



**HAL**  
open science

## Stability estimates for an inverse scattering problem at high frequencies

Habib Ammari, Hajer Bahouri, David dos Santos Ferreira, Isabelle Gallagher

► **To cite this version:**

Habib Ammari, Hajer Bahouri, David dos Santos Ferreira, Isabelle Gallagher. Stability estimates for an inverse scattering problem at high frequencies. 2012. hal-00702325

**HAL Id: hal-00702325**

**<https://hal.science/hal-00702325>**

Preprint submitted on 30 May 2012

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

# STABILITY ESTIMATES FOR AN INVERSE SCATTERING PROBLEM AT HIGH FREQUENCIES

HABIB AMMARI, HAJER BAHOURI, DAVID DOS SANTOS FERREIRA,  
AND ISABELLE GALLAGHER

ABSTRACT. We consider an inverse scattering problem and its near-field approximation at high frequencies. We first prove, for both problems, Lipschitz stability results for determining the low-frequency component of the potential. Then we show that, in the case of a radial potential supported sufficiently near the boundary, infinite resolution can be achieved from measurements of the near-field operator in the monotone case.

## CONTENTS

Introduction	1
1. Far field pattern	3
2. Near field pattern	13
3. The case of a potential located near the boundary	15
4. Concluding remarks	19
Appendix A. Bessel functions	20
References	22

## INTRODUCTION

The first aim of this paper is to establish Lipschitz stability results for the inverse scattering problem of determining the low-frequency component (lower than the operating frequency) of the compactly supported potential from scattering or near-field measurements. It is known that, in general, the problem is exponentially unstable [4, 5, 6, 36, 40]. However, taking advantage of a priori information may improve stability and give accurate reconstruction algorithms [8, 12, 13]. The Lipschitz stability results proved in this paper together with the recent analysis of the local convergence of the nonlinear Landweber iteration in [33] show that the low-frequency component of the potential can be determined from the data in a linearly stable

---

*Date:* May 30, 2012.

*2000 Mathematics Subject Classification.* 35R30, 35B30.

*Key words and phrases.* Inverse scattering problem, Lipschitz stability, resolution.

This work was supported by the ERC Advanced Grant Project MULTIMOD-267184 and the Institut Universitaire de France.

way. Moreover, they precisely quantify the resolution limit, which is defined as the characteristic size of the smallest oscillations in the potential that can be stably recovered from the data. Since Rayleigh's work, it has been admitted that the resolution limit in inverse scattering is of order  $\pi$  over the operating frequency [19]. This is nothing else than a direct application of the uncertainty principle in inverse scattering [18, 24, 28, 55]. It is well-known that if the support of the potential is a point support, then the reconstructed location of the point potential from the scattering data has finite size of order of the Rayleigh resolution limit [7, 18]. Having this in mind, the results of this paper prove that the Fourier transform of the potential can be reconstructed in a linearly stable way for all frequencies (dual variable to the space one) smaller than the operating frequency, and therefore, justify the notion of resolution limit. More intriguingly, again in view of [33], they prove that the stability of the reconstruction of the potential increases at high operating frequencies.

The second aim of the paper is to show that infinite resolution can be achieved from near-field measurements. Here, the near-field operator approximates Sommerfeld's radiation condition and is equivalent to the measurements of the Cauchy data at a finite distance. Moreover, if the potential is supported near the boundary, then infinite resolution can be achieved in the monotone case. In fact, a Lipschitz stability result holds for both the low and high frequency components of the potential. It should be noted that the scattering amplitude can be recovered from the near-field operator. However, approximating the near-field operator from the scattering amplitude is a severely ill-posed problem [37, 47, 48, 59] and therefore cannot be of any practical and realistic use. It was shown in [37] that in order to compute the near-field operator from the scattering amplitude one needs to differentiate the scattering amplitude an infinite number of times.

The results of this paper extend to medium scattering the recent results in [9, 10, 63], where a stability and resolution analysis was performed for linearized conductivity and wave imaging problems. They can be also used to justify the hopping (or continuation in the frequency) reconstruction algorithms proposed in [14, 24, 25, 26].

