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# GENERALIZED NAVIER-STOKES FLOWS

MARC ARNAUDON AND ANA BELA CRUZEIRO

ABSTRACT. We introduce a notion of generalized Navier-Stokes flows on manifolds, that extends to the viscous case the one defined by Brenier. Their kinetic energy extends the kinetic energy for classical Brownian flows, defined as the  $L^2$  norm of their drift. We prove that there exists a generalized flow which realizes the infimum of kinetic energies among all generalized flows with prescribed initial and final configuration. Finally we construct generalized flows with prescribed drift and kinetic energy smaller than the  $L^2$  norm of the drift.

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## 1. INTRODUCTION

Consider the Euler equations describing the velocity of incompressible non viscous (perfect) fluids,

$$(1.1) \quad \frac{\partial}{\partial t} u = -(u \cdot \nabla) u - \nabla p, \quad \operatorname{div} u = 0.$$

The corresponding Lagrangian flows  $g(t)$ , namely the integral curves for  $u$ , solutions of  $\frac{d}{dt} g(t)(x) = u(t, g(t)(x))$ ,  $g(0)(x) = x$ , can be characterized as geodesics on the infinite dimensional "manifold" of measure preserving diffeomorphisms of the underlying configuration space (c.f. [2], [3]). In particular such solutions  $g(t)$  minimize the following action functional, defined in the time interval  $[0, T]$ ,

$$(1.2) \quad S[g] = \frac{1}{2} \int_0^T \int \left| \frac{d}{dt} g(t)(x) \right|^2 dx$$

and the corresponding Euler-Lagrange equations are precisely equations (1.1).

The work of Ebin and Marsden ([7]) showed that, given a final condition  $g(T)$  with some suitable Sobolev regularity (and, in particular, smooth) lying in a small corresponding neighborhood of the identity, existence and uniqueness of a local minimal geodesic can be obtained. However, in general, there may be situations

where such a geodesic is not defined (c.f.[13]). The main difficulty lies in that the topology induced by the energy is not strong enough to deal with the regularity of the maps.

In [4] Y. Brenier introduced the notion of generalized solutions for the minimal action principle, in the spirit of the Monge-Kantorovich problem. These solutions are probability measures defined on the set of Lagrangian trajectories. This weaker variational approach allows to consider measure-preserving maps  $g(t)$  with possible splitting and crossing during the evolution. He proved, in particular, that classical solutions can be regarded as generalized solutions and that there exist generalized solutions which do not correspond to classical flows.

In the viscous case, where the velocity obeys the Navier-Stokes system

$$(1.3) \quad \frac{\partial}{\partial t} u = -(u \cdot \nabla) u + \nu \Delta u - \nabla p, \quad \operatorname{div} u = 0$$

with a viscosity coefficient  $\nu > 0$ , it is not so clear how to define a corresponding variational principle. Following the initial ideas in [11] and [15] we have considered a stochastic variational principle defined on stochastic Lagrangian flows. The classical action is defined for these flows and we have characterized the stochastic processes which are critical for the action as processes whose drift satisfies Navier-Stokes equation (c.f. [5] for flows living on the flat torus and [1] for flows in a general compact Riemannian manifold). The abovementioned difficulties encountered in the Euler case to prove existence of critical paths remain in this setting.

So, in the spirit of Brenier's work, we introduce and study here a concept of generalized Lagrangian flows for the Navier-Stokes problem.

We mention that another approach to Lagrangian trajectories for the Navier-Stokes equation as geodesics in a different geometric framework was considered in [14]. There the approach is deterministic: what is deformed to pass from Euler to Navier-Stokes is the geometry.

## 2. STOCHASTIC GENERALIZED FLOWS

Let  $(M, \mathbf{g})$  be a compact oriented Riemannian manifold of dimension  $d \geq 2$ , without boundary. Denote by  $\rho$  the Riemannian distance in  $M$ .

We shall write  $dx$  for integration with respect to the (normalized) volume measure on  $M$ .

Let  $a \in \Gamma(TM \otimes TM)$  satisfying  $a(x, x) = \mathbf{g}^{-1}(x)$  for all  $x \in M$ . Assume there exists a separable Hilbert space  $H$  and a  $C^2$  map  $\sigma \in \Gamma(\operatorname{Hom}(H, TM))$  such that  $a(x, y) = \sigma(x)\sigma^*(y)$  for all  $x, y \in M$ . Assume furthermore that

$$(2.4) \quad \operatorname{tr} \nabla_{\sigma(\cdot)} \sigma(\cdot) = 0 \quad \text{and} \quad \forall v \in H, \operatorname{div} \sigma(v)(\cdot) = 0.$$

**Remark 2.1.** Our main example concerns  $M = \mathbb{T} = \mathbb{R}/2\pi\mathbb{Z} \times \mathbb{R}/2\pi\mathbb{Z}$  the two dimensional torus,  $H$  the Hilbert space of real-valued sequences indexed by  $\mathbb{Z}^2 \oplus \mathbb{Z}^2$ ,  $\sigma$  the map defined by

$$(2.5) \quad \sigma((k_1, k_2) + (0, 0))(\theta) =: A_{(k_1, k_2)}(\theta) = (k_2, -k_1) \cos k \cdot \theta$$

and

$$(2.6) \quad \sigma((0, 0) + (k_1, k_2))(\theta) =: B_{(k_1, k_2)}(\theta) = (k_2, -k_1) \sin k \cdot \theta.$$

In this situation it has been proved in [5] that condition (2.4) is fulfilled.

Another example is given by any compact semi-simple Lie group  $G$  endowed with the metric given by the opposite of the Killing form. Then one can take  $H = T_e G$  where  $e$  is the identity of  $G$  and for  $x \in G$ ,  $\sigma(x) = TL_x$  with  $L_x$  the left translation.

In the same spirit many examples can be constructed by projection on symmetric spaces of compact type.

Let  $\eta$  a probability measure on  $M \times M$  with marginals equal to  $dx$ : in particular it disintegrates as  $\eta(dx, dy) = dx\eta_x(dy)$ . Consider semimartingale flows  $g(t)(x)$  on  $M$ , defined on time interval  $[0, T]$  for some fixed  $T > 0$ , with the following properties:

- (1)  $g(0)(x) \equiv x$  and for all  $x \in M$ ,  $g(T)(x)$  has law  $\eta_x$ ;
- (2)  $g(\cdot)(x)$  satisfies the Itô equation

$$(2.7) \quad dg(t)(x) = \sigma(g(t)(x)) dW_t + b(t, g(t)(x), \omega) dt$$

where  $W_t$  is a cylindrical Brownian motion in  $H$  and  $(t, x, \omega) \mapsto b(t, x, \omega) \in T_x M$  is a time-dependent adapted drift with locally bounded variation in  $x$  (in the sense of distributions). Recall that if  $P(g(x))_t : T_x M \rightarrow T_{g(t)(x)} M$  is the parallel transport along  $g(t)(x)$ , then

$$dg(t)(x) = P(g(x))_t d \left( \int_0^t P(g(x))_s^{-1} \circ dg(s)(x) \right)_t ;$$

- (3) the kinetic energy of  $g$

$$(2.8) \quad \mathcal{E}(g) := \frac{1}{2} \mathbb{E} \left[ \int_M \left( \int_0^T \|b(t, x, \cdot)\|^2 dt \right) dx \right]$$

is finite.

- (4) almost surely for all  $t \in [0, T]$ ,  $\operatorname{div} b(t, \cdot, \omega) = 0$ . This together with (2.4) implies that the flow is incompressible, i.e. for all  $t, \omega$  a.s. for all  $f \in C(M)$ ,

$$(2.9) \quad \int_M f(g(t)(x)(\omega)) dx = \int_M f(x) dx.$$

**Remark 2.2.** In (2.7) we could have replaced Itô equation by Stratonovitch equation since  $\operatorname{tr} \nabla_{\sigma(\cdot)} \sigma(\cdot) = 0$ .

**Definition 2.3.** We denote by  $\mathcal{H} = \mathcal{H}_2(\sigma, \eta, T)$  the space of laws of such semimartingale flows.

Notice that  $\mathcal{H}$  is a convex set.

When the viscosity parameter is zero, namely in the case where  $\sigma = 0$  we can consider  $\eta_x = \delta_{h(x)}$  and this notion of semimartingale flow coincides with the one of generalized flow introduced in [4].

