denoted by $\mathcal{B}^\infty(\mathbb{R})/\sim$. We finish this paragraph by stating the following important fact on the properties of the map $f \mapsto \widetilde{f}$. The proof of it is left to the reader.

Proposition 2.9. The map $f \mapsto \widetilde{f}$ is an isometric isomorphism between the Banach spaces $\mathcal{B}^p(\mathbb{R})/\sim$ and $L^p(\mathbf{b}\mathbb{R})$ for any $p \in [1, +\infty]$

Furthermore, we point out that a suitable extension of Lebesgue and Fatou's convergence results is obtained in [38].

Gelfand-Raikov-Chilov Theorem 2.4 allows us to define the Bohr compactification of $\mathbb{R}$. But, in the Harmonic Analysis, it is well known that the Pontrygain Theorem gives an alternative definition. We briefly recall it here and we refer the reader to the large literature on the subject [50] and for a deeper discussion on the relation between the almost periodic functions and the Bohr compactification of $\mathbb{R}$ to [32], [23].

Let G be locally compact Abelian group and let $\widehat{G}$ be its dual, i.e., $\widehat{G}$ is the set of the characters endowed with the topology inherited from G. Let $\widehat{G}_d$ be $\widehat{G}$ with the discrete topology. Then $\widehat{\widehat{G_d}} \stackrel{\text{def}}{=} \mathbf{b}G$ is the Bohr compactification of G. $\mathbf{b}G$ is a compact group such that G is a dense subset of $\mathbf{b}G$.

We end this section by stating and proving the classical result on the ergodicity of the action of $\mathbb{R}$ by translation on $\mathbf{b}\mathbb{R}$. We recall that the action of $\mathbb{R}$ by translation is defined by $\tau_x(\kappa) = \kappa + x$ where the extended addition is given by Theorem 2.4. Clearly, the family $(\tau_x)_{x\in\mathbb{R}}$ is a flow acting on $\mathbf{b}\mathbb{R}$ since $\tau_x(\tau_{x'}(\kappa)) = \tau_{x+x'}(\kappa)$ and the Haar measure is invariant under translation.

Theorem 2.10. The action of $\mathbb{R}$ on $\mathbf{b}\mathbb{R}$ is ergodic, that is, for any Borel set $A \subset \mathbf{b}\mathbb{R}$ which is invariant under the translation action we have $h(A) \in \{0, 1\}$, where $h(A)$ denotes the normalized Haar measure of A. Moreover $h(\mathbb{R}) = 0$.

Proof. Let $A \subset \mathbf{b}\mathbb{R}$ be an invariant Borel set. We have

$$h(A) = h(A \cap \tau_x A) = \int_{\mathbf{b}\mathbb{R}} \mathbb{1}_A(\beta)\, \mathbb{1}_A(\beta + x) dh(\beta).$$

Now, translations are strongly continuous on $L^2(\mathbb{R})$. Indeed, this is a standard consequence of the density of $C(\mathbf{b}\mathbb{R})$ in $L^2(\mathbf{b}\mathbb{R})$, which follows from Theorem 2.4, and the invariance of the Haar measure. Therefore, the right-hand side is a continuous function of x, and so the identity still holds with $x \in \mathbf{b}\mathbb{R}$. Hence we get, using Fubini theorem and the invariance of the Haar measure,

$$\begin{aligned} h(A) &= \int_{\mathbf{b}\mathbb{R}} \int_{\mathbf{b}\mathbb{R}} \mathbb{1}_A(\kappa)\, \mathbb{1}_A(\kappa + \xi) dh(\kappa) dh(\xi) \\ &= \int_{\mathbf{b}\mathbb{R}} \mathbb{1}_A(\kappa) dh(\kappa) \int_{\mathbf{b}\mathbb{R}} \mathbb{1}_A(\xi) dh(\xi) = h(A)^2, \end{aligned}$$

from which it follows that $h(A) \in \{0, 1\}$, as asserted. It remains to show that $h(\mathbb{R}) = 0$. First we observe that $\mathbb{R}$ is a Borel subset of $\mathbf{b}\mathbb{R}$, since it is the union of a countable family of compact sets, e.g., the images of the intervals $[-k, k]$, $k \in \mathbb{N}$. Since $\mathbb{R}$ is invariant under the translation action we have $h(\mathbb{R}) \in \{0, 1\}$. But, for any $\kappa \in \mathbf{b}\mathbb{R} \backslash \mathbb{R}$, $\kappa + \mathbb{R}$ is also an invariant Borel set and $\mathbb{R} \bigcap \{\kappa + \mathbb{R}\} = \varnothing$. By the invariance of the Haar measure $h(\kappa + \mathbb{R}) = h(\mathbb{R})$. Hence, we conclude that $h(\mathbb{R}) = 0$ and the proof is achieved. □

3. Generalized Riesz Products on $\mathbf{b}\mathbb{R}$

Riesz products were discovered in 1918 by F. Riesz [46] to answer affirmatively a special question in the theory of Fourier series, namely, whether there exists a continuous measure whose Fourier coefficients do not vanish at infinity. Roughly speaking, the Riesz products are a kind of measures on the circle constructed inductively. The pioneer Riesz product construction gives a concrete example. Since then, the Riesz construction proved to be the source of powerful ideas that can be used to produce concrete counterexample of measures with a number of desired properties (controllability of the convergence of the Fourier coefficients being the goal of the original construction).

Later, A. Zygmund extended Riesz construction and introduced what it is nowadays called classical Riesz products [53, p.208].

In 1975, that Riesz products appear as a spectral type of some dynamical systems was shown by F. Ledrappier [36]. Ten years later, M. Queffelec [43], inspired by the work of Coquet-Kamae and Mandes-France [14], showed that the specific generalized Riesz products are the right tool to describe the spectrum of the class of dynamical systems arising from the substitution (see [44] and the references therein). In 1991, B. Host, J.-F. Méla, F. Parreau in

[27] realized a large class of Riesz products as the maximal spectral type of the unitary operator associated with a non-singular dynamical system and a cocycle. Finally, in the more general setting, J. Bourgain established the connections between some class of generalized Riesz products on the circle and the maximal spectral type of a class of maps called rank one maps [10]. One year later, an alternative proof is given by Choksi-Nadkarni using the Host-Méla-Parreau argument [12],[39] and at the same time a simple proof is obtained by Klemes-Reinhold [34] using the standard Fourier analysis argument.

In [3], el Abdalaoui-Lemańczyk-Lesigne and Ulcigrai proved that the generalized Riesz products analogous to Peyriére-Riesz products can be realized as a spectral type of some class of rank one flows.

Here our aim is to extend the notion of generalized Riesz products to the Bohr compactification of $\mathbb{R}$. It turns out that such generalization can be done directly. More precisely, in the study of the spectrum of some class of rank one flows, the following trigonometric polynomials on $\mathbb{R}$ appears

$$P_k(\theta) = \frac{1}{\sqrt{p_k}} \sum\nolimits_{j=0}^{p_k-1} e^{i\theta(jh_k+\check{s}_j)}, \tag{4}$$

$$\text{where } \check{s}_0 = 0 \text{ and } \check{s}_j = \sum_{l=0}^{j-1} s_{k,l},\ j \geqslant 1,$$

with $(p_k)_{k\in\mathbb{N}}$ is a sequence of positive integers greater than 1 and $(s_{k,j})_{j=0,\cdots,p_k}$ is a sequence of positive real numbers with $s_{k,0} = 0$ for any $k \in \mathbb{N}$; the sequence (h_k) is defined inductively by

$$h_0 = 1 \text{ and } h_{k+1} = p_k h_k + \sum_{l=0}^{p_k} s_{k,l}. \tag{5}$$

For simplicity, for any $p, q \in \{0, \cdots, p_k - 1\}$ we introduce the following sequence of real numbers

$$\check{s}_{n,p,q} = \sum_{j=\min(p,q)}^{\max(p,q)-1} s_{n,j}, \tag{6}$$

and for any real number t, we put

$$e(t) = e^{it}.$$

Theorem 3.1 (Generalized Riesz Products on $\mathbf{b}\mathbb{R}$)**.** Let $(P_n)_{n\in\mathbb{N}}$ be a family of trigonometric polynomials given by (4). Then the weak limit of the sequence of probability measures on $\mathbf{b}\mathbb{R}$

