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STRONG EXISTENCE AND UNIQUENESS FOR STOCHASTIC DIFFERENTIAL EQUATION WITH HÖLDER DRIFT AND DEGENERATE NOISE

P.E. CHAUDRU DE RAYNAL

ABSTRACT. In this paper, we prove pathwise uniqueness for stochastic degenerate systems with a Hölder drift, for a Hölder exponent larger than the critical value $2/3$. This work extends to the degenerate setting the earlier results obtained by Zvonkin [Zvo74], Veretennikov [Ver80], Krylov and Röckner [KR05] from non-degenerate to degenerate cases. The existence of a threshold for the Hölder exponent in the degenerate case may be understood as the price to pay to balance the degeneracy of the noise. Our proof relies on regularization properties of the associated PDE, which is degenerate in the current framework and is based on a parametrix method.

1 Introduction

Let T belongs to $\mathbb{R}^{+,*}$ and d to \mathbb{N}^* , we consider the following $\mathbb{R}^d \times \mathbb{R}^d$ system for any t in $[0, T]$:

$$\begin{cases} dX_t^1 = F_1(t, X_t^1, X_t^2)dt + \sigma(t, X_t^1, X_t^2)dW_t, & X_0^1 = x_1, \\ dX_t^2 = F_2(t, X_t^1, X_t^2)dt, & X_0^2 = x_2, \end{cases} \quad (1.1)$$

where $(W_t, t \geq 0)$ is a standard d -dimensional Brownian motion defined on some filtered probability space $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t)_{t \geq 0})$ and $F_1, F_2, \sigma : [0, T] \times \mathbb{R}^d \times \mathbb{R}^d \rightarrow \mathbb{R}^d \times \mathbb{R}^d \times \mathcal{M}_d(\mathbb{R})$ (the set of real $d \times d$ matrices) are measurable functions. The diffusion matrix $a := \sigma\sigma^*$ is supposed to be uniformly elliptic. The notation “ $*$ ” stands for the transpose.

In this paper, we investigate the well posedness of (1.1) outside the Cauchy-Lipschitz framework. Notably, we are interested in the strong posedness, *i.e.* strong existence and uniqueness of a solution. Strong existence means that there exists a process $(X_t^1, X_t^2, 0 \leq t \leq T)$ adapted to the filtration generated by the Brownian motion $(W_t, 0 \leq t \leq T)$ which satisfies (1.1). Strong uniqueness means that, if two processes satisfy this equation with the same initial conditions, their trajectories are almost surely indistinguishable. Here, we show that under a suitable Hölder assumption on the drift coefficients and Lipschitz condition on the diffusion matrix, the strong well-posedness holds for (1.1).

It may be a real challenge to prove the existence of a unique solution for a differential system without Lipschitz conditions on the coefficients. For example, in [DL89], DiPerna and Lions showed that under integrability conditions on $b, \nabla b$ and $\operatorname{div}(b)$, the integral equation: $Y_t = \int_0^t b(s, Y_s)ds, Y_0 = y$ admits a unique solution defined as a regular Lagrangian flow (see [DL89] for the definition of such a solution).

In a stochastic case, the first result in this direction is due to Zvonkin. In [Zvo74], the author showed that the strong well-posedness holds for the one-dimensional system

$$Y_t = \int_0^t b(s, Y_s)ds + W_t, Y_0 = y \quad t \in [0, T], \quad (1.2)$$

for a measurable function b in \mathbb{L}^∞ . Then, Veretennikov [Ver80] generalized this result to the multi-dimensional case and Krylov and Röckner showed in [KR05] the strong well-posedness for b in

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$\mathbb{L}_{\text{loc}}^p$, $p > d$. There are some extensions of these works and we refer the reader to the paper of Zhang [Zha05] and references therein for a summary of the results. Finally, when b is a measurable and bounded function, Davie showed in [Dav07] that for almost every Brownian path, there exists a unique solution for the system (1.2). We emphasize that this result implies the strong uniqueness, but the converse is not true. Indeed, in such a case, there exists an a priori set $\Omega' \subset \Omega$ with $\mathbb{P}(\Omega') = 1$ such that for all ω in Ω' the solution of (1.2) is unique.

All these results rely on the regularization of differential systems by adding a non-degenerate noise, and we refer to [Fla11] for a partial revue on this subject. In (1.1), the noise added is completely degenerate w.r.t the degenerate component X^2 . This sort of system has also been studied by Veretenikov. When the drift is measurable and bounded and the diffusion matrix is Lipschitz w.r.t the non-degenerate component X^1 , besides if they are two times continuously differentiable functions with bounded derivatives w.r.t the degenerate component, the author showed in [Ver83] that the strong well-posedness holds. Therefore, he did not consider the regularization in the degenerate direction.

Their proofs rely on the deep connection between SDEs and PDEs (see [Bas98] or [Fri06] for a partial revue in the elliptic and parabolic cases). The generator associated to the Markov process Y is a linear partial differential operator of second order (usually denoted by \mathcal{L}) with the transition density of Y as fundamental solution. As explained by Fedrizzi and Flandoli in [FF11]: “if we have a good theory for the PDE:

$$\frac{\partial}{\partial t}u + \mathcal{L}u = \Phi, \text{ on } [0, T] \quad u_T = \mathbf{0}, \quad (1.3)$$

where the source term Φ has the same regularity as the drift, then, we have the main tools to prove strong uniqueness”.

In this paper, we show that the noise regularizes, even in the degenerate direction, by means of the random drift. Unfortunately, there is a price to pay to balance the degeneracy of the noise. First, the drift must be at least $2/3$ -Hölder continuous w.r.t the degenerate component. We do not know how sharp is this critical value, but it is consistent with our approach. Secondly, the drift F_2 of the second component must be Lipschitz continuous w.r.t the first component and its derivative in this direction has to be uniformly non degenerate: this allows the drift to regularize.

Our proof also relies on regularization properties of the associated PDE, and the “good theory” is here a “strong theory”: a Lipschitz bound on the solution of (1.3) and on its derivative w.r.t the first component. In our case, the generator \mathcal{L} is given by: for all ψ in $C^{1,2,1}([0, T] \times \mathbb{R}^d \times \mathbb{R}^d, \mathbb{R}^d)$:

$$\begin{aligned} \mathcal{L}\psi(t, x_1, x_2) &= \frac{1}{2}Tr(a(t, x_1, x_2))D_{x_1}^2\psi(t, x_1, x_2) + [F_1(t, x_1, x_2)] \cdot [D_{x_1}\psi(t, x_1, x_2)] \\ &\quad + [F_2(t, x_1, x_2)] \cdot [D_{x_2}\psi(t, x_1, x_2)]. \end{aligned} \quad (1.4)$$

where $Tr(a)$ stands for the trace of the matrix a and “ \cdot ” denotes the standard Euclidean inner product on \mathbb{R}^d and where for any z in \mathbb{R}^d , the notation D_z means the derivative w.r.t the variable z . Here, the operator is non uniformly parabolic. When the coefficients are smooth and the Lie algebra generated by the vector fields spans the whole space such an operator admits a smooth fundamental solution (see [Hö67]): it is said to be hypoelliptic. The assumption on the uniform non-degeneracy of the derivative of the drift F_2 w.r.t x_1 can be understood as a sort of weak Hörmander condition.

In our case, the form of the degeneracy is a non-linear generalization of Kolmogorov’s degeneracy, in reference to the first work [Kol34] of Kolmogorov in this direction. Degenerate operators of this form have been studied by many authors see e.g. the work of Di Francesco and Polidoro [DFP06], and Delarue and Menozzi [DM10]. We also emphasize that, in [Men11], Menozzi deduced from the

regularization property exhibited in ([DM10]) the well weak posedness of a generalization of (1.1). Nevertheless, to the best of our knowledge, there does not exist a strong theory, in the sense defined above, for the PDE (1.3) when \mathcal{L} is defined by (1.4). We investigate it by using the so called parametrix approach (see [Fri64] for partial revue in the elliptic setting).

In comparison with the works of Veretennikov [Ver80, Ver83], Krylov and Röckner [KR05], and Flandoli and Fedrizzi [FF11], asking for F_1 to be only in \mathbb{L}^p , $p > d$ might appear as the right framework. Since the parametrix is a perturbation method and we are interested in \mathbb{L}^∞ estimates, we suppose the drift F_1 to be Hölder continuous w.r.t x_1 .

1.1 Organization of this paper

Subsection 1.2 states useful notations, detailed assumptions and the main result of this paper: strong existence and uniqueness for (1.1). In Subsection 1.3, we provide the strategy to prove this result, which includes the regularization properties of the associated PDE. Finally, our main result is proved in Subsection 1.4. The remainder of this paper is dedicated to the proof of the regularization properties of the associated PDE.

The strategy is exposed in Subsection 1.5: it is based on a smooth approximation of the coefficients and the parametrix. Existence and uniqueness of a smooth solution for the PDE with smooth coefficients is given in Subsection 1.6.

Section 2 explains the proof of the regularization properties in a simple case and allows to understand our assumptions and how the proof in the general case can be achieved. Section 3 defines the mathematical tools and the proof of the regularization properties of the PDE is provided in Section 4. This is the technical part of this paper.

1.2 Main Result

Notations. In order to simplify the notations, we adopt the following convention: x, y, z, ξ , etc.. denote the $2d$ -dimensional real variables $(x_1, x_2), (y_1, y_2), (z_1, z_2), (\xi_1, \xi_2)$, etc.. Consequently, each component of the d -dimensional variables x_k , $k = 1, 2$ is denoted by x_{kl} , $l = 1, \dots, d$. We denote by $g(t, X_t)$ any function $g(t, X_t^1, X_t^2)$ from $[0, T] \times \mathbb{R}^d \times \mathbb{R}^d$ to \mathbb{R}^N , $N \in \mathbb{N}$. Here, $X_t = (X_t^1, X_t^2)$ and then $F(t, X_t)$ is the \mathbb{R}^{2d} valued function $(F_1(t, X_t^1, X_t^2), F_2(t, X_t^1, X_t^2))^*$. We rewrite the system (1.1) in a shortened form:

$$dX_t = F(t, X_t)dt + B\sigma(t, X_t)dW_t, \quad (1.5)$$

where B is the $2d \times d$ matrix: $B = (\text{Id}, 0_{\mathbb{R}^d \times \mathbb{R}^d})^*$. “Id” stands for the identity matrix of $\mathcal{M}_d(\mathbb{R})$, the set of real $d \times d$ matrices. When it is necessary, we write $(X_s^{t,x})_{t \leq s \leq T}$ the process in (1.1) which starts from x at time t , i.e $X_t^{t,x} = x$.

We denote by $\text{GL}_d(\mathbb{R})$ the set of $d \times d$ invertible matrices with real coefficients and by ϕ a measurable function from $[0, T] \times \mathbb{R}^d \times \mathbb{R}^d$ to \mathbb{R}^2 . Each one-dimensional component of this function is denoted by ϕ_i , $i = 1, 2$ and plays the role of one coordinate of F_i . Hence, ϕ_i satisfies the same regularity as F_i given latter. We recall that a denotes the square of the diffusion matrix σ : $a := \sigma\sigma^*$. Subsequently, we denote by c, C, C', C'' or C''' a positive constant, depending only on known parameters in **(H)**, given just below, and may change from line to line and from an equation to another.

We recall that the canonical Euclidean inner product on \mathbb{R}^d is denoted by “.” and the notation D_z means the derivative w.r.t the variable z . Hence, for all integer n , D_z^n is the n^{th} derivative w.r.t z and for all integer m the $n \times m$ cross differentiations w.r.t z, y are denoted by $D_z^n D_y^m$. Furthermore, the partial derivative ∂/∂_t is denoted by ∂_t .

Hypotheses. *(H).* We say that assumptions **(H)** hold if the coefficients satisfy:

(H1): *Regularity of the coefficients.* There exist $0 < \beta_i^j < 1$, $1 \leq i, j \leq 2$ and three positive constants C_1, C_2, C_σ such that for all (t, x_1, x_2) and (t, y_1, y_2) in $[0, T] \times \mathbb{R}^d \times \mathbb{R}^d$,

$$\begin{aligned} |F_1(t, x_1, x_2) - F_1(t, y_1, y_2)| &\leq C_1(|x_1 - y_1|^{\beta_1^1} + |x_2 - y_2|^{\beta_1^2}) \\ |F_2(t, x_1, x_2) - F_2(t, y_1, y_2)| &\leq C_2(|x_1 - y_1| + |x_2 - y_2|^{\beta_2^2}) \\ |\sigma(t, x_1, x_2) - \sigma(t, y_1, y_2)| &\leq C_\sigma(|x_1 - y_1| + |x_2 - y_2|). \end{aligned}$$

Moreover, the coefficients are supposed to be continuous w.r.t the time and $\beta_i^2 > 2/3$, $i = 1, 2$. Thereafter, we set $\beta_2^1 = 1$ for notational convenience.

(H2): *Uniform ellipticity of $\sigma\sigma^*$.* The function $\sigma\sigma^*$ also satisfies the uniform ellipticity hypothesis:

$$\exists \Lambda > 1 \text{ such that } \forall \zeta \in \mathbb{R}^{2d}, \quad \Lambda^{-1}|\zeta|^2 \leq [\sigma\sigma^*(t, x_1, x_2)\zeta] \cdot \zeta \leq \Lambda|\zeta|^2,$$

for all $(t, x_1, x_2) \in [0, T] \times \mathbb{R}^d \times \mathbb{R}^d$.

(H3-a): *Differentiability and regularity of $x_1 \mapsto F_2(\cdot, x_1, \cdot)$.* For all $(t, x_2) \in [0, T] \times \mathbb{R}^d$, the function $F_2(t, \cdot, x_2) : x_1 \mapsto F_2(t, x_1, x_2)$ is continuously differentiable and there exist $0 < \alpha^1 < 1$ and a positive constant \bar{C}_2 such that, for all (t, x_2) in $[0, T] \times \mathbb{R}^d$ and x_1, y_1 in \mathbb{R}^d ,

$$|D_{x_1}F_2(t, x_1, x_2) - D_{x_1}F_2(t, y_1, x_2)| \leq \bar{C}_2|x_1 - y_1|^{\alpha^1}.$$

(H3-b): *Non degeneracy of $(D_{x_1}F_2)(D_{x_1}F_2)^*$.* There exists a closed convex subset $\mathcal{E} \subset \text{GL}_d(\mathbb{R})$ such that for all t in $[0, T]$ and (x_1, x_2) in \mathbb{R}^{2d} the matrix $D_{x_1}F_2(t, x_1, x_2)$ belongs to \mathcal{E} . We emphasize that this implies that

$$\exists \bar{\Lambda} > 1 \text{ such that } \forall \zeta \in \mathbb{R}^{2d}, \quad \bar{\Lambda}^{-1}|\zeta|^2 \leq [(D_{x_1}F_2)(D_{x_1}F_2)^*(t, x_1, x_2)\zeta] \cdot \zeta \leq \bar{\Lambda}|\zeta|^2,$$

for all $(t, x_1, x_2) \in [0, T] \times \mathbb{R}^d \times \mathbb{R}^d$.

Remark. Consequently, the sentence “known parameters in **(H)**” refers to the parameters belonging to these assumptions.

The reason for the existence of the critical value $2/3$ for the Hölder regularity of the drift in **(H1)** and the particular “convexity” assumption **(H3-b)** are discussed in Section 2.

The following Theorem is the main result of this paper and regards the strong well-posedness of the system (1.1).

Theorem 1.1. *Let $T > 0$ and suppose that assumptions **(H)** hold true. Then, strong existence and uniqueness hold for (1.1).*

1.3 Strategy of proof

Let us expose the basic arguments to prove Theorem 1.1. Existence of a weak solution is straightforward and follows from the regularity of the coefficients, see e.g. [SV79]. Then, if the strong uniqueness holds, the strong existence follows. The main issue consists in proving the strong uniqueness. As we already mentioned, the strategy relies on regularization properties of the linear system of PDEs:

$$\begin{cases} \partial_t u_i(t, x) + \mathcal{L}u_i(t, x) = F_i(t, x), & \text{for } (t, x) \in [0, T] \times \mathbb{R}^{2d}, \\ u_i(T, x) = 0_{\mathbb{R}^d}, & i = 1, 2. \end{cases} \quad (1.6)$$

This works as follows: suppose that there exists a unique $C_b^{1,2,1}([0, T] \times \mathbb{R}^d \times \mathbb{R}^d, \mathbb{R}^d)$ solution $u = (u^1, u^2)^*$ of this system such that u and $D_x u$ are C_T Lipschitz continuous, where C_T is small as T is small. Thanks to Itô's formula, for all t in $[0, T]$:

$$\int_0^t F(s, X_s) ds = u(t, X_t) + \int_0^t D_x u(s, X_s) B \sigma(s, X_s) dB_s.$$

Then, $\int_0^t F(s, X_s) ds$ is Lipschitz continuous w.r.t X , with Lipschitz constant C_T . The uniqueness can be proved for small T by a circular argument and the result can be deduced by iterating this strategy on sufficiently small intervals.

The main issue here is that the PDE (1.6) does not admit a $C_b^{1,2,1}([0, T] \times \mathbb{R}^d \times \mathbb{R}^d, \mathbb{R}^d)$ solution. Nevertheless, we do not need to obtain the existence of a regular solution but only the existence of Lipschitz bounds for u and $D_x u$ depending only on known parameters in **(H)**. Therefore, we investigate these bounds in a smooth setting. Indeed, thanks to assumptions **(H)**, there exists a sequence of mollified coefficients $(a^n, F_1^n, F_2^n)_{n \geq 0}$ with bounded derivatives of any order such that:

$$(a^n, F_1^n, F_2^n) \xrightarrow{n \rightarrow +\infty} (a, F_1, F_2), \quad (1.7)$$

uniformly on compact subsets of $[0, T] \times \mathbb{R}^d \times \mathbb{R}^d$ and such that (a^n, F_1^n, F_2^n) satisfy **(H)** uniformly (in n). Let us denote by \mathcal{L}^n the regularized version of \mathcal{L} (that is the version of \mathcal{L} with mollified coefficients), one has:

Lemma 1.2. *Let n in \mathbb{N} . The PDE,*

$$\partial_t u_i^n(t, x) + \mathcal{L}^n u_i^n(t, x) = F_i^n(t, x), \quad \text{for } (t, x) \in [0, T] \times \mathbb{R}^{2d} \quad u_i^n(T, x) = 0_{\mathbb{R}^d}, \quad i = 1, 2,$$

admits a unique solution $u^n = (u_1^n, u_2^n)^$, which is infinitely differentiable with bounded derivatives of any order.*

Moreover the solutions u^n , $n \geq 0$ satisfy:

Proposition 1.3. *For T small enough, there exists a positive constant C_T depending only on T and known parameters in **(H)** and not on n such that:*

$$\|D_{x_1} u^n\|_\infty + \|D_{x_2} u^n\|_\infty + \|D_{x_1}^2 u^n\|_\infty + \|D_{x_1} D_{x_2} u^n\|_\infty \leq C_T$$

and $C_T \rightarrow 0$ when $T \rightarrow 0$.

We emphasize that the estimates on the solutions u^n , $n \geq 0$ are obtained uniformly in n (that is independently of the procedure of regularization) and we do not have to solve the limit PDE problem. Besides, the terminal condition $u^n(T, \cdot) = 0$ is very important: it guarantees that the solution and its derivatives vanish at time T . Hence, it allows to control the Lipschitz constant of u^n by a constant small as T is small.