It is not clear how to obtain the existence of critical points for our variational principles within the space of measures  $\mathcal{H}$ , which are measures supported on real (semimartingale) flows  $g$ , with, in particular,  $g(0)(x) = x$ . Therefore the transports will be considered. To a semimartingale flow  $g(t)(x) \in \mathcal{H}_2(\sigma, \eta, T)$  we can associate a transport  $\Theta$ , defined as a map which to two elements  $\varphi, \phi \in L^2(M)$  associates the process

$$(2.10) \quad \Theta_t(\varphi, \phi) = \int_M \varphi(x) \phi(g(t)(x)) dx.$$

In the sequel we always take  $\varphi, \phi \in C^\infty(M)$ . In this situation the process  $\Theta_t$  is a real valued semimartingale which satisfies

$$(2.11) \quad \begin{aligned} \Theta_t(\varphi, \phi) &= (\varphi, \phi)_{L^2(M)} + \int_0^t \sum_{i \geq 1} \Theta_s(\varphi, \operatorname{div}(\phi \sigma_i)) dW_s^i \\ &+ \int_0^t \Theta_s(\varphi, \operatorname{div}(\phi b(s, \cdot, \omega))) ds + \frac{1}{2} \int_0^t \Theta_s(\varphi, \Delta \phi) ds \end{aligned}$$

where  $W_t^i = \langle W_t, \alpha_i \rangle$  and  $\sigma_i = \sigma(\alpha_i)$  for a fixed orthonormal basis  $(\alpha_i)_{i \geq 1}$  in  $H$ . Clearly

$$(2.12) \quad \Theta_t(\varphi, \phi) = (\theta^\varphi(t, \cdot), \phi)_{L^2(M)}$$

where  $x \mapsto \theta^\varphi(t, x)(\omega)$  is the function  $M \rightarrow \mathbb{R}$  defined by

$$(2.13) \quad \theta^\varphi(t, x) = \varphi((g(t)(\cdot)(\omega))^{-1}(x)).$$

Notice that for  $A$  a Borelian set in  $M$  and  $\varphi = 1_A$  we have  $\theta^\varphi(t, x) = 1_{g(t)(A)}(x)$ .

From equation (2.13) we immediately see that if  $h : \mathbb{R} \rightarrow \mathbb{R}$  is a smooth function then

$$(2.14) \quad \theta^{h \circ \varphi} = h \circ \theta^\varphi.$$

We have

$$(2.15) \quad \int_M \theta^\varphi(t, x) dx = \int_M \varphi(x) dx.$$

Moreover, using the fact that for all  $x$   $g(T)(x)$  has law  $\eta_x$ ,

$$(2.16) \quad \begin{aligned} \mathbb{E}[\Theta_T(\varphi, \phi)] &= \mathbb{E} \left[ \int_M \varphi(x) \phi(g(T)(x)) dx \right] \\ &= \int_M \varphi(x) \mathbb{E}[\phi(g(T)(x))] dx \\ &= \int_M \varphi(x) \left( \int_M \phi(y) \eta_x(dy) \right) dx \\ &= \int_{M \times M} \varphi(x) \phi(y) \eta(dx, dy). \end{aligned}$$

Now if  $\varphi_1, \phi_1, \varphi_2, \phi_2$  are smooth functions of  $M$ , the covariance of  $\Theta(\varphi_1, \phi_1)$  and  $\Theta(\varphi_2, \phi_2)$  is given by

$$(2.17) \quad d[\Theta(\varphi_1, \phi_1), \Theta(\varphi_2, \phi_2)]_t = \sum_{i \geq 1} (\Theta_t(\varphi_1, \operatorname{div}(\phi \sigma_i)) (\Theta_t(\varphi_2, \operatorname{div}(\phi \sigma_i))) dt.$$

In particular

$$\begin{aligned}
& d[\Theta(\varphi, \phi), \Theta(\varphi, \phi)]_t \\
&= \sum_{i \geq 1} \left( \int_M \theta^\varphi(t, x) \operatorname{div}(\phi \sigma_i)(x) dx \right)^2 dt \\
&= \sum_{i \geq 1} \left( \int_M \theta^\varphi(t, x) \langle d\phi, \sigma_i \rangle_x dx \right)^2 dt \quad \text{since } \operatorname{div} \sigma_i = 0 \\
&\leq \left( \int_M (\theta^\varphi(t, x))^2 dx \right) \left( \int_M \sum_{i \geq 1} \langle d\phi, \sigma_i \rangle_x^2 dx \right) dt \\
&= \left( \int_M (\theta^\varphi(t, x))^2 dx \right) \left( \int_M \|\operatorname{grad} \phi(x)\|^2 dx \right) dt \quad \text{since } \sigma(x) \sigma^*(x) = h^{-1}(x) \\
&= \int_M (\varphi(g(t)(\cdot)^{-1}(x)))^2 dx \|\operatorname{grad} \phi\|_{L^2(M)}^2 dt.
\end{aligned}$$

Since  $g(t)(\cdot)^{-1}$  preserves the Lebesgue measure on  $M$  we obtain

$$(2.18) \quad d[\Theta(\varphi, \phi), \Theta(\varphi, \phi)]_t \leq \|\varphi\|_{L^2(M)}^2 \|\operatorname{grad} \phi\|_{L^2(M)}^2 dt.$$

Notice that  $\Theta_t$  is nonnegative, that is  $\Theta_t(\varphi, \phi) \geq 0$  whenever  $\varphi$  and  $\phi$  are non-negative functions.

Denote by  $D\Theta_t(\varphi, \phi)$  the time derivative of the drift of the semimartingale  $\Theta_t(\varphi, \phi)$ . It is given by

$$(2.19) \quad D\Theta_t(\varphi, \phi) := \Theta_t \left( \varphi, \operatorname{div}(\phi b)(t, \cdot, \omega) + \frac{1}{2} \Delta \phi \right).$$

Define

$$(2.20) \quad \tilde{\Theta}_t(\varphi, \phi) := \Theta_t(\varphi, \phi) - \frac{1}{2} \int_0^t \Theta_s(\varphi, \Delta \phi) ds$$

and the kinetic energy of  $\Theta$  as

$$(2.21) \quad \mathcal{E}'(\Theta) = \frac{1}{2} \sup \left\{ \sum_{j=1}^m \sum_{k=1}^\ell \mathbb{E} \left[ \int_0^T dt \frac{\left( D\tilde{\Theta}_t(\varphi^j, \phi^k) \right)^2}{\Theta_t(\varphi^j, 1)} \right], m, \ell \geq 1, \right.$$

$$\left. \varphi^j, \phi^k \in C^\infty(M), \varphi^j \geq 0, \sum_{j=1}^m \varphi^j = 1, \phi^k \text{ s.t. } \forall v \in TM, \sum_{k=1}^\ell \langle \operatorname{grad} \phi^k, v \rangle^2 \leq \|v\|^2 \right\},$$

where  $D\tilde{\Theta}_t(\varphi^j, \phi^k)$  denotes the time derivative of the drift of  $\tilde{\Theta}_t(\varphi^j, \phi^k)$ . Notice that  $\Theta_t(\varphi^j, 1) = \int_M \varphi^j(x) dx$  by (2.15).

We are ready to define generalized flows.

**Definition 2.4.** A generalized flow with diffusion coefficient  $\sigma$  and final configuration  $\eta$  is a bilinear map  $\Theta$  which to  $\varphi, \phi \in C^\infty(M)$  associates a continuous semimartingale  $t \mapsto \Theta_t(\varphi, \phi)$  with the following properties :

(1) for all  $\varphi, \phi \in C^\infty(M)$ ,

$$(2.22) \quad \mathbb{E} [\Theta_T(\varphi, \phi)] = \int_{M \times M} \varphi(x) \phi(y) \eta(dx, dy);$$

(2) for all  $\varphi, \phi \in C^\infty(M)$ ,

$$(2.23) \quad \Theta_t(\varphi, 1) \equiv \int_M \varphi(x) dx \quad \text{and} \quad \Theta_t(1, \phi) \equiv \int_M \phi(x) dx \quad \text{a.s. for all } t;$$

(3) for all  $\varphi_1, \phi_1, \varphi_2, \phi_2 \in C^\infty(M)$

$$(2.24) \quad d[\Theta(\varphi_1, \phi_1), \Theta(\varphi_2, \phi_2)]_t = \sum_{i \geq 1} (\Theta_t(\varphi_1, \text{div}(\phi_1 \sigma_i)) (\Theta_t(\varphi_2, \text{div}(\phi_2 \sigma_i))) dt.$$

(4) for all  $\varphi, \phi \in C^\infty(M)$

$$(2.25) \quad d[\Theta(\varphi, \phi), \Theta(\varphi, \phi)]_t \leq \|\varphi\|_{L^2(M)}^2 \|\text{grad } \phi\|_{L^2(M)}^2 dt.$$

(5) for all  $\varphi, \phi \in C^\infty(M)$ , the semimartingale  $\tilde{\Theta}(\varphi, \phi)$  as defined in (2.20) has absolute continuous drift with time derivative  $D\tilde{\Theta}(\varphi, \phi)$ , and the energy  $\mathcal{E}'(\Theta)$  of  $\Theta$  as defined in (2.21) is finite. In particular

$$(2.26) \quad \mathbb{E} \left[ \int_0^T dt (D\tilde{\Theta}_t(\varphi, \phi))^2 \right] \leq 2\mathcal{E}'(\Theta) \|\varphi\|_{L^2(M)}^2 \|\text{grad } \phi\|_{L^\infty(M)}^2.$$

(6) for all  $\varphi, \phi \in C^\infty(M)$ ,

$$(2.27) \quad \Theta_0(\varphi, \phi) = (\varphi, \phi)_{L^2(M)}.$$

(7)  $\Theta$  is nonnegative, that is for all nonnegative  $\varphi, \phi \in C^\infty(M)$ ,  $\Theta(\varphi, \phi)$  is a nonnegative process.