$$\prod_{k=0}^{n} \Big|P_k(\theta)\Big|^2 \, dh(\theta),$$

exists and is denoted by

$$\prod_{k=0}^{\infty}\Big|P_k(\theta)\Big|^2\,dh(\theta),$$

Proof. Let $R_n(t)=P_0(t)\cdots P_n(t)$, $Q_n(t)=|R_n(t)|^2$ and $\sigma_n=|R_n(\kappa)|^2dh(\kappa)$. By the definition of P_n we have

$$\big|P_n(t)\big|^2=1+\Delta_n(t)\text{ with }\Delta_n(t)=\frac{1}{p_n}\sum\nolimits_{p\neq q}e\Big(i\big((p-q)h_n+\breve{s}_{n,p,q}\big)t\Big)$$

and it is obvious that

$$\fint_{\mathbb{R}}Q_n dt=1.$$

Therefore, σ_n is a probability measure on $\mathbf{b}\mathbb{R}$. In addition, for any $t\in\mathbb{R}$, we have

$$Q_{n+1}(t)=Q_n(t)P_n(t)=Q_n(t)+Q_n(t)\Delta_n(t)$$

Hence

$$\widehat{\sigma_{n+1}}\big(\lambda\big)=\widehat{Q_{n+1}}\big(\{\lambda\}\big)=\widehat{Q_n}\big(\{\lambda\}\big)+\widehat{Q_n}*\widehat{\Delta_n}(\lambda)\geqslant\widehat{Q_n}\big(\{\lambda\}\big)=\widehat{\sigma_n}\big(\lambda\big)$$

Consequently the limit r_λ of the sequence $(\widehat{\sigma_n}\big(\lambda\big))$ exists. Now, since $\mathbf{b}\mathbb{R}$ is compact and (σ_n) is a sequence of probability measure on $\mathbf{b}\mathbb{R}$ we can extract a subsequence (σ_{n_k}) which converge weakly to some probability measure on $\mathbf{b}\mathbb{R}$. We deduce that the limit of (σ_n) exists in the weak topology and this finishes the proof. □

The proof above is largely inspired by lemma 2.1 in [34]; it gives more, namely, the polynomials P_n can be chosen with positive coefficients and satisfying

$$\fint\Big|P_n(t)\Big|^2dt=1\text{ and }\fint\prod_{j=1}^{n}\Big|P_j(t)\Big|^2dt=1.$$

We mention also that we have

$$\fint\prod_{j=1}^{k}\Big|P_{n_j}(t)\Big|^2dt=1, \tag{7}$$

for any given sequence of positive integers $n_1<n_2<\cdots<n_k, k\in\mathbb{N}^*$.

We can now formulate our main result whose proof occupies all Section 6.

Theorem 3.2 (Main result)**.** Let $(p_m)_{m\in\mathbb{N}}$ be a sequence of positive integers greater than 1 and $((s_{m,j})_{j=0}^{p_m-1})_{m\in\mathbb{N}}$ be a sequence of positive real numbers. Assume that there exists a sequence of positive integers $m_1<m_2<\cdots$,

such that for any positive integer j, the numbers $(h_{m_j}, s_{m_j,0}, \cdots, s_{m_j,p_{m_j}-1})$ are rationally independent. Then the generalized Riesz product

$$\mu = \prod_{k=0}^{n} \left|P_k(\theta)\right|^2 dh(\theta),$$

where

$$P_k(\theta) = \frac{1}{\sqrt{p_k}} \sum\nolimits_{j=0}^{p_k-1} e\big(i\theta\big(jh_k + \breve{s}_{k,0,j}\big)\big).$$

is singular with respect to the Haar measure on $\mathbf{b}\mathbb{R}$.

4. On the Kakutani criterion and the Bourgain singularity criterion of the Riesz products on the Bohr compactification of $\mathbb{R}$

The famous dichotomy theorem of Kakutani has a rather long history. In his 1948 celebrated paper [30], Kakutani established a purity law for infinite product measures. Precisely, if $\mathbb{P} = \bigotimes_{i=1}^{+\infty} \mathbb{P}_i$ and $\mathbb{Q} = \bigotimes_{i=1}^{+\infty} \mathbb{Q}_i$ are a infinite product measures, where $\mathbb{P}_i, \mathbb{Q}_i$ are probability measures such that $\mathbb{P}_i$ is absolutely continuous with respect to $\mathbb{Q}_i$, for each positive integer i, then

$$\mathbb{P} \ll \mathbb{Q} \text{ or } \mathbb{P} \perp \mathbb{Q} \text{ according as } \prod_i \int \left(\frac{d\mathbb{P}_i}{d\mathbb{Q}_i}\right) d\mathbb{Q}_i \text{ converges or diverges.}$$

There are a several proofs of Kakutani criterion in literature (see [9] and the references given there). For a proof based on the Hellinger integral we refer the reader to [17, p.60].

Kakutani's Theorem was specialized to the Gaussian measures on Hilbert space with identical correlation operators in [22] and it was extended to Gaussian measures with non-identical correlation operators by Segal [51], Hajek [25], Feldman [20] and Rozanov [49]. Later, in 1979, Ritter in [47], [48] generalized Kakutani's Theorem to a certain non-products measures with application to the classical Riesz products.

Here, applying the Bourgain methods, we obtain a new extension of Kakutani's Theorem to the class of generalized Riesz products on the Bohr compactification of $\mathbb{R}$. Indeed, we show that the independence along subsequence suffices to prove the singularity.

Nevertheless, our strategy is similar to the strategy of [3] and it is based on the extension of Bourgain methods to the generalized Riesz products on the Bohr compactification of $\mathbb{R}$ combined with the Central Limit tools introduced in [2].

Moreover, having in mind applications beyond the context of this paper, we shall state and prove a Guenais sufficient condition on the L^1 flatness of the polynomials which implies the existence of generalized Riesz products on $\mathbf{b}\mathbb{R}$ with Haar component. We recall that the generalized Riesz products

μ is given by

$$\mu = \prod_{k=0}^{+\infty} \left|P_k(\theta)\right|^2 dh(\theta)$$

where

$$P_k(t) = \frac{1}{\sqrt{p_k}} \sum\nolimits_{j=0}^{p_k-1} e\big(i\theta(jh_k + \check{s}_{k,j})\big) \text{ and } \check{s}_{k,j} = \sum_{l=0}^{j-1} s_{k,l}$$

Theorem 4.1 (**b**$\mathbb{R}$ version of Bourgain criterion)**.** The following are equivalent

(i) μ is singular with respect to Haar measure.

(ii) $\inf \left\{ \int_{\mathbf{b}\mathbb{R}} \prod_{\ell=1}^{L} \left|P_{n_\ell}\right| dh \ : \ L \in \mathbb{N},\ n_1 < n_2 < \ldots < n_L \right\} = 0.$

The proof of **b**$\mathbb{R}$ version of Bourgain criterion is based on the following lemma.

Lemma 4.2. The following are equivalent

(1) $\displaystyle\int_{\mathbf{b}\mathbb{R}} \prod_{k=0}^{N} \left|P_k\right| dh \xrightarrow[N\to+\infty]{} 0.$

(2) $\inf \left\{ \int_{\mathbf{b}\mathbb{R}} \prod_{\ell=1}^{L} |P_{n_\ell}| \, dh \ : \ L \in \mathbb{N},\ n_1 < n_2 < \ldots < n_L \right\} = 0.$

Proof. The proof is a simple application of Cauchy-Schwarz inequality. Consider $n_1 < n_2 < \ldots < n_L$ and $N \geqslant n_L$. Denote $\mathcal{N} = \{n_1 < n_2 < \ldots < n_L\}$ and $\mathcal{N}^c$ its complement in $\{1, \cdots, N\}$. Let $a < b$ be two real numbers and define a probability measure on $\mathbb{R}$ by

$$d\lambda_{a,b}(t) = \frac{\mathbb{1}_{[a,b]}(t)}{b-a} dt, \text{ where } dt \text{ is the Lebesgue measure.}$$

