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STRONG EXISTENCE AND UNIQUENESS FOR STOCHASTIC DIFFERENTIAL EQUATION WITH HÖLDER DRIFT AND DEGENERATE NOISE

P.E. CHAUDRU DE RAYNAL

ABSTRACT. In this paper, we prove pathwise uniqueness for stochastic degenerate systems with a Hölder drift, for a Hölder exponent larger than the critical value $2/3$. This work extends to the degenerate setting the earlier results obtained by Zvonkin [Zvo74], Veretennikov [Ver80], Krylov and Röckner [KR05] from non-degenerate to degenerate cases. The existence of a threshold for the Hölder exponent in the degenerate case may be understood as the price to pay to balance the degeneracy of the noise. Our proof relies on regularization properties of the associated PDE, which is degenerate in the current framework and is based on a parametrix method.

1 Introduction

Let T be a positive number and d be a positive integer, we consider the following $\mathbb{R}^d \times \mathbb{R}^d$ system for any t in $[0, T]$:

$$\begin{cases} dX_t^1 = F_1(t, X_t^1, X_t^2)dt + \sigma(t, X_t^1, X_t^2)dW_t, & X_0^1 = x_1, \\ dX_t^2 = F_2(t, X_t^1, X_t^2)dt, & X_0^2 = x_2, \end{cases} \quad (1.1)$$

where x_1, x_2 belong to \mathbb{R}^d , $(W_t, t \geq 0)$ is a standard d -dimensional Brownian motion defined on some filtered probability space $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t)_{t \geq 0})$ and $F_1, F_2, \sigma : [0, T] \times \mathbb{R}^d \times \mathbb{R}^d \rightarrow \mathbb{R}^d \times \mathbb{R}^d \times \mathcal{M}_d(\mathbb{R})$ (the set of real $d \times d$ matrices) are measurable functions. The diffusion matrix $a := \sigma\sigma^*$ is supposed to be uniformly elliptic. The notation “*” stands for the transpose.

In this paper, we investigate the well posedness of (1.1) outside the Cauchy-Lipschitz framework. Notably, we are interested in the strong posedness, *i.e.* strong existence and uniqueness of a solution. Strong existence means that there exists a process $(X_t^1, X_t^2, 0 \leq t \leq T)$ adapted to the filtration generated by the Brownian motion $(W_t, 0 \leq t \leq T)$ which satisfies (1.1). Strong uniqueness means that, if two processes satisfy this equation with the same initial conditions, their trajectories are almost surely indistinguishable. Here, we show that under a suitable Hölder assumption on the drift coefficients and Lipschitz condition on the diffusion matrix, the strong well-posedness holds for (1.1).

It may be a real challenge to prove the existence of a unique solution for a differential system without Lipschitz conditions on the coefficients. For example, in [DL89], DiPerna and Lions showed that under integrability conditions on $b, \nabla b$ and $\operatorname{div}(b)$, the integral equation: $Y_t = \int_0^t b(s, Y_s)ds, Y_0 = y$ admits a unique solution defined as a regular Lagrangian flow (see [DL89] for the definition of such a solution).

In a stochastic case, the first result in this direction is due to Zvonkin. In [Zvo74], the author showed that the strong well-posedness holds for the one-dimensional system

$$Y_t = \int_0^t b(s, Y_s)ds + W_t, Y_0 = y \quad t \in [0, T], \quad (1.2)$$

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for a measurable function b in \mathbb{L}^∞ . Then, Veretennikov [Ver80] generalized this result to the multi-dimensional case and Krylov and Röckner showed in [KR05] the strong well-posedness for b in $\mathbb{L}_{\text{loc}}^p$, $p > d$. There are some extensions of these works and we refer the reader to the paper of Zhang [Zha05] and references therein for a summary of the results. Finally, when b is a measurable and bounded function, Davie showed in [Dav07] that for almost every Brownian path, there exists a unique solution for the system (1.2). We emphasize that this result implies the strong uniqueness, but the converse is not true. Indeed, in such a case, there exists an a priori set $\Omega' \subset \Omega$ with $\mathbb{P}(\Omega') = 1$ such that for all ω in Ω' the solution of (1.2) is unique.

All these results rely on the regularization of differential systems by adding a non-degenerate noise, and we refer to [Fla11] for a partial review on this subject. In (1.1), the noise added is completely degenerate w.r.t the degenerate component X^2 . This sort of system has also been studied by Veretennikov in [Ver83] but without considering any regularization in the degenerate direction. Indeed, the author showed that strong well-posedness holds when the drift is measurable and bounded and the diffusion matrix is Lipschitz w.r.t the non-degenerate component X^1 and when both the drift and the diffusion matrix are twice continuously differentiable functions with bounded derivatives w.r.t the degenerate component.

Their proofs rely on the deep connection between SDEs and PDEs (see [Bas98] or [Fri06] for a partial review in the elliptic and parabolic cases). The generator associated to the Markov process Y is a linear partial differential operator of second order (usually denoted by \mathcal{L}) with the transition density of Y as fundamental solution. As explained by Fedrizzi and Flandoli in [FF11]: “if we have a good theory for the PDE:

$$\frac{\partial}{\partial t}u + \mathcal{L}u = \Phi, \text{ on } [0, T] \quad u_T = \mathbf{0}, \quad (1.3)$$

where the source term Φ has the same regularity as the drift, then, we have the main tools to prove strong uniqueness”.

In this paper, we show that the noise regularizes, even in the degenerate direction, by means of the random drift. Unfortunately, there is a price to pay to balance the degeneracy of the noise. First, the drift must be at least 2/3-Hölder continuous w.r.t the degenerate component. We do not know how sharp is this critical value, but it is consistent with our approach. Secondly, the drift F_2 of the second component must be Lipschitz continuous w.r.t the first component and its derivative in this direction has to be uniformly non degenerate: this allows the drift to regularize.

Our proof also relies on regularization properties of the associated PDE, and the “good theory” is here a “strong theory”: a Lipschitz bound on the solution of (1.3) and on its derivative w.r.t the first component. In our case, the generator \mathcal{L} is given by: for all ψ in $C^{1,2,1}([0, T] \times \mathbb{R}^d \times \mathbb{R}^d, \mathbb{R}^d)$ ¹:

$$\begin{aligned} \mathcal{L}\psi(t, x_1, x_2) &= \frac{1}{2}\text{Tr}(a(t, x_1, x_2)D_{x_1}^2\psi(t, x_1, x_2)) + [F_1(t, x_1, x_2)] \cdot [D_{x_1}\psi(t, x_1, x_2)] \\ &\quad + [F_2(t, x_1, x_2)] \cdot [D_{x_2}\psi(t, x_1, x_2)]. \end{aligned} \quad (1.4)$$

where $\text{Tr}(a)$ stands for the trace of the matrix a and “ \cdot ” denotes the standard Euclidean inner product on \mathbb{R}^d and where for any z in \mathbb{R}^d , the notation D_z means the derivative w.r.t the variable z . Here, the operator is not uniformly parabolic. When the coefficients are smooth and when the Lie algebra generated by the vector fields spans the whole space, such an operator admits a smooth fundamental solution (see [Hö67]): in this case it is said to be hypoelliptic and the coefficients are said to satisfied a Hörmander condition. The assumption on the uniform non-degeneracy of the derivative of the drift

¹*i.e. continuously differentiable w.r.t. the first variable, twice continuously differentiable w.r.t. the second variable and once continuously differentiable w.r.t. the third variable.*

F_2 w.r.t x_1 can be understood as a sort of weak Hörmander condition.

In our case, the form of the degeneracy is a non-linear generalization of Kolmogorov’s degeneracy, in reference to the first work [Kol34] of Kolmogorov in this direction. Degenerate operators of this form have been studied by many authors see e.g. the work of Di Francesco and Polidoro [DFP06], and Delarue and Menozzi [DM10]. We also emphasize that, in [Men11], Menozzi deduced from the regularization property exhibited in ([DM10]) the well weak posedness of a generalization of (1.1). Nevertheless, to the best of our knowledge, there does not exist a strong theory, in the sense defined above, for the PDE (1.3) when \mathcal{L} is defined by (1.4). We investigate it by using the so called parametrix approach (see [Fri64] for partial review in the elliptic setting).

In comparison with the works of Veretennikov [Ver80, Ver83], Krylov and Röckner [KR05], and Flandoli and Fedrizzi [FF11], asking for F_1 to be only in \mathbb{L}^p , $p > d$ might appear as the right framework. Since the parametrix is a perturbation method and we are interested in \mathbb{L}^∞ estimates, we suppose the drift F_1 to be Hölder continuous w.r.t x_1 .

1.1 Organization of this paper

Subsection 1.3 states useful notations, detailed assumptions and the main result of this paper: strong existence and uniqueness for (1.1). In Subsection 1.4, we expose the strategy to prove this result. It is based on the regularization properties of the associated PDE which are, in fact, the main contribution of this work. These properties are given in Subsection 1.5. Finally, our main result is proved in Subsection 1.6. The remainder of this paper is dedicated to the proof of the regularization properties of the associated PDE.

We present in Section 2 the linear and Brownian Heuristic. It explains the proof of the regularization properties in a simple case and allows to understand our assumptions and how the proof in the general case can be achieved. Then, we give in Section 3 the mathematical tools and the proof of the regularization properties of the PDE is given in Section 4. This is the technical part of this paper.

1.2 Notations.

In order to simplify the notations, we adopt the following convention: x, y, z, ξ , etc.. denote the $2d$ -dimensional real variables $(x_1, x_2), (y_1, y_2), (z_1, z_2), (\xi_1, \xi_2)$, etc.. Consequently, each component of the d -dimensional variables x_k , $k = 1, 2$ is denoted by x_{kl} , $l = 1, \dots, d$. We denote by $g(t, X_t)$ any function $g(t, X_t^1, X_t^2)$ from $[0, T] \times \mathbb{R}^d \times \mathbb{R}^d$ to \mathbb{R}^N , $N \in \mathbb{N}$. Here, $X_t = (X_t^1, X_t^2)$ and then $F(t, X_t)$ is the \mathbb{R}^{2d} valued function $(F_1(t, X_t^1, X_t^2), F_2(t, X_t^1, X_t^2))^*$. We rewrite the system (1.1) in a shortened form:

$$dX_t = F(t, X_t)dt + B\sigma(t, X_t)dW_t, \quad (1.5)$$

where B is the $2d \times d$ matrix: $B = (\text{Id}, 0_{\mathbb{R}^d \times \mathbb{R}^d})^*$. “Id” stands for the identity matrix of $\mathcal{M}_d(\mathbb{R})$, the set of real $d \times d$ matrices. When necessary, we write $(X_s^{t,x})_{t \leq s \leq T}$ the process in (1.1) which starts from x at time t , i.e $X_t^{t,x} = x$.

We denote by $C^{1,2,1}$ the space of functions that are continuously differentiable w.r.t. the first variable, twice continuously differentiable w.r.t. the second variable and once continuously differentiable w.r.t. the third variable.

We denote by $\text{GL}_d(\mathbb{R})$ the set of $d \times d$ invertible matrices with real coefficients and by ϕ a measurable function from $[0, T] \times \mathbb{R}^d \times \mathbb{R}^d$ to \mathbb{R}^2 . Each one-dimensional component of this function is denoted by ϕ_i , $i = 1, 2$ and plays the role of one coordinate of F_i . Hence, ϕ_i satisfies the same regularity as F_i given latter. We recall that a denotes the square of the diffusion matrix σ : $a := \sigma\sigma^*$. Subsequently,

we denote by c, C, C', C'' or C''' a positive constant, depending only on known parameters in **(H)**, given just below, and may change from line to line and from an equation to another.

We recall that the canonical Euclidean inner product on \mathbb{R}^d is denoted by “.” and the notation D_z means the derivative w.r.t the variable z . Hence, for all integer n , D_z^n is the n^{th} derivative w.r.t z and for all integer m the $n \times m$ cross differentiations w.r.t z, y are denoted by $D_z^n D_y^m$. Furthermore, the partial derivative ∂/∂_t is denoted by ∂_t .

1.3 Main Result

Assumptions (H). We say that assumptions **(H)** hold if the following assumptions are satisfied:

(H1): regularity of the coefficients. There exist $0 < \beta_i^j < 1$, $1 \leq i, j \leq 2$ and three positive constants C_1, C_2, C_σ such that for all t in $[0, T]$ and all (x_1, x_2) and (y_1, y_2) in $\mathbb{R}^d \times \mathbb{R}^d$

$$\begin{aligned} |F_1(t, x_1, x_2) - F_1(t, y_1, y_2)| &\leq C_1(|x_1 - y_1|^{\beta_1^1} + |x_2 - y_2|^{\beta_1^2}) \\ |F_2(t, x_1, x_2) - F_2(t, y_1, y_2)| &\leq C_2(|x_1 - y_1| + |x_2 - y_2|^{\beta_2^2}) \\ |\sigma(t, x_1, x_2) - \sigma(t, y_1, y_2)| &\leq C_\sigma(|x_1 - y_1| + |x_2 - y_2|). \end{aligned}$$

Moreover, the coefficients are supposed to be continuous w.r.t the time and the exponents β_i^j , $i = 1, 2$ are supposed to be strictly greater than $2/3$. Thereafter, we set $\beta_2^1 = 1$ for notational convenience.

(H2): uniform ellipticity of $\sigma\sigma^*$. The function $\sigma\sigma^*$ satisfies the uniform ellipticity hypothesis:

$$\exists \Lambda > 1, \forall \zeta \in \mathbb{R}^{2d}, \quad \Lambda^{-1}|\zeta|^2 \leq [\sigma\sigma^*(t, x_1, x_2)\zeta] \cdot \zeta \leq \Lambda|\zeta|^2,$$

for all $(t, x_1, x_2) \in [0, T] \times \mathbb{R}^d \times \mathbb{R}^d$.

(H3-a): differentiability and regularity of $x_1 \mapsto F_2(\cdot, x_1, \cdot)$. For all $(t, x_2) \in [0, T] \times \mathbb{R}^d$, the function $F_2(t, \cdot, x_2) : x_1 \mapsto F_2(t, x_1, x_2)$ is continuously differentiable and there exist $0 < \alpha^1 < 1$ and a positive constant \bar{C}_2 such that, for all (t, x_2) in $[0, T] \times \mathbb{R}^d$ and x_1, y_1 in \mathbb{R}^d

$$|D_{x_1} F_2(t, x_1, x_2) - D_{x_1} F_2(t, y_1, x_2)| \leq \bar{C}_2 |x_1 - y_1|^{\alpha^1}.$$

(H3-b): non degeneracy of $(D_{x_1} F_2)(D_{x_1} F_2)^*$. There exists a closed convex subset $\mathcal{E} \subset \text{GL}_d(\mathbb{R})$ (the set of $d \times d$ invertible matrices with real coefficients) such that for all t in $[0, T]$ and (x_1, x_2) in \mathbb{R}^{2d} the matrix $D_{x_1} F_2(t, x_1, x_2)$ belongs to \mathcal{E} . We emphasize that this implies that

$$\exists \bar{\Lambda} > 1, \forall \zeta \in \mathbb{R}^{2d}, \quad \bar{\Lambda}^{-1}|\zeta|^2 \leq [(D_{x_1} F_2)(D_{x_1} F_2)^*(t, x_1, x_2)\zeta] \cdot \zeta \leq \bar{\Lambda}|\zeta|^2,$$

for all $(t, x_1, x_2) \in [0, T] \times \mathbb{R}^d \times \mathbb{R}^d$.

Remark. The reason for the existence of the critical value $2/3$ for the Hölder regularity of the drift in **(H1)** and the particular “convexity” assumption **(H3-b)** are discussed in Section 2. In the following, the sentence “known parameters in **(H)**” refers to the parameters belonging to these assumptions.

The following Theorem is the main result of this paper and regards the strong well-posedness of the system (1.1).