1.4 Proof of Theorem 1.1

As we discussed, we only have to prove the strong uniqueness, since the weak existence holds. Thereafter, we denote by “**1**” the $2d \times 2d$ matrix:

$$\begin{pmatrix} \text{Id} & 0_{\mathbb{R}^d \times \mathbb{R}^d} \\ 0_{\mathbb{R}^d \times \mathbb{R}^d} & 0_{\mathbb{R}^d \times \mathbb{R}^d} \end{pmatrix}.$$

Let $(X_t, t \geq 0)$ and $(Y_t, t \geq 0)$ be two solutions of (1.1) with the same initial condition x in \mathbb{R}^{2d} . Let u^n be the solution of the linear system of PDEs (1.6). By using Lemma 1.2 we can apply Itô's formula for $u^n(t, X_t) - X_t$ and we obtain:

$$\begin{aligned} u^n(t, X_t) - X_t &= \int_0^t [\partial_t u^n + \mathcal{L}u^n](s, X_s) ds - \int_0^t F(s, X_s) ds + u^n(0, x) - x \\ &\quad + \int_0^t [D_x u^n - \mathbf{1}] B\sigma(s, X_s) dW_s. \end{aligned}$$

In order to use the fact that $\partial_t u^n + \mathcal{L}^n u^n = F^n$, we rewrite:

$$\begin{aligned} u^n(t, X_t) - X_t &= \int_0^t [\partial_t u^n + \mathcal{L}^n u^n](s, X_s) ds + \int_0^t (\mathcal{L} - \mathcal{L}^n) u^n(s, X_s) ds \\ &\quad - \int_0^t F(s, X_s) ds + u^n(0, x) - x \\ &\quad + \int_0^t [D_x u^n - \mathbf{1}] B\sigma(s, X_s) dW_s, \end{aligned}$$

and then,

$$\begin{aligned} u^n(t, X_t) - X_t &= \int_0^t (\mathcal{L} - \mathcal{L}^n) u^n(s, X_s) ds + \int_0^t (F^n(s, X_s) - F(s, X_s)) ds + u^n(0, x) - x \\ &\quad + \int_0^t [D_x u^n - \mathbf{1}] B\sigma(s, X_s) dW_s, \end{aligned}$$

By the same argument, we obtain:

$$\begin{aligned} u^n(t, Y_t) - Y_t &= \int_0^t (\mathcal{L} - \mathcal{L}^n) u^n(s, Y_s) ds + \int_0^t (F^n(s, Y_s) - F(s, Y_s)) ds + u^n(0, x) - x \\ &\quad + \int_0^t [D_x u^n - \mathbf{1}] B\sigma(s, Y_s) dW_s. \end{aligned}$$

By taking the expectation of the supremum over t of the square norm of the difference of the two equalities above, we get:

$$\begin{aligned} \mathbb{E} \left[\sup_{t \in [0, T]} |X_t - Y_t|^2 \right] &\leq C \mathbb{E} \left[\sup_{t \in [0, T]} |u^n(t, X_t) - u^n(t, Y_t)|^2 \right] \\ &\quad + C \mathbb{E} \left[\int_0^T |[D_x u^n B](s, X_s) - [D_x u^n B](s, Y_s)|^2 |\sigma(s, Y_s)|^2 ds \right] \\ &\quad + C \mathbb{E} \left[\int_0^T (\|D_x u^n B\|_\infty + 1) |\sigma(s, Y_s) - \sigma(s, X_s)|^2 ds \right] \\ &\quad + C \mathcal{R}(n, T), \end{aligned}$$

where

$$\begin{aligned} \mathcal{R}(n, T) &= CT \left(\mathbb{E} \left[\sup_{t \in [0, T]} |F^{(n)}(t, Y_t) - F(t, Y_t)|^2 \right] + \mathbb{E} \left[\sup_{t \in [0, T]} |(\mathcal{L}^n - \mathcal{L})u^n(t, Y_t)|^2 \right] \right) \\ &\quad + CT \left(\mathbb{E} \left[\sup_{t \in [0, T]} |F^{(n)}(t, X_t) - F(t, X_t)|^2 \right] + \mathbb{E} \left[\sup_{t \in [0, T]} |(\mathcal{L}^n - \mathcal{L})u^n(t, X_t)|^2 \right] \right). \end{aligned}$$

First, note that from (1.7), for both Y_t and X_t , we have:

$$\mathbb{E} \left[\sup_{t \in [0, T]} |F^n(t, X_t) - F(t, X_t)|^2 \right] + \mathbb{E} \left[\sup_{t \in [0, T]} |(\mathcal{L}^n - \mathcal{L})u^n(t, X_t)|^2 \right] \rightarrow 0, \text{ as } n \rightarrow \infty,$$

so that $\mathcal{R}(n, T) \rightarrow 0$ when $n \rightarrow \infty$. Secondly, we know from Proposition 1.3, that for T small enough and for all $t \in [0, T]$, the functions $u^n(t, \cdot, \cdot)$ and $D_{x_1} u^n(t, \cdot, \cdot)$ are Lipschitz continuous, with a Lipschitz constant independent of n . Since $D_x u^n B = (D_{x_1} u^n, 0_{\mathbb{R}^d \times \mathbb{R}^d})$, by letting $n \rightarrow \infty$ and using the two arguments above, we deduce that for T small enough:

$$\mathbb{E} \left[\sup_{t \in [0, T]} |X_t - Y_t|^2 \right] \leq C(T) \left\{ \mathbb{E} \left[\sup_{t \in [0, T]} |X_t - Y_t|^2 \right] + \mathbb{E} \left[\int_0^T |X_s - Y_s|^2 ds \right] \right\},$$

where $C(T) \rightarrow 0$ when $T \rightarrow 0$. Then, the strong uniqueness holds for T small enough. By iterating this computation, the same result holds on any finite intervals and so on $[0, \infty)$. \square

1.5 Strategy of proof of Lemmas 1.2 and Proposition 1.3

First of all, each coordinate of the vectorial solution u_i of the decoupled linear PDE (1.6) can be described by the PDE

$$\partial_t u_i(t, x) + \mathcal{L}^n u_i(t, x) = \phi_i(t, x), \quad \text{for } (t, x) \in [0, T] \times \mathbb{R}^{2d}, \quad u_i(T, x) = 0, \quad (1.8)$$

where $\phi_i : \mathbb{R}^{2d} \rightarrow \mathbb{R}$ satisfies the same hypotheses as F_i given in (H). Therefore, we only have to prove Lemma 1.2 and Proposition 1.3 for (1.8) instead of (1.6).

Secondly, we do not solve the limit PDE problem (1.6). The estimates in Proposition 1.3 are obtained for mollified coefficients $a^n, F_1^n, F_2^n, \phi_1^n, \phi_2^n$, $n \geq 0$ ¹ but depend only on known parameters in (H). Consequently, we forget the superscript “ n ” which arises from the mollifying procedure, and we further assume that:

Hypotheses. (HR): We say that assumptions (HR) hold if: Assumptions (H) hold true and F_1, F_2, ϕ_1, ϕ_2 and a are infinitely differentiable bounded functions with bounded derivatives.

The existence of a smooth solution under (HR) is established in Subsection 1.6 below. Then, the estimates on this solution are obtained by using the parametrix method (see [Fri64] for a revue in the elliptic setting).

1.6 Proof of Lemma 1.2

Strategy. We show that under (HR), there exists a unique solution u of the linear system of PDEs (1.8) which is infinitely differentiable with bounded derivatives of any order. Existence and regularity are proved by adopting a viscosity solution approach, and uniqueness by using the Feynman-Kac representation. This proof is decomposed as follows:

We propose a candidate u_i as:

$$u_i(t, x_1, x_2) = \mathbb{E} \left[\int_t^T \phi_i(s, X_s^{1,t,x}, X_s^{2,t,x}) ds \right], \quad (1.9)$$

¹here, $(\phi_1^n, \phi_2^n)_{n \geq 0}$ denotes the sequence of mollified coefficients (ϕ_1, ϕ_2) : they are infinitely differentiable with bounded derivatives of any order and they satisfy the same hypotheses as the (ϕ_1, ϕ_2) uniformly in n .

where the process (X^1, X^2) satisfies (1.1).

(i) We show that u_i is a viscosity solution of the linear systems (1.8).

(ii) We show that for all t in $[0, T]$, the function $u_i(t, \cdot, \cdot) : (x_1, x_2) \in \mathbb{R}^{2d} \mapsto u_i(t, x_1, x_2)$ is infinitely differentiable with bounded derivatives, thanks to the property of differentiation of the mapping $(X^{1,\cdot}, X^{2,\cdot}) : x \in \mathbb{R}^{2d} \mapsto (X^{1,x}, X^{2,x})$.

(iii) From the Markov property, we deduce that, for all (x_1, x_2) in \mathbb{R}^{2d} , the function $u_i(\cdot, x_1, x_2) : t \in [0, T] \mapsto u_i(t, x_1, x_2)$ is infinitely differentiable with bounded derivatives.

(iv) Since a smooth viscosity solution is a classical solution, we obtain the existence and regularity of a classical solution of (1.8).

(v) We conclude the proof of uniqueness by using the Feynman-Kac representation of the solution.

Proof. Let $i \in \{1, 2\}$ and consider the function u_i defined in (1.9):

(i) Under **(HR)** we deduce from [Kun82] and Theorem 70 of [Pro04] that there exists a unique strong solution $X = (X^1, X^2)$ of the stochastic system and an a.s. continuous version of this process $(X_s^{1,t,x}, X_s^{2,t,x})_{t \leq s \leq T}$. From the regularity of ϕ_i , we deduce that u_i is continuous. By using Theorem 5.1 p 69 of [FS06], we conclude that u_i is a sub and super viscosity solution.

(ii) Thanks to [Kun82], we know that for all t in $[0, T]$, for all s in $[t, T]$, the mapping $X_s^{t,x} : x \mapsto X_s^{t,x}$ is a.s. infinitely differentiable and, for all k in \mathbb{N}^* , for all $(i_1, \dots, i_k) \in \{1, 2\}^k$ the tangent process $(D_{x_{i_1}, \dots, x_{i_k}}^k X_s^{t,x})_{t \leq s \leq T}$ satisfies:

$$\mathbb{E} \left[\sup_{s \in [t, T]} \left| D_{x_{i_1}, \dots, x_{i_k}}^k X_s^{t,x} \right| \right] \leq K.$$

Since ϕ_i is Lipschitz continuous, it satisfies the domination property:

$$|\phi_i(X_T^{t,x}) - \phi_i(X_T^{t,z})| \leq K'|x - z|,$$

where K' is a random constant with finite moments of all order according to the Kolmogorov's Theorem. Then, one can apply the Lebesgue differentiation Theorem on the right hand side of (1.9) and, for all t in $[0, T]$, we deduce that $D_{x_j} u_i(t, x_1, x_2)$, $j = 1, 2$ exist and satisfy

$$D_{x_j} u_i(t, x_1, x_2) = \mathbb{E} \left[\sum_{l=1}^2 D_{x_j} \phi_i(X_T^{t,x}) D_{x_j} X_T^{l,t,x} \right]. \quad (1.10)$$

By iterating this argument, we obtain that u_i is infinitely differentiable w.r.t the space variables (x_1, x_2) and its derivatives are bounded.

(iii) On a first hand, we know that $(X_s^{1,t,x}, X_s^{2,t,x})$ is continuous w.r.t t (see Lemma 4.6.1 of [Kun82]). So that for all x in \mathbb{R}^{2d} , the function $u_i(\cdot, x_1, x_2) : t \in [0, T] \mapsto u_i(t, x_1, x_2)$ is continuous. By using the Markov property we deduce that, for all $0 < h < t$:

$$u_i(t-h, x_1, x_2) = \mathbb{E} \left[u_i(t, X_t^{1,t-h,x}, X_t^{2,t-h,x}) \right] - \mathbb{E} \left[\int_{t-h}^t \phi_i(s, X_s^{1,t-h,x}, X_s^{2,t-h,x}) ds \right]. \quad (1.11)$$

On the other hand, by applying Itô's formula on $\mathbb{R}^d \times \mathbb{R}^d$ for $u_i(t, X_t^{1,t-h,x}, X_t^{2,t-h,x})$ and taking the expectation we have:

$$\begin{aligned} & \mathbb{E} \left[u_i(t, X_t^{1,t-h,x}, X_t^{2,t-h,x}) \right] \\ &= u_i(t, x_1, x_2) + \mathbb{E} \left[\int_{t-h}^t \mathcal{L}u_i(t, X_r^{1,t-h,x}, X_r^{2,t-h,x}) dr \right] \\ &+ \mathbb{E} \left[\int_{t-h}^t D_x u_i(t, X_r^{1,t-h,x}, X_r^{2,t-h,x}) B \sigma(r, X_r^{1,t-h,x}, X_r^{2,t-h,x}) dW_r \right], \end{aligned} \quad (1.12)$$

where the last term in the right hand side is equal to 0.

By summing (1.11) and (1.12), we obtain:

$$\begin{aligned} & \frac{u_i(t-h, x_1, x_2) - u_i(t, x_1, x_2)}{h} \\ &= \frac{1}{h} \mathbb{E} \left[\int_{t-h}^t \mathcal{L}u_i(t, X_r^{1,t-h,x}, X_r^{2,t-h,x}) - \phi_i(r, X_r^{1,t-h,x}, X_r^{2,t-h,x}) dr \right]. \end{aligned}$$

The continuity of u_i w.r.t t and letting h tends to 0 give:

$$-\frac{\partial u_i}{\partial t}(t, x_1, x_2) = \mathcal{L}u_i(t, x_1, x_2) - \phi_i(t, x_1, x_2).$$

(iv) Then, by iterating this argument and using the boundedness of the tangent process at every order we deduce that the function u_i defined in (1.9) is infinitely differentiable with bounded derivatives of any order. Since a smooth viscosity solution is a classical solution, this concludes the existence part.

(v) Let v_i be such a solution, by applying Itô's formula for $v_i(T, X_T^{1,t,x}, X_T^{2,t,x})$, where $(X^{1,t,x}, X^{2,t,x})$ is a solution of the SDE (1.1) such that $(X_t^{1,t,x}, X_t^{2,t,x}) = (x_1, x_2)$ a.s. and taking the expectation we have:

$$v_i(t, x_1, x_2) = \mathbb{E} \left[\int_t^T \phi_i(s, X_s^{1,t,x}, X_s^{2,t,x}) ds \right],$$

then, $v_i = u_i$ and the uniqueness follows. This concludes the proof of Lemma 1.2. \square

2 The linear and Brownian heuristic

This section introduces the main issue when proving Proposition 1.3 in a simple case. Furthermore, it allows to understand the role of some of the assumptions in **(H)** and to present in a simple form the effects of the degeneracy. By “simple”, we mean that the following assumptions hold:

Hypotheses. (HL) We say that hypotheses **(HL)** hold if **(H)** and **(HR)** hold with : $F_1 \equiv 0_{\mathbb{R}^d}$, $\sigma \equiv \text{Id}$, $F_2(s, x_1, x_2) = \bar{F}_2(x_2) + \Gamma_s x_1$ where Γ_s belongs to the convex subset \mathcal{E} in $\text{GL}_d(\mathbb{R})$.

The SDE (1.1) becomes:

$$\begin{cases} dX_s^1 = dW_s, & X_t^1 = x_1, \\ dX_s^2 = (\bar{F}_2(X_s^2) + \Gamma_s X_s^1) ds, & X_t^2 = x_2, \end{cases} \quad (2.1)$$

for all $t < s$ in $[0, T]$, x in \mathbb{R}^d , and under **(HL)** this system admits a unique strong solution X . We recall that the associated PDE is:

$$\begin{cases} \partial_t u_i(t, x) + \mathcal{L}u_i(t, x) = \phi_i(t, x), & \text{for } (t, x) \in [0, T] \times \mathbb{R}^{2d} \\ u_i(T, x) = 0, & i = 1, 2 \end{cases} \quad (2.2)$$

In this section, we explain the parametrix method for (2.2) and show how it permits to prove Proposition 1.3 when the assumption $\beta_2^2 > 2/3$ is satisfied.

2.1 The frozen system and parametrix

Kolmogorov's example. Let $d = 1$, in [Kol34] Kolmogorov showed that the solution of: $dY_t = \alpha W_t dt$, ($\alpha \neq 0$), admits a density. Notably, this density is Gaussian and given by:

$$p(0, x_1, x_2; t, y_1, y_2) = \frac{\sqrt{3}}{\alpha\pi t^2} \exp\left(-\frac{1}{2} \left| K_t^{-1/2} (y_1 - x_1, y_2 - x_2 - t\alpha x_1)^* \right|^2\right), \quad (2.3)$$

with the following covariance matrix K_t :

$$K_t := \begin{pmatrix} t & (1/2)\alpha t^2 \\ (1/2)\alpha t^2 & (1/3)\alpha^2 t^3 \end{pmatrix}. \quad (2.4)$$

This example illustrates the behaviour of the system in small time: it is not diffusive. The first coordinate oscillates with fluctuations of order $1/2$, while the second one oscillates with fluctuations of order $3/2$. As a direct consequence, the transport of the initial condition of the first coordinate has a key role in the second one.

The frozen system. The parametrix method consists in freezing the coefficients. As the Kolmogorov's example suggested, we have to freeze the system along the curve $\theta_{t,s} = (\theta_{t,s}^1, \theta_{t,s}^2)^*$, s in $(t, T]$ that solves the ODE:

$$\frac{d}{ds} \theta_{t,s} = (0_{\mathbb{R}^d}, \bar{F}_2(\theta_{t,s}^2(\xi)) + \Gamma_s \theta_{t,s}^1(\xi))^*, \quad \theta_{t,t}(\xi) = \xi,$$

for all ξ in \mathbb{R}^{2d} . This permits to keep track of the transport of the initial condition. This curve can be written as:

$$\theta_{t,s}^1(\xi) = \xi^1, \quad \theta_{t,s}^2(\xi) = \xi^2 + \int_t^s [\Gamma_r \xi^1 + \bar{F}_2(\theta_{t,r}^2(\xi))] dr. \quad (2.5)$$

The frozen system is:

$$\begin{cases} d\bar{X}_s^1 = dW_s, & \bar{X}_t^1 = x_1, \\ d\bar{X}_s^2 = (\bar{F}_2(\theta_{t,s}^2(\xi)) + \Gamma_s \bar{X}_s^1) ds, & \bar{X}_t^2 = x_2, \end{cases} \quad (2.6)$$

for all s in $(t, T]$. This is our candidate to approximate (2.1).

2.2 Existence of a density for the frozen system

In this case, the crucial point is the specific form of the covariance matrix $\bar{\Sigma}_{t,s}$ of \bar{X}_s , standard computations show that:

$$\bar{\Sigma}_{t,s} = \begin{pmatrix} (s-t) & \int_t^s \int_t^r \Gamma_u du dr \\ \int_t^s \int_t^r \Gamma_u du dr & \int_t^s (\int_t^r \Gamma_u du) (\int_t^r \Gamma_u du)^* dr \end{pmatrix}, \quad (2.7)$$

for all s in $(t, T]$. We emphasize that the existence of a transition density of \bar{X} results from the non-degeneracy of this matrix. In [DM10], the authors proved:

Lemma 2.1. *Suppose that assumptions (HL) hold true, then, the solution of (2.6) admits a transition density \bar{q} given by, for all s in $[t, T]$:*

$$\bar{q}(t, x_1, x_2; s, y_1, y_2) = \frac{1}{(2\pi)^{d/2}} \det(\bar{\Sigma}_{t,s})^{-1/2} \exp\left(-|\bar{\Sigma}_{t,s}^{-1/2} (y_1 - x_1, y_2 - m_{t,s}^{2,\xi}(x))^*|^2\right), \quad (2.8)$$

where

$$m_{t,s}^{2,\xi}(x) = x_2 + \int_t^s \Gamma_r x_1 dr + \int_t^s \bar{F}_2(\theta_{t,r}^2(\xi)) dr,$$

and where $\bar{\Sigma}_{t,s}$ is the uniformly non-degenerate matrix given by (2.7).

As shown in [DM10], the spectrum is controlled uniformly in $\Gamma \in \mathcal{E}$ which explains the assumption **(H3-b)**. We can show that the transition density \bar{q} satisfies:

Lemma 2.2. *Suppose that assumptions **(HL)** hold true, then, for T small enough, the transition kernel \bar{q} and its derivatives admit a Gaussian-type bound: there exists a positive constant c depending only on known parameters in **(H)** such that for all ξ in \mathbb{R}^{2d} :*

$$\begin{aligned} & \left| D_t^{N^t} D_{x_1}^{N^{x_1}} D_{x_2}^{N^{x_2}} D_{y_1}^{N^{y_1}} \bar{q}(t, x_1, x_2; s, y_1, y_2) \right| \\ & \leq (s-t)^{-[3(N^{x_2}+N^t)+N^{x_1}+N^{y_1}]/2} \frac{c}{(s-t)^{2d}} \exp \left(-c \left(\frac{|y_1 - x_1|^2}{s-t} + \frac{|y_2 - m_{t,s}^{2,\xi}(x)|^2}{(s-t)^3} \right) \right), \end{aligned} \quad (2.9)$$

for all s in $(t, T]$, any $N^t, N^{x_1}, N^{x_2}, N^{y_1}$ less than 2.

Proof. The proof is given in Section 3 for a more general framework, see Proposition 3.1. \square

This Lemma says that each differentiation of the transition kernel w.r.t. the diffusive component gives a singularity of order 1/2 while the differentiation w.r.t. the degenerate component gives a singularity of order 3/2. Note that the mean $m_{t,T}^{2,\xi}(x)$ of \bar{X}_T satisfies the ODE:

$$\frac{d}{ds} \varphi_s = \bar{F}_2(\theta_{t,s}^2(\xi)) + \Gamma_s(\varphi_s - \theta_{t,s}^2(\xi)), \quad \varphi_0 = x_2. \quad (2.10)$$

Under **(HL)**, when $\xi = x$, the forward transport function $(\theta_{t,s}^2(x))_{t \leq s \leq T}$ is also a solution of (2.10), so that: $\theta_{t,s}(x) = m_{t,s}^x(x)$ for all s in $[t, T]$ and all x in \mathbb{R}^{2d} .

2.3 Representation of the solution

By standard computations, it can be checked that the transition density (2.8) of the frozen process \bar{X} is the fundamental solution of the heat equation:

$$\partial_t \bar{q}(t, x; T, y) + \bar{\mathcal{L}} \bar{q}(t, x; T, y) = 0, \quad \bar{q}(T, x; T, y) = \delta_y(x),$$

where $\bar{\mathcal{L}} = (1/2)\Delta_{x_1} + [\bar{F}_2(\theta_{t,T}^2(\xi)) + \Gamma_T x_1] \cdot D_{x_2}$. Note that the PDE (2.2) reads:

$$\begin{aligned} \partial_t u_i(t, x) + \bar{\mathcal{L}} u_i(t, x) &= \phi_i(t, x) + (\bar{\mathcal{L}} - \mathcal{L}) u_i(t, x), \quad \text{for } (t, x) \in [0, T) \times \mathbb{R}^{2d} \\ u_i(T, x) &= 0, \quad i = 1, 2. \end{aligned}$$

So that, the unique solution of this PDE can be written as:

$$\begin{aligned} & u_i(t, x_1, x_2) \\ &= \int_t^T \int_{\mathbb{R}^{2d}} \phi_i(s, y_1, y_2) \bar{q}(t, x_1, x_2; s, y_1, y_2) dy_1 dy_2 ds \\ & \quad - \int_t^T \int_{\mathbb{R}^{2d}} [\bar{F}_2(y_2) - \bar{F}_2(\theta_{t,T}^2(\xi))] \cdot D_{x_2} u_i(s, y_1, y_2) \bar{q}(t, x_1, x_2; s, y_1, y_2) dy_1 dy_2 ds. \end{aligned}$$

2.4 A priori estimates

For sake of simplicity, we suppose throughout this subsection that $\bar{F}_2 \equiv 0$, so that

$$u_i(t, x_1, x_2) = \int_t^T \int_{\mathbb{R}^{2d}} \phi_i(s, y_1, y_2) \bar{q}(t, x_1, x_2; s, y_1, y_2) dy_1 dy_2 ds. \quad (2.11)$$

In order to prove Proposition 1.3, we need to obtain estimates of the supremum norm of the first and second order derivatives of the u_i , $i = 1, 2$. As shown in Lemma 2.2, the differentiation of the transition density gives a time-singularity, so that it is not obvious that Lebesgue differentiation Theorem can be applied in (2.11).