Then we have

$$\begin{aligned}
\int \prod_{k=0}^{N} |P_k| \, d\lambda_{a,b} &= \int \prod_{k\in\mathcal{N}} |P_k|^{\frac{1}{2}} \times \prod_{k\in\mathcal{N}^c} |P_k|^{\frac{1}{2}} \prod_{k=0}^{N} |P_k|^{\frac{1}{2}} \, d\lambda_{a,b} \\
&\leqslant \left(\int \prod_{k\in\mathcal{N}} |P_k| \, d\lambda_{a,b} \right)^{\frac{1}{2}} \left(\int \prod_{k\in\mathcal{N}^c} |P_k| \times \prod_{k=0}^{N} |P_k| \, d\lambda_{a,b} \right)^{\frac{1}{2}} \\
&\leqslant \left(\int \prod_{k\in\mathcal{N}} |P_k| \, d\lambda_{a,b} \right)^{\frac{1}{2}} \left(\int \prod_{k\in\mathcal{N}^c} |P_k|^2 \, d\lambda_{a,b} \right)^{\frac{1}{4}} \left(\int \prod_{k=0}^{N} |P_k|^2 \, d\lambda_{a,b} \right)^{\frac{1}{4}}
\end{aligned}$$

By letting $b-a$ goes to infinity, we get

$$\fint \prod_{k=0}^{N} |P_k|\, dt \leqslant \left(\fint \prod_{k\in\mathcal{N}} |P_k|\, dt\right)^{\frac{1}{2}} \left(\fint \prod_{k\in\mathcal{N}^c} |P_k|^2\, dt\right)^{\frac{1}{4}} \left(\fint \prod_{k=0}^{N} |P_k|^2\, dt\right)^{\frac{1}{4}}$$
$$= \left(\fint \prod_{k\in\mathcal{N}} |P_k|\, dt\right)^{\frac{1}{2}}.$$

The last equality follows from (7).

□

Proof of Theorem 4.1. Assume that (i) holds. To prove that μ is singular , it suffices to show that for any $\epsilon > 0$, there is a Borel set E with $h(E) < \epsilon$ and $\mu(E^c) < \epsilon$. Let $0 < \epsilon < 1$.

Fix N_0 such that for any $N > N_0$, we have $\displaystyle\int_{\mathbf{b}\mathbb{R}} \prod_{k=0}^{N} |P_k|\, dh < \epsilon^2$. The set $E = \left\{\omega \in \mathbf{b}\mathbb{R} \ : \ \prod_{k=0}^{N} |P_k(\omega)| \geqslant \epsilon\right\}$ satisfies:

$$h(E) \leqslant \frac{1}{\epsilon} \left\| \prod_{k=0}^{N} P_k \right\|_1 \leqslant \epsilon^2/\epsilon = \epsilon,$$

and since E^c is open set, it follows from the Portmanteau Theorem that we have

$$\begin{aligned}
\mu(E^c) &\leqslant \liminf_{M\to+\infty} \int_{E^c} \prod_{k=0}^{M} |P_k|^2\, dh \\
&\leqslant \liminf_{M\to+\infty} \int_{E^c} \prod_{k=0}^{N} |P_k|^2 \prod_{k=N+1}^{M} |P_k|^2\, dh \\
&\leqslant \epsilon^2 \lim_{M\to+\infty} \int_{\mathbf{b}\mathbb{R}} \prod_{k=N+1}^{M} |P_k|^2\, dh = \epsilon^2 < \epsilon.
\end{aligned}$$

For the converse, given $0 < \epsilon < 1$, there exists a continuous function φ on $\mathbf{b}\mathbb{R}$ such that:

$$0 \leqslant \varphi \leqslant 1, \qquad \mu(\{\varphi \neq 0\}) \leqslant \epsilon \qquad \text{and} \quad h(\{\varphi \neq 1\}) \leqslant \epsilon.$$

Let $f_N = \prod_{k=1}^{N} |P_k|$. By Cauchy-Schwarz inequality, we have

$$\begin{aligned}\int f_N \, dh &= \int_{\{\varphi\neq 1\}} f_N \, dh + \int_{\{\varphi=1\}} f_N \, dh \\ &\leqslant h(\{\varphi\neq 1\})^{1/2} \left(\int_{\mathbf{b}\mathbb{R}} f_N^2 \, dh\right)^{1/2} + \left(\int_{\{\varphi=1\}} f_N^2 \, dh\right)^{1/2} h(\{\varphi=1\})^{1/2} \\ &\leqslant \sqrt{\epsilon} + \left(\int_{\mathbf{b}\mathbb{R}} f_N^2 \, \varphi \, dh\right)^{1/2}.\end{aligned}$$

Since μ is the weak limit of $f_N^2 dh$, we have

$$\lim_{N\to\infty} \int_{\mathbf{b}\mathbb{R}} f_N^2 \, \varphi \, dh = \int_{\mathbf{b}\mathbb{R}} \varphi \, d\mu \leqslant \mu(\{\varphi\neq 0\}) \leqslant \epsilon.$$

Thus, $\limsup \int_{\mathbf{b}\mathbb{R}} f_N \, dh \leqslant 2\sqrt{\epsilon}$. Since ϵ is arbitrary, we get $\lim_{N\to\infty} \int_{\mathbf{b}\mathbb{R}} f_N \, dh = 0$, and this completes the proof. □

From now on, let $\mathcal{M}$ be a sequence of positive integers for which $(h_m, (s_{m,j})_{j=0}^{p_m-1})$ are linearly independent over the rationals and let us fix some subsequence $\mathcal{N} = \{n_1 < n_2 < \ldots < n_k\}$ of $\mathcal{M}$, $k \in \mathbb{N}$ and $m \in \mathcal{M}$ with $m > n_k$. Put

$$Q(t) = \prod_{i=1}^{k} \left| P_{n_i}(t) \right|.$$

We define the degree of any trigonometric polynomial f by

$$\deg(f) = \max\{|\xi| : \widehat{f}(\{\xi\}) \neq 0\}, \tag{8}$$

and we denote $d_m \overset{\text{def}}{=} \deg(P_m)$. From equations (5) and (6), we have

$$d_m = h_{m+1} - h_m - s_{m,p_m} < h_{m+1}, \tag{9}$$

$$h_m \leqslant h_{m+1}/p_m \leqslant h_{m+1}/2. \tag{10}$$

Since $n_j + 1 \leqslant n_{j+1}$, telescoping we get

$$\begin{aligned} q_k \overset{\text{def}}{=} \deg(Q_k) &= d_{n_1} + d_{n_2} + \cdots + d_{n_k} \\ &\leqslant (h_{n_1+1} - h_{n_1}) + (h_{n_2+1} - h_{n_2}) + \cdots + (h_{n_k+1} - h_{n_k}) \\ &< h_{n_k+1}. \end{aligned} \tag{11}$$

In the same spirit as above it is easy to see the following lemma. The proof of it in the case of the torus is given in [10] and [2].

Lemma 2. With the above notations we have

$$\int_{\mathbf{b}\mathbb{R}} Q\big|P_m\big|dh \leqslant \frac{1}{2}\Big(\int_{\mathbf{b}\mathbb{R}} Qdh + \int_{\mathbf{b}\mathbb{R}} Q\big|P_m\big|^2 dh(\omega)\Big) - \frac{1}{8}\Big(\int_{\mathbf{b}\mathbb{R}} Q\,\big|\big|P_m\big|^2 - 1\big|dh\Big)^2.$$

The following proposition is a simple extension of Proposition 2.4 in [2].

Proposition 4.3. We have

$$\lim_{m\to\infty}\int_{\mathbf{b}\mathbb{R}} Q\big|P_m\big|^2 dh = \int_{\mathbf{b}\mathbb{R}} Qdh$$

Proof. The sequence of probability measures $\big|P_m(z)\big|^2 dh$ converges weakly to the Haar measure. □

From the proposition 4.3 and Lemma 2 we deduce the following

Proposition 4.4. With the above notations we have

$$\liminf \int_{\mathbf{b}\mathbb{R}} Q\big|P_m\big|dh \;\leqslant\; \int_{\mathbf{b}\mathbb{R}} Qdh \;-\; \frac{1}{8}\Big(\limsup \int_{\mathbf{b}\mathbb{R}} Q\big|\big|P_m\big|^2 - 1\big|dh\Big)^2.$$

Now, in the following lemma, we state a sufficient condition for the existence of an absolutely continuous component with respect to the Haar measure for the given generalized Riesz product . In the case of $\mathbb{Z}$ action, the lemma is due to Mélanie Guenais [24], and the proof is similar.

Lemma 4.5. If $\sum_{k=1}^{+\infty}\sqrt{1-\Big(\int_{\mathbf{b}\mathbb{R}} |P_k|\,dh\Big)^2} < \infty$, then μ admits an absolutely continuous component.

