Efficiency in auctions with crossholdings

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Abstract

We study the impact of crossholdings on the efficiency of the standard auction formats. The ascending auction is not equivalent to the second-price auction. In a class of examples, the ascending auction is the only efficient standard auction format.

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1 Introduction

In many cases, firms seal an agreement, a collaboration or the creation of a joint-venture by exchanging shares. Regulation authorities do not control such an exchange provided that it "does not in itself give sole control of one party over the other or create a situation of common control" (European Council Regulation (1989)). However, these crossholdings affect the preferences of the agents. We prove, through the study of a paradigmatic economic interaction, the auction process[1], that these crossholdings matter and that they should be taken into account. We consider a framework in which two of the bidders have crossholdings and study how this affects the efficiency of the auction. Both the first-price and the second-price auction are inefficient. However, the ascending auction, which is not identical to the second-price auction in this context, is efficient.

2 The model

An indivisible good is auctioned to 3 risk-neutral bidders[2], 1, 2 and 3. Bidder i's valuation, i =1, 2,3, denoted by v_i, is private information to i. Each valuation is drawn independently from an interval $[0, 1]$ according to the same strictly increasing distribution function F with corresponding density f. F is common knowledge among bidders.

Bidder 1 owns a fraction θ of the capital of bidder 2 who symmetrically owns a fraction θ of the capital of bidder 1, with $\theta \in (0, \frac{1}{2})$. We assume that any additional profit of a bidder goes to its shareholders in proportion to their stakes.

In order to define bidders' utilities, we introduce the following notations: p_i is the probability that bidder i obtains the good and x_i is the expected payment of bidder i.

Bidder 3's utility function can be defined as follows: $U_3 = p_3 v_3 - x_3$

[1]For the study of the impact of shareholdings in the context of a Cournot model, see Reynolds and Snapp (1986).

[2]All our results can be extended to the case with $n \geq 3$ bidders, 2 bidders with crossholdings and other bidders without any shareholding.

Now, to define the two other bidders' utility functions, let us examine how a profit by bidder 1 affects bidder 2's utility and vice-versa. Suppose that bidder 1 makes a profit of π. Bidder 2 owns a fraction θ of bidder 1, then, he gets back $\theta\pi$. Now, bidder 1 owns a fraction θ of bidder 2, he consequently gets back $\theta^2\pi$ from bidder 2's profit. This mechanism continues *ad infinitum* so that the total profit of bidder 1 is $\Sigma_{k=0}^{\infty}(\theta^2)^k\pi = \frac{\pi}{1-\theta^2}$ and the total profit of bidder 2 is $\Sigma_{k=0}^{\infty}\theta(\theta^2)^k\pi = \frac{\theta\pi}{1-\theta^2}$. Thus, up to a rescaling of payoffs, we can represent bidders 1 and 2 as if they were maximizing utility functions defined as follows:

$$\begin{aligned} U_1 &= p_1v_1 - x_1 + \theta(p_2v_2 - x_2) \\ U_2 &= p_2v_2 - x_2 + \theta(p_1v_1 - x_1) \end{aligned}$$

3 Efficiency of the auction formats

3.1 The first-price auction

We directly obtain an impossibility result without computing the equilibria of the first-price auction.

Proposition 1 *There is no efficient equilibrium of the first-price auction.*

Proof: Suppose that an efficient equilibrium exists. Then, all the bidders must bid according to the same strictly increasing bidding function, b. Since b is a best response for bidder 3 and for bidder 1, $\forall v \in (0,1)$, we must have, in $u = v$:

$$\frac{\partial[(v-b(u))F^2(u)]}{\partial u} = 0 \tag{1}$$

$$\frac{\partial[(v-b(u))F^2(u) + \theta\int_u^1(v-b(t))F(t)f(t)dt]}{\partial u} = 0 \tag{2}$$

Taking the difference between (1) and (2), we derive:

$$\forall v \in (0,1),\ (v-b(v))F(v)f(v) = 0 \text{ and thus } b(v) = v$$

If bidders submit their valuations, their utilities are always equal to zero. Any bidder i can profitably deviate by submitting $\frac{v_i}{2}$. Therefore, this cannot be an equilibrium. Q.E.D.

