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ABSTRACT

In [1], the authors have approached the sphere-like surfaces using the tensor product of an algebraic cubic spline quasi-interpolant with a 2π-periodic Uniform Algebraic Trigonometric B-splines (UAT B-splines) of order four. In this paper, we improve the results given in [1], by introducing a new quasi-interpolant based on the tensor product of an algebraic cubic spline quasi-interpolant with a periodic cubic spline quasi-interpolant, obtained by the periodization of an algebraic cubic spline quasi-interpolant. Our approach allows us to obtain an approximating surface which is of class $\mathcal{C}^2$ and with an approximation order $O(h^4)$. We show that this method is particularly well designed to render $3D$ closed surfaces, and it has been successfully applied to reconstruct human organs such as the left ventricle of the heart.

Index Terms— B-spline, Quasi-interpolant, Sphere-like surface, Medical imaging.

1. INTRODUCTION

Let $\mathcal{S}$ be a sphere-like surface, i.e. a surface of $\mathbb{R}^3$ which is topologically equivalent to a sphere. Assume that we have a set of scattered points $P_1, \ldots, P_d$ located on $\mathcal{S}$, along with real numbers $r_1, \ldots, r_d$ associated with each of these points. As in practice these numbers are derived, in general, from experiments and are almost always noisy, we are interested in this paper to find a function F defined on $\mathcal{S}$ which approximates them in the sense that

$$F(P_i) \simeq r_i, \quad 1 \leq i \leq d. \tag{1}$$

Data fitting problems where the underlying domain is a sphere-like surface arise in many areas, including e.g. geophysics and meteorology where $\mathcal{S}$ is taken as a model of earth (see [6]), and in medical modeling where $\mathcal{S}$ may be the surface of a human organ like heart, lung, bladder, kidney, etc. In most of these applications, the function F is constructed so that its associated closed surface $\mathcal{S}_F = \{F(s)s,\ s \in \mathcal{S}\}$ is at least continuous. In some cases, the $\mathcal{C}^1$ or $\mathcal{C}^2$ continuity is required.

Several methods have been proposed in the literature in the past fifteen years for fitting scattered data on sphere-like surfaces. For a survey of these methods on the sphere see [6] and references therein. Some other methods based on specific techniques have been developed in [7], [8], [9], [10], [11] and [12].

Among the methods developed for solving Problem (1) and based on quasi-interpolants, we encounter tensor spline methods. The principle of these methods consists in converting Problem (1) to one defined on a rectangle. More specifically and without loss of generality, when the surface $\mathcal{S}$ is the unit sphere, it can be identified with the rectangle $\mathcal{H} = [-\frac{\pi}{2}, \frac{\pi}{2}] \times [0, 2\pi]$ by the mapping χ defined by

$$\begin{array}{rccc} \chi: & \mathcal{H} & \longrightarrow & \mathcal{S} \\ & (\theta, \phi) & \longrightarrow & \begin{pmatrix} cos(\theta)cos(\phi) \\ cos(\theta)sin(\phi) \\ sin(\theta) \end{pmatrix}. \end{array}$$

The representation f of F in polar coordinates, defined on $\mathcal{H}$ by $f = Fo\chi$, is identical to that of F, i.e. $\mathcal{S}_f = \mathcal{S}_F = \{f(\theta, \phi)\chi(\theta, \phi),\ (\theta, \phi) \in \mathcal{H}\}$. However, the smoothness properties of f are not equivalent to those of its corresponding closed surface $\mathcal{S}_f$. More specifically, the surface $\mathcal{S}_f = \mathcal{S}_F$ is of class $\mathcal{C}^p$, $p = 0,\ 1,\ 2$, on $\mathcal{S}$ if and only if $f \in \mathcal{C}^p$ and satisfies the following $(2p+2)$ conditions :

