Observer for Lipschitz nonlinear systems: mean value theorem and sector nonlinearity transformation

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***Abstract*— The problem of observer design for nonlinear Lipschitz systems is dealt with in this work. An emphasis is put on the maximization of the admissible Lipschitz constant for which the observer design is possible. This problem is tackled using a Takagi-Sugeno modeling approach. The idea is to re-write the state estimation error dynamics as an autonomous Takagi-Sugeno system, using the Mean Value Theorem and the sector nonlinearity transformation. State estimation error dynamics stability is studied with the Lyapunov theory by choosing a non-quadratic Lyapunov function and by computing its variation between m consecutive samples. The interest of these manipulations is to obtain LMI conditions admitting solutions for large values of the Lipschitz constant. Finally, illustrative examples are provided in order to highlight the performances of the proposed approach.**

I. INTRODUCTION

Many problems in control and monitoring need the knowledge of the state variables and the parameters of the system. However, measuring these variables is faced to two major problems. The first one concerns the technical and economical reasons, indeed, nowadays many sensors are very expensive and bulky. The second problem is that some state variables are not accessible for measure. Therefore, the problems of state estimation and observer design become the heart of control and monitoring design systems.

Most of the developed state estimation methods are based on linear models of the studied systems [14], [10], [8]. However, linear models only describe the behavior of the system around a specific operating point which leads to degraded performances far from this particular point. In order to increase the system performances the use of nonlinear models seems very interesting and appropriate because it allows an accurate representation of the system on a wide operating range. Despite the accurate system description, the disadvantage of the nonlinear approach is the lack of a unified and general solution for observer design. The existing results are dedicated to specific classes of nonlinear systems, such as Lipschitz systems or bilinear systems. Many approaches have been then elaborated, for example, those based on the nonlinear transformation of the original nonlinear system into a linear one such as using immersion, Lie algebraic transformations, etc [6], [5], [11]. It is pointed out in many works that this kind of approaches is very difficult to achieve due to the strong conditions under which these transformations exist. When these transformations fail to solve the problem, it has also been proposed to use the "high gain observer". The key point in this type of observers is to determine the gains in order to counteract the effect of the nonlinearities (see [22]). However, high gains penalize the state estimation in the presence of measurement noises.

Recently, observer design for Lipschitz systems has intensively been studied since many nonlinearities encountered in practical systems satisfy the Lipschitz condition, at least locally. Firstly, in [22], the problem of observer design for Lipschitz systems is studied and sufficient conditions are established to guarantee the asymptotic stability of the state estimation error dynamics. However, a design method is not proposed to determine the observer gain. In [19], an iterative design approach is proposed to find the observer gain by solving an algebraic Riccati equation. Unfortunately, this algorithm may fail to provide a solution even if the observability condition is satisfied. In [20], necessary and sufficient existence conditions of an observer are proposed, by formulating them as an $\mathscr{H}_\infty$ standard problem. However, it is pointed out that this standard problem may have no solution because the regularity assumption is not satisfied. This work is extended in [18] by transforming the problem in order to satisfy the regularity assumption required in the $\mathscr{H}_\infty$ optimization.

The main idea of the work cited above is still to compute the observer gain in order that the linear part counteract the effect of the nonlinear part; the major problem is that if the Lipschitz constant of the nonlinearity is greater than an admissible value, the design methods cannot be applied. In some recent works [23], the mean value theorem (MVT) is used to write the state estimation error as a linear parameter varying (LPV) system. Contrarily to the other methods, the use of the MVT allows to obtain a solution, even for large Lipschitz constant. Many of the cited works are extended to discrete time case [4], [12], [3].

In this work, the problem of observer design for Lipschitz nonlinear system is considered. An approach combining the MVT and the sector nonlinearity transformation is proposed. First, based on a Lipschitz assumption, the state estimation dynamics is written as a Linear Parameter Varying (LPV) system. Secondly, the sector nonlinearity transformation is used to transform the LPV system into a Takagi-Sugeno (T-S) system. The aim of this last step is to apply the recent developments in stability analysis and stabilizing controller design [13]. As pointed out above, the proposed result may have a solution, even for large values of the Lipschitz

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constant. Moreover, if no solution exists, it is possible to relax the conditions by using the results proposed in [13] for T-S systems.

