



HAL
open science

REGULARIZATION BY STOCHASTIC DRIFT

Paul-Eric Chaudru de Raynal

► **To cite this version:**

Paul-Eric Chaudru de Raynal. REGULARIZATION BY STOCHASTIC DRIFT. 2012. hal-00702532v1

HAL Id: hal-00702532

<https://hal.science/hal-00702532v1>

Preprint submitted on 30 May 2012 (v1), last revised 8 Mar 2017 (v4)

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

REGULARIZATION BY STOCHASTIC DRIFT

P.E. CHAUDRU de RAYNAL

May 30, 2012

Abstract

We here prove pathwise (strong) uniqueness for degenerate systems with Hölder drift, for Hölder exponent larger than the critical value $c_c = 2/3$. This work extends the one by Veretennikov [Ver80], Krylov and Röckner [KR05] and Flandoli [Fla11] from non-degenerate to degenerate cases. In comparison with, the non trivial value for c_c is here the price to pay to balance the degeneracy of the noise. The main tool for proving strong solvability relies on regularization property of the associated PDE, which is degenerated in the current framework.

Key words: Kolmogorov operator, Strong uniqueness, Hölder drift, Degeneracy, Stochastic Differential Equation.

1 Introduction

Given a real $T > 0$ and a function b from \mathbb{R} to \mathbb{R} , we consider the following deterministic system:

$$dX_t = b(t, X_t)dt \quad \text{on } (0, T], \quad \text{and } X_0 = x_0. \quad (1.1)$$

When b is at least Lipschitz-continuous, the Cauchy-Lipschitz Theorem provides strong existence and uniqueness for the system. When b is less than Lipschitz-continuous, strong uniqueness may be a real challenge. For example, under integrability condition on b , ∇b and $\text{div}(b)$, a famous work of DiPerna and Lions [DL89] shows that for almost every initial condition, there exists a unique flow that satisfies (1.1). Nevertheless, uniqueness does not hold pathwise (so called strong uniqueness). A possible way to regularize a deterministic system consists in adding a “microscopic” noise *i-e* by considering the stochastic system:

$$dX_t = b(t, X_t)dt + dW_t \quad \text{on } (0, T], \quad \text{and } X_0 = x_0, \quad (1.2)$$

defined on a filtered probability space $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t)_{0 \leq t \leq T})$ endowed with a Brownian motion $(W_t, 0 \leq t \leq T)$. The first work in that direction is due to A.N. Zvonkin. In [Zvo74], he showed that strong solvability holds for b in \mathbb{L}^∞ in the one dimensional case, by using a transformation that allows to get ride of the drift part. Since, strong solvability of stochastic system (1.2) has motivated many authors. Veretennikov [Ver80] generalized the result to the multidimensional case and N.V. Krylov and M. Röckner showed in [KR05] that pathwise existence and uniqueness hold for b in $\mathbb{L}_{\text{loc}}^p$, $p > d$. Then, X. Zhang (see [Zha05]) extended the result for multiplicative noise with Sobolev matrix diffusion and F. Flandoli and E. Fedrizzi gave in [FF11] another proof

of the result of N.V. Krylov and M. Röckner. All of these works rely on the deep connection between SDEs and PDEs (see [Bas98] or [Fri06] for a partial revue): the generator associated to the Markov process X is a linear partial differential operator of second order (usually denoted by L) with the transition density of X as fundamental solution. The proof of the strong solvability of the SDE then relies on regularization properties of parabolic (or elliptic) operators.

The result that we present here is in the same spirit of these works: let $F_1, F_2, \sigma : [0, T] \times \mathbb{R}^d \times \mathbb{R}^d \rightarrow \mathbb{R}^d \times \mathbb{R}^d \times \mathcal{M}_d(\mathbb{R})$ be measurable functions, and for all t in $[0, T]$ and s in $(t, T]$, define:

$$\begin{cases} dX_s^1 = F_1(s, X_s^1, X_s^2)ds + \sigma(s, X_s^1, X_s^2)dW_s, & X_t^1 = x_1, \\ dX_s^2 = F_2(s, X_s^1, X_s^2)ds, & X_t^2 = x_2. \end{cases} \quad (1.3)$$

This is the generalization of (1.1) where we add a *degenerate* noise, or a “macroscopic” noise if one focuses on the second component. We here show that strong solvability holds for (1.3) outside the Cauchy-Lipschitz framework: only suitable Hölder assumption on the drift coefficients are needed. Our approach relies on [Fla11] and, then, on the connection between SDE and PDE. The key is to obtain Lipschitz bounds on the solution of the associated PDE and on its derivative with respect to (w.r.t.) the non degenerate component.

The main issue is that the PDE associated to such a system is non-uniformly parabolic. In that case, the generator L of the process $(X_t^1, X_t^2)_{t \geq 0}$ is given by: for all ψ in $C^{1,2,1}([0, T] \times \mathbb{R}^d \times \mathbb{R}^d)^1$:

$$\begin{aligned} L\psi(t, x_1, x_2) &= \frac{1}{2}Tr(a(t, x_1, x_2))D_{x_1}^2\psi(t, x_1, x_2) + [F_1(t, x_1, x_2)] \cdot [D_{x_1}\psi(t, x_1, x_2)] \\ &\quad + [F_2(t, x_1, x_2)] \cdot [D_{x_2}\psi(t, x_1, x_2)]. \end{aligned} \quad (1.4)$$

where “ \cdot ” denotes the inner product and $a = \sigma\sigma^*$. A popular result, due to Hörmander [Hör67], says that such an operator admits a fundamental solution when the coefficients are smooth (say C^∞) and when the Lie algebra generated by the vector field spans the whole space. This is referred to as “Hypoellipticity”. In our case, the form of the degeneracy can be seen as a (particular) generalization of Kolmogorov degeneracy, in reference to the first work [Kol34] of Kolmogorov in that direction. Degenerate operators of the form $\mathcal{A} = \sum_{i=1}^{N_0} a_{ij}(x)\partial_{ij} + \sum_{i=1}^N b_i(x)\partial_{x_i}$, $N > N_0$ have been studied by many authors. When the degenerate part is linear (b_i linear for $i > N_0$), Di Francesco and Polidoro in [DFP06] obtain Schauder estimates and Harnack type inequality. Also, they succeed to give a Gaussian lower bound for the fundamental solution by using *parametrix*. Delarue and Menozzi consider in [DM10] a noise propagating in a n cascade of d -dimensional oscillators ($N_0 = d$, $N = nd$, $n > 1$). Under Lipschitz assumption on the drift and Hölder condition on the diffusion matrix, they show that the solution of such a system admits a density which is upper and lower bounded by a Gaussian like type bound. The approach is based on a truncated *parametrix* expansion of the transition density and related stochastic control problem. Thanks to their Aronson type control on their transition density, Menozzi [Men11] shows that the system admits a unique weak solution.

We prove that adding degenerate noise restores strong solvability. We emphasize that our assumptions ensure the existence of a (weak) solution of (1.3) (see [SV79]). Hence, strong solvability follows from strong uniqueness. As we discussed, the proof relies on the regularization

¹*i.e.* C^1 in time, C^2 for the diffusive component, and C^1 for the degenerate component.

properties of L . We investigate the PDE: $\partial_t u + Lu = \Phi$, $u_T = \mathbf{0}$, when the source term Φ has the same regularity as the drift. We show that under our assumptions, this PDE enjoys suitable regularization properties. Namely, the solution and its derivative w.r.t. the diffusive component are Lipschitz continuous. By using Itô's Formula with u , we get rid of the irregular drift coefficients that appear in (1.3). The resulting SDE involves the solution and its derivative. Thus, we recover the Lipschitz property and, then, strong uniqueness. Unfortunately, there is a price to pay to balance the degeneracy: Firstly, the function $x_2 \mapsto F(\cdot, \cdot, x_2)$ must be at least c_c -Hölder continuous, $c_c > 2/3$. Secondly, the function $x_1 \mapsto F_2(\cdot, x_1, \cdot)$ must be Lipschitz and its derivative $D_{x_1} F_2$ uniformly non degenerate. Finally, the drift coefficient F_1 is supposed to be Hölder continuous w.r.t. x_1 . We refer the reader to Section 3 for further discussion on the existence of a critical value c_c . The second assumption is quite natural. This allows the noise to propagate in the second component. That is, a sort of weak Hörmander condition. Further details can be found in Section 3. The third assumption is a direct consequence of our approach. In comparison with the works of Veretennikov [Ver80], Krylov and Röckner [KR05], and Flandoli and Fedrizzi [FF11], asking for F_1 to be in \mathbb{L}^p only might appear as the right framework. Unfortunately, we are not able to establish an \mathbb{L}^∞ estimate on the second order derivative of the solution w.r.t. the non degenerate component in this case. Also, the PDE admits only generalized solution, so that, Itô's Formula does not apply and one has to derive an Itô-Krylov type Formula. Consequently, we have to deal with Calderón Zygmund estimate on the solution and to handle the second order derivative by using Krylov inequality (see [Kry09]). Recently in [BZ11] M. Bramanti and M. Zhu exhibit \mathbb{L}^p estimates for general class of Hörmander vector field with drift. But, to the best of our knowledge, there is no Krylov inequality for such a degenerate process.

1.1 Overview

The natural generalization of such a result is its extension to the case of a random noise propagating through a chain of n differential equations:

$$\begin{aligned} dX_s^1 &= F_1(s, X_s^1, \dots, X_s^n)ds + \sigma(s, X_s^1, \dots, X_s^n)dW_s \\ dX_s^2 &= F_2(s, X_s^1, \dots, X_s^n)ds \\ dX_s^3 &= F_3(s, X_s^2, \dots, X_s^n)ds \\ &\vdots \\ dX_s^n &= F_n(s, X_s^{n-1}, X_s^n)ds \end{aligned}$$

where $(F_j)_{1 \leq j \leq n}$ are some measurable functions from $[0, T] \times \mathbb{R}^{(n-j+2)d}$ to \mathbb{R}^d . By using Gaussian estimates from [DM10], Menozzi shows in [Men11] that weak existence and uniqueness hold.

Nevertheless, we failed to generalize the result to this n -case. This again comes from the regularization properties of the associated generator: the classical way for studying these operators is the parametrix. This approach relies on the associated frozen system (that is the system with constant coefficients). The solution of the PDE can be seen as a time-space convolution of a perturbed kernel with the frozen density. In this case, the frozen system is Gaussian and its $nd \times nd$ covariance matrix is homogeneous to:

$$\begin{pmatrix} t^1/2\text{Id} & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & t^{3/2}\text{Id} & \cdots & \mathbf{0} \\ \vdots & \mathbf{0} & \ddots & \mathbf{0} \\ \mathbf{0} & \cdots & \mathbf{0} & t^{d-1/2}\text{Id} \end{pmatrix}.$$

Then, the differentiation of the frozen transition density over the j -th component generates a time-singularity of order $j - 1/2$, which is not integrable. Unfortunately, the regularity of the perturbed kernel does not prevent such a singularity.

1.1.1 Application

For example, Eq. (1.3) describes the dynamics of some Hamiltonian systems (see e.g. Soize [Soi94] for a general overview or the more specific works by Talay [Tal02] and Hérau and Nier [HN04] for questions of convergence to equilibrium). Also, it corresponds to the dynamics used in mathematical finance to price an Asian option (see for example [BPV01] for a specific discussion of the regularity of the price in such a degenerate case).

1.2 Organization of the paper

In Section 2, we state the detailed assumptions and the main Theorem of this paper: strong existence and uniqueness hold for the system (1.3). Then, we explain how the proof works and introduce the mathematical tools. Also, we give the regularisation properties of L . Thanks to these properties, we prove our main result. In Section 3, we illustrate the effect of the degeneracy by investigating the case when X is a Brownian motion. We also make some comments about our result and our assumptions. This study allows to understand how the degenerate system behaves, and how the analysis must be achieved. The remainder of the paper is dedicated to the investigation of the smoothing properties of L . This is done under regularization procedure, but the estimates are obtained uniformly. In Section 4 the main tools for studying these properties are presented: the parametrix and the linearization of the system play a central role. The proof is derived in Section 5. This is the technical part of this paper.

2 General setting and main result

2.1 Framework

Let T belongs to $\mathbb{R}^{+,*}$, t to $[0, T]$, and d to \mathbb{N}^* . Consider the following $\mathbb{R}^d \times \mathbb{R}^d$ system for any s in $(t, T]$:

$$\begin{cases} dX_s^1 = F_1(s, X_s^1, X_s^2)ds + \sigma(s, X_s^1, X_s^2)dW_s, & X_t^1 = x_1, \\ dX_s^2 = F_2(s, X_s^1, X_s^2)ds, & X_t^2 = x_2, \end{cases} \quad (2.1)$$

where $(W_t, t \geq 0)$ is a standard d -dimensional Brownian motion defined on some filtered probability space $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t)_{t \geq 0})$. F_1 , F_2 and σ are some measurable functions satisfying:

Hypotheses. (H1). For all $(t, x_1) \in [0, T] \times \mathbb{R}^d$, the function $F_2(t, \cdot, x_2) : x_1 \mapsto F_2(t, x_1, x_2)$ is continuously differentiable and there exist $0 < \beta_i^j < 1$, $1 \leq i, j \leq 2$, $0 < \alpha^i < 1$, $i = 1, 2$ and

three positive constants C_1, C_2, C_a such that for all (t, x_1, x_2) and (t, y_1, y_2) in $[0, T] \times \mathbb{R}^d \times \mathbb{R}^d$,

$$\begin{aligned} |F_1(t, x_1, x_2) - F_1(t, y_1, y_2)| &\leq C_1(|x_1 - y_1|^{\beta_1^1} + |x_2 - y_2|^{\beta_1^2}) \\ |F_2(t, x_1, x_2) - F_2(t, y_1, y_2)| &\leq C_2(|x_1 - y_1|^{\beta_2^1} + |x_2 - y_2|^{\beta_2^2}) \\ |D_{x_1}F_2(t, x_1, x_2) - D_{x_1}F_2(t, y_1, y_2)| &\leq C_2(|x_1 - y_1|^{\alpha^1} + |x_2 - y_2|^{\alpha^2}) \\ |a(t, x_1, x_2) - a(t, y_1, y_2)| &\leq C_a(|x_1 - y_1| + |x_2 - y_2|) \end{aligned}$$

and there are continuous time functions. Moreover, $\beta_2^1 = 1$, $\beta_i^2 > c_c$, $i = 1, 2$, where $c_c = 2/3$. The function σ also satisfies the uniform parabolic hypothesis:

$$\exists \Lambda > 1 \text{ such that } \forall \zeta \in \mathbb{R}^{2d}, \quad \Lambda^{-1}|\zeta|^2 \leq [\sigma\sigma^*(t, x_1, x_2)\zeta] \cdot \zeta \leq \Lambda|\zeta|^2,$$

for all $(t, x_1, x_2) \in [0, T] \times \mathbb{R}^d \times \mathbb{R}^d$, where “ $*$ ” stands for the transpose.

Finally, we also suppose the following assumption: there exists a closed convex subset $\mathcal{E} \subset \text{GL}_d(\mathbb{R})$ such that for all t in $[0, T]$ and (x_1, x_2) in \mathbb{R}^{2d} the matrix $D_{x_1}F_2(t, x_1, x_2)$ belongs to \mathcal{E} .

In the sequel, the sentence “known parameters in **H1**” refers to the parameters in this hypothesis.

Notations. Since the notation can become a bit heavy, we rewrite the system (1.3) in a shortened form:

$$dX_t = F(t, X_t)dt + B\sigma(t, X_t)dW_t, \quad (2.2)$$

where $X_t = (X_t^1, X_t^2)$, $F(t, X_t)$ is the \mathbb{R}^{2d} valued function $(F_1(t, X_t^1, X_t^2), F_2(t, X_t^1, X_t^2))^*$ and B is the $2d \times d$ matrix: $B = (\text{Id}, 0_{\mathbb{R}^d \times \mathbb{R}^d})^*$. Here, Id stands for the identity matrix of $\mathcal{M}_d(\mathbb{R})$, the set of real $d \times d$ matrix.

We sometimes denote by $g(t, X_t)$ the function $g(t, X_t^1, X_t^2)$ from $[0, T] \times \mathbb{R}^d \times \mathbb{R}^d$ to \mathbb{R}^{2d} . In the same way, we often denote by x, y or ξ the $2d$ -dimensional variable (x_1, x_2) , (y_1, y_2) and (ξ_1, ξ_2) . Consequently, each component of the d -dimensional variables x_k , $k = 1, 2$ are denoted by x_{kl} , $l = 1, \dots, d$. We denote by Φ (*resp.* ϕ) a measurable function from $[0, T] \times \mathbb{R}^d \times \mathbb{R}^d$ to \mathbb{R}^{2d} (*resp.* \mathbb{R}^2). Each d -dimensional component of this function is denoted by Φ_i (*resp.* ϕ_i), $i = 1, 2$. We emphasize that we often switch from one notation to another in the following.

The inner product is denoted by “ \cdot ”. In the sequel, we denote by C a positive constant, depending only on known parameters in **H1**, which may change from line to line and from one equation to another.

2.2 Main Result.

We have the following:

Theorem 2.1. *Let $T > 0$. Suppose that assumption **H1** is in force. Then, strong existence and uniqueness hold for the 2-system of d -dimensional SDEs:*

$$\begin{aligned} dX_s^1 &= F_1(s, X_s^1, X_s^2)ds + \sigma(s, X_s^1, X_s^2)dW_s, \\ dX_s^2 &= F_2(s, X_s^1, X_s^2)ds, \end{aligned} \quad (2.3)$$

for all s in $(t, T]$, any t in $[0, T]$, and for any initial condition (x_1, x_2) in \mathbb{R}^{2d} at time t .

2.3 Strategy of proof.

Existence of a weak solution is straightforward: it follows from the regularity of the coefficients (see [SV79]). Then, if strong uniqueness holds, strong existence follows. The main issue consists in proving strong uniqueness. As mentioned before, the strategy for proving strong uniqueness consists in investigating the regularization properties of L . This is done by assuming that the coefficients of (2.3) are smooth. Indeed, under **H1**, one can find a sequence of mollified coefficients $(a^n, F_1^n, F_2^n, \Phi_1^n, \Phi_2^n)_{n \geq 0}$ such that $(a^n, F_1^n, F_2^n, \Phi_1^n, \Phi_2^n)$ tends to $(a, F_1, F_2, \Phi_1, \Phi_2)$ uniformly on compact subsets of $[0, T] \times \mathbb{R}^d \times \mathbb{R}^d$. This is referred to as “regularization procedure” in the following. For all $n \geq 0$, we connect the degenerate system (2.1) with the systems of PDEs:

$$\begin{cases} \partial_t u_i^{(n)}(t, x) + L^{(n)} u_i^{(n)}(t, x) = \Phi_i^{(n)}(t, x), & \text{for } (t, x) \in [0, T] \times \mathbb{R}^{2d} \\ u_i^{(n)}(T, x) = 0_{\mathbb{R}^d}, & i = 1, 2, \end{cases} \quad (2.4)$$

where $L^{(n)}$ denotes the regularized version of L , and where the $(\Phi_i^{(n)})_{i=1,2}$ are the regularized versions of $(\Phi_i)_{i=1,2}$. In the whole paper, the functions $(\Phi_i)_{i=1,2}$ are supposed to be at least regular as F_1 in **H1**. We emphasize that the derivative w.r.t x_2 is only of order one. We have:

Theorem 2.2. *Let $T > 0$ be “small enough”. Suppose that assumption **H1** is in force. Then, for all $n \geq 0$, the linear systems (2.4) admit a unique solution $u^{(n)} = (u_1^{(n)}, u_2^{(n)})^* \in C^{1,2,1}([0, T] \times \mathbb{R}^{2d}, \mathbb{R}^{2d})$.*

Moreover, there exist two constants C_T and C'_T , only depending on T and on known parameters in **H1** such that, for all $n \geq 0$, the following inequalities hold: for all (x_1, y_1) and (x_2, y_2) in \mathbb{R}^{2d} ,

$$\sup_{t \in [0, T]} |u_i^{(n)}(t, x_1, x_2) - u_i^{(n)}(t, y_1, y_2)| \leq C_T (|x_1 - y_1| + |x_2 - y_2|), \quad (2.5)$$

$$\sup_{t \in [0, T]} |D_{x_1} u_i^{(n)}(t, x_1, x_2) - D_{x_1} u_i^{(n)}(t, y_1, y_2)| \leq C'_T (|x_1 - y_1| + |x_2 - y_2|), \quad (2.6)$$

for $i = 1, 2$ and where C_T and $C'_T \rightarrow 0$ when $T \rightarrow 0$.

The key point is that the estimates of the solution in Theorem 2.2 are obtained uniformly in n . The terminal condition in (2.4) is very important: it guarantees that the solution and its derivatives vanish at time T .