Set i in $\{1, 2\}$ and let us focus on the worst case in Proposition 1.3, that is, the cross derivative $D_{x_1}D_{x_2}u_i$. Having in mind to use the Lebesgue differentiation Theorem, we focus on the cross differentiation of \bar{q} , which gives a time singularity of order 2. In order to invert the integral and the differentiation operator, we split the integral: for any $\epsilon > 0$, we can write:

$$\begin{aligned} u_i(t, x_1, x_2) &= \int_{t+\epsilon}^T \int_{\mathbb{R}^{2d}} \phi_i(s, y_1, y_2) \bar{q}(t, x_1, x_2; s, y_1, y_2) dy_1 dy_2 ds \\ &\quad + \int_t^{t+\epsilon} \int_{\mathbb{R}^{2d}} \phi_i(s, y_1, y_2) \bar{q}(t, x_1, x_2; s, y_1, y_2) dy_1 dy_2 ds, \end{aligned}$$

and thanks to Lebesgue differentiation Theorem we have:

$$\begin{aligned} D_{x_1}D_{x_2}u_i(t, x_1, x_2) &= \int_{t+\epsilon}^T \int_{\mathbb{R}^{2d}} \phi_i(s, y_1, y_2) D_{x_1}D_{x_2}\bar{q}(t, x_1, x_2; s, y_1, y_2) dy_1 dy_2 ds \\ &\quad + D_{x_1}D_{x_2} \left[\int_t^{t+\epsilon} \int_{\mathbb{R}^{2d}} \phi_i(s, y_1, y_2) \bar{q}(t, x_1, x_2; s, y_1, y_2) dy_1 dy_2 ds \right]. \end{aligned} \quad (2.12)$$

On a first hand, the last term in the right hand side reads²:

$$\begin{aligned} &D_{x_1}D_{x_2} \left[\int_t^{t+\epsilon} \int_{\mathbb{R}^{2d}} \phi_i(s, y_1, y_2) \bar{q}(t, x_1, x_2; s, y_1, y_2) dy_1 dy_2 ds \right] \\ &= D_{x_1}D_{x_2} \mathbb{E} \left[\int_t^{t+\epsilon} \phi_i(s, \bar{X}_s^{1,t,x}, \bar{X}_s^{2,t,x}) ds \right]. \end{aligned}$$

It follows from the proof of Lemma 1.2 in Section 1.6 that Lebesgue differentiation Theorem applies and there exists a positive constant K such that

$$\sup_{(t,x) \in [0,T] \times \mathbb{R}^{2d}} \left| \mathbb{E} \left[\int_t^{t+\epsilon} D_{x_1}D_{x_2} \phi_i(s, \bar{X}_s^{1,t,x}, \bar{X}_s^{2,t,x}) ds \right] \right| \leq K\epsilon. \quad (2.13)$$

On the other hand, for all s in $[t, T]$ we have:

$$\begin{aligned} &\int_{\mathbb{R}^{2d}} \phi_i(s, y_1, y_2) D_{x_1}D_{x_2}\bar{q}(t, x_1, x_2; s, y_1, y_2) dy_1 dy_2 \\ &= \int_{\mathbb{R}^{2d}} (\phi_i(s, y_1, y_2) - \phi_i(s, y_1, \theta_{t,s}^2(\xi))) D_{x_1}D_{x_2}\bar{q}(t, x_1, x_2; s, y_1, y_2) dy_1 dy_2 \\ &\quad + \int_{\mathbb{R}^{2d}} \phi_i(s, y_1, \theta_{t,s}^2(\xi)) D_{x_1}D_{x_2}\bar{q}(t, x_1, x_2; s, y_1, y_2) dy_1 dy_2, \end{aligned}$$

and the last term in the right hand side is equal to 0. In the sequel, we refer this argument as *the centering argument*. Combining this argument and the estimate for $D_{x_1}D_{x_2}\bar{q}$ in Lemma 2.2, we have, for T small enough and all s in $[t, T]$:

$$\begin{aligned} &\int_{\mathbb{R}^{2d}} (\phi_i(s, y_1, y_2) - \phi_i(s, y_1, \theta_{t,s}^2(\xi))) D_{x_1}D_{x_2}\bar{q}(t, x_1, x_2; s, y_1, y_2) dy_1 dy_2 \\ &\leq \int_{\mathbb{R}^{2d}} \left\{ (s-t)^{-2} |\phi_i(s, y_1, y_2) - \phi_i(s, y_1, \theta_{t,s}^2(\xi))| \right. \\ &\quad \left. \times \frac{c}{(s-t)^{2d}} \exp \left(-c \left(\frac{|y_1 - x_1|^2}{s-t} + \frac{|y_2 - m_{t,s}^{2,\xi}(x)|^2}{(s-t)^3} \right) \right) \right\} dy_1 dy_2, \end{aligned}$$

²The superscript “ t, x ” stands for the starting time and point of the process \bar{X} .

where c depends only on known parameters in **(H)**. Using Hölder regularity of ϕ_i supposed in **(H1)**, we have:

$$\begin{aligned} & \left| \int_{\mathbb{R}^{2d}} (s-t)^{-2} |\phi_i(s, y_1, y_2) - \phi_i(s, y_1, \theta_{t,s}^2(\xi))| \bar{q}(t, x_1, x_2; s, y_1, y_2) dy_1 dy_2 \right| \\ & \leq C \int_{\mathbb{R}^{2d}} \left\{ (s-t)^{-2+3\beta_i^2/2} \frac{|y_2 - \theta_{t,s}^2(\xi)|^{\beta_i^2}}{(s-t)^{3\beta_i^2/2}} \frac{c}{(s-t)^{2d}} \right. \\ & \quad \left. \times \exp \left(-c \left(\frac{|y_1 - x_1|^2}{s-t} + \frac{|y_2 - m_{t,s}^{2,\xi}(x)|^2}{(s-t)^3} \right) \right) \right\} dy_1 dy_2. \end{aligned}$$

Now, we use the off-diagonal decay of the Gaussian exponential: by letting $\xi = x$ (and then $\theta_{t,s}^2(x) = m_{t,s}^{2,x}(x)$), for all $\eta > 0$, there exists a constant $\bar{C} > 0$ such that³

$$\left(\frac{|y_2 - m_{t,s}^{2,x}(x)|}{(s-t)^{3/2}} \right)^{\beta_i^2} \times \exp \left(-\eta \left(\frac{|y_1 - x_1|^2}{s-t} + \frac{|y_2 - m_{t,s}^{2,x}(x)|^2}{(s-t)^3} \right) \right) \leq \bar{C},$$

where \bar{C} depends on η and β_i^2 only. Thus, by increasing the constant c in the exponential, we obtain the following estimate:

$$\begin{aligned} & \left| \int_{\mathbb{R}^{2d}} (s-t)^{-2} |\phi_i(s, y_1, y_2) - \phi_i(s, y_1, \theta_{t,s}^2(x))| \bar{q}(t, x_1, x_2; s, y_1, y_2) dy_1 dy_2 \right| \tag{2.14} \\ & \leq C' \int_{\mathbb{R}^{2d}} (s-t)^{-2+3\beta_i^2/2} \frac{c}{(s-t)^{2d}} \exp \left(-c \left(\frac{|y_1 - x_1|^2}{s-t} + \frac{|y_2 - m_{t,s}^{2,\xi}(x)|^2}{(s-t)^3} \right) \right) dy_1 dy_2. \end{aligned}$$

Therefore, by choosing the value of $\beta_i^2 > 2/3$, the singularity $(s-t)^{-2+3\beta_i^2/2}$ becomes integrable. From (2.14), letting ϵ tends to 0 in (2.12) and using (2.13), we deduce that:

$$\|D_{x_1} D_{x_2} u_i\|_{\infty} \leq C'' T^{-1+3\beta_i^2/2}.$$

From this discussion, one can also see the specific choice of the freezing curve as the one that matches the off-diagonal decay of the exponential in \bar{q} when $\xi = x$.

3 Mathematical tools

In this section, we introduce the ingredients for the proof of Proposition 1.3.

3.1 The frozen system

Consider the system:

$$\begin{aligned} d\tilde{X}_s^{1,t,x} &= F_1(s, \theta_{t,s}(\xi)) ds + \sigma(s, \theta_{t,s}(\xi)) dW_s \\ d\tilde{X}_s^{2,t,x} &= F_2(s, \theta_{t,s}(\xi)) ds + D_{x_1} F_2(s, \theta_{t,s}(\xi)) (\tilde{X}_s^{1,t,x} - \theta_{t,s}^1(\xi)) ds \end{aligned} \tag{3.1}$$

for all s in $(t, T]$, any t in $[0, T]$, and for any initial condition x in \mathbb{R}^{2d} at time t and any $\xi \in \mathbb{R}^{2d}$ and where $(\theta_{t,s}(\xi))_{t \leq s \leq T}$ is defined by:

$$\frac{d}{ds} \theta_{t,s}(\xi) = F(s, \theta_{t,s}(\xi)), \quad \theta_{t,t}(\xi) = \xi. \tag{3.2}$$

The following Proposition holds:

³By using the inequality: $\forall \eta > 0, \forall q > 0, \exists \bar{C} > 0$ s.t. $\forall \sigma > 0, \sigma^q e^{-\eta\sigma} \leq \bar{C}$.

Proposition 3.1. *Suppose that assumptions **(HR)** hold, then:*

(i) *There exists a unique (strong) solution of (3.1) with mean*

$$(m_{t,s}^\xi)_{t \leq s \leq T} = (m_{t,s}^{1,\xi}, m_{t,s}^{2,\xi})_{t \leq s \leq T},$$

where

$$\begin{aligned} m_{t,s}^{1,\xi}(x) &= x_1 + \int_t^s F_1(r, \theta_{t,r}(\xi)) dr \\ m_{t,s}^{2,\xi}(x) &= x_2 + \int_t^s \left[F_2(r, \theta_{t,r}(\xi)) + D_{x_1} F_2(r, \theta_{t,r}(\xi))(x_1 - \theta_{t,r}^1(\xi)) \right. \\ &\quad \left. + D_{x_1} F_2(r, \theta_{t,r}(\xi)) \int_t^r F_1(v, \theta_{t,v}(\xi)) dv \right] dr, \end{aligned} \quad (3.3)$$

and an uniformly non-degenerate covariance matrix $(\tilde{\Sigma}_{t,s})_{t \leq s \leq T}$:

$$\tilde{\Sigma}_{t,s} = \begin{pmatrix} \int_t^s \sigma \sigma^*(r, \theta_{t,r}(\xi)) dr & \int_t^s R_{r,s}(\xi) \sigma \sigma^*(r, \theta_{t,r}(\xi)) dr \\ \int_t^s \sigma \sigma^*(r, \theta_{t,r}(\xi)) R_{r,s}^*(\xi) dr & \int_t^s R_{t,r}(\xi) \sigma \sigma^*(r, \theta_{t,r}(\xi)) R_{t,r}^*(\xi) dr \end{pmatrix}, \quad (3.4)$$

where:

$$R_{t,r}(\xi) = \left[\int_t^r D_{x_1} F_2(v, \theta_{t,v}(\xi)) dv \right], \quad t \leq r \leq s \leq T.$$

(ii) *This solution is a Gaussian process with transition density:*

$$\begin{aligned} \tilde{q}(t, x_1, x_2; s, y_1, y_2) & \\ &= \frac{3^{d/2}}{(2\pi)^{d/2}} (\det[\tilde{\Sigma}_{t,s}])^{-1/2} \exp \left(-\frac{1}{2} |\tilde{\Sigma}_{t,s}^{-1/2} (y_1 - m_{t,s}^{1,\xi}(x), y_2 - m_{t,s}^{2,\xi}(x))^*|^2 \right), \end{aligned} \quad (3.5)$$

for all s in $[t, T]$.

(iii) *This transition density \tilde{q} is the fundamental solution of the PDE driven by $\tilde{\mathcal{L}}^{t,\xi}$ and given by:*

$$\begin{aligned} \tilde{\mathcal{L}}^{t,\xi} &:= \frac{1}{2} Tr \left[a(t, \theta_{t,s}(\xi)) D_{x_1}^2 \right] + [F_1(t, \theta_{t,s}(\xi))] \cdot D_{x_1} \\ &\quad + [F_2(t, \theta_{t,s}(\xi)) + D_{x_1} F_2(t, \theta_{t,s}(\xi)) (x_1 - \theta_{t,s}^1(\xi))] \cdot D_{x_2}. \end{aligned} \quad (3.6)$$

(iv) *For T small enough, there exist two positive constants c and C , depending only on known parameters in **(H)**, such that*

$$\tilde{q}(t, x_1, x_2; s, y_1, y_2) \leq C \hat{q}_c(t, x_1, x_2; s, y_1, y_2),$$

where

$$\hat{q}_c(t, x_1, x_2; s, y_1, y_2) = \frac{c}{(s-t)^{2d}} \exp \left(-c \left(\frac{|y_1 - m_{t,s}^{1,\xi}(x)|^2}{s-t} + \frac{|y_2 - m_{t,s}^{2,\xi}(x)|^2}{(s-t)^3} \right) \right),$$

and:

$$\begin{aligned} &\left| D_t^{N^t} D_{x_1}^{N^{x_1}} D_{x_2}^{N^{x_2}} D_{y_1}^{N^{y_1}} \tilde{q}(t, x_1, x_2; s, y_1, y_2) \right| \\ &\leq C (s-t)^{-[3(N^{x_2} + N^t) + N^{x_1} + N^{y_1}]/2} \hat{q}_c(t, x_1, x_2; s, y_1, y_2), \end{aligned} \quad (3.7)$$

for all s in $[t, T]$ and any integers $N^t, N^{x_1}, N^{x_2}, N^{y_1}$ less than 2.

Proof. (i) First of all, note that, under **(HR)**, the ODE: $[d/ds]\theta_{t,s}(\xi) = F(s, \theta_{t,s}(\xi))$, $\theta_{t,t}(\xi) = \xi$ admits a unique solution and (3.1) admits a unique strong solution \tilde{X} . One can write (3.1) as:

$$\begin{aligned}\tilde{X}_s^{1,t,x} &= x_1 + \int_t^s F_1(r, \theta_{t,r}(\xi))dr + \int_t^s \sigma(r, \theta_{t,r}(\xi))dW_r, \\ \tilde{X}_s^{2,t,x} &= x_2 + \int_t^s \left[F_2(r, \theta_{t,r}(\xi)) + D_{x_1}F_2(r, \theta_{t,r}(\xi))(x_1 - \theta_{t,r}^1(\xi)) \right. \\ &\quad \left. + D_{x_1}F_2(r, \theta_{t,r}(\xi)) \int_t^r F_1(v, \theta_{t,v}(\xi))dv \right] dr \\ &\quad + \int_t^s D_{x_1}F_2(r, \theta_{t,r}(\xi)) \int_t^r \sigma(v, \theta_{t,v}(\xi))dW_v dr\end{aligned}$$

Then, the expressions of the mean (3.3) and the variance (3.4) follow from the stochastic Fubini Theorem and standard computations. The uniform non-degeneracy of $(\tilde{\Sigma}_{t,s})_{t < s \leq T}$ arises from assumptions **(H)** and Proposition 3.1 in [DM10].

(ii)-(iii) These assertions result from standard computations.

(iv) For all s in $(t, T]$, we know from Proposition 3.1 in [DM10] that the matrix $\tilde{\Sigma}_{t,s}$ is symmetric and uniformly non degenerate. Besides, from Subsection 2.3 and Proposition 3.4 in [DM10] there exists a constant C depending only on known parameters in **(H)** such that: for all $s \in [t, T]$, for all $(x, y, \xi) \in \mathbb{R}^{2d} \times \mathbb{R}^{2d} \times \mathbb{R}^{2d}$,

$$\begin{aligned}& - \left[\tilde{\Sigma}_{t,s}^{-1}(y_1 - m_{t,s}^{1,\xi}(x), y_2 - m_{t,s}^{2,\xi}(x))^* \right] \cdot \left[(y_1 - m_{t,s}^{1,\xi}(x), y_2 - m_{t,s}^{2,\xi}(x))^* \right] \\ & \leq -C \left[\left(\frac{y_1 - m_{t,s}^{1,\xi}(x)}{(s-t)^{1/2}}, \frac{y_2 - m_{t,s}^{2,\xi}(x)}{(s-t)^{3/2}} \right)^* \right] \cdot \left[\left(\frac{y_1 - m_{t,s}^{1,\xi}(x)}{(s-t)^{1/2}}, \frac{y_2 - m_{t,s}^{2,\xi}(x)}{(s-t)^{3/2}} \right)^* \right].\end{aligned}$$

For $i, j = 1, 2$, let $[\tilde{\Sigma}_{t,s}^{-1}]_{i,j}$ denotes the block of size $d \times d$ of the matrix $\tilde{\Sigma}_{t,s}^{-1}$ at the $(i-1)d+1, (j-1)d+1$ rank. We can deduce from (3.4) that there exists a positive constant C depending only on known parameters in **(H)** such that (we also refer the reader to Lemma 3.6 and to the proof of Lemma 5.5 in [DM10] for more details), for all s in $(t, T]$, for all ζ in \mathbb{R}^d :

$$\begin{cases} |[\tilde{\Sigma}_{t,s}^{-1}]_{1,1}\zeta| \leq C(s-t)^{-1} |\zeta|, \\ |[\tilde{\Sigma}_{t,s}^{-1}]_{1,2}\zeta| + |[\tilde{\Sigma}_{t,s}^{-1}]_{2,1}\zeta| \leq C(s-t)^{-2} |\zeta|, \\ |[\tilde{\Sigma}_{t,s}^{-1}]_{2,2}\zeta| \leq C(s-t)^{-3} |\zeta|, \end{cases} \quad (3.8)$$

hence, $\tilde{\Sigma}_{t,\cdot}^{-1}$ has the same structure as $K_{-\cdot}^{-1}$ in (2.4).

Now, we compute the derivatives w.r.t. each component:

$$\begin{aligned}& |D_{x_2} \tilde{q}(t, x_1, x_2; s, y_1, y_2)| \\ &= \left| \left(-2[\tilde{\Sigma}_{t,s}^{-1}]_{2,1}(y_1 - m_{t,s}^{1,\xi}(x)) - 2[\tilde{\Sigma}_{t,s}^{-1}]_{2,2}(y_2 - m_{t,s}^{2,\xi}(x)) \right) \tilde{q}(t, x_1, x_2; s, y_1, y_2) \right| \\ &\leq C(s-t)^{-3/2} \left(\left| \frac{(y_1 - m_{t,s}^{1,\xi}(x))}{(s-t)^{1/2}} \right| + \left| \frac{(y_2 - m_{t,s}^{2,\xi}(x))}{(s-t)^{3/2}} \right| \right) \tilde{q}(t, x_1, x_2; s, y_1, y_2) \\ &\leq C'(s-t)^{-3/2} \hat{q}_c(t, x_1, x_2; s, y_1, y_2).\end{aligned}$$

Note that the symmetry $D_{x_2} \tilde{q}(t, x_1, x_2; s, y_1, y_2) = -D_{y_2} \tilde{q}(t, x_1, x_2; s, y_1, y_2)$ holds. Now, we have

$$\begin{aligned}
& |D_{y_1} \tilde{q}(t, x_1, x_2; s, y_1, y_2)| \\
&= \left| \left(2[\tilde{\Sigma}_{t,s}^{-1}]_{1,1}(y_1 - m_{t,s}^{1,\xi}(x)) + 2[\tilde{\Sigma}_{t,s}^{-1}]_{1,2}(y_2 - m_{t,s}^{2,\xi}(x)) \right) \tilde{q}(t, x_1, x_2; s, y_1, y_2) \right| \\
&\leq C(s-t)^{-1/2} \hat{q}_c(t, x_1, x_2; s, y_1, y_2).
\end{aligned}$$

Unfortunately, the transport of the initial condition of the diffusive component in the degenerate component breaks the symmetry and $D_{x_1} \tilde{q}(t, x_1, x_2; s, y_1, y_2) \neq -D_{y_1} \tilde{q}(t, x_1, x_2; s, y_1, y_2)$. Indeed

$$\begin{aligned}
& D_{x_1} \tilde{q}(t, x_1, x_2; s, y_1, y_2) \\
&= \left(-2[\tilde{\Sigma}_{t,s}^{-1}]_{1,1}(y_1 - m_{t,s}^{1,\xi}(x)) - 2[\tilde{\Sigma}_{t,s}^{-1}]_{1,2}(y_2 - m_{t,s}^{2,\xi}(x)) \right. \\
&\quad \left. -2[\tilde{\Sigma}_{t,s}^{-1}]_{1,2} \left[(R_{t,s}(\xi)) (y_1 - m_{t,s}^{1,\xi}(x)) \right] \right. \\
&\quad \left. -2[\tilde{\Sigma}_{t,s}^{-1}]_{2,2} \left[(R_{t,s}(\xi)) (y_2 - m_{t,s}^{2,\xi}(x)) \right] \right) \tilde{q}(t, x_1, x_2; s, y_1, y_2).
\end{aligned}$$

Since the term $R_{t,s}(\xi)$ is of order $(s-t)$ (this is the transport of the initial condition from time t to s), we deduce that

$$\begin{aligned}
& |D_{x_1} \tilde{q}(t, x_1, x_2; s, y_1, y_2)| \\
&\leq C(s-t)^{-1/2} \left\{ \left| \frac{(y_1 - m_{t,s}^{2,\xi}(x))}{(s-t)^{1/2}} \right| + \left| \frac{(y_2 - m_{t,s}^{2,\xi}(x))}{(s-t)^{3/2}} \right| + \left| \frac{(y_1 - m_{t,s}^{2,\xi}(x))}{(s-t)^{1/2}} \right| \right. \\
&\quad \left. + \left| \frac{(y_2 - m_{t,s}^{2,\xi}(x))}{(s-t)^{3/2}} \right| \right\} \tilde{q}(t, x_1, x_2; s, y_1, y_2) \\
&\leq C'(s-t)^{-1/2} \hat{q}_c(t, x_1, x_2; s, y_1, y_2).
\end{aligned}$$

Finally,

$$\begin{aligned}
& D_{x_1}^2 \tilde{q}(t, x_1, x_2; s, y_1, y_2) \\
&= \left(-2[\tilde{\Sigma}_{t,s}^{-1}]_{1,1} D_{x_1} m_{t,s}^{1,\xi}(x) - 2[\tilde{\Sigma}_{t,s}^{-1}]_{1,2} D_{x_1} m_{t,s}^{2,\xi}(x) - 2[\tilde{\Sigma}_{t,s}^{-1}]_{1,2} \left[(R_{t,s}(\xi)) D_{x_1} m_{t,s}^{1,\xi}(x) \right] \right. \\
&\quad \left. -2[\tilde{\Sigma}_{t,s}^{-1}]_{2,2} \left[(R_{t,s}(\xi)) D_{x_1} m_{t,s}^{2,\xi}(x) \right] \right) \tilde{q}(t, x_1, x_2; s, y_1, y_2) \\
&+ \left(-2[\tilde{\Sigma}_{t,s}^{-1}]_{1,1}(y_1 - m_{t,s}^{1,\xi}(x)) - 2[\tilde{\Sigma}_{t,s}^{-1}]_{1,2}(y_2 - m_{t,s}^{2,\xi}(x)) \right. \\
&\quad \left. -2[\tilde{\Sigma}_{t,s}^{-1}]_{1,2} \left[(R_{t,s}(\xi)) (y_1 - m_{t,s}^{1,\xi}(x)) \right] \right. \\
&\quad \left. -2[\tilde{\Sigma}_{t,s}^{-1}]_{2,2} \left[(R_{t,s}(\xi)) (y_2 - m_{t,s}^{2,\xi}(x)) \right] \right)^2 \tilde{q}(t, x_1, x_2; s, y_1, y_2).
\end{aligned}$$

Note that, from (3.3) we have $D_{x_1} m_{t,s}^{\xi}(x) = (\text{Id}, R_{t,s}(\xi))^*$, so that,

$$|D_{x_1}^2 \tilde{q}(t, x_1, x_2; s, y_1, y_2)| \leq C(s-t)^{-1} \hat{q}_c(t, x_1, x_2; s, y_1, y_2).$$

Since \tilde{q} satisfies the Fokker-Planck equation

$$\partial_t \tilde{q}(t, x, T, y) + \tilde{\mathcal{L}}^{t,\xi} \tilde{q}(t, x, T, y) = 0, \quad \tilde{q}(T, x, T, y) = \delta_y(x), \quad x, y \in \mathbb{R}^{2d},$$

we deduce from previous estimates:

$$|\partial_t \tilde{q}(t, x_1, x_2; s, y_1, y_2)| \leq C(s-t)^{-3/2} \hat{q}_c(t, x_1, x_2; s, y_1, y_2).$$

The other derivatives can be deduced from these computations and the estimate (3.7) follows. \square

Remark. From this proof, one can deduce that the symmetry $D_{x_2} \tilde{q} = -D_{y_2} \tilde{q}$ holds. Therefore, for all t in $[0, T]$, all s in $[t, T]$ and y_1, x_1, x_2 in \mathbb{R}^d ,

$$\int_{\mathbb{R}^d} D_{x_2} \tilde{q}(t, x_1, x_2; s, y_1, y_2) dy_2 = 0. \quad (3.9)$$

This argument is very useful in the sequel.