The paper is organized as follows. Some preliminaries are detailed and the addressed problem is stated in Section II. The transformation of a Lipschitz system into a T-S one and the observer design for the latter are exposed in Section III. Before concluding, illustrative examples are provided in Section IV.

Notations. The symbol $*$ denotes the terms induced by symmetry in a block matrix or in a matrix product (*i.e.*, for any matrices M and N, $*MN$ stands for $N^T MN$). The notation $\xi_{s,i}$ is used to denote the i^{th} vector of the canonical basis of $\mathbb{R}^s$ (*i.e.* with all components are null, except the i^{th} which is equal to "1").

II. Preliminaries and problem statement

Let us consider the discrete-time nonlinear system given by

$$x(k+1) = Ax(k) + f(x(k),u(k)) \tag{1}$$
$$y(k) = Cx(k) \tag{2}$$

where $x(k) \in \mathbb{R}^n$, $u(k) \in \mathbb{R}^{n_u}$, $y(k) \in \mathbb{R}^{n_y}$ are the system state, input and output respectively and $f(x(k),u(k))$ is a function of $x(k)$ and $u(k)$ containing nonlinearities of the system. Assume that $f(x(k),u(k))$ is globally Lipschitz or at least locally Lipschitz in a region $\mathscr{D}$ including the origin with respect to $x(k)$ and uniformly in $u(k)$. So, we have

$$\|f(x_1,u) - f(x_2,u)\| \leq \gamma \|x_1 - x_2\| \tag{3}$$

$$\begin{cases} \forall x_1, x_2 \in \mathbb{R}^n & \text{globally Lipschitz} \\ \forall x_1, x_2 \in \mathscr{D} & \text{locally Lipschitz} \end{cases} \tag{4}$$

The parameter $\gamma > 0$ is the Lipschitz constant and $\|.\|$ is the 2-norm. The state observer for the system (1)-(2) is given in the form

$$\hat{x}(k+1) = A\hat{x}(k) + f(\hat{x}(k),u(k)) + L(y(k) - \hat{y}(k)) \tag{5}$$
$$\hat{y}(k) = C\hat{x}(k) \tag{6}$$

Defining the state estimation error $e(k) = x(k) - \hat{x}(k)$, from (1)-(2) and (5)-(6), the state estimation error dynamics is given by

$$e(k+1) = (A - LC)\,e(k) + f(x(k),u(k)) - f(\hat{x}(k),u(k)) \tag{7}$$

The goal consists on determining the observer gain L such that the observer error dynamics is asymptotically stable ($\lim_{k\to+\infty} e(k) = 0$).

Many approaches are proposed in the literature in order to cope with this problem [4], [2], [1], [19], [20]. In particular, design methods based on LMI conditions are studied but it is known that in general, the Lipschitz constant γ appears in the LMI conditions, like in the following lemma.

Lemma 1: [4] The state estimation error converges asymptotically to zero if there exists two matrices $K \in R^{n\times n_y}$, $P = P^T > 0 \in R^{n\times n}$ and a positive scalar τ such that the following LMI holds

$$\begin{pmatrix} -P + \tau\gamma^2 I & A^T P - C^T K^T & A^T P - C^T K^T \\ * & P - \tau I & 0 \\ * & * & -P \end{pmatrix} < 0 \tag{8}$$

The observer gain L is then obtained by $L = P^{-1}K$.

These approaches are based on finding the gain L in order that the linear part of the state estimation error dynamics counteracts the effect of the nonlinear part, but if the Lipschitz constant is greater than a maximal admissible value, denoted γ_{ad}, the approaches fail to provide the gain L.

In this work, a new approach is proposed in order to increase the value of the admissible Lipschitz constant by transforming this problem into a stability relaxation one for T-S system, using the MVT and sector nonlinearity transformation. The stability is then studied using a Lyapunov theory and a non-quadratic Lyapunov function. The stability conditions are formulated in term of linear matrix inequalities which have a solution for large values of the Lipschitz constant.