Proof. We denote by $\|\cdot\|_p$ the norm in $L^p(h)$. For all functions P and Q in $L^2(h)$, by Cauchy-Schwarz inequality we have

$$\|P\|_1\|Q\|_1 - \|PQ\|_1 = -\int (|P| - \|P\|_1)\,(|Q| - \|Q\|_1)\;dh \leqslant \big\||P| - \|P\|_1\big\|_2\,\big\||Q| - \|Q\|_1\big\|_2, \tag{12}$$

and by assumption,

$$\sum_{k=1}^{+\infty}\sqrt{1-\|P_k\|_1^2} < \infty, \tag{13}$$

hence $\sum_{k=1}^{+\infty} 1-\|P_k\|_1^2<\infty$ and the infinite product $\prod_k \|P_k\|_1$ is convergent:

$$\prod_{k=1}^{+\infty} \|P_k\|_1 > 0. \tag{14}$$

Let $n_0 \leqslant n$ be a positive integers and take $P=P_n$ and $Q=\prod_{k=n_0}^{n-1} P_k$, then $\||P|-\|P\|_1\|_2=\sqrt{1-\|P\|_1^2}$ and $\||Q|-\|Q\|_1\|_2 \leqslant 1$; hence by (12) we have

$$\|PQ\|_1 \geqslant \|P\|_1\|Q\|_1-\sqrt{1-\|P\|_1^2}.$$

Using the fact that $\|P_k\|_1 \leqslant 1$, we obtain by induction

$$\left\|\prod_{k=n_0}^{n} P_k\right\|_1 \geqslant \prod_{k=n_0}^{n}\|P_k\|_1-\sum_{k=n_0}^{n}\sqrt{1-\|P_k\|_1^2} \geqslant \prod_{k=n_0}^{n}\|P_k\|_1-\sum_{k=n_0}^{+\infty}\sqrt{1-\|P_k\|_1^2}\,.$$

From (13) combined with (14) we deduce that for large enough n_0

$$\lim_{n\to+\infty}\prod_{k=n_0}^{n}\|P_k\|_1-\sum_{k=n_0}^{+\infty}\sqrt{1-\|P_k\|_1^2}>0,$$

hence the sequence $\left(\prod_{k=n_0}^{n} P_k\right)$ does not go to zero in L^1-norm. It follows from Bourgain criterion that the generalized Riesz product $\prod_{k=n_0}^{+\infty}|P_k|^2$ is not purely singular. As $\prod_{k=1}^{n_0-1}|P_k|^2$ has only countably many zeros, we conclude that μ admits also an absolutely continuous component with respect to the Haar measure. □

5. On the Kac Central Limit Theorem

The Kac Central Limit Theorem in the context of the Bohr compactification of $\mathbb{R}$ is stated and proved in [28]. For sake of completeness we prove it here using the standard probability arguments.

Definition 5.1. The real numbers $\omega_1,\omega_2,\cdots,\omega_r$ are called rationally independent if they are linearly independent over $\mathbb{Z}$, i.e. for all $n_1,\cdots,n_r\in\mathbb{Z}$,

$$n_1\omega_1+\cdots+n_r\omega_r=0 \Longrightarrow n_1=\cdots=n_r=0.$$

Theorem 5.2 (M. Kac [28])**.** Let $\omega_1,\omega_2,\cdots,\omega_n,\cdots$ be rationally independent. Then, the functions $\cos(\omega_1 t),\cos(\omega_2 t),\cdots,\cos(\omega_n t),\cdots$ are stochastically independent with respect to the Haar measure of the Bohr compactification of $\mathbb{R}$.

Proof. It is sufficient to show that for any positive integer k and for a given positive integers $l_1, l_2, \cdots, l_k$, we have

$$\begin{aligned}
&\fint_{\mathbb{R}} \Big(\cos(\omega_1 t)\Big)^{l_1} \Big(\cos(\omega_2 t)\Big)^{l_2} \cdots \Big(\cos(\omega_k t)\Big)^{l_k} dt \\
&= \fint_{\mathbb{R}} \Big(\cos(\omega_1 t)\Big)^{l_1} dt \fint_{\mathbb{R}} \Big(\cos(\omega_2 t)\Big)^{l_k} dt \cdots \fint_{\mathbb{R}} \Big(\cos(\omega_k t)\Big)^{l_k} dt
\end{aligned}$$

Write

$$\cos(\omega_j t) = \frac{1}{2}\Big(e^{i\omega_j t} + e^{-\omega_j t}\Big),\ j = 1, 2, \cdots, t \in \mathbb{R},$$

and recall that

$$\fint_{\mathbb{R}} e^{i\alpha t} dt = \lim_{T \longrightarrow +\infty} \frac{1}{2T} \int_{-T}^{T} e^{i\alpha t} dt = \begin{cases} 1, & \text{if } \alpha = 0 \\ 0, & \text{if not .} \end{cases}$$

Hence

$$\begin{aligned}
\Big(\cos(\omega_1 t)\Big)^{l_1} \cdots \Big(\cos(\omega_k t)\Big)^{l_k} &= \prod_{j=1}^{k} \frac{1}{2^{l_j}} \prod_{j=1}^{k} \Big(e^{i\omega_j t} + e^{-i\omega_j t}\Big)^{l_j} \\
&= \prod_{j=1}^{k} \frac{1}{2^{l_j}} \prod_{j=1}^{k} \Big(\sum\nolimits_{r_j=0}^{l_j} \binom{l_j}{r_j} e^{i(2r_j - l_j)\omega_j t}\Big).
\end{aligned}$$

Whence

$$\prod_{j=1}^{k} \Big(\cos(\omega_j t)\Big)^{l_j} = \prod_{j=1}^{k} \frac{1}{2^{l_j}} \Big(\sum\nolimits_{r_1, r_2, \cdots, r_k = 0}^{l_1, l_2, \cdots, l_k} \binom{l_1}{r_1}\binom{l_2}{r_2} \cdots \binom{l_k}{r_k} e^{i(\sum_{j=1}^{k}(2r_j - l_j)\omega_j)t}\Big).$$

Because of linear independence,

$$\sum\nolimits_{j=1}^{k} (2r_j - l_j)\omega_j$$

can be zero only if $2r_i = l_i$, for any $i = 1, \cdots, k$, and thus it follows that

$$\fint_{\mathbb{R}} \prod_{j=1}^{k} \Big(\cos(\omega_j t)\Big)^{l_j} dt = \begin{cases} \displaystyle\prod_{j=1}^{k} \frac{1}{2^{l_j}} \binom{l_j}{\frac{l_j}{2}}, & \text{if all } l_i \text{ are even} \\ 0, & \text{otherwise.} \end{cases}$$

We conclude that

$$\fint_{\mathbb{R}} \prod_{j=1}^{k} \Big(\cos(\omega_j t)\Big)^{l_j} dt = \prod_{j=1}^{k} \fint_{\mathbb{R}} \Big(\cos(\omega_j t)\Big)^{l_j} dt$$

and this finish the proof of the theorem. □

In the following we recall the classical well known multidimensional Central Limit Theorem in probability theory [16, p.81] stated in the following forms

Theorem 5.3 (Multidimensional CLT Theorem)**.** Let $(Z_{n,k})_{1\leqslant k\leqslant k_n,\, n\geqslant 1}$ be a triangle array of random variables vectors in $\mathbb{C}^d$ and put

$$Z_{n,k} = (Z^1_{n,k}, Z^2_{n,k}, \dots, Z^d_{n,k}).$$

Suppose that

(1) The random variables $Z^j_{n,k}$ are square integrable.
(2) For each $n \in \mathbb{N}$, the $Z_{n,k}, 1 \leqslant k \leqslant k_n$, are independents.
(3) For any $\varepsilon > 0$, $\lim\limits_{n\to\infty} \sum\limits_{k=1}^{k_n} \mathbb{P}\Big(\|Z_{n,k}\| \geqslant \varepsilon\sqrt{k_n}\Big) = 0.$
(4) $\lim\limits_{n\to\infty} \dfrac{1}{\sqrt{k_n}} \sum\limits_{k=1}^{k_n} \mathbb{E}(Z_{n,k}) = 0.$
(5) For each $j, l \in \{1, \cdots, d\}$,
   (a) $\lim\limits_{n\to\infty} \dfrac{1}{k_n} \sum\limits_{k=1}^{k_n} \mathrm{Cov}(Z^j_{n,k}, Z^l_{n,k}) = \gamma_{j,l}.$
   (b) $\lim\limits_{n\to\infty} \dfrac{1}{k_n} \sum\limits_{k=1}^{k_n} \mathbb{E}(Z^j_{n,k}.Z^l_{n,k}) = 0.$

Then Γ is a hermitian non-negative definite matrix and the sequence of random vectors

$$\left(\frac{1}{\sqrt{k_n}} \sum_{k=1}^{k_n} Z_{n,k}\right)_{n\geqslant 1}$$

converges in distribution to the complex Gaussian measure $\mathcal{N}_{\mathbb{C}}(0,\Gamma)$ on $\mathbb{C}^d$.

