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To cite this version:

Jean-Yves Chemin, Isabelle Gallagher, Chloé Mullaert. The role of spectral anisotropy in the resolution of the three-dimensional Navier-Stokes equations. 2012. hal-00702784

HAL Id: hal-00702784

https://hal.science/hal-00702784v1

Preprint submitted on 31 May 2012

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THE ROLE OF SPECTRAL ANISOTROPY IN THE RESOLUTION OF THE THREE-DIMENSIONAL NAVIER-STOKES EQUATIONS

JEAN-YVES CHEMIN, ISABELLE GALLAGHER, AND CHLOÉ MULLAERT

Abstract. We present different classes of initial data to the three-dimensional, incompressible Navier-Stokes equations, which generate a global in time, unique solution though they may be arbitrarily large in the end-point function space in which a fixed-point argument may be used to solve the equation locally in time. The main feature of these initial data is an anisotropic distribution of their frequencies. One of those classes is taken from [5]-[6], and another one is new.

1. Introduction

In this article, we are interested in the construction of global smooth solutions which cannot be obtained in the framework of small data. Let us recall what the incompressible Navier-Stokes (with constant density) is:

$$(NS)\quad \begin{cases} \partial_t u + u\cdot\nabla u - \Delta u = -\nabla p \quad \text{in} \quad \mathbb{R}^+\times\mathbb{R}^3 \\ \operatorname{div} u = 0 \\ u_{|t=0} = u_0. \end{cases}$$

In all this paper $x = (x_h, x_3) = (x_1, x_2, x_3)$ will denote a generic point of $\mathbb{R}^3$ and we shall write $u = (u^h, u^3) = (u^1, u^2, u^3)$ for a vector field on $\mathbb{R}^3 = \mathbb{R}^2_h \times \mathbb{R}_v$. We also define the horizontal differentiation operators $\nabla^h \stackrel{\text{def}}{=} (\partial_1, \partial_2)$ and $\operatorname{div}_h \stackrel{\text{def}}{=} \nabla^h\cdot$, as well as $\Delta_h \stackrel{\text{def}}{=} \partial_1^2 + \partial_2^2$.

First, let us recall the history of global existence results for small data. In his seminal work [15], J. Leray proved in 1934 that if $\|u_0\|_{L^2}\|\nabla u_0\|_{L^2}$ is small enough, then there exists a global regular solution of (NS). Then in [8], H. Fujita and T. Kato proved in 1964 that if

$$\|u_0\|_{\dot H^{\frac12}} \stackrel{\text{def}}{=} \Big(\int_{\mathbb{R}^3} |\xi|\,|\widehat{u}_0(\xi)|^2 d\xi\Big)^{\frac12}$$

is small enough, then there exists a unique global solution in the space

$$C_b(\mathbb{R}^+;\dot H^{\frac12}) \cap L^4(\mathbb{R}^+;\dot H^1).$$

After works of many authors on this question (see in particular [11], [13], [17],and [3]), the optimal norm to express the smallness of the initial data was found on 2001 by H. Koch and D. Tataru in [14]. This is the BMO^{-1} norm. We are not going to define precisely this norm here. Let us simply notice that this norm is in between two Besov norms which can be easily defined. More precisely we have

$$\|u_0\|_{\dot B^{-1}_{\infty,\infty}} \lesssim \|u_0\|_{BM0^{-1}} \lesssim \|u_0\|_{\dot B^{-1}_{\infty,2}} \quad \text{with}$$

Key words and phrases. Navier-Stokes equations, global wellposedness, anisotropy.

$$\|u_0\|_{\dot B^{-1}_{\infty,\infty}} \stackrel{\text{def}}{=} \sup_{t>0} t^{\frac12}\|e^{t\Delta}u_0\|_{L^\infty} \quad \text{and} \quad \|u_0\|_{\dot B^{-1}_{\infty,2}} \stackrel{\text{def}}{=} \|e^{t\Delta}u_0\|_{L^2(\mathbb{R}^+;L^\infty)}.$$

Fisrt of all, let us mention that $\dot H^{\frac12}$ is continuously embedded in $\dot B^{-1}_{\infty,2}$. To have a more precise idea of what these spaces mean, let us observe that the space $\dot B^{-1}_{\infty,\infty}$ we shall denote by $\dot C^{-1}$ from now on, contains all the derivatives of order 1 of bounded functions. Let us give some examples. If we consider a divergence free vector field of the type

$$u_{\varepsilon,0}(x) = \frac{1}{\varepsilon}\cos\Big(\frac{x_3}{\varepsilon}\Big)\big(-\partial_2\phi(x), \partial_1\phi(x), 0\big)$$

for some given function ϕ in the Schwartz class of $\mathbb{R}^3$, then we have

$$\|u_{\varepsilon,0}\|_{\dot B^{-1}_{\infty,2}} \sim \|u_{\varepsilon,0}\|_{\dot C^{-1}} \sim 1 \quad \text{and} \quad \|u_{\varepsilon,0}\|_{\dot H^{\frac12}} \sim \varepsilon^{-\frac32}.$$

Another example which will be a great interest for this paper is the case when

$$u_{\varepsilon,0}(x) = \phi_0(\varepsilon x_3)\big(-\partial_2\phi(x_h), \partial_1\phi(x_h), 0\big).$$

As claimed by Proposition 1.1 of [5], we have, for small enough ε,

$$\|u_{\varepsilon,0}\|_{\dot C^{-1}} \geq \frac12\|\phi\|_{\dot C^{-1}(\mathbb{R}^2)}\|\phi_0\|_{L^\infty(\mathbb{R})}. \tag{1.1}$$

In this paper, we are going to consider inital data the regularity of which will be (at least) $\dot H^{\frac12}$. Our interest is focused on the size of the initial data measured in the $\dot C^{-1}$ norm.

Let us define $\mathcal{G}$ the set of divergence free vector fields in $\dot H^{\frac12}(\mathbb{R}^3)$ generating global smooth solutions to (NS) and let us recall some known results about the geometry of $\mathcal{G}$.

First of all, Fujita-Kato' theorem [8] can be interpreted as follows: the set $\mathcal{G}$ contains a ball of positive radius. Next let us assume that $\mathcal{G}$ is not the whole space $\dot H^{\frac12}$ (in other words, we assume that an initial data exists which generates singularities in finite time). Then there exists a critical radius ρ_c such that if u_0 is an initial data such that $\|u_0\|_{\dot H^{\frac12}} < \rho_c$, then u_0 generates a global regular solution and for any $\rho > \rho_c$, there exists an intial data of $\dot H^{\frac12}$ norm ρ which generates a singularity at finite time. Using the theory of profiles introduced in the context of Navier-Stokes equations by the second author (see [9]), W. Rusin and V. Sverak prove in [16] that the set (where $\mathcal{G}^c$ denotes the complement of $\mathcal{G}$ in $\dot H^{\frac12}$)

$$\mathcal{G}^c \cap \big\{u_0 \in \dot H^{\frac12} \,/\, \|u_0\|_{\dot H^{\frac12}} = \rho_c\big\}$$

is non empty and compact up to dilations and translations.

In collaboration with P. Zhang, the first two authors prove in [6] that any point u_0 of $\mathcal{G}$, is at the center of an interval I included in $\mathcal{G}$ and such that the length of I measured in the $\dot C^{-1}$ norm is arbitrary large. In other words for any u_0 in $\mathcal{G}$, there exist arbitrary large (in $\dot C^{-1}$) perturbations of this initial data that generate global solutions. As we shall see, the perturbations are strongly anisotropic.

Our aim is to give a new point of view about the important role played by anisotropy in the resolution of the Cauchy problem for (NS).

The first result we shall present shows that as soon as enough anisotropy is present in the initial data (where the degree of anisotropy is given by the norm of the data only), then it generates a global unique solution. A similar result can be found in [2, Theorem 1].

Theorem 1. *A constant c_0 exists which satisfies the following. If $(u_{\varepsilon,0})_{\varepsilon>0}$ is a family of divergence free vector field in $\dot{H}^{\frac{1}{2}}$ such that $\|u_{\varepsilon,0}\|_{\dot{H}^{\frac{1}{2}}} \leq \rho$ and satisfying*

$$\forall \xi \in \operatorname{Supp} \widehat{u}_{\varepsilon,0}\,, \quad \textit{either} \quad |\xi_h| \leq \varepsilon|\xi_3| \quad \textit{or} \quad |\xi_3| \leq \varepsilon|\xi_h|\,, \tag{1.2}$$

then, if $\varepsilon^4\|u_{\varepsilon,0}\|_{\dot{H}^{\frac{1}{2}}}$ is less than c_0, $u_{\varepsilon,0}$ belongs to $\mathcal{G}$.

Let us remark that this result has little to do with the precise structure of the equations: as will appear clearly in its proof in Section 2, it can actually easily be recast as a small data theorem, the smallness being measured in anisotropic Sobolev spaces. It is therefore of a different nature than the next Theorems 2 and 3, whose proofs on the contrary rely heavily on the structure of the nonlinearity (more precisely on the fact that the two-dimensional equations are globally well-posed).

The next theorem shows that as soon as the initial data has slow variations in one direction, then it generates a global solution, which, roughly speaking, corresponds to the case when the support in Fourier space of the initial data lies in the region where $|\xi_3| \leq \varepsilon|\xi_h|$. Furthermore, one can add to any initial data in $\mathcal{G}$ any such slowly varying data, and the superposition still generates a global solution (provided the variation is slow enough and the profile vanishes at zero).

Theorem 2 ([5],[6])**.** *Let $v_0^h = (v_0^1, v_0^2)$ be a horizontal, smooth divergence free vector field on $\mathbb{R}^3$ (i.e. v_0^h is in $L^2(\mathbb{R}^3)$ as well as all its derivatives), belonging, as well as all its derivatives, to $L^2(\mathbb{R}_{x_3}; \dot{H}^{-1}(\mathbb{R}^2))$; let w_0 be a smooth divergence free vector field on $\mathbb{R}^3$. Then, there exists a positive ε_0 depending on norms of v_0^h and w_0 such that, if $\varepsilon \leq \varepsilon_0$, then the following initial data belongs to $\mathcal{G}$:*

$$v_{\varepsilon,0}(x) \stackrel{\text{def}}{=} (v_0^h + \varepsilon w_0^h, w_0^3)(x_1, x_2, \varepsilon x_3)\,.$$

If moreover $v_0^h(x_1, x_2, 0) = w_0^3(x_1, x_2, 0) = 0$ for all $(x_1, x_2) \in \mathbb{R}^2$, and if u_0 belongs to $\mathcal{G}$, then there exists a positive number ε_0' depending on u_0 and on norms of v_0^h and w_0 such that if $\varepsilon \leq \varepsilon_0'$, the following initial data belongs to $\mathcal{G}$:

$$u_{\varepsilon,0} \stackrel{\text{def}}{=} u_0 + v_{\varepsilon,0}\,.$$

One can assume that v_0^h and w_0^3 have frequency supports in a given ring of $\mathbb{R}^3$, so that (1.2) holds. Nevertheless Theorem 1 not apply since $v_{\varepsilon,0}$ is of the order of $\varepsilon^{-\frac{1}{2}}$ in $\dot{H}^{\frac{1}{2}}$. Actually the proof of Theorem 2 is deeper than that of Theorem 1, as it uses the structure of the quadratic term in (NS). The proof of Theorem 2 may be found in [5] and [6], we shall not give it here. Note that Inequality (1.1) implies that $v_{\varepsilon,0}$ may be chosen arbitrarily large in $\dot{C}^{-1}$.

One formal way to translate the above result is that the vertical frequencies of the initial data $v_{\varepsilon,0}$ are actually very small, compared with the horizontal frequencies. The following theorem gives a statement in terms of frequency sizes, in the spirit of Theorem 1. However as already pointed out, Theorem 1 again does not apply because the initial data is too large in $\dot{H}^{\frac{1}{2}}$. Notice also that the assumption made in the statement of Theorem 2 that the profile should vanish at $x_3 = 0$ is replaced here by a smallness assumption in $L^2(\mathbb{R}^2)$.

Theorem 3. *Let $(v_{\varepsilon,0})_\varepsilon$ be a family of smooth divergence free vector field, uniformly bounded in the space $L^\infty(\mathbb{R}; \dot{H}^s(\mathbb{R}^2))$ for all $s \geq -1$, such that $(\sqrt{\varepsilon}\, v_{\varepsilon,0})_\varepsilon$ is uniformly bounded in the*

space $L^2(\mathbb{R}_{x_3};\dot{H}^s(\mathbb{R}^2))$ *for* $s\geq -1$, *and satisfying*

$$\forall\varepsilon\in]0,1[\,,\quad \forall\xi\in\operatorname{Supp}\widehat{v}_{\varepsilon,0}\,,\quad |\xi_3|\leq\varepsilon|\xi_h|\,.$$

Then there exists a positive number ε_0 *such that for all* $\varepsilon\leq\varepsilon_0$, *the data* $v_{\varepsilon,0}$ *belongs to* $\mathcal{G}$.

Moreover if u_0 *belongs to* $\mathcal{G}$, *then there are positive constants* c_0 *and* ε_0' *such that if*

$$\|v_{\varepsilon,0}(\cdot,0)\|_{L^2(\mathbb{R}^2_h)}\leq c_0$$

then for all $\varepsilon\leq\varepsilon_0'$, *the following initial data belongs to* $\mathcal{G}$:

$$u_{\varepsilon,0}\stackrel{\text{def}}{=}u_0+v_{\varepsilon,0}\,.$$

Let us remark that as in [5], the data $v_{\varepsilon,0}$ may be arbitrarily large in $\dot{C}^{-1}$. Note that Theorems 2 and 3, though of similar type, are not comparable (unless one imposes the spectrum of the initial profiles in Theorem 2 to be included in a ring of $\mathbb{R}^3$, in which case the result follows from Theorem 3).

The paper is organized as follows. In the second section, we introduce anisotropic Sobolev spaces and as a warm up, we prove Theorem 1.

The rest of the paper is devoted to the proof of Theorem 3. In the thid section, we define a (global) approximated solution and prove estimates on this approximated solutions and prove Theorem 3.

The last section is devoted to the proof of a propagation result for a linear transport diffusion equation we admit in the preceeding section. Let us point out that we make the choice not to use the technology anisotropic paradifferential calculus and to present an elementary proof.