The following lemmas are used in the remainder of the paper.

Lemma 2: [17] For any matrices Q and A of appropriate dimensions, finding $P = P^T > 0$ satisfying (9) is equivalent to finding $P = P^T > 0$ satisfying (10) and also to finding $P = P^T > 0$ and G satisfying (11)

$$A^T PA - Q < 0 \tag{9}$$

$$\begin{pmatrix} -Q & A^T P \\ PA & -P \end{pmatrix} < 0 \tag{10}$$

$$\begin{pmatrix} -Q & A^T G^T \\ GA & -G - G^T + P \end{pmatrix} < 0 \tag{11}$$

Lemma 3: [9] For any matrices T_1, T_2, T_3 and A of appropriate dimensions, finding $P = P^T > 0$ satisfying (12) is equivalent to finding $P = P^T > 0$ and G satisfying (13)

$$\begin{pmatrix} T_3 & * \\ T_2 & T_1 + A^T PA \end{pmatrix} < 0 \tag{12}$$

$$\begin{pmatrix} T_3 & * & * \\ T_2 & T_1 & * \\ 0 & GA & P - G - G^T \end{pmatrix} < 0 \tag{13}$$

Lemma 4: (Mean value theorem) Consider $g(z) : \mathbb{R}^n \to \mathbb{R}$. Let $a, b \in \mathbb{R}^n$. If $g(z)$ is differentiable on $[a,b]$ then there exists a vector $\tilde{z} \in \mathbb{R}^n$ with $\tilde{z} \in]a,b[$ (i.e. $\tilde{z}_i \in]a_i, b_i[$, for $i = 1,\ldots,n$), such that

$$g(a) - g(b) = \frac{\partial g}{\partial z}(\tilde{z})(a - b) \tag{14}$$

Lemma 5: (Sector nonlinearity approach) [21], [16] Any nonlinear function $g(z) : \mathbb{R} \to \mathbb{R}$ satisfying

$$\underline{g} \leq g(z) \leq \overline{g},\ \forall z \tag{15}$$

can be written as

$$g(z) = \mu_1(z)\underline{g} + \mu_2(z)\overline{g} \tag{16}$$

where

$$\mu_1(z) = \frac{\overline{g} - g(z)}{\overline{g} - \underline{g}} \qquad \mu_2(z) = \frac{g(z) - \underline{g}}{\overline{g} - \underline{g}} \tag{17}$$

and the functions $\mu_i(z)$ satisfy the convex sum property *i.e.* $\mu_1(z)+\mu_2(z)=1$ and $0 \leq \mu_i(z) \leq 1,\ \forall z$.

III. New observer design algorithm

In order to compute the gain of the observer (5)-(6), the state estimation error dynamics (7) is transformed into a T-S system. For that purpose, the last two terms in (7) are studied. Let us denote $z^T(k)=[x^T(k)\ u^T(k)]$ and $\hat{z}^T(k)=[\hat{x}^T(k)\ u^T(k)]$. Since the function $f(z)$ is differentiable, using lemma 4, there exists n constant vectors $\tilde{z}^i(k)$, with $\tilde{z}^i_j(k) \in]z_j(k)\ \hat{z}_j(k)[$ for $j=1,\ldots,n+n_u$, such that

$$f(z(k))-f(\hat{z}(k))=\sum_{i=1}^{n}\sum_{j=1}^{n+n_u}\xi_{n,i}\xi^T_{n+n_u,j}\frac{\partial f_i}{\partial z_j(k)}(\tilde{z}^i)(z(k)-\hat{z}(k)) \tag{18}$$

where

$$\xi_{n,i}=\begin{pmatrix} \underset{1}{0} & \cdots & \underset{i-1}{0} & \underset{i}{1} & \underset{i+1}{0} & \cdots & \underset{n}{0} \end{pmatrix} \tag{19}$$

which means that $\xi_{n,i}$ is a vector of dimension n that the component i is "1" and the others are zero. Since the n_u last components of $(z-\hat{z})$ are null, it follows

$$f(z(k))-f(\hat{z}(k))=\sum_{i=1}^{n}\sum_{j=1}^{n}\xi_{n,i}\xi^T_{n,j}\frac{\partial f_i}{\partial x_j(k)}(\tilde{z}^i(k))(x(k)-\hat{x}(k)) \tag{20}$$