We now expose the basics arguments for proving our main result. Let the source term Φ be the drift term F . Thanks to Itô’s Formula, one can replace the drift of the stochastic process which solves (2.3) by the solution of (2.4). By splitting the interval $[0, T]$ on sufficiently small intervals, one can recover the Lipschitz property of the coefficients of the SDE on each small time-interval (see Theorem 2.2). By splitting the whole interval in a sufficiently small subdivision, one deduces strong uniqueness for (2.3).

Finally, the most technical part of the proof consists in showing smoothing properties of the generator L defined by (1.4). This is done in Section 4 by using the so-called *parametrix* approach (see [Fri06] for a partial revue).

2.4 Proof of the main result: application of Theorem 2.2 to the system (2.3)

In the following, “ $\mathbf{1}$ ” denotes the $2d \times 2d$ matrix:

$$\begin{pmatrix} \text{Id} & 0_{\mathbb{R}^d \times \mathbb{R}^d} \\ 0_{\mathbb{R}^d \times \mathbb{R}^d} & 0_{\mathbb{R}^d \times \mathbb{R}^d} \end{pmatrix}.$$

Let $(X_t, t \geq 0)$ and $(Y_t, t \geq 0)$ be two solutions of (2.3) for the same initial condition x_0 in \mathbb{R}^{2d} . Let $u^{(n)}$ be the solution of the linear systems of PDEs (2.4) with $\Phi^{(n)} = F^{(n)}$. Then, for T small enough, we know from Theorem 2.2 that one can apply Itô's Formula to both $u^{(n)}(t, X_t) - X_t$ and $u^{(n)}(t, Y_t) - Y_t$. One gets:

$$\begin{aligned} u^{(n)}(t, X_t) - X_t &= \int_0^t \left[\frac{\partial}{\partial t} u^{(n)} + L^{(n)} u^{(n)} \right] (s, X_s) ds - \int_0^t F(s, X_s) ds + u^{(n)}(0, x_0) - x_0 \\ &+ \int_0^t [D_x u^{(n)} - \mathbf{1}] B \sigma(s, X_s) dW_s, \end{aligned} \quad (2.7)$$

and

$$\begin{aligned} u^{(n)}(t, Y_t) - Y_t &= \int_0^t \left[\frac{\partial}{\partial t} u^{(n)} + L u^{(n)} \right] (s, Y_s) ds - \int_0^t F(s, Y_s) ds + u^{(n)}(0, x_0) - x_0 \\ &+ \int_0^t [D_x u^{(n)} - \mathbf{1}] B \sigma(s, Y_s) dW_s. \end{aligned} \quad (2.8)$$

Since $u^{(n)}$ is a solution of (2.4), the first term in the right hand side in the two equalities is equal to $F^{(n)}$. By taking the expectation of the supremum over t of the square norm of the difference between (2.7) and (2.8) we get:

$$\begin{aligned} \mathbb{E} \sup_{t \in [0, T]} |X_t - Y_t|^2 &\leq C \sup_{t \in [0, T]} \mathbb{E} |u^{(n)}(t, X_t) - u^{(n)}(t, Y_t)|^2 + CT \mathbb{E} \sup_{t \in [0, T]} |F^{(n)}(t, X_t) - F(t, X_t)|^2 \\ &+ CT \mathbb{E} \sup_{t \in [0, T]} |F^{(n)}(t, Y_t) - F(t, Y_t)|^2 \\ &+ C \mathbb{E} \int_0^T |[D_x u^{(n)} B - B](s, X_s) - [D_x u^{(n)} B - B](s, Y_s)|^2 |\sigma(s, Y_s)|^2 ds \\ &+ C \mathbb{E} \int_0^T \|D_x u^{(n)} B\|_\infty^2 |\sigma(s, Y_s) - \sigma(s, X_s)|^2 ds. \end{aligned}$$

Note that $D_x u^{(n)} B = (D_{x_1} u^{(n)}, 0_{\mathbb{R}^d \times \mathbb{R}^d})$ and that, for both Y_t and X_t ,

$$\mathbb{E} \sup_{t \in [0, T]} |F^{(n)}(t, X_t) - F(t, X_t)|^2 \rightarrow 0, \text{ as } n \rightarrow \infty.$$

So that, from estimates (2.5) and (2.6) of Theorem 2.2, and by letting n tends to the infinity, we deduce that:

$$\mathbb{E} \sup_{t \in [0, T]} |X_t - Y_t|^2 \leq C(T) \left\{ \mathbb{E} \sup_{t \in [0, T]} |X_t - Y_t|^2 + \mathbb{E} \int_0^T |X_s - Y_s|^2 ds \right\},$$

where $C(T) \rightarrow 0$ when $T \rightarrow 0$. Then, strong uniqueness holds for T small enough. By iterating this computation over a sufficiently small subdivision of the interval, the result follows.

Remark 1. *From now, we omit the superscript “(n)” that follows from the regularization procedure. We only work with smooth coefficients and show how to obtain the regularization properties of L depending only on known parameters in **H1**.*

3 The linear and Brownian heuristic.

This Section introduces the main issue when solving (2.3) in a simple case. Furthermore, it allows to understand some assumptions in **H1** and to present in a simple form the effect of the degeneracy.

Let us introduce the *Kolmogorov example*: in [Kol34] Kolmogorov showed that the solution of: $dY_t = \alpha W_t dt$, ($\alpha \neq 0$), admits a density, which is Gaussian and whose covariance matrix is given by:

$$\begin{pmatrix} t & (1/2)\alpha t^2 \\ (1/2)\alpha t^2 & (1/3)\alpha^2 t^3 \end{pmatrix}.$$

This simple example illustrates the behaviour of the system in small time. The diffusive coordinate oscillates with fluctuation of order $1/2$, while the degenerate one oscillates with fluctuation of order $3/2$. As a direct consequence, the transport of the initial condition of the first coordinate has a key role in the second one. This observation is crucial in the following.

Suppose now that $F_1 \equiv 0$, $\sigma \equiv \text{Id}$ and F_2 is homogeneous and linear w.r.t x_1 i-e $F_2(s, x_1, x_2) = \bar{F}_2(x_2) + \Gamma_s x_1$ for all s in $[t, T]$ and where Γ_s belongs to the set $\mathcal{M}_d(\mathbb{R})$ in (2.3). The SDE (2.3) becomes:

$$\begin{cases} dX_s^1 = dW_s, & X_t^1 = x_1, \\ dX_s^2 = (\bar{F}_2(X_s^2) + \Gamma_s X_s^1) ds, & X_t^2 = x_2, \end{cases} \quad (3.1)$$

for all s in $(t, T]$.

As we said, we investigate the regularization properties of the generator L of (3.1) by using the *parametrix* approach. It is a perturbation method. It consists of a Gaussian approximation of the fundamental solution of L by a McKean-Singer expansion (see [MS67]). Here, our approach is a first order expansion or, similarly, a variation of parameter approach. We approximate the original operator L by a Gaussian operator \tilde{L} for which the fundamental solution enjoys well known properties. This is done by considering a Gaussian stochastic system of generator \tilde{L} that approximates the system (3.1). In a uniform elliptic case, a classical way for deriving such a system consists in freezing the coefficients of the original SDE at the starting point. Here, as required by the degeneracy, the choice of the “freezing” point for the *parametrix* strategy must be done carefully. Indeed, the degeneracy of the second variable breaks down the diffusive time scale, so that the transport of the initial condition of the first component into the second one can not be neglected. Then, the freezing curve $\theta_{t,s} = (\theta_{t,s}^1, \theta_{t,s}^2)^*$, s in $(t, T]$ solves the ODE:

$$\frac{d}{ds} \theta_{t,s} = (0_{\mathbb{R}^d}, \bar{F}_2(\theta_{t,s}^2(\xi)) + \xi_1)^*, \quad \theta_{t,t}(\xi) = \xi,$$

for all ξ in \mathbb{R}^{2d} . It is of the implicit form:

$$\theta_{t,s}(\xi) = \int_t^s [U_r \xi + \bar{F}(\theta_{t,r}(\xi))] dr + \xi, \quad (3.2)$$

where,

$$U_s = \begin{pmatrix} 0_{\mathbb{R}^d \times \mathbb{R}^d} & 0_{\mathbb{R}^d \times \mathbb{R}^d} \\ \Gamma_s & 0_{\mathbb{R}^d \times \mathbb{R}^d} \end{pmatrix}. \quad (3.3)$$

The Gaussian frozen system is:

$$\begin{cases} d\bar{X}_s^1 = dW_s, & \bar{X}_t^1 = x_1, \\ d\bar{X}_s^2 = (\bar{F}_2(\theta_{t,s}^2(\xi)) + \bar{X}_s^1) ds, & \bar{X}_t^2 = x_2. \end{cases} \quad (3.4)$$

This is our candidate to approximate (3.1). In the reminder of this Section, we first illustrate the existence of the critical value c_c in **H1**. Then, we explain why the derivative of F_2 w.r.t x_1 is supposed to be in a convex set of $\text{GL}_d(\mathbb{R})$.

(i) Suppose that, for all s in $(t, T]$, $\Gamma_s \equiv \text{Id}$. We have:

Lemma 3.1. *The system (3.4) has a unique solution that admits a Gaussian transition density \bar{q} given by:*

$$\bar{q}(t, x_1, x_2; s, y_1, y_2) = \frac{1}{(2\pi)^{d/2}} \det(K_{s-t})^{-1/2} \exp\left(-|K_{s-t}^{-1/2}(y_1 - x_1, y_2 - m_{t,s}^{2,\xi}(x))^*|^2\right), \quad (3.5)$$

where $m_{t,s}^{2,\xi}(x) = x_2 + (s-t)x_1 + \int_t^s \bar{F}_2(\theta_{t,r}^2(\xi)) dr$ and where K_{s-t} is the uniformly non-degenerate matrix:

$$K_{s-t} := \begin{pmatrix} (s-t)\text{Id} & (1/2)(s-t)^2\text{Id} \\ (1/2)(s-t)^2\text{Id} & (1/3)(s-t)^3\text{Id} \end{pmatrix}. \quad (3.6)$$

Moreover, the transition kernel \bar{q} and its derivatives admit a Gaussian-type bound:

$$\begin{aligned} & \left| D_t^{N^t} D_{x_1}^{N^{x_1}} D_{x_2}^{N^{x_2}} D_{y_1}^{N^{y_1}} \bar{q}(t, x_1, x_2; s, y_1, y_2) \right| \\ & \leq (s-t)^{-[3(N^{x_2} + N^t) + N^{x_1} + N^{y_1}]/2} \frac{c}{(s-t)^{2d}} \exp\left(-c^{-1} |\mathbb{T}_{t,s}^{-1}(y_1 - x_1, y_2 - m_{t,s}^{2,\xi}(x))^*|^2\right), \end{aligned} \quad (3.7)$$

for any t in $[0, T]$, all s in $(t, T]$, any $N^t, N^{x_1}, N^{x_2}, N^{y_1}$ in \mathbb{N} and where c denotes a positive constant depending only on known parameters in **H1**. Here,

$$\mathbb{T}_{T-t}^{-1} = \begin{pmatrix} (T-t)^{-1/2}\text{Id} & 0_{\mathbb{R}^d \times \mathbb{R}^d} \\ 0_{\mathbb{R}^d \times \mathbb{R}^d} & (T-t)^{-3/2}\text{Id} \end{pmatrix}, \quad (3.8)$$

is called the time-scale matrix of the system (3.1) and gives the order of the fluctuation of each component.

The deterministic ODE associated with $\bar{X}^{t,x}$ has the form:

$$\frac{d}{dt} \bar{\psi}_t = (0_{\mathbb{R}^d}, \bar{F}_2(\theta_{t,s}^2(\xi)))^* + U_t \bar{\psi}_t, \quad (3.9)$$

where U_t is now the sub-diagonal identity matrix². The system (3.9) is the deterministic counterparts of (3.1), for which the mean of the Gaussian process is a solution. Note that, when

² U_t is given by: $U_t = \begin{pmatrix} 0_{\mathbb{R}^d \times \mathbb{R}^d} & 0_{\mathbb{R}^d \times \mathbb{R}^d} \\ \text{Id} & 0_{\mathbb{R}^d \times \mathbb{R}^d} \end{pmatrix}$.

the freezing point and the initial condition are the same (*i-e* when $\xi = x$), the frozen curve $(\theta_{t,s}(x))_{s \geq t}$ defines the flow of the ODE (3.9). So that, the freezing curve $(\theta_{t,s}(x))_{s \geq t}$ matches the mean of the Gaussian process \bar{X} , *i-e*: $\forall s \in [t, T], \forall x \in \mathbb{R}^{2d} : \theta_{t,s}(x) = m_{t,s}^x(x)$.

The transition density (3.5) is the fundamental solution of the Kolmogorov degenerate PDE: $\partial_t \bar{q}(t, x; T, y) + \mathcal{L}^K \bar{q}(t, x; T, y) = 0$, $\bar{q}(T, x; T, y) = \delta_y(x)$, where $\mathcal{L}^K = (1/2)\Delta_{x_1} + [\bar{F}_2(\theta_{t,T}^2(\xi)) + x_1] \cdot D_{x_2}$. Then, any solution of:

$$\frac{\partial}{\partial t} u(t, x) + \mathcal{L}^K u(t, x) = \phi(t, x), \quad (3.10)$$

writes:

$$u(t, x_1, x_2) = \int_t^T \int_{\mathbb{R}^{2d}} \phi(s, y_1, y_2) \bar{q}(t, x_1, x_2; s, y_1, y_2) dy_1 dy_2 ds. \quad (3.11)$$

As the proof of the Theorem 2.1 shows (see Subsection 2.4), one can replace the drift coefficients of $(X_s^{1,t,x}, X_s^{2,t,x}, s \geq t)$ by the solution of the PDE (3.10) with source term $\phi = (0_{\mathbb{R}^d}, \bar{F}_2)^*$. In order to obtain estimates of Theorem 2.2, one needs to bound the supremum norms of $D_{x_1} u$, $D_{x_2} u$, $D_{x_1}^2 u$ and $D_{x_1 x_2}^2 u$ independently of the regularization procedure. As representation (3.11) shows, the differentiability of u can be seen as the time-space convolution of the source term ϕ with the derivative of the fundamental solution of \mathcal{L}^K . Lemma 3.1 shows that such a differentiation gives a time-singularity, which is not always integrable. Moreover, it shows that the cross derivative $D_{x_1 x_2}^2$ generates the time-singularity of higher order, that is of order 2. It must be compensated by the regularity of \bar{F}_2 under **H1**:

$$\begin{aligned} & (s-t)^{-2} |\bar{F}_2(y_2) - \bar{F}_2(\theta_{t,s}^2(\xi))| \bar{q}(t, x; s, y) \\ & \leq C(s-t)^{-2+3\beta_2^2/2} \frac{|y_2 - \theta_{t,s}^2(\xi)|^{\beta_2^2}}{(s-t)^{3\beta_2^2/2}} \frac{c}{(s-t)^{2d}} \exp\left(-c^{-1} |\mathbb{T}_{t,s}^{-1}(y_1 - x_1, y_2 - m_{t,s}^{2,\xi}(x))^*|^2\right), \end{aligned}$$

from Lemma 3.1. By letting $\xi = x$, there exists a $\bar{C} > 0$ such that³

$$\frac{|y_2 - \theta_{t,s}^2(x)|^{\beta_2^2}}{(s-t)^{3\beta_2^2/2}} \times \exp\left(-c^{-1} |\mathbb{T}_{t,s}^{-1}(y_1 - x_1, y_2 - m_{t,s}^{2,x}(x))^*|^2\right) \leq \bar{C}.$$

Thus, by damaging the constant c in the exponential, one obtains:

$$\begin{aligned} & (s-t)^{-2} |\bar{F}_2(y_2) - \bar{F}_2(\theta_{t,s}^2(x))| \bar{q}(t, x; s, y) \\ & \leq C(s-t)^{-2+3\beta_2^2/2} \frac{c}{(s-t)^{2d}} \exp\left(-c^{-1} |\mathbb{T}_{t,s}^{-1}(y_1 - x_1, y_2 - m_{t,s}^{2,x}(x))^*|^2\right). \end{aligned}$$

Then, the value of β_2^2 must be such that $(s-t)^{-2+3\beta_2^2/2}$ is integrable, and one gets: $\beta_2^2 > 2/3 = c_c$. This is the reason of the existence of the critical value. From this discussion, one can also see the specific choice of the freezing curve as the one that matches the “off-diagonal” decay of the exponential in \bar{q} when $\xi = x$.

Proof of Lemma 3.1 One can write the second component as:

$$\bar{X}_s^2 = x_2 + (s-t)x_1 + \underbrace{\int_t^s \bar{F}_2(\theta_{t,r}^2(\xi)) dr}_{:=m_{t,s}^{2,\xi}(x)} + \int_t^T (T-s) dW_s. \quad (3.12)$$

³By using the inequality: $\forall q > 0, \exists \bar{C} > 0$ s.t. $\forall \sigma > 0, \sigma^q e^{-\sigma} \leq \bar{C}$.

We deduce that the $2d \times 2d$ covariance matrix of the process (\bar{X}^1, \bar{X}^2) is given by:

$$K_{s-t} := \begin{pmatrix} (s-t)\text{Id} & (1/2)(s-t)^2\text{Id} \\ (1/2)(s-t)^2\text{Id} & (1/3)(s-t)^3\text{Id} \end{pmatrix}.$$

Basic computation shows that $K_{s-t}^{-1} = \mathbb{T}_{s-t}^{-1} \hat{K}^{-1} \mathbb{T}_{s-t}^{-1}$, for all s in $(t, T]$, where:

$$\hat{K}^{-1} = \begin{pmatrix} 4 & -6 \\ -6 & 12 \end{pmatrix}.$$

As a generalization of the *Kolmogorov example* the Gaussian transition density is of the form, for all s in $(t, T]$:

$$\bar{q}(t, x_1, x_2; s, y_1, y_2) = \frac{1}{(2\pi)^{d/2}} \det(K_{s-t})^{-1/2} \exp\left(-|K_{s-t}^{-1/2}(y_1 - x_1, y_2 - m_{t,s}^{2,\xi}(x))^*|^2\right). \quad (3.13)$$

The remainder of the proof follows from easy computations. For details, we refer to the proof of Proposition 4.8 below in a more general case. \square

(ii) Let us now suppose that $\Gamma_s \neq \text{Id}$. The frozen system writes, in a shortened form:

$$d\bar{X}_s = [\bar{F}(\theta_{t,s}^2(\xi)) + U_s \bar{X}_s] ds + B dW_s, \quad (3.14)$$

for all s in $(t, T]$, with the initial condition $\bar{X}_t = x$, where $\bar{F} = (0_{\mathbb{R}^d}, \bar{F}_2)^*$, where U_s is given by (3.3) and where $(\theta_{t,s})_{t \leq s \leq T}$ is the forward curve (3.2). In this case, the crucial point is the specific form of the covariance matrix $\bar{\Sigma}_{t,s}$ of $\bar{X}_s^{t,x}$. As the proof of Lemma 3.1 above shows, it is given by:

$$\bar{\Sigma}_{t,s} = \begin{pmatrix} (s-t) & \int_t^s \bar{\Gamma}_{t,r} dr \\ \int_t^s \bar{\Gamma}_{t,r} dr & \int_t^s \int_t^r \bar{\Gamma}_{t,r} \bar{\Gamma}_{t,r}^* dr \end{pmatrix},$$

for all s in $(t, T]$ and where: $\bar{\Gamma}_{t,r} := \int_t^r \Gamma_u du$. Existence of a transition density of \bar{X} follows from the non-degeneracy of this matrix, this is what we investigate in the following.

Let φ be a function in the space $\mathbb{L}_2([0, T], \mathbb{R}^d)$, the deterministic counterpart of (3.14) is:

$$\frac{d}{dt} \bar{\psi}_t = (0_{\mathbb{R}^d}, \bar{F}_2(\theta_{t,T}^2(\xi)))^* + U_t \bar{\psi}_t + B \varphi_t, \quad (3.15)$$

that is, the controlled version of (3.9). In [DM10] (see Proposition 3.1), the authors show that controllability of the system above is equivalent to $\det(\bar{\Sigma}_{t,T}) > 0$. Also, they prove that $\det(\Gamma_s) > 0$ for *a.e* s in $[t, T]$ is a sufficient condition for having controllability of (3.15) and then, $\det(\bar{\Sigma}_{t,T}) > 0$. Then, the right choice of Γ seems to be Γ in the set of real $d \times d$ invertible matrix: $\text{GL}_d(\mathbb{R})$. Unfortunately, this condition is not sufficient: whereas it preserves the non-degeneracy of $\bar{\Sigma}_{t,T}$ for a given Γ , it does not ensure a uniform control in $\text{GL}_d(\mathbb{R})$. Let $t = 0$ and $T = 1$: in [DM10], the authors show that (see example 3.5 p.22) one can find a sequence of $(\Gamma_s^m)_{m \geq 0}$, $s \in [0, 1]$, such that $\det(\Gamma_s^m) = 1$ for all $m \geq 1$ and for which the variance of the second component converges towards 0 as $O(m^{-2})$. So that, $\det(\bar{\Sigma}_{0,1})$ vanishes although $\det(\Gamma_s^m) = 1$ is constant. This could be resume as follows: the mapping $U \in \mathbb{L}_2([0, 1], \mathcal{M}_{2d}(\mathbb{R})) \mapsto \bar{\Sigma}_{0,1}$ is continuous for the weak topology. This ensures the controllability of (3.15), so that, $\det(\bar{\Sigma}_{0,1}) > 0$. Nevertheless, as the example above shows, the set of invertible matrix is not closed for the weak topology, but the convex set \mathcal{E} in **H1** is.