Theorem 1.1. *Under **(H)**, strong existence and uniqueness hold for (1.1) for any positf T .*

1.4 Strategy of proof

Let us expose the basic arguments to prove Theorem 1.1. Existence of a weak solution follows from a compactness argument. Then, if the strong uniqueness holds, the strong existence follows. The

main issue consists in proving the strong uniqueness. As we already mentioned, the strategy relies on regularization properties of the linear system of PDEs:

$$\begin{cases} \partial_t u_i(t, x) + \mathcal{L}u_i(t, x) = F_i(t, x), & \text{for } (t, x) \in [0, T] \times \mathbb{R}^{2d}, \\ u_i(T, x) = 0_{\mathbb{R}^d}, & i = 1, 2. \end{cases} \quad (1.6)$$

This works as follows: suppose that there exists a unique $C^{1,2,1}([0, T] \times \mathbb{R}^d \times \mathbb{R}^d, \mathbb{R}^d)$ solution $u = (u^1, u^2)^*$ of this system. Thanks to Itô's formula, for all t in $[0, T]$ we have that

$$\int_0^t F(s, X_s) ds = u(t, X_t) - \int_0^t D_x u(s, X_s) B \sigma(s, X_s) dB_s.$$

Hence, if the functions u and $D_x u B = (D_{x_1} u, 0_{\mathbb{R}^d \times \mathbb{R}^d})^*$ are C_T Lipschitz continuous, where C_T is a positive constant that tends to 0 when T goes to 0, the drift $\int_0^t F(s, X_s) ds$ is Lipschitz continuous w.r.t the argument X , with Lipschitz constant C_T . The uniqueness then follows for T small enough by classical and circular arguments and the result can be deduced by iterating this strategy on sufficiently small partitions of any positive interval.

The main issue here is that, to the best of our knowledge, it is not known that the PDE (1.6) admits a $C^{1,2,1}([0, T] \times \mathbb{R}^d \times \mathbb{R}^d, \mathbb{R}^d)$ solution. The point is that we do not need to obtain the existence of a regular solution, but only the existence of the Lipschitz bounds for u and $D_{x_1} u$ depending only on known parameters in **(H)**.

Therefore, we investigate these bounds in a smooth setting. Thanks to assumptions **(H)**, there exists a sequence of smooth mollified coefficients $(a^n, F_1^n, F_2^n)_{n \geq 1}$ with bounded derivatives of any order, satisfying **(H)** uniformly (in n), such that this sequence converges in supremum norm to (a, F_1, F_2) . More details on the regularization procedure are given in Subsection 1.5 below.

Thanks to a parametrix expansion of the (smooth) solution u^n of the regularized PDE (1.6) (see *e.g.* [Fri64]), we show that for T small enough there exists a positive constant C_T , which is independent of the regularization procedure, such that the supremum norm of $D_{x_1} u^n$, $D_{x_2} u^n$, $D_{x_1}^2 u^n$ and $D_{x_2} D_{x_1} u^n$ are bounded by C_T . We then recover the Lipschitz regularity of the drift in small time and we obtain uniqueness by letting the regularization procedure tends to the infinity.

1.5 PDE's results

We here state the PDE results that we need for proving Theorem 1.1.

The mollifying procedure. Let us first detail how the smooth approximation of the coefficients a, F_1, F_2 works. For all positive integer n , we set:

$$F_2^n(t, x) = \int F_2(t-s, x-y) \varphi_1^n(y) \varphi_2^n(s) dy ds,$$

where $\varphi_1^n(\cdot) = c_1 n^{2d} \varphi(n|\cdot|)$ and $\varphi_2^n(\cdot) = c_2 n \varphi(n|\cdot|)$ for c_1, c_2 two constants of normalization and for a smooth function φ with support in the unit ball. For example $\varphi : z \in \mathbb{R} \mapsto \exp\left(-\frac{1}{1-z^2}\right) \mathbf{1}_{]-1,1[}(z)$ and $|\cdot| = \|\cdot\|_\infty$.

By defining $(F_1^n)_{n \geq 1}$ and $(a^n)_{n \geq 1}$ with the same procedure, it is then clear that for every n the mollified coefficients a^n, F_1^n, F_2^n are infinitely differentiable with bounded derivatives of any order and such that

$$(a^n, F_1^n, F_2^n) \xrightarrow{n \rightarrow +\infty} (a, F_1, F_2), \quad (1.7)$$

uniformly on $[0, T] \times \mathbb{R}^d \times \mathbb{R}^d$. Moreover, it is well-seen that they satisfy the same assumptions as (a, F_1, F_2) uniformly in n .

Let us just check the non-degeneracy assumption **(H3-a)** on $D_{x_1}F_2^n$. For all $\delta > 0$, one can find a positive integer $N(\delta)$ and a sequence of rectangles $(R_k)_{1 \leq k \leq N(\delta)}$ having sides of length less than δ and a family of points $\{(s_k, y_k) \in R_k, 1 \leq k \leq N(\delta)\}$ such that, for all $(t, x) \in [0, T] \times \mathbb{R}^{2d}$:

$$D_{x_1}F_2^n(t, x) = \lim_{\delta \rightarrow 0} \sum_{k=1}^{N(\delta)} D_{x_1}F_2(t - s_k, x - y_k) \int_{R_k} \varphi_1^n(y) \varphi_2^n(s) dy ds.$$

Since $D_{x_1}F_2$ belongs to the closed convex subset \mathcal{E} , it is clear that $D_{x_1}F_2^n$ belongs to \mathcal{E} .

The regularized PDE. As we said, we do not solve the limit PDE problem (1.6). The investigations are done with mollified coefficients $(a^n, F_1^n, F_2^n)_{n \geq 1}$ defined above. Let us denote by \mathcal{L}^n the regularized version of \mathcal{L} (that is the version of \mathcal{L} with mollified coefficients). We have from Section 2.1 Chapter II of [Fre85] (note that the time dependence here is not a problem to do so):

Lemma 1.2. *Let n be a positive integer. The PDE,*

$$\partial_t u_i^n(t, x) + \mathcal{L}^n u_i^n(t, x) = F_i^n(t, x), \quad \text{for } (t, x) \in [0, T] \times \mathbb{R}^{2d} \quad u_i^n(T, x) = 0_{\mathbb{R}^d}, \quad i = 1, 2,$$

admits a unique solution $u^n = (u_1^n, u_2^n)^$, which is infinitely differentiable with bounded derivatives of any order.*

Besides, the terminal condition $u^n(T, \cdot) = 0$ is very important: it guarantees that the solution and its derivatives vanish at time T . Hence, it allows to control the Lipschitz constant of u^n by a constant small as T is small. Indeed, we show that the solutions u^n , $n \geq 1$ satisfy:

Proposition 1.3. *There exist a positive \mathcal{T} , a positive $\delta_{1.3}$ and a positive constant C depending only on known parameters in **(H)** and not on n such that, for all positive T less than \mathcal{T} :*

$$\|D_{x_1} u^n\|_\infty + \|D_{x_2} u^n\|_\infty + \|D_{x_1}^2 u^n\|_\infty + \|D_{x_1} D_{x_2} u^n\|_\infty \leq CT^{\delta_{1.3}}.$$

In order to prove these results, we emphasize that each coordinate of the vectorial solution u_i^n of the decoupled linear PDE (1.2) can be described by the PDE

$$\partial_t u_i^n(t, x) + \mathcal{L}^n u_i^n(t, x) = \phi_i^n(t, x), \quad \text{for } (t, x) \in [0, T] \times \mathbb{R}^{2d}, \quad u_i^n(T, x) = 0, \quad i = 1, 2, \quad (1.8)$$

where $\phi_i^n : \mathbb{R}^{2d} \rightarrow \mathbb{R}$ denotes the sequence of mollified (by the procedure described above) coefficients ϕ_i : functions that satisfies the same regularity assumptions as F_i given in **(H1)** (these functions play the role of one of the coordinates of F_i). Therefore, we only have to prove Lemma 1.2 and Proposition 1.3 for (1.8) instead of (1.2).

Since the estimates on the solutions u^n , $n \geq 1$ are obtained uniformly in n (that is independently of the procedure of regularization), when we investigate the properties of the solution of the PDE (1.8) in the following sections, we forget the superscript “ n ” which arises from the mollifying procedure, and we further assume that the following assumptions hold:

Assumptions (HR). We say that assumptions **(HR)** hold if: Assumptions **(H)** hold true and $F_1, F_2, \phi_1, \phi_2, a$ are infinitely differentiable functions with bounded derivatives of any order.

1.6 Proof of Theorem 1.1

We now from Theorem 6.1.7 of [SV79] that the system (1.1) admits a weak solution (we emphasize that this result remains valid under the linear growth conditions assumed on the coefficients). Hence, we only have to prove the strong uniqueness. Thereafter, we denote by “**1**” the $2d \times 2d$ matrix:

$$\begin{pmatrix} \text{Id} & 0_{\mathbb{R}^d \times \mathbb{R}^d} \\ 0_{\mathbb{R}^d \times \mathbb{R}^d} & 0_{\mathbb{R}^d \times \mathbb{R}^d} \end{pmatrix}. \quad (1.9)$$

Let $(X_t, t \geq 0)$ and $(Y_t, t \geq 0)$ be two solutions of (1.1) with the same initial condition x in \mathbb{R}^{2d} . Let u^n be the solution of the linear system of PDEs (1.6). By using Lemma 1.2, we can apply Itô's formula for $u^n(t, X_t) - X_t$ and we obtain

$$\begin{aligned} u^n(t, X_t) - X_t &= \int_0^t [\partial_t u^n + \mathcal{L}u^n](s, X_s) ds - \int_0^t F(s, X_s) ds + u^n(0, x) - x \\ &\quad + \int_0^t [D_x u^n - \mathbf{1}] B \sigma(s, X_s) dW_s. \end{aligned}$$

In order to use the fact that $\partial_t u^n + \mathcal{L}^n u^n = F^n$, we rewrite

$$\begin{aligned} u^n(t, X_t) - X_t &= \int_0^t [\partial_t u^n + \mathcal{L}^n u^n](s, X_s) ds + \int_0^t (\mathcal{L} - \mathcal{L}^n) u^n(s, X_s) ds \\ &\quad - \int_0^t F(s, X_s) ds + u^n(0, x) - x + \int_0^t [D_x u^n - \mathbf{1}] B \sigma(s, X_s) dW_s, \end{aligned}$$

and then,

$$\begin{aligned} u^n(t, X_t) - X_t &= \int_0^t (\mathcal{L} - \mathcal{L}^n) u^n(s, X_s) ds + \int_0^t (F^n(s, X_s) - F(s, X_s)) ds + u^n(0, x) - x \\ &\quad + \int_0^t [D_x u^n - \mathbf{1}] B \sigma(s, X_s) dW_s, \end{aligned}$$

By the same argument, we obtain:

$$\begin{aligned} u^n(t, Y_t) - Y_t &= \int_0^t (\mathcal{L} - \mathcal{L}^n) u^n(s, Y_s) ds + \int_0^t (F^n(s, Y_s) - F(s, Y_s)) ds + u^n(0, x) - x \\ &\quad + \int_0^t [D_x u^n - \mathbf{1}] B \sigma(s, Y_s) dW_s. \end{aligned}$$

By taking the expectation of the supremum over t of the square norm of the difference of the two equalities above, we get from Doob inequality that

$$\begin{aligned} \mathbb{E} \left[\sup_{t \in [0, T]} |X_t - Y_t|^2 \right] &\leq C \left\{ \mathbb{E} \left[\sup_{t \in [0, T]} |u^n(t, X_t) - u^n(t, Y_t)|^2 \right] \right. \\ &\quad + \mathbb{E} \left[\int_0^T |[D_x u^n B](s, X_s) - [D_x u^n B](s, Y_s)|^2 |\sigma(s, Y_s)|^2 ds \right] \\ &\quad \left. + \mathbb{E} \left[\int_0^T (\|D_x u^n B\|_\infty + \mathbf{1}) |\sigma(s, Y_s) - \sigma(s, X_s)|^2 ds \right] + \mathcal{R}(n, T) \right\}, \end{aligned}$$

where

$$\begin{aligned} \mathcal{R}(n, T) &= T \left(\mathbb{E} \left[\sup_{t \in [0, T]} |F^{(n)}(t, Y_t) - F(t, Y_t)|^2 \right] + \mathbb{E} \left[\sup_{t \in [0, T]} |(\mathcal{L}^n - \mathcal{L})u^n(t, Y_t)|^2 \right] \right. \\ &\quad \left. + \mathbb{E} \left[\sup_{t \in [0, T]} |F^{(n)}(t, X_t) - F(t, X_t)|^2 \right] + \mathbb{E} \left[\sup_{t \in [0, T]} |(\mathcal{L}^n - \mathcal{L})u^n(t, X_t)|^2 \right] \right). \end{aligned}$$

First, note that from (1.7), for both Y_t and X_t , we have:

$$\mathbb{E} \left[\sup_{t \in [0, T]} |F^n(t, X_t) - F(t, X_t)|^2 \right] + \mathbb{E} \left[\sup_{t \in [0, T]} |(\mathcal{L}^n - \mathcal{L})u^n(t, X_t)|^2 \right] \rightarrow 0, \text{ as } n \rightarrow \infty,$$

so that $\mathcal{R}(n, T) \rightarrow 0$ when $n \rightarrow \infty$. Secondly, we know from Proposition 1.3, that there exists a positive \mathcal{T} such that for T less than \mathcal{T} and for all $t \in [0, T]$, the functions $u^n(t, \dots)$ and $D_{x_1} u^n(t, \dots)$ are Lipschitz continuous, with a Lipschitz constant independent of n . Since $D_x u^n B = (D_{x_1} u^n, 0_{\mathbb{R}^d \times \mathbb{R}^d})$, by letting $n \rightarrow \infty$ and using the two arguments above, we deduce that for $T \leq \mathcal{T}$:

$$\mathbb{E} \left[\sup_{t \in [0, T]} |X_t - Y_t|^2 \right] \leq C(T) \left\{ \mathbb{E} \left[\sup_{t \in [0, T]} |X_t - Y_t|^2 \right] + \mathbb{E} \left[\int_0^T |X_s - Y_s|^2 ds \right] \right\},$$

where $C(T)$ tends to 0 when T tends to 0. One can now find another positive \mathcal{T}' , depending only on known parameters in **(H)**, such that strong uniqueness holds for T less than \mathcal{T}' . By iterating this computation, the same result holds on any finite intervals and so on $[0, \infty)$. \square

2 The linear and Brownian heuristic

This section introduces the main issue when proving Proposition 1.3 in a simple case. It allows to understand our strategy and the role of some of the assumptions in **(H)**. Furthermore, this presents in a simple form the effects of the degeneracy. By “simple”, we mean that the assumptions **(HL)** below hold true.

Assumptions (HL). We say that hypotheses **(HL)** hold if **(H)** and **(HR)** hold with : $F_1 \equiv 0_{\mathbb{R}^d}$, $\sigma \equiv \text{Id}$, for all $(t < s, x_1, x_2) \in [0, T]^2 \times \mathbb{R}^{2d}$, $F_2(s, x_1, x_2) = \bar{F}_2(x_2) + \Gamma_s x_1$. This implies in particular that for all s in $[0, T]$, Γ_s belongs to the convex subset \mathcal{E} of $\text{GL}_d(\mathbb{R})$.

Under **(HL)**, the SDE (1.1) becomes:

$$\begin{cases} dX_s^1 = dW_s, & X_t^1 = x_1, \\ dX_s^2 = (\bar{F}_2(X_s^2) + \Gamma_s X_s^1) ds, & X_t^2 = x_2, \end{cases} \quad (2.1)$$

for all $t < s$ in $[0, T]^2$, x in \mathbb{R}^{2d} and admits a unique strong solution X . We recall that the associated PDE is

$$\begin{cases} \partial_t u_i(t, x) + \mathcal{L}u_i(t, x) = \phi_i(t, x), & \text{for } (t, x) \in [0, T] \times \mathbb{R}^{2d} \\ u_i(T, x) = 0, & i = 1, 2, \end{cases} \quad (2.2)$$

where \mathcal{L} is the generator of (2.1).

Our strategy to study the solution of the PDE (2.2) rests upon parametrization. This method is based on the following observation: in small time, the generator of the solution of an SDE with smooth and variable coefficients and the generator of the solution of the same SDE with fixed (frozen at some point) coefficients are “closed”. The variable generator is then seen as a perturbation of the frozen generator, which usually enjoys well known properties.

Here, we know the explicit form of the fundamental solution of the frozen generator (which is the transition density of the solution of the frozen SDE). Especially, we can prove that this fundamental solution and its derivatives admit Gaussian type bounds. Hence, thanks to the parametrization, we write the solution of the PDE (2.2) in term of this fundamental solution and we can study it.

We emphasize that the choice of the freezing point for the coefficients plays a central role: the perturbation done in the parametrization has to be of the order of the typical trajectories of the process associated to the frozen operator.