In connection with our results in this paper, we also refer to the works by Isakov [34] and Isakov and Kindermann [35], Bao, Lin, and Triki [15, 16], Nagayasu, Uhlmann and Wang [45, 46], as well as Derveaux, Papanicolaou, and Tsogka [27]. In [34, 35], an evidence of increasing stability in wave imaging when frequency is growing was given. In [15], stability estimates for the inverse source problem were established and the conversion of the logarithmic type stability to a Lipschitz one first proved. Numerical results to illustrate the stability of the source reconstruction problem were presented in [16]. In [17], Lipschitz stability estimates for the time-dependent wave equation were obtained. In [45], a stability estimate for a linearized conductivity problem was derived and its dependence on the depth of the

inclusion highlighted. In [46], it is shown that the ill-posedness of the inverse acoustic problem decreases when the frequency increases and the stability estimate changes from logarithmic type for low frequencies to a Lipschitz type for large frequencies. In [27], the enhancement of resolution in the near-field was studied and numerically illustrated. Our results in this paper confirm these important observations in a quite general situation and precisely quantify them.

Our paper is organized as follows. Section 1 is devoted to the stability of the reconstruction of the potential from the scattering amplitude (called also far-field pattern) in the high frequency regime. Theorem 2 proves that the low-frequency component of the potential can be determined in a stable way from the scattering amplitude. The threshold frequency determines the resolution limit. Section 2 extends the results of Section 1 to the near-field measurements. Theorem 3 shows that the same results as those in Section 1 hold for reconstructing the potential from measurements of the near-field operator. In Section 3 we show that we gain infinite resolution for potentials supported near the boundary. If the potential is supported near the boundary, then infinite resolution can be achieved in the monotone case. Theorem 4 provides a Lipschitz stability result for both the low and high frequency components of the potential. Finally, in Appendix A, we provide useful results on Bessel's functions.

Finally, we mention that the letter  $C$  will be used to denote a universal constant which may vary from line to line. We also use  $A \lesssim B$  to denote an estimate of the form  $A \leq CB$  for some constant  $C$ . We also use the classical notation  $\langle x \rangle = \sqrt{1 + |x|^2}$ .

**Acknowledgements.** We are very grateful to J. Sjöstrand for taking the time to discuss some properties of the Laplace transform with us.

## 1. FAR FIELD PATTERN

**1.1. Definitions and notations.** Let  $\Omega$  be a bounded domain in the Euclidean space  $\mathbb{R}^d$  of dimension  $d \geq 2$ , let  $q \in C_0^\infty(\mathbb{R}^d)$  be a real-valued potential supported in  $\Omega$ . We use the classical notation  $D = -i\partial$  for derivatives and consider the Helmholtz equation with potential

$$(1.1) \quad D^2u - \lambda^2u + qu = 0$$

at frequency  $\lambda \in \mathbb{R}_+^* := \mathbb{R}_+ \setminus \{0\}$ . Plane waves  $e^{i\lambda x \cdot \omega}$  propagating along the direction  $\omega$  in  $S^{d-1}$  are solutions of the free Helmholtz equation

$$(1.2) \quad D^2u - \lambda^2u = 0.$$

Here  $S^{d-1}$  denotes the unit sphere in  $\mathbb{R}^d$ . More generally, plane waves generate the set of solutions in the space of tempered distributions,  $\mathcal{S}'(\mathbb{R}^d)$ , of the free Helmholtz equation: all solutions of (1.2) with polynomial growth are superpositions of elementary plane waves  $e^{i\lambda x \cdot \omega}$  when  $\omega$  varies on the

sphere  $S^{d-1}$ . It will be useful later on to adopt Melrose's notation in [42] to designate these solutions: for  $g \in \mathcal{C}^\infty(S^{d-1})$ , we shall write

$$\Phi_0(\lambda)g(x) = \int_{S^{d-1}} e^{i\lambda x \cdot \omega} g(\omega) d\omega.$$

Obviously  $\Phi_0(\lambda)g$  is a solution of (1.2) which belongs to  $\mathcal{S}'(\mathbb{R}^d)$ .