$$f(\theta, 0) = f(\theta, 2\pi), \tag{$\mathcal{C}_1$}$$
$$f(\pm\frac{\pi}{2}, \phi) = c_{\pm}, \tag{$\mathcal{C}_2$}$$
$$\frac{\partial f}{\partial \phi}(\theta, 0) = \frac{\partial f}{\partial \phi}(\theta, 2\pi), \tag{$\mathcal{C}_3$}$$
$$\frac{\partial f}{\partial \theta}(\pm\frac{\pi}{2}, \phi) = a_{\pm}cos(\phi) + b_{\pm}sin(\phi), \tag{$\mathcal{C}_4$}$$
$$\frac{\partial^2 f}{\partial \phi^2}(\theta, 0) = \frac{\partial^2 f}{\partial \phi^2}(\theta, 2\pi), \tag{$\mathcal{C}_5$}$$
$$\frac{\partial^2 f}{\partial \theta^2}(\pm\frac{\pi}{2}, \phi) = d_{\pm} + e_{\pm}cos(2\phi) + f_{\pm}sin(2\phi), \tag{$\mathcal{C}_6$}$$

with $a_{\pm},\ b_{\pm},\ c_{\pm},\ d_{\pm},\ e_{\pm}$ and $f_{\pm}$ are real constants (see [2]).

Now, if we set $\mathcal{F}_p = \{f \in \mathcal{C}^p(\mathcal{H}) :\ \text{conditions } (\mathcal{C}_1) - (\mathcal{C}_{2p+2}) \text{ hold}\}$, then the problem of finding F such that $\mathcal{S}_F$ is of class $\mathcal{C}^p$ and satisfies $F(P_i) \simeq r_i,\quad 1 \leq i \leq d$, becomes equivalent to finding f belonging to $\mathcal{F}_p$ and satisfies

$f(\theta_i, \phi_i) \simeq r_i, \;\; 1 \leq i \leq d$, where (θ_i, ϕ_i) are the polar coordinates of P_i, i.e. $\chi(\theta_i, \phi_i) = P_i$.
Since the problem is now posed on a rectangular domain, it is natural to use a tensor spline method. Then, in this case, the quasi-interpolant emanating from such method for approximating f has the following form

$$\widetilde{f}(\theta, \phi) = \sum_{i=1}^{m} \sum_{j=1}^{n} c_{i,j} v_i(\theta) \widetilde{v_j}(\phi) \tag{2}$$

where $\{v_1, \ldots, v_m\}$ (resp. $\{\widetilde{v_1}, \ldots, \widetilde{v_n}\}$) is a linearly independent set of functions on $[-\frac{\pi}{2}, \frac{\pi}{2}]$ (resp. on $[0, 2\pi]$).

In Section 2, we give some preliminary results on cubic polynomial B-splines and we construct an associated quasi-interpolant of order four. In Section 3, we construct the periodic quasi-interpolant based on cubic polynomial B-splines of order four. Section 4, is devoted to the construction of a local quasi-interpolant on the sphere. Finally, in Section 5, we illustrate the performance of the method with some numerical tests (see Figure 2 and Figure 3) and applications to the left ventricle of the heart (see Figure 5).

2. QUASI-INTERPOLANT BASED ON CUBIC POLYNOMIAL B-SPLINES

In this section we construct a quasi-interpolant based on cubic polynomial B-splines which will be used in Section 4. For $I = [-\frac{\pi}{2}, \frac{\pi}{2}]$ and a given positive integer n, let $\Theta_n = \{\theta_i\}_{i=-3}^{n+3}$, with mesh length $h = \frac{\pi}{n}$, be a uniform partition of the interval I defined by

$$\begin{cases} \theta_i = -\frac{\pi}{2} + ih \;\text{ for }\; i = 0, ..., n \\ \theta_{-3} = \theta_{-2} = \theta_{-1} = -\frac{\pi}{2} \\ \theta_{n+1} = \theta_{n+2} = \theta_{n+3} = \frac{\pi}{2} \end{cases}$$