3.2 Representation and differentiation of the solution

Lemma 3.2. Suppose that assumptions (HR) hold, then, for all t in $[0, T]$, the solution $u = (u_1, u_2)^*$ of the PDE (1.8) can be written as:

$$\begin{aligned} u_i(t, x_1, x_2) &= \int_t^T \int_{\mathbb{R}^{2d}} \phi_i(s, y_1, y_2) \tilde{q}(t, x_1, x_2; s, y_1, y_2) dy_1 dy_2 ds \\ &\quad + \int_t^T \int_{\mathbb{R}^{2d}} (\tilde{\mathcal{L}}^{t,\xi} - \mathcal{L}) u_i(s, y_1, y_2) \tilde{q}(t, x_1, x_2; s, y_1, y_2) dy_1 dy_2 ds, \end{aligned} \quad (3.10)$$

for $i = 1, 2$. Moreover, this solution is infinitely differentiable and for all $\epsilon > 0$ all integer n and all multi-index $(j_1, \dots, j_n) \in \{1, 2\}^n$, we have:

$$\begin{aligned} &D_{x_{j_1}} \cdots D_{x_{j_n}} u_i(t, x_1, x_2) \\ &= \int_{t+\epsilon}^T \int_{\mathbb{R}^{2d}} \phi_i(s, y_1, y_2) D_{x_{j_1}} \cdots D_{x_{j_n}} \tilde{q}(t, x_1, x_2; s, y_1, y_2) dy_1 dy_2 ds \\ &\quad + \int_{t+\epsilon}^T \int_{\mathbb{R}^{2d}} (\tilde{\mathcal{L}}^{t,\xi} - \mathcal{L}) u_i(s, y_1, y_2) D_{x_{j_1}} \cdots D_{x_{j_n}} \tilde{q}(t, x_1, x_2; s, y_1, y_2) dy_1 dy_2 ds + O(\epsilon), \end{aligned} \quad (3.11)$$

for $i = 1, 2$ and where $\tilde{\mathcal{L}}^{t,\xi}$ is given in (3.6).

Proof. We recall that the PDE (1.8) is given by:

$$\partial_t u_i(t, x_1, x_2) + \mathcal{L} u_i(t, x_1, x_2) = \phi_i(t, x_1, x_2), \quad u_i(T, x_1, x_2) = 0, \quad i = 1, 2, \quad (3.12)$$

and it can be rewritten as:

$$\begin{aligned} \partial_t u_i(t, x_1, x_2) + \tilde{\mathcal{L}}^{t,\xi} u_i(t, x_1, x_2) &= (\tilde{\mathcal{L}}^{t,\xi} u_i(t, x_1, x_2) - \mathcal{L} u_i(t, x_1, x_2)) + \phi_i(t, x_1, x_2), \\ u_i(T, x_1, x_2) &= 0, \quad i = 1, 2. \end{aligned} \quad (3.13)$$

From Proposition 3.1, we know that the fundamental solution of $\tilde{\mathcal{L}}^{t,\xi}$ is \tilde{q} . Then, any solution of (3.13) reads:

$$\begin{aligned} u_i(t, x_1, x_2) &= \int_t^T \int_{\mathbb{R}^{2d}} \phi_i(s, y_1, y_2) \tilde{q}(t, x_1, x_2; s, y_1, y_2) dy_1 dy_2 ds \\ &\quad + \int_t^T \int_{\mathbb{R}^{2d}} (\tilde{\mathcal{L}}^{t,\xi} - \mathcal{L}) u_i(t, x_1, x_2) \tilde{q}(t, x_1, x_2; s, y_1, y_2) dy_1 dy_2 ds. \end{aligned}$$

Given $\epsilon > 0$, we have

$$\begin{aligned}
u_i(t, x_1, x_2) &= \int_{t+\epsilon}^T \int_{\mathbb{R}^{2d}} \phi_i(s, y_1, y_2) \tilde{q}(t, x_1, x_2; s, y_1, y_2) dy_1 dy_2 ds \\
&+ \int_{t+\epsilon}^T \int_{\mathbb{R}^{2d}} (\tilde{\mathcal{L}}^{t,\xi} - \mathcal{L}) u_i(t, x_1, x_2) \tilde{q}(t, x_1, x_2; s, y_1, y_2) dy_1 dy_2 ds \\
&+ \int_t^{t+\epsilon} \int_{\mathbb{R}^{2d}} \phi_i(s, y_1, y_2) \tilde{q}(t, x_1, x_2; s, y_1, y_2) dy_1 dy_2 ds \\
&+ \int_t^{t+\epsilon} \int_{\mathbb{R}^{2d}} (\tilde{\mathcal{L}}^{t,\xi} - \mathcal{L}) u_i(t, x_1, x_2) \tilde{q}(t, x_1, x_2; s, y_1, y_2) dy_1 dy_2 ds.
\end{aligned} \tag{3.14}$$

The two last terms in the right hand side read:

$$\int_t^{t+\epsilon} \mathbb{E} \left[\phi_i(s, \tilde{X}_s^{1,t,x}, \tilde{X}_s^{2,t,x}) + (\tilde{\mathcal{L}}^{t,\xi} - \mathcal{L}) u_i(t, \tilde{X}_s^{1,t,x}, \tilde{X}_s^{2,t,x}) \right] ds.$$

Let n be a positive integer, for all multi-index $(j_1, \dots, j_n) \in \{1, 2\}^n$, by following the proof of Lemma 1.2, it follows from the regularity of each ϕ_i and u_i under **(HR)** that there exists a positive constant $K(n)$ such that

$$\left| D_{x_{j_1}} \cdots D_{x_{j_n}} \mathbb{E} \left[\phi_i(s, \tilde{X}_s^{1,t,x}, \tilde{X}_s^{2,t,x}) + (\tilde{\mathcal{L}}^{t,\xi} - \mathcal{L}) u_i(t, \tilde{X}_s^{1,t,x}, \tilde{X}_s^{2,t,x}) \right] \right| \leq K(n).$$

The claim follows from Lebesgue differentiation Theorem. \square

4 Proof of Proposition 1.3

4.1 From parametrix to uniform Lipschitz estimates

We give preliminary results in order to prove the Lipschitz bounds. These bounds are obtained under **(HR)** but depend only on known parameters in **(H)**.

Some notations. For all $t < s \in [0, T]$, for all $\xi \in \mathbb{R}^{2d}$, we denote by $\Delta_{t,s}(\xi)$ the perturbation operator defined by:

$$\forall (s, y) \in [0, T] \times \mathbb{R}^{2d}, \Delta_{t,s}(\xi) g(s, y) = g(s, y) - g(s, \theta_{t,s}(\xi)),$$

for all measurable function g of $[0, T] \times \mathbb{R}^d \times \mathbb{R}^d$. By analogy with our standard notations, we also define for all $y \in \mathbb{R}^{2d}$: $\Delta_{t,s}^1(\xi) g(s, y_1, y_2) = g(s, y_1, y_2) - g(s, \theta_{t,s}^1(\xi), y_2)$ and $\Delta_{t,s}^2(\xi) g(s, y_1, y_2) = g(s, y_1, y_2) - g(s, y_1, \theta_{t,s}^2(\xi))$ with the convention $y \in \mathbb{R}^{2d} = (y_1, y_2) \in \mathbb{R}^d \times \mathbb{R}^d$.

Considering these notations, (3.10) becomes: for all $x \in \mathbb{R}^{2d}$ and $t \in [0, T]$,

$$\begin{aligned}
u_i(t, x) &= \int_t^T \int_{\mathbb{R}^{2d}} \phi_i(s, y) \tilde{q}(t, x; s, y) dy ds \\
&- \int_t^T \int_{\mathbb{R}^{2d}} \frac{1}{2} \text{Tr} \left[\Delta_{t,s}(\xi) a(s, y) D_{x_1}^2 u_i(s, y) \right] \tilde{q}(t, x; s, y) dy ds \\
&- \int_t^T \int_{\mathbb{R}^{2d}} [\Delta_{t,s}(\xi) F_1(s, y)] \cdot D_{x_1} u_i(s, y) \tilde{q}(t, x; s, y) dy ds \\
&+ \int_t^T \int_{\mathbb{R}^{2d}} [D_{x_1} F_2(s, \theta_{t,s}(\xi)) \Delta_{t,s}^1(\xi) y_1 - \Delta_{t,s}(\xi) F_2(s, y)] \cdot D_{x_2} u_i(s, y) \tilde{q}(t, x; s, y) dy ds \\
&:= H_i^1(t; t, x) + H_i^2(t; t, x) + H_i^3(t; t, x) + H_i^4(t; t, x),
\end{aligned} \tag{4.1}$$

for $i = 1, 2$, with the notation $\tilde{q}(t, x; s, y)$ for $\tilde{q}(t, x_1, x_2; s, y_1, y_2)$. When splitting the time integral as in (3.14), we write for all $\epsilon > 0$:

$$u_i(t, x) = H_i^1(t + \epsilon; t, x) + H_i^2(t + \epsilon; t, x) + H_i^3(t + \epsilon; t, x) + H_i^4(t + \epsilon; t, x) + O(\epsilon), \quad (4.2)$$

where the terms $H_i^j(t + \epsilon; t, x)$, $j = 1, \dots, 4$ are properly defined by identifying (3.11) with (4.1).

Proof of Proposition 1.3. We prove Proposition 1.3 by using a circular argument since the representation (4.1) of each u_i , $i = 1, 2$ involves the derivatives themselves. In the following, u_i denotes the i^{th} component of the solution $u = (u_1, u_2)^*$ of the linear system of PDE (1.8). The following Lemmas hold for $i = 1, 2$.

Lemma 4.1. *Suppose assumptions (HR) hold. Then, for T small enough there exist two positive reals $\delta_{4.1}$ and $\bar{\delta}_{4.1}$ and a positive constant C depending only on known parameters in (H) such that:*

$$\|D_{x_1}^2 u_i\|_\infty \leq T^{\delta_{4.1}} C (1 + \|D_{x_2} u_i\|_\infty),$$

and

$$\|D_{x_1} u_i\|_\infty \leq T^{\bar{\delta}_{4.1}} C (1 + \|D_{x_2} u_i\|_\infty).$$

Lemma 4.2. *Suppose assumptions (HR) hold. Then, for T small enough, there exist a real $\delta_{4.2} > 0$ and a positive constant C depending only on known parameters in (H), such that:*

$$\|D_{x_2} u_i\|_\infty \leq CT^{\delta_{4.2}} (1 + \|D_{x_1} D_{x_2} u_i\|_\infty).$$

Lemma 4.3. *Suppose assumptions (HR) hold. Then, for T small enough, there exist a positive real $\delta_{4.3}$ and a positive constant C , depending only on known parameters in (H) such that:*

$$\|D_{x_1} D_{x_2} u_i\|_\infty \leq CT^{\bar{\delta}_{4.3}}.$$

Proposition 1.3 follows from Lemmas 4.1, 4.2 and 4.3. \square

4.2 Proof of Lemma 4.1

Here we prove that for T small enough, there exists a positive constant C depending only on known parameters in (H) such that:

$$\begin{aligned} (i) \quad \|D_{x_1}^2 u_i\|_\infty &\leq C \left\{ \left(T^{3\beta_2^2/2} + T^{(1+\alpha^1)/2} \right) \|D_{x_2} u_i\|_\infty \right. \\ &\quad \left. + \left(T^{\beta_1^1/2} + T^{3\beta_1^2/2} \right) \|D_{x_1} u_i\|_\infty + T^{\beta_i^1/2} + T^{3\beta_i^2/2} \right\} \\ (ii) \quad \|D_{x_1} u_i\|_\infty &\leq C \left\{ \left(T^{1/2+3\beta_2^2/2} + T^{(3+\alpha^1)/2} \right) \|D_{x_2} u_i\|_\infty \right. \\ &\quad \left. + (T + T^2) \|D_{x_1}^2 u_i\|_\infty + T^{(1+\beta_i^1)/2} + T^{(1+3\beta_i^2)/2} \right\}. \end{aligned}$$

Proof of (i). Let $\epsilon > 0$, from the representation (4.1) and Lemma 3.2 we have:

$$D_{x_1}^2 u_i(t, x_1, x_2) = \sum_{j=1}^4 D_{x_1}^2 H_i^j(t + \epsilon; t, x_1, x_2) + O(\epsilon). \quad (4.3)$$

First, note that $D_{x_1}^2 H_i^1(t + \epsilon; t, x_1, x_2)$ reads:

$$\begin{aligned} D_{x_1}^2 H_i^1(t + \epsilon; t, x_1, x_2) &= D_{x_1}^2 \int_{t+\epsilon}^T \int_{\mathbb{R}^{2d}} \Delta_{t,s}(\xi) \phi_i(s, y_1, y_2) \tilde{q}(t, x_1, x_2; s, y_1, y_2) dy_1 dy_2 ds \\ &\quad + D_{x_1}^2 \int_{t+\epsilon}^T \int_{\mathbb{R}^{2d}} \phi_i(s, \theta_{t,s}^1(\xi), \theta_{t,s}^2(\xi)) \tilde{q}(t, x_1, x_2; s, y_1, y_2) dy_1 dy_2 ds. \end{aligned} \quad (4.4)$$

After integrating w.r.t y_1 and y_2 , the last term in the right hand side does not depend on x_1 and so is equal to 0.

Secondly, from (4.1) and Lemma 3.2, $D_{x_1}^2 H_i^4(t + \epsilon; t, x_1, x_2)$ reads:

$$\begin{aligned} D_{x_1}^2 H_i^4(t + \epsilon; t, x_1, x_2) &= \int_{t+\epsilon}^T \int_{\mathbb{R}^{2d}} [D_{x_1} F_2(s, \theta_{t,s}^1(\xi), \theta_{t,s}^2(\xi)) \Delta_{t,s}^1(\xi) y_1 - \Delta_{t,s}(\xi) F_2(s, y_1, y_2)] \\ &\quad \cdot D_{x_2} u_i(s, y_1, y_2) [D_{x_1}^2 \tilde{q}(t, x_1, x_2; s, y_1, y_2)] dy_1 dy_2 ds. \end{aligned}$$

Note that for all $s \in [t, T]$ and y, ξ in \mathbb{R}^{2d} , the following estimate holds:

$$\begin{aligned} &|[\Delta_{t,s}(\xi) F_2(s, y_1, y_2) - D_{x_1} F_2(s, \theta_{t,s}^1(\xi), \theta_{t,s}^2(\xi)) \Delta_{t,s}^1(\xi) y_1]| \\ &\leq C \left(|\Delta_{t,s}^2(\xi) y_2|^{\beta_2^2} + |\Delta_{t,s}^1(\xi) y_1|^{1+\alpha^1} \right). \end{aligned} \quad (4.5)$$

Indeed, by applying a Taylor expansion with integrable remainder, for all s in $[t, T]$ we have:

$$F_2(s, \theta_{t,s}^1(\xi), \theta_{t,s}^2(\xi)) = F_2(s, y_1, \theta_{t,s}^2(\xi)) + \int_{y_1}^{\theta_{t,s}^1(\xi)} D_{x_1} F_2(s, \lambda, \theta_{t,s}^2(\xi)) d\lambda,$$

and plugging into:

$$[F_2(t, y_1, y_2) - F_2(s, \theta_{t,s}^1(\xi), \theta_{t,s}^2(\xi)) - D_{x_1} F_2(t, \theta_{t,s}^1(\xi), \theta_{t,s}^2(\xi)) (y_1 - \theta_{t,s}^1(\xi))],$$

we obtain:

$$\begin{aligned} &\left| \left[F_2(s, y_1, y_2) - F_2(s, y_1, \theta_{t,s}^2(\xi)) \right. \right. \\ &\quad \left. \left. - \int_{y_1}^{\theta_{t,s}^1(\xi)} D_{x_1} F_2(s, \lambda, \theta_{t,s}^2(\xi)) d\lambda - D_{x_1} F_2(s, \theta_{t,s}^1(\xi), \theta_{t,s}^2(\xi)) (y_1 - \theta_{t,s}^1(\xi)) \right] \right| \\ &\leq |F_2(s, y_1, y_2) - F_2(s, x_1, \theta_{t,s}^2(\xi))| \\ &\quad + \int_{y_1}^{\theta_{t,s}^1(\xi)} |D_{x_1} F_2(s, \lambda, \theta_{t,s}^2(\xi)) - D_{x_1} F_2(s, \theta_{t,s}^1(\xi), \theta_{t,s}^2(\xi))| d\lambda \\ &\leq C_2 |y_2 - \theta_{t,s}^2(\xi)|^{\beta_2^2} + \bar{C}_2 \int_{y_1}^{\theta_{t,s}^1(\xi)} |\lambda - \theta_{t,s}^1(\xi)|^{\alpha^1} d\lambda \\ &\leq C' \left(|y_2 - \theta_{t,s}^2(\xi)|^{\beta_2^2} + |(y_1 - \theta_{t,s}^1(\xi))|^{1+\alpha^1} \right), \end{aligned}$$

this proves (4.5).