2.1 The frozen system

Kolmogorov's example. To understand how the frozen system could be, we go back to the work of Kolmogorov [Kol34] where the author studied the prototype system (1.1). When $d = 1$, Kolmogorov showed that the solution of $dY_s^1 = dW_s$, $dY_s^2 = \alpha Y_s^1 ds$, ($\alpha \neq 0$), with initial condition $(Y_0^1, Y_0^2) = (x_1, x_2) \in \mathbb{R}^2$, admits a density. Notably, this density is Gaussian and given by, for all s in $(0, T]$ and $(y_1, y_2) \in \mathbb{R}^2$

$$p(0, x_1, x_2; s, y_1, y_2) = \frac{\sqrt{3}}{\alpha\pi s^2} \exp\left(-\frac{1}{2} \left| K_s^{-1/2} (y_1 - x_1, y_2 - x_2 - s\alpha x_1)^* \right|^2\right), \quad (2.3)$$

with the following covariance matrix

$$K_s := \begin{pmatrix} s & (1/2)\alpha s^2 \\ (1/2)\alpha s^2 & (1/3)\alpha^2 s^3 \end{pmatrix}. \quad (2.4)$$

This example illustrates the behaviour of the system in small time: it is not diffusive. The first coordinate oscillates with fluctuations of order $1/2$, while the second one oscillates with fluctuations of order $3/2$. As a direct consequence, the transport of the initial condition of the first coordinate has a key role in the second one. This observation is crucial when freezing the coefficients.

The frozen system. As Kolmogorov's example suggests, we have to keep track of the transport of the initial condition when we freeze the coefficients. This allows us to reproduce a perturbation of the order of the typical trajectories of the frozen process. Then, we freeze the system (2.1) along the curve $\theta_{t,s} = (\theta_{t,s}^1, \theta_{t,s}^2)^*$, s in $[t, T]$ that solves the ODE:

$$\frac{d}{ds} \theta_{t,s} = (0_{\mathbb{R}^d}, \bar{F}_2(\theta_{t,s}^2(\xi)) + \Gamma_s \theta_{t,s}^1(\xi))^*, \quad \theta_{t,t}(\xi) = \xi, \quad (2.5)$$

for all ξ in \mathbb{R}^{2d} (we emphasize that in the regularized setting this ODE is well-posed). Hence, we obtain the following frozen system

$$\begin{cases} d\bar{X}_s^1 = dW_s, & \bar{X}_t^1 = x_1, \\ d\bar{X}_s^2 = (\bar{F}_2(\theta_{t,s}^2(\xi)) + \Gamma_s \bar{X}_s^1) ds, & \bar{X}_t^2 = x_2, \end{cases} \quad (2.6)$$

for all s in $(t, T]$. This is our candidate to approximate (2.1). Obviously, in order to reproduce the typical trajectories of the frozen process, the variable ξ in (2.5) will be chosen as the initial condition x of the solution of the SDE (2.6).

2.2 Existence and Gaussian bound of the density of the frozen system

In this case, the crucial point is the specific form of the covariance matrix $\bar{\Sigma}_t$ of \bar{X} . For any s in $(t, T]$, standard computations show that

$$\bar{\Sigma}_{t,s} = \begin{pmatrix} (s-t) & \int_t^s \int_t^r \Gamma_u du dr \\ \int_t^s \int_t^r \Gamma_u du dr & \int_t^s \left(\int_t^r \Gamma_u du \right) \left(\int_t^r \Gamma_u du \right)^* dr \end{pmatrix}. \quad (2.7)$$

Therefore, the existence and the Gaussian estimates of the transition density of \bar{X}_s^x stem from the control of the spectrum of $\bar{\Sigma}_{t,s}$. Such an investigation has been already done by Delarue and Menozzi in [DM10]. The two following Lemmas shortly describe some of their results that are useful for us. The proofs are not given. For further details, we refer to Section 3 pp 18-24 of their paper. They prove that

Lemma 2.1. *Suppose that assumptions (HL) hold true, then, a sufficient condition for the non-degeneracy of the variance matrix $\bar{\Sigma}_{t,s}$, s in $(t, T]$ is given by*

$$\det[\Gamma_r] > 0 \text{ for a.e. } r \in [t, s].$$

In that case, the solution of (2.6) admits a transition density \bar{q} given by, for all s in $(t, T]$:

$$\bar{q}(t, x_1, x_2; s, y_1, y_2) = \frac{1}{(2\pi)^{d/2}} \det(\bar{\Sigma}_{t,s})^{-1/2} \exp\left(-|\bar{\Sigma}_{t,s}^{-1/2}(y_1 - x_1, y_2 - m_{t,s}^{2,\xi}(x))^*|^2\right), \quad (2.8)$$

where

$$m_{t,s}^{2,\xi}(x) = x_2 + \int_t^s \Gamma_r x_1 dr + \int_t^s \bar{F}_2(\theta_{t,r}^2(\xi)) dr,$$

and where $\bar{\Sigma}_{t,s}$ is the uniformly non-degenerate matrix given by (2.7).

From this expression, we can give the following Gaussian type estimate on the transition density of the solution of the EDS (2.6) and on its derivatives (the Gaussian bounds on the derivatives are proven in a more general case in the proof of Proposition 3.1):

Lemma 2.2. *Suppose that assumptions (HL) hold true, then, the transition kernel \bar{q} and its derivatives admit a Gaussian-type bound: there exists a positive constant c depending only on known parameters in (H) such that for all ξ in \mathbb{R}^{2d} :*

$$\begin{aligned} & |D_{x_1}^{N^{x_1}} D_{x_2}^{N^{x_2}} D_{y_1}^{N^{y_1}} \bar{q}(t, x_1, x_2; s, y_1, y_2)| \\ & \leq (s-t)^{-[3N^{x_2} + N^{x_1} + N^{y_1}]/2} \frac{c}{(s-t)^{2d}} \exp\left(-c\left(\frac{|y_1 - x_1|^2}{s-t} + \frac{|y_2 - m_{t,s}^{2,\xi}(x)|^2}{(s-t)^3}\right)\right), \end{aligned} \quad (2.9)$$

for all s in $(t, T]$, any $N^{x_1}, N^{x_2}, N^{y_1}$ less than 2.

We emphasize that the constant c that appears in the exponential in estimate (2.9) does not depend on Γ , as suggested by Lemma 2.1. This uniform control is not obvious and is related to the ‘‘closed convex’’ assumption (H3-b).

If the control is not uniform, Delarue and Menozzi show in [DM10] (see Example 3.5) that one can find a sequence of matrix $(\Gamma_1^n)_{n \geq 0}$ with positive constant determinant such that $\det[\Sigma_{0,1}^n]$ converges towards 0 as n tends to the infinity. The crucial point in their example is that the sequence of functions $(t \in [0, 1] \mapsto \Gamma_t^n)_{n \geq 0}$ weakly converges towards 0. Hence, to overcome this problem, the authors need some closure for the weak topology. The closed convex assumption (H3-b) allows them to obtain compactness for the weak topology.

Note that for all s in $[t, T]$, the mean $(x_1, m_{t,s}^{2,x}(x))$ of \bar{X}_s^x satisfies the ODE (2.5) with initial condition $\xi = x$. Since this equation admits a unique solution under (HL), we deduce that for all s in $[t, T]$, the forward transport function with the starting point x as initial condition is equal to the mean: $\theta_{t,s}(x) = m_{t,s}^x(x)$.

Finally, as we will show in the proof of Proposition 3.1 in Section 3, the transition density \bar{q} enjoys the following symmetry property:

$$\forall (t < s, x, y) \in [0, T]^2 \times \mathbb{R}^d \times \mathbb{R}^d, D_{x_2} \bar{q}(t, x; s, y) = -D_{y_2} \bar{q}(t, x; s, y). \quad (2.10)$$

This plays a crucial role in the proof of the Lipschitz estimates of the solution u of the PDE (2.2) and of its derivative $D_{x_1} u$.

2.3 Representation of the solution by parametrix

Let $\bar{\mathcal{L}} = (1/2)\Delta_{x_1} + [\bar{F}_2(\theta_{t,T}^2(\xi)) + \Gamma_T x_1] \cdot D_{x_2}$ be the generator of \bar{X} . We can write the PDE (2.2)

$$\begin{cases} \partial_t u_i(t, x) + \bar{\mathcal{L}} u_i(t, x) = \phi_i(t, x) + (\bar{\mathcal{L}} - \mathcal{L}) u_i(t, x), & \text{for } (t, x) \in [0, T] \times \mathbb{R}^{2d} \\ u_i(T, x) = 0, & i = 1, 2, \end{cases}$$

So that, for all (t, x) in $[0, T] \times \mathbb{R}^{2d}$, the unique solution of this PDE can be written as:

$$u_i(t, x) = \int_t^T \int_{\mathbb{R}^{2d}} \left\{ \phi_i(s, y) - [\bar{F}_2(y_2) - \bar{F}_2(\theta_{t,T}^2(\xi))] \cdot D_{x_2} u_i(s, y) \right\} \bar{q}(t, x; s, y) dy ds. \quad (2.11)$$

2.4 A priori estimates

In order to prove the Lipschitz estimates of u and $D_{x_1} u$ of Proposition 1.3, we need to obtain estimates of the supremum norm of the first and second order derivatives of the u_i , $i = 1, 2$. Having in mind to invert differentiation and integral operators, any differentiation of the solution can be seen as an integral of a certain function against the derivative of the degenerate gaussian kernel \bar{q} .

As shown in Lemma 2.2, the differentiation of this kernel generates a time-singularity. Each differentiation of the transition kernel w.r.t. the diffusive component gives a time-singularity of order 1/2 while the differentiation w.r.t. the degenerate component gives a time-singularity of order 3/2.

The main issue consists in smoothing this singularity by using the regularity of the coefficients assumed in **(H)** together with gaussian decay in \bar{q} by letting the freezing point ξ be the starting point of the process.

Let us illustrate the computations with the worst case in Proposition 1.3, that is, the cross derivative $D_{x_1} D_{x_2} u_i$ (which gives a time singularity of order 2, see Lemma 2.2). Let (t, x) in $[0, T] \times \mathbb{R}^{2d}$. In order to invert the integral and the differentiation operator without choosing a particular ξ *a priori*, we split the time integral: for i in $\{1, 2\}$, for any $\epsilon > 0$, we write

$$\begin{aligned} u_i(t, x) &= \int_{t+\epsilon}^T \int_{\mathbb{R}^{2d}} \left\{ \phi_i(s, y) - [\bar{F}_2(y_2) - \bar{F}_2(\theta_{t,T}^2(\xi))] \cdot D_{x_2} u_i(s, y) \right\} dy ds \\ &\quad + \int_t^{t+\epsilon} \int_{\mathbb{R}^{2d}} \left\{ \phi_i(s, y) - [\bar{F}_2(y_2) - \bar{F}_2(\theta_{t,T}^2(\xi))] \cdot D_{x_2} u_i(s, y) \right\} \bar{q}(t, x; s, y) dy ds. \end{aligned}$$

Then, by switching the differentiation and the integral sign we have:

$$\begin{aligned} D_{x_1} D_{x_2} u_i(t, x) & \tag{2.12} \\ &= \int_{t+\epsilon}^T \int_{\mathbb{R}^{2d}} \left\{ \phi_i(s, y) - [\bar{F}_2(y_2) - \bar{F}_2(\theta_{t,T}^2(\xi))] \cdot D_{x_2} u_i(s, y) \right\} D_{x_1} D_{x_2} \bar{q}(t, x; s, y) dy ds \\ &\quad + D_{x_1} D_{x_2} \left[\int_t^{t+\epsilon} \int_{\mathbb{R}^{2d}} \left\{ \phi_i(s, y) - [\bar{F}_2(y_2) - \bar{F}_2(\theta_{t,T}^2(\xi))] \cdot D_{x_2} u_i(s, y) \right\} \bar{q}(t, x; s, y) dy ds \right]. \end{aligned}$$

On a one hand, the second term in the right hand side of the equation above reads

$$D_{x_1} D_{x_2} \mathbb{E} \left[\int_t^{t+\epsilon} \left\{ \phi_i(s, \bar{X}_s^{t,x}) - [\bar{F}_2(\bar{X}_s^{2,t,x}) - \bar{F}_2(\theta_{t,T}^2(\xi))] \cdot D_{x_2} u_i(s, \bar{X}_s^{1,t,x}, \bar{X}_s^{2,t,x}) \right\} ds \right],$$

and it is clear that, from the regularity of the solution of (2.6) (see e.g. [Kun82]), the regularity of u_i (Lemma 1.2) and of ϕ_i under **(HL)**, there exists a positive constants K depending on parameters in **(HL)** (and hence on the regularization procedure) such that this term is bounded by $K\epsilon$.

On the other hand, we deal with the first term in the right hand side of (2.12). Let us first deal with the first term in the integrand. For all s in $(t, T]$ we have:

$$\begin{aligned} & \int_{\mathbb{R}^{2d}} \phi_i(s, y_1, y_2) D_{x_1} D_{x_2} \bar{q}(t, x_1, x_2; s, y_1, y_2) dy_1 dy_2 \\ &= \int_{\mathbb{R}^{2d}} (\phi_i(s, y_1, y_2) - \phi_i(s, y_1, \theta_{t,s}^2(\xi))) D_{x_1} D_{x_2} \bar{q}(t, x_1, x_2; s, y_1, y_2) dy_1 dy_2 \\ &\quad + \int_{\mathbb{R}^{2d}} \phi_i(s, y_1, \theta_{t,s}^2(\xi)) D_{x_1} D_{x_2} \bar{q}(t, x_1, x_2; s, y_1, y_2) dy_1 dy_2, \end{aligned}$$

where, thanks to the symmetry (2.10) and integration by parts argument, the last term in the right hand side is equal to 0. In the sequel, we refer to this argument as *the centering argument*. Combining this argument and the estimate of $D_{x_1}D_{x_2}\bar{q}$ in Lemma 2.2, we have, for all s in $(t, T]$:

$$\begin{aligned} & \left| \int_{\mathbb{R}^{2d}} (\phi_i(s, y_1, y_2) - \phi_i(s, y_1, \theta_{t,s}^2(\xi))) D_{x_1}D_{x_2}\bar{q}(t, x_1, x_2; s, y_1, y_2) dy_1 dy_2 \right| \\ & \leq \int_{\mathbb{R}^{2d}} \left\{ (s-t)^{-2} |\phi_i(s, y_1, y_2) - \phi_i(s, y_1, \theta_{t,s}^2(\xi))| \right. \\ & \quad \left. \times \frac{c}{(s-t)^{2d}} \exp\left(-c\left(\frac{|y_1 - x_1|^2}{s-t} + \frac{|y_2 - m_{t,s}^{2,\xi}(x)|^2}{(s-t)^3}\right)\right) \right\} dy_1 dy_2, \end{aligned}$$

where c depends only on known parameters in **(H)**. Using Hölder regularity of ϕ_i assumed in **(H1)** we have

$$\begin{aligned} & \left| \int_{\mathbb{R}^{2d}} (s-t)^{-2} |\phi_i(s, y_1, y_2) - \phi_i(s, y_1, \theta_{t,s}^2(\xi))| \bar{q}(t, x_1, x_2; s, y_1, y_2) dy_1 dy_2 \right| \\ & \leq C \int_{\mathbb{R}^{2d}} \left\{ (s-t)^{-2+3\beta_i^2/2} \frac{|y_2 - \theta_{t,s}^2(\xi)|^{\beta_i^2}}{(s-t)^{3\beta_i^2/2}} \right. \\ & \quad \left. \times \frac{c}{(s-t)^{2d}} \exp\left(-c\left(\frac{|y_1 - x_1|^2}{s-t} + \frac{|y_2 - m_{t,s}^{2,\xi}(x)|^2}{(s-t)^3}\right)\right) \right\} dy_1 dy_2. \end{aligned}$$

Now, we use the off-diagonal decay of the Gaussian exponential: by letting $\xi = x$ (and then $\theta_{t,s}^2(x) = m_{t,s}^{2,x}(x)$), for all $\eta > 0$, there exists a constant $\bar{C} > 0$ such that²

$$\left(\frac{|y_2 - m_{t,s}^{2,x}(x)|}{(s-t)^{3/2}} \right)^{\beta_i^2} \times \exp\left(-\eta\left(\frac{|y_1 - x_1|^2}{s-t} + \frac{|y_2 - m_{t,s}^{2,x}(x)|^2}{(s-t)^3}\right)\right) \leq \bar{C},$$

where \bar{C} depends on η and β_i^2 only. Thus, by increasing the constant c in the exponential, we obtain the following estimate:

$$\begin{aligned} & \left| \int_{\mathbb{R}^{2d}} (s-t)^{-2} |\phi_i(s, y_1, y_2) - \phi_i(s, y_1, \theta_{t,s}^2(x))| \bar{q}(t, x_1, x_2; s, y_1, y_2) dy_1 dy_2 \right| \tag{2.13} \\ & \leq C' \int_{\mathbb{R}^{2d}} (s-t)^{-2+3\beta_i^2/2} \frac{c}{(s-t)^{2d}} \exp\left(-c\left(\frac{|y_1 - x_1|^2}{s-t} + \frac{|y_2 - m_{t,s}^{2,\xi}(x)|^2}{(s-t)^3}\right)\right) dy_1 dy_2. \end{aligned}$$

Therefore, by choosing the value of β_i^2 strictly greater than $2/3$, the singularity $(s-t)^{-2+3\beta_i^2/2}$ becomes integrable. By applying the same procedure with the term

$$\int_{t+\epsilon}^T \int_{\mathbb{R}^{2d}} \left\{ \phi_i(s, y) - [\bar{F}_2(y_2) - \bar{F}_2(\theta_{t,T}^2(\xi))] \cdot D_{x_2}u_i(s, y) \right\} dy ds,$$

by letting ϵ tends to 0 in (2.12) we can deduce that

$$\|D_{x_1}D_{x_2}u_i\|_\infty \leq C'''(T^{-1+3\beta_i^2/2} + T^{-1+3\beta_2^2/2} \|D_{x_2}u_i\|_\infty). \tag{2.14}$$

The main problem here is that the supremum norm of $D_{x_2}u_i$ appears in the bound, so that we also have to estimate this quantity in term on known parameters in **(H)** to overcome the problem. It is well seen that the same arguments lead to

$$\|D_{x_2}u_i\|_\infty \leq C''''(T^{(-1+3\beta_i^2)/2} + T^{(-1+3\beta_2^2)/2} \|D_{x_2}u_i\|_\infty).$$

²By using the inequality: $\forall \eta > 0, \forall q > 0, \exists \bar{C} > 0$ s.t. $\forall \sigma > 0, \sigma^q e^{-\eta\sigma} \leq \bar{C}$.