(8) for all  $\varphi, \phi \in C^\infty(M)$ , a.s. for all  $t \in [0, T]$ ,

$$(2.28) \quad |\Theta_t(\varphi, \phi)| \leq \|\varphi\|_{L^2(M)} \|\phi\|_{L^2(M)}.$$

Notice that (3) and (8) imply (4).

**Definition 2.5.** The kinetic energy  $\mathcal{E}'(\Theta)$  of a generalized flow  $\Theta$  is given by formula (2.21). The set of laws of generalized flows with finite kinetic energy, diffusion coefficient  $\sigma$  and final configuration  $\eta$  will be denoted by  $\mathcal{H}' = \mathcal{H}'(\sigma, \eta, T)$ .

From the discussion above and equations (2.12), (2.13) to any semimartingale flow we can associate a map  $\Theta = \Theta^g$ .

**Proposition 2.6.** *We have*

$$(2.29) \quad \mathcal{E}'(\Theta^g) = \mathcal{E}(g).$$

As a consequence, the map  $g \mapsto \Theta^g$  yields a natural inclusion  $\mathcal{H}'(\sigma, \eta, T) \subset \mathcal{H}(\sigma, \eta, T)$ .

*Proof.* We begin with the inequality  $\mathcal{E}'(\Theta^g) \leq \mathcal{E}(g)$ . For this it is sufficient to prove that taking  $\Theta = \Theta^g$ ,

$$(2.30) \quad \frac{1}{2} \sum_{j=1}^m \sum_{k=1}^{\ell} \mathbb{E} \left[ \int_0^T dt \frac{(D\tilde{\Theta}_t(\varphi^j, \phi^k))^2}{\Theta_t(\varphi^j, 1)} \right] \leq \mathcal{E}(g)$$

for all  $m, \ell, \varphi^j, \phi^k$  as in the definition. But from (2.12) and (2.13),

$$\begin{aligned}
& \sum_{j=1}^m \sum_{k=1}^{\ell} \mathbb{E} \left[ \int_0^T dt \frac{\left( D\tilde{\Theta}_t(\varphi^j, \phi^k) \right)^2}{\Theta_t(\varphi^j, 1)} \right] \\
&= \sum_{j=1}^m \sum_{k=1}^{\ell} \mathbb{E} \left[ \int_0^T dt \frac{1}{\Theta_t(\varphi^j, 1)} \left( \int_M \theta^{\varphi^j}(t, x) \operatorname{div}(\phi^k b(t, \cdot, \omega))(x) dx \right)^2 \right] \\
&\leq \sum_{j=1}^m \sum_{k=1}^{\ell} \mathbb{E} \left[ \int_0^T dt \int_M \theta^{\varphi^j}(t, x) \left( \operatorname{div}(\phi^k b(t, \cdot, \omega))(x) \right)^2 dx \right] \\
&\leq \int_0^T \mathbb{E} \left[ \int_M \sum_{j=1}^m \left( \varphi^j(g(t)(\cdot)^{-1}(x)) \sum_{k=1}^{\ell} \langle d\phi^k, b(t, \cdot, \omega) \rangle_x^2 \right) dx \right] dt
\end{aligned}$$

since  $\operatorname{div} b(t, \cdot, \omega) \equiv 0$ , so

$$\begin{aligned}
& \sum_{j=1}^m \sum_{k=1}^{\ell} \mathbb{E} \left[ \int_0^T dt \frac{\left( D\tilde{\Theta}_t(\varphi^j, \phi^k) \right)^2}{\Theta_t(\varphi^j, 1)} \right] \\
&\leq \int_0^T \mathbb{E} \left[ \int_M \sum_{j=1}^m \varphi^j(g(t)(\cdot)^{-1}(x)) \|b(t, x, \omega)\|^2 dx \right] dt \quad \text{by definition of } \phi_k \\
&= \int_0^T \mathbb{E} \left[ \int_M \sum_{j=1}^m \varphi^j(x) \|b(t, g(t)(x), \omega)\|^2 dx \right] dt \\
&= \int_0^T \mathbb{E} \left[ \int_M \|b(t, g(t)(x), \omega)\|^2 dx \right] dt = 2\mathcal{E}(g).
\end{aligned}$$

Let us now prove the converse inequality. We have

$$\begin{aligned}
& \mathcal{E}(g) - \frac{1}{2} \sum_{j=1}^m \sum_{k=1}^{\ell} \mathbb{E} \left[ \int_0^T dt \frac{\left( D\tilde{\Theta}_t(\varphi^j, \phi^k) \right)^2}{\Theta_t(\varphi^j, 1)} \right] \\
&= \frac{1}{2} \sum_{j=1}^m \mathbb{E} \left[ \int_0^T dt \left( \int_M \|b(t, g(t)(y), \omega)\|^2 \varphi^j(y) dy \right. \right. \\
&\quad \left. \left. - \frac{1}{\int_M \varphi^j(x) dx} \sum_{k=1}^{\ell} \left( \int_M \varphi^j(y) \langle d\phi^k, b(t, \cdot, \omega) \rangle_{g(t)(y)} dy \right)^2 \right) \right]
\end{aligned}$$

which rewrites as

$$\begin{aligned}
& \frac{1}{2} \sum_{j=1}^m \mathbb{E} \left[ \int_0^T dt \frac{1}{\int_M \varphi^j(x) dx} \int_{M \times M} \varphi^j(y) \varphi^j(z) \left( \|b(t, g(t)(y), \omega)\|^2 \right. \right. \\
&\quad \left. \left. - \sum_{k=1}^{\ell} \langle d\phi^k, b(t, \cdot, \omega) \rangle_{g(t)(y)} \langle d\phi^k, b(t, \cdot, \omega) \rangle_{g(t)(z)} \right) dy dz \right]
\end{aligned}$$



and this equals

$$\begin{aligned} & \frac{1}{2} \sum_{j=1}^m \mathbb{E} \left[ \int_0^T dt \frac{1}{\int_M \varphi^j(x) dx} \int_M \varphi^j(y) \varphi^j(z) \left( \|b(t, g(t)(y), \omega)\|^2 \right. \right. \\ & \quad \left. \left. - \sum_{k=1}^{\ell} \langle d\phi^k, b(t, \cdot, \omega) \rangle_{g(t)(y)}^2 \right) dy \right] \\ & + \frac{1}{2} \sum_{j=1}^m \mathbb{E} \left[ \int_0^T dt \frac{1}{\int_M \varphi^j(x) dx} \int_{M \times M} \varphi^j(y) \varphi^j(z) \sum_{k=1}^{\ell} \langle d\phi^k, b(t, \cdot, \omega) \rangle_{g(t)(y)} \right. \\ & \quad \left. \times \left( \langle d\phi^k, b(t, \cdot, \omega) \rangle_{g(t)(y)} - \langle d\phi^k, b(t, \cdot, \omega) \rangle_{g(t)(z)} \right) dy dz \right]. \end{aligned}$$

Choosing  $(\phi^1, \dots, \phi^\ell)$  an isometric embedding of  $M$  into  $\mathbb{R}^\ell$ , the first term in the right vanishes. Then with Cauchy-Schwartz inequality and the first part of the proof we obtain

$$\begin{aligned} & \mathcal{E}(g) - \frac{1}{2} \sum_{j=1}^m \sum_{k=1}^{\ell} \mathbb{E} \left[ \int_0^T dt \frac{\left( D\tilde{\Theta}_t(\varphi^j, \phi^k) \right)^2}{\Theta_t(\varphi^j, 1)} \right] \\ & \leq \frac{1}{\sqrt{2}} \mathcal{E}(g)^{1/2} \left( \sum_{k=1}^{\ell} \mathbb{E} \left[ \int_0^T dt \int_{M \times M} \sum_{j=1}^m \frac{1}{\int_M \varphi^j(x) dx} \varphi^j(y) \varphi^j(z) \right. \right. \\ & \quad \left. \left. \times \left( \langle d\phi^k, b(t, \cdot, \omega) \rangle_{g(t)(y)} - \langle d\phi^k, b(t, \cdot, \omega) \rangle_{g(t)(z)} \right)^2 dy dz \right] \right)^{1/2}. \end{aligned}$$