To guarantee the uniqueness of solutions to the Helmholtz equation (1.1), one can impose conditions on the behavior of solutions at infinity. More precisely, we are interested in solutions which can be decomposed

$$(1.3) \quad u = e^{i\lambda x \cdot \omega} + u^{\text{scat}} = u^{\text{in}} + u^{\text{scat}}$$

as the sum of an incoming planar wave and a scattered wave satisfying Sommerfeld's radiation condition

$$(1.4) \quad \left| \left( \frac{\partial}{\partial |x|} - i\lambda \right) u^{\text{scat}} \right| = o \left( \frac{1}{|x|^{\frac{d-1}{2}}} \right) \quad \text{as } |x| \rightarrow +\infty,$$

uniformly with respect to the direction  $\theta = \frac{x}{|x|}$  at fixed frequency  $\lambda \in \mathbb{R}_+^*$ . The following result from [42, Lemma 2.4] holds.

**Proposition 1.1.** *There exists a unique solution to the Helmholtz equation (1.1) with potential  $q \in \mathcal{C}_0^\infty(\Omega)$  of the form*

$$(1.5) \quad \varphi_q(x, \omega, \lambda) = e^{i\lambda x \cdot \omega} + \varphi_q^{\text{scat}}(x, \omega, \lambda),$$

where the scattered wave  $\varphi_q^{\text{scat}}$  satisfies Sommerfeld's radiation condition (1.4) and which is given by

$$(1.6) \quad \varphi_q^{\text{scat}} = -R_q(\lambda)(e^{i\lambda x \cdot \omega} q).$$

Here  $R_q$  denotes the meromorphic continuation of the perturbed resolvent. Furthermore  $\varphi_q$  depends smoothly on  $(x, \omega, \lambda) \in \mathbb{R}^d \times S^{d-1} \times \mathbb{R}_+^*$  and is bounded.

We choose to denote

$$\Phi_q(\lambda)g = \int_{S^{d-1}} \varphi_q(x, \omega, \lambda)g(\omega) d\omega$$

the operator with kernel  $\varphi_q(x, \omega, \lambda)$  given by (1.5).

**Theorem 1.** *The scattered wave in the solution (1.3) to the Helmholtz equation (1.1) given by Proposition 1.1 assumes the form*

$$(1.7) \quad \varphi_q^{\text{scat}}(x) = \frac{e^{i\lambda|x|}}{|x|^{\frac{d-1}{2}}} a_q \left( \frac{x}{|x|}, \omega, \lambda \right) + \mathcal{O} \left( \frac{1}{|x|^{\frac{d+1}{2}}} \right) \quad \text{as } |x| \rightarrow +\infty,$$

where  $a_q$  is a smooth function on  $S^{d-1} \times S^{d-1} \times \mathbb{R}_+^*$  and

$$a_q \left( \frac{x}{|x|}, \omega, \lambda \right) = -\frac{1}{2i\lambda} \left( \frac{\lambda}{2\pi i} \right)^{\frac{d-1}{2}} \int q(y) \varphi_q(y, \omega, \lambda) e^{-i\lambda \frac{x}{|x|} \cdot y} dy.$$

*Proof.* We consider the Green function  $G_\lambda$  corresponding to the free Helmholtz equation

$$(1.8) \quad (D_y^2 - \lambda^2)G_\lambda(x, y) = \delta(x - y),$$

subject to Sommerfeld's radiation condition (1.4), with  $\delta$  being the Dirac delta function.