4 Smoothing of the linear systems of PDEs

The regularization properties of (2.4) are the key of the proof. We also recall that, outside the regularization procedure, the Φ_i , $i = 1, 2$ in Theorem 2.2 are supposed to satisfy the same regularity as F_1 in **H1**. For the reader convenience, we study the scalar version of systems (2.4):

$$\begin{cases} \partial_t u_i(t, x) + Lu_i(t, x) = \phi_i(t, x), & \text{for } (t, x) \in [0, T] \times \mathbb{R}^{2d} \\ u_i(T, x) = 0, & i = 1, 2, \end{cases} \quad (4.1)$$

where $(\phi_i)_{i=1,2}$ are functions from $[0, T] \times \mathbb{R}^{2d}$ to \mathbb{R} , that is, ϕ_i is the scalar version of Φ_i . We give a preliminary version of Theorem 2.2:

Proposition 4.1. *There exists a unique solution $u = (u_1, u_2)^* \in C^{1,2,1}([0, T] \times \mathbb{R}^{2d}, \mathbb{R}^2)$ of the systems (4.1). Moreover, for T small enough, there exist two reals $\delta_{4.1}, \bar{\delta}_{4.1} > 0$ and two constants $C_{4.1}, \bar{C}_{4.1}$ depending only on known parameters in **H1** such that:*

$$\sup_{t \in [0, T]} |u_i(t, x_1, x_2) - u_i(t, y_1, y_2)| \leq C_{4.1} T^{\delta_{4.1}} (|x_1 - y_1| + |x_2 - y_2|), \quad (4.2)$$

$$\sup_{t \in [0, T]} |D_{x_1} u_i(t, x_1, x_2) - D_{x_1} u_i(t, y_1, y_2)| \leq C_{4.1} T^{\bar{\delta}_{4.1}} (|x_1 - y_1| + |x_2 - y_2|), \quad (4.3)$$

for $i = 1, 2$.

We emphasize that Theorem 2.2 easily follows from this Proposition. Indeed, each coordinate of the vectorial solution of the decoupled linear systems of PDEs (2.4) has the properties described above.

Strategy of proof: The proof of Proposition 4.1 is done in four steps:

Step 1: We first show that there exists a unique solution u of the linear systems (4.1), which is a C_b^∞ function. Existence and regularity are proved by adopting a viscosity solution approach. We propose a candidate u as:

$$u(t, x_1, x_2) = \mathbb{E} \int_t^T \phi(r, X_r^{1,t,x}, X_r^{2,t,x}) dr,$$

where the process (X^1, X^2) satisfies (2.3). Then, we show that u is a viscosity solution of the linear systems (4.1) and that it is a smooth function. Since a smooth viscosity solution is a classical solution, we get existence and regularity of a classical solution of (4.1). One concludes by proving uniqueness of the Feynman-Kac representation of u , thanks to Itô's Formula.

Step 2: As we explained in Section 3, we approximate the original operator L by a Gaussian operator \tilde{L} for which the fundamental solution enjoys well known properties. Then, we rewrite the systems of PDEs (4.1) as:

$$\begin{cases} \partial_t u_i(t, x) + \tilde{L}u_i(t, x) = (\tilde{L} - L)u_i(t, x) + \phi_i(t, x), & \text{for } (t, x) \in [0, T] \times \mathbb{R}^{2d} \\ u_i(T, x) = 0, & i = 1, 2. \end{cases}$$

The solution $u = (u_1, u_2)^*$ of these systems writes:

$$u_i(t, x) = \int_t^T \int_{\mathbb{R}^{2d}} (\tilde{L} - L) u_i(s, y) \tilde{q}(t, x; T, y) dy ds + \int_t^T \int_{\mathbb{R}^{2d}} \phi_i(s, y) \tilde{q}(t, x; T, y) dy ds, \quad (4.4)$$

for $i = 1, 2$. The parametrix consists in investigating the regularizations properties of the perturbed operator $(\tilde{L} - L)$. The aim of this step is to get the “good” operator \tilde{L} that approximates L . This is done by considering the frozen stochastic system of generator \tilde{L} .

We recover the Gaussian framework by a zero-order Taylor expansion of F_2 . The Brownian heuristic shows (see Section 3) that the linearization has to be done around the frozen forward curve that solves:

$$\frac{d}{dt} \theta_{t,s}(\xi) = F(s, \theta_{t,s}(\xi)), \quad \theta_{t,t}(\xi) = \xi.$$

We obtain the Gaussian SDE,

$$d\tilde{X}_s^{t,x} = F(s, \theta_{t,s}(\xi)) ds + \bar{B} D_x F(s, \theta_{t,s}(\xi)) (\tilde{X}_s^{t,x} - \theta_{t,s}(\xi)) ds + B \sigma(s, \theta_{t,s}(\xi)) dW_s, \quad (4.5)$$

where \bar{B} is the real $2d \times d$ matrix $(0_{\mathbb{R}^d \times \mathbb{R}^d}, Id)^*$. Then, we check the existence of a transition density of \tilde{X} . We denote by $\tilde{L}^{t,\xi}$ the generator⁴ of (4.5) and $\tilde{q}^{t,\xi}$ its fundamental solution. This is our candidate to approximate L .

Step 3: We focus on the properties of the frozen system generated by $\tilde{L}^{t,\xi}$. In Section 4.3, we give the explicit form of the frozen transition density $\tilde{q}^{t,\xi}$. From this form and discussion of Section 3, we deduce a Gaussian type bound with the “right-scales” for $\tilde{q}^{t,\xi}$ and its derivatives.

Step 4: This last step consists in investigating the smoothing properties of the perturbed operator $(\tilde{L} - L)$. We derive the Lipschitz estimates on the u_i , $i = 1, 2$, depending only on known parameters in **H1**. These estimates follow from the boundedness of the derivatives of the representation (4.4). This representation can be seen as an expectation of C^∞ functions (thanks to *step 1*), so that, it is differentiable uniformly in the freezing point ξ . Thanks to the smoothing properties of the perturbed kernel, by differentiating and by letting $\xi = x$ (the freezing point to be the started point), we deduce the Lipschitz bounds. The strategy is presented in Section 4.4 and we give the detailed proof in Section 5.

4.1 Regularity of the solution in smooth case

Here, we prove the following result:

Proposition 4.2. *There exists a unique C_b^∞ solution of (4.1) which is given by the function $u : [0, T] \times \mathbb{R}^{2d} \rightarrow \mathbb{R}^2$:*

$$u(t, x_1, x_2) = \mathbb{E} \int_t^T \phi(r, X_r^{1,t,x}, X_r^{2,t,x}) dr.$$

⁴Here, the superscript “ t, ξ ” stands for the dependence on the freezing point.

Proof of Proposition 4.2

Lemma 4.3. *let:*

$$u(t, x_1, x_2) = \mathbb{E} \left[\int_t^T \phi(s, X_s^{1,t,x}, X_s^{2,t,x}) ds \right]. \quad (4.6)$$

Then, u is a viscosity solution of (4.1).

Proof. First we have to prove the continuity of u . From regularity of the coefficients, there exists an a.s. continuous version of the process $(X_s^{1,t,x}, X_s^{2,t,x})_{s \geq t}$ (see [Kun86]). Since $(\phi_i)_{i=1,2}$ are smooth, continuity follows. We show now that u is both a sub and super viscosity solution: this follows from Theorem 5.1 p 69 of [FS06]. □

Lemma 4.4. *There exists a unique strong solution $X = (X^1, X^2)$ of the stochastic system:*

$$\begin{cases} dX_s^1 = F_1(s, X_s^1, X_s^2) ds + \sigma(s, X_s^1, X_s^2) dW_s, & X_t^1 = x_1, \\ dX_s^2 = F_2(s, X_s^1, X_s^2) ds, & X_t^2 = x_2, \end{cases}$$

$s \in [t, T]$. It is infinitely differentiable w.r.t x_i , $i = 1, 2$, and, for all k in \mathbb{N}^* , for all $(i_1, \dots, i_k) \in \{1, 2\}^k$ the process $D_{x_{i_1}, \dots, x_{i_k}}^k X_s^{t,x}$ satisfies:

$$\mathbb{E} \left[\sup_{s \in [t, T]} \left| D_{x_{i_1}, \dots, x_{i_k}}^k X_s^{t,x} \right| \right] \leq K.$$

Proof. The result follows from the regularity of the coefficients, from [Kun86] and from Theorem 70 of [Pro04]. □

Lemma 4.5. *Let f be a C_b^∞ function from \mathbb{R}^{2d} to \mathbb{R} . Let $v(t, x) = \mathbb{E}f(X_T^{t,x})$, where $X_T^{t,x}$ is a solution of (2.1). Then, for any $i = 1, 2$, $D_{x_i}v(t, x)$ exists, is continuous, and:*

$$D_{x_i}v(t, x) = \mathbb{E} \left[\sum_{j=1}^2 D_{x_j} f(X_T^{t,x}) D_{x_i} X_T^{j,t,x} \right]. \quad (4.7)$$

Proof. Thanks to Kolmogorov criterion, there exists a a.s. continuous version of the process which solves (2.1). Moreover, it follows from Lemma 4.4 that, for all t in $[0, T]$, for all s in $[t, T]$, the mapping $X_s^{t,\cdot} : x \mapsto X_s^{t,x}$ is a.s. continuously differentiable. Since f is Lipschitz, f always satisfies the domination property:

$$|f(X_T^{t,x}) - f(X_T^{t,z})| \leq K|x - z|,$$

where K is a random constant with finite moment of all order by Kolmogorov Theorem. One can apply the Lebesgue differentiation Theorem so that $D_{x_i}v(t, x)$, $i = 1, 2$ exists and satisfies (4.7). □

Lemma 4.6. *Let u be the function defined in Lemma 4.3. Then u is $C^{1,2,1}$ and:*

$$\frac{\partial u}{\partial t}(t, x_1, x_2) = -Lu(t, x_1, x_2) + \phi(t, x_1, x_2).$$

Proof. We know that $(X_s^{1,t,x}, X_s^{2,t,x})$ is continuous w.r.t t (see Lemma 4.6.1 of [Kun86]). So that for all x in \mathbb{R}^{2d} , the function $u(\cdot, x_1, x_2) : t \in [0, T] \mapsto u(t, x_1, x_2)$ is continuous. Using the Markov property we deduce that, for all $0 < h < t$:

$$u(t-h, x_1, x_2) = \mathbb{E}u(t, X_t^{1,t-h,x}, X_t^{2,t-h,x}) - \mathbb{E} \int_{t-h}^t \phi(s, X_s^{1,t-h,x}, X_s^{2,t-h,x}) ds.$$

By iterating Lemma 4.5, one deduces that u is two times differentiable w.r.t the space variables. Combing with Lemma 4.4, one deduces that the space derivatives of u are bounded. Then, applying Itô’s Formula on $\mathbb{R}^d \times \mathbb{R}^d$ to $u(t, X_t^{1,t-h,x}, X_t^{2,t-h,x})$ and taking the expectation lead to:

$$\begin{aligned} \mathbb{E}u(t, X_t^{1,t-h,x}, X_t^{2,t-h,x}) &= u(t, x_1, x_2) + \mathbb{E} \int_{t-h}^t Lu(t, X_r^{1,t-h,x}, X_r^{2,t-h,x}) dr \\ &\quad + \mathbb{E} \int_{t-h}^t D_x u(t, X_r^{1,t-h,x}, X_r^{2,t-h,x}) B\sigma(r, X_r^{1,t-h,x}, X_r^{2,t-h,x}) dW_r, \end{aligned}$$

where the last term in the right hand side is equal to 0. Then:

$$\frac{u(t-h, x_1, x_2) - u(t, x_1, x_2)}{h} = \frac{1}{h} \mathbb{E} \int_{t-h}^t [Lu(t, X_r^{1,t-h,x}, X_r^{2,t-h,x}) + \phi(r, X_r^{1,t-h,x}, X_r^{2,t-h,x})] dr.$$

By the continuity of u w.r.t t and by letting h tends to 0 we deduce that:

$$\frac{\partial u}{\partial t}(t, x_1, x_2) = -Lu(t, x_1, x_2) + \phi(t, x_1, x_2).$$

□

Then, by iterating Lemma 4.5 and by using the boundedness of the tangent process at every order from Lemma 4.4, we deduce that the function u defined in (4.6) is C_b^∞ for all t in $[0, T]$. Combining Lemmas 4.3 and 4.6 and existence of a smooth classical solution of (4.1) follows. Let u be such a solution, by applying Itô’s Formula on $u(T, X_T^{1,t,x_1}, X_T^{2,t,x_2})$, where $(X_T^{1,t,x_1}, X_T^{2,t,x_2})$ is a solution of the SDE (2.1) such that $(X_t^{1,t,x_1}, X_t^{2,t,x_2}) = (x_1, x_2)$ a.s., and taking the expectation we have:

$$u(t, x_1, x_2) = \mathbb{E} \left[\int_t^T \phi(s, X_s^{1,t,x}, X_s^{2,t,x}) ds \right],$$

and then, uniqueness follows. This conclude the proof of Proposition 4.2. □

4.2 Linearized frozen system and “Parametrix”

4.2.1 Linearized frozen system

Here, we give the Gaussian frozen system that approximates (2.3). We refer the reader to the discussion in Section 3 or to the beginning of Section 4 for further details.

Let x_1, x_2 belong to \mathbb{R}^d and t to $[0, T]$. Recall that we are interested in the following non-linear and degenerate system of SDEs:

$$\begin{cases} dX_s^{1,t,x} = F_1(s, X_s^{1,t,x}, X_s^{2,t,x}) ds + \sigma(s, X_s^{1,t,x}, X_s^{2,t,x}) dW_s, & X_t^{1,t,x} = x_1, \\ dX_s^{2,t,x} = F_2(s, X_s^{1,t,x}, X_s^{2,t,x}) ds, & X_t^{2,t,x} = x_2, \end{cases} \quad (4.8)$$

for all s in $(t, T]$.

We must linearize the system around some well chosen “freezing curve”. This forward transport function solves:

$$\frac{d}{ds}\theta_{t,s}(\xi) = F(s, \theta_{t,s}(\xi)), \quad \theta_{t,t}(\xi) = \xi. \quad (4.9)$$

The linearized system is:

$$\begin{cases} d\tilde{X}_s^{1,t,x} = F_1(s, \theta_{t,s}(\xi))ds + \sigma(s, \theta_{t,s}(\xi))dW_s, \\ d\tilde{X}_s^{2,t,x} = F_2(s, \theta_{t,s}(\xi))ds + D_{x_1}F_2(s, \theta_{t,s}(\xi))(\tilde{X}_s^{1,t,x} - \theta_{t,s}^1(\xi))ds. \end{cases} \quad (4.10)$$

The deterministic counterparts of (4.10) writes

$$\frac{d}{ds}\varphi_s = F(s, \theta_{t,s}(\xi)) + \bar{B}D_x F(s, \theta_{t,s}(\xi))(\varphi_s - \theta_{t,s}(\xi)), \quad \varphi_0 = x, \quad (4.11)$$

where \bar{B} is given by (4.5). In Section 3, we showed that the controllability of (4.11) is a sufficient condition for the non-degeneracy of the covariance matrix of \tilde{X} and, then, for the existence of transition density. Also, we showed that the controllability of (4.11) follows from the specific non-degeneracy of $(D_{x_1}F_2)(D_{x_1}F_2)^*$. Thanks to **H1**, Proposition 3.1 and 3.4 in [DM10], we know that this system is controllable. So that, the covariance matrix of the solution of (4.10) is invertible. The transition density \tilde{q} of $\tilde{X}^{t,x}$ is given⁵ by:

$$\tilde{q}(t, x_1, x_2; T, y_1, y_2) = \frac{3^{d/2}}{(2\pi)^{d/2}} (\det[\tilde{\Sigma}_{t,T}])^{-1/2} \exp\left(-|\tilde{\Sigma}_{t,T}^{-1/2}(y_1 - m_{t,T}^{1,\xi}, y_2 - m_{t,T}^{2,\xi})^*|^2\right),$$

where:

$$\begin{aligned} m_{t,T}^{1,\xi}(x) &= x_1 + \int_t^T F_1(s, \theta_{t,s}(\xi))ds \\ m_{t,T}^{2,\xi}(x) &= x_2 + \int_t^T \left[D_{x_1}F_2(s, \theta_{t,s}(\xi))(x_1 - \theta_{t,s}^1(\xi)) \right. \\ &\quad \left. + F_2(s, \theta_{t,s}(\xi)) + D_{x_1}F_2(s, \theta_{t,s}(\xi)) \int_t^s F_1(u, \theta_{t,u}(\xi))du \right] ds, \end{aligned} \quad (4.12)$$

and where the covariance matrix $\tilde{\Sigma}_{t,T}$ is given by:

$$\tilde{\Sigma}_{t,T} = \begin{pmatrix} \int_t^T \sigma\sigma^*(s, \theta_{t,s}(\xi))ds & \int_t^T R_{s,T}(\xi)\sigma\sigma^*(s, \theta_{t,s}(\xi))ds \\ \int_t^T \sigma\sigma^*(s, \theta_{t,s}(\xi))R_{s,T}^*(\xi)ds & \int_t^T R_{s,T}(\xi)\sigma\sigma^*(s, \theta_{t,s}(\xi))R_{0,s}(\xi)^*ds \end{pmatrix}, \quad (4.13)$$

with:

$$R_{s,T}(\xi) = \left[\int_s^T D_{x_1}F_2(r, \theta_{t,r}(\xi))dr \right].$$

Note that the mean $m_{t,T}^\xi(x) = (m_{t,T}^{1,\xi}(x), m_{t,T}^{2,\xi}(x))^*$, of $\tilde{X}_T^{t,x}$ satisfies the ODE (4.11). As discussed in Section 3, when $\xi = x$ the forward transport function $\theta_{t,T}(x)$ is also a solution of (4.11), so that: $\theta_{t,T}(x) = m_{t,T}^x(x)$, for all x in \mathbb{R}^{2d} .

To analyse the smoothing effect, we have:

⁵For the sake of simplicity, we forget the superscript “ t, ξ ” on the frozen transition density. When a specific choice of ξ is done, we mention it.

Lemma 4.7.