Since  $M$  is compact there exists a finite number of charts covering  $M$ , whose image is equal to  $B(0, 1)$ . For each  $n \geq 1$ , letting  $\varepsilon = \frac{1}{2^n}$  there exist functions  $\varphi_\varepsilon^1, \dots, \varphi_\varepsilon^{m_\varepsilon}$  all with support included in a ball of radius  $\varepsilon$ :  $\text{supp}(\varphi_\varepsilon^j) \subset B(x_j, \varepsilon)$ . We can assume that there exists  $\delta > 0$  independent of  $n$  such that for all  $j = 1, \dots, m_\varepsilon$ ,

$$(2.31) \quad \int_M \varphi_\varepsilon^j(x) dx \geq \delta \varepsilon^d :$$

this needs some work but it is quite intuitive. Start with covering  $M$  with a finite number of  $B(x_j, \varepsilon)$ , such that for each point  $y$  in  $M$  the number of balls containing  $y$  is bounded above by a constant non depending on  $\varepsilon$ . A candidate for  $\varphi_\varepsilon^j$  would be  $1_{B(x_j, \varepsilon)}$  but it is not smooth and the sum is not equal to 1. However it is possible to smoothen it keeping the same support and almost the same integral. Normalizing by dividing by the sum of all obtained functions, we get the  $\varphi_\varepsilon^j$  satisfying (2.31). For  $(U, \psi)$  one of the charts, let us denote by  $I_U$  the set of indices  $j$  such that  $B(x_j, \varepsilon) \subset U$ . Since the number of charts is finite, for  $\varepsilon$  sufficiently small any ball of radius  $\varepsilon$  is included in the domain of a chart. Let us prove that for all  $k = 1, \dots, \ell$ ,

$$\begin{aligned} & \mathbb{E} \left[ \int_0^T dt \int_{M \times M} \sum_{j \in I_U} \frac{1}{\int_M \varphi_\varepsilon^j(x) dx} \varphi_\varepsilon^j(y) \varphi_\varepsilon^j(z) \left( \langle d\phi^k, b(t, \cdot, \omega) \rangle_{g(t)(y)} \right. \right. \\ & \quad \left. \left. - \langle d\phi^k, b(t, \cdot, \omega) \rangle_{g(t)(z)} \right)^2 dy dz \right] \rightarrow 0 \end{aligned}$$

as  $n \rightarrow \infty$ .

If we restrict the sum to  $j \in I_U$  then the integral on  $M \times M$  can be replaced by an integral on  $U \times U$ . Making the change of variables  $(v, w) = (\psi(y), \psi(z))$  this integral becomes

$$\int_{B(0,1) \times B(0,1)} \sum_{j \in I_U} \frac{1}{\int_M \varphi_\varepsilon^j(x) dx} \varphi_\varepsilon^j \circ \psi^{-1}(v) \varphi_\varepsilon^j \circ \psi^{-1}(w) \left( \langle d\phi^k, b(t, \cdot, \omega) \rangle_{g(t)(\psi^{-1}(v))} - \langle d\phi^k, b(t, \cdot, \omega) \rangle_{g(t)(\psi^{-1}(w))} \right)^2 \kappa(v, w) dv dw$$

where  $\kappa$  is the absolute value of the determinant of the Jacobian of  $(\psi^{-1}, \psi^{-1})$ . Let  $C_1 > 0$  an upper bound for  $\kappa$  and  $C_2 > 0$  such that

$$\begin{aligned} & \langle d\phi^k, b(t, \cdot, \omega) \rangle_{g(t)(\psi^{-1}(v))} - \langle d\phi^k, b(t, \cdot, \omega) \rangle_{g(t)(\psi^{-1}(w))} \\ & \leq C_2 \|b(t, g(t) \circ \psi^{-1}(v), \omega) - b(t, g(t) \circ \psi^{-1}(w), \omega)\|^2. \end{aligned}$$

Denote by

$$b'(v) = b'(t, v, \omega) = b(t, g(t) \circ \psi^{-1}(v), \omega)$$

which is clearly an  $L^2$  function in  $v$  for almost all  $(t, \omega)$ . The same integral is bounded by

$$\begin{aligned} & \frac{C_1 C_2}{\delta \varepsilon^d} \int_{B(0,1) \times B(0,1)} \sum_{j \in I_U} \varphi_\varepsilon^j \circ \psi^{-1}(v) \varphi_\varepsilon^j \circ \psi^{-1}(w) \|b'(v) - b'(w)\|^2 dv dw \\ & = \frac{C_1 C_2}{\delta \varepsilon^d} \sum_{j \in I_U} \int_{\psi(B(x_j, \varepsilon))} dv \varphi_\varepsilon^j \circ \psi^{-1}(v) \\ & \quad \times \left( \int_{\psi(B(x_j, \varepsilon)) - v} da \varphi_\varepsilon^j \circ \psi^{-1}(v+a) \|b'(v) - b'(v+a)\|^2 \right). \end{aligned}$$

There exists  $C_3 > 0$  independent of  $j$  and  $\varepsilon$  such that  $\psi(B(x_j, \varepsilon)) \leq B(\psi(x_j), C_3 \varepsilon)$ . Extending  $\varphi_\varepsilon^j \circ \psi^{-1}$  and  $b' \circ \psi^{-1}$  by 0 outside  $B(0, 1)$ , we can bound the above expression by

$$\begin{aligned} & \frac{C_1 C_2}{\delta \varepsilon^d} \sum_{j \in I_U} \int_{2B(0, C_3 \varepsilon)} da \left( \int_{\psi(B(x_j, \varepsilon))} dv \varphi_\varepsilon^j \circ \psi^{-1}(v) \|b'(v) - b'(v+a)\|^2 \right) \\ & \leq \frac{C_1 C_2}{\delta \varepsilon^d} \int_{2B(0, C_3 \varepsilon)} da \left( \int_{\mathbb{R}^d} dv \|b'(v) - b'(v+a)\|^2 \right) \\ & \leq \frac{C_1 C_2 C_3^d}{\delta} \sup_{a \in 2B(0, C_3 \varepsilon)} \int_{\mathbb{R}^d} dv \|(b' - \tau_{-a} b')(v)\|^2 \end{aligned}$$

where  $\tau_a b(v) = b(v - a)$ . By continuity in  $L^2$  of  $a \mapsto \tau_a b'$  the above expression is bounded by a function  $\delta(t, \omega) > 0$  converging to 0  $dt \otimes d\mathbb{P}(\omega)$  a.e. as  $\varepsilon \rightarrow 0$ .

On the other hand

$$\begin{aligned}
& \int_{M \times M} \sum_{j=1}^{m_\varepsilon} \frac{1}{\int_M \varphi_\varepsilon^j(x) dx} \varphi_\varepsilon^j(y) \varphi_\varepsilon^j(z) \\
& \quad \times \left( \langle d\phi^k, b(t, \cdot, \omega) \rangle_{g(t)(y)} - \langle d\phi^k, b(t, \cdot, \omega) \rangle_{g(t)(z)} \right)^2 dydz \Big] \\
(2.32) \quad & \leq 2 \int_{M \times M} \sum_{j=1}^{m_\varepsilon} \frac{1}{\int_M \varphi_\varepsilon^j(x) dx} \varphi_\varepsilon^j(y) \varphi_\varepsilon^j(z) \\
& \quad \times \left( \langle d\phi^k, b(t, \cdot, \omega) \rangle_{g(t)(y)}^2 + \langle d\phi^k, b(t, \cdot, \omega) \rangle_{g(t)(z)}^2 \right) dydz \Big] \\
& = 4 \int_M \langle d\phi^k, b(t, \cdot, \omega) \rangle_{g(t)(y)}^2 dy
\end{aligned}$$

which is integrable with respect to  $(k, t, \omega)$  and does not depend on  $\varepsilon$ . We conclude with the dominated convergence theorem on the integrals with respect to  $(k, t, \omega)$ .  $\square$

**Remark 2.7.** It is clear from definitions 2.4 and 2.5 that  $\mathcal{H}'$  is a convex set.

### 3. EXISTENCE OF GENERALIZED MINIMAL FLOWS

In this section we prove our main result, the existence of generalized minimal flows with prescribed final configuration.