2. Preliminaries: notation and anisotropic function spaces

In this section we recall the definition of the various function spaces we shall be using in this paper, namely anisotropic Lebesgue and Sobolev spaces.

We denote by $L^p_hL^q_v$ (resp. $L^q_v(L^p_h)$) the space $L^p(\mathbb{R}^2_h;L^q(\mathbb{R}_v))$ (resp. $L^q(\mathbb{R}_v;L^p(\mathbb{R}^2_h))$ equipped with the norm

$$\|f\|_{L^p_hL^q_v}\stackrel{\text{def}}{=}\left(\int_{\mathbb{R}^2_h}\Big(\int_{\mathbb{R}_v}|f(x_h,x_3)|^q\,dx_3\Big)^{\frac{p}{q}}dx_h\right)^{\frac{1}{p}}$$

and similarly $\dot{H}^{s,\sigma}$ is the space $\dot{H}^s(\mathbb{R}^2;\dot{H}^\sigma(\mathbb{R}))$ with

$$\|f\|_{\dot{H}^{s,\sigma}}\stackrel{\text{def}}{=}\left(\int_{\mathbb{R}^3}|\xi_h|^{2s}|\xi_3|^{2\sigma}|\widehat{f}(\xi_h,\xi_3)|^2\,d\xi_hd\xi_3\right)^{\frac{1}{2}}$$

where $\widehat{f}=\mathcal{F}f$ is the Fourier transform of f. Note that $\dot{H}^{s,\sigma}$ is a Hilbert space as soon as $s<1$ and $\sigma<1/2$. We define also

$$\|f\|_{\dot{H}^{s_1,s_2,s_3}}\stackrel{\text{def}}{=}\left(\int_{\mathbb{R}^3}|\xi_1|^{2s_1}|\xi_2|^{2s_2}|\xi_3|^{2s_3}|\widehat{f}(\xi_1,\xi_2,\xi_3)|^2\,d\xi_1d\xi_2d\xi_3\right)^{\frac{1}{2}}.$$

This is a Hilbert space if all s_j are less than 1/2. Finally we shall often usethe spaces $L^p_v \dot{H}^s_h = L^p(\mathbb{R}_v; \dot{H}^s(\mathbb{R}^2_h))$. Let us notice that $L^2_v \dot{H}^s_h = \dot{H}^{s,0}$ The following result, proved by D. Iftimie in [12] is the basis of the proof of Theorem 1.

Theorem 4. *There is a constant ε_0 such that the following result holds. Let $(s_i)_{1\leq i\leq 3}$ be such that $s_1+s_2+s_3 = 1/2$ and $-1/2 < s_i < 1/2$. Then any divergence free vector field of norm smaller than ε_0 in $\dot{H}^{s_1,s_2,s_3}$ generates a global smooth solution to (NS).*

This theorem implies obviously the following corollary, since $\dot{H}^{s,\frac{1}{2}-s}$ is continuously embedded in $\dot{H}^{\frac{s}{2},\frac{s}{2},\frac{1}{2}-s}$ as soon as $0 < s < 1/2$. More precisely, we have that the space $\dot{H}^{s,\frac{1}{2}-s}$ is the space $\dot{H}^{s,0,\frac{1}{2}-s} \cap \dot{H}^{0,s,\frac{1}{2}-s}$.

Corollary 2.1. *There is a constant ε_0 such that the following result holds. Let s be given in $]0,1/2[$. Then any divergence free vector field of norm smaller than ε_0 in $\dot{H}^{s,\frac{1}{2}-s}$ generates a global smooth solution to* (NS).

Proof of Theorem 1. Let us decompose u_0 into two parts, namely we write $u_0 = v_0 + w_0$, with

$$v_0 \stackrel{\text{def}}{=} \mathcal{F}^{-1}\big(\mathbf{1}_{|\xi_h|\leq\varepsilon|\xi_3|}\widehat{u}_0(\xi)\big) \quad \text{and} \quad w_0 \stackrel{\text{def}}{=} \mathcal{F}^{-1}\big(\mathbf{1}_{|\xi_3|\leq\varepsilon|\xi_h|}\widehat{u}_0(\xi)\big)\,.$$

Let $0 < s < 1/2$ be given. On the one hand we have

$$\|v_0\|^2_{\dot{H}^{s,\frac{1}{2}-s}} = \int_{|\xi_3|\leq\varepsilon|\xi_h|} |\xi_h|^{2s}|\xi_3|^{1-2s}|\widehat{u}_0(\xi)|^2\,d\xi$$

hence since $s < 1/2$,

$$\begin{aligned}\|v_0\|^2_{\dot{H}^{s,\frac{1}{2}-s}} &\leq \varepsilon^{1-2s}\int |\xi_h|\,|\widehat{u}_0(\xi)|^2\,d\xi\\ &\leq \varepsilon^{1-2s}\|u_0\|^2_{\dot{H}^{\frac{1}{2}}}\,.\end{aligned}$$

Identical computations give, since $s > 0$,

$$\begin{aligned}\|w_0\|^2_{\dot{H}^{s,\frac{1}{2}-s}} &= \int_{|\xi_h|\leq\varepsilon|\xi_3|} |\xi_h|^{2s}|\xi_3|^{1-2s}|\widehat{u}_0(\xi)|^2\,d\xi\\ &\leq \varepsilon^{2s}\int |\xi_3|\,|\widehat{u}_0(\xi)|^2\,d\xi\\ &\leq \varepsilon^{2s}\|u_0\|^2_{\dot{H}^{\frac{1}{2}}}\,.\end{aligned}$$

To conclude we can choose $s = 1/4$, which gives

$$\|u_0\|_{\dot{H}^{\frac{1}{4},\frac{1}{4}}} \leq \varepsilon^{\frac{1}{4}}\|u_0\|_{\dot{H}^{\frac{1}{2}}}\,.$$

Then, the result follows by the wellposedness of (NS) in $\dot{H}^{\frac{1}{4},\frac{1}{4}}$ given by Corollary 2.1. □

Remark 2.1. The proof of Theorem 1 does not use the special structure of the nonlinear term in (NS) as it reduces to checking that the initial data is small in an adequate scale-invariant space.

3. Proof of Theorem 3

In this section we shall prove the second part of Theorem 3: we consider an initial data $u_0+v_{\varepsilon,0}$ satisfying the assumptions of the theorem and we prove that for $\varepsilon > 0$ small enough, it generates a global, unique solution to (NS). It will be clear from the proof that in the case when $u_0 \equiv 0$ (which amounts to the first part of Theorem 3), the assumption that $v_{\varepsilon,0}(x_h, 0)$ is small in $L^2(\mathbb{R}^2)$ is not necessary. Thus the proof of the whole of Theorem 3 will be obtained.

3.1. Decomposition of the initial data.

The first step of the proof consists in decomposing the initial data as follows.

Proposition 3.1. *Let $v_{\varepsilon,0}$ be a divergence free vector field satisfying*

$$\forall \varepsilon \in]0,1[\,, \quad \forall \xi \in \operatorname{Supp} \widehat{v}_{\varepsilon,0}\,, \quad |\xi_3| \le \varepsilon |\xi_h|\,.$$

Then there exist two divergence free vector fields $\big(\overline{v}^h_{\varepsilon,0}, 0\big)$ and $w_{\varepsilon,0}$ the spectrum of which is included in that of $v_{\varepsilon,0}$, and such that

$$v_{\varepsilon,0} = (\overline{v}^h_{\varepsilon,0}, 0) + w_{\varepsilon,0} \quad \textit{with} \quad \big|\widehat{w}^h_{\varepsilon,0}\big| \le \varepsilon \big|\widehat{w}^3_{\varepsilon,0}\big|\,.$$

Proof. Let $\mathbb{P}_h \stackrel{\text{def}}{=} \mathrm{Id} - \nabla_h \Delta_h^{-1} \mathrm{div}_h$ be the Leray projector onto horizontal divergence free vector fields and define

$$\overline{v}^h_{\varepsilon,0} \stackrel{\text{def}}{=} \mathbb{P}_h\, v^h_{\varepsilon,0} \quad \text{and} \quad w_{\varepsilon,0} \stackrel{\text{def}}{=} v_{\varepsilon,0} - (\overline{v}^h_{\varepsilon,0}, 0)\,. \tag{3.3}$$

The estimate on $w_{\varepsilon,0}$ simply comes from the fact that obviously

$$\widehat{w}^h_{\varepsilon,0}(\xi) = \frac{\xi_h \cdot \widehat{v}^h_{\varepsilon,0}}{|\xi_h|^2}\, \xi_h\,,$$

and therefore since $v_{\varepsilon,0}$ is divergence free and using the spectral assumption we find

$$|\widehat{w}^h_{\varepsilon,0}(\xi)| \le \frac{|\xi_h \cdot \widehat{v}^h_{\varepsilon,0}|}{|\xi_h|} = \frac{|\xi_3 \widehat{v}^3_{\varepsilon,0}|}{|\xi_h|} \le \varepsilon |\widehat{v}^3_{\varepsilon,0}| = \varepsilon |\widehat{w}^3_{\varepsilon,0}(\xi)|\,.$$

That proves the proposition. □

3.2. Construction of an approximate solution and end of the proof of Theorem 3.

The construction of the approximate solution follows closely the ideas of [5]-[6]. We write indeed

$$v^{app}_\varepsilon \stackrel{\text{def}}{=} (\overline{v}^h_\varepsilon, 0) + w_\varepsilon \quad \text{and} \quad u^{app}_\varepsilon \stackrel{\text{def}}{=} u + v^{app}_\varepsilon\,,$$

where u is the global unique solution associated with u_0 and $\overline{v}^h_\varepsilon$ solves the two dimensional Navier-Stokes equations for each given x_3:

$$(\mathrm{NS2D})_{x_3} \quad \begin{cases} \partial_t \overline{v}^h_\varepsilon + \overline{v}^h_\varepsilon \cdot \nabla^h \overline{v}^h_\varepsilon - \Delta_h \overline{v}^h_\varepsilon = -\nabla^h \overline{p}_\varepsilon \quad \text{in} \quad \mathbb{R}^+ \times \mathbb{R}^2 \\ \mathrm{div}_h\, \overline{v}^h_\varepsilon = 0 \\ \overline{v}^h_{\varepsilon|t=0} = \overline{v}^h_{\varepsilon,0}(\cdot, x_3)\,, \end{cases}$$

while w_ε solves the linear transport-diffusion type equation

$$(\mathrm{T}) \quad \begin{cases} \partial_t w_\varepsilon + \overline{v}^h_\varepsilon \cdot \nabla^h w_\varepsilon - \Delta w_\varepsilon = -\nabla q_\varepsilon \quad \text{in} \quad \mathbb{R}^+ \times \mathbb{R}^3 \\ \mathrm{div}\, w_\varepsilon = 0 \\ w_{\varepsilon|t=0} = w_{\varepsilon,0}\,. \end{cases}$$

Those vector fields satisfy the following bounds (see Paragraph 3.3 for a proof).

Lemma 3.1. *Under the assumptions of Theorem 3, the family u_ε^{app} is uniformly bounded in $L^2(\mathbb{R}^+;L^\infty(\mathbb{R}^3))$, and $\nabla u_\varepsilon^{app}$ is uniformly bounded in $L^2(\mathbb{R}^+;L^\infty_v L^2_h)$.*

Now define u_ε the solution associated with the initial data $u_0+v_{\varepsilon,0}$, which a priori has a finite life span, depending on ε. Consider

$$R_\varepsilon \stackrel{\text{def}}{=} u_\varepsilon - u_\varepsilon^{app}\,,$$

which satisfies the following property (see Paragraph 3.4 for a proof).

Lemma 3.2. *For any positive δ there exists $\varepsilon(\delta)$ and $c(\delta)$ such that if*

$$\varepsilon \le \varepsilon(\delta) \quad \text{and if} \quad \|v_{\varepsilon,0}(\cdot,0)\|_{L^2_h} \le c(\delta),$$

then the vector field $R^\varepsilon \stackrel{\text{def}}{=} u_\varepsilon - u_\varepsilon^{app}$ satisfies the equation

$$(\mathrm{E}_\varepsilon)\begin{cases} \partial_t R_\varepsilon + R_\varepsilon\cdot\nabla R_\varepsilon - \Delta R_\varepsilon + u_\varepsilon^{app}\cdot\nabla R_\varepsilon + R_\varepsilon\cdot\nabla u_\varepsilon^{app} = F_\varepsilon - \nabla\widetilde{q}_\varepsilon \\ \operatorname{div} R_\varepsilon = 0 \\ R_{\varepsilon|t=0} = 0 \end{cases}$$

with $\|F_\varepsilon\|_{L^2(\mathbb{R}^+;\dot H^{-\frac12}(\mathbb{R}^3))} \le \delta$.

Assuming those two lemmas to be true, the end of the proof of Theorem 3 follows very easily using the method given in [5, Section 2]: an energy estimate in $\dot H^{\frac12}(\mathbb{R}^3)$ on (E_ε), using the fact that the forcing term is as small as needed and that the initial data is zero, gives that R_ε is unique, and uniformly bounded in $L^\infty(\mathbb{R}^+;\dot H^{\frac12})\cap L^2(\mathbb{R}^+;\dot H^{\frac32})$. Since the approximate solution is also unique and globally defined, Theorem 3 is proved. □

3.3. **Proof of the estimates on the approximate solution (Lemma 3.1).** As noted in [6, Appendix B], the global solution u associated with $u_0\in\dot H^{\frac12}$ belongs to $L^2(\mathbb{R}^+;L^\infty(\mathbb{R}^3))$, and ∇u belongs to $L^2(\mathbb{R}^+;L^\infty_v L^2_h)$. So we just need to study v_ε^{app}, which we shall do in two steps: first $\overline{v}_\varepsilon^h$, then w_ε.