The associated polynomial spline space of order 4 is defined by

$$\mathcal{S}_4(I, \Theta_n) = \{s \in \mathcal{C}^2 : \; s_{|[\theta_i, \theta_{i+1}]} \in \mathbb{P}_3\},$$

where $\mathbb{P}_3$ is the polynomial space of degree ≤ 3. The classical normalized cubique B-splines B_i^4 satisfy $supp(B_i^4) = [\theta_i, \theta_{i+4}]$ and $B_i^4(\theta) > 0$, for $\theta_i < \theta < \theta_{i+4}$. They form a partition of unity, i.e. $\sum_{i=1}^{n+3} B_i^4(\theta) = 1$ and the family $\{B_i^4, \; i = 1, \ldots, n+3\}$ forms a basis of $\mathcal{S}_4(I, \Theta_n)$.
We now construct a local linear operator $\mathcal{Q}_1$ which maps a given function f onto the cubic spline space $\mathcal{S}_4(I, \Theta_n)$ and which has an optimal approximation order. This operator is the $\mathcal{C}^2$ cubic spline quasi-interpolant defined by

$$\mathcal{Q}_1 f := \sum_{i=1}^{n+3} \mu_i(f) B_i^4, \tag{3}$$

where the coefficients $\mu_i(f)$ are defined as linear combinations of some values and derivatives of f on the set Θ_n in order to have the exactness of the quasi-interpolant $\mathcal{Q}_1$ on $\mathbb{P}_3$, i.e. $\mathcal{Q}_1 p = p$, for all $p \in \mathbb{P}_3$.
More specifically, these coefficients are defined as follows

$$\begin{aligned}
&\mu_1(f) = f(-\frac{\pi}{2}), \\
&\mu_2(f) = f(-\frac{\pi}{2}) + \frac{h}{3} f'(-\frac{\pi}{2}), \\
&\mu_3(f) = f(-\frac{\pi}{2}) + h f'(-\frac{\pi}{2}) + \frac{h^2}{3} f''(-\frac{\pi}{2}), \\
&\mu_i(f) = \frac{1}{6}(-f(\theta_{i-3}) + 8f(\theta_{i-2}) - f(\theta_{i-1})), 4 \leq i \leq n, \\
&\mu_{n+1}(f) = f(\frac{\pi}{2}) - h f'(\frac{\pi}{2}) + \frac{h^2}{3} f''(\frac{\pi}{2}), \\
&\mu_{n+2}(f) = f(\frac{\pi}{2}) - \frac{h}{3} f'(\frac{\pi}{2}), \\
&\mu_{n+3}(f) = f(\frac{\pi}{2}).
\end{aligned}$$

Using classical theorems of approximation, see for example [3], we can easily prove that

$$\|\mathcal{Q}_1 f - f\|_{\infty, I} = O(h^4).$$

3. PERIODIC QUASI-INTERPOLANT BASED ON CUBIC POLYNOMIAL B-SPLINES

In this section we construct a periodic quasi-interpolant based on cubic polynomial B-splines. It is obtained by the periodization of the quasi-interpolant introduced in Section 2. More precisely, for $J = [0, 2\pi]$ and a given positive integer m, let $\Phi_m = \{\phi_j\}_{j=-3}^{m+3}$, with mesh length $k = \frac{2\pi}{m}$, be a uniform partition of the interval J defined by

$$\begin{cases} \phi_j = jk \;\text{ for }\; j = 0, ..., m \\ \phi_{-3} = \phi_{m-3}, \; \phi_{-2} = \phi_{m-2}, \; \phi_{-1} = \phi_{m-1} \\ \phi_{m+3} = \phi_3, \; \phi_{m+2} = \phi_2, \; \phi_{m+1} = \phi_1 \end{cases}$$

Then, this quasi-interpolant, can be written in the form

$$\mathcal{Q}_2 f := \sum_{j=1}^{m+3} \nu_j(f) B_j^4, \tag{4}$$

where

$$\nu_j(f) = \frac{1}{6}(-f_{j-3} + 8f_{j-2} - f_{j-1}), \;\; \text{for } 1 \leq j \leq m+3,$$

and $f_j = f(\phi_j), \forall j$.
It easy to see that for any periodic fonction $f \in C^4(J)$ of period 2π we have

$$\|\mathcal{Q}_2 f - f\|_{\infty, J} \leq C k^4 \|f^{(4)}\|_{\infty, J},$$

where C is a constant independent of m (see [4]).