Plugging (4.4) into (4.3), we have:

$$\begin{aligned}
& D_{x_1}^2 u_i(t, x_1, x_2) \\
&= \int_{t+\epsilon}^T \int_{\mathbb{R}^{2d}} \Delta_{t,s}(\xi) \phi_i(s, y_1, y_2) [D_{x_1}^2 \tilde{q}(t, x_1, x_2; s, y_1, y_2)] dy_1 dy_2 ds \\
&\quad - \frac{1}{2} \int_{t+\epsilon}^T \int_{\mathbb{R}^{2d}} \text{Tr} [\Delta_{t,s}(\xi) a(s, y_1, y_2) D_{x_1}^2 u_i(s, y_1, y_2)] [D_{x_1}^2 \tilde{q}(t, x_1, x_2; s, y_1, y_2)] dy_1 dy_2 ds \\
&\quad - \int_{t+\epsilon}^T \int_{\mathbb{R}^{2d}} [\Delta_{t,s}(\xi) F_1(s, y_1, y_2)] \cdot D_{x_1}^2 u_i(s, y_1, y_2) [D_{x_1}^2 \tilde{q}(t, x_1, x_2; s, y_1, y_2)] dy_1 dy_2 ds \\
&\quad + \int_{t+\epsilon}^T \int_{\mathbb{R}^{2d}} [D_{x_1} F_2(s, \theta_{t,s}^1(\xi), \theta_{t,s}^2(\xi)) \Delta_{t,s}^1(\xi) y_1 - \Delta_{t,s}(\xi) F_2(s, y_1, y_2)] \\
&\quad \cdot D_{x_2} u_i(s, y_1, y_2) [D_{x_1}^2 \tilde{q}(t, x_1, x_2; s, y_1, y_2)] dy_1 dy_2 ds + O(\epsilon).
\end{aligned}$$

From Proposition 3.1, we know that for all s in $[t, T]$, $D_{x_1}^2 \tilde{q}(t, x_1, x_2; s, y_1, y_2) \leq C(s-t)^{-1} \hat{q}_c(t, x_1, x_2; s, y_1, y_2)$. Combining this argument with the regularity of the coefficients given in **(H)**, and thanks to (4.5), we obtain:

$$\begin{aligned}
& |D_{x_1}^2 u_i(t, x_1, x_2)| \\
&\leq C' \left\{ \int_{t+\epsilon}^T \left\{ (s-t)^{-1} \int_{\mathbb{R}^{2d}} \sum_{j=1}^2 (s-t)^{(j-1/2)\beta_i^j} \left| \frac{\Delta_{t,s}^j(\xi) y_1}{(s-t)^{(j-1/2)}} \right|^{\beta_i^j} \right. \right. \\
&\quad \left. \left. \times \hat{q}_c(t, x_1, x_2; s, y_1, y_2) \right\} dy_1 dy_2 ds \right. \\
&\quad + \int_{t+\epsilon}^T \left\{ (s-t)^{-1} \|D_{x_1}^2 u_i\|_\infty \int_{\mathbb{R}^{2d}} \sum_{j=1}^2 (s-t)^{(j-1/2)} \left| \frac{\Delta_{t,s}^j(\xi) y_1}{(s-t)^{(j-1/2)}} \right| \right. \\
&\quad \left. \times \hat{q}_c(t, x_1, x_2; s, y_1, y_2) \right\} dy_1 dy_2 ds \\
&\quad + \int_{t+\epsilon}^T \left\{ (s-t)^{-1} \|D_{x_1} u_i\|_\infty \int_{\mathbb{R}^{2d}} \sum_{j=1}^2 (s-t)^{(j-1/2)\beta_1^j} \left| \frac{\Delta_{t,s}^j(\xi) y_1}{(s-t)^{(j-1/2)}} \right|^{\beta_1^j} \right. \\
&\quad \left. \times \hat{q}_c(t, x_1, x_2; s, y_1, y_2) \right\} dy_1 dy_2 ds \\
&\quad + \int_{t+\epsilon}^T \left\{ (s-t)^{-1} \|D_{x_2} u_i\|_\infty \int_{\mathbb{R}^{2d}} \left[(s-t)^{3\beta_2^2/2} \left| \frac{\Delta_{t,s}^2(\xi) y_2}{(s-t)^{3/2}} \right|^{\beta_2^2} \right. \right. \\
&\quad \left. \left. + (s-t)^{(1+\alpha^1)/2} \left| \frac{\Delta_{t,s}^1(\xi) y_1}{(s-t)^{1/2}} \right|^{1+\alpha^1} \right] \hat{q}_c(t, x_1, x_2; s, y_1, y_2) \right\} dy_1 dy_2 ds \Big\} + O(\epsilon).
\end{aligned}$$

Set $\xi = x$, by using the off-diagonal decay of the Gaussian exponential in \hat{q}_c (see Subsection 2.4) and by integrating w.r.t the space variables we have:

$$\begin{aligned}
& |D_{x_1}^2 u_i(t, x_1, x_2)| \\
& \leq C'' \left\{ \int_{t+\epsilon}^T \sum_{j=1}^2 (s-t)^{-1+(j-1/2)\beta_i^j} ds \right. \\
& \quad + \int_{t+\epsilon}^T \left((s-t)^{-1/2} + (s-t)^{1/2} \right) \|D_{x_1}^2 u_i\|_\infty ds \\
& \quad + \int_{t+\epsilon}^T \sum_{j=1}^2 (s-t)^{(j-1/2)\beta_2^j-1} \|D_{x_1} u_i\|_\infty ds \\
& \quad \left. + \int_{t+\epsilon}^T \left((s-t)^{-1+3\beta_1^2/2} + (s-t)^{(\alpha^1-1)/2} \right) \|D_{x_2} u_i\|_\infty ds \right\} \\
& + O(\epsilon).
\end{aligned}$$

By letting $\epsilon \rightarrow 0$, we finally obtain:

$$\begin{aligned}
\|D_{x_1}^2 u_i\|_\infty & \leq C''' \left\{ \left(T^{3\beta_2^2/2} + T^{(1+\alpha^1)/2} \right) \|D_{x_2} u_i\|_\infty + \left(T^{\beta_1^1/2} + T^{3\beta_1^2/2} \right) \|D_{x_1} u_i\|_\infty \right. \\
& \quad \left. + \left(T^{1/2} + T^{3/2} \right) \|D_{x_1}^2 u_i\|_\infty + T^{\beta_i^1/2} + T^{3\beta_i^2/2} \right\}.
\end{aligned}$$

Taking T small enough such that $C'''(T^{1/2} + T^{3/2}) = 1/2$, the assertion (i) follows from a circular argument. The proof of the second statement (ii) can be done by the same arguments and concludes the proof of Lemma 4.1. \square

4.3 Proof of Lemma 4.2

We first derive a representation formula for H_i^2 and H_i^3 to handle the singularity of the derivative of the kernel \tilde{q} . These formulas are given in the following claim and serve for the proof of Lemma 4.2.

Claim 4.4. *For all $(t, x_1, x_2) \in [0, T] \times \mathbb{R}^{2d}$, for all $\epsilon > 0$, we have:*

$$\begin{aligned}
& D_{x_2} H_i^2(t + \epsilon; t, x_1, x_2) \tag{4.6} \\
& = -\frac{1}{2} \int_{t+\epsilon}^T \int_{\mathbb{R}^{2d}} \text{Tr} \left[[\Delta_{t,s}^2(\xi) a(s, y_1, y_2)] D_{x_1}^2 u_i(s, y_1, y_2) \right] [D_{x_2} \tilde{q}(t, x_1, x_2; s, y_1, y_2)] dy_1 dy_2 ds \\
& \quad - \frac{1}{2} \int_{t+\epsilon}^T \left\{ \left[\sum_{l=1}^d \int_{\mathbb{R}^{2d}} \left[\frac{\partial}{\partial y_{1l}} a_l(s, y_1, \theta_{t,s}^2(\xi)) \right] \cdot [\Delta_{t,s}^2(\xi) D_{x_1} u_i(s, y_1, y_2)] \right] \right. \\
& \quad \left. \times [D_{x_2} \tilde{q}(t, x_1, x_2; s, y_1, y_2)] \right\} dy_1 dy_2 ds \\
& \quad - \frac{1}{2} \sum_{l=1}^d \int_{t+\epsilon}^T \int_{\mathbb{R}^{2d}} [\Delta_{t,s}^1(\xi) a_l(s, y_1, \theta_{t,s}^2(\xi))] \cdot [\Delta_{t,s}^2(\xi) D_{x_1} u_i(s, y_1, y_2)] \\
& \quad \times \left[D_{x_2} \left(\frac{\partial}{\partial y_{1l}} \tilde{q}(t, x_1, x_2; s, y_1, y_2) \right) \right] dy_1 dy_2 ds,
\end{aligned}$$

where “ a_l ” denotes the l^{th} line of the matrix a , and

$$\begin{aligned}
& D_{x_2} H_i^3(t + \epsilon; t, x_1, x_2) \\
&= - \int_{t+\epsilon}^T \int_{\mathbb{R}^{2d}} \left\{ \left([\Delta_{t,s}^2(\xi) F_1(s, y_1, y_2)] \cdot D_{x_1} u_i(s, y_1, y_2) \right) \right. \\
&\quad \left. \times D_{x_2} \tilde{q}(t, x_1, x_2; s, y_1, y_2) \right\} dy_1 dy_2 ds \\
&\quad - \int_{t+\epsilon}^T \left\{ \int_{\mathbb{R}^{2d}} \left([\Delta_{t,s}^1(\xi) F_1(s, y_1, \theta_{t,s}^2(\xi))] \cdot [\Delta_{t,s}^2(\xi) D_{x_1} u_i(s, y_1, y_2)] \right) \right. \\
&\quad \left. \times D_{x_2} \tilde{q}(t, x_1, x_2; s, y_1, y_2) \right\} dy_1 dy_2 ds.
\end{aligned} \tag{4.7}$$

Proof of Claim 4.4: Start with (4.6):

$$\begin{aligned}
& H_i^2(t + \epsilon; t, x_1, x_2) \\
&= - \int_{t+\epsilon}^T \int_{\mathbb{R}^{2d}} \left\{ \left(\frac{1}{2} \text{Tr} [\Delta_{t,s}(\xi) a(s, y_1, y_2)] D_{x_1}^2 u_i(s, y_1, y_2) \right) \right. \\
&\quad \left. \times \tilde{q}(t, x_1, x_2; s, y_1, y_2) \right\} dy_1 dy_2 ds \\
&= - \int_{t+\epsilon}^T \int_{\mathbb{R}^{2d}} \left\{ \left(\frac{1}{2} \text{Tr} [\Delta_{t,s}^2(\xi) a(s, y_1, y_2)] D_{x_1}^2 u_i(s, y_1, y_2) \right) \right. \\
&\quad \left. \times \tilde{q}(t, x_1, x_2; s, y_1, y_2) \right\} dy_1 dy_2 ds \\
&\quad - \int_{t+\epsilon}^T \int_{\mathbb{R}^{2d}} \left\{ \left(\frac{1}{2} \text{Tr} [\Delta_{t,s}^1(\xi) a(s, y_1, \theta_{t,s}^2(\xi))] D_{x_1}^2 u_i(s, y_1, y_2) \right) \right. \\
&\quad \left. \times \tilde{q}(t, x_1, x_2; s, y_1, y_2) \right\} dy_1 dy_2 ds.
\end{aligned} \tag{4.8}$$

Now, we focus on the term (4.8): we know from **(H)** that the coefficient a is Lipschitz continuous, so that, it is $a.e$ differentiable. By an integration by parts argument we have:

$$\begin{aligned}
& - \int_{t+\epsilon}^T \int_{\mathbb{R}^{2d}} \frac{1}{2} \text{Tr} [\Delta_{t,s}^1(\xi) a(s, y_1, \theta_{t,s}^2(\xi))] D_{x_1}^2 u_i(s, y_1, y_2) \tilde{q}(t, x_1, x_2; s, y_1, y_2) dy_1 dy_2 ds \\
&= - \frac{1}{2} \sum_{l=1}^d \int_{t+\epsilon}^T \int_{\mathbb{R}^{2d}} \left[\frac{\partial}{\partial y_{1l}} \Delta_{t,s}^1(\xi) a_l(s, y_1, \theta_{t,s}^2(\xi)) \right] \cdot D_{x_1} u_i(s, y_1, y_2) \tilde{q}(t, x_1, x_2; s, y_1, y_2) dy_1 dy_2 ds \\
&\quad - \frac{1}{2} \sum_{l=1}^d \int_{t+\epsilon}^T \int_{\mathbb{R}^{2d}} [\Delta_{t,s}^1(\xi) a_l(s, y_1, \theta_{t,s}^2(\xi))] \cdot D_{x_1} u_i(s, y_1, y_2) \frac{\partial}{\partial y_{1l}} \tilde{q}(t, x_1, x_2; s, y_1, y_2) dy_1 dy_2 ds.
\end{aligned} \tag{4.9}$$

Note that, for all $l \in \{1, \dots, d\}$, $[\partial/\partial y_{1l}] \Delta_{t,s}^1(\xi) a_l(s, y_1, \theta_{t,s}^2(\xi)) = [\partial/\partial y_{1l}] a_l(s, y_1, \theta_{t,s}^2(\xi))$. One can center the two terms in the right hand side w.r.t the derivative $D_{x_1} u_i$ as follows:

$$\begin{aligned}
& - \int_{t+\epsilon}^T \int_{\mathbb{R}^{2d}} \left\{ \frac{1}{2} \text{Tr} \left[[\Delta_{t,s}^1(\xi) a(s, y_1, \theta_{t,s}^2(\xi))] D_{x_1}^2 u_i(s, y_1, y_2) \right] \right\} \\
& \quad \times \tilde{q}(t, x_1, x_2; s, y_1, y_2) \Big\} dy_1 dy_2 ds \\
& = - \frac{1}{2} \sum_{l=1}^d \int_{t+\epsilon}^T \int_{\mathbb{R}^{2d}} \left\{ \left[\frac{\partial}{\partial y_{1l}} a_l(s, y_1, \theta_{t,s}^2(\xi)) \right] \cdot [\Delta_{t,s}^2(\xi) D_{x_1} u_i(s, y_1, y_2)] \right. \\
& \quad \times \tilde{q}(t, x_1, x_2; s, y_1, y_2) \Big\} dy_1 dy_2 ds \\
& - \frac{1}{2} \sum_{l=1}^d \int_{t+\epsilon}^T \int_{\mathbb{R}^{2d}} \left\{ [\Delta_{t,s}^1(\xi) a_l(s, y_1, \theta_{t,s}^2(\xi))] \cdot [\Delta_{t,s}^2(\xi) D_{x_1} u_i(s, y_1, y_2)] \frac{\partial}{\partial y_{1l}} \right. \\
& \quad \times \tilde{q}(t, x_1, x_2; s, y_1, y_2) \Big\} dy_1 dy_2 ds \\
& - \frac{1}{2} \sum_{l=1}^d \int_{t+\epsilon}^T \int_{\mathbb{R}^{2d}} \left\{ \left[\frac{\partial}{\partial y_{1l}} a_l(s, y_1, \theta_{t,s}^2(\xi)) \right] \cdot D_{x_1} u_i(s, y_1, \theta_{t,s}^2(\xi)) \right. \\
& \quad \tilde{q}(t, x_1, x_2; s, y_1, y_2) \Big\} dy_1 dy_2 ds \\
& - \frac{1}{2} \sum_{l=1}^d \int_{t+\epsilon}^T \int_{\mathbb{R}^{2d}} \left\{ [\Delta_{t,s}^1(\xi) a_l(s, y_1, \theta_{t,s}^2(\xi))] \cdot D_{x_1} u_i(s, y_1, \theta_{t,s}^2(\xi)) \right. \\
& \quad \times \frac{\partial}{\partial y_{1l}} \tilde{q}(t, x_1, x_2; s, y_1, y_2) \Big\} dy_1 dy_2 ds.
\end{aligned}$$

Thanks to (3.9), after integrating w.r.t y_2 and differentiating w.r.t x_2 the two last terms in the right hand side are equal to 0. Thus, (4.6) follows.

Proof of (4.7). Note that H_i^3 can be written as:

$$\begin{aligned}
& H_i^3(t + \epsilon; t, x_1, x_2) \\
& = - \int_{t+\epsilon}^T \int_{\mathbb{R}^{2d}} [\Delta_{t,s}^2(\xi) F_1(s, y_1, y_2)] \cdot D_{x_1} u_i(s, y_1, y_2) \tilde{q}(t, x_1, x_2; s, y_1, y_2) dy_1 dy_2 ds \\
& - \int_{t+\epsilon}^T \int_{\mathbb{R}^{2d}} \left\{ [\Delta_{t,s}^1(\xi) F_1(s, y_1, \theta_{t,s}^2(\xi))] \cdot [\Delta_{t,s}^2(\xi) D_{x_1} u_i(s, y_1, y_2)] \right. \\
& \quad \times \tilde{q}(t, x_1, x_2; s, y_1, y_2) \Big\} dy_1 dy_2 ds \\
& - \int_{t+\epsilon}^T \int_{\mathbb{R}^{2d}} [\Delta_{t,s}^1(\xi) F_1(s, y_1, \theta_{t,s}^2(\xi))] \cdot D_{x_1} u_i(s, y_1, \theta_{t,s}^2(\xi)) \tilde{q}(t, x_1, x_2; s, y_1, y_2) dy_1 dy_2 ds.
\end{aligned}$$

From (3.9), by integrating w.r.t y_2 and differentiating w.r.t x_2 , we deduce that the last term in the right hand side is equal to 0 and (4.7) follows. This concludes the proof of Claim 4.4. \square

Proof of Lemma 4.2. It follows from the representation (4.1) and Lemma 3.2 that, for all $\epsilon > 0$:

$$D_{x_2} u_i(t, x_1, x_2) = \sum_{j=1}^4 D_{x_2} H_i^j(t + \epsilon; t, x_1, x_2) + O(\epsilon) \quad (4.10)$$

We bound each $D_{x_2} H_i^j$. We recall from Proposition 3.1 that there exists a positive constant C depending only on known parameters in **(H)** such that, for all s in $[t, T]$:

$$D_{x_2} \tilde{q}(t, x_1, x_2; s, y_1, y_2) \leq C(s-t)^{-3/2} \hat{q}_c(t, x_1, x_2; s, y_1, y_2). \quad (4.11)$$

Bound of $D_{x_2} H_i^1$. We have:

$$\begin{aligned} D_{x_2} H_i^1(t+\epsilon; t, x_1, x_2) &= \int_{t+\epsilon}^T \int_{\mathbb{R}^{2d}} \Delta_{t,s}^2(\xi) \phi_i(s, y_1, y_2) [D_{x_2} \tilde{q}(t, x_1, x_2; s, y_1, y_2)] dy_1 dy_2 ds \\ &\quad + \int_{t+\epsilon}^T D_{x_2} \left[\int_{\mathbb{R}^{2d}} \phi_i(s, y_1, \theta_{t,s}^2(\xi)) \tilde{q}(t, x_1, x_2; s, y_1, y_2) dy \right] ds, \end{aligned}$$

and by using (3.9) the last term is equal to 0. We deduce that:

$$\begin{aligned} D_{x_2} H_i^1(t+\epsilon; t, x_1, x_2) &= \int_{t+\epsilon}^T \int_{\mathbb{R}^{2d}} \Delta_{t,s}^2(\xi) \phi_i(s, y_1, y_2) [D_{x_2} \tilde{q}(t, x_1, x_2; s, y_1, y_2)] dy_1 dy_2 ds. \end{aligned} \quad (4.12)$$

So,

$$\begin{aligned} &|D_{x_2} H_i^1(t+\epsilon, x_1, x_2)| \\ &\leq C' \int_{t+\epsilon}^T \int_{\mathbb{R}^{2d}} (s-t)^{-3/2(1-\beta_i^2)} \left| \frac{\Delta_{t,s}^2(\xi) y_2}{(s-t)^{3/2}} \right|^{\beta_i^2} \hat{q}_c(t, x_1, x_2; s, y_1, y_2) dy_1 dy_2 ds \end{aligned}$$

By setting $\xi = x$, we have

$$|D_{x_2} H_i^1(t+\epsilon; t, x_1, x_2)| \leq C'' \int_{t+\epsilon}^T (s-t)^{-3/2(1-\beta_i^2)} ds. \quad (4.13)$$

By letting $\epsilon \rightarrow 0$, we obtain

$$\|D_{x_2} H_i^1\|_\infty \leq C''' T^{1/2(3\beta_i^2-1)}. \quad (4.14)$$

Bound of $D_{x_2} H_i^2$. Thanks to (4.6),

$$D_{x_2} H_i^2(t+\epsilon; t, x_1, x_2) \quad (4.15)$$

$$\begin{aligned} &= -\frac{1}{2} \int_{t+\epsilon}^T \int_{\mathbb{R}^{2d}} \text{Tr} [\Delta_{t,s}^2(\xi) a(s, y_1, y_2) D_{x_1}^2 u_i(s, y_1, y_2)] [D_{x_2} \tilde{q}(t, x_1, x_2; s, y_1, y_2)] dy_1 dy_2 ds \\ &\quad - \frac{1}{2} \sum_{l=1}^d \int_{t+\epsilon}^T \int_{\mathbb{R}^{2d}} \left\{ \left[\frac{\partial}{\partial y_{1l}} a_l(s, y_1, \theta_{t,s}^2(\xi)) \right] \cdot [\Delta_{t,s}^2(\xi) D_{x_1} u_i(s, y_1, y_2)] \right. \\ &\quad \left. \times [D_{x_2} \tilde{q}(t, x_1, x_2; s, y_1, y_2)] \right\} dy_1 dy_2 ds \\ &\quad + \frac{1}{2} \sum_{l=1}^d \int_{t+\epsilon}^T \int_{\mathbb{R}^{2d}} [\Delta_{t,s}^1(\xi) a_l(s, y_1, \theta_{t,s}^2(\xi))] \cdot [\Delta_{t,s}^2(\xi) D_{x_1} u_i(s, y_1, y_2)] \\ &\quad \times \left[D_{x_2} \left(\frac{\partial}{\partial y_{1l}} \tilde{q}(t, x_1, x_2; s, y_1, y_2) \right) \right] dy_1 dy_2 ds, \end{aligned} \quad (4.16)$$

and using Mean Value Theorem, we have:

$$|\Delta_{t,s}^2(\xi) D_{x_1} u_i(s, y_1, y_2)| \leq \|D_{x_1} D_{x_2} u_i\|_\infty |\Delta_{t,s}^2(\xi) y_2|. \quad (4.17)$$

Since from Proposition 3.1 we have: for all s in $[t, T]$, for all l in $\{1, \dots, d\}$

$$|D_{x_2}([\partial/\partial y_{1l}]\tilde{q}(t, x_1, x_2; s, y_1, y_2))| \leq C'(s-t)^{-2}\hat{q}_c(t, x_1, x_2; s, y_1, y_2).$$

It follows from the Lipschitz regularity of a and (4.17) that:

$$\begin{aligned} & |D_{x_2}H_i^2(t+\epsilon; t, x_1, x_2)| \\ & \leq C''' \left\{ \|D_{x_1}^2 u_i\|_\infty \int_{t+\epsilon}^T \int_{\mathbb{R}^{2d}} \left| \frac{\Delta_{t,s}^2(\xi)y_2}{(s-t)^{3/2}} \right| \hat{q}_c(t, x_1, x_2; s, y_1, y_2) dy_1 dy_2 ds \right. \\ & + \|D_{x_1} D_{x_2} u_i\|_\infty \int_{t+\epsilon}^T \int_{\mathbb{R}^{2d}} \left| \frac{\Delta_{t,s}^2(\xi)y_2}{(s-t)^{3/2}} \right| \hat{q}_c(t, x_1, x_2; s, y_1, y_2) dy_1 dy_2 ds \\ & \left. + \|D_{x_1} D_{x_2} u_i\|_\infty \int_{t+\epsilon}^T \int_{\mathbb{R}^{2d}} \left| \frac{\Delta_{t,s}^1(\xi)y_1}{(s-t)^{1/2}} \right| \left| \frac{\Delta_{t,s}^2(\xi)y_2}{(s-t)^{3/2}} \right| \hat{q}_c(t, x_1, x_2; s, y_1, y_2) dy_1 dy_2 ds \right\}. \end{aligned} \quad (4.18)$$