Let  $R > 0$  be large enough so that the ball of radius  $R$  contains the support of  $q$ . By definition of the Green function, for all  $|x| \leq R$  we have

$$u(x) = \int_{|y| \leq R} (D_y^2 - \lambda^2)G_\lambda(x, y)u(y) \, dy,$$

and if  $u$  is a solution to the Helmholtz equation (1.1) we deduce by Green's formula that for any  $|x| \leq R$

$$\begin{aligned} u(x) &= - \int_{|y| \leq R} G_\lambda(x, y)q(y)u(y) \, dy \\ &\quad - \int_{|y|=R} \left( \frac{\partial G_\lambda}{\partial r}(x, y) u(y) - G_\lambda(x, y) \frac{\partial u}{\partial r}(y) \right) \, d\sigma(y). \end{aligned}$$

Along the same lines, it is possible to derive a similar identity for the plane wave  $u^{\text{in}}(x) = e^{i\lambda\omega \cdot x}$

$$u^{\text{in}}(x) = - \int_{|y|=R} \left( \frac{\partial G_\lambda}{\partial r}(x, y) u^{\text{in}}(y) - G_\lambda(x, y) \frac{\partial u^{\text{in}}}{\partial r}(y) \right) \, d\sigma(y),$$

taking into account the fact that  $(D^2 - \lambda^2)u^{\text{in}} = 0$ . Subtracting the two identities gives the following representation formula for the scattered wave  $u^{\text{scat}} = u - u^{\text{in}}$

$$\begin{aligned} u^{\text{scat}}(x) &= - \int_{|y| \leq R} G_\lambda(x, y) q(y) u(y) \, dy \\ &\quad - \int_{|y|=R} \left( \frac{\partial G_\lambda}{\partial r}(x, y) u^{\text{scat}}(y) - G_\lambda(x, y) \frac{\partial u^{\text{scat}}}{\partial r}(y) \right) \, d\sigma(y). \end{aligned}$$

The Green function of the free Helmholtz equation is explicitly given by

$$(1.9) \quad G_\lambda(x, y) = \frac{1}{4i} \left( \frac{\lambda}{2\pi} \right)^{\frac{d-2}{2}} |x - y|^{-\frac{d-2}{2}} H_{d/2-1}^{(1)}(\lambda|x - y|),$$

where  $H_{d/2-1}^{(1)}$  is the Hankel function of first kind and order  $d/2 - 1$  (cf. Appendix A). The asymptotic behavior of Hankel functions (A.3) implies that for  $|y|$  large enough

$$(1.10) \quad G_\lambda(x, y) = \frac{1}{2i\lambda} \left( \frac{\lambda}{2\pi i} \right)^{\frac{d-1}{2}} e^{i\lambda|x-y|} |x - y|^{-\frac{d-1}{2}} \left( 1 + \mathcal{O}\left( \frac{1}{\lambda|x-y|} \right) \right).$$

Since we have

$$|x - y| = |y| - \frac{x \cdot y}{|y|} + \mathcal{O}\left( \frac{|x|^2}{|y|} \right)$$

we find for  $|y|$  large enough and fixed  $\lambda$ :

$$e^{i\lambda|x-y|} = e^{i\lambda|y|} e^{-i\lambda\frac{y}{|y|}\cdot x} \left( 1 + \mathcal{O}\left(\frac{|x|^2}{|y|}\right) \right).$$

We therefore obtain for fixed  $x, \lambda$  and large  $R = |y|$

$$(1.11) \quad G_\lambda(x, y) = \frac{1}{2i\lambda} \left( \frac{\lambda}{2\pi i} \right)^{\frac{d-1}{2}} |y|^{-\frac{d-1}{2}} e^{i\lambda|y|} e^{-i\lambda\frac{y}{|y|}\cdot x} \left( 1 + \mathcal{O}\left(\frac{1}{R}\right) \right).$$