By choosing T sufficiently small (*e.g.* such that $C'''T^{(-1+3\beta_2^2)/2}$ is less than $1/2$) we obtain that

$$\|D_{x_2}u_i\|_\infty \leq 2C'''T^{(-1+3\beta_2^2)/2}. \quad (2.15)$$

We refer to this argument as the *circular argument* in the following. By plugging this bound in (2.14) and by applying the same strategy with $D_{x_1}u_i$ and $D_{x_1}^2u_i$, Proposition 1.3 under (HL) follows for T less than $(1/(2C'''))^{2/(-1+3\beta_2^2)}$.

From this discussion, one can also see the specific choice of the freezing curve as the one that matches the off-diagonal decay of the exponential in \bar{q} when $\xi = x$.

3 Mathematical tools

In this section, we introduce the ingredients for the proof of Proposition 1.3.

3.1 The frozen system

Consider the frozen system:

$$\begin{aligned} d\tilde{X}_s^{1,t,x} &= F_1(s, \theta_{t,s}(\xi))ds + \sigma(s, \theta_{t,s}(\xi))dW_s \\ d\tilde{X}_s^{2,t,x} &= F_2(s, \theta_{t,s}(\xi))ds + D_{x_1}F_2(s, \theta_{t,s}(\xi))(\tilde{X}_s^{1,t,x} - \theta_{t,s}^1(\xi))ds \end{aligned} \quad (3.1)$$

for all s in $(t, T]$, any t in $[0, T]$, and for any initial condition x in \mathbb{R}^{2d} at time t and any $\xi \in \mathbb{R}^{2d}$, linearized around the transport of the initial condition $(\theta_{t,s}(\xi))_{t \leq s \leq T}$ defined by

$$\frac{d}{ds}\theta_{t,s}(\xi) = F(s, \theta_{t,s}(\xi)), \quad \theta_{t,t}(\xi) = \xi. \quad (3.2)$$

The following Proposition holds:

Proposition 3.1. *Suppose that assumptions (HR) hold, then:*

(i) *There exists a unique (strong) solution of (3.1) with mean*

$$(m_{t,s}^\xi)_{t \leq s \leq T} = (m_{t,s}^{1,\xi}, m_{t,s}^{2,\xi})_{t \leq s \leq T},$$

where

$$\begin{aligned} m_{t,s}^{1,\xi}(x) &= x_1 + \int_t^s F_1(r, \theta_{t,r}(\xi))dr, \\ m_{t,s}^{2,\xi}(x) &= x_2 + \int_t^s \left[F_2(r, \theta_{t,r}(\xi)) + D_{x_1}F_2(r, \theta_{t,r}(\xi))(x_1 - \theta_{t,r}^1(\xi)) \right. \\ &\quad \left. + D_{x_1}F_2(r, \theta_{t,r}(\xi)) \int_t^r F_1(v, \theta_{t,v}(\xi))dv \right] dr, \end{aligned} \quad (3.3)$$

and uniformly non-degenerate covariance matrix $(\tilde{\Sigma}_{t,s})_{t \leq s \leq T}$:

$$\tilde{\Sigma}_{t,s} = \begin{pmatrix} \int_t^s \sigma \sigma^*(r, \theta_{t,r}(\xi))dr & \int_t^s R_{r,s}(\xi) \sigma \sigma^*(r, \theta_{t,r}(\xi))dr \\ \int_t^s \sigma \sigma^*(r, \theta_{t,r}(\xi)) R_{r,s}^*(\xi)dr & \int_t^s R_{t,r}(\xi) \sigma \sigma^*(r, \theta_{t,r}(\xi)) R_{t,r}^*(\xi)dr \end{pmatrix}, \quad (3.4)$$

where:

$$R_{t,r}(\xi) = \left[\int_t^r D_{x_1}F_2(v, \theta_{t,v}(\xi))dv \right], \quad t \leq r \leq s \leq T.$$

(ii) *This solution is a Gaussian process with transition density:*

$$\tilde{q}(t, x_1, x_2; s, y_1, y_2) = \frac{3^{d/2}}{(2\pi)^{d/2}} (\det[\tilde{\Sigma}_{t,s}])^{-1/2} \exp\left(-\frac{1}{2}|\tilde{\Sigma}_{t,s}^{-1/2}(y_1 - m_{t,s}^{1,\xi}(x), y_2 - m_{t,s}^{2,\xi}(x))^*|^2\right), \quad (3.5)$$

for all s in $(t, T]$.

(iii) This transition density \tilde{q} is the fundamental solution of the PDE driven by $\tilde{\mathcal{L}}^{t,\xi}$ and given by:

$$\begin{aligned} \tilde{\mathcal{L}}^{t,\xi} &:= \frac{1}{2} \text{Tr} \left[a(t, \theta_{t,s}(\xi)) D_{x_1}^2 \right] + [F_1(t, \theta_{t,s}(\xi))] \cdot D_{x_1} \\ &\quad + [F_2(t, \theta_{t,s}(\xi)) + D_{x_1} F_2(t, \theta_{t,s}(\xi)) (x_1 - \theta_{t,s}^1(\xi))] \cdot D_{x_2}. \end{aligned} \quad (3.6)$$

(iv) There exist two positive constants c and C , depending only on known parameters in (\mathbf{H}) , such that

$$\tilde{q}(t, x_1, x_2; s, y_1, y_2) \leq C \hat{q}_c(t, x_1, x_2; s, y_1, y_2), \quad (3.7)$$

where

$$\hat{q}_c(t, x_1, x_2; s, y_1, y_2) = \frac{c}{(s-t)^{2d}} \exp \left(-c \left(\frac{|y_1 - m_{t,s}^{1,\xi}(x)|^2}{s-t} + \frac{|y_2 - m_{t,s}^{2,\xi}(x)|^2}{(s-t)^3} \right) \right),$$

and:

$$\left| D_{x_1}^{N^{x_1}} D_{x_2}^{N^{x_2}} D_{y_1}^{N^{y_1}} \tilde{q}(t, x_1, x_2; s, y_1, y_2) \right| \leq C (s-t)^{-[3N^{x_2} + N^{x_1} + N^{y_1}]/2} \hat{q}_c(t, x_1, x_2; s, y_1, y_2), \quad (3.8)$$

for all s in $(t, T]$ and any integers $N^{x_1}, N^{x_2}, N^{y_1}$ less than 2.

Proof. (i) First of all, note that, under (\mathbf{HR}) , the ODE: $[d/ds]\theta_{t,s}(\xi) = F(s, \theta_{t,s}(\xi))$, $\theta_{t,t}(\xi) = \xi$ admits a unique solution and (3.1) admits a unique strong solution \tilde{X} . By rewriting (3.1) in integral form and by plugging the obtained representation of \tilde{X}^1 in \tilde{X}^2 , it is easily seen that the expressions of the mean (3.3) and the variance (3.4) follow from the stochastic Fubini Theorem and standard computations. The uniform non-degeneracy of $(\tilde{\Sigma}_{t,s})_{t < s \leq T}$ arises from assumptions (\mathbf{H}) and Proposition 3.1 in [DM10].

(ii)-(iii) These assertions result from standard computations.

(iv) For all s in $(t, T]$, we know from Proposition 3.1 in [DM10] that the matrix $\tilde{\Sigma}_{t,s}$ is symmetric and uniformly non degenerate. Besides, from Subsection 2.3 and Proposition 3.4 in [DM10] there exists a constant C depending only on known parameters in (\mathbf{H}) such that: for all $s \in (t, T]$, for all $(x, y, \xi) \in \mathbb{R}^{2d} \times \mathbb{R}^{2d} \times \mathbb{R}^{2d}$,

$$\begin{aligned} & - \left[\tilde{\Sigma}_{t,s}^{-1}(y_1 - m_{t,s}^{1,\xi}(x), y_2 - m_{t,s}^{2,\xi}(x))^* \right] \cdot \left[(y_1 - m_{t,s}^{1,\xi}(x), y_2 - m_{t,s}^{2,\xi}(x))^* \right] \\ & \leq -C \left[\left(\frac{y_1 - m_{t,s}^{1,\xi}(x)}{(s-t)^{1/2}}, \frac{y_2 - m_{t,s}^{2,\xi}(x)}{(s-t)^{3/2}} \right)^* \right] \cdot \left[\left(\frac{y_1 - m_{t,s}^{1,\xi}(x)}{(s-t)^{1/2}}, \frac{y_2 - m_{t,s}^{2,\xi}(x)}{(s-t)^{3/2}} \right)^* \right]. \end{aligned}$$

For $i, j = 1, 2$, let $[\tilde{\Sigma}_{t,s}^{-1}]_{i,j}$ denotes the block of size $d \times d$ of the matrix $\tilde{\Sigma}_{t,s}^{-1}$ at the $(i-1)d+1, (j-1)d+1$ rank. We can deduce from (3.4) that there exists a positive constant C depending only on known parameters in (\mathbf{H}) such that (we also refer the reader to Lemma 3.6 and to the proof of Lemma 5.5 in [DM10] for more details), for all s in $(t, T]$, for all ζ in \mathbb{R}^d :

$$\begin{aligned} \left| [\tilde{\Sigma}_{t,s}^{-1}]_{1,1} \zeta \right| &\leq C (s-t)^{-1} |\zeta|, \\ \left| [\tilde{\Sigma}_{t,s}^{-1}]_{1,2} \zeta \right| + \left| [\tilde{\Sigma}_{t,s}^{-1}]_{2,1} \zeta \right| &\leq C (s-t)^{-2} |\zeta|, \\ \left| [\tilde{\Sigma}_{t,s}^{-1}]_{2,2} \zeta \right| &\leq C (s-t)^{-3} |\zeta|, \end{aligned} \quad (3.9)$$

hence, $\tilde{\Sigma}_{t,\cdot}^{-1}$ has the same structure as $K_{\cdot,t}^{-1}$ in (2.4).

Now, we compute the derivatives w.r.t. each component and estimate it with the help of (3.9). Let $(t < s, x, y) \in [0, T]^2 \times \mathbb{R}^{2d} \times \mathbb{R}^{2d}$, we have:

$$\begin{aligned} & |D_{x_2} \tilde{q}(t, x_1, x_2; s, y_1, y_2)| \\ &= \left| \left(-2[\tilde{\Sigma}_{t,s}^{-1}]_{2,1}(y_1 - m_{t,s}^{1,\xi}(x)) - 2[\tilde{\Sigma}_{t,s}^{-1}]_{2,2}(y_2 - m_{t,s}^{2,\xi}(x)) \right) \tilde{q}(t, x_1, x_2; s, y_1, y_2) \right| \\ &\leq C(s-t)^{-3/2} \left(\left| \frac{(y_1 - m_{t,s}^{2,\xi}(x))}{(s-t)^{1/2}} \right| + \left| \frac{(y_2 - m_{t,s}^{2,\xi}(x))}{(s-t)^{3/2}} \right| \right) \tilde{q}(t, x_1, x_2; s, y_1, y_2) \\ &\leq C'(s-t)^{-3/2} \hat{q}_c(t, x_1, x_2; s, y_1, y_2). \end{aligned}$$

Note that the symmetry $D_{x_2} \tilde{q}(t, x_1, x_2; s, y_1, y_2) = -D_{y_2} \tilde{q}(t, x_1, x_2; s, y_1, y_2)$ holds. Now, we have

$$\begin{aligned} & |D_{y_1} \tilde{q}(t, x_1, x_2; s, y_1, y_2)| \\ &= \left| \left(2[\tilde{\Sigma}_{t,s}^{-1}]_{1,1}(y_1 - m_{t,s}^{1,\xi}(x)) + 2[\tilde{\Sigma}_{t,s}^{-1}]_{1,2}(y_2 - m_{t,s}^{2,\xi}(x)) \right) \tilde{q}(t, x_1, x_2; s, y_1, y_2) \right| \\ &\leq C(s-t)^{-1/2} \hat{q}_c(t, x_1, x_2; s, y_1, y_2). \end{aligned}$$

Unfortunately, the transport of the initial condition of the diffusive component in the degenerate component breaks the symmetry and $D_{x_1} \tilde{q}(t, x_1, x_2; s, y_1, y_2) \neq -D_{y_1} \tilde{q}(t, x_1, x_2; s, y_1, y_2)$. Indeed

$$\begin{aligned} D_{x_1} \tilde{q}(t, x_1, x_2; s, y_1, y_2) &= \left(-2[\tilde{\Sigma}_{t,s}^{-1}]_{1,1}(y_1 - m_{t,s}^{1,\xi}(x)) - 2[\tilde{\Sigma}_{t,s}^{-1}]_{1,2}(y_2 - m_{t,s}^{2,\xi}(x)) \right. \\ &\quad \left. -2[\tilde{\Sigma}_{t,s}^{-1}]_{1,2} \left[(R_{t,s}(\xi)) (y_1 - m_{t,s}^{1,\xi}(x)) \right] \right. \\ &\quad \left. -2[\tilde{\Sigma}_{t,s}^{-1}]_{2,2} \left[(R_{t,s}(\xi)) (y_2 - m_{t,s}^{2,\xi}(x)) \right] \right) \tilde{q}(t, x_1, x_2; s, y_1, y_2). \end{aligned}$$

Since the term $R_{t,s}(\xi)$ is of order $(s-t)$ (this is the transport of the initial condition from time t to s), we deduce that

$$\begin{aligned} |D_{x_1} \tilde{q}(t, x_1, x_2; s, y_1, y_2)| &\leq C(s-t)^{-1/2} \left\{ \left| \frac{(y_1 - m_{t,s}^{2,\xi}(x))}{(s-t)^{1/2}} \right| + \left| \frac{(y_2 - m_{t,s}^{2,\xi}(x))}{(s-t)^{3/2}} \right| \right. \\ &\quad \left. + \left| \frac{(y_1 - m_{t,s}^{2,\xi}(x))}{(s-t)^{1/2}} \right| + \left| \frac{(y_2 - m_{t,s}^{2,\xi}(x))}{(s-t)^{3/2}} \right| \right\} \tilde{q}(t, x_1, x_2; s, y_1, y_2) \\ &\leq C'(s-t)^{-1/2} \hat{q}_c(t, x_1, x_2; s, y_1, y_2). \end{aligned}$$

Finally,

$$\begin{aligned} & D_{x_1}^2 \tilde{q}(t, x_1, x_2; s, y_1, y_2) \\ &= \left(-2[\tilde{\Sigma}_{t,s}^{-1}]_{1,1} D_{x_1} m_{t,s}^{1,\xi}(x) - 2[\tilde{\Sigma}_{t,s}^{-1}]_{1,2} D_{x_1} m_{t,s}^{2,\xi}(x) - 2[\tilde{\Sigma}_{t,s}^{-1}]_{1,2} \left[(R_{t,s}(\xi)) D_{x_1} m_{t,s}^{1,\xi}(x) \right] \right. \\ &\quad \left. -2[\tilde{\Sigma}_{t,s}^{-1}]_{2,2} \left[(R_{t,s}(\xi)) D_{x_1} m_{t,s}^{2,\xi}(x) \right] \right) \tilde{q}(t, x_1, x_2; s, y_1, y_2) \\ &+ \left(-2[\tilde{\Sigma}_{t,s}^{-1}]_{1,1}(y_1 - m_{t,s}^{1,\xi}(x)) - 2[\tilde{\Sigma}_{t,s}^{-1}]_{1,2}(y_2 - m_{t,s}^{2,\xi}(x)) \right. \\ &\quad \left. -2[\tilde{\Sigma}_{t,s}^{-1}]_{1,2} \left[(R_{t,s}(\xi)) (y_1 - m_{t,s}^{1,\xi}(x)) \right] \right. \\ &\quad \left. -2[\tilde{\Sigma}_{t,s}^{-1}]_{2,2} \left[(R_{t,s}(\xi)) (y_2 - m_{t,s}^{2,\xi}(x)) \right] \right)^2 \tilde{q}(t, x_1, x_2; s, y_1, y_2). \end{aligned}$$

Note that, from (3.3) we have $D_{x_1} m_{t,s}^\xi(x) = (\text{Id}, R_{t,s}(\xi))^*$, so that,

$$|D_{x_1}^2 \tilde{q}(t, x_1, x_2; s, y_1, y_2)| \leq C(s-t)^{-1} \hat{q}_c(t, x_1, x_2; s, y_1, y_2).$$

The other derivatives can be deduced from these computations and estimate (3.8) follows. \square

Remark. From this proof, one can deduce that the symmetry $D_{x_2} \tilde{q} = -D_{y_2} \tilde{q}$ holds. Therefore, for all t in $[0, T]$, all s in $[t, T]$ and y_1, x_1, x_2 in \mathbb{R}^d ,

$$\int_{\mathbb{R}^d} D_{x_2} \tilde{q}(t, x_1, x_2; s, y_1, y_2) dy_2 = 0. \quad (3.10)$$

This argument is very useful in the sequel.