By setting $\xi = x$ and letting $\epsilon \rightarrow 0$ we deduce that:

$$\|D_{x_2}H_i^2\|_\infty \leq C'''T (\|D_{x_1} D_{x_2} u_i\|_\infty + \|D_{x_1}^2 u_i\|_\infty). \quad (4.19)$$

Bound of $D_{x_2}H_i^3$. From (4.7):

$$D_{x_2}H_i^3(t+\epsilon; t, x_1, x_2) \quad (4.20)$$

$$= - \int_{t+\epsilon}^T \int_{\mathbb{R}^{2d}} [\Delta_{t,s}^2(\xi)F_1(s, y_1, y_2)] \cdot D_{x_1} u_i(s, y_1, y_2) D_{x_2} \tilde{q}(t, x_1, x_2; s, y_1, y_2) dy_1 dy_2 ds$$

$$\begin{aligned} & - \int_{t+\epsilon}^T \int_{\mathbb{R}^{2d}} \left\{ [\Delta_{t,s}^1(\xi)F_1(s, y_1, \theta_{t,s}^2(\xi))] \cdot [\Delta_{t,s}^2(\xi)D_{x_1} u_i(s, y_1, y_2)] \right. \\ & \left. \times D_{x_2} \tilde{q}(t, x_1, x_2; s, y_1, y_2) \right\} dy_1 dy_2 ds. \end{aligned} \quad (4.21)$$

Then, by using the regularity of F_1 , (4.11) and (4.17), we have:

$$\begin{aligned} & |D_{x_2}H_i^3(t+\epsilon; t, x_1, x_2)| \quad (4.22) \\ & \leq C' \|D_{x_1} u_i\|_\infty \int_{t+\epsilon}^T \left\{ (s-t)^{-3/2(1-\beta_1^2)} \right. \\ & \quad \left. \times \int_{\mathbb{R}^{2d}} \left| \frac{\Delta_{t,s}^2(\xi)y_2}{(s-t)^{3/2}} \right|^{\beta_1^2} \hat{q}_c(t, x_1, x_2; s, y_1, y_2) \right\} dy_1 dy_2 ds \\ & + C' \|D_{x_1} D_{x_2} u_i\|_\infty \int_{t+\epsilon}^T \left\{ (s-t)^{\beta_1^{1/2}} \right. \\ & \quad \left. \times \int_{\mathbb{R}^{2d}} \left| \frac{\Delta_{t,s}^1(\xi)y_1}{(s-t)^{1/2}} \right|^{\beta_1^1} \left| \frac{\Delta_{t,s}^2(\xi)y_2}{(s-t)^{3/2}} \right| \hat{q}_c(t, x_1, x_2; s, y_1, y_2) \right\} dy_1 dy_2 ds. \end{aligned}$$

By setting $\xi = x$, we obtain

$$\begin{aligned} & |D_{x_2}H_i^3(t+\epsilon; t, x_1, x_2)| \\ & \leq C' \int_{t+\epsilon}^T \|D_{x_1} u_i\|_\infty (s-t)^{-3/2(1-\beta_1^2)} + \|D_{x_1} D_{x_2} u_i\|_\infty (s-t)^{\beta_1^{1/2}} ds, \end{aligned} \quad (4.23)$$

so that, by letting $\epsilon \rightarrow 0$ the following bound holds:

$$\|D_{x_2} H_i^3\|_\infty \leq C''' \left(T^{(3\beta_1^2-1)/2} + T^{1+\beta_1^2/2} \right) \|D_{x_1} D_{x_2} u_i\|_\infty. \quad (4.24)$$

Bound of $D_{x_2} H_i^4$. From (4.1),

$$\begin{aligned} & D_{x_2} H_i^4(t + \epsilon; t, x_1, x_2) \\ &= - \int_{t+\epsilon}^T \int_{\mathbb{R}^{2d}} [\Delta_{t,s}^1(\xi) F_2(s, y_1, \theta_{t,s}^2(\xi)) - D_{x_1} F_2(s, \theta_{t,s}^1(\xi), \theta_{t,s}^2(\xi)) \Delta_{t,s}^1(\xi) y_1] \\ & \quad \cdot [D_{x_2} u_i(s, y_1, y_2)] D_{x_2} \tilde{q}(t, x_1, x_2; s, y_1, y_2) dy_1 dy_2 ds \\ & - \int_{t+\epsilon}^T \int_{\mathbb{R}^{2d}} [\Delta_{t,s}^2(\xi) F_2(s, y_1, y_2)] \cdot D_{x_2} u_i(s, y_1, y_2) D_{x_2} \tilde{q}(t, x_1, x_2; s, y_1, y_2) dy_1 dy_2 ds. \end{aligned}$$

So that, by using the regularity of F_2 , (4.5) and (4.11), we have:

$$\begin{aligned} & |D_{x_2} H_i^4(t + \epsilon; t, x_1, x_2)| \\ & \leq C' \|D_{x_2} u_i\|_\infty \int_{t+\epsilon}^T (s-t)^{-1+\alpha^1/2} \int_{\mathbb{R}^{2d}} \left| \frac{\Delta_{t,s}^1(\xi) y_1}{(s-t)^{1/2}} \right|^{1+\alpha^1} \hat{q}_c(t, x_1, x_2; s, y_1, y_2) dy_1 dy_2 ds \\ & + C' \|D_{x_2} u_i\|_\infty \int_{t+\epsilon}^T (s-t)^{-3(1-\beta_2^2)/2} \int_{\mathbb{R}^{2d}} \left| \frac{\Delta_{t,s}^2(\xi) y_2}{(s-t)^{3/2}} \right|^{\beta_2^2} \hat{q}_c(t, x_1, x_2; s, y_1, y_2) dy_1 dy_2 ds. \end{aligned}$$

By setting $\xi = x$ and letting $\epsilon \rightarrow 0$, we obtain

$$\begin{aligned} & |D_{x_2} H_i^4(t, x_1, x_2)| \\ & \leq C''' \int_t^T \|D_{x_2} u_i\|_\infty \left((s-t)^{-1+\alpha^1/2} + (s-t)^{-3(1-\beta_2^2)/2} \right) ds, \end{aligned} \quad (4.25)$$

so that

$$\|D_{x_2} H_i^4\|_\infty \leq C''' \left(T^{(3\beta_2^2-1)/2} + T^{\alpha^1/2} \right) \|D_{x_2} u_i\|_\infty. \quad (4.26)$$

Combining (4.14), (4.19), (4.24), (4.26) and using estimates of $\|D_{x_1} u_i\|_\infty$ and $\|D_{x_1}^2 u_i\|_\infty$ from Lemma 4.1, we can find a non negative real $\delta_{4.2}$ such that, for T small enough:

$$\|D_{x_2} u_i\|_\infty \leq CT^{\delta_{4.2}} (1 + \|D_{x_1} D_{x_2} u_i\|_\infty),$$

where C and $\delta_{4.2}$ only depend on known parameters in **(H)**. This concludes the proof of Lemma 4.2. \square

4.4 Proof of Lemma 4.3

We first derive a representation formula for H_i^4 to handle the singularity of the derivative of the kernel \tilde{q} . This formula is given in the following claims and serves for the proof of Lemma 4.2.

Claim 4.5. For all $\epsilon > 0$, we have:

$$D_{x_2} H_i^4(t + \epsilon; t, x_1, x_2) = \tag{4.27}$$

$$\begin{aligned} & - \int_{t+\epsilon}^T \int_{\mathbb{R}^{2d}} \left([\Delta_{t,s}^1(\xi) F_2(s, y_1, \theta_{t,s}^2(\xi)) - D_{x_1} F_2(s, \theta_{t,s}^1(\xi), \theta_{t,s}^2(\xi)) \Delta_{t,s}^1(\xi) y_1] \right. \\ & \quad \cdot \Delta_{t,s}^2(\xi) D_{x_2} u_i(s, y_1, y_2) \left. \right) D_{x_2} \tilde{q}(t, x_1, x_2; s, y_1, y_2) dy_1 dy_2 ds \\ & - \int_{t+\epsilon}^T \int_{\mathbb{R}^{2d}} [\Delta_{t,s}^2(\xi) F_2(s, y_1, y_2)] \cdot D_{x_2} u_i(s, y_1, y_2) D_{x_2} \tilde{q}(t, x_1, x_2; s, y_1, y_2) dy_1 dy_2 ds, \end{aligned} \tag{4.28}$$

Proof of Claim 4.5: From (4.1) and Lemma 3.2, we can center the term $D_{x_2} H_i^4$ w.r.t the derivative $D_{x_2} u_i$ as follows:

$$\begin{aligned} & D_{x_2} H_i^4(t + \epsilon; t, x_1, x_2) \\ & = - \int_{t+\epsilon}^T \int_{\mathbb{R}^{2d}} [\Delta_{t,s}^1(\xi) F_2(s, y_1, \theta_{t,s}^2(\xi)) - D_{x_1} F_2(s, \theta_{t,s}^1(\xi), \theta_{t,s}^2(\xi)) \Delta_{t,s}^1(\xi) y_1] \\ & \quad \cdot [\Delta_{t,s}^2(\xi) D_{x_2} u_i(s, y_1, y_2)] D_{x_2} \tilde{q}(t, x_1, x_2; s, y_1, y_2) dy_1 dy_2 ds \\ & - D_{x_2} \left[\int_{t+\epsilon}^T \int_{\mathbb{R}^{2d}} [\Delta_{t,s}^1(\xi) F_2(s, y_1, \theta_{t,s}^2(\xi)) - D_{x_1} F_2(s, \theta_{t,s}^1(\xi), \theta_{t,s}^2(\xi)) \Delta_{t,s}^1(\xi) y_1] \right. \\ & \quad \cdot D_{x_2} u_i(s, y_1, \theta_{t,s}^2(\xi)) \tilde{q}(t, x_1, x_2; s, y_1, y_2) dy_1 dy_2 ds \left. \right] \\ & - \int_{t+\epsilon}^T \int_{\mathbb{R}^{2d}} [\Delta_{t,s}^2(\xi) F_2(s, y_1, y_2)] \cdot D_{x_2} u_i(s, y_1, y_2) D_{x_2} \tilde{q}(t, x_1, x_2; s, y_1, y_2) dy_1 dy_2 ds. \end{aligned}$$

By (3.9), the second term in the right hand side is equal to 0. This proves Claim 4.5. \square

In order to prove Lemma 4.3, we need to obtain an estimate on the regularity of $D_{x_2} u_i$ w.r.t. x_2 . This estimate is given in the following claim.

Claim 4.6. Under the assumptions of Lemma 4.2 for all $\gamma < 3 \inf\{\beta_1^2, \beta_2^2\} - 1$, the semi-Hölder norm of exponent $\gamma/3$ of $D_{x_2} u_i$ w.r.t x_2 is bounded uniformly in t and x_1 :

$$\|D_{x_2} u_i\|_{\infty, \infty, \gamma/3} := \sup_{w_1, w_2, w'_2 \in \mathbb{R}^d, t \in [0, T]} \frac{|D_{x_2} u_i(t, w_1, w_2) - D_{x_2} u_i(t, w_1, w'_2)|}{|w_2 - w'_2|^{\gamma/3}} \leq C_T$$

where C_T , depending only known parameters in (\mathbf{H}) and T , is small as T is small.

Proof of Claim 4.6: Let us first introduce the quantity:

$$M(D_{x_2} u_i, T) := \sup_{w_1, w_2, w'_2 \in \mathbb{R}^d, t \in [0, T]} \frac{|D_{x_2} u_i(t, w_1, w_2) - D_{x_2} u_i(t, w_1, w'_2)|}{|w_2 - w'_2|^{\gamma/3} + |w_2 - w'_2|^{\beta_2^2} + |w_2 - w'_2|^{\beta_1^2} + |w_2 - w'_2|}.$$

From (4.1) and Lemma 3.2, for all (t, x_1) in $[0, T] \times \mathbb{R}^d$ and (x_2, z_2) in $\mathbb{R}^d \times \mathbb{R}^d$ we have:

$$\begin{aligned} & |D_{x_2} u_i(t, x_1, x_2) - D_{x_2} u_i(t, x_1, z_2)| \\ & \leq \left| \sum_{j=1}^4 \left(D_{x_2} H_i^j \right) (t + \epsilon; t, x_1, x_2) - \left(D_{x_2} H_i^j \right) (t + \epsilon; t, x_1, z_2) \right| + O(\epsilon). \end{aligned} \tag{4.29}$$

We recall that $(H_i^j, j = 1, \dots, 4)$ depend on the freezing point $\xi = (\xi_1, \xi_2)$ of the process which started from x_1, x_2 and x_1, z_2 at time t . Here, we choose the same freezing point “ ξ ” for the two processes

(with different initial conditions). Let us note that, from (4.1) each $D_{x_2}H_i^j$ can be written as:

$$D_{x_2}H_i^j(t + \epsilon; t, x_1, x_2) = \int_{t+\epsilon}^T \int_{\mathbb{R}^{2d}} F_i^j(s, y_1, y_2, \theta_{t,s}^1(\xi), \theta_{t,s}^2(\xi)) D_{x_2} \tilde{q}(t, x_1, x_2; s, y_1, y_2) dy_1 dy_2 ds,$$

where F_i^j is some function properly defined by (4.1). We set $\mathcal{S} = \{s \in [t, T] \text{ s.t. } |x_2 - z_2| < (s - t)^{3/2}\}$ and $\mathcal{S}^c = \{s \in [t, T] \text{ s.t. } |x_2 - z_2| \geq (s - t)^{3/2}\}$. We have:

$$\begin{aligned} & \sum_{j=1}^4 \left(D_{x_2} H_i^j \right) (t + \epsilon; t, x_1, x_2) - \left(D_{x_2} H_i^j \right) (t + \epsilon; t, x_1, z_2) \\ &= \sum_{j=1}^4 \int_{t+\epsilon}^T \mathbf{1}_{\mathcal{S}} \int_{\mathbb{R}^{2d}} \left\{ F_i^j(s, y_1, y_2, \theta_{t,s}^1(\xi), \theta_{t,s}^2(\xi)) \right. \\ & \quad \left. \left(D_{x_2} \tilde{q}(t, x_1, x_2; s, y_1, y_2) - D_{x_2} \tilde{q}(t, x_1, z_2; s, y_1, y_2) \right) \right\} dy_1 dy_2 ds \\ & \quad + \sum_{j=1}^4 \int_{t+\epsilon}^T \mathbf{1}_{\mathcal{S}^c} \int_{\mathbb{R}^{2d}} \left\{ F_i^j(s, y_1, y_2, \theta_{t,s}^1(\xi), \theta_{t,s}^2(\xi)) \right. \\ & \quad \left. \left(D_{x_2} \tilde{q}(t, x_1, x_2; s, y_1, y_2) - D_{x_2} \tilde{q}(t, x_1, z_2; s, y_1, y_2) \right) \right\} dy_1 dy_2 ds \\ & := \sum_{j=1}^4 P_i^j(t + \epsilon; t, x; z, \mathcal{S}) + \sum_{j=1}^4 P_i^j(t + \epsilon; t, x; z, \mathcal{S}^c). \end{aligned} \quad (4.30)$$

As a first step, we bound the sum $\sum_{j=1}^4 P_i^j(t + \epsilon; t, x; z, \mathcal{S})$ in (4.30). We first prove that for all s in \mathcal{S} the following inequality holds:

$$\begin{aligned} & |D_{x_2} \tilde{q}(t, x_1, x_2; s, y_1, y_2) - D_{x_2} \tilde{q}(t, x_1, z_2; s, y_1, y_2)| \\ & \leq C(s - t)^{-(3+\gamma)/2} \hat{q}_c(t, x_1, x_2; s, y_1, y_2) |x_2 - z_2|^{\gamma/3}, \end{aligned} \quad (4.31)$$

where c and C depend only on known parameters in **(H)**.

By using Mean Value Theorem and the Gaussian estimate of $D_{x_2}^2 \tilde{q}$ from Proposition 3.1 we have:

$$\begin{aligned} & |D_{x_2} \tilde{q}(t, x_1, x_2; s, y_1, y_2) - D_{x_2} \tilde{q}(t, x_1, z_2; s, y_1, y_2)| \\ & \leq \sup_{\rho \in (0,1)} \left| D_{x_2}^2 \tilde{q}(t, x_1, x_2 + \rho(x_2 - z_2); s, y_1, y_2) \right| |x_2 - z_2| \\ & \leq C'(s - t)^{-3} \sup_{\rho \in (0,1)} \hat{q}_{\bar{c}}(t, x_1, x_2 + \rho(x_2 - z_2); s, y_1, y_2) |x_2 - z_2|, \end{aligned} \quad (4.32)$$

where \bar{c} is a positive constant depending only on known parameters in **(H)**. Note that on \mathcal{S} , the following inequality holds:

$$\sup_{\rho \in (0,1)} \hat{q}_{\bar{c}}(t, x_1, x_2 + \rho(x_2 - z_2); s, y_1, y_2) \leq C'' \hat{q}_c(t, x_1, x_2; s, y_1, y_2), \quad (4.33)$$

Combining (4.32) and (4.33), we obtain:

$$\begin{aligned} & |D_{x_2} \tilde{q}(t, x_1, x_2; s, y_1, y_2) - D_{x_2} \tilde{q}(t, x_1, z_2; s, y_1, y_2)| \\ & \leq C'''(s - t)^{-3} \hat{q}_c(t, x_1, x_2; s, y_1, y_2) |x_2 - z_2|, \end{aligned}$$

Let $0 < \gamma < 1$ and rewrite $|x_2 - z_2| = |x_2 - z_2|^{1-\gamma/3} |x_2 - z_2|^{\gamma/3}$. Since $|x_2 - z_2| < (s - t)^{3/2}$ we have $|x_2 - z_2| < (s - t)^{3/2-\gamma/2} |x_2 - z_2|^{\gamma/3}$ and (4.31) follows.

Bound of $\sum_{j=1}^3 \left| P_i^j(t + \epsilon; t, x; z, \mathcal{S}) \right|$. Following the proof of Lemma 4.2 and using (4.31) instead of (4.11) when bounding the terms (4.12), (4.15) and (4.20) and setting $\xi = x$ we deduce that

$$\begin{aligned} & \sum_{j=1}^3 \left| P_i^j(t + \epsilon; t, x; z, \mathcal{S}) \right| \\ & \leq C \int_{t+\epsilon}^T \left\{ (s-t)^{-(3+\gamma-3\beta_1^2)/2} + \|D_{x_1}^2 u_i\|_\infty (s-t)^{-\gamma/2} + \|D_{x_1} D_{x_2} u_i\|_\infty (s-t)^{-\gamma/2} \right. \\ & \quad \left. + \|D_{x_1} u_i\|_\infty (s-t)^{-(3+\gamma-3\beta_1^2)/2} + \|D_{x_1} D_{x_2} u_i\|_\infty (s-t)^{-(\gamma-\beta_1^1)/2} \right\} ds |x_2 - z_2|^{\gamma/3}, \end{aligned} \quad (4.34)$$

for all $\gamma < 3 \inf(\beta_1^2, \beta_2^2) - 1$.

Bound of $P_i^4(t + \epsilon; t, x; z, \mathcal{S})$. Thanks to (4.27), we have:

$$\begin{aligned} & P_i^4(t + \epsilon; t, x; z, \mathcal{S}) \\ & = \int_{t+\epsilon}^T \mathbf{1}_{\mathcal{S}} \int_{\mathbb{R}^{2d}} \left([\Delta_{t,s}^1(\xi) F_2(s, y_1, \theta_{t,s}^2(\xi)) - D_{x_1} F_2(s, \theta_{t,s}^1(\xi), \theta_{t,s}^2(\xi)) \Delta_{t,s}^1(\xi) y_1] \right. \\ & \quad \left. \cdot \Delta_{t,s}^2(\xi) D_{x_2} u_i(s, y_1, y_2) \right) [D_{x_2} \tilde{q}(t, x_1, x_2; t, y_1, y_2) - D_{x_2} \tilde{q}(t, x_1, z_2; t, y_1, y_2)] dy_1 dy_2 ds \\ & + \int_{t+\epsilon}^T \mathbf{1}_{\mathcal{S}} \int_{\mathbb{R}^{2d}} [\Delta_{t,s}^2(\xi) F_2(s, y_1, y_2)] \cdot D_{x_2} u_i(s, y_1, y_2) \\ & \quad \times [D_{x_2} \tilde{q}(t, x_1, x_2; t, y_1, y_2) - D_{x_2} \tilde{q}(t, x_1, z_2; t, y_1, y_2)] dy_1 dy_2 ds. \end{aligned} \quad (4.35)$$

From (4.5) and (4.31), we deduce that:

$$\begin{aligned} & |P_i^4(t + \epsilon; t, x; z, \mathcal{S})| \\ & \leq C' \int_{t+\epsilon}^T \mathbf{1}_{\mathcal{S}} \int_{\mathbb{R}^{2d}} \left\{ \|D_{x_2} u_i\|_\infty (s-t)^{3(\beta_2^2-1-\gamma/3)/2} \left| \frac{\Delta_{t,s}^2(\xi) y_2}{(s-t)^{3/2}} \right|^{\beta_2^2} \right. \\ & \quad + M(D_{x_2} u_i, T) (s-t)^{-1-\gamma/2+\alpha^1/2} \left| \frac{\Delta_{t,s}^1(\xi) y_1}{(s-t)^{1/2}} \right|^{1+\alpha^1} \left((s-t)^{\gamma/2} \left| \frac{y_2 - \theta_{t,s}^2(\xi)}{(s-t)^{3/2}} \right|^{\gamma/3} \right. \\ & \quad \left. + (s-t)^{3\beta_1^2/2} \left| \frac{y_2 - \theta_{t,s}^2(\xi)}{(s-t)^{3/2}} \right|^{\beta_1^2} + (s-t)^{3\beta_2^2/2} \left| \frac{y_2 - \theta_{t,s}^2(\xi)}{(s-t)^{3/2}} \right|^{\beta_2^2} \right. \\ & \quad \left. \left. + (s-t)^{3/2} \left| \frac{y_2 - \theta_{t,s}^2(\xi)}{(s-t)^{3/2}} \right| \right) \hat{q}_c(t, x_1, x_2; s, y_1, y_2) dy_1 dy_2 \right\} ds |x_2 - z_2|^{\gamma/3}, \end{aligned} \quad (4.36)$$

for all $\gamma < 3\beta_2^2 - 1$.