Then, the state estimation error dynamics (7) obtained with the system (1)-(2) and the observer (5)-(6) becomes

$$e(k+1)=\left(\sum_{i=1}^{n}\sum_{j=1}^{n}\xi_{n,i}\xi^T_{n,j}\frac{\partial f_i}{\partial x_j(k)}(\tilde{z}^i(k))+A-LC\right)e(k) \tag{21}$$

Since the function $f(x,u)$ is Lipschitz with respect to x, its derivatives are bounded

$$\tilde{a}_{ij2} \leq \frac{\partial f_i}{\partial x_j}(x,u) \leq \tilde{a}_{ij1},\quad \forall x,u,\quad i,j=1,...,n \tag{22}$$

where $\tilde{a}_{ij1}$ and $\tilde{a}_{ij2}$ are known constants. With the lemma 5, each derivative can be written as

$$\frac{\partial f_i}{\partial x_j(k)}(z(k))=\sum_{l=1}^{2}v^l_{ij}(z(k))\tilde{a}_{ijl} \tag{23}$$

where the functions v^1_{ij} and v^2_{ij} are defined by

$$v^1_{ij}(z(k))=\frac{\frac{\partial f_i}{\partial x_j(k)}(z(k))-\tilde{a}_{ij2}}{\tilde{a}_{ij1}-\tilde{a}_{ij2}},\quad v^2_{ij}(z(k))=\frac{\tilde{a}_{ij1}-\frac{\partial f_i}{\partial x_j(k)}(z(k))}{\tilde{a}_{ij1}-\tilde{a}_{ij2}} \tag{24}$$

$$\sum_{l=1}^{2}v^l_{ij}(z(k))=1,\quad 0\leq v^l_{ij}(z(k))\leq 1,\quad l=1,2 \tag{25}$$

Using (23), the dynamics of the state estimation error can now be written as

$$e(k+1)=\left(A-LC+\sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{l=1}^{2}v^l_{ij}(z(k))\xi_{n,i}\xi^T_{n,j}\tilde{a}_{ijl}\right)e(k) \tag{26}$$

Following the sector nonlinearity transformation [21], where the functions v^l_{ij} are factorized, it is possible to rewrite (26) under the form

$$\sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{l=1}^{2}v^l_{ij}(z(k))\xi_{n,i}\xi^T_{n,j}\tilde{a}_{ijl}=\sum_{i=1}^{q}h_i(z(k))\mathscr{A}_i \tag{27}$$

where $q\leq 2^{n^2}$ and where the different components of the matrices $\mathscr{A}_i$ are given by the parameters $\tilde{a}_{ijl}$. Since the functions $h_i(z(k))$ are defined by the products of some functions v^1_{ij} and v^2_{ij}, they satisfy the convex sum property

$$\sum_{l=1}^{q}h_i(z(k))=1,\quad 0\leq h_i(z(k))\leq 1,\quad \forall k\in\mathbb{N},\ i=1,\ldots,q \tag{28}$$

From (28), it follows that $A-LC=\sum_{i=1}^{q}h_i(z(k))(A-LC)$. Then, defining $\Pi_i=\mathscr{A}_i+A-LC$, the dynamics of the state estimation error (26) is written in the following form

$$e(k+1)=\sum_{i=1}^{q}h_i(z(k))\Pi_i e(k) \tag{29}$$

Note that a general sector nonlinearity approach has been recently proposed in [16] allowing an adequate choice of additional parameters in order to ensure the observability of the local models (namely, the pairs (A_i,C_i)). In our problem, this result can be used in order to guarantee the observability of the pairs $(\mathscr{A}_i+A,C)$.