Analogously, we have

$$\frac{\partial G_\lambda}{\partial r}(x, y) = i\lambda c_d(\lambda) |y|^{-\frac{d-1}{2}} e^{i\lambda|y|} e^{-i\lambda\frac{y}{|y|}\cdot x} \left( 1 + \mathcal{O}\left(\frac{1}{R}\right) \right),$$

with  $c_d(\lambda) = \frac{1}{2i\lambda} (\lambda/2\pi i)^{\frac{d-1}{2}}$ . This leads to

$$\begin{aligned} u^{\text{scat}}(x) &= - \int_{|y|\leq R} G_\lambda(x, y) q(y) u(y) dy \\ &\quad + \frac{c_d(\lambda)}{R^{\frac{d-1}{2}}} \int_{|y|=R} \left( \frac{\partial u^{\text{scat}}}{\partial r} - i\lambda u^{\text{scat}} \right) e^{i\lambda R - i\lambda\frac{y}{|y|}\cdot x} \left( 1 + \mathcal{O}\left(\frac{1}{R}\right) \right) d\sigma(y). \end{aligned}$$

Sommerfeld's radiation condition implies that the second right-hand side term tends to zero when  $R$  tends to infinity, so we get

$$u^{\text{scat}}(x) = - \int G_\lambda(x, y) q(y) u(y) dy.$$

Using once again the asymptotic formula (1.11) together with the fact that the Green function is symmetric we get

$$u^{\text{scat}}(x) = - \frac{1}{2i\lambda} \left( \frac{\lambda}{2\pi i} \right)^{\frac{d-1}{2}} \frac{e^{i\lambda|x|}}{|x|^{\frac{d-1}{2}}} \int q(y) u(y) e^{-i\lambda\frac{x}{|x|}\cdot y} \left( 1 + \mathcal{O}\left(\frac{|y|^2}{|x|}\right) \right) dy.$$

To summarize, we have that

$$u^{\text{scat}}(x) = \frac{e^{i\lambda|x|}}{|x|^{\frac{d-1}{2}}} a_q \left( \frac{x}{|x|}, \omega, \lambda \right) + \mathcal{O}\left(\frac{1}{|x|^{\frac{d+1}{2}}}\right),$$

with

$$a_q \left( \frac{x}{|x|}, \omega, \lambda \right) = - \frac{1}{2i\lambda} \left( \frac{\lambda}{2\pi i} \right)^{\frac{d-1}{2}} \int q(y) u(y) e^{-i\lambda\frac{x}{|x|}\cdot y} dy.$$

This proves that the scattered part of any solution of Helmholtz' equation subject to Sommerfeld's radiation condition takes the form announced in Theorem 1.  $\square$

**Definition 1.2.** We define the scattering amplitude associated with the potential  $q \in C_0^\infty(\mathbb{R}^d)$  by the smooth function  $a_q : S^{d-1} \times S^{d-1} \times \mathbb{R}_+^* \rightarrow \mathbb{C}$  given by

$$(1.12) \quad a_q(\theta, \omega, \lambda) = - \frac{1}{2i\lambda} \left( \frac{\lambda}{2\pi i} \right)^{\frac{d-1}{2}} \int q(y) \varphi_q(y, \omega, \lambda) e^{-i\lambda\theta\cdot y} dy.$$

We denote

$$A_q(\lambda)g(\theta) = \int_{S^{d-1}} a_q(\theta, \omega, \lambda)g(\omega) d\omega, \quad \theta \in S^{d-1},$$

the corresponding operator with kernel  $a_q(\theta, \omega, \lambda)$ .

It is easy to get an asymptotic expansion of

$$\Phi_q(\lambda)g(x) = \int_{S^{d-1}} (e^{i\lambda x \cdot \omega} + \varphi_q^{\text{scat}}(x, \omega, \lambda))g(\omega) d\omega$$

as  $|x| \rightarrow +\infty$  using the stationary phase and Theorem 1:

$$(1.13) \quad \Phi_q(\lambda)g = \left(\frac{2\pi}{\lambda|x|}\right)^{\frac{d-1}{2}} \left( e^{-i\lambda|x|} e^{i(d-1)\frac{\pi}{4}} g(-\theta) \right. \\ \left. + e^{i\lambda|x|} \left( e^{-i(d-1)\frac{\pi}{4}} g(\theta) + \left(\frac{\lambda}{2\pi}\right)^{\frac{d-1}{2}} A_q(\lambda)g(\theta) \right) \right) + \mathcal{O}\left(\frac{1}{|x|}\right)$$

with  $\theta = x/|x| \in S^{d-1}$ . The operator which maps the coefficient of  $e^{-i\lambda|x|}$  into the coefficient of  $e^{i\lambda|x|}$  is given by

$$g(-\theta) \mapsto i^{-d+1} \left( g(\theta) + \left(\frac{\lambda i}{2\pi}\right)^{\frac{d-1}{2}} A_q(\lambda)g(\theta) \right).$$

This is, after renormalization and composition with the antipodal map, the scattering matrix [42].