For  $g$  a semimartingale flow let  $Dg(t)(x) = b(t, x)$  denote its drift. Notice that

$$Dg(t)(x) = \lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon} \mathbb{E}_t \left[ \exp_{g(t)(x)}^{-1} g(t + \varepsilon \wedge \tau(t, x, r))(x) \right]$$

where  $\mathbb{E}_t$  denotes conditional expectation with respect to the past filtration,  $\tau(t, x, r)$  is the exit time of  $s \mapsto g(t + s)(x)$  from a small ball  $B(x, r)$  ( $r > 0$ ).

The kinetic energy of  $g$  has already been defined as

$$(3.33) \quad \mathcal{E}(g) = \frac{1}{2} \mathbb{E} \left[ \int_M \left( \int_0^T \|b(t, x, \cdot)\|^2 dt \right) dx \right].$$

When the vector field  $u$  is smooth and satisfies Navier-Stokes equations, the process  $dg_u(t) = \sigma(g_u(t))dW(t) + u(t, g_u(t))dt$  will be critical for the kinetic energy functional (c.f. [5] and [1]).

Again we notice that when the viscosity is zero the semimartingales are paths of bounded variation and the kinetic energy reduces to (1.2).

If  $\eta$  is a final configuration at time  $T$  one can define

$$(3.34) \quad \mathcal{E}(\sigma, \eta, T) = \inf \{ \mathcal{E}(g), \text{Law}(g) \in \mathcal{H}_2(\sigma, \eta, T) \}$$

where by convention  $\mathcal{E}(\sigma, \eta, T) = \infty$  if there is no semimartingale flow with configuration  $\eta$  at time  $T$ .

Similarly we define

$$(3.35) \quad \mathcal{E}'(\sigma, \eta, T) = \inf \{ \mathcal{E}'(\Theta), \text{Law}(\Theta) \in \mathcal{H}'(\sigma, \eta, T) \}$$

with  $\mathcal{E}'(\sigma, \eta, T) = \infty$  if there is no generalized flow with configuration  $\eta$  at time  $T$ .

From Proposition 2.6 we know that

$$(3.36) \quad \mathcal{E}'(\sigma, \eta, T) \leq \mathcal{E}(\sigma, \eta, T).$$

In this section we shall prove the following

**Theorem 3.1.** *If  $\mathcal{E}'(\sigma, \eta, T) < \infty$  then there exists a generalized flow  $\Theta$  such that  $\mathcal{E}'(\Theta) = \mathcal{E}'(\sigma, \eta, T)$ .*

*Proof.* Assume  $\mathcal{E}'(\sigma, \eta, T) < \infty$ . Let  $(\Theta^n)_{n \geq 1}$  be a sequence of generalized flows with laws in  $\mathcal{H}'(\sigma, \eta, T)$  satisfying

$$(3.37) \quad \lim_{n \rightarrow \infty} \mathcal{E}'(\Theta^n) = \mathcal{E}'(\sigma, \eta, T).$$

Consider a sequence  $(\tilde{\varphi}^j)_{j \geq 1}$  of elements of  $C^\infty(M)$  dense for the topology of uniform convergence, and a sequence  $(\tilde{\phi}^k)_{k \geq 1}$  of elements of  $C^\infty(M)$  dense for the topology of uniform convergence of functions and their first and second order derivatives.

For fixed  $j, k \geq 1$  and  $n \geq 1$ , the semimartingale

$$\tilde{\Theta}_t^n(\tilde{\varphi}^j, \tilde{\phi}^k) = \Theta_t^n(\tilde{\varphi}^j, \tilde{\phi}^k) - \frac{1}{2} \int_0^t \Theta_s^n(\tilde{\varphi}^j, \Delta \tilde{\phi}^k) ds$$

has starting point  $(\tilde{\varphi}^j, \tilde{\phi}^k)_{L^2(M)}$ . Its drift  $D\tilde{\Theta}_t^n(\tilde{\varphi}^j, \tilde{\phi}^k)$  satisfies

$$(3.38) \quad \mathbb{E} \left[ \int_0^T dt \left( D\tilde{\Theta}_t^n(\tilde{\varphi}^j, \tilde{\phi}^k) \right)^2 \right] \leq 2 \|\tilde{\varphi}^j\|_{L^\infty(M)}^2 \|\text{grad } \tilde{\phi}^k\|_{L^\infty(M)}^2 \mathcal{E}'(\Theta^n).$$

On the other hand the bracket of  $\tilde{\Theta}^n(\tilde{\varphi}^j, \tilde{\phi}^k)$  satisfies

$$(3.39) \quad d \left[ \tilde{\Theta}^n(\tilde{\varphi}^j, \tilde{\phi}^k), \tilde{\Theta}^n(\tilde{\varphi}^j, \tilde{\phi}^k) \right]_t \leq \|\tilde{\varphi}^j\|_{L^2(M)}^2 \|\text{grad } \tilde{\phi}^k\|_{L^2(M)}^2 dt.$$

We also have

$$(3.40) \quad \mathbb{E} \left[ \int_0^T \left| \Theta_s^n(\tilde{\varphi}^j, \Delta \tilde{\phi}^k) \right|^2 ds \right] \leq T \|\tilde{\varphi}^j\|_{L^\infty(M)}^2 \|\Delta \tilde{\phi}^k\|_{L^\infty(M)}^2.$$

With Theorem 3 in [16] we can conclude that the sequence

$$(3.41) \quad \left( Y^n := \left( \tilde{\Theta}^n(\tilde{\varphi}^j, \tilde{\phi}^k), \frac{1}{2} \int_0^\cdot \Theta_s^n(\tilde{\varphi}^j, \Delta \tilde{\phi}^k) ds \right) \right)_{n \geq 1}$$

is tight and that we can extract a subsequence which converges in law. In fact, we can extract a subsequence such that all

$$\left( \left( \tilde{\Theta}^n(\tilde{\varphi}^j, \tilde{\phi}^k), \frac{1}{2} \int_0^\cdot \Theta_s^n(\tilde{\varphi}^j, \Delta \tilde{\phi}^k) ds \right) \right)_{n \geq 1}, \quad j, k \geq 1$$

together with all their covariances simultaneously converge in law to a limit for which all linear combinations are preserved. For simplicity we denote by  $(Y^n)_{n \geq 1}$  this subsequence.

Let  $(\tilde{\Theta} \cdot (\tilde{\varphi}^j, \tilde{\phi}^k), A \cdot (\tilde{\varphi}^j, \tilde{\phi}^k))$  be the limit of  $(Y^n)_{n \geq 1}$  and

$$(3.42) \quad \Theta_t(\tilde{\varphi}^j, \tilde{\phi}^k) := \tilde{\Theta}_t(\tilde{\varphi}^j, \tilde{\phi}^k) + A_t(\tilde{\varphi}^j, \tilde{\phi}^k).$$

By Theorem 10 in [10], we have

$$(3.43) \quad \mathbb{E} \left[ \int_0^T dt \left( D\tilde{\Theta}_t(\tilde{\varphi}^j, \tilde{\phi}^k) \right)^2 \right] \leq \limsup_{n \rightarrow \infty} \mathbb{E} \left[ \int_0^T dt \left( D\tilde{\Theta}_t^n(\tilde{\varphi}^j, \tilde{\phi}^k) \right)^2 \right]$$

which implies

$$(3.44) \quad \mathbb{E} \left[ \int_0^T dt \left( D\tilde{\Theta}_t(\tilde{\varphi}^j, \tilde{\phi}^k) \right)^2 \right] \leq 2 \|\tilde{\varphi}^j\|_{L^\infty(M)}^2 \|\text{grad } \tilde{\phi}^k\|_{L^\infty(M)}^2 \mathcal{E}'(\sigma, \eta, T).$$

Inequality (3.39) also extends to the limiting process:

$$(3.45) \quad d \left[ \tilde{\Theta}(\tilde{\varphi}^j, \tilde{\phi}^k), \tilde{\Theta}(\tilde{\varphi}^j, \tilde{\phi}^k) \right]_t \leq \|\tilde{\varphi}^j\|_{L^2(M)}^2 \|\text{grad } \tilde{\phi}^k\|_{L^2(M)}^2 dt.$$

As for the starting point we have

$$(3.46) \quad \tilde{\Theta}_0(\tilde{\varphi}^j, \tilde{\phi}^k) = (\tilde{\varphi}^j, \tilde{\phi}^k)_{L^2(M)}.$$

Furthermore, by bilinearity of all the the  $\tilde{\Theta}^n$ , (3.44), (3.45) and (3.46) are still true with functions  $\varphi$  and  $\phi$  which are linear combinations of functions  $\tilde{\varphi}^j$  and  $\tilde{\phi}^k$ . Moreover  $\tilde{\Theta}$  is bilinear for these combinations.