For all x, ξ in \mathbb{R}^{2d} ,

$$\begin{aligned} & \left| \left[F_2(t, x_1, x_2) - F_2(t, \theta_{t,T}(\xi)) - D_{x_1} F_2(t, \theta_{t,T}(\xi))(x_1 - \theta_{t,T}^1(\xi)) \right] \right| \\ & \leq C|x_2 - \theta_{t,T}^2(\xi)|^{\beta_2^2} + C|(x_1 - \theta_{t,T}^1(\xi))|^{1+\alpha^1}. \end{aligned}$$

Proof. We first apply a Taylor expansion with integrable remainder:

$$F_2(t, \theta_{t,T}^1(\xi), \theta_{t,T}^2(\xi)) = F_2(t, x_1, \theta_{t,T}^2(\xi)) + \int_{x_1}^{\theta_{t,T}^1(\xi)} D_{x_1} F_2(t, y_1, \theta_{t,T}^2(\xi)) dy_1. \quad (4.14)$$

And plugging into:

$$\left[F_2(t, x_1, x_2) - F_2(t, \theta_{t,T}^1(\xi), \theta_{t,T}^2(\xi)) - D_{x_1} F_2(t, \theta_{t,T}^1(\xi), \theta_{t,T}^2(\xi))(x_1 - \theta_{t,T}^1(\xi)) \right],$$

we get:

$$\begin{aligned} & \left| \left[F_2(t, x_1, x_2) - F_2(t, \theta_{t,T}(\xi)) - D_{x_1} F_2(t, \theta_{t,T}(\xi))(x_1 - \theta_{t,T}^1(\xi)) \right] \right| \\ & \leq \left| F_2(t, x_1, x_2) - F_2(t, x_1, \theta_{t,T}^2(\xi)) \right| \\ & \quad + \int_{x_1}^{\theta_{t,T}^1(\xi)} \left| D_{x_1} F_2(t, y_1, \theta_{t,T}^2(\xi)) - D_{x_1} F_2(t, \theta_{t,T}^1(\xi), \theta_{t,T}^2(\xi)) \right| dy_1, \\ & \leq C|x_2 - \theta_{t,T}^2(\xi)|^{\beta_2^2} + C \int_{x_1}^{\theta_{t,T}^1(\xi)} |y_1 - \theta_{t,T}^1(\xi)|^{\alpha^1} dy_1, \\ & \leq C|x_2 - \theta_{t,T}^2(\xi)|^{\beta_2^2} + C|(x_1 - \theta_{t,T}^1(\xi))|^{1+\alpha^1}. \end{aligned}$$

We will often use this property. □

4.2.2 The Parametrix

Remember we are interested in:

$$\frac{\partial u_i}{\partial t}(t, x_1, x_2) + Lu_i(t, x_1, x_2) = \phi_i(t, x_1, x_2), \quad i = 1, 2. \quad (4.15)$$

The operator $(\tilde{L}_s^{t, \xi_1, \xi_2})_{t \leq s \leq T}$ associated to the Gaussian process $(\tilde{X}_s^1, \tilde{X}_s^2, t \leq s \leq T)$ in (4.10) writes:

$$\begin{aligned} \tilde{L}_s^{t, \xi_1, \xi_2} & := \frac{1}{2} Tr \left[a(t, \theta_{t,s}(\xi)) D_{x_1}^2 \right] + [F_1(t, \theta_{t,s}(\xi))] \cdot D_{x_1} \\ & \quad + [F_2(t, \theta_{t,s}(\xi)) + D_{x_1} F_2(t, \theta_{t,s}(\xi))(x_1 - \theta_{t,s}^1(\xi))] \cdot D_{x_2}. \end{aligned} \quad (4.16)$$

The fundamental solution is \tilde{q} . We rewrite the systems of PDEs (4.15) as:

$$\frac{\partial u_i}{\partial t}(t, x_1, x_2) + \tilde{L}^{t, \xi} u_i(t, x_1, x_2) = (\tilde{L}^{t, \xi} u_i(t, x_1, x_2) - Lu_i(t, x_1, x_2)) + \phi_i(t, x_1, x_2), \quad i = 1, 2,$$

The solution is given by, for $i = 1, 2$:

$$\begin{aligned}
u_i(t, x_1, x_2) &= \\
&\int_t^T \int_{\mathbb{R}^{2d}} \phi_i(s, y_1, y_2) \tilde{q}(t, x_1, x_2; s, y_1, y_2) dy_1 dy_2 ds \\
&+ \int_t^T \int_{\mathbb{R}^{2d}} \frac{1}{2} \text{Tr} \left[a(s, \theta_{t,s}^1(\xi), \theta_{t,s}^2(\xi)) - a(s, y_1, y_2) D_{x_1}^2 u_i(s, y_1, y_2) \right] \tilde{q}(t, x_1, x_2; s, y_1, y_2) dy_1 dy_2 ds \\
&+ \int_t^T \int_{\mathbb{R}^{2d}} [F_1(s, \theta_{t,s}^1(\xi), \theta_{t,s}^2(\xi)) - F_1(s, y_1, y_2)] \cdot D_{x_1} u_i(s, y_1, y_2) \tilde{q}(t, x_1, x_2; s, y_1, y_2) dy_1 dy_2 ds \\
&+ \int_t^T \int_{\mathbb{R}^{2d}} [F_2(s, \theta_{t,s}^1(\xi), \theta_{t,s}^2(\xi)) + D_{x_1} F_2(s, \theta_{t,s}^1(\xi), \theta_{t,s}^2(\xi))(y_1 - \theta_{t,s}^1(\xi)) - F_2(s, y_1, y_2)] \\
&\quad \cdot D_{x_2} u_i(s, y_1, y_2) \tilde{q}(t, x_1, x_2; s, y_1, y_2) dy_1 dy_2 ds \\
&:= H_i^1(t, x_1, x_2) + H_i^2(t, x_1, x_2) + H_i^3(t, x_1, x_2) + H_i^4(t, x_1, x_2). \tag{4.17}
\end{aligned}$$

4.3 The frozen system

Proposition 4.8. *There exists a Gaussian type function \hat{q}_c defined by:*

$$\begin{aligned}
\hat{q}_c(t, x_1, x_2; T, y_1, y_2) &= \frac{c}{(T-t)^{2d}} \\
&\times \exp \left(-c^{-1} \left| \mathbb{T}_{t,T}^{-1} (y_1 - m_{t,T}^{1,\xi}(x), y_2 - m_{t,T}^{2,\xi}(x))^* \right|^2 \right),
\end{aligned}$$

where c is a positive constant depending only on known parameters in **H1**, and where $\mathbb{T}_{t,T}^{-1}$ is the $2d \times 2d$ scaling matrix given by (3.8), such that:

$$\tilde{q}(t, x_1, x_2; T, y_1, y_2) \leq \hat{q}_c(t, x_1, x_2; T, y_1, y_2),$$

and:

$$\left| D_t^{N^t} D_{x_1}^{N^{x_1}} D_{x_2}^{N^{x_2}} D_{y_1}^{N^{y_1}} \tilde{q}(t, x_1, x_2; s, y_1, y_2) \right| \leq C(s-t)^{-[3(N^{x_2}+N^t)+N^{x_1}+N^{y_1}]/2} \hat{q}_c(t, x_1, x_2; s, y_1, y_2), \tag{4.18}$$

for any $N^t, N^{x_1}, N^{x_2}, N^{y_1}$ in \mathbb{N} .

Proof. We know from **H1**, arguments of Section 3 (controllability of (4.11)) and previous discussion, that the matrix $\tilde{\Sigma}_{t,T}$ is symmetric and uniformly non degenerate. By definition (4.13), one deduces that there exists a constant C depending only on known parameters in **H1** such that:

$$\forall \zeta \in \mathbb{R}^{2d}, \quad - \left[\tilde{\Sigma}_{t,T}^{-1} \zeta \right] \cdot \zeta \leq -C \left[K_{T-t}^{-1} \zeta \right] \cdot \zeta,$$

where K_{T-t}^{-1} is defined by (3.6). Then, from Lemma 3.1 in Section 3:

$$\forall \zeta \in \mathbb{R}^{2d}, \quad - \left[\tilde{\Sigma}_{t,T}^{-1} \zeta \right] \cdot \zeta \leq -C \left[\mathbb{T}_{t,T}^{-1} \zeta \right] \cdot \left[\mathbb{T}_{t,T}^{-1} \zeta \right].$$

So: $\forall (t, x, y, \xi) \in [0, T] \times \mathbb{R}^{2d} \times \mathbb{R}^{2d} \times \mathbb{R}^{2d}$,

$$\begin{aligned}
& - \left[\tilde{\Sigma}_{t,T}^{-1}(y_1 - m_{t,T}^{1,\xi}(x), y_2 - m_{t,T}^{2,\xi}(x))^* \right] \cdot \left[(y_1 - m_{t,T}^{1,\xi}(x), y_2 - m_{t,T}^{2,\xi}(x))^* \right] \\
& \leq -C \left[\mathbb{T}_{T-t}^{-1}(y_1 - m_{t,T}^{1,\xi}(x), y_2 - m_{t,T}^{2,\xi}(x))^* \right] \cdot \left[\mathbb{T}_{T-t}^{-1}(y_1 - m_{t,T}^{1,\xi}(x), y_2 - m_{t,T}^{2,\xi}(x))^* \right].
\end{aligned}$$

For $i, j = 1, 2$, let $[\tilde{\Sigma}_{t,s}^{-1}]_{i,j}$ denote the block of size $d \times d$ of the matrix $\tilde{\Sigma}_{t,s}^{-1}$ at the $(i-1)d + 1, (j-1)d + 1$ rank. Then, it follows from the definition (4.13) of $\tilde{\Sigma}$ that for all s in $(t, T]$, for all ζ in \mathbb{R}^d :

$$\begin{cases}
\left| [\tilde{\Sigma}_{t,s}^{-1}]_{j,j} \zeta \right| \leq C(s-t)^{-j+1/2} \left| [\mathbb{T}_{s-t}^{-1}]_j \zeta \right|, \\
\left| [\tilde{\Sigma}_{t,s}^{-1}]_{i,j} \zeta \right| \leq C(s-t)^{-\min(i,j)+1/2} \left| [\mathbb{T}_{s-t}^{-1}]_{\max(i,j)} \zeta \right|, \\
\left| [\tilde{\Sigma}_{t,s}^{-1}]_{i,j} \zeta \right| \leq C(s-t)^{-\max(i,j)+1/2} \left| [\mathbb{T}_{s-t}^{-1}]_{\min(i,j)} \zeta \right|,
\end{cases} \quad (4.19)$$

where $[\mathbb{T}_{s-t}^{-1}]_j$ stands for $[\mathbb{T}_{s-t}^{-1}]_{j,j}$ with the notation above. Compute now:

$$\begin{aligned}
|D_{x_2} \tilde{q}(t, x_1, x_2; s, y_1, y_2)| &= C \left| -2[\tilde{\Sigma}_{t,s}^{-1}]_{2,1}(y_1 - m_{t,s}^{1,\xi}(x)) - 2[\tilde{\Sigma}_{t,s}^{-1}]_{2,2}(y_2 - m_{t,s}^{2,\xi}(x)) \right| \\
&\quad \times \tilde{q}(t, x_1, x_2; s, y_1, y_2) \\
&\leq C(s-t)^{-3/2} \left\{ \left| [\mathbb{T}_{s-t}^{-1}]_1(y_1 - m_{t,s}^{2,\xi}(x)) \right| + \left| [\mathbb{T}_{s-t}^{-1}]_2(y_2 - m_{t,s}^{2,\xi}(x)) \right| \right\} \\
&\quad \times \tilde{q}(t, x_1, x_2; s, y_1, y_2) \\
&\leq C(s-t)^{-3/2} \hat{q}_c(t, x_1, x_2; s, y_1, y_2).
\end{aligned}$$

$$\begin{aligned}
|D_{y_1} \tilde{q}(t, x_1, x_2; s, y_1, y_2)| &= C \left| 2[\tilde{\Sigma}_{t,s}^{-1}]_{1,1}(y_1 - m_{t,s}^{1,\xi}(x)) + 2[\tilde{\Sigma}_{t,s}^{-1}]_{1,2}(y_2 - m_{t,s}^{2,\xi}(x)) \right| \\
&\quad \times \tilde{q}(t, x_1, x_2; s, y_1, y_2) \\
&\leq C(s-t)^{-1/2} \hat{q}_c(t, x_1, x_2; s, y_1, y_2).
\end{aligned}$$

$$\begin{aligned}
& |D_{x_1} \tilde{q}(t, x_1, x_2; s, y_1, y_2)| \\
&= \left| -2[\tilde{\Sigma}_{t,s}^{-1}]_{1,1}(y_1 - m_{t,s}^{1,\xi}(x)) - 2[\tilde{\Sigma}_{t,s}^{-1}]_{1,2}(y_2 - m_{t,s}^{2,\xi}(x)) \right. \\
&\quad \left. - 2[\tilde{\Sigma}_{t,s}^{-1}]_{1,2} \left[(R_{t,s}(\xi))(y_1 - m_{t,s}^{1,\xi}(x)) \right] - 2[\tilde{\Sigma}_{t,s}^{-1}]_{2,2} \left[(R_{t,s}(\xi))(y_2 - m_{t,s}^{2,\xi}(x)) \right] \right| \\
&\quad \times \tilde{q}(t, x_1, x_2; s, y_1, y_2) \\
&\leq C(s-t)^{-1/2} \left\{ \left| [\mathbb{T}_{s-t}^{-1}]_1(y_1 - m_{t,s}^{1,\xi}(x)) \right| + \left| [\mathbb{T}_{s-t}^{-1}]_2(y_2 - m_{t,s}^{2,\xi}(x)) \right| \right. \\
&\quad \left. + C(s-t) \left((s-t)^{-3/2} \left| [\mathbb{T}_{s-t}^{-1}]_1(y_1 - m_{t,s}^{1,\xi}(x)) \right| + (s-t)^{-3/2} \left| [\mathbb{T}_{s-t}^{-1}]_2(y_2 - m_{t,s}^{2,\xi}(x)) \right| \right) \right\} \\
&\quad \times \tilde{q}(t, x_1, x_2; s, y_1, y_2) \\
&\leq C(s-t)^{-1/2} \hat{q}_c(t, x_1, x_2; s, y_1, y_2).
\end{aligned}$$

Note that, from the definition (4.12) of $m_{t,s}^\xi(x)$, one has:

$$\text{For } i, j = 1, 2, \quad \left| D_{x_i} m_{t,s}^{j,\xi}(x) \right| \leq C |(s-t)^{\min(i,j)-1} \mathbf{1}_{i \geq j} \text{Id}|. \quad (4.20)$$

Then, combining with the estimate above,

$$\begin{aligned} & |D_{x_2}^2 \tilde{q}(t, x_1, x_2; s, y_1, y_2)| \\ & \leq \left| -2[\tilde{\Sigma}_{t,s}^{-1}]_{1,1} D_{x_1} m_{t,s}^{1,\xi}(x) - 2[\tilde{\Sigma}_{t,s}^{-1}]_{1,2} D_{x_1} m_{t,s}^{2,\xi}(x) \right. \\ & \quad \left. - 2[\tilde{\Sigma}_{t,s}^{-1}]_{1,2} \left[(R_{t,s}(\xi)) D_{x_1} m_{t,s}^{1,\xi}(x) \right] - 2[\tilde{\Sigma}_{t,s}^{-1}]_{2,2} \left[(R_{t,s}(\xi)) D_{x_1} m_{t,s}^{2,\xi}(x) \right] \right| \\ & \quad \times \tilde{q}(t, x_1, x_2; s, y_1, y_2) \\ & \quad + \left| -2[\tilde{\Sigma}_{t,s}^{-1}]_{1,1} (y_1 - m_{t,s}^{1,\xi}(x)) - 2[\tilde{\Sigma}_{t,s}^{-1}]_{1,2} (y_2 - m_{t,s}^{2,\xi}(x)) \right. \\ & \quad \left. - 2[\tilde{\Sigma}_{t,s}^{-1}]_{1,2} \left[(R_{t,s}(\xi)) (y_1 - m_{t,s}^{1,\xi}(x)) \right] - 2[\tilde{\Sigma}_{t,s}^{-1}]_{2,2} \left[(R_{t,s}(\xi)) (y_2 - m_{t,s}^{2,\xi}(x)) \right] \right| \\ & \quad \times |D_{x_1} \tilde{q}(t, x_1, x_2; s, y_1, y_2)| \\ & \leq C(s-t)^{-1} \hat{q}_c(t, x_1, x_2; s, y_1, y_2). \end{aligned}$$

Since \tilde{q} satisfies the Fokker-Planck equation $[\partial/\partial t]\tilde{q}(t, x, T, y) + \tilde{L}^{t,\xi}\tilde{q}(t, x, T, y) = 0$, $\tilde{q}(T, x, T, y) = \delta_y(x)$, $x, y \in \mathbb{R}^{2d}$, one deduces from the previous estimates:

$$\left| \frac{\partial}{\partial t} \tilde{q}(t, x_1, x_2; s, y_1, y_2) \right| \leq C(s-t)^{-3/2} \hat{q}_c(t, x_1, x_2; s, y_1, y_2).$$

The other derivatives can be deduced from the computations above and (4.20). The estimate (4.18) follows. \square

We emphasize that the frozen density transition \tilde{q} satisfies:

Remark 2. For all smooth function f from $[0, T] \times \mathbb{R}^{2d}$ to \mathbb{R} , for all z_2 in \mathbb{R}^d :

$$\frac{\partial}{\partial x_{2l}} \left[\int_t^T \int_{\mathbb{R}^{2d}} f(t, y_1, z_2) \tilde{q}(t, x_1, x_2; s, y_1, y_2) dy_1 dy_2 ds \right] = 0, \quad \forall l = 1, \dots, d. \quad (4.21)$$

Also, note that the marginal law of \tilde{X}^2 involves the initial condition of \tilde{X}^1 .

4.4 From parametrix to uniform Lipschitz estimates

In the following, we denote by $C(T)$ a constant, depending only on known parameters in **H1** and T , that tends to 0 with T . This constant may change from line to line or from one equation to another. We give here the strategy for obtaining Lipschitz bounds on u_i , $i = 1, 2$ and its derivative, depending only on known parameters in **H1**. That is, the estimates (4.2) in Proposition 4.1. We refer the reader to Section 5 below for the detailed proof.

Before running into the strategy of the proof, let us observe that the representation (4.17) is differentiable w.r.t. the space variable, uniformly in the freezing point ξ , and that Lebesgue differentiation Theorem applies. So that, the derivatives of the solutions u_i , $i = 1, 2$ are the time-space convolution of the perturbed kernel $(\tilde{L} - L)u$ with the derivatives of \tilde{q} . As we

show in Proposition 4.8, the differentiation of this function generates a time singularity which is not always integrable. The main idea to restore integrability consists in “smoothing” this singularity. This can be done by using the regularity of the coefficients and the exponential decay of \tilde{q} . Let us illustrate this argument by computing the second order derivative over x_1 of the integrand of H_i^3 defined by (4.17):

$$\begin{aligned} & \left| D_{x_1}^2 \text{integrand}[\dots] \right| \\ & \leq C(s-t)^{-1} |F_1(s, y) - F_1(s, \theta_{t,s}(\xi))| \|D_{x_1} u_i\|_\infty \hat{q}_c(t, x; s, y) \\ & \leq C \|D_{x_1} u_i\|_\infty \left\{ (s-t)^{-(1+\beta_1)/2} \underbrace{\frac{|y_1 - \theta_{t,s}^1(\xi)|^{\beta_1}}{(s-t)^{\beta_1/2}}}_{(a)} + (s-t)^{-(1+3\beta_1^2)/2} \underbrace{\frac{|y_2 - \theta_{t,s}^2(\xi)|^{\beta_1^2}}{(s-t)^{3\beta_1^2/2}}}_{(b)} \right\} \hat{q}_c(t, x; s, y). \end{aligned}$$

The inequality above holds for all ξ in \mathbb{R}^{2d} . By letting $\xi = x$, we know that the freezing curve matches the mean of the Gaussian process: $\forall s \in [t, T], \forall x \in \mathbb{R}^{2d}: \theta_{t,s}(x) = m_{t,s}^x(x)$. Then the terms (a) and (b) are compensated by the exponential decay in \hat{q}_c (see Proposition 4.8 and Section 3), and this term is integrable. This is how the regularity of the coefficients counterbalances the singularity generated by the differentiation of the frozen kernel. We emphasize that the control depends only on known parameters in **H1** and on the supremum norm of the derivative of the solution. Moreover, it is small as T is small. In the following, this observation is crucial.

The representation (4.17) of each u_i , $i = 1, 2$, involves the derivatives of the solution itself. In order to obtain the Lipschitz bounds, one has to estimate each derivative that appears in (4.17) and use a circular argument. In the following, u_i denotes the i^{th} component of the solution $u = (u_1, u_2)^*$ of the linear systems of PDE (4.1). The steps of the proof that are described below hold for $i = 1, 2$.