Bidders do not have identical preferences. Bidder 3, if he loses the auction, derives a utility zero. In contrast, bidder 1, if he loses the auction, may derive a strictly positive utility in case bidder 2 wins the auction and makes a strictly positive profit. As a result, bidder 1 and bidder 3 do not bid the same and the auction cannot be efficient.

3.2 The second-price auction

The second-price auction was originally designed in order to obtain efficiency in a private value framework, even if bidders were ex-ante asymmetric in terms of valuation distribution. The following proposition shows that the efficiency property is not robust to the specific asymmetry we consider here.

Proposition 2 *There is no efficient equilibrium of the second-price auction.*

Proof: Suppose that an efficient equilibrium exists. Then, all the bidders must bid according to the same strictly increasing bidding function, b. If this bidding function is not the identity function, bidder 3 can profitably deviate by always submitting his valuation. Therefore, we must have $b = Id$.

Now, let us prove that submitting his valuation is not a best response for bidder 1 to bidders 2 and 3 submitting their valuations. If that were the case then $\forall v \in [0,1]$, the derivative in u of the following expression:

$$2\int_0^u (v-t)F(t)dF(t) + \theta[F(u)\int_u^1 (t-u)dF(t) + \int_u^1 \int_u^t (t-s)dF(s)dF(t)]$$

should be equal to zero for $u = v$.

Since

$$\frac{\partial \int_0^u (v-t)F(t)dF(t)}{\partial u} = (v-u)f(u)F(u)$$

Then, $\forall v \in [0,1]$, for $u = v$, the first part of the expression is equal to zero. Now, let us consider the second part of the expression: $g(u) = F(u)\int_u^1 (t-u)dF(t) + \int_u^1 \int_u^t (t-s)dF(s)dF(t)$. g is continuous and differentiable. Besides

$g(0) > 0$ and $g(1) = 0$, then g' cannot be equal to zero everywhere on the interval $[0, 1]$. Therefore, always submitting his valuation cannot be a best response for bidder 1. Q.E.D.

Bidders do not have identical motivations. While it is a dominant strategy for bidder 3 to submit his valuation, bidders 1 and 2 prefer to shade their bids in order to lower the price conditional on their losing the auction. Bidders have different bidding functions and the allocation is not efficient.

Our results do not allow to compare the first-price auction and the second-price auction in this context. This issue awaits future research.

3.3 The ascending auction

For the sake of simplicity, from now on, we restrict our study to piecewise continuous bidding functions and to equilibria with undominated strategies in which bidders who are ex-ante identical have identical strategies.

In the ascending auction, bidders observe the behaviors of their opponents while competing in the auction. Thus, bidders' strategies can depend on who is still active in the auction process. In the standard independent private value case, this information is irrelevant. Therefore, the second-price auction and the ascending auction are equivalent. Here, bidders use this information since they care about who wins if they do not. Hence the differences between the ascending auction and the second-price auction.

Proposition 3 *There is a unique equilibrium of the ascending auction. It is defined as follows.*

Bidder 3 quits the auction when the current price is equal to v_3.

Bidder 1 and 2: if bidder 3 is still active, they remain active as long as the current price is below their valuations and leave the auction when the current price is equal to their valuations for the good. If, for a price $\widehat{p} \geq 0$, *bidder 3 leaves the auction and bidder 1 and bidder 2 are both still active, then, for* $i = 1, 2$, *bidder* i *quits the auction at the price:*

$$v_i - \int_{\widehat{p}}^{v_i} \left(\frac{1 - F(v_i)}{1 - F(t)}\right)^{\frac{1-\theta}{\theta}} dt.$$

Proof: For bidder i, a strategy is a function $b_{i,j(\widehat{p})}(v)$ which defines the price for which bidder i, if his valuation is v, leaves the auction if j already left at a price p. By convention, we consider that $b_{i,0(0)}(v)$ (that we will also write $b_i(v)$) defines the price for which bidder i, if his valuation is v, leaves the auction if no bidder has left the auction yet.