3.2 Definitions and rules of calculus

We introduce some definitions and rules of computations that will be useful in the following section. Let us begin by the following definition:

Definition 3.2. For all ζ in \mathbb{R}^{2d} we denote by $\Delta(\zeta)$ the perturbation operator around ζ acting on any function f from $[0, T] \times \mathbb{R}^{2d}$ as follows:

$$\forall (s, y) \in [0, T] \times \mathbb{R}^{2d}, \quad (\Delta(\zeta)f)(s, y) = f(s, y) - f(s, \zeta),$$

and for $j = 1, 2$, for all ζ in \mathbb{R}^{2d} , we denote by $\Delta^j(\zeta)$ the perturbation operator around ζ_j acting on any function f from $[0, T] \times \mathbb{R}^{2d}$ as follows:

$$\forall (s, y) \in [0, T] \times \mathbb{R}^{2d}, \quad (\Delta^1(\zeta)f)(s, y_1) = f(s, y_1, \zeta_2) - f(s, \zeta_1, \zeta_2),$$

and

$$\forall (s, y) \in [0, T] \times \mathbb{R}^{2d}, \quad (\Delta^2(\zeta)f)(s, y_1, y_2) = f(s, y_1, y_2) - f(s, y_1, \zeta_2).$$

Epecially, the notation $\Delta^j(\zeta)y_j$ stands for $y_j - \zeta_j$.

Given this definition we can give a generic *centering argument*, as introduced in Subsection 2.3 in the linear and Brownian heuristic:

Claim 3.3. Let \tilde{q} be the function defined by (3.5) in Proposition 3.1 and let f and g be two continuous functions defined on $[0, T] \times \mathbb{R}^{2d}$. For all N^1, N^2 in \mathbb{N} , for all $t < s \in [0, T]^2$, $x \in \mathbb{R}^{2d}$ and $\zeta \in \mathbb{R}^{2d}$ we have that:

- (a) $D_{x_1}^{N^1} D_{x_2}^{N^2} \int_{\mathbb{R}^{2d}} f(s, y) \tilde{q}(t, x; s, y) dy = D_{x_1}^{N^1} D_{x_2}^{N^2} \int_{\mathbb{R}^{2d}} (\Delta(\zeta)f)(s, y) \tilde{q}(t, x; s, y) dy$,
if $N_1 + N_2 > 0$,
- (b) $D_{x_1}^{N^1} D_{x_2}^{N^2} \int_{\mathbb{R}^{2d}} f(s, y) \tilde{q}(t, x; s, y) dy = D_{x_1}^{N^1} D_{x_2}^{N^2} \int_{\mathbb{R}^{2d}} (\Delta^2(\zeta)f)(s, y) \tilde{q}(t, x; s, y) dy$,
if $N_2 > 0$,
- (c) $D_{x_1}^{N^1} D_{x_2}^{N^2} \int_{\mathbb{R}^{2d}} (\Delta^1(\zeta)f)(s, y) g(s, y) \tilde{q}(t, x; s, y) dy$
 $= D_{x_1}^{N^1} D_{x_2}^{N^2} \int_{\mathbb{R}^{2d}} (\Delta^1(\zeta)f)(s, y) (\Delta^2(\zeta)g)(s, y) \tilde{q}(t, x; s, y) dy$, if $N_2 > 0$.

Proof. Let f and g be defined as in Claim 3.3 and let $t < s \in [0, T]^2$, $x \in \mathbb{R}^{2d}$. We have, by Definition 3.2 of Δ :

$$\begin{aligned} D_{x_1}^{N^1} D_{x_2}^{N^2} \int_{\mathbb{R}^{2d}} f(s, y) \tilde{q}(t, x; s, y) dy &= D_{x_1}^{N^1} D_{x_2}^{N^2} \int_{\mathbb{R}^{2d}} (\Delta(\zeta)f)(s, y) \tilde{q}(t, x; s, y) dy \\ &\quad + D_{x_1}^{N^1} D_{x_2}^{N^2} \int_{\mathbb{R}^{2d}} f(s, \zeta) \tilde{q}(t, x; s, y) dy, \end{aligned}$$

for all ζ in \mathbb{R}^{2d} . The last term in the right hand side is equal to 0 since it does not depend on x . This concludes the proof of (a). Now, we prove (c). For all ζ in \mathbb{R}^{2d} we have

$$\begin{aligned} & D_{x_1}^{N^1} D_{x_2}^{N^2} \int_{\mathbb{R}^{2d}} (\Delta^1(\zeta)f)(s, y_1)g(s, y_1, y_2)\tilde{q}(t, x_1, x_2; s, y_1, y_2)dy_1dy_2 \\ &= D_{x_1}^{N^1} D_{x_2}^{N^2} \int_{\mathbb{R}^{2d}} (\Delta^1(\zeta)f)(s, y_1)(\Delta^2(\zeta)g)(s, y_1, y_2)\tilde{q}(t, x_1, x_2; s, y_1, y_2)dy_1dy_2 \\ & \quad + D_{x_1}^{N^1} D_{x_2}^{N^2} \int_{\mathbb{R}^{2d}} (\Delta^1(\zeta)f)(s, y_1)g(s, y_1, \zeta_2)\tilde{q}(t, x_1, x_2; s, y_1, y_2)dy_1dy_2. \end{aligned}$$

By using again differentiation under the integral sign Theorem, Remark 3.10 (since $N^2 > 0$), the last term in the right hand side is equal to 0. Finally, assertion (b) follows from the same arguments. This concludes the proof of the Claim. \square

3.3 Representation and differentiation of the solution of the PDE (1.8)

Lemma 3.4. *Suppose that assumptions (HR) hold, then, the solution $u = (u_1, u_2)^*$ of the PDE (1.8) can be written as*

$$u_i(t, x) = \mathbb{E} \left[\int_t^T \phi_i(s, \tilde{X}_s^{t,x}) - (\mathcal{L} - \tilde{\mathcal{L}}^{t,\xi})u_i(s, \tilde{X}_s^{t,x})ds \right] \quad (3.11)$$

for all $x \in \mathbb{R}^{2d}$ and $t \in [0, T]$. It is infinitely differentiable and for all $\xi \in \mathbb{R}^{2d}$, for all $\epsilon > 0$, all N^1, N^2 in $\mathbb{N} \setminus \{0\}$, we have

$$D_{x_1}^{N^1} D_{x_2}^{N^2} u_i(t, x) = \int_{t+\epsilon}^T D_{x_1}^{N^1} D_{x_2}^{N^2} \mathbb{E} \left[\phi_i(s, \tilde{X}_s^{t,x}) - (\mathcal{L} - \tilde{\mathcal{L}}^{t,\xi})u_i(s, \tilde{X}_s^{t,x}) \right] ds + \mathcal{O}(\epsilon) \quad (3.12)$$

for $i = 1, 2$.

Proof. We recall that from Lemma 1.2, the PDE (1.8) is well posed and can be rewritten as

$$\begin{aligned} \partial_t u_i(t, x) + \tilde{\mathcal{L}}^{t,\xi} u_i(t, x) &= -(\mathcal{L} - \tilde{\mathcal{L}}^{t,\xi})u_i(t, x) + \phi_i(t, x), \quad (t, x) \in [0, T] \times \mathbb{R}^{2d} \\ u_i(T, x) &= 0, \quad i = 1, 2, \end{aligned} \quad (3.13)$$

so that (3.11) follows from Feynman-Kac representation. Next, given $\epsilon > 0$, we have

$$\begin{aligned} u_i(t, x) &= \int_{t+\epsilon}^T \mathbb{E} \left[\phi_i(s, \tilde{X}_s^{t,x}) - (\mathcal{L} - \tilde{\mathcal{L}}^{t,\xi})u_i(s, \tilde{X}_s^{t,x}) \right] ds \\ & \quad + \int_t^{t+\epsilon} \mathbb{E} \left[\phi_i(s, \tilde{X}_s^{t,x}) - (\mathcal{L} - \tilde{\mathcal{L}}^{t,\xi})u_i(s, \tilde{X}_s^{t,x}) \right] ds. \end{aligned}$$

Under (HR), the coefficients of \mathcal{L} , $\tilde{\mathcal{L}}^{t,\xi}$ and the functions ϕ_i , u_i are smooth (Lemma 1.2) and have bounded derivatives. From (3.11), differentiation under the integral sign Theorem and classical regularity results on the solution of the SDE (2.6) (see e.g. [Kun82]) we can deduce that the second term in the right hand side above is bounded by $K\epsilon$ where K is a positive constant depending on parameters in (HR). The proof follows. \square

Note that for all $\epsilon > 0$ it follows from Lemma 3.4 that for all $x \in \mathbb{R}^{2d}$ and $t \in [0, T]$

$$\begin{aligned}
u_i(t, x) &= \int_t^T \int_{\mathbb{R}^{2d}} \phi_i(s, y) \tilde{q}(t, x; s, y) dy ds \\
&\quad - \int_t^T \int_{\mathbb{R}^{2d}} \frac{1}{2} \text{Tr} [\Delta(\theta_{t,s}(\xi)) a(s, y) D_{x_1}^2 u_i(s, y)] \tilde{q}(t, x; s, y) dy ds \\
&\quad - \int_t^T \int_{\mathbb{R}^{2d}} [\Delta(\theta_{t,s}(\xi)) F_1(s, y)] \cdot D_{x_1} u_i(s, y) \tilde{q}(t, x; s, y) dy ds \\
&\quad - \int_t^T \int_{\mathbb{R}^{2d}} [\Delta(\theta_{t,s}(\xi)) F_2(s, y) - D_{x_1} F_2(s, \theta_{t,s}(\xi)) \Delta^1(\theta_{t,s}(\xi)) y_1] \cdot D_{x_2} u_i(s, y) \tilde{q}(t, x; s, y) dy ds \\
&=: H_i^1(t + \epsilon; t, x) + H_i^2(t + \epsilon; t, x) + H_i^3(t + \epsilon; t, x) + H_i^4(t + \epsilon; t, x) \\
&\quad + \int_{t+\epsilon}^T \mathbb{E} \left[\phi_i(s, \tilde{X}_s^{t,x}) - (\mathcal{L} - \tilde{\mathcal{L}}^{t,\xi}) u_i(s, \tilde{X}_s^{t,x}) \right] ds. \tag{3.14}
\end{aligned}$$

We now derive a representation formula for the derivatives of H_i^1 , H_i^2 , H_i^3 and H_i^4 (defined by (3.14)) that involve a differentiation w.r.t. the degenerate variable. This allows to handle the singularity of the derivative of the kernel \tilde{q} (see assertion (iv) in Proposition 3.1).

Lemma 3.5. *For all i in $\{1, 2\}$, all (t, x) in $[0, T] \times \mathbb{R}^{2d}$, n in $\{0, 1\}$ and all $\epsilon > 0$, we have:*

$$\begin{aligned}
D_{x_1}^n D_{x_2} u_i(t, x) &= D_{x_1}^n D_{x_2} H_i^1(t + \epsilon; t, x) + D_{x_1}^n D_{x_2} H_i^2(t + \epsilon; t, x) + D_{x_1}^n D_{x_2} H_i^3(t + \epsilon; t, x) \\
&\quad + D_{x_1}^n D_{x_2} H_i^4(t + \epsilon; t, x) + \mathcal{O}(\epsilon), \tag{3.15}
\end{aligned}$$

where the terms $D_{x_1}^n D_{x_2} H_i^j(t + \epsilon; t, x)$, $j = 1, 2, 3, 4$ can be written as:

$$D_{x_1}^n D_{x_2} H_i^1(t + \epsilon; t, x) = - \int_{t+\epsilon}^T \int_{\mathbb{R}^{2d}} \Delta^2(\theta_{t,s}(\xi)) F_i(s, y) [D_{x_1}^n D_{x_2} \tilde{q}(t, x; s, y)] dy ds, \tag{3.16}$$

and

$$\begin{aligned}
&D_{x_1}^n D_{x_2} H_i^2(t + \epsilon; t, x_1, x_2) \tag{3.17} \\
&= -\frac{1}{2} \int_{t+\epsilon}^T \int_{\mathbb{R}^{2d}} \text{Tr} [\Delta^2(\theta_{t,s}(\xi)) a(s, y_1, y_2)] D_{x_1}^2 u_i(s, y_1, y_2) [D_{x_1}^n D_{x_2} \tilde{q}(t, x_1, x_2; s, y_1, y_2)] dy_1 dy_2 ds \\
&\quad + \frac{1}{2} \int_{t+\epsilon}^T \left\{ \left[\sum_{l=1}^d \int_{\mathbb{R}^{2d}} \left[\frac{\partial}{\partial y_{1l}} a_l(s, y_1, \theta_{t,s}^2(\xi)) \right] \cdot [\Delta^2(\theta_{t,s}(\xi)) D_{x_1} u_i(s, y_1, y_2)] \right] \right. \\
&\quad \quad \left. \times [D_{x_1}^n D_{x_2} \tilde{q}(t, x_1, x_2; s, y_1, y_2)] \right\} dy_1 dy_2 ds \\
&\quad + \frac{1}{2} \sum_{l=1}^d \int_{t+\epsilon}^T \int_{\mathbb{R}^{2d}} [\Delta^1(\theta_{t,s}(\xi)) a_l(s, y_1, \theta_{t,s}^2(\xi))] \cdot [\Delta^2(\theta_{t,s}(\xi)) D_{x_1} u_i(s, y_1, y_2)] \\
&\quad \quad \times \left[D_{x_1}^n D_{x_2} \left(\frac{\partial}{\partial y_{1l}} \tilde{q}(t, x_1, x_2; s, y_1, y_2) \right) \right] dy_1 dy_2 ds,
\end{aligned}$$

where “ a_l .” denotes the l^{th} line of the matrix a , and

$$\begin{aligned}
& D_{x_1}^n D_{x_2} H_i^3(t + \epsilon; t, x_1, x_2) \\
&= - \int_{t+\epsilon}^T \int_{\mathbb{R}^{2d}} \left\{ \left([\Delta^2(\theta_{t,s}(\xi)) F_1(s, y_1, y_2)] \cdot D_{x_1} u_i(s, y_1, y_2) \right) \right. \\
&\quad \times D_{x_1}^n D_{x_2} \tilde{q}(t, x_1, x_2; s, y_1, y_2) \left. \right\} dy_1 dy_2 ds \\
&\quad - \int_{t+\epsilon}^T \left\{ \int_{\mathbb{R}^{2d}} \left([\Delta^1(\theta_{t,s}(\xi)) F_1(s, y_1, \theta_{t,s}^2(\xi))] \cdot [\Delta^2(\theta_{t,s}(\xi)) D_{x_1} u_i(s, y_1, y_2)] \right) \right. \\
&\quad \times D_{x_1}^n D_{x_2} \tilde{q}(t, x_1, x_2; s, y_1, y_2) \left. \right\} dy_1 dy_2 ds,
\end{aligned} \tag{3.18}$$

and finally:

$$\begin{aligned}
& D_{x_1}^n D_{x_2} H_i^4(t + \epsilon; t, x_1, x_2) = \\
&\quad - \int_{t+\epsilon}^T \int_{\mathbb{R}^{2d}} \left\{ [\Delta^1(\theta_{t,s}(\xi)) F_2(s, y_1, \theta_{t,s}^2(\xi)) - D_{x_1} F_2(s, \theta_{t,s}^1(\xi), \theta_{t,s}^2(\xi)) \Delta^1(\theta_{t,s}(\xi)) y_1] \right. \\
&\quad \cdot [\Delta^2(\theta_{t,s}(\xi)) D_{x_2} u_i(s, y_1, y_2)] \left. \right\} D_{x_1}^n D_{x_2} \tilde{q}(t, x_1, x_2; s, y_1, y_2) dy_1 dy_2 ds \\
&\quad - \int_{t+\epsilon}^T \int_{\mathbb{R}^{2d}} [\Delta^2(\theta_{t,s}(\xi)) F_2(s, y_1, y_2)] \cdot D_{x_2} u_i(s, y_1, y_2) D_{x_1}^n D_{x_2} \tilde{q}(t, x_1, x_2; s, y_1, y_2) dy_1 dy_2 ds,
\end{aligned} \tag{3.19}$$