3.3.1. *Estimates on $\overline{v}_\varepsilon^h$.* Due to the spectral assumption on $\overline{v}_{\varepsilon,0}^h$, it is easy to see that

$$\forall\alpha=(\alpha_h,\alpha_3)\in\mathbb{N}^2\times\mathbb{N}\,,\quad \varepsilon^{\frac12-\alpha_3}\partial^\alpha\overline{v}_{\varepsilon,0}^h \quad\text{is uniformly bounded in } L^2_v\dot H^s_h\,,$$
$$\text{and}\quad \varepsilon^{-\alpha_3}\partial^\alpha\overline{v}_{\varepsilon,0}^h \quad\text{is uniformly bounded in } L^\infty_v\dot H^s_h\,.$$

Indeed the definition of $\overline{v}_{\varepsilon,0}^h$ given in (3.3), and the spectral assumption as well as the a priori bounds on $v_{\varepsilon,0}$, give directly the first result. To prove the second result one uses first the Gagliardo-Nirenberg inequality:

$$\|\partial^\alpha\overline{v}_{\varepsilon,0}^h\|^2_{L^\infty_v\dot H^s_h} \le \|\partial^\alpha\overline{v}_{\varepsilon,0}^h\|_{L^2_v\dot H^s_h}\|\partial_3\partial^\alpha\overline{v}_{\varepsilon,0}^h\|_{L^2_v\dot H^s_h}\,,$$

and then the same arguments. The proof of [5, Lemma 3.1 and Corollary 3.1] enables us to infer from those bounds the following result.

Proposition 3.2. *Under the assumptions of Theorem 3, for all real numbers $s > -1$ and all $\alpha = (\alpha_h, \alpha_3) \in \mathbb{N}^2 \times \mathbb{N}$ there is a constant $\overline{C}$ such that the vector field $\overline{v}^h_\varepsilon$ satisfies the following bounds:*

$$\|\partial^\alpha \overline{v}^h_\varepsilon(t)\|^2_{L^\infty_v \dot{H}^s_h} + \sup_{x_3\in\mathbb{R}} \int_0^t \|\partial^\alpha \nabla^h \overline{v}^h_\varepsilon(t')\|^2_{\dot{H}^s_h}\, dt' \\ + \varepsilon\Big(\|\partial^\alpha \nabla^h \overline{v}^h_\varepsilon(t)\|^2_{L^2_v \dot{H}^s_h} + \int_0^t \|\partial^\alpha \overline{v}^h_\varepsilon(t')\|^2_{L^2(\mathbb{R}^3)}\, dt'\Big) \le \overline{C}\, \varepsilon^{2\alpha_3}\,.$$

3.3.2. *Estimates on w_ε.* The definition of $w_{\varepsilon,0}$ given in (3.3), along with the spectral assumption on $(v_{\varepsilon,0})_{\varepsilon>0}$ lead to

$$\forall \varepsilon \in]0,1[\,, \quad \forall \xi \in \operatorname{Supp} \widehat{w}_{\varepsilon,0}\,, \quad |\xi_3| \le \varepsilon |\xi_h| \quad \text{and} \quad \big|\widehat{w}^h_{\varepsilon,0}(\xi)\big| \le \varepsilon \big|\widehat{w}^3_{\varepsilon,0}(\xi)\big|\,.$$

The proof of the following result is technical and postponed to section 4.

Proposition 3.3. *Under the assumptions of Theorem 3, w^3_ε and $\varepsilon^{-1} w^h_\varepsilon$ are uniformly bounded in the space $L^\infty(\mathbb{R}^+;\, L^\infty_v L^2_h) \cap L^2(\mathbb{R}^+;\, L^\infty_v \dot{H}^s_h)$ for all $s \ge 0$. Moreover $\varepsilon^{\frac12 - \alpha_3} \partial^\alpha w_\varepsilon$ is uniformly bounded in $L^\infty(\mathbb{R}^+;\, L^2_v \dot{H}^s_h) \cap L^2(\mathbb{R}^+;\, L^2_v \dot{H}^s_h)$ for all $s \ge 0$ and all $\alpha = (\alpha_h, \alpha_3) \in \mathbb{N}^2 \times \mathbb{N}$.*

The Gagliardo-Nirenberg inequality and Sobolev embeddings lead to Lemma 3.1.

3.4. **Proof of the estimates on the remainder (Lemma 3.2).** Substracting the equation on u^{app}_ε from the equation on u one finds directly that

$$F_\varepsilon = (\partial_3^2 \overline{v}^h_\varepsilon, \partial_3 \overline{p}_\varepsilon) + w_\varepsilon \cdot \nabla v^{app}_\varepsilon + u \cdot \nabla v^{app}_\varepsilon + v^{app}_\varepsilon \cdot \nabla u\,,$$

which we decompose into $F_\varepsilon = G_\varepsilon + H_\varepsilon$ with

$$G_\varepsilon \stackrel{\text{def}}{=} (\partial_3^2 \overline{v}^h_\varepsilon, \partial_3 \overline{p}_\varepsilon) + w_\varepsilon \cdot \nabla v^{app}_\varepsilon \quad \text{and} \quad H_\varepsilon \stackrel{\text{def}}{=} u \cdot \nabla v^{app}_\varepsilon + v^{app}_\varepsilon \cdot \nabla u\,.$$

Lemma 3.2 follows from the two following propositions.

Proposition 3.4. *There is a positive constant C such that for all ε in $]0,1[$,*

$$\|G_\varepsilon\|_{L^2(\mathbb{R}^+;\dot{H}^{-\frac12}(\mathbb{R}^3))} \le C\varepsilon^{\frac12}\,.$$

Proof. Let us start by splitting G_ε in three parts: $G_\varepsilon = G^1_\varepsilon + G^2_\varepsilon + G^3_\varepsilon$ with

$$G^1_\varepsilon \stackrel{\text{def}}{=} (\partial_3^2 \overline{v}^h_\varepsilon, 0)\,, \; G^2_\varepsilon \stackrel{\text{def}}{=} (0, \partial_3 \overline{p}_\varepsilon)\,, \quad \text{and} \quad G^3_\varepsilon \stackrel{\text{def}}{=} w_\varepsilon \cdot \nabla v^{app}_\varepsilon\,.$$

On the one hand we have obviously

$$\|G^1_\varepsilon\|_{L^2(\mathbb{R}^+;\dot{H}^{-\frac12}(\mathbb{R}^3))} \le \|\partial_3 \overline{v}^h_\varepsilon\|_{L^2(\mathbb{R}^+;\dot{H}^{\frac12}(\mathbb{R}^3))}\,.$$

Proposition 3.2 applied with $\alpha = (0,1)$, $\alpha = (0,2)$ and $\alpha = (\alpha_h, 1)$ with $|\alpha_h| = 1$ gives

$$\int_0^t \|\partial_3 \overline{v}_\varepsilon(t',\cdot)\|^2_{L^2} dt' \lesssim \varepsilon \quad \text{and} \quad \int_0^t \|\partial_3 \nabla \overline{v}_\varepsilon(t',\cdot)\|^2_{L^2} dt' \lesssim \varepsilon.$$

By interpolation, we infer that

$$\|G^1_\varepsilon\|_{L^2(\mathbb{R}^+;\dot{H}^{-\frac12}(\mathbb{R}^3))} \lesssim \varepsilon^{\frac12}\,. \tag{3.4}$$

To estimate G^2_ε we use the fact that

$$-\Delta_h \overline{p}_\varepsilon = \sum_{j,k=1}^{2} \partial_j \partial_k (\overline{v}^j_\varepsilon \overline{v}^k_\varepsilon)$$

and since $(-\Delta_h)^{-1}\partial_j\partial_k$ is a Fourier multiplier of order 0 for each (j,k) in $\{1,2\}^2$ we get

$$\|G^2_\varepsilon\|_{L^2(\mathbb{R}^+;\dot{H}^{-\frac{1}{2}}(\mathbb{R}^3))} \lesssim \sum_{j,k=1}^{2} \left\|\overline{v}^j_\varepsilon \partial_3 \overline{v}^k_\varepsilon\right\|_{L^2(\mathbb{R}^+;\dot{H}^{-\frac{1}{2}}(\mathbb{R}^3))} .$$

As $L^2_v \dot{H}^{-\frac{1}{2}}_h \hookrightarrow \dot{H}^{-\frac{1}{2}}(\mathbb{R}^3)$, we get

$$\|G^2_\varepsilon\|_{L^2(\mathbb{R}^+;\dot{H}^{-\frac{1}{2}}(\mathbb{R}^3))} \lesssim \sum_{j,k=1}^{2} \left\|\overline{v}^j_\varepsilon \partial_3 \overline{v}^k_\varepsilon\right\|_{L^2(\mathbb{R}^+;L^2_v \dot{H}^{-\frac{1}{2}}_h)} .$$

Using the Sobolev embedding $L^{\frac{4}{3}}_h \hookrightarrow \dot{H}^{-\frac{1}{2}}_h$ and Hölder's inequality gives

$$\begin{aligned}\|G^2_\varepsilon\|_{L^2(\mathbb{R}^+;\dot{H}^{-\frac{1}{2}}(\mathbb{R}^3))} &\lesssim \sum_{j,k=1}^{2} \left\|\overline{v}^j_\varepsilon \partial_3 \overline{v}^k_\varepsilon\right\|_{L^2(\mathbb{R}^+;L^2_v L^{\frac{4}{3}}_h)} \\ &\leq C\|\overline{v}^h_\varepsilon\|_{L^\infty(\mathbb{R}^+;L^\infty_v L^{\frac{8}{3}}_h)} \|\partial_3 \overline{v}^h_\varepsilon\|_{L^\infty(\mathbb{R}^+;L^\infty_v L^{\frac{8}{3}}_h)}\end{aligned}$$

so the Sobolev embedding $\dot{H}^{\frac{1}{4}}_h \hookrightarrow L^{\frac{8}{3}}_h$ gives finally

$$\|G^2_\varepsilon\|_{L^2(\mathbb{R}^+;\dot{H}^{-\frac{1}{2}}(\mathbb{R}^3))} \lesssim C\|\overline{v}^h_\varepsilon\|_{L^\infty(\mathbb{R}^+;L^\infty_v \dot{H}^{\frac{1}{4}}_h)} \|\partial_3 \overline{v}^h_\varepsilon\|_{L^2(\mathbb{R}^+;L^2_v \dot{H}^{\frac{1}{4}}_h)} .$$

The result follows again from Proposition 3.2: choosing $s = 1/4$ and $\alpha = 0$ we get that $\overline{v}^h_\varepsilon$ is uniformly bounded in $L^\infty(\mathbb{R}^+; L^\infty_v \dot{H}^{\frac{1}{4}}_h)$, while $s = -3/4$ and $\alpha = (\alpha_h, 1)$ with $|\alpha_h| = 1$ gives

$$\|\partial_3 \overline{v}^h_\varepsilon\|_{L^2(\mathbb{R}^+;L^2_v \dot{H}^{\frac{1}{4}}_h)} \lesssim \varepsilon^{\frac{1}{2}} .$$

We infer finally that

$$\|G^2_\varepsilon\|_{L^2(\mathbb{R}^+;\dot{H}^{-\frac{1}{2}}(\mathbb{R}^3))} \lesssim \varepsilon^{\frac{1}{2}} . \tag{3.5}$$

To end the proof of the proposition let us estimate G^3_ε. We simply use two-dimensional product laws, which gives

$$\begin{aligned}\|G^3_\varepsilon\|_{L^2(\mathbb{R}^+;\dot{H}^{-\frac{1}{2}}(\mathbb{R}^3))} &= \|w_\varepsilon \cdot \nabla v^{app}_\varepsilon\|_{L^2(\mathbb{R}^+;\dot{H}^{-\frac{1}{2}}(\mathbb{R}^3))} \\ &\lesssim \|w^h_\varepsilon\|_{L^\infty(\mathbb{R}^+;L^\infty_v \dot{H}^{\frac{1}{4}}_h)} \|\nabla^h v^{app}_\varepsilon\|_{L^\infty(\mathbb{R}^+;L^2_v \dot{H}^{\frac{1}{4}}_h)} \\ &\qquad + \|w^3_\varepsilon\|_{L^\infty(\mathbb{R}^+;L^\infty_v \dot{H}^{\frac{1}{4}}_h)} \|\partial_3 v^{app}_\varepsilon\|_{L^\infty(\mathbb{R}^+;L^2_v \dot{H}^{\frac{1}{4}}_h)} \lesssim \varepsilon^{\frac{1}{2}} ,\end{aligned}$$

due to Propositions 3.2 and 3.3. Together with Inequalities (3.4) and (3.5) that proves Proposition 3.4. □

Proposition 3.5. *Let $\delta > 0$ be given. There are positive constants $\varepsilon(\delta)$ and $c(\delta)$ such that if $\varepsilon \leq \varepsilon(\delta)$ and if $\|v_{\varepsilon,0}(\cdot,0)\|_{L^2_h} \leq c(\delta)$, then*

$$\|H_\varepsilon\|_{L^2(\mathbb{R}^+;\dot{H}^{-\frac{1}{2}}(\mathbb{R}^3))} \leq \delta.$$

Proof. First, we approximate H_ε, and then we estimate this approximation.