The stability of such type of systems has been extensively studied these recent years. Hence, useful results exist such as the quadratic stability obtained by a quadratic Lyapunov function. Thereafter, some relaxed stability conditions are provided with particular Lyapunov functions [13].

The stability of the system (29) is studied, in this section, in order to determine the gain L which stabilizes the state estimation error dynamics. We start with a result for stability analysis with quadratic Lyapunov function. The result is then extended using a new type of Lyapunov function for relaxed stability conditions.

A. Classic Lyapunov function for observer design

Based on the stability analysis of the autonomous T-S system generating the state estimation error (29), sufficient observer existence conditions are derived in the following theorem.

Theorem 1: The state estimation error in equation (29) converges asymptotically towards zero if there exists a symmetric positive definite matrix $P\in R^{n\times n}$ and a matrix $K\in R^{n\times n_y}$ such that the following LMIs hold $\forall i=1,...,q$

$$\begin{pmatrix} -P & A^TP+\mathscr{A}_i^TP-C^TK^T \\ PA+P\mathscr{A}_i-KC & -P \end{pmatrix}<0 \tag{30}$$

The gain L of the observer is computed from $L=P^{-1}K$.

Proof: Consider the quadratic Lyapunov function defined by

$$V(e(k))=e^T(k)Pe(k),\quad P=P^T>0 \tag{31}$$

Its discrete time derivative is defined by

$$\Delta V(e(k))=V(e(k+1))-V(e(k)) \tag{32}$$

$$=\sum_{i=1}^{q}h_i(z(k))e^T(k)\left(\Pi_i^TP\Pi_i-P\right)e(k)<0 \tag{33}$$

Obviously, $\Delta V(e(k)) < 0$ holds if

$$\sum_{i=1}^{q} h_i(z(k)) \left(\Pi_i^T P \Pi_i - P\right) < 0 \quad (34)$$

With the Schur complement, (34) is equivalent to

$$\sum_{i=1}^{q} h_i(z(k)) \begin{pmatrix} -P & \Pi_i^T P \\ P\Pi_i & -P \end{pmatrix} < 0 \quad (35)$$

where Π_i is defined in (29). Finally, using the convex sum property (28) and the variable change $K = PL$, the LMI (30) in theorem 1 are obtained. ■

It is known that if the number of sub-models increases then it becomes difficult or even impossible to obtain a common matrix P satisfying the LMIs proposed in theorem 1. This conservatism has been largely studied and some results have been established to reduce it. In the next section, the approach proposed by [13] for controller design is adapted to observer design.

B. Multi-samples Lyapunov function for observer design

The main idea is to compute the variation of the Lyapunov function between the samples k and $k+m$ where $m > 1$. It is proved that increasing m relaxes the obtained LMI conditions. Obviously, setting $m = 1$ will lead to the result given in the theorem 1. The observer existence conditions derived with this approach are given in the next theorem.

Theorem 2: If there exists symmetric and positive definite matrices $P_i \in R^{n\times n}$ and matrices $G \in R^{n\times n}$ and $K \in R^{n\times n_y}$ such that the LMIs (36) hold, then the state estimation error converges asymptotically towards zero.

$$\begin{pmatrix} -P_0 & \Phi_{i_0} & 0 & 0 & \cdots & 0 \\ * & \Omega_{0,1} & \Phi_{i_1} & 0 & \cdots & 0 \\ * & * & \Omega_{1,2} & \ddots & \ddots & \vdots \\ * & * & * & \ddots & \Phi_{i_{m-2}} & 0 \\ * & * & * & * & \Omega_{m-2,m-1} & \Phi_{i_{m-1}} \\ * & * & * & * & * & \Omega_{m-1} \end{pmatrix} < 0 \quad (36)$$

$$i_0 = 1,...,q \ / \ i_1 = 1,...,q \ ... \ i_{m-1} = 1,...,q$$

where

$$\Omega_{i,i+1} = -G - G^T + P_i - P_{i+1}, \quad i = 0,\ldots,m-2$$
$$\Omega_{m-1} = -G - G^T + P_{m-1}$$
$$\Phi_{i_j} = A^T G^T + \mathscr{A}_{i_j}^T G^T - C^T K^T, \quad j = 0,...,m-1$$