Proof. Equation (3.15) follows from Lemma 3.4. Representation (3.16) is a direct consequence of assertion (b) in Claim 3.3. Next, we deal with (3.17). By using first the decomposition $\Delta = \Delta^1 + \Delta^2$ and by integrating by parts, we have

$$\begin{aligned}
& H_i^2(t + \epsilon; t, x_1, x_2) \\
&= - \int_{t+\epsilon}^T \int_{\mathbb{R}^{2d}} \left(\frac{1}{2} \text{Tr} \left[[\Delta^2(\theta_{t,s}(\xi)) a(s, y_1, y_2)] D_{y_1}^2 u_i(s, y_1, y_2) \right] \right) \tilde{q}(t, x_1, x_2; s, y_1, y_2) dy_1 dy_2 ds \\
&\quad + \frac{1}{2} \sum_{l=1}^d \int_{t+\epsilon}^T \int_{\mathbb{R}^{2d}} \left[\frac{\partial}{\partial y_{1l}} \Delta^1(\theta_{t,s}(\xi)) a_l(s, y_1, \theta_{t,s}^2(\xi)) \right] \cdot D_{y_1} u_i(s, y_1, y_2) \tilde{q}(t, x_1, x_2; s, y_1, y_2) dy_1 dy_2 ds \\
&\quad + \frac{1}{2} \sum_{l=1}^d \int_{t+\epsilon}^T \int_{\mathbb{R}^{2d}} \left[\Delta^1(\theta_{t,s}(\xi)) a_l(s, y_1, \theta_{t,s}^2(\xi)) \right] \cdot D_{y_1} u_i(s, y_1, y_2) \frac{\partial}{\partial y_{1l}} \tilde{q}(t, x_1, x_2; s, y_1, y_2) dy_1 dy_2 ds.
\end{aligned}$$

Note that, for all $l \in \{1, \dots, d\}$, $[\partial/\partial y_{1l}] \Delta^1(\theta_{t,s}(\xi)) a_l(s, y_1, \theta_{t,s}^2(\xi)) = [\partial/\partial y_{1l}] a_l(s, y_1, \theta_{t,s}^2(\xi))$. We conclude by differentiating and then by applying assertion (c) of Claim 3.3 with $f = a$ and $g = D_{x_1} u_i$. This gives (3.17).

By using again the fact that $\Delta = \Delta^1 + \Delta^2$, assertions (3.18) and (3.19) are immediate consequences of assertions (b) and (c) of Claim 3.3. This concludes the proof of Lemma 3.5. \square

4 Proof of Proposition 1.3

4.1 From intermediate gradient estimates to Proposition 1.3

We here give the intermediate estimates that allow to prove Proposition 1.3. In the following, u_i denotes the i^{th} component of the solution $u = (u_1, u_2)^*$ of the linear system of PDE (1.8). The following arguments and Lemmas hold for $i = 1, 2$.

Since the representation (3.14) of each u_i involves its derivatives, we prove Proposition 1.3 by using a *circular argument* (see (2.15)). We first show that

Lemma 4.1. *Suppose that assumptions (HR) hold. Then, there exist a positive $\mathcal{T}_{4.1}$, two positive numbers $\delta_{4.1}$ and $\bar{\delta}_{4.1}$ and a positive constant C , depending only on known parameters in (H) such that:*

$$\|D_{x_1}^2 u_i\|_\infty \leq T^{\delta_{4.1}} C (1 + \|D_{x_2} u_i\|_\infty),$$

and

$$\|D_{x_1} u_i\|_\infty \leq T^{\bar{\delta}_{4.1}} C (1 + \|D_{x_2} u_i\|_\infty),$$

for all T less than $\mathcal{T}_{4.1}$.

Then, we estimate the gradients that involve the derivatives w.r.t. the degenerate variable “ x_2 ”. As shown in Subsection 2.4, this differentiation generates the worst singularity. In this more general case, we have to use the regularity of the solution itself, in addition to the regularity of the coefficients assumed in (H), in order to smooth this singularity. Notably, we need to estimate the Hölder regularity of $D_{x_2} u_i$. Hence, we first prove the following sort of Hölder estimate on $D_{x_2} u_i$:

Lemma 4.2. *Suppose that assumptions (HR) hold and let*

$$M(D_{x_2} u_i, T) := \sup_{w_1, w_2 \neq w'_2 \in \mathbb{R}^d, t \in [0, T]} \frac{|D_{x_2} u_i(t, w_1, w_2) - D_{x_2} u_i(t, w_1, w'_2)|}{|w_2 - w'_2|^{\gamma/3} + |w_2 - w'_2|^{\beta_2^2} + |w_2 - w'_2|^{\beta_1^2} + |w_2 - w'_2|}, \quad (4.1)$$

for some positive number γ . There exist a positive $\mathcal{T}_{4.2}$, a positive constant C and a positive number $\delta_{4.2}$, depending only on known parameters in (H), such that for all $0 < \gamma < 3 \inf\{\beta_1^2, \beta_2^2\} - 1$,

$$M(D_{x_2} u_i, T) \leq CT^{\delta_{4.2}} (\|D_{x_2} u_i\|_\infty + \|D_{x_1} D_{x_2} u_i\|_\infty),$$

for all T less than $\mathcal{T}_{4.2}$.

This allows us to obtain the following estimates

Lemma 4.3. *Suppose that assumptions (HR) hold and let n in $\{0, 1\}$. Then, there exist a positive $\mathcal{T}_{4.3}(n)$, a positive number $\delta_{4.3}(n)$ and a positive constant $C(n)$, depending only on known parameters in (H) and n , such that:*

$$\|D_{x_1}^n D_{x_2} u_i\|_\infty \leq C(n) T^{\delta_{4.3}(n)},$$

for all T less than $\mathcal{T}_{4.3}(n)$.

Finally, we can deduce Proposition 1.3 from Lemmas 4.1 and 4.3. □

4.2 Gradient estimates

4.2.1 Proof of Lemma 4.1

Here we prove that there exists a positive constant C depending only on known parameters in (H) such that:

$$\begin{aligned} \text{(i)} \quad & \|D_{x_1}^2 u_i\|_\infty \leq C \left\{ \left(T^{3\beta_2^2/2} + T^{(1+\alpha^1)/2} \right) \|D_{x_2} u_i\|_\infty + \left(T^{\beta_1^1/2} + T^{3\beta_1^2/2} \right) \|D_{x_1} u_i\|_\infty + T^{\beta_1^1/2} + T^{3\beta_1^2/2} \right\}, \\ \text{(ii)} \quad & \|D_{x_1} u_i\|_\infty \leq C \left\{ \left(T^{1/2+3\beta_2^2/2} + T^{(3+\alpha^1)/2} \right) \|D_{x_2} u_i\|_\infty + (T + T^2) \|D_{x_1}^2 u_i\|_\infty + T^{(1+\beta_1^1)/2} + T^{(1+3\beta_1^2)/2} \right\}, \end{aligned}$$

for all T less than some positive $\mathcal{T}_{4.1}$.

We first show (i). Let $\epsilon > 0$, from the representation (3.14) and Lemma 3.4 we have:

$$D_{x_1}^2 u_i(t, x) = \sum_{j=1}^4 D_{x_1}^2 H_i^j(t + \epsilon; t, x) + \mathcal{O}(\epsilon), \quad (4.2)$$

and thanks to assertion (a) in Claim 3.3 applied on $D_{x_1}^2 H_i^1$:

$$\begin{aligned}
D_{x_1}^2 u_i(t, x) &= \int_{t+\epsilon}^T \int_{\mathbb{R}^{2d}} \Delta(\theta_{t,s}(\xi)) \phi_i(s, y) [D_{x_1}^2 \tilde{q}(t, x; s, y)] dy ds \\
&\quad - \frac{1}{2} \int_{t+\epsilon}^T \int_{\mathbb{R}^{2d}} \text{Tr} [\Delta(\theta_{t,s}(\xi)) a(s, y) D_{x_1}^2 u_i(s, y)] [D_{x_1}^2 \tilde{q}(t, x; s, y)] dy ds \\
&\quad - \int_{t+\epsilon}^T \int_{\mathbb{R}^{2d}} [\Delta(\theta_{t,s}(\xi)) F_1(s, y)] \cdot D_{x_1}^2 u_i(s, y) [D_{x_1}^2 \tilde{q}(t, x; s, y)] dy ds \\
&\quad + \int_{t+\epsilon}^T \int_{\mathbb{R}^{2d}} [\Delta^2(\theta_{t,s}(\xi)) F_2(s, y) - D_{x_1} F_2(s, \theta_{t,s}^1(\xi), \theta_{t,s}^2(\xi)) \Delta(\theta_{t,s}(\xi)) y_1] \\
&\quad \cdot [D_{x_2} u_i(s, y)] [D_{x_1}^2 \tilde{q}(t, x; s, y)] dy ds + \mathcal{O}(\epsilon).
\end{aligned} \tag{4.3}$$

Then, note that thanks to a Taylor expansion of order 0 with integrable remainder of the mapping $y_1 \in \mathbb{R}^d \mapsto F_2(s, y_1, \theta_{t,s}^2(\xi))$ around $\theta_{t,s}^1(\xi)$ and **(H3-a)**, we have that:

$$\begin{aligned}
&|[\Delta(\theta_{t,s}(\xi)) F_2(s, y) - D_{x_1} F_2(s, \theta_{t,s}(\xi)) (\Delta^1(\theta_{t,s}(\xi)) y_1)] \cdot D_{x_2} u_i(s, y)| \\
&\leq C \|D_{x_2} u_i\|_\infty \left(|\Delta^2(\theta_{t,s}(\xi)) y_2|^{\beta_2^2} + |\Delta^1(\theta_{t,s}(\xi)) y_1|^{1+\alpha^1} \right).
\end{aligned} \tag{4.4}$$

From Proposition 3.1, we know that for all s in $(t, T]$ and $y \in \mathbb{R}^{2d}$, $|D_{x_1}^2 \tilde{q}(t, x; s, y)| \leq C'(s-t)^{-1} \hat{q}_c(t, x; s, y)$. Plugging this estimate in (4.3), together with the regularity of the coefficients given in **(H)** and (4.4), we obtain that

$$\begin{aligned}
|D_{x_1}^2 u_i(t, x_1, x_2)| &\leq C'' \int_{t+\epsilon}^T (s-t)^{-1} \int_{\mathbb{R}^{2d}} \left\{ \sum_{j=1}^2 \left\{ (s-t)^{(j-1/2)\beta_j^i} \left| \frac{\Delta^j(\theta_{t,s}(\xi)) y_j}{(s-t)^{(j-1/2)}} \right|^{\beta_j^i} + \|D_{x_1}^2 u_i\|_\infty \right. \right. \\
&\quad \left. \left. \times (s-t)^{(j-1/2)} \left| \frac{\Delta^j(\theta_{t,s}(\xi)) y_j}{(s-t)^{(j-1/2)}} \right| + \|D_{x_1} u_i\|_\infty (s-t)^{(j-1/2)\beta_j^i} \left| \frac{\Delta^j(\theta_{t,s}(\xi)) y_j}{(s-t)^{(j-1/2)}} \right|^{\beta_j^i} \right\} \right. \\
&\quad \left. + \|D_{x_2} u_i\|_\infty \left[(s-t)^{3\beta_2^2/2} \left| \frac{\Delta^2(\theta_{t,s}(\xi)) y_2}{(s-t)^{3/2}} \right|^{\beta_2^2} + (s-t)^{(1+\alpha^1)/2} \right. \right. \\
&\quad \left. \left. \times \left| \frac{\Delta^1(\theta_{t,s}(\xi)) y_1}{(s-t)^{1/2}} \right|^{1+\alpha^1} \right] \right\} \hat{q}_c(t, x; s, y) dy ds + \mathcal{O}(\epsilon).
\end{aligned}$$

Set $\xi = x$, by using the off-diagonal decay of the Gaussian exponential in \hat{q}_c (see the computations in Subsection 2.4) and by integrating w.r.t the space variables we have:

$$\begin{aligned}
&|D_{x_1}^2 u_i(t, x_1, x_2)| \\
&\leq C''' \int_{t+\epsilon}^T \left\{ \sum_{j=1}^2 \left\{ (s-t)^{-1+(j-1/2)\beta_j^i} + \left((s-t)^{-1/2} + (s-t)^{1/2} \right) \|D_{x_1}^2 u_i\|_\infty \right. \right. \\
&\quad \left. \left. + (s-t)^{(j-1/2)\beta_j^i-1} \|D_{x_1} u_i\|_\infty \right\} + \left((s-t)^{-1+3\beta_1^2/2} + (s-t)^{(\alpha^1-1)/2} \right) \|D_{x_2} u_i\|_\infty \right\} ds + \mathcal{O}(\epsilon).
\end{aligned}$$

By letting $\epsilon \rightarrow 0$, we finally obtain:

$$\begin{aligned} \|D_{x_1}^2 u_i\|_\infty &\leq C''' \left\{ \left(T^{3\beta_2^2/2} + T^{(1+\alpha^1)/2} \right) \|D_{x_2} u_i\|_\infty + \left(T^{\beta_1^1/2} + T^{3\beta_1^2/2} \right) \|D_{x_1} u_i\|_\infty \right. \\ &\quad \left. + \left(T^{1/2} + T^{3/2} \right) \|D_{x_1}^2 u_i\|_\infty + T^{\beta_i^1/2} + T^{3\beta_i^2/2} \right\}. \end{aligned}$$

Then, by setting $\mathcal{T}_{4.1}^{(i)} = \sup \{T > 0, \text{ such that } C'''(T^{1/2} + T^{3/2}) \leq 1/2\}$ we deduce the assertion (i) for all T less than $\mathcal{T}_{4.1}^{(i)}$ from a *circular argument* (see (2.15)). The proof of the second statement (ii) can be done by the same arguments. This concludes the proof of Lemma 4.1. \square

4.2.2 Proof of Lemma 4.3

We know from Lemma 3.5 that, for all $\epsilon > 0$

$$D_{x_1}^n D_{x_2} u_i(t, x) = \sum_{j=1}^4 D_{x_1}^n D_{x_2} H_i^j(t + \epsilon; t, x) + \mathcal{O}(\epsilon), \quad (4.5)$$

where the $D_{x_1}^n D_{x_2} H_i^j$, $j = 1, \dots, 4$, are respectively given by (3.16), (3.17), (3.18) and (3.19). Since from Proposition 3.1 we have that for all s in $(t, T]$ and $y \in \mathbb{R}^{2d}$,

$$|D_{x_1}^n D_{x_2} \tilde{q}(t, x; s, y)| \leq C(s-t)^{-(3+n)/2} \hat{q}_c(t, x; s, y), \quad (4.6)$$

we deduce that each $D_{x_1}^n D_{x_2} H_i^j$, $j = 1, \dots, 4$, can be bounded by a time-space integral of some integrand against the degenerate Gaussian kernel \hat{q}_c multiplied by a time singularity of order $(3+n)/2$.

As done in the previous subsection and in Subsection 2.4, we use the regularity of the coefficients together with the regularity of the solution u_i and its derivatives to smooth this singularity.