Using [10, Theorem 2.1] we get

$$\lim_{t\to\infty} \|u\,(t,.)\,\|_{\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)} = 0$$

so we can approximate u in $L^\infty(\mathbb{R}^+,\dot{H}^{\frac{1}{2}})$: for all $\eta > 0$, there exists an integer N, real numbers $(t_j)_{0\leq j\leq N}$ and smooth, compactly supported, divergence free functions $(\phi_j)_{1\leq j\leq N}$ such that

$$\widetilde{u}_\eta(t) \stackrel{\text{def}}{=} \sum_{j=1}^{N} \mathbf{1}_{[t_{j-1},t_j]}\,(t)\,\phi_j$$

is uniformly bounded in $L^\infty(\mathbb{R}^+,\dot{H}^{\frac{1}{2}}) \cap L^2(\mathbb{R}^+,\dot{H}^{\frac{3}{2}})$ and satisfies

$$\|u - \widetilde{u}_\eta\|_{L^\infty(\mathbb{R}^+,\dot{H}^{\frac{1}{2}}(\mathbb{R}^3))} \leq \eta\,. \tag{3.6}$$

We split H_ε into two contributions

$$H_\varepsilon = H_{\varepsilon,\eta} + (\widetilde{u}_\eta - u)\cdot\nabla v_\varepsilon^{app} + v_\varepsilon^{app}\cdot\nabla(\widetilde{u}_\eta - u)$$

with $H_{\varepsilon,\eta} \stackrel{\text{def}}{=} \widetilde{u}_\eta\cdot\nabla v_\varepsilon^{app} + v_\varepsilon^{app}\cdot\nabla\widetilde{u}_\eta\,.$

As v_ε^{app} and $\widetilde{u}_\eta - u$ are divergence free vector fields,

$$H_\varepsilon - H_{\varepsilon,\eta} = \operatorname{div}\big((\widetilde{u}_\eta - u)\otimes v_\varepsilon^{app} + v_\varepsilon^{app}\otimes(\widetilde{u}_\eta - u)\big)\,.$$

Thanks to [6, Lemma 3.3] we get

$$\|H_\varepsilon - H_{\varepsilon,\eta}\|_{\dot{H}^{-\frac{1}{2}}} \lesssim \|\widetilde{u}_\eta - u\|_{\dot{H}^{\frac{1}{2}}}\big(\|\nabla^h v_\varepsilon^{app}\|_{L^\infty_v L^2_h} + \|v_\varepsilon^{app}\|_{L^\infty} + \|\partial_3 v_\varepsilon^{app}\|_{L^2_v\dot{H}^{\frac{1}{2}}_h}\big)$$

and Proposition 3.2 along with (3.6) lead to

$$\|H_\varepsilon - H_{\varepsilon,\eta}\|_{L^2(\mathbb{R}^+,\dot{H}^{-\frac{1}{2}}(\mathbb{R}^3))} \lesssim \eta\,.$$

It remains to estimate $H_{\varepsilon,\eta} = \widetilde{u}_\eta\cdot\nabla v_\varepsilon^{app} + v_\varepsilon^{app}\cdot\nabla\widetilde{u}_\eta$. By Propositions 3.2 and 3.3 we have

$$\begin{aligned}\|\widetilde{u}_\eta^3\partial_3 v_\varepsilon^{app}\|_{L^2(\mathbb{R}^+,\dot{H}^{-\frac{1}{2}}(\mathbb{R}^3))} &\lesssim \|\widetilde{u}_\eta^3\|_{L^\infty(\mathbb{R}^+,\dot{H}^{\frac{1}{2}}(\mathbb{R}^3))}\|\partial_3 v_\varepsilon^{app}\|_{L^2(\mathbb{R}^+,\dot{H}^{\frac{1}{2}}(\mathbb{R}^3))}\\ &\lesssim \|\widetilde{u}_\eta^3\|_{L^\infty(\mathbb{R}^+,\dot{H}^{\frac{1}{2}}(\mathbb{R}^3))}\,\varepsilon^{\frac{1}{2}}.\end{aligned}$$

Since $\widetilde{u}_\eta$ is uniformly bounded in $L^\infty(\mathbb{R}^+,\dot{H}^{\frac{1}{2}}(\mathbb{R}^3))$, we infer that

$$\lim_{\varepsilon\to 0}\|\widetilde{u}_\eta^3\partial_3 v_\varepsilon^{app}\|_{L^2(\mathbb{R}^+,\dot{H}^{-\frac{1}{2}}(\mathbb{R}^3))} = 0\,.$$

Lemma 3.4 of [6] claims that

$$\|ab\|_{\dot{H}^{-\frac{1}{2}}} \leq C\|a\|_{L^2_v\dot{H}^{\frac{1}{2}}_h}\|b(\cdot,0)\|_{L^2_h} + C\|x_3 a\|_{L^2}\|\partial_3 b\|_{L^\infty_v\dot{H}^{\frac{1}{2}}_h}.$$

So we get

$$\|\widetilde{u}_\eta^h\cdot\nabla^h v_\varepsilon^{app}\|_{\dot{H}^{-\frac{1}{2}}} \lesssim \|\widetilde{u}_\eta^h\|_{L^2_v\dot{H}^{\frac{1}{2}}_h}\|\nabla^h v_\varepsilon^{app}\,(\cdot,0)\,\|_{L^2_h} + \|x_3\widetilde{u}_\eta^h\|_{L^2}\|\partial_3\nabla^h v_\varepsilon^{app}\|_{L^\infty_v\dot{H}^{\frac{1}{2}}_h}$$

and

$$\|v_\varepsilon^{app}\cdot\nabla\widetilde{u}_\eta\|_{\dot H^{-\frac12}}\lesssim\|\nabla\widetilde{u}_\eta\|_{L^2_v\dot H^{\frac12}_h}\|v_\varepsilon^{app}(\cdot,0)\|_{L^2_h}+\|x_3\nabla\widetilde{u}_\eta\|_{L^2}\|\partial_3 v_\varepsilon^{app}\|_{L^\infty_v\dot H^{\frac12}_h}\,.$$

Propositions 3.2 and 3.3 lead to

$$\lim_{\varepsilon\to0}\int_{\mathbb{R}+}\|x_3\widetilde{u}_\eta^h(t)\|^2_{L^2(\mathbb{R}^3)}\|\partial_3\nabla^h v_\varepsilon^{app}(t)\|^2_{L^\infty_v\dot H^{\frac12}_h(\mathbb{R}^3)}\,dt=0$$

and

$$\lim_{\varepsilon\to0}\int_{\mathbb{R}+}\|x_3\nabla\widetilde{u}_\eta(t)\|^2_{L^2(\mathbb{R}^3)}\|\partial_3 v_\varepsilon^{app}(t)\|^2_{L^\infty_v\dot H^{\frac12}_h(\mathbb{R}^3)}\,dt=0\,.$$

Now we recall that $\widetilde{u}_\eta$ is uniformly bounded in $L^\infty(\mathbb{R}^+,\dot H^{\frac12})\cap L^2(\mathbb{R}^+,\dot H^{\frac32})$, hence $\widetilde{u}_\eta$ is uniformly bounded in $L^\infty(\mathbb{R}^+,L^2_v\dot H^{\frac12}_h)$ and $\nabla\widetilde{u}_\eta$ is uniformly bounded in $L^2(\mathbb{R}^+,L^2_v\dot H^{\frac12}_h)$. So in order to to conclude we just have to estimate

$$\|v_\varepsilon^{app}(\cdot,0)\|_{L^\infty(\mathbb{R}^+,L^2_h(\mathbb{R}^2))}+\|\nabla^h v_\varepsilon^{app}(\cdot,0)\|_{L^2(\mathbb{R}^+,L^2_h(\mathbb{R}^2))}\,.$$

This is done in the following proposition, which concludes the proof of Proposition 3.5. □

Proposition 3.6. *For all $\delta>0$ there are positive constants $\varepsilon(\delta)$ and $c(\delta)$ such that for all $0<\varepsilon\le\varepsilon(\delta)$, if $\|u_{\varepsilon,0}(\cdot,0)\|_{L^2_h}\le c(\delta)$ then*

$$\|v_\varepsilon^{app}(\cdot,0)\|_{L^\infty(\mathbb{R}^+,L^2_h(\mathbb{R}^2))}+\|\nabla^h v_\varepsilon^{app}(\cdot,0)\|_{L^2(\mathbb{R}^+,L^2_h(\mathbb{R}^2))}\le\delta\,.$$

Proof. First, we estimate $\overline{v}_\varepsilon^h$ and w_ε^h. For all $\varepsilon>0$, an energy estimate in L^2_h gives

$$\frac12\|\overline{v}_\varepsilon^h(t,\cdot,0)\|^2_{L^2_h}+\int_0^t\|\nabla^h\overline{v}_\varepsilon^h(t',\cdot,0)\|^2_{L^2_h}\,\mathrm{d}t'=\frac12\|v_{\varepsilon,0}(\cdot,0)\|^2_{L^2_h}\,. \tag{3.7}$$

Then, for all $\delta>0$ there is a constant $c(\delta)$ such that if $\|v_{\varepsilon,0}(\cdot,0)\|_{L^2_h}\le c(\delta)$ then

$$\|\overline{v}_\varepsilon^h(\cdot,0)\|_{L^\infty(\mathbb{R}^+,L^2_h(\mathbb{R}^2))}+\|\nabla^h\overline{v}_\varepsilon^h(\cdot,0)\|_{L^2(\mathbb{R}^+,L^2_h(\mathbb{R}^2))}\le\delta\,.$$

Moreover, by Proposition 3.3 we have

$$\|w_\varepsilon^h(\cdot,0)\|_{L^\infty(\mathbb{R}^+,L^2_h)}+\|\nabla^h w_\varepsilon^h(\cdot,0)\|_{L^2(\mathbb{R}^+,L^2_h)}\lesssim\varepsilon\,.$$

It remains to estimate w_ε^3. According to Proposition 3.3, w_ε and $\nabla^h w_\varepsilon$ are uniformly bounded respectively in $L^\infty(\mathbb{R}^+,L^\infty_v\dot H_h^{-\frac12})$ and $L^2(\mathbb{R}^+,L^\infty_v\dot H_h^{-\frac12})$, so we shall get the result by proving that for all $\delta>0$ there are positive constants $\varepsilon(\delta)$ and $c(\delta)$ such that if $\varepsilon\le\varepsilon(\delta)$ and $\|u_{\varepsilon,0}(\cdot,0)\|_{L^2_h}\le c(\delta)$ then

$$\|w_\varepsilon^3(\cdot,0)\|_{L^\infty(\mathbb{R}^+,\dot H^{\frac12}_h)}+\|\nabla^h w_\varepsilon^3(\cdot,0)\|_{L^2(\mathbb{R}^+,\dot H^{\frac12}_h)}\le\delta\,.$$

Recall that w_ε^3 satisfies

$$\begin{cases}\partial_t w_\varepsilon^3+\overline{v}_\varepsilon^h\cdot\nabla^h w_\varepsilon^3-\Delta_h w_\varepsilon^3=\partial_3^2 w_\varepsilon^3-\partial_3 q_\varepsilon\\ w^3_{\varepsilon|t=0}=w^3_{\varepsilon,0}\,.\end{cases}$$

Define $T_\varepsilon \stackrel{\text{def}}{=} \partial_3^2 w_\varepsilon^3 - \partial_3 q_\varepsilon$. An energy estimate in $\dot H_h^{\frac12}$ gives

$$\begin{aligned}\|w_\varepsilon^3\,(t,0)\,\|^2_{\dot H_h^{\frac12}} + \int_0^t \|\nabla^h w_\varepsilon^3\,(t',0)\,\|^2_{\dot H_h^{\frac12}}\mathrm{d}t' \\ \lesssim \|w^3_{\varepsilon,0}(\cdot,0)\|^2_{\dot H_h^{\frac12}} + \|T_\varepsilon(\cdot,0)\|^2_{L^2(\mathbb{R}^+,\dot H_h^{-\frac12})} + \int_0^t \big|\langle \overline{v}_\varepsilon^h\cdot\nabla^h w_\varepsilon^3, w_\varepsilon^3\rangle_{\dot H_h^{\frac12}}\big|(t',0)\mathrm{d}t'\,.\end{aligned} \tag{3.8}$$

Using [4, Lemma 1.1] we get for each fixed x_3

$$\big|\langle \overline{v}_\varepsilon^h\cdot\nabla^h w_\varepsilon^3, w_\varepsilon^3\rangle_{\dot H_h^{\frac12}}(x_3)\big| \lesssim \|\nabla^h\overline{v}_\varepsilon^h(x_3)\|_{L^2_h}\|\nabla^h w_\varepsilon^3(x_3)\|_{\dot H_h^{\frac12}}\|w_\varepsilon^3(x_3)\|_{\dot H_h^{\frac12}}.$$

In particular, using (3.7), we get

$$\begin{aligned}\int_0^t \big|\langle \overline{v}_\varepsilon^h\cdot\nabla^h w_\varepsilon^3, w_\varepsilon^3\rangle_{\dot H_h^{\frac12}}\,(t',0)\,\big|\mathrm{d}t' \\ \lesssim \|\nabla^h\overline{v}_\varepsilon^h(\cdot,0)\|_{L^2(\mathbb{R}^+,L^2_h)}\|\nabla^h w_\varepsilon^3(\cdot,0)\|_{L^2(\mathbb{R}^+,\dot H_h^{\frac12})}\|w_\varepsilon^3(\cdot,0)\|_{L^\infty(\mathbb{R}^+,\dot H_h^{\frac12})} \\ \lesssim \|\overline{v}^h_{\varepsilon,0}(\cdot,0)\|_{L^2_h}\|\nabla^h w_\varepsilon^3(\cdot,0)\|_{L^2(\mathbb{R}^+,\dot H_h^{\frac12})}\|w_\varepsilon^3(\cdot,0)\|_{L^\infty(\mathbb{R}^+,\dot H_h^{\frac12})}\end{aligned}$$

Then we infer that

$$\begin{aligned}\int_0^t \big|\langle \overline{v}_\varepsilon^h\cdot\nabla^h w_\varepsilon^3, w_\varepsilon^3\rangle_{\dot H_h^{\frac12}}\,(t',\cdot,0)\,\big|\mathrm{d}t' \lesssim \|u_{\varepsilon,0}(\cdot,0)\|_{L_h} \\ \times\Big(\|\nabla^h w_\varepsilon^3(\cdot,0)\|^2_{L^2(\mathbb{R}^+,\dot H_h^{\frac12})} + \|w_\varepsilon^3(\cdot,0)\|^2_{L^\infty(\mathbb{R}^+,\dot H_h^{\frac12})}\Big)\,.\end{aligned}$$

Plugging this inequality into (3.8) we obtain that there is a constant C such that

$$\begin{aligned}\|w_\varepsilon^3(\cdot,0)\|^2_{L^\infty(\mathbb{R}^+,\dot H_h^{\frac12})} + \big(1 - C\|u_{\varepsilon,0}(\cdot,0)\|_{L^2_h}\big)\|\nabla^h w_\varepsilon^3(\cdot,0)\|^2_{L^2(\mathbb{R}^+,\dot H_h^{\frac12})} \\ \lesssim \|w^3_{\varepsilon,0}(\cdot,0)\|^2_{\dot H_h^{\frac12}} + \|T_\varepsilon(\cdot,0)\|^2_{L^2(\mathbb{R}^+,\dot H_h^{-\frac12})} \\ \lesssim \|w^3_{\varepsilon,0}(\cdot,0)\|_{L^2_h}\|w^3_{\varepsilon,0}(\cdot,0)\|_{\dot H^1_h} + \|T_\varepsilon(\cdot,0)\|^2_{L^2(\mathbb{R}^+,\dot H_h^{-\frac12})}\,.\end{aligned}$$