The observer gain is computed by

$$L = G^{-1}K \quad (37)$$

Proof: Let us define the Lyapunov function candidate $\mathscr{V}_m(E_m(k))$ by

$$\mathscr{V}_m(E_m(k)) = \sum_{i=0}^{m-1} e^T(k+i) P_i e(k+i) \quad (38)$$

where the matrices P_i are symmetric and positive definite and $E_m(k)$ is defined by

$$E_m(k) = \begin{bmatrix} e^T(k) & \ldots & e^T(k+m) \end{bmatrix}^T \quad (39)$$

Defining Ψ_k by

$$\Psi_k = \sum_{i=1}^{q} h_i(z(k))\Pi_i \quad (40)$$

the equation (29) leads to

$$e(k+1) = \Psi_k e(k) \quad (41)$$
$$e(k+2) = \Psi_{k+1}\Psi_k e(k) \quad (42)$$
$$\vdots$$
$$e(k+m) = \Psi_{k+m-1}\cdots\Psi_{k+2}\Psi_{k+1}\Psi_k e(k) \quad (43)$$

The variation of the Lyapunov function is given by

$$\Delta\mathscr{V}_m(E_m(k)) = \sum_{j=0}^{m-1} e^T(k+j+1) P_j e(k+j+1) - \sum_{j=0}^{m-1} e^T(k+j) P_j e(k+j) \quad (44)$$

With (43), the following stability condition is obtained

$$e^T(k) \left(\sum_{j=0}^{m-1} (*) P_j \Psi_{k+j} \times \cdots \times \Psi_k - \sum_{j=0}^{m-1} (*) P_j \Psi_{k+j-1} \times \cdots \times \Psi_k \right) e(k) < 0 \quad (45)$$

After calculation, (45) becomes

$$\sum_{j=0}^{m-2} (*) \left(P_j - P_{j+1}\right) \Psi_{k+j} \times \cdots \times \Psi_k \quad (46)$$
$$+ (*) P_{m-1} \Psi_{k+m-1} \times \cdots \times \Psi_k - P_0 < 0$$

For $\ell = 1,\ldots,m-1$, let us define Γ_ℓ by

$$\Gamma_\ell = (*) P_{m-1} \Psi_{k+m-1} \times \cdots \times \Psi_{k+\ell} + \sum_{j=\ell}^{m-2} (*) \left(P_j - P_{j+1}\right) \Psi_{k+j} \times \cdots \times \Psi_{k+\ell} + P_{\ell-1} - P_\ell \quad (47)$$

From this definition, it can be seen that Γ_ℓ satisfies

$$\Gamma_\ell = \Psi_{k+\ell}^T \Gamma_{\ell+1} \Psi_{k+\ell} + P_{\ell-1} - P_\ell \quad (48)$$

The inequality (46) can be written in the form

$$\Psi_k^T \Gamma_1 \Psi_k - P_0 < 0 \quad (49)$$

Using Lemma 2, (49) is equivalent to

$$\begin{pmatrix} -P_0 & (G\Psi_k)^T \\ G\Psi_k & -G - G^T + \Gamma_1 \end{pmatrix} < 0 \quad (50)$$

With Lemma 3 and (48), (50) becomes

$$\begin{pmatrix} -P_0 & (G\Psi_k)^T & 0 \\ G\Psi_k & -G - G^T + P_0 - P_1 & (G\Psi_{k+1})^T \\ 0 & G\Psi_{k+1} & \Gamma_2 - G - G^T \end{pmatrix} < 0 \quad (51)$$

Finally, repeating this procedure $(m-2)$ times, with the variable change $K = GL$ and since the functions $h_i(z(t))$ satisfy (28), the LMI (36) of the theorem 2 are obtained. ■

IV. Illustrative simulations

In this section, illustrative examples are studied. In the first example, the proposed approach is compared to existing ones. The chosen criterion is the maximum value of the Lipschitz constant for which a solution exists. In the second one, the observer proposed in theorem 2 is applied to estimate the state of a nonlinear model.