(i) *Bound on $D_{x_1} u_i$ and $D_{x_2}^2 u_i$:* In these cases, the regularity of the coefficients allows to compensate the singularity generated by the differentiation of \tilde{q} . By using a circular argument, we obtain in Section 5.1 an estimate of the supremum norm of $D_{x_1} u_i$ and $D_{x_2}^2 u_i$ in terms of the supremum norm of $D_{x_2} u_i$:

$$\left\| D_{x_2}^2 u_i \right\|_\infty \leq C(T) (1 + \|D_{x_2} u_i\|_\infty), \quad (4.22)$$

$$\|D_{x_1} u_i\|_\infty \leq C(T) (1 + \|D_{x_2} u_i\|_\infty). \quad (4.23)$$

(ii) *Bound on $D_{x_2} u_i$:* In that case, the singularity induced by the differentiation the frozen kernel is of order $3/2$ (see Proposition 4.8). The first way for smoothing this singularity consists in using the regularity of the coefficients from **H1**. Nevertheless, the regularity of the coefficients “ F_1 ” and “ a ” do not settle the problem, indeed:

$$(s-t)^{-3/2} |F_1(s, y) - F_1(s, \theta_{t,s}(\xi))| \leq C(s-t)^{-(3-\beta_1)} \frac{|y_1 - \theta_{t,s}^1(\xi)|^{\beta_1^2/2}}{(s-t)^{\beta_1/2}} + C(s-t)^{-3(1-\beta_1^2)} \frac{|y_2 - \theta_{t,s}^2(\xi)|^{\beta_1^2/2}}{(s-t)^{3\beta_1^2/2}}.$$

Because of the specific choice of β_1^2 , the second term in the right hand side is integrable. But the first term is not integrable. A way for overcoming this difficulty consists in using the *a priori* regularity of the solution. This can be done by centering the term as:

$$\begin{aligned}
& \int_t^T \int_{\mathbb{R}^{2d}} [F_1(s, \theta_{t,s}^1(\xi), \theta_{t,s}^2(\xi)) - F_1(s, y_1, y_2)] \cdot D_{x_1} u_i(s, y_1, y_2) \tilde{q}(t, x_1, x_2; s, y_1, y_2) dy_1 dy_2 ds \\
&= \int_t^T \int_{\mathbb{R}^{2d}} [F_1(s, y_1, \theta_{t,s}^2(\xi)) - F_1(s, y_1, y_2)] \cdot D_{x_1} u_i(s, y_1, y_2) \\
&\quad \times \tilde{q}(t, x_1, x_2; s, y_1, y_2) dy_1 dy_2 ds \\
&\quad + \int_t^T \int_{\mathbb{R}^{2d}} [F_1(s, \theta_{t,s}^1(\xi), \theta_{t,s}^2(\xi)) - F_1(s, y_1, \theta_{t,s}^2(\xi))] \cdot [D_{x_1} u_i(s, y_1, y_2) - D_{x_1} u_i(s, y_1, \theta_{t,s}(\xi))] \\
&\quad \times \tilde{q}(t, x_1, x_2; s, y_1, y_2) dy_1 dy_2 ds \\
&\quad + \int_t^T \int_{\mathbb{R}^{2d}} [F_1(s, \theta_{t,s}^1(\xi), \theta_{t,s}^2(\xi)) - F_1(s, y_1, \theta_{t,s}^2(\xi))] \cdot D_{x_1} u_i(s, y_1, \theta_{t,s}(\xi)) \\
&\quad \times \tilde{q}(t, x_1, x_2; s, y_1, y_2) dy_1 dy_2 ds.
\end{aligned}$$

By differentiating w.r.t x_2 , the last term in the right hand side is equal to 0 (see Remark 2). Having in mind that the cross derivative of $D_{x_1 x_2}^2 u_i$ is bounded, we use the Lipschitz regularity of $D_{x_1} u_i$ w.r.t the degenerate component. This allows to smooth the singularity: by the Mean Value Theorem:

$$(s-t)^{-3/2} |D_{x_1} u_i(s, y_1, y_2) - D_{x_1} u_i(s, y_1, \theta_{t,s}(\xi))| \leq \|D_{x_1 x_2}^2 u_i\|_\infty \frac{|y_2 - \theta_{t,s}^2(\xi)|}{(s-t)^{3/2}},$$

which is compensated by the exponential decay of the Gaussian transition density when $\xi = x$. The argument does not work with the diffusive term H_i^2 in (4.17). Because the derivative of the solution that appears in this term is $D_{x_1}^2 u_i$. The idea is to use an integration by parts argument, in order to recover the same framework as H_i^3 . Then, by using the centering argument described above, one can differentiate the “centered” version of H_i^2 w.r.t x_2 and get ride of the singularity. Same type of argument as in (i) allows to control the derivative of the term H_i^4 . Then, we deduce in Section 5.2 a bound on the supremum norm of $D_{x_2} u_i$ depending only on T and on supremum norms of $D_{x_1} u_i$, $D_{x_1}^2 u_i$, $D_{x_1 x_2}^2 u_i$. By a circular argument and (i) (the constants behind the estimates are small as T is small), one has:

$$\|D_{x_2} u_i\|_\infty \leq C(T) (1 + \|D_{x_1 x_2}^2 u_i\|_\infty). \quad (4.24)$$

(iii) *Bound on $D_{x_1 x_2}^2 u_i$* : In order to conclude the circular argument, it remains to bound the supremum norm of $D_{x_1 x_2}^2 u_i$. Proposition 4.8 shows that the singularity that follows from the cross differentiation of \tilde{q} w.r.t x_1 and x_2 is of order 2. We emphasize that the procedure describe in (ii) allows to estimate the terms $D_{x_1 x_2}^2 H_i^1$, $D_{x_1 x_2}^2 H_i^2$, $D_{x_1 x_2}^2 H_i^3$ but not $D_{x_1 x_2}^2 H_i^4$. Indeed, the derivative of u_i that appears in this term is $D_{x_2} u_i$ and we can not use the same argument as in (ii), since we were not able to estimate $D_{x_2}^2 u_i$. Then, we investigate the Hölder-regularity of $D_{x_2} u_i$ w.r.t x_2 . In Section 5.3, we obtain the following estimate: for all t in $[0, T]$ and x_1 in \mathbb{R}^d ,

$$\begin{aligned} \forall (x_2, z_2) \in \mathbb{R}^{2d} |D_{x_2} u_i(t, x_1, x_2) - D_{x_2} u_i(t, x_1, z_2)| \\ \leq C(T) \left(1 + \left\| D_{x_1}^2 u_i \right\|_{\infty}\right) |x_2 - z_2|^{\gamma/3}, \end{aligned}$$

for any $\gamma < 1$. Then, by differentiating w.r.t x_1 and x_2 the centering versions of (4.17), we deduce in Section 5.4 the following bound:

$$\left\| D_{x_1 x_2}^2 u_i \right\|_{\infty} \leq C(T). \quad (4.25)$$

(iv) Summarizing estimates (4.22) and (4.23), and combining with (4.24) and (4.25), this completes the proof of the circular argument and gives the required uniform bounds for Proposition 4.1.

5 Boundedness and regularity of the solution in small time: proof

This part is devoted to the investigation of the regularity of the solution and its derivatives given by Proposition 4.1. We recall that these estimates are obtained under the regularization procedure, but only depend on T and on known parameters in **H1**.

5.1 Supremum norms of $D_{x_1}^2 u$ and $D_{x_1} u$ in terms of $D_{x_2} u$:

We can give the following bound:

Proposition 5.1. *Let $u = (u_1, u_2)^*$ be the solution of (4.1) defined by (4.17). Then, for T small enough, there exist two positive reals $\delta_{5.1}$ and $\bar{\delta}_{5.1}$ and a constant C depending only on known parameter in **H1** such that:*

$$\left\| D_{x_1}^2 u_i \right\|_{\infty} \leq T^{\delta_{5.1}} 5.1 C (1 + \|D_{x_2} u_i\|_{\infty}),$$

and

$$\|D_{x_1} u_i\|_{\infty} \leq T^{\bar{\delta}_{5.1}} 5.1 C (1 + \|D_{x_2} u_i\|_{\infty}),$$

for $i = 1, 2$.

Proof. We have the following Lemma:

Lemma 5.2. *Under our assumption, it holds:*

$$\begin{aligned} \left\| D_{x_1}^2 u_i \right\|_{\infty} &\leq C \left\{ \left(T^{3\beta_2^2/2} + T^{(1+\alpha^1)/2} \right) \|D_{x_2} u_i\|_{\infty} \right. \\ &\quad \left. + \left(T^{\beta_1^1/2} + T^{3\beta_1^2/2} \right) \|D_{x_1} u_i\|_{\infty} + T^{\beta_i^1/2} + T^{3\beta_i^2/2} \right\} \\ \|D_{x_1} u_i\|_{\infty} &\leq C \left\{ \left(T^{1/2+3\beta_2^2/2} + T^{(3+\alpha^1)/2} \right) \|D_{x_2} u_i\|_{\infty} \right. \\ &\quad \left. + (T + T^2) \left\| D_{x_1}^2 u_i \right\|_{\infty} + T^{(1+\beta_i^1)/2} + T^{(1+3\beta_i^2)/2} \right\}, \end{aligned}$$

for $i = 1, 2$ and where C is a positive constant depending only on known parameters in **H1**.

From Lemma 5.2, we easily deduce a bound for the supremum norms of $D_{x_1}^2 u_i$ and $D_{x_1} u_i$ only depending on known parameters in **H1** and on the supremum norm of $D_{x_2} u_i$. Such a bound is small as T is small enough. \square

Proof of Lemma 5.2: Let $i \in \{1, 2\}$. From the representation (4.17), one has:

$$\left[D_{x_1}^2 u_i(t, x_1, x_2) \right]_{(x_1, x_2) = (\xi_1, \xi_2)} \quad (5.1)$$

$$\begin{aligned} &= \int_t^T \int_{\mathbb{R}^{2d}} \phi_i(s, y_1, y_2) \left[D_{x_1}^2 \tilde{q}(t, x_1, x_2; s, y_1, y_2) \right]_{(x_1, x_2) = (\xi_1, \xi_2)} dy_1 dy_2 ds \quad (5.2) \\ &+ \frac{1}{2} \int_t^T \int_{\mathbb{R}^{2d}} \text{Tr} \left[(a(s, \theta_{t,s}^1(\xi), \theta_{t,s}^2(\xi)) - a(s, y_1, y_2)) D_{x_1}^2 u_i(s, y_1, y_2) \right] \\ &\quad \times \left[D_{x_1}^2 \tilde{q}(t, x_1, x_2; s, y_1, y_2) \right]_{(x_1, x_2) = (\xi_1, \xi_2)} dy_1 dy_2 ds \\ &+ \int_t^T \int_{\mathbb{R}^{2d}} [F_1(s, \theta_{t,s}^1(\xi), \theta_{t,s}^2(\xi)) - F_1(s, y_1, y_2)] \cdot D_{x_1}^2 u_i(s, y_1, y_2) \\ &\quad \times \left[D_{x_1}^2 \tilde{q}(t, x_1, x_2; s, y_1, y_2) \right]_{(x_1, x_2) = (\xi_1, \xi_2)} dy_1 dy_2 ds \\ &+ \int_t^T \int_{\mathbb{R}^{2d}} [F_2(s, \theta_{t,s}^1(\xi), \theta_{t,s}^2(\xi)) + D_{x_1} F_2(s, \theta_{t,s}^1(\xi), \theta_{t,s}^2(\xi))(y_1 - \theta_{t,s}^1(\xi)) - F_2(s, y_1, y_2)] \\ &\quad \cdot D_{x_2} u_i(s, y_1, y_2) \left[D_{x_1}^2 \tilde{q}(t, x_1, x_2; s, y_1, y_2) \right]_{(x_1, x_2) = (\xi_1, \xi_2)} dy_1 dy_2 ds. \end{aligned}$$

From Remark 2, one can center the term (5.2) as follows:

$$\begin{aligned} &\left[D_{x_1}^2 \int_t^T \int_{\mathbb{R}^{2d}} \phi_i(s, y_1, y_2) \tilde{q}(t, x_1, x_2; s, y_1, y_2) dy_1 dy_2 ds \right]_{(x_1, x_2) = (\xi_1, \xi_2)} \\ &= \left[D_{x_1}^2 \int_t^T \int_{\mathbb{R}^{2d}} (\phi_i(s, y_1, y_2) - \phi_i(s, \theta_{t,s}^1(\xi), \theta_{t,s}^2(\xi))) \tilde{q}(t, x_1, x_2; s, y_1, y_2) dy_1 dy_2 ds \right]_{(x_1, x_2) = (\xi_1, \xi_2)} \\ &\quad + \left[D_{x_1}^2 \int_t^T \int_{\mathbb{R}^{2d}} \phi_i(s, \theta_{t,s}^1(\xi), \theta_{t,s}^2(\xi)) \tilde{q}(t, x_1, x_2; s, y_1, y_2) dy_1 dy_2 ds \right]_{(x_1, x_2) = (\xi_1, \xi_2)} \\ &= \left[D_{x_1}^2 \int_t^T \int_{\mathbb{R}^{2d}} (\phi_i(s, y_1, y_2) - \phi_i(s, \theta_{t,s}^1(\xi), \theta_{t,s}^2(\xi))) \tilde{q}(t, x_1, x_2; s, y_1, y_2) dy_1 dy_2 ds \right]_{(x_1, x_2) = (\xi_1, \xi_2)} \\ &= \int_t^T \int_{\mathbb{R}^{2d}} (\phi_i(s, y_1, y_2) - \phi_i(s, \theta_{t,s}^1(\xi), \theta_{t,s}^2(\xi))) \\ &\quad \times \left[D_{x_1}^2 \tilde{q}(t, x_1, x_2; s, y_1, y_2) \right]_{(x_1, x_2) = (\xi_1, \xi_2)} dy_1 dy_2 ds. \quad (5.3) \end{aligned}$$

Plugging (5.3) into (5.1), using Proposition 4.8, Lemma 4.7 and regularity of the coefficients (hypothesis **H1**), we get:

$$\begin{aligned}
& \left| \left[D_{x_1}^2 u_i(t, x_1, x_2) \right]_{(x_1, x_2) = (\xi_1, \xi_2)} \right| \\
& \leq C \int_t^T (s-t)^{-1} \int_{\mathbb{R}^{2d}} \sum_{j=1}^2 (s-t)^{(j-1/2)\beta_i^j} \left| [\mathbb{T}_{s-t}^{-1}]_j(y_j - \theta_{t,s}^j(\xi)) \right|^{\beta_i^j} \hat{q}_c(t, \xi_1, \xi_2; s, y_1, y_2) dy_1 dy_2 ds. \\
& + C \int_t^T \left[(s-t)^{-1} \left\| D_{x_1}^2 u_i \right\|_{\infty} \right. \\
& \quad \times \left. \int_{\mathbb{R}^{2d}} \sum_{j=1}^2 (s-t)^{(j-1/2)} \left| [\mathbb{T}_{s-t}^{-1}]_j(y_j - \theta_{t,s}^j(\xi)) \right| \hat{q}_c(t, \xi_1, \xi_2; s, y_1, y_2) dy_1 dy_2 \right] ds \\
& + C \int_t^T \left[(s-t)^{-1} \left\| D_{x_1} u_i \right\|_{\infty} \right. \\
& \quad \times \left. \int_{\mathbb{R}^{2d}} \sum_{j=1}^2 (s-t)^{(j-1/2)\beta_1^j} \left| [\mathbb{T}_{s-t}^{-1}]_j(y_j - \theta_{t,s}^j(\xi)) \right|^{\beta_1^j} \hat{q}_c(t, \xi_1, \xi_2; s, y_1, y_2) dy_1 dy_2 \right] ds \\
& + C \int_t^T \left[(s-t)^{-1} \left\| D_{x_2} u_i \right\|_{\infty} \right. \\
& \quad \times \left. \int_{\mathbb{R}^{2d}} (s-t)^{3\beta_2^2/2} \left| [\mathbb{T}_{s-t}^{-1}]_2(y_2 - \theta_{t,s}^2(\xi)) \right|^{\beta_2^2} + (s-t)^{(1+\alpha^1)/2} \left| [\mathbb{T}_{s-t}^{-1}]_1(y_1 - \theta_{t,s}^1(\xi)) \right|^{1+\alpha^1} \right. \\
& \quad \left. \times \hat{q}_c(t, \xi_1, \xi_2; s, y_1, y_2) dy_1 dy_2 \right] ds.
\end{aligned}$$

So that, by integrating w.r.t the space variables:

$$\begin{aligned}
& \left| \left[D_{x_1}^2 u_i(t, x_1, x_2) \right]_{(x_1, x_2) = (\xi_1, \xi_2)} \right| \\
& \leq C \int_t^T \sum_{j=1}^2 (s-t)^{-1+(j-1/2)\beta_i^j} ds + C \int_t^T \left((s-t)^{-1/2} + (s-t)^{1/2} \right) \left\| D_{x_1}^2 u_i \right\|_{\infty} ds \\
& + C \int_t^T \sum_{j=1}^2 (s-t)^{(j-1/2)\beta_2^j-1} \left\| D_{x_1} u_i \right\|_{\infty} ds \\
& + C \int_t^T \left((s-t)^{-1+3\beta_1^2/2} + (s-t)^{(\alpha^1-1)/2} \right) \left\| D_{x_2} u_i \right\|_{\infty} ds.
\end{aligned}$$

Finally, we obtain:

$$\begin{aligned}
\left\| D_{x_1}^2 u_i \right\|_{\infty} & \leq C \left\{ \left(T^{3\beta_2^2/2} + T^{(1+\alpha^1)/2} \right) \left\| D_{x_2} u_i \right\|_{\infty} \right. \\
& \quad + \left(T^{\beta_1^1/2} + T^{3\beta_1^2/2} \right) \left\| D_{x_1} u_i \right\|_{\infty} \\
& \quad \left. + \left(T^{1/2} + T^{3/2} \right) \left\| D_{x_1}^2 u_i \right\|_{\infty} + T^{\beta_1^1/2} + T^{3\beta_1^2/2} \right\},
\end{aligned}$$

and the result follows for T small enough. The proof of the second statement of Lemma 5.2 can be done by the same arguments.

5.2 Supremum norm of $D_{x_2}u$ in term of supremum norm of $D_{x_1x_2}^2u$:

Proposition 5.3. *Let $u = (u_1, u_2)^*$ be the solution of (4.1) defined by (4.17). Then, there exist a real $\delta_{5.3} > 0$ and a constant C depending only on known parameters in **H1**, such that, for T small enough:*

$$\|D_{x_2}u_i\|_\infty \leq CT^{\delta_{5.3}} (1 + \|D_{x_1x_2}^2u_i\|_\infty),$$

for $i = 1, 2$.

Before proving this result, we first give the following:

Lemma 5.4. *Let $u = (u_1, u_2)^*$ be the solution of (4.1) defined by (4.17). Then,*

$$\begin{aligned} & [D_{x_2}u_i(t, x_1, x_2)] \\ &= D_{x_2}H_i^1(t, x_1, x_2) + D_{x_2}H_i^2(t, x_1, x_2) + D_{x_2}H_i^3(t, x_1, x_2) + D_{x_2}H_i^4(t, x_1, x_2), \end{aligned}$$

for $i = 1, 2$ and where the $(H_i^j)_{j=1, \dots, 4}$ are defined by (4.17). Moreover, for all $\xi \in \mathbb{R}^{2d}$, all $(t, x_1, x_2) \in [0, T] \times \mathbb{R}^{2d}$ one can write:

$$\begin{aligned} & D_{x_2}H_i^1(t, x_1, x_2) \\ &= \int_t^T \int_{\mathbb{R}^{2d}} (\phi_i(s, y_1, y_2) - \phi_i(s, y_1, \theta_{t,s}^2(\xi))) [D_{x_2}\tilde{q}(t, x_1, x_2; s, y_1, y_2)] dy_1 dy_2 ds. \end{aligned} \quad (5.4)$$

$$\begin{aligned} & D_{x_2}H_i^2(t, x_1, x_2) \tag{5.5} \\ &= \frac{1}{2} \int_t^T \int_{\mathbb{R}^{2d}} \text{Tr} \left[(a(s, y_1, \theta_{t,s}^2(\xi)) - a(s, y_1, y_2)) D_{x_1}^2 u_i(s, y_1, y_2) \right] \\ & \quad \times [D_{x_2}\tilde{q}(t, x_1, x_2; s, y_1, y_2)]_{(x_1, x_2) = (\xi_1, \xi_2)} dy_1 dy_2 ds \\ & \quad - \frac{1}{2} \sum_{l=1}^d \int_t^T \int_{\mathbb{R}^{2d}} \left[\frac{\partial}{\partial y_{1l}} a_l(s, y_1, \theta_{t,s}^2(\xi)) \right] \cdot [D_{x_1}u_i(s, y_1, y_2) - D_{x_1}u_i(s, y_1, \theta_{t,s}^2(\xi))] \\ & \quad \times [D_{x_2}\tilde{q}(t, x_1, x_2; s, y_1, y_2)] dy_1 dy_2 ds \\ & \quad + \frac{1}{2} \sum_{l=1}^d \int_t^T \int_{\mathbb{R}^{2d}} [a_l(s, \theta_{t,s}^1(\xi), \theta_{t,s}^2(\xi)) - a_l(s, y_1, \theta_{t,s}^2(\xi))] \\ & \quad \cdot [D_{x_1}u_i(s, y_1, y_2) - D_{x_1}u_i(s, y_1, \theta_{t,s}^2(\xi))] \left[D_{x_2} \left(\frac{\partial}{\partial y_{1l}} \tilde{q}(t, x_1, x_2; s, y_1, y_2) \right) \right] dy_1 dy_2 ds, \end{aligned}$$

where a_l denotes the l^{th} line of the matrix a .

$$\begin{aligned}
& D_{x_2} H_i^3(t, x_1, x_2) \tag{5.6} \\
&= \int_t^T \int_{\mathbb{R}^{2d}} [F_1(s, y_1, \theta_{t,s}^2(\xi)) - F_1(s, y_1, y_2)] \cdot D_{x_1} u_i(s, y_1, y_2) \\
&\quad \times D_{x_2} \tilde{q}(t, x_1, x_2; s, y_1, y_2) dy_1 dy_2 ds \\
&\quad + \int_t^T \int_{\mathbb{R}^{2d}} [F_1(s, \theta_{t,s}^1(\xi), \theta_{t,s}^2(\xi)) - F_1(s, y_1, \theta_{t,s}^2(\xi))] \cdot [D_{x_1} u_i(s, y_1, y_2) - D_{x_1} u_i(s, y_1, \theta_{t,s}^2(\xi))] \\
&\quad \times D_{x_2} \tilde{q}(t, x_1, x_2; s, y_1, y_2) dy_1 dy_2 ds.
\end{aligned}$$