We also need the following important and classical fact from Probability Theory connected to the notion of the uniform integrability.

Definition 5.4. The sequence $\{X_n, n \geqslant 1\}$ of random variables is said to be uniformly integrable if and only if

$$\lim_{c\longrightarrow+\infty} \int_{\{|X_n|>c\}} \big|X_n\big| d\mathbb{P} = 0 \text{ uniformly in } n.$$

It is well-known that if

$$\sup_{n\in\mathbb{N}}\left(\mathbb{E}\big(\big|X_n\big|^{1+\varepsilon}\big)\right) < +\infty, \tag{15}$$

for some ε positive, then $\{X_n\}$ are uniformly integrable.
Let us mention that the convergence in distribution or probability does not in general imply that the moments converge (even if they exist). The useful condition to ensure the convergence of the moments is the uniform integrability. Indeed, we have

Theorem 5.5. If the sequence of random variables $\{X_n\}$ converges in distribution to some random variable X and for some $p > 0$, $\sup_{n\in\mathbb{N}}\big(\mathbb{E}(|X_n|^p)\big) = M < +\infty$, then for each $r < p$,

$$\lim_{n\longrightarrow+\infty}\mathbb{E}\left(\big|X_n\big|^r\right) = \mathbb{E}\left(\big|X\big|^r\right)$$

For the proof of Theorem 5.5 we refer the reader to [5, p.32-33] or [13, p.100].

Now let us state and prove the Kac Central limit Theorem.

Theorem 5.6 (Kac CLT [28])**.** Let $(\lambda_n)_{n\in\mathbb{N}}$ be a sequence of rationally independent real numbers. Then the functions $\cos(\lambda_n t) + i\sin(\lambda_n t), n = 1, \cdots$, are stochastically independent under the Haar measure of the Bohr compactification of $\mathbb{R}$ and converge in distribution to the complex Gaussian measure $\mathcal{N}_{\mathbb{C}}(0,1)$ on $\mathbb{C}$.

Proof. By Theorem 5.2 the functions $\cos(\lambda_n t) + i\sin(\lambda_n t), n = 1, \cdots$, are stochastically independent under the Haar measure of $\mathbf{b}\mathbb{R}$ and it is straightforward to verify that the hypotheses of Central Limit Theorem 5.3 are satisfied. We conclude that the sequence

$$\left(\frac{1}{\sqrt{n}}\sum_{k=1}^{n} e^{i\lambda_k t}\right)_{n\geqslant 1}$$

converges in distribution to the complex Gaussian measure $\mathcal{N}_{\mathbb{C}}(0,1)$ on $\mathbb{C}$. □

6. Proof of the main result (Theorem 3.2)

Using the analogous lemma of Féjer's Lemma [53, p.49] combined with the CLT methods introduced in [2], we shall give a direct proof of the singularity of a large class of generalized Riesz products on $\mathbf{b}\mathbb{R}$. Therefore, our strategy is slightly different from the strategy of the proofs given by many authors in the case of the torus [10], [33], [34], [2]. Indeed, the crucial argument in their proofs is to estimate the following quantity

$$\int Q\big|\big|P_m\big|^2 - 1\big|dh. \tag{16}$$

Precisely, they showed that the weak limit point of the sequence $\big(\big||P_m|^2-1\big|\big)$ is bounded below by a positive constant and it is well-known that this implies the singularity of the generalized Riesz products (see for instance [2] or [33]).

Let us start our proof by proving the following lemma analogous to Féjer's Lemma [53, p.49]

Lemma 3. With the above notations we have

$$\limsup_{\substack{m\longrightarrow+\infty\\ m\in\mathcal{M}}}\int_{\mathbf{b}\mathbb{R}} Q\big|P_m\big|dh=\Big(\int_{\mathbf{b}\mathbb{R}} Qdh\Big)\Big(\limsup_{\substack{m\longrightarrow+\infty\\ m\in\mathcal{M}}}\int_{\mathbf{b}\mathbb{R}}|P_m|dh\Big).$$

Proof. By our assumption the sequence $\big(h_m,(s_{m,p_m-1})\big)_{m\in\mathcal{M}}$ is rationally independent. Hence, by Kac Theorem 5.2, for $m>h_{n_{k+1}}$, the function Q and P_m are stochastically independent. This allows us to write

$$\int_{\mathbf{b}\mathbb{R}} Q\big|P_m\big|dh=\int_{\mathbf{b}\mathbb{R}} Qdh\int_{\mathbf{b}\mathbb{R}}|P_m|dh,$$

which proves the lemma. □

Proof of Theorem 3.2. Applying Lemma 3, we proceed to construct inductively the sequence $n_1<n_2<n_k<\cdots$, such that, for any $k\geqslant 1$, we have

$$\int_{\mathbf{b}\mathbb{R}}\prod_{j=1}^{k+1}\big|P_{n_j}\big|dh\leqslant\frac{\sqrt{51\pi}}{100}\int_{\mathbf{b}\mathbb{R}}\prod_{j=1}^{k}\big|P_{n_j}\big|dh. \tag{17}$$

Indeed, by our assumption combined with Kac CLT 5.2, it follows that (P_m) converges in distribution to the complex Gaussian measure $\mathcal{N}_{\mathbb{C}}(0,1)$ on $\mathbb{C}$. But, according to (15), (P_m) is uniformly integrable. Hence, from Theorem 5.5, we get

$$\lim_{m\longrightarrow+\infty}\int_{\mathbf{b}\mathbb{R}}|P_m|dh=\int_{\mathbb{C}}\big|z\big|d\mathcal{N}_{\mathbb{C}}(0,1)(z)=\frac{\sqrt{\pi}}{2}. \tag{18}$$

We remind that the density of the standard complex normal distribution $\mathcal{N}_{\mathbb{C}}(0,1)$ is given by

$$f(z)=\frac{1}{\pi}e^{-|z|^2}.$$

Now assume that we have already construct $n_1<n_2<n_3<\cdots<n_k$ and apply (18) with $\varepsilon=\dfrac{\sqrt{\pi}}{100}$ combined with Lemma 3 to get $m>n_k$ such that

$$\int_{\mathbf{b}\mathbb{R}} Q|P_m|dh\leqslant\frac{51\sqrt{\pi}}{100}\int_{\mathbf{b}\mathbb{R}} Qdh.$$

Put $n_{k+1} = m$. Therefore the inequality (17) holds and by letting $k \longrightarrow +\infty$ we conclude that

$$\int_{\mathbf{b}\mathbb{R}} \prod_{j=1}^{k} \big|P_{n_j}\big| dh \xrightarrow[k\to+\infty]{} 0.$$

which yields by Bourgain criterion that μ is singular with respect to Haar measure and completes the proof. □

Remark 6.1. The fundamental argument in the proof above is based on Lemma 3 and therefore strongly depended on the assumption that along subsequence the positive real numbers $\big(h_m, (s_{m,p_m-1})\big)$ are linearly independent over the rationals. We argue that in the general case, one may use the methods of Bourgain [10], Klemes-Reinhold [33], Klemes [33] and el Abdalaoui [2] to establish the singularity of a large class of generalized Riesz products on $\mathbf{b}\mathbb{R}$. In particular, the case when (p_m) is bounded. In the forthcoming paper, we will show how to extend a classical results from the torus and real line setting to the generalized Riesz products on the Bohr compactification of $\mathbb{R}$.

Appendix.I. On The Flatness problem on $\mathbf{b}\mathbb{R}$.

We are concerned here with the flat polynomials issue in $\mathbf{b}\mathbb{R}$. First, we recall briefly the relevant fact on the flatness problem in the torus $\mathbb{T}$.