As $w_{\varepsilon,0}$ is uniformly bounded in $L^\infty_v\dot H^1_h$, it remains to prove that

$$\lim_{\varepsilon\to 0}\|T_\varepsilon(\cdot,0)\|_{L^2(\mathbb{R}^+,\dot H_h^{-\frac12})} = 0\,.$$

As $\partial_3^2 w_\varepsilon^3 = -\partial_3 \mathrm{div}_h w_\varepsilon^h$, we get

$$\|\partial_3^2 w_\varepsilon^3(\cdot,0)\|_{L^2(\mathbb{R}^+,\dot H_h^{-\frac12})} \le \|\partial_3\nabla^h w_\varepsilon^h(\cdot,0)\|_{L^2(\mathbb{R}^+,\dot H_h^{-\frac12})} \le \|\partial_3\nabla^h w_\varepsilon^h\|_{L^2(\mathbb{R}^+,L^\infty_v\dot H_h^{-\frac12})}\,.$$

The bounds on w_ε given in Proposition 3.3 along with the Gagliardo-Nirenberg inequality lead to

$$\begin{aligned}\|\partial_3^2 w_\varepsilon^3(\cdot,0)\|_{L^2(\mathbb{R}^+,\dot H_h^{-\frac12})} &\le \|\partial_3\nabla^h w_\varepsilon^h\|^{\frac12}_{L^2(\mathbb{R}^+,L^2_v\dot H_h^{-\frac12})}\|\partial_3^2\nabla^h w_\varepsilon^h\|^{\frac12}_{L^2(\mathbb{R}^+,L^2_v\dot H_h^{-\frac12})} \\ &\lesssim \varepsilon^2\,.\end{aligned}$$

Now let us turn to the pressure term. Recall that

$$-\Delta q_\varepsilon = \mathrm{div}\,N_\varepsilon\,,\quad \text{with}\quad N_\varepsilon \stackrel{\text{def}}{=} \overline{v}_\varepsilon^h\cdot\nabla^h w_\varepsilon = \mathrm{div}_h\,(\overline{v}_\varepsilon^h\otimes w_\varepsilon)$$

since $\overline{v}^h_\varepsilon$ is divergence free. To estimate $\partial_3 q_\varepsilon(\cdot,0)$ we use Gagliardo-Nirenberg's inequality, according to which it suffices to estimate $\partial_3 q_\varepsilon$ in L^2_v and in $\dot H^1_v$.

Since $(-\Delta)^{-1}\mathrm{div}_h\,\mathrm{div}$ is a zero order Fourier multiplier, we have

$$\|\partial_3 q_\varepsilon\|_{L^2(\mathbb{R}^+,H^{-\frac12,1})} \lesssim \|\partial_3(\overline{v}^h_\varepsilon\otimes w_\varepsilon)\|_{L^2(\mathbb{R}^+,H^{-\frac12,1})}\,.$$

On the one hand we write

$$\|w_\varepsilon\partial_3\overline{v}^h_\varepsilon\|_{L^2(\mathbb{R}^+,L^2_v\dot H_h^{-\frac12})} \lesssim \|w_\varepsilon\|_{L^2(\mathbb{R}^+,L^2_v\dot H_h^{\frac12})}\|\partial_3\overline{v}^h_\varepsilon\|_{L^\infty(\mathbb{R}^+,L^\infty_v L^2_h)} \quad\lesssim \varepsilon^{\frac12}$$

by Propositions 3.2 and 3.3, and similarly

$$\|\overline{v}^h_\varepsilon\partial_3 w_\varepsilon\|_{L^2(\mathbb{R}^+,L^2_v\dot H_h^{-\frac12})} \lesssim \|\partial_3 w_\varepsilon\|_{L^2(\mathbb{R}^+,L^2_v\dot H_h^{\frac12})}\|\overline{v}^h_\varepsilon\|_{L^\infty(\mathbb{R}^+,L^\infty_v L^2_h)} \quad\lesssim \varepsilon^{\frac12}\,.$$

In the same way we find that

$$\|\partial_3(\overline{v}^h_\varepsilon\otimes w_\varepsilon)\|_{L^2(\mathbb{R}^+,\dot H^{-\frac12,1})} \lesssim \varepsilon^{\frac32}\,.$$

This ends the proof of Proposition 3.6. □

4. Estimates on the linear transport-diffusion equation

In this appendix we shall prove Proposition 3.3. It turns out to be convenient to rescale w_ε. Thus we define the vector field

$$W_\varepsilon(t,x) \stackrel{\text{def}}{=} \big(\frac{w^h_\varepsilon}{\varepsilon}, w^3_\varepsilon\big)(t,x_h,\varepsilon^{-1}x_3)$$

which satisfies

$$\begin{cases} \partial_t W_\varepsilon + \overline{V}^h_\varepsilon\cdot\nabla^h W_\varepsilon - \Delta_h W_\varepsilon - \varepsilon^2\,\partial_3^2 W_\varepsilon &= -\big(\nabla^h Q_\varepsilon, \varepsilon^2\,\partial_3 Q_\varepsilon\big)\\ \mathrm{div}\, W_\varepsilon &= 0\\ W_\varepsilon\,(0,.) &= W_{\varepsilon,0}\end{cases}$$

where

$$\overline{V}^h_\varepsilon(t,x) \stackrel{\text{def}}{=} \overline{v}^h_\varepsilon(t,x_h,\varepsilon^{-1}x_3) \qquad\text{and}\qquad Q_\varepsilon(t,x)\stackrel{\text{def}}{=}\varepsilon^{-1}q_\varepsilon(t,x_h,\varepsilon^{-1}x_3)\,.$$

Note that thanks to Proposition 3.2, the vector field $\partial^\alpha\overline{V}^h_\varepsilon$ is uniformly bounded in the space $L^\infty(\mathbb{R}^+,L^2_v\dot H^s_h)\cap L^2(\mathbb{R}^+,L^2_v\dot H^{s+1}_h)$ for each $\alpha\in\mathbb{N}^3$ and any $s>-1$, and hence also in $L^\infty(\mathbb{R}^+,L^\infty_v\dot H^s_h)\cap L^2(\mathbb{R}^+,L^\infty_v\dot H^{s+1}_h)$.

Similary we have defined

$$W_{\varepsilon,0}(x)\stackrel{\text{def}}{=}\big(\frac{w^h_{\varepsilon,0}}{\varepsilon}, w^3_{\varepsilon,0}\big)(x_h,\varepsilon^{-1}x_3)$$

and by construction it is bounded in $\dot H^s(\mathbb{R}^3)$ for all $s\geq -1$.

Proposition 3.3 is a corollary of the next statement.

Proposition 4.1. *Under the assumptions of Theorem 3, the following results hold.*

1. *For all $s>-1$, and all $\alpha\in\mathbb{N}^3$, $\partial^\alpha W_\varepsilon$ is bounded in $L^\infty(\mathbb{R}^+,L^2_v\dot H^s_h)\cap L^2(\mathbb{R}^+,L^2_v\dot H^{s+1}_h)$; in particular $\partial^\alpha W_\varepsilon$ is bounded in $L^\infty(\mathbb{R}^+,L^\infty_v\dot H^s_h)\cap L^2(\mathbb{R}^+,L^\infty_v\dot H^{s+1}_h)$.*
2. *For all $\alpha\in\mathbb{N}^3$, $\partial^\alpha W_\varepsilon$ is bounded in $L^2(\mathbb{R}^+,L^2)$, hence in particular in $L^2(\mathbb{R}^+,L^\infty_v L^2_h)$.*

Proof. Let us start by proving the first statement of the proposition. We notice that it is enough to prove the result for $s \in]-1,1[$, and we shall argue by induction on α.

• Let us start by considering the case $\alpha = 0$. An energy estimate in $L^2_v \dot{H}^s_h$ on the equation satisfied by W_ε gives

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|W_\varepsilon\|^2_{L^2_v\dot{H}^s_h} + \|\nabla^h W_\varepsilon\|^2_{L^2_v\dot{H}^s_h} + \varepsilon^2\|\partial_3 W_\varepsilon\|^2_{L^2_v\dot{H}^s_h}$$
$$= -\langle \overline{V}^h_\varepsilon \cdot \nabla^h W_\varepsilon, W_\varepsilon\rangle_{L^2_v\dot{H}^s_h} - \langle \nabla^h Q_\varepsilon, W^h_\varepsilon\rangle_{L^2_v\dot{H}^s_h} - \langle \varepsilon^2\,\partial_3 Q_\varepsilon, W^3_\varepsilon\rangle_{L^2_v\dot{H}^s_h}\,.$$

For the non-linear term we have, by [4, Lemma 1.1] and for each given t and x_3,

$$\big|\langle \overline{V}^h_\varepsilon \cdot \nabla^h W_\varepsilon, W_\varepsilon\rangle_{\dot{H}^s_h}(t,x_3)\big| \lesssim \|\nabla^h \overline{V}^h_\varepsilon(t,x_3)\|_{L^2_h}\|\nabla^h W_\varepsilon(t,x_3)\|_{\dot{H}^s_h}\|W_\varepsilon(t,x_3)\|_{\dot{H}^s_h}$$
$$\leq \frac{1}{4}\|\nabla^h W_\varepsilon(t,x_3)\|^2_{\dot{H}^s_h} + C\|\nabla^h \overline{V}^h_\varepsilon(t,x_3)\|^2_{L^2_h}\|W_\varepsilon(t,x_3)\|^2_{\dot{H}^s_h}$$

so after integration over x_3, we find

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|W_\varepsilon\|^2_{L^2_v\dot{H}^s_h} + \frac{3}{4}\|\nabla^h W_\varepsilon\|^2_{L^2_v\dot{H}^s_h} + \varepsilon^2\|\partial_3 W_\varepsilon\|^2_{L^2_v\dot{H}^s_h}$$
$$\leq C\|\nabla^h \overline{V}^h_\varepsilon\|^2_{L^\infty_v L^2_h}\|W_\varepsilon\|^2_{L^2_v\dot{H}^s_h} - \langle \nabla^h Q_\varepsilon, W^h_\varepsilon\rangle_{L^2_v\dot{H}^s_h} - \langle \varepsilon^2\,\partial_3 Q_\varepsilon, W^3_\varepsilon\rangle_{L^2_v\dot{H}^s_h}\,.$$

Now let us study the pressure term. As W_ε is a divergence free vector field we have

$$-\langle \nabla^h Q_\varepsilon, W^h_\varepsilon\rangle_{L^2_v\dot{H}^s_h} - \langle \varepsilon^2\,\partial_3 Q_\varepsilon, W^3_\varepsilon\rangle_{L^2_v\dot{H}^s_h} = (\varepsilon^2 - 1)\langle \nabla^h Q_\varepsilon, W^h_\varepsilon\rangle_{L^2_v\dot{H}^s_h}\,.$$

We claim that

$$\big|\langle \nabla^h Q_\varepsilon(t), W^h_\varepsilon(t)\rangle_{L^2_v\dot{H}^s_h}\big| \leq \frac{1}{4}\|\nabla^h W_\varepsilon(t)\|^2_{L^2_v\dot{H}^s_h} + C_\varepsilon(t)\|W_\varepsilon(t)\|^2_{L^2_v\dot{H}^s_h} \tag{4.1}$$

where C_ε is uniformly bounded in $L^1(\mathbb{R}^+)$. Assuming that claim to be true, we infer (up to changing C_ε) that

$$\frac{\mathrm{d}}{\mathrm{d}t}\|W_\varepsilon(t)\|^2_{L^2_v\dot{H}^s_h} + \|\nabla^h W_\varepsilon(t)\|^2_{L^2_v\dot{H}^s_h} + \varepsilon^2\|\partial_3 W_\varepsilon(t)\|^2_{L^2_v\dot{H}^s_h} \lesssim C_\varepsilon(t)\|W_\varepsilon(t)\|^2_{L^2_v\dot{H}^s_h}\,.$$

Thanks to Gronwall's lemma this gives

$$\|W_\varepsilon(t)\|^2_{L^2_v\dot{H}^s_h} + \int_0^t \|\nabla^h W_\varepsilon(t')\|^2_{L^2_v\dot{H}^s_h}\,dt' \lesssim \|W_{\varepsilon,0}\|^2_{L^2_v\dot{H}^s_h}\,,$$

and the conclusion of Proposition 4.1 (1), for $\alpha = 0$ and $-1 < s < 1$, comes from the a priori bounds on $W_{\varepsilon,0}$. It remains to prove the claim (4.1). For all real numbers r, we have

$$\big|\langle \nabla^h Q_\varepsilon(t), W^h_\varepsilon(t)\rangle_{L^2_v\dot{H}^s_h}\big| \leq \|\nabla^h Q_\varepsilon(t)\|_{L^2_v\dot{H}^r_h}\|W^h_\varepsilon(t)\|_{L^2_v\dot{H}^{2s-r}_h}\,.$$

As W_ε is a divergence free vector field we can write

$$\mathrm{div}\,(\overline{V}^h_\varepsilon \cdot \nabla^h W_\varepsilon) = -\Delta_h Q_\varepsilon - \varepsilon^2\,\partial_3^2 Q_\varepsilon\,.$$

Then we define

$$M^h_\varepsilon \stackrel{\mathrm{def}}{=} \overline{V}^h_\varepsilon \cdot \nabla^h W^h_\varepsilon + \partial_3(W^3_\varepsilon \overline{V}^h_\varepsilon)$$

and using the fact that $\overline{V}^h_\varepsilon$ is divergence free, we have

$$\mathrm{div}(\overline{V}^h_\varepsilon \cdot \nabla^h W_\varepsilon) = \mathrm{div}_h\, M^h_\varepsilon\,.$$

It follows that

$$Q_\varepsilon = (-\Delta_h - \varepsilon^2\,\partial_3^2)^{-1}\mathrm{div}_h\, M_\varepsilon^h\,, \tag{4.2}$$

and since $\nabla^h(-\Delta_h - \varepsilon^2\,\partial_3^2)^{-1}\mathrm{div}_h$ is a zero-order Fourier multiplier, we infer that for all real numbers r,

$$\|\nabla^h Q_\varepsilon\|_{L^2_v\dot H^r_h} \leq \|M_\varepsilon^h\|_{L^2_v\dot H^r_h}\,,$$

and therefore

$$|\langle \nabla^h Q_\varepsilon(t), W_\varepsilon^h(t)\rangle_{L^2_v\dot H^s_h}| \leq \|M_\varepsilon^h(t)\|_{L^2_v\dot H^r_h}\|W_\varepsilon^h(t)\|_{L^2_v\dot H^{2s-r}_h}\,. \tag{4.3}$$