Proof. Let $i \in \{1, 2\}$. The H_i^1 case: from (4.17) we know that, for all $\xi \in \mathbb{R}^{2d}$, all $(t, x_1, x_2) \in [0, T] \times \mathbb{R}^{2d}$:

$$H_i^1(t, x_1, x_2) = \int_t^T \int_{\mathbb{R}^{2d}} \phi_i(s, y_1, y_2) \tilde{q}(t, x_1, x_2; s, y_1, y_2) dy_1 dy_2 ds,$$

so that,

$$\begin{aligned}
& D_{x_2} H_i^1(t, x_1, x_2) \\
&= \int_t^T \int_{\mathbb{R}^{2d}} (\phi_i(s, y_1, y_2) - \phi_i(s, y_1, \theta_{t,s}^2(\xi))) [D_{x_2} \tilde{q}(t, x_1, x_2; s, y_1, y_2)] dy_1 dy_2 ds \\
&\quad + \int_t^T D_{x_2} \left[\int_{\mathbb{R}^{2d}} \phi_i(s, y_1, \theta_{t,s}^2(\xi)) \tilde{q}(t, x_1, x_2; s, y_1, y_2) dy_1 dy_2 \right] ds.
\end{aligned}$$

By differentiating w.r.t x_2 and using Lemma (2), we get:

$$\begin{aligned}
& D_{x_2} H_i^1(t, x_1, x_2) \\
&= \int_t^T \int_{\mathbb{R}^{2d}} (\phi_i(s, y_1, y_2) - \phi_i(s, y_1, \theta_{t,s}^2(\xi))) [D_{x_2} \tilde{q}(t, x_1, x_2; s, y_1, y_2)] dy_1 dy_2 ds.
\end{aligned}$$

The H_i^2 case: here, we deal with the term H_i^2 by using ‘‘centering’’ and ‘‘integration by parts’’ arguments as discussed in Subsection 4.4. Note that H_i^2 can be written as:

$$\begin{aligned}
H_i^2(t, x_1, x_2) &= \int_t^T \int_{\mathbb{R}^{2d}} \frac{1}{2} \text{Tr} \left[(a(s, \theta_{t,s}^1(\xi), \theta_{t,s}^2(\xi)) - a(s, y_1, y_2)) D_{x_1}^2 u_i(s, y_1, y_2) \right] \\
&\quad \times \tilde{q}(t, x_1, x_2; s, y_1, y_2) dy_1 dy_2 ds \\
&= \int_t^T \int_{\mathbb{R}^{2d}} \frac{1}{2} \text{Tr} \left[(a(s, y_1, \theta_{t,s}^2(\xi)) - a(s, y_1, y_2)) D_{x_1}^2 u_i(s, y_1, y_2) \right] \\
&\quad \times \tilde{q}(t, x_1, x_2; s, y_1, y_2) dy_1 dy_2 ds \tag{5.7}
\end{aligned}$$

$$\begin{aligned}
& + \int_t^T \int_{\mathbb{R}^{2d}} \frac{1}{2} \text{Tr} \left[(a(s, \theta_{t,s}^1(\xi), \theta_{t,s}^2(\xi)) - a(s, y_1, \theta_{t,s}^2(\xi))) D_{x_1}^2 u_i(s, y_1, y_2) \right] \\
&\quad \times \tilde{q}(t, x_1, x_2; s, y_1, y_2) dy_1 dy_2 ds. \tag{5.8}
\end{aligned}$$

Now focus on the term (5.8): we know from **H1** that the coefficient a is Lipschitz continuous, so that, it is *a.e* differentiable. By an integration by parts argument we get:

$$\begin{aligned}
 & \int_t^T \int_{\mathbb{R}^{2d}} \frac{1}{2} \text{Tr} \left[\left(a(s, \theta_{t,s}^1(\xi), \theta_{t,s}^2(\xi)) - a(s, y_1, \theta_{t,s}^2(\xi)) \right) D_{x_1}^2 u_i(s, y_1, y_2) \right] \\
 & \quad \times \tilde{q}(t, x_1, x_2; s, y_1, y_2) dy_1 dy_2 ds \\
 &= -\frac{1}{2} \sum_{l=1}^d \int_t^T \int_{\mathbb{R}^{2d}} \left[\frac{\partial}{\partial y_{1l}} a_l(s, y_1, \theta_{t,s}^2(\xi)) \right] \cdot D_{x_1} u_i(s, y_1, y_2) \\
 & \quad \times \tilde{q}(t, x_1, x_2, s, y_1, y_2) dy_1 dy_2 ds \tag{5.9}
 \end{aligned}$$

$$\begin{aligned}
 & +\frac{1}{2} \sum_{l=1}^d \int_t^T \int_{\mathbb{R}^{2d}} \left[a_l(s, \theta_{t,s}^1(\xi), \theta_{t,s}^2(\xi)) - a_l(s, y_1, \theta_{t,s}^2(\xi)) \right] \cdot D_{x_1} u_i(s, y_1, y_2) \\
 & \quad \times \frac{\partial}{\partial y_{1l}} \tilde{q}(t, x_1, x_2, s, y_1, y_2) dy_1 dy_2 ds. \tag{5.10}
 \end{aligned}$$

The terms (5.9) and (5.10) can be centered (w.r.t the derivative $D_{x_1} u_i$) as:

$$\begin{aligned}
 & -\frac{1}{2} \sum_{l=1}^d \int_t^T \int_{\mathbb{R}^{2d}} \left[\frac{\partial}{\partial y_{1l}} a_l(s, y_1, \theta_{t,s}^2(\xi)) \right] \cdot \left[D_{x_1} u_i(s, y_1, y_2) - D_{x_1} u_i(s, y_1, \theta_{t,s}^2(\xi)) \right] \\
 & \quad \times \tilde{q}(t, x_1, x_2, s, y_1, y_2) dy_1 dy_2 ds \\
 & +\frac{1}{2} \sum_{l=1}^d \int_t^T \int_{\mathbb{R}^{2d}} \left[a_l(s, \theta_{t,s}^1(\xi), \theta_{t,s}^2(\xi)) - a_l(s, y_1, \theta_{t,s}^2(\xi)) \right] \cdot \left[D_{x_1} u_i(s, y_1, y_2) - D_{x_1} u_i(s, y_1, \theta_{t,s}^2(\xi)) \right] \\
 & \quad \times \frac{\partial}{\partial y_{1l}} \tilde{q}(t, x_1, x_2, s, y_1, y_2) dy_1 dy_2 ds \tag{5.11} \\
 & -\frac{1}{2} \sum_{l=1}^d \int_t^T \int_{\mathbb{R}^{2d}} \left[\frac{\partial}{\partial y_{1l}} a_l(s, y_1, \theta_{t,s}^2(\xi)) \right] \cdot D_{x_1} u_i(s, y_1, \theta_{t,s}^2(\xi)) \\
 & \quad \times \tilde{q}(t, x_1, x_2, s, y_1, y_2) dy_1 dy_2 ds \\
 & +\frac{1}{2} \sum_{l=1}^d \int_t^T \int_{\mathbb{R}^{2d}} \left[a_l(s, \theta_{t,s}^1(\xi), \theta_{t,s}^2(\xi)) - a_l(s, y_1, \theta_{t,s}^2(\xi)) \right] \cdot D_{x_1} u_i(s, y_1, \theta_{t,s}^2(\xi)) \\
 & \quad \times \frac{\partial}{\partial y_{1l}} \tilde{q}(t, x_1, x_2, s, y_1, y_2) dy_1 dy_2 ds.
 \end{aligned}$$

The functions which appear in the two last terms of the equality above do not depend on the variable “ y_2 ”. From Remark 2, by differentiating w.r.t x_2 , these terms are equal to 0. Then, plugging (5.11) into the expression for H_t^2 and by differentiating w.r.t x_2 we get:

$$\begin{aligned}
& D_{x_2}H_i^2(t, x_1, x_2) \\
&= \frac{1}{2} \int_t^T \int_{\mathbb{R}^{2d}} \text{Tr} \left[(a(s, y_1, \theta_{t,s}^2(\xi)) - a(s, y_1, y_2)) D_{x_1}^2 u_i(s, y_1, y_2) \right] \\
&\quad \times [D_{x_2} \tilde{q}(t, x_1, x_2; s, y_1, y_2)] dy_1 dy_2 ds \\
&\quad - \frac{1}{2} \sum_{l=1}^d \int_t^T \int_{\mathbb{R}^{2d}} \left[\frac{\partial}{\partial y_{1l}} a_l(s, y_1, \theta_{t,s}^2(\xi)) \right] \cdot [D_{x_1} u_i(s, y_1, y_2) - D_{x_1} u_i(s, y_1, \theta_{t,s}^2(\xi))] \\
&\quad \times [D_{x_2} \tilde{q}(t, x_1, x_2; s, y_1, y_2)] dy_1 dy_2 ds \\
&\quad + \frac{1}{2} \sum_{l=1}^d \int_t^T \int_{\mathbb{R}^{2d}} [a_l(s, \theta_{t,s}^1(\xi), \theta_{t,s}^2(\xi)) - a_l(s, y_1, \theta_{t,s}^2(\xi))] \\
&\quad \cdot [D_{x_1} u_i(s, y_1, y_2) - D_{x_1} u_i(s, y_1, \theta_{t,s}^2(\xi))] \left[D_{x_2} \left(\frac{\partial}{\partial y_{1l}} \tilde{q}(t, x_1, x_2; s, y_1, y_2) \right) \right] dy_1 dy_2 ds.
\end{aligned}$$

The H_i^3 case: note that H_i^3 can be written as:

$$\begin{aligned}
& H_i^3(t, x_1, x_2) \\
&= \int_t^T \int_{\mathbb{R}^{2d}} [F_1(s, y_1, \theta_{t,s}^2(\xi)) - F_1(s, y_1, y_2)] \cdot D_{x_1} u_i(s, y_1, y_2) \\
&\quad \times \tilde{q}(t, x_1, x_2; s, y_1, y_2) dy_1 dy_2 ds \\
&\quad + \int_t^T \int_{\mathbb{R}^{2d}} [F_1(s, \theta_{t,s}^1(\xi), \theta_{t,s}^2(\xi)) - F_1(s, y_1, \theta_{t,s}^2(\xi))] \cdot [D_{x_1} u_i(s, y_1, y_2) - D_{x_1} u_i(s, y_1, \theta_{t,s}^2(\xi))] \\
&\quad \times \tilde{q}(t, x_1, x_2; s, y_1, y_2) dy_1 dy_2 ds \\
&\quad + \int_t^T \int_{\mathbb{R}^{2d}} [F_1(s, \theta_{t,s}^1(\xi), \theta_{t,s}^2(\xi)) - F_1(s, y_1, \theta_{t,s}^2(\xi))] \cdot D_{x_1} u_i(s, y_1, \theta_{t,s}^2(\xi)) \\
&\quad \times \tilde{q}(t, x_1, x_2; s, y_1, y_2) dy_1 dy_2 ds.
\end{aligned}$$

By differentiating w.r.t x_2 and by Remark 2, one has:

$$\begin{aligned}
& D_{x_2}H_i^3(t, x) \\
&= \int_t^T \int_{\mathbb{R}^{2d}} [F_1(s, y_1, \theta_{t,s}^2(\xi)) - F_1(s, y_1, y_2)] \cdot D_{x_1} u_i(s, y_1, y_2) \\
&\quad \times D_{x_2} \tilde{q}(t, x_1, x_2; s, y_1, y_2) dy_1 dy_2 ds \\
&\quad + \int_t^T \int_{\mathbb{R}^{2d}} [F_1(s, \theta_{t,s}^1(\xi), \theta_{t,s}^2(\xi)) - F_1(s, y_1, \theta_{t,s}^2(\xi))] \cdot [D_{x_1} u_i(s, y_1, y_2) - D_{x_1} u_i(s, y_1, \theta_{t,s}^2(\xi))] \\
&\quad \times D_{x_2} \tilde{q}(t, x_1, x_2; s, y_1, y_2) dy_1 dy_2 ds.
\end{aligned}$$

□

Proof of Proposition 5.3. Let $i \in \{1, 2\}$. The H_i^1 case: from Lemma (5.4), one can write:

$$\begin{aligned}
& D_{x_2}H_i^1(t, x_1, x_2) \\
&= \int_t^T \int_{\mathbb{R}^{2d}} (\phi_i(s, y_1, y_2) - \phi_i(s, y_1, \theta_{t,s}^2(\xi))) [D_{x_2} \tilde{q}(t, x_1, x_2; s, y_1, y_2)] dy_1 dy_2 ds.
\end{aligned}$$

Then, using Proposition 4.8 we have:

$$\begin{aligned}
 & \left| [D_{x_2} H_i^1(t, x)]_{(x_1, x_2) = (\xi_1, \xi_2)} \right| \\
 & \leq \int_t^T \int_{\mathbb{R}^{2d}} C(s-t)^{3\beta_i^2/2} |[\mathbb{T}_{s-t}^{-1}]_2(y_2 - \theta_{t,s}^2(\xi))|^{\beta_i^2} \\
 & \quad \times \left| [D_{x_2} \tilde{q}(t, x_1, x_2; s, y_1, y_2)]_{(x_1, x_2) = (\xi_1, \xi_2)} \right| dy_1 dy_2 ds \\
 & \leq C \int_t^T (s-t)^{-3/2(1-\beta_i^2)} ds.
 \end{aligned} \tag{5.12}$$

And then,

$$\|D_{x_2} H_i^1\|_\infty \leq CT^{1/2(3\beta_i^2-1)}. \tag{5.13}$$

The H_i^2 case: from Lemma (5.4), one can write:

$$\begin{aligned}
 & D_{x_2} H_i^2(t, x_1, x_2) \\
 & = \frac{1}{2} \int_t^T \int_{\mathbb{R}^{2d}} \text{Tr} \left[(a(s, y_1, \theta_{t,s}^2(\xi)) - a(s, y_1, y_2)) D_{x_1^2}^2 u_i(s, y_1, y_2) \right] \\
 & \quad \times [D_{x_2} \tilde{q}(t, x_1, x_2; s, y_1, y_2)]_{(x_1, x_2) = (\xi_1, \xi_2)} dy_1 dy_2 ds \\
 & - \frac{1}{2} \sum_{l=1}^d \int_t^T \int_{\mathbb{R}^{2d}} \left[\frac{\partial}{\partial y_{1l}} a_l(s, y_1, \theta_{t,s}^2(\xi)) \right] \cdot [D_{x_1} u_i(s, y_1, y_2) - D_{x_1} u_i(s, y_1, \theta_{t,s}(\xi))] \\
 & \quad \times [D_{x_2} \tilde{q}(t, x_1, x_2; s, y_1, y_2)] dy_1 dy_2 ds \\
 & + \frac{1}{2} \sum_{l=1}^d \int_t^T \int_{\mathbb{R}^{2d}} [a_l(s, \theta_{t,s}^1(\xi), \theta_{t,s}^2(\xi)) - a_l(s, y_1, \theta_{t,s}^2(\xi))] \\
 & \quad \cdot [D_{x_1} u_i(s, y_1, y_2) - D_{x_1} u_i(s, y_1, \theta_{t,s}^2(\xi))] \left[D_{x_2} \left(\frac{\partial}{\partial y_{1l}} \tilde{q}(t, x_1, x_2; s, y_1, y_2) \right) \right] dy_1 dy_2 ds.
 \end{aligned}$$

Using Mean Value Theorem, we have:

$$|D_{x_1} u_i(s, y_1, y_2) - D_{x_1} u_i(s, y_1, \theta_{t,s}(\xi))| \leq \|D_{x_1 x_2}^2 u_i\|_\infty |y_2 - \theta_{t,s}^2(\xi)|. \tag{5.14}$$

Combining (5.14), Lipschitz regularity of a from **H1** and Proposition 4.8, we obtain:

$$\begin{aligned}
 & \left| [D_{x_2} H_i^2(t, x_1, x_2)]_{(x_1, x_2) = (\xi_1, \xi_2)} \right| \\
 & \leq C \|D_{x_1^2}^2 u_i\|_\infty \int_t^T \int_{\mathbb{R}^{2d}} |[\mathbb{T}_{s-t}^{-1}]_2(y_2 - \theta_{t,s}^2(\xi))| \hat{q}_c(t, \xi_1, \xi_2; s, y_1, y_2) dy_1 dy_2 ds \\
 & + C \|D_{x_1 x_2}^2 u_i\|_\infty \int_t^T \int_{\mathbb{R}^{2d}} |[\mathbb{T}_{s-t}^{-1}]_2(y_2 - \theta_{t,s}^2(\xi))| \hat{q}_c(t, \xi_1, \xi_2; s, y_1, y_2) dy_1 dy_2 ds \\
 & + C \|D_{x_1 x_2}^2 u_i\|_\infty \int_t^T \int_{\mathbb{R}^{2d}} |[\mathbb{T}_{s-t}^{-1}]_1(y_1 - \theta_{t,s}^1(\xi))| |[\mathbb{T}_{s-t}^{-1}]_2(y_2 - \theta_{t,s}^2(\xi))| \\
 & \quad \times \hat{q}_c(t, \xi_1, \xi_2; s, y_1, y_2) dy_1 dy_2 ds.
 \end{aligned} \tag{5.15}$$

Then:

$$\|D_{x_2}H_i^2\|_\infty \leq CT \left(\|D_{x_1x_2}^2u_i\|_\infty + \|D_{x_1}^2u_i\|_\infty \right). \quad (5.16)$$

The H_i^3 case: from Lemma (5.4), one can write:

$$\begin{aligned} & D_{x_2}H_i^3(t, x_1, x_2) \\ &= \int_t^T \int_{\mathbb{R}^{2d}} [F_1(s, y_1, \theta_{t,s}^2(\xi)) - F_1(s, y_1, y_2)] \cdot D_{x_1}u_i(s, y_1, y_2) \\ &\quad \times D_{x_2}\tilde{q}(t, x_1, x_2; s, y_1, y_2) dy_1 dy_2 ds \\ &+ \int_t^T \int_{\mathbb{R}^{2d}} [F_1(s, \theta_{t,s}^1(\xi), \theta_{t,s}^2(\xi)) - F_1(s, y_1, \theta_{t,s}^2(\xi))] \cdot [D_{x_1}u_i(s, y_1, y_2) - D_{x_1}u_i(s, y_1, \theta_{t,s}^2(\xi))] \\ &\quad \times D_{x_2}\tilde{q}(t, x_1, x_2; s, y_1, y_2) dy_1 dy_2 ds. \end{aligned}$$

From Proposition 4.8, ineq. (5.14) and **H1** we get:

$$\begin{aligned} & \left| [D_{x_2}H_i^3(t, x_1, x_2)]_{(x_1, x_2)=(\xi_1, \xi_2)} \right| \quad (5.17) \\ & \leq C \|D_{x_1}u_i\|_\infty \int_t^T (s-t)^{-3/2(1-\beta_1^2)} \int_{\mathbb{R}^{2d}} |[\mathbb{T}_{s-t}^{-1}]_2(y_2 - \theta_{t,s}^2(\xi))| \\ &\quad \times \hat{q}_c(t, \xi_1, \xi_2; s, y_1, y_2) dy_1 dy_2 ds \\ & + C \|D_{x_1x_2}^2u_i\|_\infty \int_t^T (s-t)^{\beta_1^2/2} \int_{\mathbb{R}^{2d}} |[\mathbb{T}_{s-t}^{-1}]_1(y_1 - \theta_{t,s}^1(\xi))|^{\beta_1^2} |[\mathbb{T}_{s-t}^{-1}]_2(y_2 - \theta_{t,s}^2(\xi))| \\ &\quad \times \hat{q}_c(t, \xi_1, \xi_2; s, y_1, y_2) dy_1 dy_2 ds, \end{aligned}$$

and we obtain the following bound:

$$\|D_{x_2}H_i^3\|_\infty \leq C \left(T^{(3\beta_1^2-1)/2} + T^{(\beta_1^2+2)/2} \right) \|D_{x_1x_2}^2u_i\|_\infty. \quad (5.18)$$

The H_i^4 case: recall from (4.17) that:

$$\begin{aligned} & H_i^4(t, x_{1,2}) \\ &= \int_t^T \int_{\mathbb{R}^{2d}} [F_2(s, \theta_{t,s}^1(\xi), \theta_{t,s}^2(\xi)) + D_{x_1}F_2(s, \theta_{t,s}^1(\xi), \theta_{t,s}^2(\xi))(y_1 - \theta_{t,s}^1(\xi)) - F_2(s, y_1, y_2)] \\ &\quad \cdot D_{x_2}u_i(s, y_1, y_2) \tilde{q}(t, x_1, x_2; s, y_1, y_2) dy_1 dy_2 ds, \end{aligned}$$

by using Proposition 4.8, Lemma 4.7 and **H1**, we get:

$$\begin{aligned}
& \left| D_{x_2} [H_i^4(t, x_{1,2})]_{(x_1, x_2) = (\xi_1, \xi_2)} \right| \\
& \leq \int_t^T \int_{\mathbb{R}^{2d}} |F_2(s, y_1, y_2) - F_2(s, \theta_{t,s}^1(\xi), \theta_{t,s}^2(\xi)) - D_{x_1} F_2(s, \theta_{t,s}^1(\xi), \theta_{t,s}(\xi))(y_1 - \theta_{t,s}^1)| \\
& \quad \times |D_{x_2} u_i(s, y_1, y_2)| [D_{x_2} \tilde{q}(t, x_1, x_2; s, y_1, y_2)]_{(x_1, x_2) = (\xi_1, \xi_2)} dy_1 dy_2 ds \\
& \leq C \|D_{x_2} u_i\|_\infty \int_t^T \int_{\mathbb{R}^{2d}} \left[(s-t)^{-3/2(1-\beta_2^2)} |[\mathbb{T}_{s-t}^{-1}]_2(y_2 - \theta_{t,s}^2(\xi))|^{\beta_2^2} \right. \\
& \quad \left. + (s-t)^{-(2-\alpha^1)/2} |[\mathbb{T}_{s-t}^{-1}]_1(y_1 - \theta_{t,s}^1(\xi))|^{1+\alpha^1} \right] \hat{q}_c(t, \xi_1, \xi_2; s, y_1, y_2) dy_1 dy_2 ds \\
& \leq C \left(T^{(3\beta_2^2-1)/2} + T^{\alpha^1/2} \right) \|D_{x_2} u_i\|_\infty.
\end{aligned}$$

So that:

$$\|D_{x_2} H_i^4\|_\infty \leq C \left(T^{(3\beta_2^2-1)/2} + T^{\alpha^1/2} \right) \|D_{x_2} u_i\|_\infty. \quad (5.19)$$

Combining (5.13), (5.16), (5.18), (5.19), Proposition 5.1, one can find a real $\delta_{5.3} > 0$ such that, for T small enough:

$$\|D_{x_2} u_i\|_\infty \leq CT^{\delta_{5.3}} (1 + \|D_{x_1 x_2}^2 u_i\|_\infty),$$

where C and $\delta_{5.3}$ only depend on known parameters in **H1**. □

5.3 Hölder estimate of $D_{x_2} u$.

We have the following:

Proposition 5.5. *Let $u = (u_1, u_2)^*$ be the solution of (4.1) defined by (4.17). Then, there exist a positive real $\delta_{5.5}$ and a positive constant C , depending only on known parameters in **H1** such that, for T small enough:*

$$\|D_{x_2} u_i\|_{C^{\dots, \gamma/3}} \leq C \left(T^{\delta_{5.5}} \|D_{x_1 x_2}^2 u_i\|_\infty + T^{(3\beta_i^2-1-\gamma)/2} \right),$$

for $i = 1, 2$ and for any $\gamma < 1$, where $\|\cdot\|_{C^{\dots, \gamma/3}}$ denotes the Hölder norm of exponent $\gamma/3$ w.r.t x_2 .