For  $\varphi, \phi \in C^\infty(M)$  there exist sequences  $(\tilde{\varphi}^{j_\ell})_{\ell \geq 1}$  and  $(\tilde{\phi}^{k_\ell})_{\ell \geq 1}$  which converge uniformly to  $\varphi$  and  $\phi$  (for the second sequence uniform convergence holds for functions and their first order derivatives). From (3.44), (3.45) and (3.46) and the bilinearity of  $\tilde{\Theta}$  we deduce that  $\tilde{\Theta}(\tilde{\varphi}^{j_\ell}, \tilde{\phi}^{k_\ell})$  converges to a semimartingale  $\tilde{\Theta}(\varphi, \phi)$  which does not depend on the sequences  $(\tilde{\varphi}^{j_\ell})_{\ell \geq 1}$  and  $(\tilde{\phi}^{k_\ell})_{\ell \geq 1}$ . Here the convergence is taken in the topology of  $L^2$  convergence of the drift and the quadratic variation (the so-called  $H_2$  topology). It is also easy to check that  $(\varphi, \phi) \mapsto \tilde{\Theta}(\varphi, \phi)$  is bilinear and that for all  $\varphi, \phi \in C^\infty(M)$ ,  $\tilde{\Theta}(\varphi, \phi)$  is the limit in law of  $(\tilde{\Theta}^n(\varphi, \phi))_{n \geq 1}$  (the last point is due to the fact that the bounds in the right of (3.38) and (3.39) can be taken independent of  $n$ , and this allows to identify any limit of a subsequence of  $(\tilde{\Theta}^n(\varphi, \phi))_{n \geq 1}$  to  $\tilde{\Theta}(\varphi, \phi)$ ). Similarly, for  $\varphi_1, \phi_1, \varphi_2, \phi_2 \in C^\infty(M)$ , the process  $[\tilde{\Theta}(\varphi_1, \phi_1), \tilde{\Theta}(\varphi_2, \phi_2)]$  is the limit in law of  $([\tilde{\Theta}^n(\varphi_1, \phi_1), \tilde{\Theta}^n(\varphi_2, \phi_2)])_{n \geq 1}$ .

As a consequence, for all  $j, k \geq 1$

$$(3.47) \quad A_t(\tilde{\varphi}^j, \tilde{\phi}^k) = \frac{1}{2} \int_0^t \Theta_s(\tilde{\varphi}^j, \Delta \tilde{\phi}^k) ds$$

and by bilinearity, for  $\varphi, \phi \in C^\infty(M)$

$$(3.48) \quad \Theta_t(\varphi, \phi) = \tilde{\Theta}_t(\varphi, \phi) + \frac{1}{2} \int_0^t \Theta(\varphi, \Delta \phi) ds.$$

It remains to prove that  $\Theta$  is an element of  $\mathcal{H}'(\sigma, \eta, T)$  and that

$$(3.49) \quad \mathcal{E}'(\Theta) \leq \mathcal{E}'(\sigma, \eta, T).$$

By passing to the limit we get (1), (2), (3), (4), (6), (7), (8) of Definition 2.4. We are left to prove (3.49). For this we need to improve (3.43). Theorem 10 in [10] says that for any  $\varphi, \phi \in C^\infty(M)$  and any  $K > 0$ , if for all  $n \geq 1$

$$\mathbb{E} \left[ \int_0^T dt \left( D\tilde{\Theta}_t^n(\varphi, \phi) \right)^2 \right] \leq K$$

then

$$\mathbb{E} \left[ \int_0^T dt \left( D\tilde{\Theta}_t(\varphi, \phi) \right)^2 \right] \leq K$$

Let

$$K = \liminf_{n \rightarrow \infty} \mathbb{E} \left[ \int_0^T dt \left( D\tilde{\Theta}_t^n(\varphi, \phi) \right)^2 \right].$$

Consider a subsequence  $\Theta^{n_\ell}$  such that

$$\lim_{\ell \rightarrow \infty} \mathbb{E} \left[ \int_0^T dt \left( D\tilde{\Theta}_t^{n_\ell}(\varphi, \phi) \right)^2 \right] = K.$$

Then fixing  $\varepsilon > 0$  and applying the above result to the sequence  $\tilde{\Theta}^{n_\ell}$  for sufficiently large  $\ell$  we obtain

$$\mathbb{E} \left[ \int_0^T dt \left( D\tilde{\Theta}_t(\varphi, \phi) \right)^2 \right] \leq K + \varepsilon.$$

Letting  $\varepsilon \rightarrow 0$  we obtain

$$(3.50) \quad \mathbb{E} \left[ \int_0^T dt \left\| D\tilde{\Theta}_t(\varphi, \phi) \right\|^2 \right] \leq \liminf_{n \rightarrow \infty} \mathbb{E} \left[ \int_0^T dt \left\| D\tilde{\Theta}_t^n(\varphi, \phi) \right\|^2 \right].$$

Letting  $\varphi^j, \phi^k$  as in (2.21), we have

$$\begin{aligned} \sum_{j=1}^m \sum_{k=1}^{\ell} \mathbb{E} \left[ \int_0^T dt \frac{\left( D\tilde{\Theta}_t(\varphi^j, \phi^k) \right)^2}{\Theta_t(\varphi^j, 1)} \right] &\leq \sum_{j=1}^m \sum_{k=1}^{\ell} \liminf_{n \rightarrow \infty} \mathbb{E} \left[ \int_0^T dt \frac{\left( D\tilde{\Theta}_t^n(\varphi^j, \phi^k) \right)^2}{\Theta_t^n(\varphi^j, 1)} \right] \\ &\leq \liminf_{n \rightarrow \infty} \sum_{j=1}^m \sum_{k=1}^{\ell} \mathbb{E} \left[ \int_0^T dt \frac{\left( D\tilde{\Theta}_t^n(\varphi^j, \phi^k) \right)^2}{\Theta_t^n(\varphi^j, 1)} \right] \\ &\leq \mathcal{E}'(\sigma, \eta, T). \end{aligned}$$

Finally taking the supremum in the left as in (2.21) yields

$$(3.51) \quad \mathcal{E}'(\Theta) \leq \mathcal{E}'(\sigma, \eta, T)$$

and this achieves the proof.  $\square$

**Remark 3.2.** At this stage several questions arise. If a flow  $g$  is a critical point of the energy in  $\mathcal{H}_2(\sigma, \eta, T)$ , does it minimize the energy? Is the equality  $\mathcal{E}'(\sigma, \eta, T) = \mathcal{E}(\sigma, \eta, T)$  true? If the law of generalized flow  $\Theta$  minimizing  $\mathcal{E}'$  in  $\mathcal{H}_2'(\sigma, \eta, T)$  is unique, is it the law of a flow  $g \in \mathcal{H}_2(\sigma, \eta, T)$ ? The next result establishes convexity for the set of laws of drifts of generalized flows with minimizing energy. Do extremal laws in this set correspond to laws of drifts of flows  $g \in \mathcal{H}_2(\sigma, \eta, T)$ ?

**Proposition 3.3.** *The set of laws of processes  $D\tilde{\Theta}$  where  $\Theta \in \mathcal{H}_2'(\sigma, \eta, T)$  minimizes  $\mathcal{E}'$  is convex.*

*Proof.* Let  $\Theta^1$  and  $\Theta^2$  minimizing  $\mathcal{E}'$  defined respectively on  $\Omega_1$  and  $\Omega_2$ . On  $\Omega_1 \times \Omega_2 \times \{1, 2\}$  endowed with product filtration and product probability (on  $\{1, 2\}$  we consider the uniform probability) define  $\Theta(\omega_1, \omega_2, i) = \Theta^i(\omega_i)$ . It is straightforward to check that  $\Theta \in \mathcal{H}_2'(\sigma, \eta, T)$  and  $\mathcal{E}'(\Theta) \leq \frac{1}{2}(\mathcal{E}'(\Theta^1) + \mathcal{E}'(\Theta^2))$ . So  $\mathcal{E}'(\Theta) = \mathcal{E}'(\sigma, \eta, T)$ .  $\square$

## 4. CONSTRUCTING GENERALIZED FLOWS WITH PRESCRIBED DRIFT

We have shown in section 2 that semimartingale flows can be regarded as generalized ones. A smooth solution of Navier-Stokes equation will thus give rise to a generalized solution of the corresponding variational problem. We shall now show that these are not the only possible generalized flows: indeed, we can define weaker solutions (which, in particular, will not necessarily correspond to semimartingale flows) of the Navier-Stokes variational problem built upon weak solutions of some transport equations.

Now consider a deterministic drift  $b(t, x)$  such that  $b \in L^1([0, T], L^2(TM))$  and  $\operatorname{div} b \equiv 0$  in the weak sense.