We can estimate $\|M_\varepsilon^h\|_{L^2_v\dot H^r_h}$ as follows, thanks to the divergence-free condition on W_ε:

$$\begin{aligned}\|M_\varepsilon^h\|_{L^2_v\dot H^r_h} &\leq \|\overline{V}_\varepsilon^h\cdot\nabla^h W_\varepsilon\|_{L^2_v\dot H^r_h} + \|\partial_3(W_\varepsilon^3\overline{V}_\varepsilon^h)\|_{L^2_v\dot H^r_h}\\ &\leq \|\overline{V}_\varepsilon^h\cdot\nabla^h W_\varepsilon\|_{L^2_v\dot H^r_h} + \|W_\varepsilon^3\,\partial_3\overline{V}_\varepsilon^h\|_{L^2_v\dot H^r_h} + \|\overline{V}_\varepsilon^h\,\mathrm{div}_h W_\varepsilon^h\|_{L^2_v\dot H^r_h}\,.\end{aligned}$$

Thanks to two-dimensional product laws, if $-1<r<0$ then we get

$$\|\overline{V}_\varepsilon^h\cdot\nabla^h W_\varepsilon\|_{L^2_v\dot H^r_h} + \|\overline{V}_\varepsilon^h\,\mathrm{div}_h W_\varepsilon^h\|_{L^2_v\dot H^r_h} \lesssim \|\overline{V}_\varepsilon^h\|_{L^\infty_v\dot H^{\frac12}_h}\|\nabla^h W_\varepsilon\|_{L^2_v\dot H^{r+\frac12}_h}$$

and

$$\|W_\varepsilon^3\partial_3\overline{V}_\varepsilon^h\|_{L^2_v\dot H^r_h} \lesssim \|\nabla\overline{V}_\varepsilon^h\|_{L^\infty_v L^2_h}\|W_\varepsilon^3\|_{L^2_v\dot H^{r+1}_h}\,.$$

So if $-1<r<0$, then

$$\|M_\varepsilon^h\|_{L^2_v\dot H^r_h} \lesssim \|\overline{V}_\varepsilon^h\|_{L^\infty_v\dot H^{\frac12}_h}\|\nabla^h W_\varepsilon\|_{L^2_v\dot H^{r+\frac12}_h} + \|\nabla\overline{V}_\varepsilon^h\|_{L^\infty_v L^2_h}\|W_\varepsilon^3\|_{L^2_v\dot H^{r+1}_h} \tag{4.4}$$

and this leads to (4.1) for $-1<s<1$, due to the following computations.

- If $0<s<1$, we choose $r=s-1$ to get

$$\|M_\varepsilon^h\|_{L^2_v\dot H^{s-1}_h} \lesssim \|\overline{V}_\varepsilon^h\|_{L^\infty_v\dot H^{\frac12}_h}\|\nabla^h W_\varepsilon\|_{L^2_v\dot H^{s-\frac12}_h} + \|\nabla\overline{V}_\varepsilon^h\|_{L^\infty_v L^2_h}\|W_\varepsilon^3\|_{L^2_v\dot H^s_h}\,,$$

so by (4.3) with $r=s-1$, we infer that

$$\begin{aligned}\big|\langle\nabla^h Q_\varepsilon, W_\varepsilon^h\rangle_{L^2_v\dot H^s_h}\big| &\lesssim \|\overline{V}_\varepsilon^h\|_{L^\infty_v\dot H^{\frac12}_h}\|\nabla^h W_\varepsilon\|_{L^2_v\dot H^{s-\frac12}_h}\|\nabla^h W_\varepsilon^h\|_{L^2_v\dot H^s_h}\\ &\quad+\frac18\|\nabla^h W_\varepsilon^h\|^2_{L^2_v\dot H^s_h} + C\|\nabla\overline{V}_\varepsilon^h\|^2_{L^\infty_v L^2_h}\|W_\varepsilon^3\|^2_{L^2_v\dot H^s_h}\,.\end{aligned} \tag{4.5}$$

We then use the interpolation inequality

$$\|\overline{V}_\varepsilon^h\|_{L^\infty_v\dot H^{\frac12}_h}\|\nabla^h W_\varepsilon\|_{L^2_v\dot H^{s-\frac12}_h} \lesssim \|\overline{V}_\varepsilon^h\|^{\frac12}_{L^\infty_v L^2_h}\|\nabla^h\overline{V}_\varepsilon^h\|^{\frac12}_{L^\infty_v L^2_h}\|W_\varepsilon\|^{\frac12}_{L^2_v\dot H^s_h}\|\nabla^h W_\varepsilon\|^{\frac12}_{L^2_v\dot H^s_h}$$

along with the convexity inequality $ab\leq\frac34 a^{4/3}+\frac14 b^4$, to get

$$\begin{aligned}\|\nabla^h W_\varepsilon\|_{L^2_v\dot H^s_h}\|\overline{V}_\varepsilon^h\|_{L^\infty_v\dot H^{\frac12}_h}\|\nabla^h W_\varepsilon\|_{L^2_v\dot H^{s-\frac12}_h} &\leq \frac18\|\nabla^h W_\varepsilon\|^2_{L^2_v\dot H^s_h}\\ &\quad+C\|\overline{V}_\varepsilon^h\|^2_{L^\infty_v L^2_h}\|\nabla\overline{V}_\varepsilon^h\|^2_{L^\infty_v L^2_h}\|W_\varepsilon\|^2_{L^2_v\dot H^s_h}\,.\end{aligned}$$

It remains to define

$$C_\varepsilon(t) \stackrel{\mathrm{def}}{=} C\|\nabla\overline{V}_\varepsilon^h(t)\|^2_{L^\infty_v L^2_h}\big(1+\|\overline{V}_\varepsilon^h(t)\|^2_{L^\infty_v L^2_h}\big) \tag{4.6}$$

to obtain from (4.5) that

$$\big|\langle\nabla^h Q_\varepsilon(t), W^h_\varepsilon(t)\rangle_{L^2_v\dot H^s_h}\big| \le \frac{1}{4}\|\nabla^h W_\varepsilon(t)\|^2_{L^2_v\dot H^s_h} + C_\varepsilon(t)\|W_\varepsilon(t)\|^2_{L^2_v\dot H^s_h}\,.$$

Notice that C_ε belongs to $L^1(\mathbb{R}^+)$ thanks to the uniform bounds on $\overline{V}^h_\varepsilon$ derived above from Proposition 3.2.

∘ If $s=0$, we choose $r=-\frac{1}{2}$ and hence by (4.3) and (4.4),

$$\Big|\langle\nabla^h Q_\varepsilon, W^h_\varepsilon\rangle_{L^2}\Big| \lesssim \|W^h_\varepsilon\|_{L^2_v\dot H^{\frac12}_h}\big(\|\overline{V}^h_\varepsilon\|_{L^\infty_v\dot H^{\frac12}_h}\|\nabla^h W_\varepsilon\|_{L^2} + \|\nabla\overline{V}^h_\varepsilon\|_{L^\infty_v L^2_h}\|W^3_\varepsilon\|_{L^2_v\dot H^{\frac12}_h}\big)\,.$$

By interpolation we infer that

$$\begin{aligned}\big|\langle\nabla^h Q_\varepsilon, W^h_\varepsilon\rangle_{L^2}\big| \lesssim{}& \|W^h_\varepsilon\|^{\frac12}_{L^2}\,\|\nabla^h W^h_\varepsilon\|^{3/2}_{L^2}\,\|\overline{V}^h_\varepsilon\|^{\frac12}_{L^\infty_v L^2_h}\,\|\nabla\overline{V}^h_\varepsilon\|^{\frac12}_{L^\infty_v L^2_h}\\ &+ \|W_\varepsilon\|_{L^2}\|\nabla^h W_\varepsilon\|_{L^2}\|\nabla\overline{V}^h_\varepsilon\|_{L^\infty_v L^2_h}\,.\end{aligned}$$

The convexity inequality $ab \le \frac34 a^{4/3} + \frac14 b^4$ implies that

$$\begin{aligned}&\|W^h_\varepsilon\|^{\frac12}_{L^2}\,\|\nabla^h W^h_\varepsilon\|^{3/2}_{L^2}\,\|\overline{V}^h_\varepsilon\|^{\frac12}_{L^\infty_v L^2_h}\|\nabla\overline{V}^h_\varepsilon\|^{\frac12}_{L^\infty_v L^2_h}\\ &\qquad\le \frac18\|\nabla^h W_\varepsilon\|^2_{L^2} + C\|W^h_\varepsilon\|^2_{L^2}\|\overline{V}^h_\varepsilon\|^2_{L^\infty_v L^2_h}\|\nabla\overline{V}^h_\varepsilon\|^2_{L^\infty_v L^2_h}\end{aligned}\tag{4.7}$$

and

$$\|W_\varepsilon\|_{L^2}\|\nabla^h W_\varepsilon\|_{L^2}\|\nabla\overline{V}^h_\varepsilon\|_{L^\infty_v L^2_h} \le \frac18\|\nabla^h W_\varepsilon\|^2_{L^2} + C\|W_\varepsilon\|^2_{L^2}\|\nabla\overline{V}^h_\varepsilon\|^2_{L^\infty_v L^2_h}\,.\tag{4.8}$$

With the above choice (4.6) for C_ε we obtain

$$\big|\langle\nabla^h Q_\varepsilon(t), W^h_\varepsilon(t)\rangle_{L^2}\big| \le \frac14\|\nabla^h W_\varepsilon(t)\|^2_{L^2} + C_\varepsilon(t)\|W_\varepsilon(t)\|^2_{L^2}\,.$$

∘ Finally if $-1<s<0$, we proceed slightly differently. We recall that

$$\mathrm{div}_h\, M^h_\varepsilon = -\Delta_h Q_\varepsilon - \varepsilon^2\,\partial_3^2 Q_\varepsilon$$

and as W_ε is divergence free, we have

$$M^h_\varepsilon = \overline{V}^h_\varepsilon\cdot\nabla^h W^h_\varepsilon - \overline{V}^h_\varepsilon\,\mathrm{div}_h W^h_\varepsilon + W^3_\varepsilon\,\partial_3\overline{V}^h_\varepsilon\,.$$

Defining

$$M^h_{\varepsilon,1} \stackrel{\text{def}}{=} \mathrm{div}_h\,(\overline{V}^h_\varepsilon\otimes W^h_\varepsilon - W^h_\varepsilon\otimes\overline{V}^h_\varepsilon) \quad\text{and}\quad M^h_{\varepsilon,2}\stackrel{\text{def}}{=} W_\varepsilon\cdot\nabla\overline{V}^h_\varepsilon\,,$$

we can split $M^h_\varepsilon = M^h_{\varepsilon,1} + M^h_{\varepsilon,2}$ and estimate each term differently.

Since $\nabla^h(-\Delta_h - \varepsilon^2\partial_3^2)\mathrm{div}_h$ is a zero-order Fourier multiplier,

$$\big|\langle\nabla^h Q_\varepsilon, W^h_\varepsilon\rangle_{L^2_v\dot H^s_h}\big| \le \|M^h_{\varepsilon,1}\|_{L^2_v\dot H^{s-1}_h}\|W^h_\varepsilon\|_{L^2_v\dot H^{s+1}_h} + \|M^h_{\varepsilon,2}\|_{L^2_v\dot H^s_h}\|W^h_\varepsilon\|_{L^2_v\dot H^s_h}\,.$$

Using two-dimensional product laws we obtain

$$\|M^h_{\varepsilon,1}\|_{L^2_v\dot H^{s-1}_h} \lesssim \|\overline{V}^h_\varepsilon\,W_\varepsilon\|_{L^2_v\dot H^s_h} \lesssim \|\overline{V}^h_\varepsilon\|_{L^\infty_v\dot H^{\frac12}_h}\|W_\varepsilon\|_{L^2_v\dot H^{s+\frac12}_h}$$

and

$$\|M^h_{\varepsilon,2}\|_{L^2_v\dot H^s_h} \lesssim \|W_\varepsilon\cdot\nabla\overline{V}^h_\varepsilon\|_{L^2_v\dot H^s_h} \lesssim \|\nabla\overline{V}^h_\varepsilon\|_{L^\infty_v L^2}\|W_\varepsilon\|_{L^2_v\dot H^{s+1}_h}\,.$$

Therefore, we get

$$
\begin{aligned}
|\langle \nabla^h Q_\varepsilon, W_\varepsilon^h\rangle_{L^2_v\dot H^s_h}| \leq \|\overline{V}_\varepsilon^h\|_{L^\infty_v \dot H_h^{\frac12}} \|W_\varepsilon\|_{L^2_v \dot H_h^{s+\frac12}} \|\nabla^h W_\varepsilon^h\|_{L^2_v\dot H^s_h} \\
+\|\nabla \overline{V}_\varepsilon^h\|_{L^\infty_v L^2} \|\nabla^h W_\varepsilon\|_{L^2_v\dot H^s_h} \|W_\varepsilon^h\|_{L^2_v\dot H^s_h}\,.
\end{aligned}
\tag{4.9}
$$

Then we use the interpolation inequality

$$
\|W_\varepsilon\|_{L^2_v \dot H_h^{s+\frac12}} \|\nabla^h W_\varepsilon^h\|_{L^2_v\dot H^s_h} \lesssim \|W_\varepsilon\|^{\frac12}_{L^2_v\dot H^s_h} \|\nabla^h W_\varepsilon^h\|^{3/2}_{L^2_v\dot H^s_h}
$$

along with the convexity inequalities $ab \leq \frac34 a^{4/3} + \frac14 b^4$ and $ab \leq \frac12 a^2 + \frac12 b^2$, to infer that again with the choice (4.6) for C_ε,

$$
\big|\langle \nabla^h Q_\varepsilon(t), W_\varepsilon^h(t)\rangle_{L^2_v\dot H^s_h}\big| \leq \frac14 \|\nabla^h W_\varepsilon(t)\|^2_{L^2_v\dot H^s_h} + C_\varepsilon(t)\|W_\varepsilon(t)\|^2_{L^2_v\dot H^s_h}\,.
$$

The first result of the proposition is therefore proved in the case when $\alpha = 0$ and $-1 < s < 1$.