Following the standard arguments for the ascending auction, we know that it is a dominant strategy for bidder 3 to bid in the following way: $\forall j = 0, 1, 2$, $\forall \widehat{p} \in R^+$ and $\forall v \in [0,1]$, $b_{3,j(\widehat{p})}(v) = v$. For the same reasons, it is also a dominant strategy for the two other bidders to have : $\forall \widehat{p} \in R^+$ and $\forall v \in [\widehat{p}, 1]$, $b_{1,2(\widehat{p})}(v) = b_{2,1(\widehat{p})}(v) = v$.

Now, let us consider b_1 and b_2. We chose to restrict our study to equilibria in which they are identical, then we can focus on b_1.

First, since it is a dominant strategy for $\forall j = 0, 1, 2$, $\forall \widehat{p} \in R^+$ to have $b_{1,2(\widehat{p})}, b_{2,1(\widehat{p})}$ and $b_{3,j(\widehat{p})}$ equal to the identity function, then it is a dominated strategy to have $b_1(v) > v$, $\forall v \in [0,1]$.

Suppose that b_1 is not increasing. Then $\exists (\underline{v}, \overline{v}) \in [0,1]^2$ with $\underline{v} < \overline{v}$ such that $b_1(\overline{v}) < b_1(\underline{v})$. Staying active in the interval $[b_1(\overline{v}), b_1(\underline{v})]$ has two possible consequences which could matter for bidder 1: raising the price paid by bidder 2 if he wins and winning the auction with a higher probability. The valuation of bidder 1 matters only for the second consequence. However if it is worth winning for a bidder with valuation $\underline{v}$, then, it is even more if his valuation is $\overline{v}$. Therefore, $b_1(\overline{v}) < b_1(\underline{v})$ cannot be part of an equilibrium. b_1 must be nondecreasing. The same type of arguments allows to rule out the possible existence of mass points.

Now, suppose that $\exists (\underline{v}, \overline{v}) \subset [0,1]$ with $\underline{v} < \overline{v}$ such that $\forall t \in (\underline{v}, \overline{v})$, $b_1(t) < t$. Since bidding functions are piecewise continuous, $\exists (\underline{u}, \overline{u}) \subset (\underline{v}, \overline{v})$, such that b_1 is continuous on $(\underline{u}, \overline{u})$. $\theta < \frac{1}{2}$, b_1 and F are continuous on $(\underline{u}, \overline{u})$, then $\forall t \in (\underline{u}, \overline{u})$, $\exists \varepsilon > 0$ such that:

$$\int_{b(t+\varepsilon)}^{t} (t-u)dF(u) - \theta \int_{b(t+\varepsilon)}^{t+\varepsilon} (t+\varepsilon-u)dF(u) > 0 \tag{3}$$

In that case, bidder 1, if his valuation is t, can profitably deviate by bidding according to $\widetilde{b}$ defined as follows:

$$\begin{aligned}
\widetilde{b}_1(t) &= b_1(t+\varepsilon) \\
\text{for } i &= 2,3, \forall x \in [0, b_1(t)],\ \widetilde{b}_{1,i(x)}(t) = b_{1,i(x)}(t) \\
\text{for } i &= 2,3, \forall x \in [b_1(t), b_1(t+\varepsilon)],\ \widetilde{b}_{1,3(x)}(t) = x \text{ and } \widetilde{b}_{1,2(x)}(t) = t
\end{aligned}$$

As a matter of fact, since we can exclude the possibility of a mass point of $b_{2,3(\widehat{p})}$, this change affects the outcome only if bidder 2 leaves the auction in the interval $[b(t), b(t+\varepsilon)]$. In that case, the following expression is an lower bound of bidder 1's net gain from this change:

$$\int_{b(t+\varepsilon)}^{t} (t-u)f(u)du - \theta \int_{b(t+\varepsilon)}^{t+\varepsilon} (t+\varepsilon-u)f(u)du \tag{4}$$

Since this expression is strictly positive,the deviation is strictly profitable and there cannot exist an interval of non-null measure on which $b_1(v) < v$.

Finally, we obtained that $b_1(v) \leq v$ and $b_1(v) \geq v$. Thus, b_1 cannot be anything else than the identity function.