By setting $\xi = x$ in (4.36) and letting $\epsilon \rightarrow 0$ in (4.34) and (4.36), we deduce that there exist two positive constants C_T and C'_T depending only on known parameters in **(H)** and T , such that:

$$\left| \sum_{j=1}^4 P_i^j(t, x; z, \mathcal{S}) \right| \leq (C'_T M(D_{x_2} u_i, T) + C_T \|D_{x_1 x_2} u_i\|_\infty) |x_2 - z_2|^{\gamma/3}, \quad (4.37)$$

for all $\gamma < 3 \inf(\beta_1^2, \beta_2^2) - 1$ and where C_T and C'_T are small as T is small.

As a second step, we bound the sum $\sum_{j=1}^4 P_i^j(t + \epsilon; t, x; z, \mathcal{S}^c)$ in (4.30). Note that this sum reads:

$$\begin{aligned}
& \sum_{j=1}^4 P_i^j(t + \epsilon; t, x; z, \mathcal{S}^c) \\
&= \sum_{j=1}^4 \left\{ \int_{t+\epsilon}^T \mathbf{1}_{\mathcal{S}^c} \int_{\mathbb{R}^{2d}} F_i^j(s, y_1, y_2, \theta_{t,s}^1(\xi), \theta_{t,s}^2(\xi)) D_{x_2} \tilde{q}(t, x_1, x_2; s, y_1, y_2) dy_1 dy_2 ds \right. \\
&\quad \left. - \int_{t+\epsilon}^T \mathbf{1}_{\mathcal{S}^c} \int_{\mathbb{R}^{2d}} F_i^j(s, y_1, y_2, \theta_{t,s}^1(\xi), \theta_{t,s}^2(\xi)) D_{x_2} \tilde{q}(t, x_1, z_2; s, y_1, y_2) dy_1 dy_2 ds \right\} \\
&:= \sum_{j=1}^4 \{ \tilde{H}_i^j(t + \epsilon; t, x_1, x_2, \mathcal{S}^c) - \tilde{H}_i^j(t + \epsilon; t, x_1, z_2, \mathcal{S}^c) \}, \tag{4.38}
\end{aligned}$$

and that for all s in \mathcal{S}^c we have:

$$1 \leq (s - t)^{-\gamma/2} |x_2 - z_2|^{\gamma/3}. \tag{4.39}$$

On a first hand, by plugging (4.39) in (4.13), (4.18) and (4.22) in the proof of Lemma 4.2 and setting $\xi = x$ we obtain that:

$$\begin{aligned}
& \sum_{j=1}^3 \left| \tilde{H}_i^j(t + \epsilon; t, x_1, x_2, \mathcal{S}^c) \right| \tag{4.40} \\
&\leq C \int_{t+\epsilon}^T \left\{ (s - t)^{-3(1+\gamma/3-\beta_1^2)/2} + \|D_{x_1}^2 u_i\|_{\infty} (s - t)^{-\gamma/2} + \|D_{x_1} D_{x_2} u_i\|_{\infty} (s - t)^{-\gamma/2} \right. \\
&\quad \left. + \|D_{x_1} u_i\|_{\infty} (s - t)^{-3(1+\gamma/3-\beta_1^2)/2} + \|D_{x_1} D_{x_2} u_i\|_{\infty} (s - t)^{-(\gamma-\beta_1^2)/2} \right\} ds |x_2 - z_2|^{\gamma/3}
\end{aligned}$$

and plugging (4.39) in (4.27) and setting $\xi = x$, we deduce that

$$\begin{aligned}
& \left| \tilde{H}_i^4(t + \epsilon; t, x_1, x_2, \mathcal{S}^c) \right| \tag{4.41} \\
&\leq C \left\{ \int_{t+\epsilon}^T \|D_{x_2} u_i\|_{\infty} (s - t)^{3(\beta_2^2-1-\gamma/3)/2} ds |x_2 - z_2|^{\gamma/3} + CM(D_{x_2} u_i, T) \right. \\
&\quad \left. \times \int_{t+\epsilon}^T (s - t)^{-1+\alpha^1/2} \left((s - t)^{\gamma/2} + (s - t)^{3\beta_1^2/2} + (s - t)^{3\beta_2^2/2} + (s - t)^{3/2} \right) ds \right\} \\
&\quad \times |x_2 - z_2|^{\gamma/3}.
\end{aligned}$$

On a second hand, we have to deal with the terms $\tilde{H}_i^j(t + \epsilon; t, x_1, z_2, \mathcal{S}^c)$, for $j \in \{1, \dots, 4\}$. Since we take the same freezing point for the two solutions with different initial conditions, we have to re-center each integrand of $\tilde{H}_i^j(t + \epsilon; t, x_1, z_2, \mathcal{S}^c)$ in order to use the off-diagonal decay of the Gaussian exponential w.r.t the degenerate component. In this case, this off-diagonal decay is given by: $|y_2 - m_{t,s}^{2,\xi}(x_1, z_2)|$ for all s in $[t, T]$.

Bound of $\tilde{H}_i^1(t + \epsilon; t, x_1, z_2, \mathcal{S}^c)$. We have:

$$\begin{aligned}
& \tilde{H}_i^1(t + \epsilon; t, x_1, z_2, \mathcal{S}^c) \\
&= \int_{t+\epsilon}^T \int_{\mathbb{R}^{2d}} \mathbf{1}_{\mathcal{S}^c} \phi_i(s, y_1, y_2) D_{x_2} \tilde{q}(t, x_1, z_2; s, y_1, y_2) dy_1 dy_2 ds \\
&= \int_{t+\epsilon}^T \int_{\mathbb{R}^{2d}} \mathbf{1}_{\mathcal{S}^c} \left(\phi_i(s, y_1, y_2) - \phi_i(s, y_1, m_{t,s}^{2,\xi}(x_1, z_2)) \right) D_{x_2} \tilde{q}(t, x_1, z_2; s, y_1, y_2) dy_1 dy_2 ds \\
&+ \int_{t+\epsilon}^T \int_{\mathbb{R}^{2d}} \mathbf{1}_{\mathcal{S}^c} \phi_i(s, y_1, m_{t,s}^{2,\xi}(x_1, z_2)) D_{x_2} \tilde{q}(t, x_1, z_2; s, y_1, y_2) dy_1 dy_2 ds.
\end{aligned}$$

By using (3.9), we know that the last term in the right hand side is equal to 0, so that:

$$\begin{aligned}
& \left| \tilde{H}_i^1(t + \epsilon; t, x_1, z_2, \mathcal{S}^c) \right| \\
&\leq C \int_{t+\epsilon}^T \left\{ \mathbf{1}_{\mathcal{S}^c} (s-t)^{-3(1-\beta_i^2+\gamma/3)/2} \int_{\mathbb{R}^{2d}} \left| \frac{y_2 - m_{t,s}^{2,\xi}(x_1, z_2)}{(s-t)^{3/2}} \right|^{\beta_i^2} \right. \\
&\quad \left. \times \hat{q}_c(t, x_1, z_2; s, y_1, y_2) \right\} dy_1 dy_2 ds |x_2 - z_2|^{\gamma/3},
\end{aligned} \tag{4.42}$$

for all $\gamma < 3\beta_i^2 - 1$.

Bound of $\tilde{H}_i^2(t + \epsilon; t, x_1, z_2, \mathcal{S}^c)$. We can split this term as:

$$\begin{aligned}
& \tilde{H}_i^2(t + \epsilon; t, x_1, z_2, \mathcal{S}^c) \\
&= \frac{1}{2} \int_{t+\epsilon}^T \mathbf{1}_{\mathcal{S}^c} \int_{\mathbb{R}^{2d}} \text{Tr} \left\{ \left[\left[a(s, y_1, y_2) - a(s, y_1, m_{t,s}^{2,\xi}(x_1, z_2)) \right] D_{x_1}^2 u_i(s, y_1, y_2) \right] \right. \\
&\quad \left. \times [D_{x_2} \tilde{q}(t, x_1, z_2; s, y_1, y_2)] \right\} dy_1 dy_2 ds \\
&+ \frac{1}{2} \int_{t+\epsilon}^T \mathbf{1}_{\mathcal{S}^c} \int_{\mathbb{R}^{2d}} \left\{ \text{Tr} \left[\left[a(s, y_1, m_{t,s}^{2,\xi}(x_1, z_2)) - a(s, \theta_{t,s}^1(\xi), \theta_{t,s}^2(\xi)) \right] D_{x_1}^2 u_i(s, y_1, y_2) \right] \right. \\
&\quad \left. \times [D_{x_2} \tilde{q}(t, x_1, z_2; s, y_1, y_2)] \right\} dy_1 dy_2 ds.
\end{aligned}$$

By using the integration by parts argument (4.9) and a centering argument w.r.t. $D_{x_1} u_i$ on the last term in the right hand side above, we obtain:

$$\begin{aligned}
& \tilde{H}_i^2(t + \epsilon; t, x_1, z_2, \mathcal{S}^c) \\
&= \frac{1}{2} \int_{t+\epsilon}^T \mathbf{1}_{\mathcal{S}^c} \int_{\mathbb{R}^{2d}} \left\{ \text{Tr} \left[\left[a(s, y_1, y_2) - a(s, y_1, m_{t,s}^{2,\xi}(x_1, z_2)) \right] D_{x_1}^2 u_i(s, y_1, y_2) \right] \right. \\
&\quad \left. \times [D_{x_2} \tilde{q}(t, x_1, z_2; s, y_1, y_2)] \right\} dy_1 dy_2 ds \\
&+ \frac{1}{2} \sum_{l=1}^d \int_{t+\epsilon}^T \mathbf{1}_{\mathcal{S}^c} \int_{\mathbb{R}^{2d}} \left\{ \left(\left[\frac{\partial}{\partial y_{1l}} a_l(s, y_1, m_{t,s}^{2,\xi}(x_1, z_2)) \right] \right. \right. \\
&\quad \left. \left. [D_{x_1} u_i(s, y_1, y_2) - D_{x_1} u_i(s, y_1, m_{t,s}^{2,\xi}(x_1, z_2))] \right) [D_{x_2} \tilde{q}(t, x_1, z_2; s, y_1, y_2)] \right\} dy_1 dy_2 ds \\
&+ \frac{1}{2} \sum_{l=1}^d \int_{t+\epsilon}^T \mathbf{1}_{\mathcal{S}^c} \int_{\mathbb{R}^{2d}} \left\{ \left[a_l(s, y_1, m_{t,s}^{2,\xi}(x_1, z_2)) - a_l(s, \theta_{t,s}^1(\xi), \theta_{t,s}^2(\xi)) \right] \right. \\
&\quad \cdot [D_{x_1} u_i(s, y_1, y_2) - D_{x_1} u_i(s, y_1, m_{t,s}^{2,\xi}(x_1, z_2))] \\
&\quad \left. \times \left[D_{x_2} \left(\frac{\partial}{\partial y_{1l}} \tilde{q}(t, x_1, z_2; s, y_1, y_2) \right) \right] \right\} dy_1 dy_2 ds, \\
&+ \frac{1}{2} \sum_{l=1}^d \int_{t+\epsilon}^T \mathbf{1}_{\mathcal{S}^c} \int_{\mathbb{R}^{2d}} \left\{ \left[a_l(s, y_1, m_{t,s}^{2,\xi}(x_1, z_2)) - a_l(s, \theta_{t,s}^1(\xi), \theta_{t,s}^2(\xi)) \right] \right. \\
&\quad \left. + \frac{\partial}{\partial y_{1l}} a_l(s, y_1, m_{t,s}^{2,\xi}(x_1, z_2)) \right] \cdot [D_{x_1} u_i(s, y_1, m_{t,s}^{2,\xi}(x_1, z_2))] \\
&\quad \left. \times \left[D_{x_2} \left(\frac{\partial}{\partial y_{1l}} \tilde{q}(t, x_1, z_2; s, y_1, y_2) \right) \right] \right\} dy_1 dy_2 ds,
\end{aligned}$$

where the last term is equal to 0 from (3.9). From (4.39), (4.4) and Proposition 3.1 we have:

$$\begin{aligned}
& \left| \tilde{H}_i^2(t + \epsilon; t, x_1, z_2, \mathcal{S}^c) \right| \tag{4.43} \\
&\leq C \int_{t+\epsilon}^T \mathbf{1}_{\mathcal{S}^c} \int_{\mathbb{R}^{2d}} \left\{ \|D_{x_1^2} u_i\|_\infty (s-t)^{-\gamma/2} \left| \frac{y_2 - m_{t,s}^{2,\xi}(x_1, z_2)}{(s-t)^{3/2}} \right| \right. \\
&\quad + \|D_{x_1} D_{x_2} u_i\|_\infty (s-t)^{-\gamma/2} \left| \frac{y_2 - m_{t,s}^{2,\xi}(x_1, z_2)}{(s-t)^{3/2}} \right| \\
&\quad + (s-t)^{-\gamma/2} \|D_{x_1} D_{x_2} u_i\|_\infty \left| \frac{y_2 - m_{t,s}^{2,\xi}(x_1, z_2)}{(s-t)^{3/2}} \right| \left| \frac{y_1 - \theta_{t,s}^1(\xi)}{(s-t)^{1/2}} \right| \\
&\quad \left. \times \hat{q}_c(t, x_1, z_2; s, y_1, y_2) dy_1 dy_2 \right\} ds |x_2 - z_2|^{\gamma/3} \\
&+ C \|D_{x_1} D_{x_2} u_i\|_\infty \int_{t+\epsilon}^T \mathbf{1}_{\mathcal{S}^c} (s-t)^{-1/2} \int_{\mathbb{R}^{2d}} \left| \frac{y_2 - m_{t,s}^{2,\xi}(x_1, z_2)}{(s-t)^{3/2}} \right| \hat{q}_c(t, x_1, z_2; s, y_1, y_2) dy_1 dy_2 ds \\
&\quad \times |m_{t,s}^{2,\xi}(x_1, z_2) - \theta_{t,s}^2(\xi)|,
\end{aligned} \tag{4.44}$$

for all $\gamma < 2$.

Bound of $\tilde{H}_i^3(t + \epsilon; t, x_1, z_2, \mathcal{S}^c)$. From (4.7), this term can be centered w.r.t. the coefficients as follows:

$$\begin{aligned} & \tilde{H}_i^3(t + \epsilon; t, x_1, z_2, \mathcal{S}^c) \\ &= \int_{t+\epsilon}^T \mathbf{1}_{\mathcal{S}^c} \int_{\mathbb{R}^{2d}} \left\{ \left[F_1(s, y_1, y_2) - F_1(s, y_1, m_{t,s}^{2,\xi}(x_1, z_2)) \right] \cdot [D_{x_1} u_i(s, y_1, y_2)] \right. \\ & \quad \left. /quad \times D_{x_2} \tilde{q}(t, x_1, z_2; s, y_1, y_2) \right\} dy_1 dy_2 ds \\ & \quad + \int_{t+\epsilon}^T \mathbf{1}_{\mathcal{S}^c} \int_{\mathbb{R}^{2d}} \left[F_1(s, y_1, m_{t,s}^{2,\xi}(x_1, z_2)) - F_1(s, \theta_{t,s}^1(\xi)_1, \theta_{t,s}^2(\xi)) \right] \\ & \quad \cdot [D_{x_1} u_i(s, y_1, y_2)] D_{x_2} \tilde{q}(t, x_1, z_2; s, y_1, y_2) dy_1 dy_2 ds, \end{aligned}$$

and then, it can be centered w.r.t. $D_{x_1} u_i$ as follows:

$$\begin{aligned} & \tilde{H}_i^3(t + \epsilon; t, x_1, z_2, \mathcal{S}^c) \\ &= \int_{t+\epsilon}^T \mathbf{1}_{\mathcal{S}^c} \int_{\mathbb{R}^{2d}} \left[F_1(s, y_1, y_2) - F_1(s, y_1, m_{t,s}^{2,\xi}(x_1, z_2)) \right] \\ & \quad \cdot [D_{x_1} u_i(s, y_1, y_2)] D_{x_2} \tilde{q}(t, x_1, z_2; s, y_1, y_2) dy_1 dy_2 ds \\ & \quad + \int_{t+\epsilon}^T \mathbf{1}_{\mathcal{S}^c} \int_{\mathbb{R}^{2d}} \left[F_1(s, y_1, m_{t,s}^{2,\xi}(x_1, z_2)) - F_1(s, \theta_{t,s}^1(\xi), \theta_{t,s}^2(\xi)) \right] \\ & \quad \cdot \left[D_{x_1} u_i(s, y_1, y_2) - D_{x_1} u(s, y_1, m_{t,s}^{2,\xi}(x_1, z_2)) \right] D_{x_2} \tilde{q}(t, x_1, z_2; s, y_1, y_2) dy_1 dy_2 ds \\ & \quad + \int_{t+\epsilon}^T \mathbf{1}_{\mathcal{S}^c} \int_{\mathbb{R}^{2d}} \left[F_1(s, y_1, m_{t,s}^{2,\xi}(x_1, z_2)) - F_1(s, \theta_{t,s}^1(\xi), \theta_{t,s}^2(\xi)) \right] \\ & \quad \cdot \left[D_{x_1} u_i(s, y_1, m_{t,s}^{2,\xi}(x_1, z_2)) \right] D_{x_2} \tilde{q}(t, x_1, z_2; s, y_1, y_2) dy_1 dy_2 ds, \end{aligned}$$

where the last term is equal to 0 from (3.9). From (4.39), (4.4) and Proposition 3.1, we can deduce the following:

$$\left| \tilde{H}_i^3(t + \epsilon; t, x_1, z_2, \mathcal{S}^c) \right| \tag{4.45}$$

$$\begin{aligned} & \leq C \int_{t+\epsilon}^T \mathbf{1}_{\mathcal{S}^c} \int_{\mathbb{R}^{2d}} \left\{ \left\| D_{x_1} u_i \right\|_{\infty} (s-t)^{-3(1-\beta_1^2+\gamma/3)/2} \left| \frac{y_2 - m_{t,s}^{2,\xi}(x_1, z_2)}{(s-t)^{3/2}} \right|^{\beta_1^2} \right. \\ & \quad + \left\| D_{x_1} D_{x_2} u_i \right\|_{\infty} (s-t)^{(\beta_1^2-\gamma)/2} \left| \frac{y_2 - m_{t,s}^{2,\xi}(x_1, z_2)}{(s-t)^{3/2}} \right| \left| \frac{y_1 - \theta_{t,s}^1(\xi)}{(s-t)^{1/2}} \right|^{\beta_1^1} \\ & \quad \left. \times \hat{q}_c(t, x_1, z_2; s, y_1, y_2) \right\} dy_1 dy_2 ds |x_2 - z_2|^{\gamma/3} \\ & \quad + C \left\| D_{x_1} D_{x_2} u_i \right\|_{\infty} \int_{t+\epsilon}^T \mathbf{1}_{\mathcal{S}^c} \int_{\mathbb{R}^{2d}} \left| \frac{y_2 - m_{t,s}^{2,\xi}(x_1, z_2)}{(s-t)^{3/2}} \right| \hat{q}_c(t, x_1, z_2; s, y_1, y_2) dy_1 dy_2 ds \\ & \quad \times |m_{t,s}^{2,\xi}(x_1, z_2) - \theta_{t,s}^2(\xi)|^{\beta_1^2} \end{aligned} \tag{4.46}$$

for all $\gamma < 3\beta_1^2 - 1$.