The problem of flatness go back to Littlewood in his 1968 famous paper [37]. In that paper, Littlewood introduce two class of complex polynomials $\mathcal{G}_n$ and $\mathcal{F}_n$ where n is a positive integer. The class $\mathcal{G}_n$ is a class of those polynomials $P(z) = \sum_{k=0}^{n} a_k z^k$ that are *unimodular*, that is, $|a_k| = 1$, for $k = \{0, \cdots, n\}$. $\mathcal{F}_n$ is the subclass of $\mathcal{G}_n$ with real coefficients, i.e., $a_k = \pm 1$, for $k = \{0, \cdots, n\}$. The polynomials P in $\mathcal{F}_n$ are nowadays called Littlewood polynomials. By Parseval's formula

$$\frac{1}{2\pi}\int_0^{2\pi} \left|P(e^{it})\right|^2 dt = n+1, \text{ for all } P \in \mathcal{G}_n.$$

Therefore, for all $P \in \mathcal{G}_n$,

$$\min_{|z|=1}\big|P(z)\big| < \sqrt{n+1} < \max_{|z|=1}\big|P(z)\big|$$

In [37] Littlewood raised the problem of the existence of a sequence (P_n) of unimodular polynomials such that

$$\max_{t\in\mathbb{R}} \frac{\big|P_n(e^{it})\big|}{\sqrt{n+1}} \xrightarrow[n\to\infty]{} 1.$$

Such sequence of unimodular polynomials are called ultraflat. Precisely, the usual definition of ultraflatness is given as follows

Definition 6.2. Let (n_k) be a sequence of positive integers and (ε_{n_k}) a sequence of positive real numbers tending to 0, we say that a sequence (P_{n_k}) of unimodular polynomials is (ε_{n_k})-ultraflat if

$$\max_{t\in\mathbb{R}}\left|\frac{\left|P_{n_k}(e^{it})\right|}{\sqrt{n_k+1}}-1\right|\leqslant\varepsilon_{n_k}.$$

The problem of existence of the ultraflat polynomials was solved affirmatively by Kahane in his 1980 paper [29]. Precisely, Kahane proved that there exists a sequence of (ε_n)-ultraflat with

$$\varepsilon_n=O\left(n^{-1/17}\sqrt{\log(n)}\right).$$

But as noted by Queffelec and Saffari [45] Kahane proof is some kind of miracle. This is due to the fact that Kahane work is inspired by Körner paper [35] and Körner paper is based on Byrnes paper [11]. But, in 1996 J. Benedetto and his student Hui-Chuan Wu discovered that Theorem 2 in Byrnes paper [11] was erroneous and as a consequence invalidated Körner main result. Fortunately Kahane proof was independent of Theorem 2 in Byrnes paper.

The problem of the existence of the ultraflat Littlewood polynomials is unsettled to this date and as pointed by Erdélyi [19] it is a common belief that there is no ultraflat sequence of Littlewood polynomials. As a consequence no long Barker sequences exist (see [8] and the references given there).

One more important class of polynomials is the class of polynomials with coefficients $a_k\in\{0,1\}$ called a class of idempotent polynomials for obvious reasons of being convolution idempotents and denoted by

$$I_n\stackrel{\text{def}}{=}\{P\in\mathbb{C}[X]\ :\ P(X)=\sum_0^{n-1}a_kX^k,a_k\in\{0,1\}\},\ n\in\mathbb{N}^*.$$

A subclass of I_n with constant term 1 is called a class of Newman polynomials and denoted by N_n.

Another extremal open problem in the class of Littewood polynomials or in the class of Newman polynomials is the problem of L^1-flatness. In the same way as the ultraflatness, the L^1-flatness is defined as follows

Definition 6.3. Let (n_k) be a sequence of positive integers and (ε_{n_k}) a sequence of positive real numbers tending to 0, we say that a sequence (P_{n_k}) of polynomials is $(\varepsilon_{n_k})-L^1$-flat if

$$\frac{\left\|P_{n_k}(e^{it})\right\|_1}{\left\|P_{n_k}(e^{it})\right\|_2}\geqslant 1-\varepsilon_{n_k}.$$

The problem of the existence of L^1-flat sequence of unimodular polynomials was solved by Newman in [40]. Later, M. Guenais inspired by Newman work constructed [24] a sequence of L^1-flat Littlewood polynomials on

$\prod_{n=0}^{+\infty} \mathbb{Z}/p_n\mathbb{Z}$ where $(p_n)_{n\geqslant 0}$ is an increasing sequence of prime numbers. As a consequence she established a existence of countable group action with simple spectrum and Haar component. Subsequently, el Abdalaoui and Lemańczyk [1] proved that the sequence of Littlewood polynomials constructed by Guenais is ultraflat. Before, J. Bourgain in his 1993 paper [10] conjectured that the supremum of the L^1 norm by L^2 norm over all idempotent polynomials on the circle must be strictly less than one. Precisely, he make the following conjecture

Conjecture 6.4 (Bourgain)**.** [10]

$$\sup_{n\geqslant 1}\left\{\sup_{P\in N_n}\left(\frac{\left\|P(e^{it})\right\|_1}{\left\|P(e^{it})\right\|_2}\right)\right\} < 1.$$

Using the Bourgain ideas, M. Guenais [24] connected the problem of the existence of L^1-flat sequence of Littlewood polynomials or Newman polynomials and the Banach problem on whether there exist a dynamical system with simple Lebesgue spectrum. Ulam in his book [52, p.76] stated the Banach problem in the following form

Question 6.5 (Banach Problem)**.** Does there exist a square integrable function $f(x)$ and a measure preserving transformation $T(x)$, $-\infty < x < \infty$, such that the sequence of functions $\{f(T^n(x)); n = 1, 2, 3, \cdots\}$ forms a complete orthogonal set in Hilbert space?

Let us formulate the Bourgain conjecture in context of Bohr compactification of $\mathbb{R}$. For that, let $\mathcal{P}(\mathbb{R})$ be the subspace of trigonometric polynomials on $\mathbb{R}$ and $(\omega_j)_{j\in\mathbb{N}}$ be an increasing sequence of real numbers. Put

$$J_{\omega,n} = \{P \in \mathcal{P}(\mathbb{R}) \ : \ P(t) = \sum_{k=0}^{n} a_k e^{i\omega_k t}, a_0 = 1, \ a_k \in \{0,1\}\}$$

Then, we can state the Bourgain conjecture in context of Bohr compactification of $\mathbb{R}$ in the following form

Conjecture 6.6 (Bourgain in $\mathbf{b}\mathbb{R}$)**.** for any increasing sequence $(\omega_j)_{j\in\mathbb{N}}$ of real numbers, we have

$$\sup_{n\geqslant 1}\left\{\sup_{P\in J_{\omega,n}}\left(\frac{\left\|P(e^{it})\right\|_1}{\left\|P(e^{it})\right\|_2}\right)\right\} < 1,$$

where $\left\|P(e^{it})\right\|_2^2 = \fint \left|\sum_{k=0}^{n-1} a_k e^{i\omega_k t}\right|^2 dt = 1 + \sum_{k=1}^{n-1} a_k^2$.

For the sequence $(\omega_j)_{j\in\mathbb{N}}$ of rationally independent numbers, we are able to prove the following proposition.

Proposition 6.7. Let $\left(\omega(j)\right)_{j=0}^{q_n-1}\Big)_{n\in\mathbb{N}}$ be a sequence of real numbers such that for any $n \in \mathbb{N}^*$, the real numbers $\left(\omega(j)\right)_{j=0}^{q_n-1}$ are rationally independent.

Then, we have

$$\fint_{\mathbb{R}} \left| \frac{1}{\sqrt{q_n}} \sum_{j=0}^{q_n-1} e^{i\omega(j)t} \right| dt \xrightarrow[n\to\infty]{} \sqrt{\frac{\pi}{2}}.$$

Since, for any $n \in \mathbb{N}^*$, the real numbers $\big(\omega(j)\big)_{j=0}^{q_n-1}$ are rationally independent, then we can apply the Kac CLT Theorem 5.2. But, the functions $\left(\frac{1}{\sqrt{q_n}} \sum_{j=0}^{q_n-1} e^{i\omega(j)t}\right)$ are in $L^2(\mathbf{b}\mathbb{R})$ we deduce that they are uniformly integrable. Hence, by Theorem 5.5, we conclude that

$$\lim_{n\longrightarrow+\infty} \fint_{\mathbb{R}} \left| \frac{1}{\sqrt{q_n}} \sum_{j=0}^{q_n-1} e^{i\omega(j)t} \right| dt = \sqrt{\frac{\pi}{2}}.$$

This finish the proof of the proposition.