Before proving this Proposition, we give the following:

Lemma 5.6. *Let H_i^4 , $i = 1, 2$ be the terms of (4.17). Then, for all $(t, x_1, x_2) \in [0, T] \times \mathbb{R}^{2d}$*

and $\xi \in \mathbb{R}^{2d}$,

$$\begin{aligned}
& D_{x_2}H_i^4(t, x_1, x_2) \tag{5.20} \\
&= \int_t^T \int_{\mathbb{R}^{2d}} [F_2(s, \theta_{t,s}^1(\xi), \theta_{t,s}^2(\xi)) + D_{x_1}F_2(s, \theta_{t,s}^1(\xi), \theta_{t,s}^2(\xi))(y_1 - \theta_{t,s}^1(\xi)) - F_2(s, y_1, y_2)] \\
&\quad \cdot D_{x_2}u_i(s, y_1, y_2) D_{x_2}\tilde{q}(t, x_1, x_2; t, y_1, y_2) dy_1 dy_2 ds \\
&= \int_t^T \int_{\mathbb{R}^{2d}} [F_2(s, \theta_{t,s}^1(\xi), \theta_{t,s}^2(\xi)) + D_{x_1}F_2(s, \theta_{t,s}^1(\xi), \theta_{t,s}^2(\xi))(y_1 - \theta_{t,s}^1(\xi)) - F_2(s, y_1, \theta_{t,s}^2(\xi))] \\
&\quad \cdot [D_{x_2}u_i(s, y_1, y_2) - D_{x_2}u_i(s, y_1, \theta_{t,s}^2(\xi))] D_{x_2}\tilde{q}(t, x_1, x_2; t, y_1, y_2) dy_1 dy_2 ds \\
&\quad + \int_t^T \int_{\mathbb{R}^{2d}} [F_2(s, y_1, \theta_{t,s}^2(\xi)) - F_2(s, y_1, y_2)] \cdot D_{x_2}u_i(s, y_1, y_2) \\
&\quad \times D_{x_2}\tilde{q}(t, x_1, x_2; t, y_1, y_2) dy_1 dy_2 ds,
\end{aligned}$$

for $i = 1, 2$. Moreover:

$$\begin{aligned}
\sup_{(x_1, x_2) \in \mathbb{R}^{2d}} |D_{x_2}H_i^4(t, x_2, x_2)| &\leq \|D_{x_2}u_i\|_{C^{\dots, \gamma/3}} \int_t^T (s-t)^{-(2+\alpha^1+\gamma)/2} ds \\
&\quad + \|D_{x_2}u_i\|_{\infty} \int_t^T (s-t)^{-3(1-\beta_2^2)/2} ds, \tag{5.21}
\end{aligned}$$

for $i = 1, 2$.

Proof. Let $i \in \{1, 2\}$. From (4.17), we can write:

$$\begin{aligned}
& H_i^4(t, x_1, x_2) \\
&= \int_t^T \int_{\mathbb{R}^{2d}} [F_2(s, \theta_{t,s}^1(\xi), \theta_{t,s}^2(\xi)) + D_{x_1}F_2(s, \theta_{t,s}^1(\xi), \theta_{t,s}^2(\xi))(y_1 - \theta_{t,s}^1(\xi)) - F_2(s, y_1, y_2)] \\
&\quad \cdot D_{x_2}u_i(s, y_1, y_2) \tilde{q}(t, x_1, x_2; t, y_1, y_2) dy_1 dy_2 ds \\
&= \int_t^T \int_{\mathbb{R}^{2d}} [F_2(s, \theta_{t,s}^1(\xi), \theta_{t,s}^2(\xi)) + D_{x_1}F_2(s, \theta_{t,s}^1(\xi), \theta_{t,s}^2(\xi))(y_1 - \theta_{t,s}^1(\xi)) - F_2(s, y_1, \theta_{t,s}^2(\xi))] \\
&\quad \cdot [D_{x_2}u_i(s, y_1, y_2) - D_{x_2}u_i(s, y_1, \theta_{t,s}^2(\xi))] \tilde{q}(t, x_1, x_2; t, y_1, y_2) dy_1 dy_2 ds \\
&\quad + \int_t^T \int_{\mathbb{R}^{2d}} [F_2(s, y_1, \theta_{t,s}^2(\xi)) - F_2(s, y_1, y_2)] \cdot D_{x_2}u_i(s, y_1, y_2) \tilde{q}(t, x_1, x_2; t, y_1, y_2) dy_1 dy_2 ds \\
&\quad - \int_t^T \int_{\mathbb{R}^{2d}} [F_2(s, y_1, \theta_{t,s}^2(\xi)) - F_2(s, \theta_{t,s}^1(\xi), \theta_{t,s}^2(\xi)) - D_{x_1}F_2(s, \theta_{t,s}^1(\xi), \theta_{t,s}^2(\xi))(y_1 - \theta_{t,s}^1(\xi))] \\
&\quad \cdot D_{x_2}u_i(s, y_1, \theta_{t,s}^2(\xi)) \tilde{q}(t, x_1, x_2; t, y_1, y_2) dy_1 dy_2 ds.
\end{aligned}$$

By differentiating w.r.t x_2 and by using Remark 2, we obtain

$$\begin{aligned}
& D_{x_2}H_i^4(t, x_1, x_2) \\
&= \int_t^T \int_{\mathbb{R}^{2d}} [F_2(s, \theta_{t,s}^1(\xi), \theta_{t,s}^2(\xi)) + D_{x_1}F_2(s, \theta_{t,s}^1(\xi), \theta_{t,s}^2(\xi))(y_1 - \theta_{t,s}^1(\xi)) - F_2(s, y_1, \theta_{t,s}^2(\xi))] \\
&\quad \cdot [D_{x_2}u_i(s, y_1, y_2) - D_{x_2}u_i(s, y_1, \theta_{t,s}^2(\xi))] D_{x_2}\tilde{q}(t, x_1, x_2; t, y_1, y_2) dy_1 dy_2 ds \\
&\quad + \int_t^T \int_{\mathbb{R}^{2d}} [F_2(s, y_1, \theta_{t,s}^2(\xi)) - F_2(s, y_1, y_2)] \cdot D_{x_2}u_i(s, y_1, y_2) \\
&\quad \times D_{x_2}\tilde{q}(t, x_1, x_2; t, y_1, y_2) dy_1 dy_2 ds,
\end{aligned}$$

and this conclude the proof of (5.20).

Finally, from (5.20) we have:

$$\begin{aligned}
 & \left[D_{x_2} H_i^4(t, x_2, x_2) \right]_{(x_1, x_2) = (\xi_1, \xi_2)} \\
 &= \int_t^T \int_{\mathbb{R}^{2d}} \left[F_2(s, \theta_{t,s}^1(\xi), \theta_{t,s}^2(\xi)) + D_{x_1} F_2(s, \theta_{t,s}^1(\xi), \theta_{t,s}^2(\xi))(y_1 - \theta_{t,s}^1(\xi)) - F_2(s, y_1, \theta_{t,s}^2(\xi)) \right] \\
 & \quad \cdot \left[D_{x_2} u_i(s, y_1, y_2) - D_{x_2} u_i(s, y_1, \theta_{t,s}^2(\xi)) \right] \left[D_{x_2} \tilde{q}(t, x_1, x_2; t, y_1, y_2) \right]_{(x_1, x_2) = (\xi_1, \xi_2)} dy_1 dy_2 ds \\
 & + \int_t^T \int_{\mathbb{R}^{2d}} \left[F_2(s, y_1, \theta_{t,s}^2(\xi)) - F_2(s, y_1, y_2) \right] \cdot D_{x_2} u_i(s, y_1, y_2) \\
 & \quad \times \left[D_{x_2} \tilde{q}(t, x_1, x_2; t, y_1, y_2) \right]_{(x_1, x_2) = (\xi_1, \xi_2)} dy_1 dy_2 ds,
 \end{aligned}$$

by assumption **H1**, Proposition 4.8 and Lemma 4.7 we get:

$$\begin{aligned}
 & \left| \left[D_{x_2} H_i^4(t, x_2, x_2) \right]_{(x_1, x_2) = (\xi_1, \xi_2)} \right| \\
 & \leq \|D_{x_2} u_i\|_{C^{\dots, \gamma/3}} \int_t^T (s-t)^{-(2+\alpha^1+\gamma)/2} \int_{\mathbb{R}^{2d}} \left| [\mathbb{T}_{s-t}^{-1}]_1(y_1 - \theta_{t,s}^1(\xi)) \right|^{1+\alpha^1} \left| [\mathbb{T}_{s-t}^{-1}]_2(y_2 - \theta_{t,s}^2(\xi)) \right|^{\gamma/3} \\
 & \quad \times \hat{q}_c(t, \xi_1, \xi_2; t, y_1, y_2) dy_1 dy_2 ds \\
 & + \|D_{x_2} u_i\|_{\infty} \int_t^T (s-t)^{-3(1-\beta_2^2)/2} \int_{\mathbb{R}^{2d}} \left| [\mathbb{T}_{s-t}^{-1}]_2(y_2 - \theta_{t,s}^2(\xi)) \right|^{\beta_2^2} \hat{q}_c(t, \xi_1, \xi_2; t, y_1, y_2) dy_1 dy_2 ds.
 \end{aligned}$$

Then,

$$\begin{aligned}
 \sup_{(x_1, x_2) \in \mathbb{R}^{2d}} \left| D_{x_2} H_i^4(t, x_2, x_2) \right| & \leq \|D_{x_2} u_i\|_{C^{\dots, \gamma/3}} \int_t^T (s-t)^{-(2+\alpha^1+\gamma)/2} ds \\
 & + \|D_{x_2} u_i\|_{\infty} \int_t^T (s-t)^{-3(1-\beta_2^2)/2} ds.
 \end{aligned}$$

□

Proof of Proposition 5.5 Let $i \in \{1, 2\}$. For all (t, x_1) in $[0, T] \times \mathbb{R}^d$ and (x_2, z_2) in \mathbb{R}^{2d} , one has:

$$\left| D_{x_2} u_i(t, x_1, x_2) - D_{x_2} u_i(t, x_1, z_2) \right| \leq \sum_{j=1}^4 \left| (D_{x_2} H_i^j)(t, x_1, x_2) - (D_{x_2} H_i^j)(t, x_1, z_2) \right|, \quad (5.22)$$

for $i = 1, 2$. We recall that the $(H_i^j)_{j=1, \dots, 4}$ depend on the freezing point of the process which started from x_1, x_2 and x_1, z_2 at time t . Here, we choose the same freezing point “ ξ ” for the two processes (with different initial conditions).

Suppose first that:

$$|x_2 - z_2| < (s-t)^{3/2}. \quad (5.23)$$

Using Mean Value Theorem we obtain:

$$\begin{aligned} & |D_{x_2}\tilde{q}(t, x_1, x_2; s, y_1, y_2) - D_{x_2}\tilde{q}(t, x_1, z_2; s, y_1, y_2)| \\ & \leq \sup_{\rho \in (0,1)} \left| D_{x_2}^2\tilde{q}(t, x_1, x_2 + \rho(x_2 - z_2); s, y_1, y_2) \right| |x_2 - z_2|, \end{aligned}$$

and we know from Proposition 4.8 that:

$$\begin{aligned} & \sup_{\rho \in (0,1)} \left| D_{x_2}^2\tilde{q}(t, x_1, x_2 + \rho(x_2 - z_2); s, y_1, y_2) \right| \\ & \leq C(s-t)^{-3} \sup_{\rho \in (0,1)} \hat{q}_{\bar{c}}(t, x_1, x_2 + \rho(x_2 - z_2); s, y_1, y_2), \end{aligned}$$

where \bar{c} is a positive constant, so that:

$$\begin{aligned} & |D_{x_2}\tilde{q}(t, x_1, x_2; s, y_1, y_2) - D_{x_2}\tilde{q}(t, x_1, z_2; s, y_1, y_2)| \\ & \leq C(s-t)^{-3} \sup_{\rho \in (0,1)} \hat{q}_{\bar{c}}(t, x_1, x_2 + \rho(x_2 - z_2); s, y_1, y_2) |x_2 - z_2|, \end{aligned} \quad (5.24)$$

for all $t \in [0, T]$ and $s \in (t, T]$ and ξ in \mathbb{R}^{2d} . Note that one can deduce from (5.23) that:

$$\sup_{\rho \in (0,1)} \hat{q}_{\bar{c}}(t, x_1, x_2 + \rho(x_2 - z_2); s, y_1, y_2) \leq \text{const.} \hat{q}_c(t, x_1, x_2; s, y_1, y_2). \quad (5.25)$$

Combining (5.24) and (5.25), one has:

$$\begin{aligned} & |D_{x_2}\tilde{q}(t, x_1, x_2; s, y_1, y_2) - D_{x_2}\tilde{q}(t, x_1, z_2; s, y_1, y_2)| \\ & \leq C(s-t)^{-3} \hat{q}_c(t, x_1, x_2; s, y_1, y_2) |x_2 - z_2|, \end{aligned}$$

for all $t \in [0, T]$, $s \in (t, T]$ and ξ in \mathbb{R}^{2d} . Rewrite: $|x_2 - z_2| = |x_2 - z_2|^{1-\gamma/3} |x_2 - z_2|^{\gamma/3}$ for any $0 < \gamma < 1$. Using (5.23) we know that:

$$|x_2 - z_2| \leq (s-t)^{3/2-\gamma/2} |x_2 - z_2|^{\gamma/3}, \quad (5.26)$$

and then:

$$\begin{aligned} & |D_{x_2}\tilde{q}(t, x_1, x_2; s, y_1, y_2) - D_{x_2}\tilde{q}(t, x_1, z_2; s, y_1, y_2)| \\ & \leq C(s-t)^{-(3+\gamma)/2} \hat{q}_c(t, x_1, x_2; s, y_1, y_2) |x_2 - z_2|^{\gamma/3}, \end{aligned} \quad (5.27)$$

The H_i^1 case: from Lemma 5.4, one can write:

$$\begin{aligned} & D_{x_2}H_i^1(t, x_1, x_2) - D_{x_2}H_i^1(t, x_1, z_2) \\ & = \int_t^T \int_{\mathbb{R}^{2d}} (\phi_i(s, y_1, y_2) - \phi_i(s, y_1, \theta_{t,s}^2(\xi))) [D_{x_2}\tilde{q}(t, x_1, x_2; s, y_1, y_2) - D_{x_2}\tilde{q}(t, x_1, z_2; s, y_1, y_2)] dy_1 dy_2 ds, \end{aligned}$$

so that,

$$\begin{aligned} & |D_{x_2}H_i^1(t, x_1, x_2) - D_{x_2}H_i^1(t, x_1, z_2)| \\ & \leq \int_t^T \int_{\mathbb{R}^{2d}} |\phi_i(s, y_1, y_2) - \phi_i(s, y_1, \theta_{t,s}^2(\xi))| | [D_{x_2}\tilde{q}(t, x_1, x_2; s, y_1, y_2) - D_{x_2}\tilde{q}(t, x_1, z_2; s, y_1, y_2)] | dy_1 dy_2 ds. \end{aligned}$$

Using Proposition 4.8, assumption **H1**, eq. (5.27) and by letting $\xi = x$, we have:

$$\begin{aligned} & |D_{x_2}H_i^1(t, x_1, x_2) - D_{x_2}H_i^1(t, x_1, z_2)| \\ & \leq \left[\int_t^T \int_{\mathbb{R}^{2d}} (s-t)^{-(3+\gamma-3\beta_i^2)/2} |[\mathbb{T}_{s-t}^{-1}]_2(y_2 - \theta_{t,s}^2(x))|^{\beta_i^2} \hat{q}_c(t, x_1, x_2; s, y_1, y_2) dy_1 dy_2 ds \right] |x_2 - z_2|^{\gamma/3}. \end{aligned}$$

Then,

$$|D_{x_2}H_i^1(t, x_1, x_2) - D_{x_2}H_i^1(t, x_1, z_2)| \leq T^{(3\beta_i^2-1-\gamma)/2} |x_2 - z_2|^{\gamma/3}, \quad (5.28)$$

for any $\gamma < 1$.

The H_i^2 case: from Lemma 5.4, one can write:

$$\begin{aligned} & D_{x_2}H_i^2(t, x_1, x_2) - D_{x_2}H_i^2(t, x_1, z_2) \\ & = \frac{1}{2} \int_t^T \int_{\mathbb{R}^{2d}} \text{Tr} \left[(a(s, y_1, \theta_{t,s}^2(\xi)) - a(s, y_1, y_2)) D_{x_1}^2 u_i(s, y_1, y_2) \right] \\ & \quad \times [D_{x_2}\tilde{q}(t, x_1, x_2; s, y_1, y_2) - D_{x_2}\tilde{q}(t, x_1, z_2; s, y_1, y_2)] dy_1 dy_2 ds \\ & \quad - \frac{1}{2} \sum_{l=1}^d \int_t^T \int_{\mathbb{R}^{2d}} \left[\frac{\partial}{\partial y_{1l}} a_{l.}(s, y_1, \theta_{t,s}^2(\xi)) \right] \cdot [D_{x_1} u_i(s, y_1, y_2) - D_{x_1} u_i(s, y_1, \theta_{t,s}^2(\xi))] \\ & \quad \times [D_{x_2}\tilde{q}(t, x_1, x_2; s, y_1, y_2) - D_{x_2}\tilde{q}(t, x_1, z_2; s, y_1, y_2)] dy_1 dy_2 ds \\ & \quad + \frac{1}{2} \sum_{l=1}^d \int_t^T \int_{\mathbb{R}^{2d}} [a_{l.}(s, \theta_{t,s}^1(\xi), \theta_{t,s}^2(\xi)) - a_{l.}(s, y_1, \theta_{t,s}^2(\xi))] \cdot [D_{x_1} u_i(s, y_1, y_2) - D_{x_1} u_i(s, y_1, \theta_{t,s}^2(\xi))] \\ & \quad \left[D_{x_2} \left(\frac{\partial}{\partial y_{1l}} \tilde{q}(t, x_1, x_2; s, y_1, y_2) \right) - D_{x_2} \left(\frac{\partial}{\partial y_{1l}} \tilde{q}(t, x_1, z_2; s, y_1, y_2) \right) \right] dy_1 dy_2 ds. \end{aligned}$$

Using Proposition 4.8, assumption **H1**, eq. (5.27) and by letting $\xi = x$, we get:

$$\begin{aligned} & |D_{x_2}H_i^2(t, x_1, x_2) - D_{x_2}H_i^2(t, x_1, z_2)| \\ & \leq C \|D_{x_1}^2 u_i\|_\infty \left[\int_t^T (s-t)^{-\gamma/2} \int_{\mathbb{R}^{2d}} \hat{q}_c(t, x_1, x_2; s, y_1, y_2) dy_1 dy_2 ds \right] |x_2 - z_2|^{\gamma/3} \\ & \quad + C \|D_{x_1 x_2}^2 u_i\|_\infty \left[\int_t^T (s-t)^{-\gamma/2} \int_{\mathbb{R}^{2d}} \hat{q}_c(t, x_1, x_2; s, y_1, y_2) dy_1 dy_2 ds \right] |x_2 - z_2|^{\gamma/3}, \end{aligned}$$

and then:

$$\begin{aligned} & |D_{x_2}H_i^2(t, x_1, x_2) - D_{x_2}H_i^2(t, x_1, z_2)| \\ & \leq CT^{1-\gamma/2} \left(\|D_{x_1}^2 u_i\|_\infty + \|D_{x_1 x_2}^2 u_i\|_\infty \right) |x_2 - z_2|^{\gamma/3}, \end{aligned} \quad (5.29)$$

for any $\gamma < 1$.