4. QUASI-INTERPOLANT ON THE SPHERE

Now, we construct in this section a quasi-interpolant $\mathcal{Q}$ obtained by the tensor product of $\mathcal{Q}_1$ and $\mathcal{Q}_2$ defined in the preceding sections. This quasi-interpolant is given by

$$\mathcal{Q}f(\theta,\phi) := \sum_{i=1}^{n+3}\sum_{j=1}^{m+3} \mu_i(\nu_j(f))B_i^4(\theta)B_j^4(\phi) \qquad (5)$$

where
$f \in \mathcal{F}_2 = \{s \in \mathcal{C}^2(\mathcal{H}) : \text{ conditions } (\mathcal{C}_1)-(\mathcal{C}_6) \text{ hold}\}$.

It is obvious that this local linear operator $\mathcal{Q}f$ is of class $\mathcal{C}^2$ on $\mathcal{H}$ and with an approximation ordre $O(h^4)$.

5. NUMERICAL RESULTS AND APPLICATIONS TO MEDICAL IMAGING

5.1. Comparaison and application to synthetic data

To test the general performance of our method on synthetic data, we use the following function f defined explicitly on the rectangular domain $\mathcal{H}$ by

$$f(\theta,\ \phi) = \sum_{i=1}^{3}(g_i(\theta,\ \phi))^{-1/2},$$

where
$g_i(\theta,\ \phi) = \left(\frac{cos(\theta)cos(\phi)}{\alpha_i}\right)^2 + \left(\frac{cos(\theta)sin(\phi)}{\alpha_{i+1}}\right)^2 + \left(\frac{sin(\theta)}{\alpha_{i+2}}\right)^2$
with
$(\alpha_1,\ \alpha_2,\ \alpha_3,\ \alpha_4,\ \alpha_5) = (5,\ 1,\ 2,\ 5,\ 1)$, (see Figure 1).
It is easy to verify that $f \in \{s \in \mathcal{C}^2(\mathcal{H}) : \text{conditions } (\mathcal{C}_i) \text{ are satisfied, } \forall i = 1,\ ...,\ 6\}$ (see [5]).

We present in the following table a comparison on the error and the computation time corresponding to the quasi-interpolant $\mathcal{Q}$ and the quasi-interpolant $\tilde{\mathcal{Q}}$ studied in [1], for different values of m and n.

m	n	Mean Square Error (MSE)		Computation time ($seconds$)	
		$\tilde{\mathcal{Q}}$	$\mathcal{Q}$	$\tilde{\mathcal{Q}}$	$\mathcal{Q}$
50	50	1.2639	0.1326×10^{-3}	1.2734	0.0432
100	100	0.8505	0.0347×10^{-3}	9.1892	0.0786
150	150	0.7126	0.0156×10^{-3}	30.2975	0.1214
200	200	0.6437	0.0089×10^{-3}	71.7041	0.1797
250	250	0.6023	0.0058×10^{-3}	139.4654	0.2431
300	300	0.5747	0.0040×10^{-3}	247.4534	0.3272
350	350	0.5550	0.0029×10^{-3}	393.1402	0.4160
400	400	0.5402	0.0023×10^{-3}	602.9566	0.5121

Table 1. The MSE and the computation time of the quasi-interpolant $\mathcal{Q}$ compared with the quasi-interpolant $\tilde{\mathcal{Q}}$ given in [1].