**Definition 1.3.** *The scattering matrix is the operator given by*

$$S_q(\lambda) = \text{Id} + \left(\frac{\lambda i}{2\pi}\right)^{\frac{d-1}{2}} A_q(\lambda).$$

Integration by parts allows to relate the values of the potential inside the domain with the scattering matrix  $S_q(\lambda)$ .

**Lemma 1.4.** *We have the following identities*

$$\int (q_1 - q_2)u_1 \overline{u_2} dx = -2i\lambda \left(\frac{2\pi}{\lambda}\right)^{d-1} \int_{S^{d-1}} g_1 \overline{g_2} - S_{q_1}(\lambda)g_1 \overline{S_{q_2}(\lambda)g_2} d\omega, \\ \int (q_1 - q_2)u_1 u_2 dx = -2i\lambda \left(\frac{2\pi}{\lambda i}\right)^{d-1} \int_{S^{d-1}} \check{g}_2 S_{q_1}(\lambda)g_1 - \check{g}_1 S_{q_2}(\lambda)g_2 d\omega,$$

for any pair of solutions<sup>1</sup>

$$u_1 = \Phi_{q_1}(\lambda)g_1, \quad u_2 = \Phi_{q_2}(\lambda)g_2,$$

to the Helmholtz equations (1.1) related to the potentials  $q_1, q_2$ .

---

<sup>1</sup>We use the notation  $\check{g}(\omega) = g(-\omega) = Pg(\omega)$  for the antipodal map.



*Proof.* Let  $R$  be large enough so that the ball of radius  $R$  contains the support of both potentials  $q_1, q_2$ . By Green's formula, we have

$$\int_{|x| \leq R} (q_1 - q_2) u_1 \overline{u_2} \, dx = \int_{|x|=R} \partial_r u_1 \overline{u_2} - u_1 \overline{\partial_r u_2} \, d\sigma(x)$$

and using the asymptotic formula (1.13) on the functions  $u_1 = \Phi_{q_1}(\lambda)g_1$  and  $u_2 = \Phi_{q_2}(\lambda)g_2$ , we deduce

$$\begin{aligned} & \partial_r u_1 \overline{u_2} - u_1 \overline{\partial_r u_2} = \\ & -2i\lambda \left( \frac{2\pi}{\lambda R} \right)^{d-1} \left( g_1(-\theta) \overline{g_2(-\theta)} - S_{q_1}(\lambda)g_1(\theta) \overline{S_{q_2}(\lambda)g_2(\theta)} \right) + \mathcal{O}\left( \frac{1}{R^d} \right), \end{aligned}$$

which implies

$$\begin{aligned} & \int_{|x| \leq R} (q_1 - q_2) u_1 \overline{u_2} \, dx \\ & = -2i\lambda \left( \frac{2\pi}{\lambda} \right)^{d-1} \int_{S^{d-1}} \left( g_1 \overline{g_2} - S_{q_1} g_1 \overline{S_{q_2} g_2} \right) \, d\theta + \mathcal{O}\left( \frac{1}{R} \right). \end{aligned}$$

Letting  $R$  tend to infinity provides the first identity.