**Theorem 4.1.** *There exists a generalized flow  $\Theta_t$  with drift  $b(t, x)$  i.e. such that for all  $\varphi, \phi \in C^\infty(M)$ ,  $(t, \omega) \in [0, T] \times \Omega$  almost everywhere*

$$(4.52) \quad D\tilde{\Theta}_t(\varphi, \phi) = \Theta_t(\varphi, \operatorname{div}(\phi b)),$$

and with kinetic energy smaller than or equal to  $\frac{1}{2} \int_0^T \left( \int_M \|b(t, x)\|^2 dx \right) dt$ .

*Proof.* For  $\varepsilon > 0$ , as in [9] Section 4.4 we regularize  $b$  by using the de Rham-Hodge semi-group on differential forms  $e^{\varepsilon \square}$  with  $\square = -(d\delta + \delta d)$ ,  $\delta$  the codifferential form of  $d$ . For a differential 1-form  $\alpha$  on  $M$  we denote by  $\alpha^\sharp$  the vector field on  $M$  associated to  $\alpha$  by the metric, and for a vector field  $A$  on  $M$  we denote by  $A^\flat$  the differential 1-form associated to  $A$  by the metric: we have

$$\langle \alpha, A \rangle = \langle \alpha^\sharp, A \rangle = \langle \alpha, A^\flat \rangle$$

where the first bracket is for duality, the second one is the scalar product in  $TM$ , the third one the scalar product in  $T^*M$ . By letting

$$(4.53) \quad (b')^\varepsilon = \left( e^{\varepsilon \square}(b^\flat) \right)^\sharp$$

we get a smooth time-dependent vector field satisfying  $\operatorname{div}(b')^\varepsilon = 0$  (see [9] Proposition 4.4.1). Then we regularize  $(b')^\varepsilon$  in time by convolution with a smooth kernel with support  $[-\varepsilon/2, \varepsilon/2]$  (for this we need to extend  $b$  by letting  $b(t, x) = 0$  for  $t < 0$  and for  $t > T$ ). Let us call  $b^\varepsilon(t, x)$  the regularized vector field. It is also divergence free, and it approximates  $b$  in  $(L^2([0, T] \times M, TM))$ . For each  $\varepsilon > 0$  we can construct a semimartingale flow as a strong solution to (2.7) where  $b(t, g(t)(x), \omega)$  has been replaced by  $b^\varepsilon(t, g(t)(x))$ . Let us denote by  $g^\varepsilon$  the solution. Letting  $(\varepsilon_n)_{n \geq 0}$  a sequence of positive numbers decreasing to 0 we let  $\Theta^n = \Theta^{g^{\varepsilon_n}}$ . Now using the fact that

$$\mathcal{E}(g^\varepsilon) = \frac{1}{2} \int_0^T \left( \int_M \|b^\varepsilon(t, x) dt\|^2 dx \right) dt \leq \frac{1}{2} \int_0^T \left( \int_M \|b(t, x) dt\|^2 dx \right) dt + 1$$

for  $\varepsilon$  sufficiently small we proceed similarly to the proof of Theorem 3.1 to establish that possibly by extracting a subsequence, there exists a generalized flow  $\Theta_t$  such that for all  $\varphi, \phi \in C^\infty(M)$   $\Theta^n(\varphi, \phi)$  converges in law to  $\Theta(\varphi, \phi)$  and  $\int_0^\cdot D\Theta_s^n(\varphi, \phi) ds$  converges in law to the drift of  $\tilde{\Theta}(\varphi, \phi)$ . We have  $D\tilde{\Theta}^n(\varphi, \phi) = \Theta^n(\varphi, \operatorname{div}(\phi b^{\varepsilon_n}))$  which is defined as  $\int_M \varphi(x) \operatorname{div}(\phi b^{\varepsilon_n} g(\cdot)(x)) dx$ . Since  $b$  is time-dependent and not smooth we have to extend this definition. We let  $(\tilde{\psi}^i)_{i \geq 1}$  be a family of smooth functions  $[0, T] \rightarrow \mathbb{R}$  such that, possibly by extending the family  $\tilde{\phi}^k$  defined in the proof of Theorem 3.1, linear combinations of functions  $(t, x) \mapsto \tilde{\psi}^i(t) \tilde{\phi}^k(x)$  with rational

coefficients are dense in  $L^2([0, T] \times M)$ . Now for all  $\beta(t, x) = \sum_{\ell=1}^L a_\ell \tilde{\psi}^{i_\ell}(t) \tilde{\phi}^{k_\ell}(x)$  with rational  $a_\ell$  we can assume that the processes

$$t \mapsto \int_0^t \Theta_s^n(\tilde{\phi}^j, \beta(s, \cdot)) ds = \sum_{\ell=1}^L a_\ell \int_0^t \Theta_s^n(\tilde{\phi}^j, \tilde{\phi}^{k_\ell}) \tilde{\psi}^{i_\ell}(s) ds$$

all together converge in law as  $n \rightarrow \infty$  to processes  $t \mapsto \int_0^t \Theta_s(\tilde{\phi}^j, \beta(s, \cdot)) ds$  which satisfy

$$(4.54) \quad \mathbb{E} \left[ \int_0^T \Theta_s(\tilde{\phi}^j, \beta(s, \cdot))^2 ds \right] \leq \|\tilde{\phi}^j\|_2^2 \|\beta\|_{L^2([0, T] \times M)}^2,$$

from the fact that for all  $n \geq 1$

$$(4.55) \quad \mathbb{E} \left[ \int_0^T \Theta_s^n(\tilde{\phi}^j, \beta(s, \cdot))^2 ds \right] \leq \|\tilde{\phi}^j\|_2^2 \|\beta\|_{L^2([0, T] \times M)}^2$$

This bound allows to define  $t \mapsto \int_0^t \Theta_s(\phi, \beta(s, \cdot)) ds$  for all smooth  $\phi$  and  $\beta \in L^2([0, T] \times M)$ . So  $t \mapsto \int_0^t \Theta_s(\varphi, \text{div}(\phi b)) ds$  is well defined, and by an argument similar to the proof of Theorem 3.1 we see that  $t \mapsto \int_0^t \Theta_s^n(\varphi, \text{div}(\phi b^{\varepsilon_n})) ds$  converges in law to  $t \mapsto \int_0^t \Theta_s(\varphi, \text{div}(\phi b)) ds$ . So we can make the identification  $\int_0^t D\tilde{\Theta}_s(\varphi, \phi) ds = \int_0^t \Theta_s(\varphi, \text{div}(\phi b)) ds$ .

We are left to prove the bound for the kinetic energy. But this is exactly similar to the first part of the proof of Proposition 2.6.  $\square$

## 5. CONSTRUCTING GENERALIZED FLOWS FROM SOLUTIONS OF FINITE VARIATION TRANSPORT EQUATIONS

In this section we aim to give an alternative construction of generalized flow with prescribed drift, using Ocone Pardoux method and weak solution of transport equations (in the sense of DiPerna and Lions).

To start with, let us consider a semimartingale flow  $g(t)(x)$  satisfying  $g(0)(x) \equiv x$  and

$$(5.56) \quad dg(t)(x) = \sigma(g(t)(x)) \circ dW_t + b(t, g(t)(x), \omega) g(t)(x) dt,$$

with the same assumptions as in the beginning of section 2. In particular the vector fields  $\sigma_i$  are divergence free. Assume that  $\sigma$  and  $b$  are  $C^1$  in the space variable. Let  $\tilde{g}(t)(x)$  be the martingale flow satisfying

$$(5.57) \quad d\tilde{g}(t)(x) = \sigma(\tilde{g}(t)(x)) \circ dW_t, \quad \tilde{g}(0)(x) \equiv x.$$

Notice that  $\tilde{g}(t)$  is measure preserving. The method of Ocone and Pardoux ([12]) consists in writing

$$(5.58) \quad g(t)(x) = \tilde{g}(t)(\psi(t)(x))$$

with  $\psi(t)(x)$  a bounded variation flow to be determined. From (5.58) we get

$$(5.59) \quad dg(t)(x) = (d\tilde{g}(t))(\psi(t)(x)) + T_{\psi(t)} \tilde{g}(t)(d\psi(t)(x))$$

which together with (5.56) and (5.57) yields

$$(5.60) \quad d\psi(t)(x) = \tilde{b}(t, \psi(t)(x), \omega) dt$$



with

$$(5.61) \quad \tilde{b}(t, y, \omega) = (T_y \tilde{g}(t)(\cdot))^{-1} (b(t, \tilde{g}(t)(y), \omega)).$$

For  $\varphi \in C^\infty(M)$  define  $\theta_t^{g, \varphi}, \theta_t^{\tilde{g}, \varphi}, \theta_t^{\psi, \varphi}$  as: for  $\phi \in C^\infty(M)$

$$(5.62) \quad \Theta^g(\varphi, \phi) = (\theta_t^{g, \varphi}, \phi)_{L^2(M)}, \quad \Theta^{\tilde{g}}(\varphi, \phi) = (\theta_t^{\tilde{g}, \varphi}, \phi)_{L^2(M)}, \quad \Theta^\psi(\varphi, \phi) = (\theta_t^{\psi, \varphi}, \phi)_{L^2(M)}$$

From (2.13) and (5.58) we get

$$(5.63) \quad \begin{aligned} \theta_t^{g, \varphi} &= \varphi \circ g(t)^{-1} \\ &= \varphi \circ \psi(t)^{-1} \circ \tilde{g}(t)^{-1}. \end{aligned}$$

and this yields

$$(5.64) \quad \Theta_t^g(\varphi, \phi) = \Theta_t^{\tilde{g}(t)}(\theta_t^{\psi, \varphi}, \phi)$$

which implies

$$(5.65) \quad \Theta_t^g(\varphi, \phi) = \Theta_t^\psi(\varphi, \phi \circ \tilde{g}(t))$$

where we used the fact that  $\tilde{g}(t)$  is measure preserving.