For $b_{1,3(\widehat{p})}$ and $b_{2,3(\widehat{p})}$, we can apply results of Ettinger (2002) which tells us that there is a unique symmetric equilibrium if two bidders with crossholdings, θ, compete in an ascending auction and valuations are distributed according to a common distribution function G[3]: for $i = 1, 2$, bidder i leaves the auction when the current price is equal to $v_i - \int_{\underline{v}}^{v_i} (\frac{1-G(v_i)}{1-G(t)})^{\frac{1-\theta}{\theta}}$. Here, we must renormalize with $G(x) = \frac{F(x)-F(\widehat{p})}{F(1)-F(\widehat{p})}$ and $\underline{v} = \widehat{p}$. That way, we obtain: $\forall i = 1, 2$, $\forall \widehat{p} \in [0,1]$ and $\forall v_i \in [\widehat{p}, 1]$, $b_{i,3(\widehat{p})}(v_i) = v_i - \int_{\widehat{p}}^{v_i} (\frac{1-F(v_i)}{1-F(t)})^{\frac{1-\theta}{\theta}} dt$.

We proved that the proposed equilibrium is the only possible equilibrium. Simple computations show that it is indeed an equilibrium. Q.E.D.

Bidder 3 has a dominant strategy: to leave the auction when the current price is equal to his valuation.

Bidder 1's case is slightly more complex.[4] He has specific incentives only if bidder 2 has a strictly positive probability to win the good. Thus, once bidder 2 quits the auction process, bidder 1 has exactly the same incentives

[3] With $G(\underline{v}) = 0$, $G(\overline{v}) = 1$, G continuous and strictly increasing on $[\underline{v}, \overline{v}]$.

[4] We will only present bidder 1's case, bidder being completely symmetric.

as any standard bidder. It is a dominant strategy for him to quit the auction process when the price is equal to his valuation.

Now, what happens if all the bidders are still active? Suppose that bidder 1 quits for a price lower than his valuation. With a strictly positive probability, bidder 3 wins the auction while his valuation is lower than v_1 (assuming that bidder 2 and 3 behave according to equilibrium strategies). Bidder 1 would have been strictly better off if he had stayed active longer and had bought the good for a price v_3. To prevent such an event from happening, bidder 1 can stay active to observe which of the two other bidders quits first. If bidder 3 quits first, bidder 1 can always drop out immediately. If bidder 2 quits first, then bidder 1 stays active until the current price is equal to his valuation. Therefore, for bidder 1, if the current price is lower than his valuation, staying active is equivalent to a costless option whose value is strictly positive. That is why, bidder 1 stays active as long as the current price is below his valuation for the good.

At last, when bidder 3 quits the auction first, the two remaining bidders, bidders 1 and 2, are symmetric. They quit the auction according to an identical bidding function decreasing in θ. For more details on this case, see Ettinger (2002).

Corollary 1 *The second-price auction and the ascending auction are not equivalent. The ascending auction is efficient. It is the only efficient format among the standard auction formats.*

In presence of crossholdings, neither the first-price auction nor the second-price auction are efficient[5]. In contrast, the ascending auction is efficient because of its dynamic specificity. During the ascending auction, bidders discover who are their direct opponent and adapt their behaviors. In a static auction such as the second-price auction, bidders with crossholdings do not know who is their direct opponent at the time they choose their bids, hence the inefficiencies.

[5] In fact, no static mechanism that treats all the bidder the same can be efficient.

4 Related literature

We observed the non equivalence of the second-price auction and the ascending auction in the presence of crossholdings. These results are related to a strand of the auction literature that compares auction formats and more specifically these two auction formats. Milgrom and Weber (1982) first noticed the difference between the two auction formats in the affiliated values case. There, the ascending auction may give a higher expected revenue because of the different possibility to extract other bidders' signals and to reassess valuations. Maskin (1992) showed that in case of interdependent valuations, with two bidders and one-dimensional signals, the ascending auction is efficient if a single crossing condition holds. Finally, Das Varma (2002) considers a framework with fixed allocative externalities[6]. In this context, he also observes that the ascending auction reveals more pay-off relevant information than the second-price auction. For some configurations, this leads to a higher expected revenue. Our setting shares some elements with this approach. However, we focus on a different issue: the efficiency. Besides, the externalities we consider are not fixed, they depend on the price. Bidders with crossholdings, if they lose the auction, do not only care about the identity of the winner. They also care about the final price. Therefore, bidders have different motivations. For instance, even when only the two *crossholders* remain active, there are no dominant strategies.

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