• To go further in the induction process, let $k \in \mathbb{N}$ be given and suppose the result proved for all $\alpha \in \mathbb{N}^3$ such that $|\alpha| \leq k$, still for $-1 < s < 1$. Now consider $\alpha \in \mathbb{N}^3$ such that $|\alpha| = k+1$. The vector field $\partial^\alpha W_\varepsilon$ solves

$$
\partial_t \partial^\alpha W_\varepsilon + \partial^\alpha(\overline{V}_\varepsilon^h \cdot \nabla^h W_\varepsilon) - \Delta_h \partial^\alpha W_\varepsilon - \varepsilon^2 \partial_3^2 \partial^\alpha W_\varepsilon = -(\nabla^h \partial^\alpha Q_\varepsilon, \varepsilon^2 \partial_3 \partial^\alpha Q_\varepsilon)\,.
$$

An energy estimate in $L^2_v \dot H^s_h$ gives

$$
\begin{aligned}
\frac12 \frac{\mathrm{d}}{\mathrm{d}t} \|\partial^\alpha W_\varepsilon\|^2_{L^2_v\dot H^s_h} + \langle \partial^\alpha(\overline{V}_\varepsilon^h \cdot \nabla^h W_\varepsilon), \partial^\alpha W_\varepsilon\rangle_{L^2_v\dot H^s_h} + \|\nabla^h \partial^\alpha W_\varepsilon\|^2_{L^2_v\dot H^s_h} \\
= -\langle \nabla^h \partial^\alpha Q_\varepsilon, \partial^\alpha W_\varepsilon^h\rangle_{L^2_v\dot H^s_h} - \varepsilon^2 \langle \partial_3 \partial^\alpha Q_\varepsilon, \partial^\alpha W_\varepsilon^3\rangle_{L^2_v\dot H^s_h}\,.
\end{aligned}
$$

We split $\langle \partial^\alpha(\overline{V}_\varepsilon^h \cdot \nabla^h W_\varepsilon), \partial^\alpha W_\varepsilon\rangle_{L^2_v\dot H^s_h}$ into two contributions:

$$
\langle \overline{V}_\varepsilon^h \cdot \nabla^h \partial^\alpha W_\varepsilon, \partial^\alpha W_\varepsilon\rangle_{L^2_v\dot H^s_h} + \sum_{0<\beta\leq\alpha} C_\beta \langle \partial^\beta \overline{V}_\varepsilon^h \cdot \nabla^h \partial^{\alpha-\beta} W_\varepsilon, \partial^\alpha W_\varepsilon\rangle_{L^2_v\dot H^s_h}\,.
\tag{4.10}
$$

The first term in (4.10) satisfies, as in [4, Lemma 1.1]

$$
\begin{aligned}
|\langle \overline{V}_\varepsilon^h \cdot \nabla^h \partial^\alpha W_\varepsilon, \partial^\alpha W_\varepsilon\rangle_{\dot H^s_h}| &\lesssim \|\nabla^h \overline{V}_\varepsilon^h\|_{L^2_h} \|\nabla^h \partial^\alpha W_\varepsilon\|_{\dot H^s_h} \|\partial^\alpha W_\varepsilon\|_{\dot H^s_h} \\
&\leq \frac14 \|\nabla^h \partial^\alpha W_\varepsilon\|^2_{\dot H^s_h} + C\|\nabla^h \overline{V}_\varepsilon^h\|^2_{L^2_h} \|\partial^\alpha W_\varepsilon\|^2_{\dot H^s_h}
\end{aligned}
$$

so

$$
|\langle \overline{V}_\varepsilon^h \cdot \nabla^h \partial^\alpha W_\varepsilon, \partial^\alpha W_\varepsilon\rangle_{L^2_v\dot H^s_h}| \leq \frac14 \|\nabla^h \partial^\alpha W_\varepsilon\|^2_{L^2_v\dot H^s_h} + C\|\nabla^h \overline{V}_\varepsilon^h\|^2_{L^\infty_v L^2_h} \|\partial^\alpha W_\varepsilon\|^2_{L^2_v\dot H^s_h}\,.
$$

For the remaining terms in (4.10), as $\overline{V}_\varepsilon^h$ is a horizontal, divergence free vector field, two-dimensional product laws give

$$
\begin{aligned}
|\langle \partial^\beta \overline{V}_\varepsilon^h \cdot \nabla^h \partial^{\alpha-\beta} W_\varepsilon, \partial^\alpha W_\varepsilon\rangle_{\dot H^s_h}| &= \big|\langle \mathrm{div}_h\big(\partial^\beta \overline{V}_\varepsilon^h \otimes \partial^{\alpha-\beta} W_\varepsilon\big), \partial^\alpha W_\varepsilon\rangle_{\dot H^s_h}\big| \\
&\lesssim \|\partial^\beta \overline{V}_\varepsilon^h \otimes \partial^{\alpha-\beta} W_\varepsilon\|_{\dot H^s_h} \|\nabla^h \partial^\alpha W_\varepsilon\|_{\dot H^s_h} \\
&\lesssim \|\partial^\beta \overline{V}_\varepsilon^h\|_{\dot H_h^{\frac{s+1}{2}}} \|\partial^{\alpha-\beta} W_\varepsilon\|_{\dot H_h^{\frac{s+1}{2}}} \|\nabla^h \partial^\alpha W_\varepsilon\|_{\dot H^s_h}
\end{aligned}
$$

so

$$|\langle\partial^\beta\overline{V}^h_\varepsilon\cdot\nabla^h\partial^{\alpha-\beta}W_\varepsilon,\partial^\alpha W_\varepsilon\rangle_{L^2_v\dot H^s_h}|\le\frac14\|\nabla^h\partial^\alpha W_\varepsilon\|^2_{L^2_v\dot H^s_h}+C\|\partial^\beta\overline{V}^h_\varepsilon\|^2_{L^\infty_v\dot H_h^{\frac{s+1}{2}}}\|\partial^{\alpha-\beta}W_\varepsilon\|^2_{L^2_v\dot H_h^{\frac{s+1}{2}}}\,.$$

Then we get

$$\begin{aligned}\frac12\frac{\mathrm{d}}{\mathrm{d}t}\|\partial^\alpha W_\varepsilon\|^2_{L^2_v\dot H^s_h}+\frac12\|\nabla^h\partial^\alpha W_\varepsilon\|^2_{L^2_v\dot H^s_h}+\varepsilon^2\|\partial_3\partial^\alpha W_\varepsilon\|^2_{L^2_v\dot H^s_h}&\lesssim\|\nabla^h\overline{V}^h_\varepsilon\|^2_{L^\infty_vL^2_h}\|\partial^\alpha W_\varepsilon\|^2_{L^2_v\dot H^s_h}\\&\quad+\big|\langle\nabla^h\partial^\alpha Q_\varepsilon,\partial^\alpha W^h_\varepsilon\rangle_{L^2_v\dot H^s_h}-\varepsilon^2\langle\partial_3\partial^\alpha Q_\varepsilon,\partial^\alpha W^3_\varepsilon\rangle_{L^2_v\dot H^s_h}\big|\\&\quad+C\sum_{0<\beta\le\alpha}\|\partial^\beta\overline{V}^h_\varepsilon\|^2_{L^\infty_v\dot H_h^{\frac{s+1}{2}}}\|\partial^{\alpha-\beta}W_\varepsilon\|^2_{L^2_v\dot H_h^{\frac{s+1}{2}}}\,.\end{aligned}$$

Now let us estimate the pressure term. We recall that

$$-\langle\nabla^h\partial^\alpha Q_\varepsilon,\partial^\alpha W^h_\varepsilon\rangle_{L^2_v\dot H^s_h}-\langle\varepsilon^2\,\partial_3\partial^\alpha Q_\varepsilon,\partial^\alpha W^3_\varepsilon\rangle_{L^2_v\dot H^s_h}=(\varepsilon^2-1)\langle\nabla^h\partial^\alpha Q_\varepsilon,\partial^\alpha W^h_\varepsilon\rangle_{L^2_v\dot H^s_h}$$

and we claim that

$$\big|\langle\nabla^h\partial^\alpha Q_\varepsilon,\partial^\alpha W^h_\varepsilon\rangle_{L^2_v\dot H^s_h}(t)\big|\le\frac14\|\nabla^h\partial^\alpha W_\varepsilon(t)\|^2_{L^2_v\dot H^s_h}+C_{1,\varepsilon}(t)+C_{2,\varepsilon}(t)\|\partial^\alpha W_\varepsilon(t)\|^2_{L^2_v\dot H^s_h}\tag{4.11}$$

with $C_{1,\varepsilon}$ and $C_{2,\varepsilon}$ uniformly bounded in $L_1(\mathbb{R}^+)$. By the induction assumption (noticing that $(s+1)/2+\alpha-1<\alpha$) we deduce that $\sum_{0<\beta\le\alpha}\|\partial^\beta\overline{V}^h_\varepsilon\|^2_{L^\infty_v\dot H_h^{\frac{s+1}{2}}}\|\partial^{\alpha-\beta}W_\varepsilon\|^2_{L^2_v\dot H_h^{\frac{s+1}{2}}}$ is uniformly bounded in $L^1(\mathbb{R}^+)$ so up to changing $C_{1,\varepsilon}$ and $C_{2,\varepsilon}$ we get

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\partial^\alpha W_\varepsilon(t)\|^2_{L^2_v\dot H^s_h}+\|\nabla^h\partial^\alpha W_\varepsilon(t)\|^2_{L^2_v\dot H^s_h}\le C_{1,\varepsilon}(t)+C_{2,\varepsilon}(t)\|\partial^\alpha W_\varepsilon(t)\|^2_{L^2_v\dot H^s_h}\,.$$

Using Gronwall's lemma in turn this implies that

$$\|\partial^\alpha W_\varepsilon(t)\|^2_{L^2_v\dot H^s_h}+\int_0^t\|\nabla^h\partial^\alpha W_\varepsilon(t')\|^2_{L^2_v\dot H^s_h}\mathrm{d}t'\lesssim\|\partial^\alpha W_{\varepsilon,0}\|^2_{L^2_v\dot H^s_h}$$

and the bounds on $W_{\varepsilon,0}$ conclude the proof if $-1<s<1$. It remains to prove the estimate (4.11) on the pressure term. We shall adapt the computations of the case $\alpha=0$. We define

$$N_{\varepsilon,\alpha,\beta}\stackrel{\mathrm{def}}{=}\partial^\beta\overline{V}^h_\varepsilon\cdot\nabla^h\partial^{\alpha-\beta}W^h_\varepsilon+\partial_3(\partial^{\alpha-\beta}W^3_\varepsilon\partial^\beta\overline{V}^h_\varepsilon)$$

and recalling (4.2) we get, since $\nabla^h(-\Delta_h-\varepsilon^2\partial_3^2)^{-1}\mathrm{div}_h$ is a Fourier multiplier of order 0,

$$\langle\nabla^h\partial^\alpha Q_\varepsilon,\partial^\alpha W^h_\varepsilon\rangle_{L^2_v\dot H^s_h}\lesssim\sum_{0\le\beta\le\alpha}\|N_{\varepsilon,\alpha,\beta}\|_{L^2_v\dot H^{r_\beta}_h}\|\partial^\alpha W^h_\varepsilon(t,\cdot)\|_{L^2_v\dot H_h^{2s-r_\beta}}$$

where r_β is any real number. Then we define

$$(*)_{\alpha,\beta}:=\|N_{\varepsilon,\alpha,\beta}\|_{L^2_v\dot H^{r_\beta}_h}\|\partial^\alpha W^h_\varepsilon(t,\cdot)\|_{L^2_v\dot H_h^{2s-r_\beta}}\,.$$

The term $(*)_{\alpha,0}$ can be treated as was done for $\alpha=0$, changing W^h_ε into $\partial^\alpha W^h_\varepsilon$. So we have, as in the proof of (4.1),

$$|(*)_{\alpha,0}|\le\frac18\|\nabla^h\partial^\alpha W_\varepsilon\|^2_{L^2_v\dot H^s_h}+C\|\partial^\alpha W_\varepsilon\|^2_{L^2_v\dot H^s_h}\|\nabla^h\partial^\alpha\overline{V}^h_\varepsilon\|^2_{L^\infty_vL^2_h}\big(1+\|\partial^\alpha\overline{V}^h_\varepsilon\|^2_{L^\infty_vL^2_h}\big)\,.\tag{4.12}$$

For the others terms we have the following estimates.

∘ If $0 < s < 1$ we choose $r_\beta = s - 1$ like in the case $\alpha = 0$, and as in (4.5) we obtain

$$\sum_{0<\beta\leq\alpha} |(*)_{\alpha,\beta}| \leq \frac{1}{8}\|\nabla^h\partial^\alpha W_\varepsilon\|^2_{L^2_v\dot{H}^s_h} + C\sum_{0<\beta\leq\alpha}\|\partial^\beta\overline{V}^h_\varepsilon\|^2_{L^\infty_v\dot{H}^{\frac{1}{2}}_h}\|\nabla^h\partial^{\alpha-\beta}W_\varepsilon\|^2_{L^2_v\dot{H}^{s-\frac{1}{2}}_h}$$
$$+ C\sum_{0<\beta\leq\alpha}\|\nabla\partial^\beta\overline{V}^h_\varepsilon\|^2_{L^\infty_v L^2_h}\|\partial^{\alpha-\beta}W^3_\varepsilon\|^2_{L^2_v\dot{H}^s_h}\,.$$

Then we define, recalling (4.12),

$$C_{1,\varepsilon} \stackrel{\text{def}}{=} C\sum_{0<\beta\leq\alpha}\|\partial^\beta\overline{V}^h_\varepsilon\|^2_{L^\infty_v\dot{H}^{\frac{1}{2}}_h}\|\nabla^h\partial^{\alpha-\beta}W_\varepsilon\|^2_{L^2_v\dot{H}^{s-\frac{1}{2}}_h} + C\sum_{0<\beta\leq\alpha}\|\nabla\partial^\beta\overline{V}^h_\varepsilon\|^2_{L^\infty_v L^2_h}\|\partial^{\alpha-\beta}W^3_\varepsilon\|^2_{L^2_v\dot{H}^s_h}$$

and

$$C_{2,\varepsilon} \stackrel{\text{def}}{=} C\|\nabla^h\partial^\alpha\overline{V}^h_\varepsilon\|^2_{L^\infty_v L^2_h}\big(1 + \|\partial^\alpha\overline{V}^h_\varepsilon\|^2_{L^\infty_v L^2_h}\big)$$

to get

$$\sum_{0\leq\beta\leq\alpha}\big|(*)_{\alpha,\beta}\big| \leq \frac{1}{4}\|\nabla^h\partial^\alpha W_\varepsilon\|^2_{L^2_v\dot{H}^s_h} + C_{1,\varepsilon} + C_{2,\varepsilon}\|\partial^\alpha W_\varepsilon\|^2_{L^2_v\dot{H}^s_h}\,.$$

Note that the famillies $(C_{1,\varepsilon})_{\varepsilon>0}$ and $(C_{2,\varepsilon})_{\varepsilon>0}$ are bounded in $L^1(\mathbb{R}^+)$ thanks to the induction assumption and Proposition 3.2.