**Lemma 5.1.** *We have for all  $\phi \in C^\infty(M)$ ,  $t \in [0, T]$ , a.s.*

$$(5.66) \quad \int_M (\operatorname{div} \tilde{b}(t, \cdot, \omega))(x) \phi(x) dx = \int_M \operatorname{div} b(t, \cdot, \omega)(y) (\phi \circ (\tilde{g}(t))^{-1})(y) dy.$$

*In particular, if  $\operatorname{div} b(t, \cdot, \omega) \equiv 0$  then  $\operatorname{div} \tilde{b}(t, \cdot, \omega) \equiv 0$ .*

*Proof.* We will write  $b(y) = b(t, y, \omega)$ ,  $\tilde{b}(x) = \tilde{b}(t, x, \omega)$  For  $\phi \in C^\infty(M)$ ,

$$\begin{aligned} \int_M (\operatorname{div} \tilde{b}) \phi &= \int_M \langle d\phi, \tilde{b} \rangle \\ &= \int_M \langle d\phi, (T\tilde{g})^{-1} \circ b \circ \tilde{g} \rangle \\ &= \int_M \langle d(\phi \circ (\tilde{g})^{-1})(\tilde{g}(x)), b(\tilde{g}(x)) \rangle dx \\ &= \int_M \langle d(\phi \circ (\tilde{g})^{-1})(y), b(y) \rangle dy \\ &= \int_M (\phi \circ (\tilde{g})^{-1})(y) (\operatorname{div} b)(y) dy \end{aligned}$$

where we used in the fourth equality the fact that  $\tilde{g}$  is measure preserving.  $\square$

Now consider a deterministic drift  $b(t, x)$  such that  $b \in L^1([0, T], L^2(TM))$  and  $\operatorname{div} b \equiv 0$  in the weak sense. It is easily seen that Lemma 5.1 is still valid for  $b$ , so we have a.s.  $\operatorname{div} \tilde{b} \equiv 0$ . Moreover a.s.  $\tilde{b} \in L^1([0, T], L^2(TM))$ . Under this condition we can apply Proposition II.1 in [6] and we deduce that a.s. the transport equation which is the weak version of (5.60), namely

$$(5.67) \quad \frac{\partial \theta_t}{\partial t} = -(\tilde{b} \cdot \nabla) \theta_t, \quad \theta_0 = \varphi, \quad \varphi \in C^\infty(M),$$

has a solution  $\theta_t^{\tilde{b}, \varphi}$  in  $\bigcap_{p \geq 1} L^\infty(0, T; L^p(M))$ . Moreover since  $\tilde{b}$  is adapted the process

$\theta_t^{\tilde{b}, \varphi}$  can also be chosen so that all  $\theta_t^{\tilde{b}, \varphi}, \varphi \in C^\infty(M)$  and also  $\tilde{g}, W$  are jointly

adapted to the same filtration for which  $W$  is still a cylindrical Brownian motion and  $\tilde{g}$  satisfies (5.57): instead of considering  $\theta_t^{\tilde{b},\varphi}$  as the limit in law of some regularized processes  $\theta_t^{\tilde{b}^\varepsilon,\varphi}$ , consider  $((\theta_t^{\tilde{b},\tilde{\varphi}^j}, \tilde{g}, W)$  as limit in law of  $((\theta_t^{\tilde{b}^\varepsilon,\tilde{\varphi}^j}, \tilde{g}, W)$  for  $(\tilde{\varphi}^j)_{j \geq 1}$  a dense subsequence in  $C^\infty(M)$ .

So by analogy to (5.64) we define

$$(5.68) \quad \Theta_t^{\sigma,b}(\varphi, \phi) = \Theta_t^{\tilde{g}}\left(\theta_t^{\tilde{b},\varphi}, \phi\right), \quad \varphi, \phi \in C^\infty(M)$$

**Proposition 5.2.** *Take a deterministic drift  $b(t, x)$  such that  $b \in L^1([0, T], L^2(TM))$  and  $\operatorname{div} b \equiv 0$  in the weak sense. Then the generalized flow  $\Theta^{\sigma,b}$  defined in equation (5.68) is a generalized flow with kinetic energy*

$$(5.69) \quad \mathcal{E}'(\Theta^{\sigma,b}) \leq \mathcal{E}(b)$$

where

$$(5.70) \quad \mathcal{E}(b) = \frac{1}{2} \int_0^T dt \int_M dx \|b(t, x)\|^2$$

*Proof.* We have

$$(5.71) \quad \begin{aligned} \Theta^{\sigma,b}(\varphi, \phi) &= \Theta_t^{\tilde{g}}\left(\theta_t^{\tilde{b},\varphi}, \phi\right) = \int_M \theta_t^{\tilde{b},\varphi}((\tilde{g}(t))^{-1}(x)) \phi(x) dx \\ &= \int_M \theta_t^{\tilde{b},\varphi}(x) \phi(\tilde{g}(t)(x)) dx \end{aligned}$$

and this implies

$$(5.72) \quad \begin{aligned} d\tilde{\Theta}^{\sigma,b}(\varphi, \phi) &= \int_M \theta_t^{\tilde{b},\varphi}(x) \langle d\phi, d^{\text{It}\tilde{\circ}}\tilde{g}(t)(x) \rangle dx + \int_M \theta_t^{\tilde{b},\varphi}(x) \operatorname{div}(\tilde{b}(\phi \circ \tilde{g}(t)))(x) dx dt \\ &= \int_M \theta_t^{\tilde{b},\varphi}(x) \langle d\phi, \sigma(\tilde{g}(t)(x)) dW_t \rangle dx + \int_M \theta_t^{\tilde{b},\varphi}(x) \langle d(\phi \circ \tilde{g}(t)), \tilde{b} \rangle(x) dx dt \\ &= \sum_{i \geq 1} \int_M \theta_t^{\tilde{b},\varphi}(x) \langle d\phi, \sigma_i \rangle(\tilde{g}(t)(x)) dx dW_t^i + \int_M \theta_t^{\tilde{b},\varphi}(x) \langle d\phi, b \rangle(\tilde{g}(t)(x)) dx dt \\ &= \sum_{i \geq 1} \int_M \theta_t^{\tilde{b},\varphi}((\tilde{g}(t))^{-1}(y)) \langle d\phi, \sigma_i \rangle(y) dy dW_t^i + \int_M \theta_t^{\tilde{b},\varphi}((\tilde{g}(t))^{-1}(y)) \langle d\phi, b \rangle(y) dy dt \\ &= \sum_{i \geq 1} \Theta^{\sigma,b}(\varphi, \langle d\phi, \sigma_i \rangle) dW_t^i + \Theta^{\sigma,b}(\varphi, \langle d\phi, b \rangle) dt \end{aligned}$$

where the first term in the right is the martingale part and the second term is the finite variation part. We prefer to write the last equality as

$$(5.73) \quad d\tilde{\Theta}^{\sigma,b}(\varphi, \phi) = \sum_{i \geq 1} \Theta^{\tilde{g}}\left(\theta_t^{\tilde{b},\varphi}, \langle d\phi, \sigma_i \rangle\right) dW_t^i + \Theta^{\tilde{g}}\left(\theta_t^{\tilde{b},\varphi}, \langle d\phi, b \rangle\right) dt.$$

From this equation the properties of a generalized flow are easily checked.

We are left to prove that  $\mathcal{E}'(\Theta^{\sigma,b}) \leq \mathcal{E}(b)$ . Again this can be done via a regularization procedure of  $\tilde{b}$  of the form  $\tilde{b}^\varepsilon = \left(e^{\varepsilon \square}(\tilde{b})\right)^\sharp$ , an extraction of subsequence, and similar estimates as before. We leave the details to the reader.  $\square$

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