To this aim let us first remark that

$$D_{x_1}^n D_{x_2} H_i^j(t + \epsilon; t, x) = \int_{t+\epsilon}^T \int_{\mathbb{R}^{2d}} F_i^j(s, y, \theta_{t,s}(\xi)) D_{x_1}^n D_{x_2} \tilde{q}(t, x; s, y) dy ds, \quad (4.7)$$

for all $j = 1, \dots, 4$, where F_i^j , $j = 1, \dots, 4$, are some function properly defined by identifying (4.7) with (3.16), (3.17), (3.18) and (3.19) respectively : F_i^j is the integrand of $D_{x_1}^n D_{x_2} H_i^j$.

Next, let us bound each F_i^j , $j = 1, \dots, 4$. From the regularity of the coefficients assumed in (H) we have that:

$$|F_i^1(s, y, \theta_{t,s}(\xi))| \leq C' |\Delta^2(\theta_{t,s}(\xi)) y_2|^{\beta_i^2}. \quad (4.8)$$

Then, we recall that from Mean Value Theorem (MVT) we have

$$|\Delta^2(\theta_{t,s}(\xi)) D_{x_1} u_i(s, y)| \leq \|D_{x_1} D_{x_2} u_i\|_\infty |\Delta^2(\theta_{t,s}(\xi)) y_2|, \quad (4.9)$$

so that

$$\begin{aligned} |F_i^2(s, y, \theta_{t,s}(\xi))| &\leq C'' \|D_{x_1}^2 u_i\|_\infty |\Delta^2(\theta_{t,s}(\xi)) y_2| + \|D_{x_1} D_{x_2} u_i\|_\infty \\ &\quad \times \left(|\Delta^2(\theta_{t,s}(\xi)) y_2| + |\Delta^1(\theta_{t,s}(\xi)) y_1| |\Delta^2(\theta_{t,s}(\xi)) y_2| \right). \end{aligned} \quad (4.10)$$

By the same way, we get that

$$\begin{aligned} |F_i^3(s, y, \theta_{t,s}(\xi))| &\leq C''' \|D_{x_1} u_i\|_\infty |\Delta^2(\theta_{t,s}(\xi)) y_2|^{\beta_1^2} \\ &\quad + \|D_{x_1} D_{x_2} u_i\|_\infty |\Delta^1(\theta_{t,s}(\xi)) y_1|^{\beta_1^1} |\Delta^2(\theta_{t,s}(\xi)) y_2|. \end{aligned} \quad (4.11)$$

And finally, since

$$\begin{aligned} & |\Delta^2(\theta_{t,s}(\xi))D_{x_2}u_i(s, y)| \\ & \leq M(D_{x_2}u_i, T) \left(|\Delta^2(\theta_{t,s}(\xi))y_2|^\gamma + |\Delta^2(\theta_{t,s}(\xi))y_2|^{\beta_1^2} + |\Delta^2(\theta_{t,s}(\xi))y_2|^{\beta_2^2} + |\Delta^2(\theta_{t,s}(\xi))y_2| \right), \end{aligned} \quad (4.12)$$

where $M(D_{x_2}u_i, T)$ is defined by (4.1) we have from the regularity of the coefficients assumed in **(H)** and (4.4) that

$$\begin{aligned} |F_i^4(s, y, \theta_{t,s}(\xi))| & \leq C'''' \|D_{x_2}u_i\|_\infty |\Delta^2(\theta_{t,s}(\xi))y_2|^{\beta_2^2} + D_M(D_{x_2}u_i, T) |\Delta^1(\theta_{t,s}(\xi))y_1|^{1+\alpha^1} \\ & \times \left(|\Delta^2(\theta_{t,s}(\xi))y_2|^\gamma + |\Delta^2(\theta_{t,s}(\xi))y_2|^{\beta_1^2} + |\Delta^2(\theta_{t,s}(\xi))y_2|^{\beta_2^2} + |\Delta^2(\theta_{t,s}(\xi))y_2| \right). \end{aligned} \quad (4.13)$$

Let us recall that for all positive κ there exist two positive constants \underline{C} and \underline{c} , depending only on known parameters in **(H)** and on κ , such that for all $(t < s, x, y) \in [0, T]^2 \times \mathbb{R}^{2d} \times \mathbb{R}^{2d}$

$$|\Delta^l(\theta_{t,s}(x))y_l|^\kappa \hat{q}_c(t, x; s, y) \leq \underline{C}(s-t)^{(j-1/2)\kappa} \hat{q}_{\underline{c}}(t, x; s, y), \quad (4.14)$$

for $l = 1, 2$.

Thus, by plugging (4.6) in each (4.7) together with the corresponding estimate (4.8), (4.10), (4.11) or (4.13) and by letting $\xi = x$ we deduce that

$$\begin{aligned} |D_{x_1}^n D_{x_2}u_i(t, x)| & \leq c' \int_{t+\epsilon}^T \int_{\mathbb{R}^{2d}} \left\{ (s-t)^{(3(\beta_i^2-1)-n)/2} + (s-t)^{-n/2} (\|D_{x_1}^2 u_i\|_\infty + \|D_{x_1} D_{x_2} u_i\|_\infty) \right. \\ & + \|D_{x_1} u_i\|_\infty (s-t)^{(3(\beta_1^2-1)-n)/2} + \|D_{x_1} D_{x_2} u_i\|_\infty (s-t)^{(\beta_1^2-n)/2} \\ & + M(D_{x_2}u_i, T) (s-t)^{(1+\alpha^1-3-n)/2} \left((s-t)^{\gamma/2} + (s-t)^{\beta_1^2/2} + (s-t)^{\beta_2^2/2} + (s-t) \right) \\ & \left. \times (\|D_{x_2}u_i\|_\infty + \|D_{x_1} D_{x_2} u_i\|_\infty) + \|D_{x_2}u_i\|_\infty (s-t)^{(3(\beta_2^2-1)-n)/2} \right\} \\ & \times \hat{q}_{c'}(t, x; s, y) dy ds + \mathcal{O}(\epsilon). \end{aligned} \quad (4.15)$$

Now, we have from Lemma 4.2 that for all T less than $\mathcal{T}_{4.2}$

$$M(D_{x_2}u_i, T) \leq c'' \mathcal{T}_{4.2} (\|D_{x_2}u_i\|_\infty + \|D_{x_1} D_{x_2} u_i\|_\infty), \quad (4.16)$$

for all $0 < \gamma < 3 \inf\{\beta_1^2, \beta_2^2\} - 1$.

By plugging (4.16) in (4.15) and by choosing n be equal to 0, after integrating in time and after letting ϵ tends to 0, we deduce from a *circular argument* (as described in (2.15)) that there exist two positive numbers $\mathcal{T}'_{4.3}(0)$, $\delta_{4.3}(0)$ and a constant \bar{C} , depending only on known parameters in **(H)**, such that

$$\|D_{x_2}u_i\|_\infty \leq \bar{C} T^{\delta_{4.3}(0)} (1 + \|D_{x_1} D_{x_2} u_i\|_\infty), \quad (4.17)$$

for all T less than $\mathcal{T}'_{4.3}(0)$. Next, by letting n be equal to 1 in (4.15) (recall that Lemma 4.2 hold for all $0 < \gamma < 3 \inf\{\beta_1^2, \beta_2^2\} - 1$, so that all the time-singularity are integrable), by using the same arguments as in the case $n = 0$ together with (4.17), we can show that there exists two positive numbers $\mathcal{T}_{4.3}(1)$, $\delta_{4.3}(1)$ and a constant \bar{C}' , depending only on known parameters in **(H)**, such that

$$\|D_{x_1} D_{x_2} u_i\|_\infty \leq \bar{C}' T^{\delta_{4.3}(1)},$$

for all T less than $\mathcal{T}_{4.3}(1)$. We conclude by plugging this estimate in (4.17).

4.2.3 Proof of Lemma 4.2

From (3.14) and Lemma 3.4, for all (t, x_1) in $[0, T] \times \mathbb{R}^d$ and $x_2 \neq z_2$ in \mathbb{R}^d we have:

$$\begin{aligned} & |D_{x_2} u_i(t, x_1, x_2) - D_{x_2} u_i(t, x_1, z_2)| \\ & \leq \left| \sum_{j=1}^4 \left(D_{x_2} H_i^j \right) (t + \epsilon; t, x_1, x_2) - \left(D_{x_2} H_i^j \right) (t + \epsilon; t, x_1, z_2) \right| + \mathcal{O}(\epsilon). \end{aligned} \quad (4.18)$$

We recall that the H_i^j , $j = 1, \dots, 4$, depend on the freezing point $\xi = (\xi_1, \xi_2)$ of the process which started from (x_1, x_2) and (x_1, z_2) at time t . Here, we choose the same freezing point “ ξ ” for the two processes (with different initial conditions). As in (4.7), we write each $D_{x_2} H_i^j$ as:

$$D_{x_2} H_i^j(t + \epsilon; t, x_1, x_2) = \int_{t+\epsilon}^T \int_{\mathbb{R}^{2d}} F_i^j(s, y_1, y_2, \theta_{t,s}^1(\xi), \theta_{t,s}^2(\xi)) D_{x_2} \tilde{q}(t, x_1, x_2; s, y_1, y_2) dy_1 dy_2 ds, \quad (4.19)$$

where we recall that the F_i^j , $j = 1, \dots, 4$, are some functions properly defined by identifying (4.19) with (3.16), (3.17), (3.18) and (3.19) respectively.

Then, we split the time interval w.r.t. the characteristic scale of the second component of the system (3.1) in order to study the perturbation on each interval separately. Hence, we set $\mathcal{S} = \{s \in (t, T] \text{ s.t. } |x_2 - z_2| < (s - t)^{3/2}\}$ and $\mathcal{S}^c = \{s \in (t, T] \text{ s.t. } |x_2 - z_2| \geq (s - t)^{3/2}\}$ and we investigate the Hölder regularity on these two sets. Since we have:

$$\begin{aligned} & \sum_{j=1}^4 \left(D_{x_2} H_i^j \right) (t + \epsilon; t, x_1, x_2) - \left(D_{x_2} H_i^j \right) (t + \epsilon; t, x_1, z_2) \\ & = \sum_{j=1}^4 \int_{t+\epsilon}^T \mathbf{1}_{\mathcal{S}} \int_{\mathbb{R}^{2d}} \left\{ F_i^j(s, y_1, y_2, \theta_{t,s}^1(\xi), \theta_{t,s}^2(\xi)) \right. \\ & \quad \left. \left((D_{x_2} \tilde{q})(t, x_1, x_2; s, y_1, y_2) - (D_{x_2} \tilde{q})(t, x_1, z_2; s, y_1, y_2) \right) \right\} dy_1 dy_2 ds \\ & \quad + \sum_{j=1}^4 \int_{t+\epsilon}^T \mathbf{1}_{\mathcal{S}^c} \int_{\mathbb{R}^{2d}} \left\{ F_i^j(s, y_1, y_2, \theta_{t,s}^1(\xi), \theta_{t,s}^2(\xi)) \right. \\ & \quad \left. \left((D_{x_2} \tilde{q})(t, x_1, x_2; s, y_1, y_2) - (D_{x_2} \tilde{q})(t, x_1, z_2; s, y_1, y_2) \right) \right\} dy_1 dy_2 ds \\ & := P_i(t + \epsilon; t, x; z, \mathcal{S}) + P_i(t + \epsilon; t, x; z, \mathcal{S}^c), \end{aligned} \quad (4.20)$$

we bound separately the first and the second term in the right hand side of the last equality above.

Estimation of $P_i(t + \epsilon; t, x; z, \mathcal{S})$. As a first step, we bound the first sum in the right hand side of (4.20). At the end of this part, it is proven that:

Claim 4.4. *For all s in \mathcal{S} , y_1, y_2 in \mathbb{R}^d , the following inequality holds:*

$$\begin{aligned} & |D_{x_2} \tilde{q}(t, x_1, x_2; s, y_1, y_2) - D_{x_2} \tilde{q}(t, x_1, z_2; s, y_1, y_2)| \\ & \leq C(s - t)^{-(3+\gamma)/2} \hat{q}_c(t, x_1, x_2; s, y_1, y_2) |x_2 - z_2|^{\gamma/3}, \end{aligned} \quad (4.21)$$

where c and C depend only on known parameters in **(H)** and where $0 < \gamma < 3$.

Next, as in the proof of Lemma 4.3 when $n = 0$, by using estimate (4.8), (4.10), (4.11), (4.13) together with (4.21) in $P_i(t + \epsilon; t, x; z, \mathcal{S})$ and by letting $\xi = x$ we deduce that

$$\begin{aligned}
& |P_i(t + \epsilon; t, x; z, \mathcal{S})| \\
& \leq C' \int_{t+\epsilon}^T \left\{ (s-t)^{-(3+\gamma-3\beta_1^2)/2} + \|D_{x_1}^2 u_i\|_\infty (s-t)^{-\gamma/2} + \|D_{x_1} D_{x_2} u_i\|_\infty (s-t)^{-\gamma/2} \right. \\
& \quad + \|D_{x_1} u_i\|_\infty (s-t)^{-(3+\gamma-3\beta_1^2)/2} + \|D_{x_1} D_{x_2} u_i\|_\infty (s-t)^{-(\gamma-\beta_1^1)/2} + \|D_{x_2} u_i\|_\infty (s-t)^{3(\beta_2^2-1-\gamma/3)/2} \\
& \quad \left. + M(D_{x_2} u_i, T) (s-t)^{-1-\gamma/2+\alpha^1/2} \left((s-t)^{\gamma/2} + (s-t)^{3\beta_1^2/2} + (s-t)^{3\beta_2^2/2} + (s-t)^{3/2} \right) \right\} ds |x_2 - z_2|^{\gamma/3},
\end{aligned}$$

for all $\gamma < 3 \inf(\beta_1^2, \beta_2^2) - 1$. By integrating in space and by using Lemma 4.1, we obtain that there exists a positive number $\delta_{4.2}^1$, depending only on known parameters in **(H)**, such that:

$$\begin{aligned}
|P_i(t + \epsilon, x; z, \mathcal{S})| & \leq C'' T^{\delta_{4.2}^1} (M(D_{x_2} u_i, T) + \|D_{x_2} u_i\|_\infty + \|D_{x_1} D_{x_2} u_i\|_\infty) \\
& \quad \times \left(|x_2 - z_2|^{\gamma/3} + |x_2 - z_2|^{\beta_2^2} + |x_2 - z_2|^{\beta_1^2} + |x_2 - z_2| \right),
\end{aligned} \tag{4.22}$$

for all $\gamma < 3 \inf(\beta_1^2, \beta_2^2) - 1$.

Estimation of $P_i(t + \epsilon; t, x; z, \mathcal{S}^c)$. As a second step, we bound the sum $P_i(t + \epsilon; t, x; z, \mathcal{S}^c)$ in (4.20). Note that this reads:

$$\begin{aligned}
& P_i(t + \epsilon; t, x; z, \mathcal{S}^c) \\
& = \sum_{j=1}^4 \int_{t+\epsilon}^T \mathbf{1}_{\mathcal{S}^c} \int_{\mathbb{R}^{2d}} F_i^j(s, y_1, y_2, \theta_{t,s}^1(\xi), \theta_{t,s}^2(\xi)) (D_{x_2} \tilde{q})(t, x_1, x_2; s, y_1, y_2) dy_1 dy_2 ds
\end{aligned} \tag{4.23}$$

$$- \sum_{j=1}^4 \int_{t+\epsilon}^T \mathbf{1}_{\mathcal{S}^c} \int_{\mathbb{R}^{2d}} F_i^j(s, y_1, y_2, \theta_{t,s}^1(\xi), \theta_{t,s}^2(\xi)) (D_{x_2} \tilde{q})(t, x_1, z_2; s, y_1, y_2) dy_1 dy_2 ds \tag{4.24}$$

and that for all s in \mathcal{S}^c we have:

$$1 \leq (s-t)^{-\gamma/2} |x_2 - z_2|^{\gamma/3}. \tag{4.25}$$

On a one hand we bound the term (4.23). Again, as in the proof of Lemma 4.3, we can use estimate (4.8), (4.10), (4.11), (4.13) together with (4.6) (with $n = 0$) and by letting $\xi = x$ we obtain

$$\begin{aligned}
& \left| \sum_{j=1}^4 \int_{t+\epsilon}^T \mathbf{1}_{\mathcal{S}^c} \int_{\mathbb{R}^{2d}} F_i^j(s, y_1, y_2, \theta_{t,s}^1(\xi), \theta_{t,s}^2(\xi)) (D_{x_2} \tilde{q})(t, x_1, x_2; s, y_1, y_2) dy_1 dy_2 ds \right| \\
& \leq C \int_{t+\epsilon}^T (s-t)^{-\gamma/2} \left\{ (s-t)^{-3(1-\beta_1^2)/2} + \|D_{x_1}^2 u_i\|_\infty + \|D_{x_1} D_{x_2} u_i\|_\infty + \|D_{x_1} u_i\|_\infty (s-t)^{-3(1-\beta_1^2)/2} \right. \\
& \quad + \|D_{x_1} D_{x_2} u_i\|_\infty (s-t)^{-\beta_1^1/2} + \|D_{x_2} u_i\|_\infty (s-t)^{3(\beta_2^2-1)/2} + M(D_{x_2} u_i, T) (s-t)^{-1+\alpha^1/2} \\
& \quad \left. \times \left((s-t)^{\gamma/2} + (s-t)^{3\beta_1^2/2} + (s-t)^{3\beta_2^2/2} + (s-t)^{3/2} \right) \right\} ds |x_2 - z_2|^{\gamma/3},
\end{aligned} \tag{4.26}$$

where we used (4.25) and where we recall that the quantity $M(D_{x_2} u_i, T)$ is defined by (4.1).