The H_i^3 case: from Lemma 5.4, one can write:

$$\begin{aligned} & D_{x_2}H_i^3(t, x_1, x_2) - D_{x_2}H_i^3(t, x_1, z_2) \\ &= \int_t^T \int_{\mathbb{R}^{2d}} [F_1(s, y_1, \theta_{t,s}^2(\xi)) - F_1(s, y_1, y_2)] \cdot D_{x_1}u_i(s, y_1, y_2) \\ &\quad \times [D_{x_2}\tilde{q}(t, x_1, x_2; s, y_1, y_2) - D_{x_2}\tilde{q}(t, x_1, z_2; s, y_1, y_2)] dy_1 dy_2 ds \\ &+ \int_t^T \int_{\mathbb{R}^{2d}} [F_1(s, \theta_{t,s}^1(\xi), \theta_{t,s}^2(\xi)) - F_1(s, y_1, \theta_{t,s}^2(\xi))] \cdot [D_{x_1}u_i(s, y_1, y_2) - D_{x_1}u_i(s, y_1, \theta_{t,s}^2(\xi))] \\ &\quad \times [D_{x_2}\tilde{q}(t, x_1, x_2; s, y_1, y_2) - D_{x_2}\tilde{q}(t, x_1, z_2; s, y_1, y_2)] dy_1 dy_2 ds. \end{aligned}$$

Using Proposition 4.8, assumption **H1**, eq. (5.27) and by letting $\xi = x$, we get:

$$\begin{aligned} & |D_{x_2}H_i^3(t, x_1, x_2) - D_{x_2}H_i^3(t, x_1, z_2)| \\ &\leq C \|D_{x_1}u_i\|_\infty \left[\int_t^T \int_{\mathbb{R}^{2d}} (s-t)^{-(3+\gamma-3\beta_1^2)/2} |[\mathbb{T}_{s-t}^{-1}]_2(y_2 - \theta_{t,s}^2(x))|^{\beta_1^2} \right. \\ &\quad \left. \times \hat{q}_c(t, x_1, x_2; s, y_1, y_2) dy_1 dy_2 ds \right] |x_2 - z_2|^{\gamma/3} \\ &+ C \|D_{x_1 x_2}^2 u_i\|_\infty \left[\int_t^T \int_{\mathbb{R}^{2d}} (s-t)^{-(\gamma-\beta_1^1)/2} |[\mathbb{T}_{s-t}^{-1}]_2(y_2 - \theta_{t,s}^2(x))| |[\mathbb{T}_{s-t}^{-1}]_1(y_1 - \theta_{t,s}^1(x))|^{\beta_1^1} \right. \\ &\quad \left. \times \hat{q}_c(t, x_1, x_2; s, y_1, y_2) dy_1 dy_2 ds \right] |x_2 - z_2|^{\gamma/3}, \end{aligned}$$

and then:

$$\begin{aligned} & |D_{x_2}H_i^3(t, x_1, x_2) - D_{x_2}H_i^3(t, x_1, z_2)| \\ &\leq C \left(T^{(-1+3\beta_1^2-\gamma)/2} \|D_{x_1}u_i\|_\infty + T^{(2+\beta_1^1-\gamma)/2} \|D_{x_1 x_2}^2 u_i\|_\infty \right) |x_2 - z_2|^{\gamma/3}, \end{aligned} \tag{5.30}$$

for any $\gamma < 1$.

The H_i^4 case: from Lemma 5.6, one can write:

$$\begin{aligned} & D_{x_2}H_i^4(t, x_1, x_2) - D_{x_2}H_i^4(t, x_1, z_2) \\ &= \int_t^T \int_{\mathbb{R}^{2d}} [F_2(s, \theta_{t,s}^1(\xi), \theta_{t,s}^2(\xi)) + D_{x_1}F_2(s, \theta_{t,s}^1(\xi), \theta_{t,s}^2(\xi))(y_1 - \theta_{t,s}^1(\xi)) - F_2(s, y_1, \theta_{t,s}^2(\xi))] \\ &\quad \cdot [D_{x_2}u_i(s, y_1, y_2) - D_{x_2}u_i(s, y_1, \theta_{t,s}^2(\xi))] \\ &\quad \times [D_{x_2}\tilde{q}(t, x_1, x_2; t, y_1, y_2) - D_{x_2}\tilde{q}(t, x_1, z_2; t, y_1, y_2)] dy_1 dy_2 ds \end{aligned} \tag{5.31}$$

$$\begin{aligned} & + \int_t^T \int_{\mathbb{R}^{2d}} [F_2(s, y_1, \theta_{t,s}^2(\xi)) - F_2(s, y_1, y_2)] \cdot D_{x_2}u_i(s, y_1, y_2) \\ &\quad \times [D_{x_2}\tilde{q}(t, x_1, x_2; t, y_1, y_2) - D_{x_2}\tilde{q}(t, x_1, z_2; t, y_1, y_2)] dy_1 dy_2 ds. \end{aligned} \tag{5.32}$$

By using (5.27), Proposition 4.8, Lemma 4.7, assumption **H1** and by letting $\xi = x$, one gets:

$$\begin{aligned}
 |(5.32)| &\leq C \int_t^T \int_{\mathbb{R}^{2d}} \left[(s-t)^{3(\beta_2^2-1-\gamma/3)/2} |[\mathbb{T}_{s-t}^{-1}]_2(y_2 - \theta_{t,s}^2(x))|^{3\beta_2^2/2} \right. \\
 &\quad \left. \times \|D_{x_2} u_i\|_\infty \hat{q}_c(t, x_1, x_2; s, y_1, y_2) |y_2 - z_2|^{\gamma/3} \right] dy_1 dy_2 ds \\
 &\leq CT^{(-1+3\beta_2^2/2-\gamma)/2} \|D_{x_2} u_i\|_\infty |y_2 - z_2|^{\gamma/3},
 \end{aligned} \tag{5.33}$$

and,

$$\begin{aligned}
 |(5.31)| &\leq C \int_t^T \int_{\mathbb{R}^{2d}} (s-t)^{(\alpha^1-2+\gamma)/2} |[\mathbb{T}_{s-t}^{-1}]_1(y_1 - \theta_{t,s}^1(x))|^{1+\alpha^1} \\
 &\quad \times |[\mathbb{T}_{s-t}^{-1}]_2(y_2 - \theta_{t,s}^2(x))|^{\gamma/3} \|D_{x_2} u_i\|_{C^{\dots, \gamma/3}} \hat{q}_c(t, x_1, x_2; s, y_1, y_2) dy_1 dy_2 ds \\
 &\leq CT^{(\alpha^1+\gamma)/2} \|D_{x_2} u_i\|_{C^{\dots, \gamma/3}}.
 \end{aligned} \tag{5.34}$$

Summarizing these estimates, one deduces:

$$\begin{aligned}
 &|D_{x_2} H_i^4(t, x_1, x_2) - D_{x_2} H_i^4(t, x_1, z_2)| \\
 &\leq CT^{(-1+3\beta_2^2/2-\gamma)/2} \|D_{x_2} u_i\|_\infty |y_2 - z_2|^{\gamma/3} + T^{(\alpha^1+\gamma)/2} \|D_{x_2} u_i\|_{C^{\dots, \gamma/3}}.
 \end{aligned} \tag{5.35}$$

Consider now the case: $|x_2 - z_2| \geq (s-t)^{3/2}$. It implies that

$$1 \leq (s-t)^{-\gamma/2} |x_2 - z_2|^{\gamma/3}. \tag{5.36}$$

From (5.22), one deduces:

$$|D_{x_2} u_i(t, x_1, x_2) - D_{x_2} u_i(t, x_1, z_2)| \leq 2 \sum_{j=1}^4 \sup_{(\zeta_1, \zeta_2) \in \mathbb{R}^{2d}} |D_{x_2} H_i^j(t, \zeta_1, \zeta_2)|.$$

From inequality (5.12) in the proof of Proposition 5.3 in Subsection 5.2, we have:

$$\sup_{(\zeta_1, \zeta_2) \in \mathbb{R}^{2d}} |D_{x_2} H_i^1(t, \zeta_1, \zeta_2)| \leq C \int_t^T (s-t)^{-3(1-\beta_i^2)/2} ds,$$

so that, by (5.36),

$$\begin{aligned}
 \sup_{(\zeta_1, \zeta_2) \in \mathbb{R}^{2d}} |D_{x_2} H_i^1(t, \zeta_1, \zeta_2)| &\leq C \int_t^T (s-t)^{-3(1+\gamma-\beta_i^2)/2} ds |x_2 - z_2|^{\gamma/3} \\
 &\leq CT^{(3\beta_i^2-\gamma-1)/2} |x_2 - z_2|^{\gamma/3},
 \end{aligned} \tag{5.37}$$

for any $\gamma < 1$. From inequality (5.15) in the proof of Proposition 5.3 in Subsection 5.2, one deduces:

$$\begin{aligned}
 \sup_{(\zeta_1, \zeta_2) \in \mathbb{R}^{2d}} |D_{x_2} H_i^2(t, \zeta_1, \zeta_2)| &\leq C \left(\|D_{x_1^2}^2 u_i\|_\infty + \|D_{x_1 x_2}^2 u_i\|_\infty \right) \int_t^T (s-t)^{-\gamma/2} ds |x_2 - z_2|^{\gamma/3} \\
 &\leq C \left(\|D_{x_1^2}^2 u_i\|_\infty + \|D_{x_1 x_2}^2 u_i\|_\infty T^{1-\gamma/2} \right) |x_2 - z_2|^{\gamma/3},
 \end{aligned} \tag{5.38}$$

for any $\gamma < 1$. From inequality (5.17) in the proof of Proposition 5.3 in Subsection 5.2, we get:

$$\begin{aligned} & \sup_{(\zeta_1, \zeta_2) \in \mathbb{R}^{2d}} |D_{x_2} H_i^3(t, \zeta_1, \zeta_2)| \\ & \leq \left(C \left\| D_{x_1}^2 u_i \right\|_{\infty} \int_t^T (s-t)^{-3/2(1-\beta_1^2+\gamma/3)} ds + \left\| D_{x_1 x_2}^2 u_i \right\|_{\infty} \int_t^T (s-t)^{-(\gamma-\beta_1^1)/2} ds \right) |x_2 - z_2|^{\gamma/3} \\ & \leq C \left(\left\| D_{x_1}^2 u_i \right\|_{\infty} T^{(3\beta_1^2-1-\gamma)/2} + \left\| D_{x_1 x_2}^2 u_i \right\|_{\infty} \right) T^{(1+\beta_1^1-\gamma)/2} |x_2 - z_2|^{\gamma/3}, \end{aligned} \quad (5.39)$$

for any $\gamma < 1$. Finally, from inequality (5.21) of Lemma 5.6, one immediately deduces:

$$\begin{aligned} \sup_{(\zeta_1, \zeta_2) \in \mathbb{R}^{2d}} |D_{x_2} H_i^4(t, \zeta_1, \zeta_2)| & \leq C \left\| D_{x_2} u_i \right\|_{C^{\dots, \gamma/3}} T^{(\alpha^1+\gamma)/2} ds \\ & \quad + \left\| D_{x_2} u_i \right\|_{\infty} \int_t^T (s-t)^{(3\beta_2^2-1-\gamma)/2} ds |x_2 - z_2|^{\gamma/3}. \end{aligned} \quad (5.40)$$

Combining (5.29), (5.30), (5.28), (5.35), (5.37), (5.38), (5.39), (5.40) with the estimates on $D_{x_1}^2 u_i$ and $D_{x_1} u_i$ of Proposition 5.1 and the estimate on the supremum norm of $D_{x_2} u_i$ from Proposition 5.3, this concludes the proof of Proposition 5.5. \square

5.4 Supremum norm of $D_{x_1 x_2}^2 u$ in small time:

We have the following:

Proposition 5.7. *Let $u = (u_1, u_2)^*$ be the solution of (4.1) defined by (4.17). Then, there exists a real $\bar{\delta}_{4.1} > 0$ such that, for T small enough, we have:*

$$\left\| D_{x_1 x_2}^2 u_i \right\|_{\infty} \leq C T^{\bar{\delta}_{4.1}},$$

for $i = 1, 2$, and where C is a constant depending only on known parameters in **H1**.

Proof. We know from Proposition 4.8 that the time-singularity that follows from the cross differentiation of \tilde{q} w.r.t x_1 and x_2 is of order 2. Let $i \in \{1, 2\}$.

The H_i^j , $j = 1, 2, 3$ case: one can proceed with the terms H_i^1, H_i^2 and H_i^3 as for the proof of Proposition 5.3 in Subsection 5.2: By using Lemma 5.4, and by differentiating again w.r.t x_1 , this leads to the same type of inequalities as: (5.12), (5.15) and (5.17), where each integrand in these inequalities is multiplied by $(s-t)^{-1/2}$. We then get the following bounds:

$$\left\| D_{x_1 x_2}^2 H_i^1 \right\|_{\infty} \leq C T^{1/2(3\beta_1^2-2)} \quad (5.41)$$

$$\left\| D_{x_1 x_2}^2 H_i^2 \right\|_{\infty} \leq C T^{1/2} \left(\left\| D_{x_1 x_2}^2 u_i \right\|_{\infty} + \left\| D_{x_1}^2 u_i \right\|_{\infty} \right) \quad (5.42)$$

$$\left\| D_{x_1 x_2}^2 H_i^3 \right\|_{\infty} \leq C \left(T^{(3\beta_1^2-2)/2} + T^{(\beta_1^1+1)/2} \right) \left\| D_{x_1 x_2}^2 u_i \right\|_{\infty}. \quad (5.43)$$

The H_i^4 case: one can proceed as in the first part of the proof of Proposition 5.5 in Subsection 5.3. By using Lemma 5.6 and by differentiating w.r.t x_1 , we get the same type of inequalities

as (5.33) and (5.34) , where each integrand in these inequalities is multiplied by $(s - t)^{-1/2}$. By integrating these terms w.r.t the space variables, and by taking the supremum in the right hand side we finally obtain:

$$\|D_{x_1 x_2}^2 H_i^1\|_\infty \leq CT^{(\alpha^1 + \gamma - 1)/2} \|D_{x_2} u_i\|_{C^{\dots, \gamma/3}} + CT^{(3\beta_2^2 - 2)/2} \|D_{x_2} u_i\|_\infty. \quad (5.44)$$

Combining (5.41), (5.42), (5.43) and (5.44), Proposition 5.3 and Proposition 5.5 , one can find a real $\bar{\delta}_{4.1} > 0$ depending only on known parameters in **H1** such that, for T small enough:

$$\|D_{x_1 x_2}^2 u_i\|_\infty \leq CT^{\bar{\delta}_{4.1}}.$$

This conclude the proof. □

Acknowledgements

I thank François Delarue for his suggestions, large comments and careful reading of the paper.

References

- [Bas98] R. F. Bass. *Diffusions and elliptic operators*. Probability and its Applications (New York). Springer-Verlag, New York, 1998.
- [BPV01] E. Barucci, S. Polidoro, and V. Vespri. Some results on partial differential equations and Asian options. *Math. Models Methods Appl. Sci.*, 11(3):475–497, 2001.
- [BZ11] M. Bramanti and M. Zhu. Lp and schauder estimates for nonvariational operators structured on hormander vector fields with drift. *arXiv:1103.5116v1.*, 2011.
- [DFP06] M. Di Francesco and S. Polidoro. Schauder estimates, harnack inequality and gaussian lower bound for kolmogorov type operators in non-divergence form. *Advances in Differential Equations*, 2006.
- [DL89] R. J. DiPerna and P.-L. Lions. Ordinary differential equations, transport theory and Sobolev spaces. *Invent. Math.*, 98(3):511–547, 1989.
- [DM10] F. Delarue and S. Menozzi. Density estimates for a random noise propagating through a chain of differential equations. *J. Funct. Anal.*, 259(6):1577–1630, 2010.
- [FF11] E. Fedrizzi and F. Flandoli. Pathwise uniqueness and continuous dependence of SDEs with non-regular drift. *Stochastics*, 83(3):241–257, 2011.
- [Fla11] F. Flandoli. *Random perturbation of PDEs and fluid dynamic models*, volume 2015 of *Lecture Notes in Mathematics*. Springer, Heidelberg, 2011. Lectures from the 40th Probability Summer School held in Saint-Flour, 2010.
- [Fri06] A. Friedman. *Stochastic differential equations and applications*. Dover Publications Inc., Mineola, NY, 2006. Two volumes bound as one, Reprint of the 1975 and 1976 original published in two volumes.
- [FS06] W. H. Fleming and H. M. Soner. *Controlled Markov processes and viscosity solutions*, volume 25 of *Stochastic Modelling and Applied Probability*. Springer, New York, second edition, 2006.
- [HN04] F. Herau and F. Nier. Isotropic hypoellipticity and trend to equilibrium for fokker-planck equation with a high-degree potential. *Arch. Ration. Mech. Anal.*, 2004.
- [Hör67] L. Hörmander. Hypoelliptic second order differential equations. *Acta Math.*, 119:147–171, 1967.
- [Kol34] A. Kolmogoroff. Zufällige Bewegungen (zur Theorie der Brownschen Bewegung). *Ann. of Math. (2)*, 35(1):116–117, 1934.
- [KR05] N. V. Krylov and M. Röckner. Strong solutions of stochastic equations with singular time dependent drift. *Probab. Theory Related Fields*, 131(2):154–196, 2005.
- [Kry09] N. V. Krylov. *Controlled diffusion processes*, volume 14 of *Stochastic Modelling and Applied Probability*. Springer-Verlag, Berlin, 2009. Translated from the 1977 Russian original by A. B. Aries, Reprint of the 1980 edition.

-
- [Kun86] H. Kunita. Limit theorems for stochastic differential equations and stochastic flows of diffeomorphisms. In *Stochastic differential systems (Bad Honnef, 1985)*, volume 78 of *Lecture Notes in Control and Inform. Sci.*, pages 239–257. Springer, Berlin, 1986.
- [Men11] S. Menozzi. Parametrix techniques and martingale problems for some degenerate Kolmogorov equations. *Electron. Commun. Probab.*, 16:234–250, 2011.
- [MS67] H. P. McKean, Jr. and I. M. Singer. Curvature and the eigenvalues of the Laplacian. *J. Differential Geometry*, 1(1):43–69, 1967.
- [Pro04] P. E. Protter. *Stochastic integration and differential equations*, volume 21 of *Applications of Mathematics (New York)*. Springer-Verlag, Berlin, second edition, 2004. Stochastic Modelling and Applied Probability.
- [Soi94] C. Soize. *The Fokker-Planck equation for stochastic dynamical systems and its explicit steady state solutions*, volume 17 of *Series on Advances in Mathematics for Applied Sciences*. World Scientific Publishing Co. Inc., River Edge, NJ, 1994.
- [SV79] D. W. Stroock and S. R. S. Varadhan. *Multidimensional diffusion processes*, volume 233 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, 1979.
- [Tal02] D. Talay. Stochastic Hamiltonian systems: exponential convergence to the invariant measure, and discretization by the implicit Euler scheme. *Markov Process. Related Fields*, 8(2):163–198, 2002. Inhomogeneous random systems (Cergy-Pontoise, 2001).
- [Ver80] A.Y. Veretennikov. On strong solutions and explicit formulas for solutions of stochastic integral equations. *Mat. Sb. (N.S.)*, 1980.
- [Zha05] X. Zhang. Strong solutions of SDES with singular drift and Sobolev diffusion coefficients. *Stochastic Process. Appl.*, 115(11):1805–1818, 2005.
- [Zvo74] A. K. Zvonkin. A transformation of the phase space of a diffusion process that will remove the drift. *Mat. Sb. (N.S.)*, 93(135):129–149, 152, 1974.