Remark 6.8. The discussion above allows us to formulate the following questions

Question 6.9. Does there exist a sequence of L^1-flat Littlewood polynomials or of the L^1-flat Newman polynomials on $\mathbf{b}\mathbb{R}$?

Subsequently

Question 6.10. Does there exist a the ultraflat Littlewood or Newman polynomials on $\mathbf{b}\mathbb{R}$?

Remark 6.11. We remind here that in [42], A. A. Prikhod'ko constructed a sequence of trigonometric polynomials $(P_n)_{n\in\mathbb{N}}$ which is locally L^1-flat, that is, for any $0 < a < b$ and $\varepsilon > 0$ there exists a positive integer $n_0 = n_0(a, b, \varepsilon)$ such that for any $n > n_0$, we have

$$\frac{1}{b-a} \int_a^b \big| \big|P_n(x)\big|^2 - 1 \big| dx < \varepsilon.$$

The polynomials $P_n,\ n \in \mathbb{N}$ are given by

$$P_n(x) = \frac{1}{\sqrt{p_n}} \sum_{p=0}^{p_n-1} e^{i\omega_n(p)x},$$

where, $\omega_n(p) = \frac{m_n.p_n}{\varepsilon_n^2} e^{\frac{\varepsilon_n}{p_n}p}$, $(m_n, p_n)_{n\in\mathbb{N}}$ is a sequence of positive integers with $p_n \xrightarrow[n\to\infty]{} +\infty$ and $m_n \xrightarrow[n\to\infty]{} +\infty$, ε_n is a sequence of rational numbers which goes to 0 as n goes to ∞.

But, this sequence is not L^1-flat on the Bohr compactification of $\mathbb{R}$. Indeed, one may show that we have

$$\fint \big|P_n(x)\big|dx \xrightarrow[n\to+\infty]{} \sqrt{\frac{\pi}{2}}.$$

In addition, it is proved in [4] that this sequence is not L^1-flat on $\mathbb{R}$.

Appendix.II. On the Bohr compactification and Stone-Čech compactification of locally Abelian group.

Let G be a Locally Abelian compact group and denoted by τ its topology. Let $\widehat{G}$ be a group of the characters on G, that is, the continuous homomorphism from G to the torus $\mathbb{T} \overset{\text{def}}{=} \{t \in \mathbb{C} \ : \ |t| = 1\}$. Put

$$\mathcal{K} = \mathbb{T}^{\widehat{G}},$$

the space of all functions from $\widehat{G}$ to $\mathbb{T}$ equipped with the product topology (i.e.; the pointwise convergence topology). Therefore, there is a canonical injective homomorphism from G to $\mathcal{K}$ given by

$$\begin{aligned} e \ : \ & G \longrightarrow \mathcal{K} \\ & x \longmapsto e(x) \ : \ h \in \mathcal{K} \longmapsto e(x)(h) = h(x). \end{aligned}$$

e is called a dual homomorphism. By abuse of notation we denote by the same letter G the image of G under e. Therefore, G is equipped with the Bohr topology denoted by τ_b and the topology inherited from G. Hence $\tau_b \subset \tau$. By taking the closure of G with respect to the Bohr topology we get the Bohr compactification of G and we denoted it by bG. Thus, by construction, bG is compact.
Let us state the following useful lemma.

Lemma 4. Let (g_n) be a sequence of elements in G. Then (g_n) converge to g with respect to τ_b if and only if, for any character $h \in \widehat{G}$ we have

$$h(g_n) \xrightarrow[n\to\infty]{} h(g).$$

Proof. By the definition of the product topology on $\mathcal{K}$, a sequence of functions (f_n) converge to some function f if and only if, for any $h \in \widehat{G}$,

$$f_n(h) \xrightarrow[n\to\infty]{} f(h).$$

By taking $f_n = e(g_n)$ and $f = e(g)$ we get

$$e(g_n)(h) \xrightarrow[n\to\infty]{} e(g)(h),$$

which means that for any $h \in \widehat{G}$,

$$h(g_n) \xrightarrow[n\to\infty]{} h(g),$$

and the proof of the lemma is complete. $\square$

We deduce from the lemma the following crucial fact about the separability of the Bohr compactification of G. We recall that the topological space is separable if it is contains a countable, dense subset.

Proposition 6.12. If G is a separable with respect to τ topology then G is separable with respect to Bohr topology.

Proof. It is straightforward since the Bohr topology is contained in the usual topology. □

In the case of $\mathbb{R}$, the lemma 4 say that the sequence of real numbers (x_n) converge in the sense of the Bohr topology to x if and only if, for any $t \in \mathbb{R}$,

$$e^{ix_n t} \xrightarrow[n\to\infty]{} e^{ixt}.$$

But since the characters on $\mathbb{R}$ are Lipschitz we deduce easy that the Bohr topology is contained in the usual topology and Q is dense in the usual topology and Bohr topology.

Nevertheless, by the Hirodata-Kakutani Theorem, $\mathbb{R}$ equipped with the Bohr topology doesn't have a countable basis and it is often that the Bohr topology doesn't have a countable basis. Precisely, Hirodata-kakutani Theorem asserts

Theorem 6.13 (Hirodata-kakutani)**.** The Bohr compactification of a given locally compact Abelian group has a countable basis if and only if the union of the spectrum of all almost periodic functions is countable.

We recall that the Fourier coefficient of f on $\chi \in \widehat{G}$ is given by

$$\widehat{f}\big(\{\chi\}\big) = \int_G f(g)\overline{\chi}(g)dh(g),$$

where h is the Haar measure on G and $\chi \in \widehat{G}$ is in the spectrum of f if the Fourier coefficients of f on χ is not zero

6.1. **The Stone-Čech compactification and Bohr compactification.** In the following the comparative board of the compactification of Stone-Čech and the Bohr compactification. For short we write LC for Locally Compact and c.f. for continuous functions.

The Stone-Čech compactification	The Bohr compactification
Let X a LC topological space	Let G a LCA topological group
Consider $C = C(X, [0,1])$ the space of c. f. : $X \to [0,1]$	$A = \widehat{G}$
Embedding X in $[0,1]^C$	embedding G in $\mathbb{T}^A$
The embedding function is the valuation $e_x(f) = f(x)$	$e_g(f) = f(g)$
Take the closure of $e(X)$	Take the closure of $e(G)$
We denote the closure of $e(X)$ by βX	and the closure of $e(G)$ by $\overline{e(G)} = bG$

Acknowledgment 1. The author wishes to express his gratitude to M. Lemańczyk, J-P. Thouvenot, E. Lesigne, A. Bouziad, J-M. Strelcyn and A. A. Prikhod'ko for fruitful discussions on the subject. It is a pleasure for him to acknowledge the warm hospitality of the Poncelet French-Russian Mathematical Laboratory in Moscow where a part of this work has been done.

References

[1] E. H. el Abdalaoui and M. Lemańczyk, *Approximately transitive dynamical systems and simple spectrum,* Arch. Math. (Basel) 97 (2011), no. 2, 187-197.

[2] E. H. el Abdalaoui, *A new class of rank-one transformations with singular spectrum,* Ergodic Theory Dynam. Systems 27 (2007), no. 5, 1541-1555.

[3] E. H. el Abdalaoui, M. Lemańczyk, E. Lesigne and C. Ulcigrai, *Spectral disjointness in some class of rank one flows*, preprint.

[4] E. H. el Abdalaoui, *On the spectral type of some class of rank one flows*, preprint.

[5] P. Billingsley, *Convergence of probability measures,* Second edition. Wiley Series in Probability and Statistics: Probability and Statistics. A Wiley-Interscience Publication. John Wiley & Sons, Inc., New York, 1999.

[6] A.S. Besicovitch, *Almost Periodic Functions,* Cambridge University Press, 1932.

[7] H. Bohr, *Almost Periodic Functions,* Chelsea Pub. Co., New York, 1947.

[8] P. Borwein and M. J. Mossinghoff, *Barker sequences and flat polynomials,* (English summary) Number theory and polynomials, 71-88, London Math. Soc. Lecture Note Ser., 352, Cambridge Univ. Press, Cambridge, 2008.