A. Example 1

Let us consider the discrete-time Lipschitz nonlinear system, proposed in [12], given by (1)-(2)

$$A = \begin{pmatrix} 0.2 & 0.01 \\ 0.1 & 0.2 \end{pmatrix}, \quad C = \begin{pmatrix} 1 & 0 \end{pmatrix} \tag{52}$$

and

$$f(x(k)) = \begin{pmatrix} 0 \\ \alpha \sin(x_1(k)) \end{pmatrix} \tag{53}$$

The nonlinear function $f(x(k))$ is Lipschitz with constant $\gamma = |\alpha|$. The observer design method in [12] admits a solution only for a Lipschitz constant $\gamma \leq 0.7916$ and the approach in [4] admits a maximal Lipschitz constant $\gamma \leq 0.81$. Then if γ is greater than these values, these methods fail to provide a solution to the LMI conditions. Using the MVT combined with the sector nonlinearity transformation, the LMI conditions are relaxed and the existence of a solution could be expected for larger values of the constant Lipschitz.

Let us compute the Jacobian matrix of $f(x(k))$ as follows

$$\frac{\partial f(x(k))}{\partial x(k)} = \begin{pmatrix} 0 & 0 \\ \alpha \cos(x_1(k)) & 0 \end{pmatrix} \tag{54}$$

Considering the premise variable $z(k) = \cos(x_1(k))$ and calculating the matrices $\mathscr{A}_i$, the parameter α giving the Lipschitz constant appears, then, in the matrices $\mathscr{A}_i$ allowing us to make a comparison between the cited approaches and the proposed one. For example, using theorem 3 with $m = 2$, the admissible Lipschitz constant is $\gamma = 101$. These results are summarized in the following table.

Method	Maximum γ
[12]	0.7916
[4]	0.81
Theorem 3 $(m = 2)$	101

B. Example 2: Rossler's system

In this second example, the proposed method is implemented to estimate the state variables of a Rossler's system, which is a nonlinear system [15]. The discrete time version with sampling time $T = 0.01$ of this system is given by the following equations

$$x(k+1) = Ax(k) + f(x(k)) + D \tag{55}$$

where

$$A = \begin{pmatrix} 1 & -0.01 & -0.01 \\ 0.01 & 1.002 & 0 \\ 0 & 0 & 0.95 \end{pmatrix}, \quad D = \begin{pmatrix} 0 \\ 0 \\ 0.02 \end{pmatrix}, \tag{56}$$

$$f(x(k)) = \begin{pmatrix} 0 \\ 0 \\ 0.01x_1(k)x_3(k) \end{pmatrix} \tag{57}$$

Assuming that only the state variable $x_2(k)$ is measured, the output equation is $y(k) = Cx(k)$ with $C = [0 \quad 1 \quad 0]$. By applying the mean value theorem, one obtains

$$f(x) - f(\hat{x}) = \frac{\partial f}{\partial x}(\tilde{x})(x - \hat{x}) \tag{58}$$

where

$$\frac{\partial f(x)}{\partial x} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0.01x_3 & 0 & 0.01x_1 \end{pmatrix} \tag{59}$$

It is known that the Rossler's system state are bounded, as it can be seen on the figure 1.

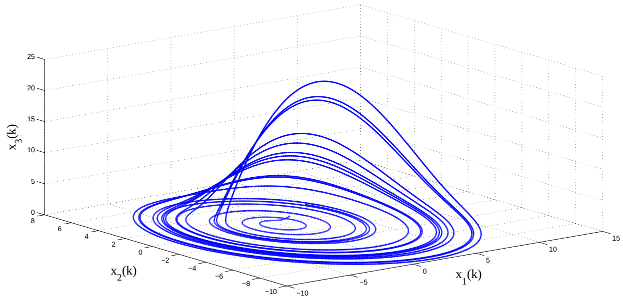

Fig. 1. State trajectory of the Rossler's nonlinear system

The lemma 5 is applied to the bounded state variables x_1 and x_3. The state estimation error dynamics can be given as follows

$$e(k+1) = \sum_{i=1}^{4} h_i(z(k)) \left(A + \mathscr{A}_i - LC\right) e(k) \tag{60}$$

where

$$\mathscr{A}_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0.20452 & 0 & 0.104 \end{pmatrix},$$

$$\mathscr{A}_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1.52 \times 10^{-4} & 0 & 0.104 \end{pmatrix},$$

$$\mathscr{A}_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0.20452 & 0 & -0.0815 \end{pmatrix},$$

$$\mathscr{A}_4 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1.52 \times 10^{-4} & 0 & -0.0815 \end{pmatrix}$$