Definition 1 *The MSE between $\mathcal{S}_f$ and $\mathcal{S}_{\mathcal{Q}f}$ was calculated as follow :*

$$MSE = \frac{1}{n.m}\sum_{i=1}^{n}\sum_{j=1}^{m}(\mathcal{Q}f(\theta_i,\phi_j) - f(\theta_i,\phi_j))^2,$$

where n and m represent the mesh sizes.

In the following figures, we illustrate the graph of the closed surface associated with f (see Figure 1), and The graph of the closed surface associated with $\mathcal{Q}f$ for $m = n = 8$ and $m = n = 16$ (see Figure 2 and Figure 3). We notice that the reconstructed surface is closer to the originale one for $m = n = 16$ than for $m = n = 8$.

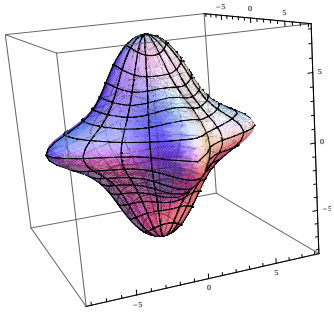

Fig. 1. The graph of the closed surface associated with f.

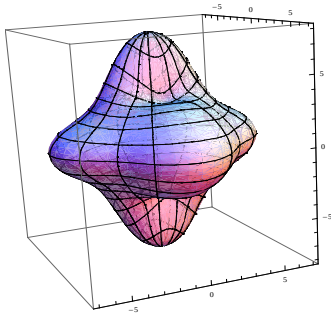

Fig. 2. The graph of the closed surface associated with $\mathcal{Q}f$ for $n = 8$ and $m = 8$.

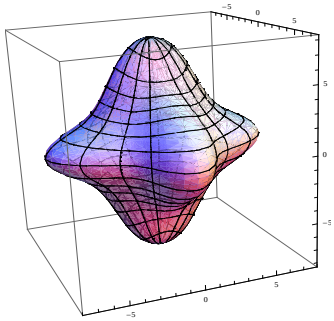

Fig. 3. The graph of the closed surface associated with $\mathcal{Q}f$ for $n = 16$ and $m = 16$.

5.2. Application to medical imaging

Medical imaging provides an ideal field to test the validity of our quasi-interpolation method. For example, the left ventricle of the human heart is a muscle hollow area that can be considered closed. In this regard, we consider 1024 points of the real surface of the left ventricle. So that we can reconstruct the closed surface associated with the left ventricle using our quasi-interpolant $\mathcal{Q}$ (see Figure 5). We first convert the data on the rectangle $\mathcal{H}$ using the application χ. Then, we apply the method of least squares that distributes these data evenly over the rectangle $\mathcal{H}$ (see Figure 4). Finally, we reconstruct the closed surface associated with these data using the quasi-interpolants $\mathcal{Q}$.

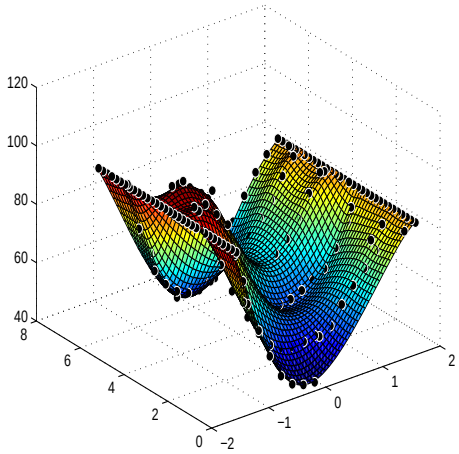

Fig. 4. Surface mesh on the rectangle $\mathcal{H}$ obtained by using the least squares method.

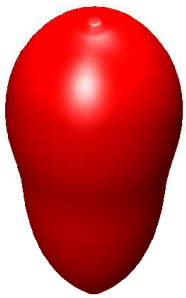

Fig. 5. The closed surface associated with the left ventricle using our quasi-interpolant $\mathcal{Q}$.

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