[9] G. Brown and W. Moran, *Products of random variables and Kakutani's criterion for orthogonality of product measures,* J. London Math. Soc. (2) 10 (1975), part 4, 401-405.

[10] J. Bourgain, *On the spectral type of Ornstein class one transformations,* Isr. J. Math. , 84 (1993), 53-63.

[11] J. S. Byrnes,*On polynomials with coefficients of modulus one,* Bull. London Math. Soc., 9 (1977), no. 2, 171-176.

[12] J. R. Choksi and M. G. Nadkarni , *The maximal spectral type of rank one transformation,* Can. Math. Bull., 37 (1) (1994), 29-36.

[13] K. L. Chung, *A course in probability theory,* Third edition. Academic Press, Inc., San Diego, CA, 2001.

[14] J. Coquet, T. Kamae and M. Mendés France, *Sur la mesure spectrale de certaines suites arithmétiques,* (French) Bull. Soc. Math. France 105 (1977), no. 4, 369-384.

[15] C. Corduneanu, *Almost periodic functions,* With the collaboration of N. Gheorghiu and V. Barbu. Translated from the Romanian by Gitta Bernstein and Eugene Tomer. Interscience Tracts in Pure and Applied Mathematics, No. 22. Interscience Publishers [John Wiley & Sons], New York-London-Sydney, 1968.

[16] D. Dacunha-Castelle and M. Duflo, *Probabilités et statistiques. Tome 2,* (French) [Probability and statistics. Vol. 2], Problèmes à temps mobile [Movable-time problems], Collection Mathématiques Appliquées pour la Maîtrise [Collection of Applied Mathematics for the Master's Degree], Masson, Paris, 1983.

[17] G. Da Prato and J. Zabczyk, *Stochastic equations in infinite dimensions,* Encyclopedia of Mathematics and its Applications, 44. Cambridge University Press, Cambridge, 1992.

[18] N. Dunford and J. T. Schwartz, *Linear Operators. I and II,* Interscience Publishers, Inc., New York, 1958, 1963.

[19] T. Erdélyi, *How far is an ultraflat sequence of unimodular polynomials from being conjugate-reciprocal?,* Michigan Math. J., 49 (2001), no. 2, 259-264.

[20] J. Feldman, *Equivalence and perpendicularity of Gaussian processes,* Pacific J. Math. 8 1958 699-708.

[21] I.M. Gelfand, D.A. Raikov and G.E. Chilov, *Commutative normed rings,* Chelsea Pub. Co., New York, 1964.

[22] U. Grenander, *Stochastic processes and statistical inference,* Ark. Mat. 1, (1950). 195-277.

[23] S. A. Grigoryan and V. T. Tonev, *Shift-invariant uniform algebras on groups,* Instytut Matematyczny Polskiej Akademii Nauk. Monografie Matematyczne (New Series) [Mathematics Institute of the Polish Academy of Sciences. Mathematical Monographs (New Series)], 68. Birkhäuser Verlag, Basel, 2006.

[24] M. Guenais, *Étude spectrale de certains produits gauches en théorie ergodique*, Thèse de doctorat, Paris 13, 1997.

[25] Y. Hajek, *On a property of normal distribution of any stochastic process,* Math. J., 8 (83) 1958, 610-618.

[26] A. Hirotada and S. Kakutani, Bohr compactifications of a locally compact Abelian group, I. Proc. Imp. Acad. Tokyo 19, (1943). 476-480

[27] B. Host, J-F. Méla and F. Parreau, *Nonsingular transformations and spectral analysis of measures,* Bull. Soc. Math. France 119 (1991), no. 1, 33-90.

[28] M. Kac, *Statistical independence in probability, analysis and number theory.* The Carus Mathematical Monographs, No. 12 Published by the Mathematical Association of America. Distributed by John Wiley and Sons, Inc., New York, (1959).

[29] J-P. Kahane, *Sur les polynômes à coefficients unimodulaires,* (French) Bull. London Math. Soc. 12 (1980), no. 5, 321-342.

[30] S. Kakutani, *On equivalence of infinite product measures,* Ann. of Math. (2) 49, (1948). 214-224.

[31] Y. Katznelson, *An introduction to harmonic analysis,* Third edition. Cambridge Mathematical Library. Cambridge University Press, Cambridge, 2004.

[32] V. P. Khavin and N. K. Nikolśkiĩ, *Commutative harmonic analysis. I. General survey. Classical aspects,* (Russian) Translation by D. Khavinson and S. V. Kislyakov, Encyclopaedia of Mathematical Sciences, 15. Springer-Verlag, Berlin, 1991.

[33] I.Klemes, *The spectral type of staircase transformations,* Thohoku Math. J., 48 (1994), 247-258.

[34] I. Klemes & K. Reinhold, *Rank one transformations with singular spectre type,* Isr. J. Math., **98** (1997), 1-14.

[35] T. W. Körner,*On a polynomial of Byrnes,* Bull. London Math. Soc., 12 (1980), no. 3, 219-224.

[36] F. Ledrappier, *Des produits de Riesz comme mesures spectrales,* (French. English summary) Ann. Inst. H. Poincaré Sect. B (N.S.) 6 (1970), 335-344.

[37] J. E. Littlewood, *On polynomials* $\sum^{n} \pm z^m, \sum^{n} e^{\alpha m i} z^m$, $z = e^{\theta i}$, J. London Math. Soc., (41) 1966, 367-376.

[38] M. Morsli, *On modular approximation property in the Besicovitch-Orlicz space of almost periodic functions,* Comment. Math. Univ. Carolin. 38 (1997), no. 3, 485-496.

[39] M. G. Nadkarni, *Spectral theory of dynamical systems*, Birkhäuser, Cambridge, MA, 1998.

[40] D. J. Newman, *An L^1 extremal problem for polynomials,* Proc. Amer. Math. Soc., 16 1965, 1287-1290.

[41] J. Peyrière, *Étude de quelques propriétés des produits de Riesz*, Ann. Inst. Fourier, Grenoble, 25 2 (1975), 127–169.

[42] A. A. Prikhod'ko, *On flat trigonometric sums and ergodic flow with simple Lebesgue spectrum,* http://arxiv.org/abs/1002.2801.

[43] M. Queffelec, *Mesures spectrales associées à certaines suites arithmétiques,* (French) Bull. Soc. Math. France 107 (1979), no. 4, 385-421.

[44] M. Queffélec, *Substitution dynamical systems-spectral analysis,* Second edition. Lecture Notes in Mathematics, 1294. Springer-Verlag, Berlin, 2010.

[45] H. Queffelec and B. Saffari,*On Bernstein's inequality and Kahane's ultraflat polynomials,* J. Fourier Anal. Appl., 2 (1996), no. 6, 519-582.

[46] F. Riesz, *Über die Fourierkoeffizienten einer stetigen Funktion von beschränkter Schwankung,* (German) Math. Z. 2 (1918), no. 3-4, 312-315.

[47] G. Ritter, *On Kakutani's dichotomy theorem for infinite products of not necessarily independent functions,* Math. Ann. 239 (1979), no. 1, 35-53.

[48] G. Ritter, *On dichotomy of Riesz products,* Math. Proc. Cambridge Philos. Soc. 85 (1979), no. 1, 79-89.

[49] J. A. Rozanov, *Certain remarks to the work "On the densities of Gaussian distributions an the Wiener-Hopf integral equations",* (Russian) Teor. Verojatnost. i Primenen., 11 1966, 545-547.

[50] W. Rudin, *Fourier analysis on groups,* Reprint of the 1962 original. Wiley Classics Library. A Wiley-Interscience Publication. John Wiley & Sons, Inc., New York, 1990.

[51] I. E. Segal, *Distributions in Hilbert space and canonical systems of operators,* Trans. Amer. Math. Soc. 88 1958 12-41.

[52] S. M. Ulam, *Problems in modern mathematics,* Science Editions John Wiley & Sons, Inc., New York 1964.

[53] A. Zygmund, *Trigonometric series* vol. I, second ed., Cambridge Univ. Press, Cambridge, 1959.

Université de Rouen-Mathématiques, Labo. de Maths Raphael SALEM UMR 60 85 CNRS, Avenue de l'Université, BP.12, 76801 Saint Etienne du Rouvray - France.

E-mail address: elhoucein.elabdalaoui@univ-rouen.fr, Gmail:elabdelh@gmail.com