Note that it is not necessary to compute the functions h_i because they are not required for observer design. Only the matrices $\mathscr{A}_i$ are needed. With these matrices, theorem 3 is applied with $m = 2$. The states and their estimates are depicted in figure 2 for $k \in [0 \; 5000]$ and the state estimation errors are illustrated in the figure 3 only for $k \in [0 \; 500]$ in order to show the forgetting of the initial conditions. The asymptotic convergence is then illustrated. The transient phenomenon can be reduced by pole clustering in a LMI region as illustrated in [7].

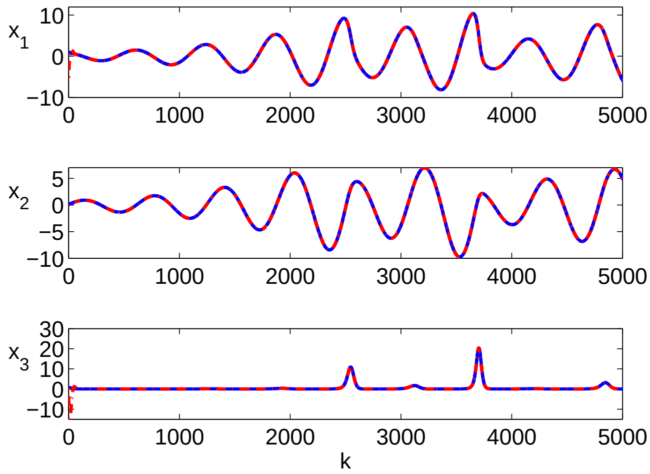

Fig. 2. State variables and their estimates

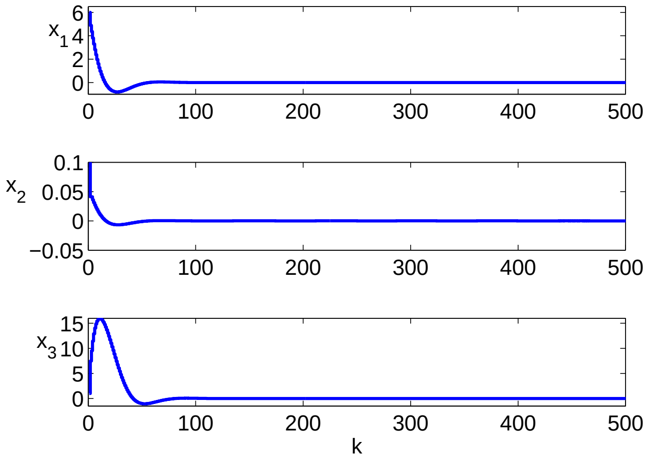

Fig. 3. State estimation errors

V. Conclusion

In this paper, a new observer design for discrete time nonlinear Lipschitz systems is proposed. It is based on the transformation of the state estimation error dynamics by the use of the Mean Value Theorem in order to use the sector nonlinearity transformation to derive an autonomous Takagi-Sugeno system. The stability of the latter is studied with the Lyapunov theory using different Lyapunov functions in order to obtain more relaxed stability conditions. LMI formulation of the stability conditions are provided. The focus is made on the fact that the problem of admissible Lipschitz constant is transformed on a relaxed stability problem. From the examples it is clear that when the classical methods fail to provide a solution the proposed one may have a solution by changing the number of samples on which the variations of the Lyapunov function are computed. In future works, the proposed observer will be extended for uncertain Lipschitz systems and unknown input estimation with an application for vehicles and motorcycles state estimation.

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