∘ If $s = 0$ then following the steps leading to (4.7)-(4.8) we choose $r_\beta = -1/2$ and write

$$\big|(*)_{\alpha,\beta}\big| \lesssim \|\partial^\alpha W^h_\varepsilon\|_{L^2_v\dot{H}^{\frac{1}{2}}_h}\big(\|\partial^\beta\overline{V}^h_\varepsilon\|_{L^\infty_v\dot{H}^{\frac{1}{2}}_h}\|\nabla^h\partial^{\alpha-\beta}W_\varepsilon\|_{L^2} + \|\nabla\partial^\beta\overline{V}^h_\varepsilon\|_{L^\infty_v L^2_h}\|\partial^{\alpha-\beta}W^3_\varepsilon\|_{L^2_v\dot{H}^{\frac{1}{2}}_h}\big)$$

so, by interpolation, we get

$$\begin{aligned}\big|(*)_{\alpha,\beta}\big| \lesssim\ & \|\partial^\alpha W^h_\varepsilon\|^{\frac{1}{2}}_{L^2}\|\partial^\alpha\nabla^h W^h_\varepsilon\|^{\frac{1}{2}}_{L^2}\|\partial^\beta\overline{V}^h_\varepsilon\|_{L^\infty_v\dot{H}^{\frac{1}{2}}_h}\|\nabla^h\partial^{\alpha-\beta}W_\varepsilon\|_{L^2}\\ &+\|\partial^\alpha W^h_\varepsilon\|^{\frac{1}{2}}_{L^2}\|\partial^\alpha\nabla^h W^h_\varepsilon\|^{\frac{1}{2}}_{L^2}\|\nabla\partial^\beta\overline{V}^h_\varepsilon\|_{L^\infty_v L^2_h}\|\partial^{\alpha-\beta}W^3_\varepsilon\|_{L^2_v\dot{H}^{\frac{1}{2}}_h}\,.\end{aligned}$$

When $\beta > 0$, the convexity inequality $abc \leq \frac{1}{4}a^4 + \frac{1}{4}b^4 + \frac{1}{2}c^2$ leads to

$$\begin{aligned}\sum_{0<\beta\leq\alpha}\big|(*)_{\alpha,\beta}\big| \leq\ & \frac{1}{8}\|\nabla^h\partial^\alpha W_\varepsilon\|^2_{L^2} + C\sum_{0<\beta\leq\alpha}\|\partial^\alpha W^h_\varepsilon\|^2_{L^2}\big(\|\partial^\beta\overline{V}^h_\varepsilon\|^4_{L^\infty_v\dot{H}^{\frac{1}{2}}_h} + \|\nabla\partial^\beta\overline{V}^h_\varepsilon\|^4_{L^\infty_v L^2_h}\big)\\ &+ C\sum_{0<\beta\leq\alpha}\big(\|\nabla^h\partial^{\alpha-\beta}W_\varepsilon\|^2_{L^2} + \|\nabla^h\partial^{\alpha-\beta}W^3_\varepsilon\|^2_{L^2_v\dot{H}^{-\frac{1}{2}}_h}\big)\,.\end{aligned}$$

We define

$$C_{1,\varepsilon} \stackrel{\text{def}}{=} C\sum_{0<\beta\leq\alpha}\big(\|\nabla^h\partial^{\alpha-\beta}W_\varepsilon\|^2_{L^2} + \|\nabla^h\partial^{\alpha-\beta}W^3_\varepsilon\|^2_{L^2_v\dot{H}^{-\frac{1}{2}}_h}\big)$$

and

$$C_{2,\varepsilon} \stackrel{\text{def}}{=} C\|\nabla^h\partial^\alpha\overline{V}^h_\varepsilon\|^2_{L^\infty_v L^2_h}\big(1 + \|\partial^\alpha\overline{V}^h_\varepsilon\|^2_{L^\infty_v L^2_h}\big) + C\sum_{0<\beta\leq\alpha}\big(\|\partial^\beta\overline{V}^h_\varepsilon\|^4_{L^\infty_v\dot{H}^{\frac{1}{2}}_h} + \|\nabla\partial^\beta\overline{V}^h_\varepsilon\|^4_{L^\infty_v L^2_h}\big)$$

to get when $s = 0$ and recalling (4.12),

$$\sum_{0<\beta\leq\alpha}\big|(*)_{\alpha,\beta}\big| \leq \frac{1}{4}\|\nabla^h\partial^\alpha W_\varepsilon\|^2_{L^2} + C_{1,\varepsilon} + C_{2,\varepsilon}\|\partial^\alpha W_\varepsilon\|^2_{L^2}\,.$$

Again note that the famillies $(C_{1,\varepsilon})_{\varepsilon>0}$ and $(C_{2,\varepsilon})_{\varepsilon>0}$ are bounded in $L^1(\mathbb{R}^+)$ thanks to the induction assumption and Proposition 3.2.

- If $-1<s<0$ then following the computations leading to (4.9), we write

$$\begin{aligned}|(*)_{\alpha,\beta}| \lesssim \|\partial^\beta \overline{V}^h_\varepsilon\|_{L^\infty_v \dot H^{\frac12}_h} \|\partial^{\alpha-\beta} W_\varepsilon\|_{L^2_v \dot H^{s+\frac12}_h} \|\nabla \partial^\alpha W^h_\varepsilon\|_{L^2_v \dot H^s_h} \\ + \|\nabla\partial^\beta \overline{V}^h_\varepsilon\|_{L^\infty_v L^2} \|\nabla \partial^{\alpha-\beta} W_\varepsilon\|_{L^2_v \dot H^s_h} \|\partial^\alpha W^h_\varepsilon\|_{L^2_v \dot H^s_h}\end{aligned}$$

so

$$\begin{aligned}\sum_{0<\beta\le\alpha} |(*)_{\alpha,\beta}| \le \ & \frac14 \|\nabla^h \partial^\alpha W_\varepsilon\|^2_{L^2_v \dot H^s_h} + \sum_{\beta<\alpha} \|\partial^\beta \overline{V}^h_\varepsilon\|^2_{L^\infty_v \dot H^{\frac12}_h} \|\partial^{\alpha-\beta} W_\varepsilon\|^2_{L^2_v \dot H^{s+\frac12}_h} \\ & + \sum_{\beta<\alpha} \|\nabla\partial^\beta \overline{V}^h_\varepsilon\|_{L^\infty_v L^2} \|\nabla \partial^{\alpha-\beta} W_\varepsilon\|_{L^2_v \dot H^s_h} \|\partial^\alpha W^h_\varepsilon\|_{L^2_v \dot H^s_h}\,.\end{aligned}$$

In this case, we define

$$C_{1,\varepsilon} \stackrel{\text{def}}{=} C \sum_{0<\beta\le\alpha} \|\partial^\beta \overline{V}^h_\varepsilon\|^2_{L^\infty_v \dot H^{\frac12}_h} \|\partial^{\alpha-\beta} W_\varepsilon\|^2_{L^2_v \dot H^{s+\frac12}_h}$$

and

$$C_{2,\varepsilon} \stackrel{\text{def}}{=} C \|\nabla^h \partial^\alpha \overline{V}^h_\varepsilon\|^2_{L^\infty_v L^2_h} \big(1 + \|\partial^\alpha \overline{V}^h_\varepsilon\|^2_{L^\infty_v L^2_h}\big) + C \sum_{0<\beta\le\alpha} \|\nabla\partial^\beta \overline{V}^h_\varepsilon\|_{L^\infty_v L^2} \|\nabla \partial^{\alpha-\beta} W_\varepsilon\|_{L^2_v \dot H^s_h}$$

which as before are bounded in $L^1(\mathbb{R}^+)$, and we obtain, recalling (4.12),

$$\sum_{0\le\beta\le\alpha} (*)_{\alpha,\beta} \le \frac14 \|\nabla^h \partial^\alpha W_\varepsilon\|^2_{L^2_v \dot H^s_h} + C_{1,\varepsilon} + C_{2,\varepsilon} \|\partial^\alpha W_\varepsilon\|^2_{L^2_v \dot H^s_h}\,.$$

The first part of the proposition is proved.

Now let us turn to the second part. As noted above, for all $\alpha\in\mathbb{N}^3$, $\partial^\alpha W_\varepsilon$ satisfies

$$\partial_t \partial^\alpha W_\varepsilon + \partial^\alpha(\overline{V}^h_\varepsilon \cdot \nabla^h W_\varepsilon) - \Delta_h \partial^\alpha W_\varepsilon - \varepsilon^2\, \partial_3^2 \partial^\alpha W_\varepsilon = -\partial^\alpha(\nabla^h Q_\varepsilon, \varepsilon^2\, \partial_3 Q_\varepsilon)\,.$$

Defining

$$g_\varepsilon \stackrel{\text{def}}{=} \overline{V}^h_\varepsilon \cdot \nabla^h W_\varepsilon + (\nabla^h Q_\varepsilon, \varepsilon^2\, \partial_3 Q_\varepsilon)\,,$$

an energy estimate in $L^2_v \dot H^{-1}_h$ gives

$$\begin{aligned}\frac12 \|\partial^\alpha W_\varepsilon(t)\|^2_{L^2_v \dot H^{-1}_h} + \int_0^t \|\partial^\alpha W_\varepsilon(t')\|^2_{L^2} \mathrm{d}t' \le \frac12 \|\partial^\alpha W_{\varepsilon,0}\|^2_{L^2_v \dot H^{-1}_h} \\ + \int_0^t \big|\, \langle \partial^\alpha g_\varepsilon, \partial^\alpha W_\varepsilon\rangle_{L^2_v \dot H^{-1}_h}(t')\big|\, dt'\,.\end{aligned} \tag{4.13}$$

We define $K_\varepsilon(t) \stackrel{\text{def}}{=} \sup_{0\le t'\le t} \|\partial^\alpha W_\varepsilon(t')\|_{\dot H^{-1}_h}$, so that

$$\frac12 K^2_\varepsilon(t) \le \frac12 \|\partial^\alpha W_{\varepsilon,0}\|^2_{L^2_v \dot H^{-1}_h} + K_\varepsilon(t) \int_0^t \|\partial^\alpha g_\varepsilon(t')\|_{L^2_v \dot H^{-1}_h}\, dt'\,.$$

This implies that

$$\frac14 K^2_\varepsilon(t) \le \frac12 \|\partial^\alpha W_{\varepsilon,0}\|^2_{L^2_v \dot H^{-1}_h} + \|\partial^\alpha g_\varepsilon\|^2_{L^1(\mathbb{R}^+, L^2_v \dot H^{-1}_h)}\,. \tag{4.14}$$

But according to (4.13) we know that

$$\int_0^t \|\partial^\alpha W_\varepsilon(t')\|_{L^2}^2\,dt' \le \frac{1}{2}\|\partial^\alpha W_{\varepsilon,0}\|^2_{L^2_v\dot H^{-1}_h} + K_\varepsilon(t)\int_0^t \|\partial^\alpha g_\varepsilon(t')\|_{L^2_v\dot H^{-1}_h}\,dt'\,,$$

so with (4.14) we infer that

$$\int_0^t \|\partial^\alpha W_\varepsilon(t')\|^2_{L^2}dt' \lesssim \|\partial^\alpha W_{\varepsilon,0}\|^2_{L^2_v\dot H^{-1}_h} + \|\partial^\alpha g_\varepsilon\|^2_{L^1(\mathbb{R}^+,L^2_v\dot H^{-1}_h)}\,.$$

It remains to estimate $\|\partial^\alpha g_\varepsilon\|_{L^1(\mathbb{R}^+,L^2_v\dot H^{-1}_h)}$. As $\overline{V}^h_\varepsilon$ is a divergence free vector field, we have

$$\begin{aligned}\|\partial^\alpha(\overline{V}^h_\varepsilon\cdot\nabla^h W_\varepsilon)\|_{L^1(\mathbb{R}^+,L^2_v\dot H^{-1}_h)} &\le \|\partial^\alpha(\overline{V}^h_\varepsilon\otimes W_\varepsilon)\|_{L^1(\mathbb{R}^+,L^2)}\\ &\lesssim \sum_{0\le\beta\le\alpha}\|\partial^\beta\overline{V}^h_\varepsilon\|_{L^2(\mathbb{R}^+;L^2_v\dot H^{1/2}_h)}\|\partial^{\alpha-\beta}W_\varepsilon\|_{L^2(\mathbb{R}^+;L^\infty_v\dot H^{1/2}_h)}\end{aligned}\tag{4.15}$$

which gives the expected bound due to Proposition 4.1 (1) proved above. On the other hand, we recall that as computed in (4.2),

$$\Delta_h Q_\varepsilon - \varepsilon^2\,\partial_3^2 Q_\varepsilon = \mathrm{div}_h\big(\overline{V}^h_\varepsilon\cdot\nabla^h W^h_\varepsilon + \partial_3(W^3_\varepsilon\,\overline{V}^h_\varepsilon)\big)\,.$$

so since $(\Delta_h-\varepsilon^2\,\partial_3^2)^{-1}\nabla_h\mathrm{div}_h$ and $(\Delta_h-\varepsilon^2\,\partial_3^2)^{-1}\varepsilon\partial_3\mathrm{div}_h$ are zero-order Fourier multipliers, the same estimates give the expected a priori bound on $(\nabla^h Q_\varepsilon, \varepsilon^2\,\partial_3 Q_\varepsilon)$, and the result follows. □

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