Bound of $\tilde{H}_i^4(t + \epsilon; t, x_1, z_2, \mathcal{S}^c)$. From (4.27), this term can be written as:

$$\begin{aligned}
& \tilde{H}_i^4(t + \epsilon; t, x_1, z_2, \mathcal{S}^c) \\
&= \int_{t+\epsilon}^T \int_{\mathbb{R}^{2d}} \left\{ \mathbf{1}_{\mathcal{S}^c} \left[F_2(s, y_1, y_2) - F_2(s, y_1, m_{t,s}^{2,\xi}(x_1, z_2)) \right] \cdot [D_{x_2} u_i(s, y_1, y_2)] \right. \\
&\quad \left. \times D_{x_2} \tilde{q}(t, x_1, z_2; s, y_1, y_2) \right\} dy_1 dy_2 ds \\
&+ \int_{t+\epsilon}^T \int_{\mathbb{R}^{2d}} \mathbf{1}_{\mathcal{S}^c} \left[F_2(s, y_1, m_{t,s}^{2,\xi}(x_1, z_2)) - F_2(s, \theta_{t,s}^1(\xi), \theta_{t,s}^2(\xi)) \right. \\
&\quad \left. - D_{x_1} F_2(s, \theta_{t,s}^1(\xi), \theta_{t,s}^2(\xi)) \Delta_{t,s}^1(\xi) y_1 \right] \cdot [D_{x_2} u_i(s, y_1, y_2)] D_{x_2} \tilde{q}(t, x_1, z_2; s, y_1, y_2) dy_1 dy_2 ds,
\end{aligned}$$

It can be centered w.r.t. $D_{x_2} u_i$ as

$$\begin{aligned}
& \tilde{H}_i^4(t + \epsilon; t, x_1, z_2, \mathcal{S}^c) \\
&= \int_{t+\epsilon}^T \int_{\mathbb{R}^{2d}} \left\{ \mathbf{1}_{\mathcal{S}^c} \left[F_2(s, y_1, y_2) - F_2(s, y_1, m_{t,s}^{2,\xi}(x_1, z_2)) \right] \cdot [D_{x_2} u_i(s, y_1, y_2)] \right. \\
&\quad \left. \times D_{x_2} \tilde{q}(t, x_1, z_2; s, y_1, y_2) \right\} dy_1 dy_2 ds \\
&+ \int_{t+\epsilon}^T \int_{\mathbb{R}^{2d}} \left\{ \mathbf{1}_{\mathcal{S}^c} \left[F_2(s, y_1, m_{t,s}^{2,\xi}(x_1, z_2)) - F_2(s, \theta_{t,s}^1(\xi), \theta_{t,s}^2(\xi)) \right. \right. \\
&\quad \left. \left. - D_{x_1} F_2(s, \theta_{t,s}^1(\xi), \theta_{t,s}^2(\xi)) \Delta_{t,s}^1(\xi) y_1 \right] \cdot [D_{x_2} u_i(s, y_1, y_2) - D_{x_2} u_i(s, y_1, m_{t,s}^{2,\xi}(x_1, z_2))] \right. \\
&\quad \left. \times D_{x_2} \tilde{q}(t, x_1, z_2; s, y_1, y_2) \right\} dy_1 dy_2 ds \\
&+ \int_{t+\epsilon}^T \int_{\mathbb{R}^{2d}} \left\{ \mathbf{1}_{\mathcal{S}^c} \left[F_2(s, y_1, m_{t,s}^{2,\xi}(x_1, z_2)) - F_2(s, \theta_{t,s}^1(\xi), \theta_{t,s}^2(\xi)) \right. \right. \\
&\quad \left. \left. - D_{x_1} F_2(s, \theta_{t,s}^1(\xi), \theta_{t,s}^2(\xi)) \Delta_{t,s}^1(\xi) y_1 \right] \cdot [D_{x_2} u_i(s, y_1, m_{t,s}^{2,\xi}(x_1, z_2))] \right. \\
&\quad \left. \times D_{x_2} \tilde{q}(t, x_1, z_2; s, y_1, y_2) \right\} dy_1 dy_2 ds,
\end{aligned}$$

where the last term is equal to 0 from (3.9). By using (4.4), (4.39) and Proposition 3.1 we have:

$$\begin{aligned}
& \left| \tilde{H}_i^4(t + \epsilon; t, x_1, z_2, \mathcal{S}^c) \right| \tag{4.47} \\
& \leq C \|D_{x_2} u_i\|_\infty \int_{t+\epsilon}^T \left\{ \mathbf{1}_{\mathcal{S}^c}(s-t)^{-3(1-\beta_2^2+\gamma/3)/2} \right. \\
& \quad \times \left. \int_{\mathbb{R}^{2d}} \left| \frac{y_2 - m_{t,s}^{2,\xi}(x_1, z_2)}{(s-t)^{3/2}} \right|^{\beta_2^2} \hat{q}_c(t, x_1, z_2; s, y_1, y_2) dy_1 dy_2 \right\} ds |x_2 - z_2|^{\gamma/3} \\
& \quad + CM(D_{x_2} u_i, T) \int_{t+\epsilon}^T \mathbf{1}_{\mathcal{S}^c}(s-t)^{-1+\alpha_1/2} \int_{\mathbb{R}^{2d}} \left\{ \left| \frac{y_2 - m_{t,s}^{2,\xi}(x_1, z_2)}{(s-t)^{3/2}} \right|^{\gamma/3} |x_2 - z_2|^{\gamma/3} \right. \\
& \quad + \left| \frac{y_2 - m_{t,s}^{2,\xi}(x_1, z_2)}{(s-t)^{3/2}} \right|^{\beta_1^2} |x_2 - z_2|^{\beta_1^2} + \left| \frac{y_2 - m_{t,s}^{2,\xi}(x_1, z_2)}{(s-t)^{3/2}} \right|^{\beta_2^2} |x_2 - z_2|^{\beta_2^2} \\
& \quad \left. + \left| \frac{y_2 - m_{t,s}^{2,\xi}(x_1, z_2)}{(s-t)^{3/2}} \right| |x_2 - z_2| \right\} \times \hat{q}_c(t, x_1, z_2; s, y_1, y_2) dy_1 dy_2 ds,
\end{aligned}$$

for all $\gamma < 3\beta_1^2/2 - 1$.

Now, note that from (3.3),

$$m_{t,s}^{2,x}(x_1, z_2) - \theta_{t,s}^2(x) = z_2 - x_2.$$

Hence, by setting $\xi = x$ and letting $\epsilon \rightarrow 0$ in (4.42), (4.43) and (4.47) and combining the resulting estimates with (4.40) and (4.41), we deduce that there exist two positive constants C_T and C'_T depending only on known parameters in **(H)** and T , such that:

$$\begin{aligned}
\left| \sum_{j=1}^4 P_i^j(t, x; z, \mathcal{S}^c) \right| & \leq (C'_T M(D_{x_2} u_i, T) + C_T \|D_{x_1} D_{x_2} u_i\|_\infty) \tag{4.48} \\
& \quad \times \left(|x_2 - z_2|^{\gamma/3} + |x_2 - z_2|^{\beta_2^2} + |x_2 - z_2|^{\beta_1^2} + |x_2 - z_2| \right),
\end{aligned}$$

for all $\gamma < 3 \inf(\beta_1^2, \beta_2^2) - 1$ and where C_T and C'_T are small as T is small.

Finally, by plugging estimates (4.37) and (4.48) in (4.29), we deduce that there exists a positive constant C_T which is small as T is small such that:

$$\begin{aligned}
& |D_{x_2} u_i(t, x_1, x_2) - D_{x_2} u_i(t, x_1, z_2)| \\
& \leq C_T (1 + C'_T M(D_{x_2} u_i, T)) \left(|x_2 - z_2|^{\gamma/3} + |x_2 - z_2|^{\beta_2^2} + |x_2 - z_2|^{\beta_1^2} + |x_2 - z_2| \right).
\end{aligned}$$

Together with the boundedness of $D_{x_2} u_i$ from Lemma 4.2, this concludes the proof of Claim 4.6. \square

Proof of Lemma 4.3. It follows from the representation (4.1) and Lemma 3.2 that, for all $\epsilon > 0$:

$$D_{x_1} D_{x_2} u_i(t, x_1, x_2) = \sum_{j=1}^4 D_{x_1} D_{x_2} H_i^j(t + \epsilon; t, x_1, x_2) + O(\epsilon) \tag{4.49}$$

We bound each $D_{x_1} D_{x_2} H_i^j$. We recall that from Proposition 3.1, there exists a positive constant C depending only on known parameters in **(H)** such that:

$$D_{x_1} D_{x_2} \tilde{q}(t, x_1, x_2; s, y_1, y_2) \leq C(s-t)^{-2} D_{x_1} D_{x_2} \hat{q}_c(t, x_1, x_2; s, y_1, y_2). \tag{4.50}$$

Bound of $D_{x_1}D_{x_2}H_i^1$. From (4.12) we have:

$$\begin{aligned} & D_{x_1}D_{x_2}H_i^1(t + \epsilon; t, x_1, x_2) \\ &= \int_{t+\epsilon}^T \int_{\mathbb{R}^{2d}} \Delta_{t,s}^2(\xi) \phi_i(s, y_1, y_2) [D_{x_1}D_{x_2}\tilde{q}(t, x_1, x_2; s, y_1, y_2)] dy_1 dy_2 ds. \end{aligned}$$

So that,

$$\begin{aligned} & |D_{x_1}D_{x_2}H_i^1(t + \epsilon; t, x_1, x_2)| \\ & \leq C' \int_{t+\epsilon}^T \int_{\mathbb{R}^{2d}} (s-t)^{-2+3/2\beta_i^2} \left| \frac{\Delta_{t,s}^2(\xi)y_2}{(s-t)^{3/2}} \right|^{\beta_i^2} \hat{q}_c(t, x_1, x_2; s, y_1, y_2) dy_1 dy_2 ds \end{aligned}$$

Since $\beta_i^2 > 2/3$, $i = 1, 2$ from **(H1)**, by setting $\xi = x$ and letting $\epsilon \rightarrow 0$ we deduce that:

$$\|D_{x_1}D_{x_2}H_i^1\|_\infty \leq C'' T^{(3\beta_i^2/2-1)}. \quad (4.51)$$

Bound of $D_{x_1}D_{x_2}H_i^2$. Thanks to (4.6), we have

$$\begin{aligned} & D_{x_1}D_{x_2}H_i^2(t + \epsilon; t, x_1, x_2) \\ &= -\frac{1}{2} \int_{t+\epsilon}^T \int_{\mathbb{R}^{2d}} \left\{ \text{Tr} [[\Delta_{t,s}^2(\xi)a(s, y_1, y_2)]D_{x_1}^2 u_i(s, y_1, y_2)] \right. \\ & \quad \left. \times [D_{x_1}D_{x_2}\tilde{q}(t, x_1, x_2; s, y_1, y_2)] \right\} dy_1 dy_2 ds \\ & \quad - \frac{1}{2} \sum_{l=1}^d \int_{t+\epsilon}^T \int_{\mathbb{R}^{2d}} \left\{ \left[\frac{\partial}{\partial y_{1l}} a_l(s, y_1, \theta_{t,s}^2(\xi)) \right] \cdot [\Delta_{t,s}^2(\xi)D_{x_1} u_i(s, y_1, y_2)] \right. \\ & \quad \left. \times [D_{x_1}D_{x_2}\tilde{q}(t, x_1, x_2; s, y_1, y_2)] \right\} dy_1 dy_2 ds \\ & \quad + \frac{1}{2} \sum_{l=1}^d \int_{t+\epsilon}^T \int_{\mathbb{R}^{2d}} \left\{ [\Delta_{t,s}^1(\xi)a_l(s, y_1, \theta_{t,s}^2(\xi))] \right. \\ & \quad \cdot [\Delta_{t,s}^2(\xi)D_{x_1} u_i(s, y_1, y_2)] \\ & \quad \left. \times \left[D_{x_1}D_{x_2} \left(\frac{\partial}{\partial y_{1l}} \tilde{q}(t, x_1, x_2; s, y_1, y_2) \right) \right] \right\} dy_1 dy_2 ds. \end{aligned}$$

By using (4.18), we deduce that:

$$\begin{aligned} & |D_{x_1}D_{x_2}H_i^2(t + \epsilon; t, x_1, x_2)| \\ & \leq C' \left\{ \|D_{x_1}^2 u_i\|_\infty \int_{t+\epsilon}^T (s-t)^{-1/2} \int_{\mathbb{R}^{2d}} \left| \frac{\Delta_{t,s}^2(\xi)y_2}{(s-t)^{3/2}} \right| \hat{q}_c(t, x_1, x_2; s, y_1, y_2) dy_1 dy_2 ds \right. \\ & \quad + \|D_{x_1}D_{x_2} u_i\|_\infty \int_{t+\epsilon}^T (s-t)^{-1/2} \int_{\mathbb{R}^{2d}} \left| \frac{\Delta_{t,s}^2(\xi)y_2}{(s-t)^{3/2}} \right| \hat{q}_c(t, x_1, x_2; s, y_1, y_2) dy_1 dy_2 ds \\ & \quad + \|D_{x_1}D_{x_2} u_i\|_\infty \int_{t+\epsilon}^T (s-t)^{-1/2} \int_{\mathbb{R}^{2d}} \left\{ \left| \frac{\Delta_{t,s}^1(\xi)y_1}{(s-t)^{1/2}} \right| \left| \frac{\Delta_{t,s}^2(\xi)y_2}{(s-t)^{3/2}} \right| \right. \\ & \quad \left. \times \hat{q}_c(t, x_1, x_2; s, y_1, y_2) \right\} dy_1 dy_2 ds \left. \right\}. \end{aligned}$$

By setting $\xi = x$ and letting $\epsilon \rightarrow 0$ we obtain:

$$\|D_{x_1} D_{x_2} H_i^2\|_\infty \leq C'' T^{1/2} (\|D_{x_1} D_{x_2} u_i\|_\infty + \|D_{x_1}^2 u_i\|_\infty). \quad (4.52)$$

Bound of $D_{x_1} D_{x_2} H_i^3$. From (4.7):

$$\begin{aligned} & D_{x_1} D_{x_2} H_i^3(t + \epsilon; t, x_1, x_2) \\ &= - \int_{t+\epsilon}^T \int_{\mathbb{R}^{2d}} \left\{ [\Delta_{t,s}^2(\xi) F_1(s, y_1, y_2)] \cdot D_{x_1} u_i(s, y_1, y_2) \right. \\ &\quad \times [D_{x_1} D_{x_2} \tilde{q}(t, x_1, x_2; s, y_1, y_2)] \left. \right\} dy_1 dy_2 ds \\ &\quad - \int_{t+\epsilon}^T \int_{\mathbb{R}^{2d}} \left\{ [\Delta_{t,s}^1(\xi) F_1(s, y_1, \theta_{t,s}^2(\xi))] \cdot [\Delta_{t,s}^2(\xi) D_{x_1} u_i(s, y_1, y_2)] \right. \\ &\quad \times [D_{x_2} \tilde{q}(t, x_1, x_2; s, y_1, y_2)] \left. \right\} dy_1 dy_2 ds. \end{aligned}$$

By using (4.22) we deduce that:

$$\begin{aligned} & |D_{x_1} D_{x_2} H_i^3(t + \epsilon; t, x_1, x_2)| \\ &\leq C' \left\{ \|D_{x_1} u_i\|_\infty \int_{t+\epsilon}^T (s-t)^{-2+3\beta_1^2/2} \int_{\mathbb{R}^{2d}} \left| \frac{\Delta_{t,s}^2(\xi) y_2}{(s-t)^{3/2}} \right|^{\beta_1^2} \hat{q}_c(t, x_1, x_2; s, y_1, y_2) dy_1 dy_2 ds \right. \\ &\quad + \|D_{x_1} D_{x_2} u_i\|_\infty \int_{t+\epsilon}^T (s-t)^{(\beta_1^1-1)/2} \int_{\mathbb{R}^{2d}} \left\{ \left| \frac{\Delta_{t,s}^1(\xi) y_1}{(s-t)^{1/2}} \right|^{\beta_1^1} \left| \frac{\Delta_{t,s}^2(\xi) y_2}{(s-t)^{3/2}} \right| \right. \\ &\quad \times \hat{q}_c(t, x_1, x_2; s, y_1, y_2) \left. \right\} dy_1 dy_2 ds \left. \right\}. \end{aligned}$$

By setting $\xi = x$ and letting $\epsilon \rightarrow 0$ we obtain the following bound:

$$\|D_{x_1} D_{x_2} H_i^3\|_\infty \leq C'' \left(T^{3\beta_1^2/2-1} + T^{(1+\beta_1^1)/2} \right) \|D_{x_1} D_{x_2} u_i\|_\infty. \quad (4.53)$$

Bound of $D_{x_1} D_{x_2} H_i^4$. Thanks to (4.27), we have

$$\begin{aligned} & D_{x_1} D_{x_2} H_i^4(t + \epsilon; t, x_1, x_2) \\ &= - \int_{t+\epsilon}^T \int_{\mathbb{R}^{2d}} [\Delta_{t,s}^1(\xi) F_2(s, y_1, \theta_{t,s}^2(\xi)) - D_{x_1} F_2(s, \theta_{t,s}^1(\xi), \theta_{t,s}^2(\xi)) \Delta_{t,s}^1(\xi) y_1] \\ &\quad \cdot [\Delta_{t,s}^2(\xi) D_{x_2} u_i(s, y_1, y_2)] D_{x_1} D_{x_2} \tilde{q}(t, x_1, x_2; s, y_1, y_2) dy_1 dy_2 ds \\ &\quad - \int_{t+\epsilon}^T \int_{\mathbb{R}^{2d}} [\Delta_{t,s}^2(\xi) F_2(s, y_1, y_2)] \cdot D_{x_2} u_i(s, y_1, y_2) D_{x_1} D_{x_2} \tilde{q}(t, x_1, x_2; s, y_1, y_2) dy_1 dy_2 ds. \end{aligned}$$

By using Claim 4.6, the regularity of F_2 , (4.5) and (4.50), we obtain:

$$\begin{aligned}
& |D_{x_1} D_{x_2} H_i^4(t + \epsilon; t, x_1, x_2)| \\
& \leq C' \|D_{x_2} u_i\|_{\infty, \infty, \gamma/3} \int_{t+\epsilon}^T (s-t)^{(-3+\gamma+\alpha^1)/2} \int_{\mathbb{R}^{2d}} \left\{ \left| \frac{\Delta_{t,s}^2(\xi) y_2}{(s-t)^{3/2}} \right|^{\gamma/3} \right. \\
& \quad \left. \times \hat{q}_c(t, x_1, x_2; s, y_1, y_2) \right\} dy_1 dy_2 ds \\
& + C' \|D_{x_2} u_i\|_{\infty} \int_{t+\epsilon}^T (s-t)^{-2+3\beta_2^2/2} \int_{\mathbb{R}^{2d}} \left| \frac{\Delta_{t,s}^2(\xi) y_2}{(s-t)^{3/2}} \right|^{\beta_2^2} \hat{q}_c(t, x_1, x_2; s, y_1, y_2) dy_1 dy_2 ds.
\end{aligned}$$

Since this inequality holds for all $\gamma < 3 \inf(\beta_2^2, \beta_1^2) - 1$, γ can be chosen such that the first term in the right hand side is integrable⁴. Then, by letting $\xi = x$ and letting $\epsilon \rightarrow 0$ we deduce that:

$$\|D_{x_1} D_{x_2} H_i^4\|_{\infty} \leq C'' \left(T^{(3\beta_2^2/2-1)} + T^{(\alpha^1+\gamma-1)/2} \right) \|D_{x_2} u_i\|_{\infty}. \quad (4.54)$$

Combining (4.51), (4.52), (4.53), (4.54) and using estimates on $\|D_{x_1} u_i\|_{\infty}$, $\|D_{x_1}^2 u_i\|_{\infty}$ and $\|D_{x_2} u_i\|_{\infty}$ given in Lemma 4.1 and Lemma 4.2, we deduce that there exists a non negative real $\bar{\delta}_{4.3}$ depending only on known parameters in **(H)** such that, for T small enough:

$$\|D_{x_1} D_{x_2} u_i\|_{\infty} \leq CT^{\bar{\delta}_{4.3}}.$$

This concludes the proof of Lemma 4.3. □

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References

- [Bas98] R. F. Bass, *Diffusions and elliptic operators*, Probability and its Applications (New York), Springer-Verlag, New York, 1998.
- [Dav07] A. M. Davie, *Uniqueness of solutions of stochastic differential equations*, International Mathematics Research Notices. IMRN (2007), no. 24, Art. ID rnm124, 26.
- [DFP06] M. Di Francesco and S. Polidoro, *Schauder estimates, harnack inequality and gaussian lower bound for kolmogorov-type operators in non-divergence form*, Advances in Differential Equations **11** (2006), no. 11, 1261–1320.
- [DL89] R. J. DiPerna and P.-L. Lions, *Ordinary differential equations, transport theory and sobolev spaces*, Inventiones Mathematicae **98** (1989), no. 3, 511–547.
- [DM10] F. Delarue and S. Menozzi, *Density estimates for a random noise propagating through a chain of differential equations*, Journal of Functional Analysis **259** (2010), no. 6, 1577–1630.
- [FF11] E. Fedrizzi and F. Flandoli, *Pathwise uniqueness and continuous dependence of SDEs with non-regular drift*, Stochastics. An International Journal of Probability and Stochastic Processes **83** (2011), no. 3, 241–257.
- [Fla11] F. Flandoli, *Random perturbation of PDEs and fluid dynamic models*, Lecture Notes in Mathematics, vol. 2015, Springer, Heidelberg, 2011, Lectures from the 40th Probability Summer School held in Saint-Flour, 2010.
- [Fri64] A. Friedman, *Partial differential equations of parabolic type*, Prentice-Hall Inc., Englewood Cliffs, N.J., 1964.
- [Fri06] ———, *Stochastic differential equations and applications*, Dover Publications Inc., Mineola, NY, 2006, Two volumes bound as one, Reprint of the 1975 and 1976 original published in two volumes.
- [FS06] W. H. Fleming and H. M. Soner, *Controlled markov processes and viscosity solutions*, vol. 25, Springer New York, 2006.
- [Hö67] L. Hörmander, *Hypoelliptic second order differential equations*, Acta Mathematica **119** (1967), 147–171.
- [Kol34] A. Kolmogorov, *Zufällige bewegungen. (zur theorie der brownschen bewegung.)*, Ann. of Math., II. Ser. **35** (1934), 116–117.
- [KR05] N. V. Krylov and M. Röckner, *Strong solutions of stochastic equations with singular time dependent drift*, Probability Theory and Related Fields **131** (2005), no. 2, 154–196.

⁴Since β_i^2 is supposed to be strictly greater than 2/3 in **(H1)**.

- [Kun82] H. Kunita, *Stochastic differential equation and stochastic flows of diffeomorphisms*, Springer-Verlag, 1982.
- [Men11] S. Menozzi, *Parametric techniques and martingale problems for some degenerate kolmogorov equations*, Electronic Communications in Probability **16** (2011), 234–250.
- [Pro04] P.E. Protter, *Stochastic integration and differential equation*, Springer, 2004.
- [SV79] D. W. Stroock and S. R. S. Varadhan, *Multidimensional diffusion processes*, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 233, Springer-Verlag, Berlin, 1979.
- [Ver80] A. Ju. Veretennikov, *Strong solutions and explicit formulas for solutions of stochastic integral equations*, Matematicheski Sbornik. Novaya Seriya **111(153)** (1980), no. 3, 434–452, 480.
- [Ver83] A. Yu. Veretennikov, *Stochastic equations with diffusion that degenerates with respect to part of the variables*, Izvestiya Akademii Nauk SSSR. Seriya Matematicheskaya **47** (1983), no. 1, 189–196.
- [Zha05] X. Zhang, *Strong solutions of SDES with singular drift and sobolev diffusion coefficients*, Stochastic Processes and their Applications **115** (2005), no. 11, 1805–1818.
- [Zvo74] A. K. Zvonkin, *A transformation of the phase space of a diffusion process that will remove the drift*, Mat. Sb. (N.S.) **93(135)** (1974), 129–149, 152.

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