On a second hand, we have to deal with the term (4.24). Since we took the same freezing point for the two solutions with different initial conditions, we have to re-center carefully each integrand $F_i^j(t + \epsilon; t, x_1, z_2, \mathcal{S}^c)$, $j = 1, \dots, 4$, in order to use the Gaussian off-diagonal decay in \tilde{q} for smoothing the time singularity. Indeed, in this case, for all s in $[t, T]$ and y_2 in \mathbb{R}^d , this off-diagonal decay is

$|y_2 - m_{t,s}^{2,\xi}(x_1, z_2)|$. Hence, we have to re-center these terms around $(m_{t,s}^{2,\xi}(x_1, z_2))_{t < s \leq T}$ with the help of Claim 3.3. Let

$$\begin{aligned} & \int_{t+\epsilon}^T \mathbf{1}_{\mathcal{S}^c} \int_{\mathbb{R}^{2d}} F_i^j(s, y_1, y_2, \theta_{t,s}^1(\xi), \theta_{t,s}^2(\xi)) (D_{x_2} \tilde{q})(t, x_1, z_2; s, y_1, y_2) dy_1 dy_2 ds \\ & := \tilde{H}_i^j(t + \epsilon; t, x_1, z_2, \mathcal{S}^c), \end{aligned} \quad (4.27)$$

for $j = 1, \dots, 4$, and note that when integrating w.r.t. the derivative of the degenerate kernel $D_{x_2} \tilde{q}$ Lemma 3.5 tells us that these integrands can also be identified with the integrands in (3.14). Below, we re-center each integrand of $\tilde{H}_i^j(t + \epsilon; t, x_1, z_2, \mathcal{S}^c)$, $j = 1, \dots, 4$ and we estimate it.

Bound of $\tilde{H}_i^1(t + \epsilon; t, x_1, z_2, \mathcal{S}^c)$. We deduce from Claim 3.3, estimate on $D_{x_2} \tilde{q}$ from Proposition 3.1, regularity of ϕ_i under **(H)** and (4.25) that:

$$\left| \tilde{H}_i^1(t + \epsilon; t, x_1, z_2, \mathcal{S}^c) \right| \leq C \int_{t+\epsilon}^T \mathbf{1}_{\mathcal{S}^c} (s-t)^{-3(1-\beta_i^2+\gamma/3)/2} ds |x_2 - z_2|^{\gamma/3},$$

for all $\gamma < 3\beta_i^2 - 1$.

Bound of $\tilde{H}_i^2(t + \epsilon; t, x_1, z_2, \mathcal{S}^c)$. We proceed as in the proof of (3.17) in Lemma 3.5: by using the decomposition $\Delta = \Delta^1 + \Delta^2$, by integrating by parts and then by applying Claim 3.3 we obtain that

$$\begin{aligned} & \tilde{H}_i^2(t + \epsilon; t, x_1, z_2, \mathcal{S}^c) \\ & = \frac{1}{2} \int_{t+\epsilon}^T \mathbf{1}_{\mathcal{S}^c} \int_{\mathbb{R}^{2d}} \left\{ \text{Tr} \left[\left[\Delta^2(m_{t,s}^{2,\xi}(x_1, z_2)) a(s, y_1, y_2) \right] D_{y_1}^2 u_i(s, y_1, y_2) \right] \right. \\ & \quad \left. \times [(D_{x_2} \tilde{q})(t, x_1, z_2; s, y_1, y_2)] \right\} dy_1 dy_2 ds \\ & + \frac{1}{2} \sum_{l=1}^d \int_{t+\epsilon}^T \mathbf{1}_{\mathcal{S}^c} \int_{\mathbb{R}^{2d}} \left\{ \left(\left[\frac{\partial}{\partial y_{1l}} a_l(s, y_1, m_{t,s}^{2,\xi}(x_1, z_2)) \right] \right. \right. \\ & \quad \left. \left. \cdot \left[\Delta^2(m_{t,s}^{2,\xi}(x_1, z_2)) D_{y_1} u_i(s, y_1, y_2) \right] \right) [(D_{x_2} \tilde{q})(t, x_1, z_2; s, y_1, y_2)] \right\} dy_1 dy_2 ds \\ & + \frac{1}{2} \sum_{l=1}^d \int_{t+\epsilon}^T \mathbf{1}_{\mathcal{S}^c} \int_{\mathbb{R}^{2d}} \left\{ \left[a_l(s, y_1, m_{t,s}^{2,\xi}(x_1, z_2)) - a_l(s, \theta_{t,s}^1(\xi), \theta_{t,s}^2(\xi)) \right] \right. \\ & \quad \left. \cdot \left[\Delta^2(m_{t,s}^{2,\xi}(x_1, z_2)) D_{y_1} u_i(s, y_1, y_2) \right] \times \left[\frac{\partial}{\partial y_{1l}} (D_{x_2} \tilde{q})(t, x_1, z_2; s, y_1, y_2) \right] \right\} dy_1 dy_2 ds. \end{aligned}$$

Therefore, from the estimates on \tilde{q} of Proposition 3.1, by using MVT (4.9), the regularity of a from **(H)** and estimate (4.25) we obtain that

$$\begin{aligned} \left| \tilde{H}_i^2(t + \epsilon; t, x_1, z_2, \mathcal{S}^c) \right| & \leq C' \int_{t+\epsilon}^T \mathbf{1}_{\mathcal{S}^c} \left\{ \left(\|D_{x_1} u_i\|_{\infty} (s-t)^{-\gamma/2} + \|D_{x_1} D_{x_2} u_i\|_{\infty} (s-t)^{-\gamma/2} \right. \right. \\ & \quad \left. \left. + (s-t)^{-\gamma/2} \|D_{x_1} D_{x_2} u_i\|_{\infty} \right) |x_2 - z_2|^{\gamma/3} + \|D_{x_1} D_{x_2} u_i\|_{\infty} \right. \\ & \quad \left. \times (s-t)^{-1/2} |m_{t,s}^{2,\xi}(x_1, z_2) - \theta_{t,s}^2(\xi)| \right\} ds, \end{aligned} \quad (4.28)$$

where we used the off-diagonal decay of \hat{q}_c and we next integrated in space. This holds for all $\gamma < 2$.

Bound of $\tilde{H}_i^3(t + \epsilon; t, x_1, z_2, \mathcal{S}^c)$. By using Claim 3.3 and representation (3.18), this term can be centered as follows

$$\begin{aligned} \tilde{H}_i^3(t + \epsilon; t, x_1, z_2, \mathcal{S}^c) &= \int_{t+\epsilon}^T \mathbf{1}_{\mathcal{S}^c} \int_{\mathbb{R}^{2d}} \left[\Delta^2(m_{t,s}^{2,\xi}(x_1, z_2)) F_1(s, y_1, y_2) \right] \\ &\quad \cdot [D_{x_1} u_i(s, y_1, y_2)] (D_{x_2} \tilde{q})(t, x_1, z_2; s, y_1, y_2) dy_1 dy_2 ds \\ &+ \int_{t+\epsilon}^T \mathbf{1}_{\mathcal{S}^c} \int_{\mathbb{R}^{2d}} \left[F_1(s, y_1, m_{t,s}^{2,\xi}(x_1, z_2)) - F_1(s, \theta_{t,s}^1(\xi), \theta_{t,s}^2(\xi)) \right] \\ &\quad \cdot \left[\Delta^2(m_{t,s}^{2,\xi}(x_1, z_2) D_{x_1} u_i(s, y_1, y_2)) \right] (D_{x_2} \tilde{q})(t, x_1, z_2; s, y_1, y_2) dy_1 dy_2 ds. \end{aligned}$$

Now, thanks to the regularity of F_1 assumed in **(H)**, if we apply the estimate on $D_{x_2} \tilde{q}$ from Proposition 3.1, MVT (4.9) and estimate (4.25), we obtain that

$$\begin{aligned} &\left| \tilde{H}_i^3(t + \epsilon; t, x_1, z_2, \mathcal{S}^c) \right| \tag{4.29} \\ &\leq C'' \int_{t+\epsilon}^T \mathbf{1}_{\mathcal{S}^c} \left\{ \left(\|D_{x_1} u_i\|_{\infty} (s-t)^{-3(1-\beta_1^2+\gamma/3)/2} + \|D_{x_1} D_{x_2} u_i\|_{\infty} (s-t)^{(\beta_1^2-\gamma)/2} \right) \right. \\ &\quad \left. \times |x_2 - z_2|^{\gamma/3} + \|D_{x_1} D_{x_2} u_i\|_{\infty} |m_{t,s}^{2,\xi}(x_1, z_2) - \theta_{t,s}^2(\xi)|^{\beta_1^2} \right\} ds \end{aligned}$$

for all $\gamma < 3\beta_1^2 - 1$.

Bound of $\tilde{H}_i^4(t + \epsilon; t, x_1, z_2, \mathcal{S}^c)$. From representation (3.14), by using the decomposition $\Delta = \Delta^1 + \Delta^2$ and then by centering w.r.t. $D_{x_2} u_i$ thanks to Claim 3.3 we can write

$$\begin{aligned} &\tilde{H}_i^4(t + \epsilon; t, x_1, z_2, \mathcal{S}^c) \\ &= \int_{t+\epsilon}^T \int_{\mathbb{R}^{2d}} \left\{ \mathbf{1}_{\mathcal{S}^c} \left[\Delta^2(m_{t,s}^{2,\xi}(x_1, z_2)) F_2(s, y_1, y_2) \right] \cdot [D_{x_2} u_i(s, y_1, y_2)] \right. \\ &\quad \left. \times (D_{x_2} \tilde{q})(t, x_1, z_2; s, y_1, y_2) \right\} dy_1 dy_2 ds \\ &+ \int_{t+\epsilon}^T \int_{\mathbb{R}^{2d}} \left\{ \mathbf{1}_{\mathcal{S}^c} \left[F_2(s, y_1, m_{t,s}^{2,\xi}(x_1, z_2)) - F_2(s, \theta_{t,s}^1(\xi), \theta_{t,s}^2(\xi)) \right. \right. \\ &\quad \left. \left. - D_{x_1} F_2(s, \theta_{t,s}^1(\xi), \theta_{t,s}^2(\xi)) \Delta^1(\theta_{t,s}(\xi)) y_1 \right] \cdot \left[\Delta^2(m_{t,s}^{2,\xi}(x_1, z_2)) D_{x_2} u_i(s, y_1, y_2) \right] \right\} \\ &\quad \times (D_{x_2} \tilde{q})(t, x_1, z_2; s, y_1, y_2) dy_1 dy_2 ds. \end{aligned}$$

By using the regularity of the coefficients from **(H)**, estimate on $D_{x_2} \tilde{q}$ from Proposition 3.1 and (4.25) we have:

$$\begin{aligned} &\left| \tilde{H}_i^4(t + \epsilon; t, x_1, z_2, \mathcal{S}^c) \right| \tag{4.30} \\ &\leq C''' \int_{t+\epsilon}^T \left\{ \mathbf{1}_{\mathcal{S}^c} \|D_{x_2} u_i\|_{\infty} (s-t)^{-3(1-\beta_2^2+\gamma/3)/2} |x_2 - z_2|^{\gamma/3} + M(D_{x_2} u_i, T) \right. \\ &\quad \left. \times (s-t)^{-1+\alpha^1/2} \left(|x_2 - z_2|^{\gamma/3} + |x_2 - z_2|^{\beta_1^2} + |x_2 - z_2|^{\beta_2^2} + |x_2 - z_2| \right) \right\} ds, \end{aligned}$$

for all $\gamma < 3\beta_1^2/2 - 1$.

Now, note that from the definition (3.3) of m ,

$$m_{t,s}^{2,x}(x_1, z_2) - \theta_{t,s}^2(x) = z_2 - x_2.$$

Hence, by letting $\xi = x$ in (4.28), (4.28), (4.29) and (4.30) and combining the resulting estimates with (4.26), we deduce that there exist a positive constant C'''' and a positive number $\delta_{4.2}^2$, depending only on known parameters in **(H)**, such that:

$$\begin{aligned} |P_i(t + \epsilon, x; z, \mathcal{S}^c)| &\leq C'''' T^{\delta_{4.2}^2} (M(D_{x_2} u_i, T) + \|D_{x_2} u_i\|_\infty + \|D_{x_1} D_{x_2} u_i\|_\infty) \\ &\quad \times \left(|x_2 - z_2|^{\gamma/3} + |x_2 - z_2|^{\beta_2^2} + |x_2 - z_2|^{\beta_1^2} + |x_2 - z_2| \right), \end{aligned} \quad (4.31)$$

for all $\gamma < 3 \inf(\beta_1^2, \beta_2^2) - 1$.

“Hölder estimate” on $D_{x_2} u_i$. Finally, by plugging estimates (4.22) and (4.31) in (4.18) and by letting ϵ tends to 0, we deduce that there exist a positive constant C and a positive number $\delta_{4.2}$ such that:

$$\begin{aligned} |D_{x_2} u_i(t, x_1, x_2) - D_{x_2} u_i(t, x_1, z_2)| &\leq CT^{\delta_{4.2}} (\|D_{x_2} u_i\|_\infty + \|D_{x_1} D_{x_2} u_i\|_\infty + M(D_{x_2} u_i, T)) \\ &\quad \times \left(|x_2 - z_2|^{\gamma/3} + |x_2 - z_2|^{\beta_2^2} + |x_2 - z_2|^{\beta_1^2} + |x_2 - z_2| \right). \end{aligned}$$

A circular argument concludes the proof of Lemma 4.2. \square

Proof of Claim 4.4. By using MVT and the Gaussian estimate of $D_{x_2}^2 \tilde{q}$ from Proposition 3.1 we have:

$$\begin{aligned} &|D_{x_2} \tilde{q}(t, x_1, x_2; s, y_1, y_2) - D_{x_2} \tilde{q}(t, x_1, z_2; s, y_1, y_2)| \\ &\leq \sup_{\rho \in (0,1)} |D_{x_2}^2 \tilde{q}(t, x_1, x_2 + \rho(x_2 - z_2); s, y_1, y_2)| |x_2 - z_2| \\ &\leq C'(s-t)^{-3} \sup_{\rho \in (0,1)} \hat{q}_{\bar{c}}(t, x_1, x_2 + \rho(x_2 - z_2); s, y_1, y_2) |x_2 - z_2|, \end{aligned} \quad (4.32)$$

where \bar{c} is a positive constant depending only on known parameters in **(H)**. Note that on \mathcal{S} :

$$\sup_{\rho \in (0,1)} \hat{q}_{\bar{c}}(t, x_1, x_2 + \rho(x_2 - z_2); s, y_1, y_2) \leq C'' \hat{q}_c(t, x_1, x_2; s, y_1, y_2). \quad (4.33)$$

Combining (4.32) and (4.33), we obtain:

$$\begin{aligned} &|D_{x_2} \tilde{q}(t, x_1, x_2; s, y_1, y_2) - D_{x_2} \tilde{q}(t, x_1, z_2; s, y_1, y_2)| \\ &\leq C'''' (s-t)^{-3} \hat{q}_c(t, x_1, x_2; s, y_1, y_2) |x_2 - z_2|. \end{aligned}$$

Rewrite $|x_2 - z_2| = |x_2 - z_2|^{1-\gamma/3} |x_2 - z_2|^{\gamma/3}$. Since $|x_2 - z_2| < (s-t)^{3/2}$ we have $|x_2 - z_2| < (s-t)^{3/2-\gamma/2} |x_2 - z_2|^{\gamma/3}$ and (4.21) follows. \square

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