The proof of the second identity is similar, since

$$\begin{aligned} \partial_r u_1 u_2 - u_1 \partial_r u_2 &= -2i\lambda \left( \frac{2\pi}{\lambda R} \right)^{d-1} \left( g_1(-\theta) S_{q_2} g_2(\theta) - S_{q_1}(\lambda)g_1(\theta) g_2(-\theta) \right) \\ &+ \mathcal{O}\left( \frac{1}{R^d} \right). \end{aligned}$$

This completes the proof of the lemma.  $\square$

Choosing  $q_1 = q_2 = q$  in Lemma 1.4, we obtain the following properties of the scattering matrix:

$${}^t S_q(\lambda) = P \circ S_q(\lambda) \circ P, \quad S_q(\lambda)^* S_q(\lambda) = \text{Id}$$

recalling that  $Pg = \check{g}$  is the antipodal map. Here,  $t$  denotes the transpose and  $*$  the transpose conjugate. Besides, using the first relation together with Lemma 1.4, we finally get

$$(1.14) \quad \int (q_1 - q_2) u_1 u_2 \, dx = 2i\lambda \left( \frac{2\pi}{\lambda i} \right)^{\frac{d-1}{2}} \int_{S^{d-1}} \check{g}_2 (A_{q_1}(\lambda) - A_{q_2}(\lambda)) g_1 \, d\omega,$$

for any pair of solutions

$$u_1 = \Phi_{q_1}(\lambda)g_1, \quad u_2 = \Phi_{q_2}(\lambda)g_2,$$

to the Helmholtz equations (1.1) with Sommerfeld's radiation condition (1.4) related to the potentials  $q_1, q_2$ .

**1.2. Stability estimates at high frequencies.** The first stability result we shall prove in this paper states that low frequencies (i.e. smaller than  $2\lambda$ ) may be recovered in a stable way from the scattering amplitude.

**Theorem 2.** *For all  $\varepsilon, M, R > 0$  and all  $\alpha > d$  there exist  $C_\varepsilon, \lambda_0 > 0$  such that the following stability estimate holds true.*

*Let  $q_1, q_2 \in C_0^\infty(\mathbb{R}^d)$  be two potentials supported in the ball centered at 0 and of radius  $R$  such that  $\|q_1\|_{L^\infty}, \|q_2\|_{L^\infty} \leq M$ . Then for all  $\lambda \geq \lambda_0$ ,*

$$(1.15) \quad \int_{|\xi| \leq (2-\varepsilon)\lambda} \langle \xi \rangle^{-\alpha} |\widehat{(q_1 - q_2)}(\xi)|^2 d\xi \leq C_\varepsilon \lambda^3 \|a_{q_1} - a_{q_2}\|_{L^2}^2 + C_\varepsilon \lambda^{-2} \|q_1 - q_2\|_{L^\infty}^2.$$

The proof of Theorem 2 relies on estimates on solutions to the Helmholtz equations stated below, as well as on the following lemma, stating that one can relate the Fourier transform of the potential and the scattering amplitude.

**Lemma 1.5.** *For all  $M, R > 0$  there exist constants  $C, \lambda_0 > 0$  such that for all potentials  $q \in C_0^\infty(\mathbb{R}^d)$  supported in the ball  $B(0, R)$  and satisfying  $\|q\|_{L^\infty} \leq M$ , we have the following approximation:*

$$(1.16) \quad \left| a_q(\theta, \omega, \lambda) + \frac{1}{2i\lambda} \left( \frac{\lambda}{2\pi i} \right)^{\frac{d-1}{2}} \widehat{q}(\lambda(\theta - \omega)) \right| \leq C \lambda^{\frac{d-5}{2}} \|q\|_{L^2}^2$$

for all  $\lambda \geq \lambda_0$ .

Lemma 1.5 is based on the following classical estimate on the free resolvent  $R_0(\lambda)$ . We refer to [21] for an exposition by Burq of an elementary proof due to Zworski. This type of *a priori* estimates have a long history and go back to the work of Agmon [2] on weighted estimates on the resolvent and the limiting absorption principle.

**Proposition 1.6.** *Let  $\chi \in C_0^\infty(\mathbb{R}^d)$ , there exists a constant  $C > 0$  such that for all  $\lambda > 1$  we have*

$$\|\chi R_0(\lambda) \chi\|_{\mathcal{L}(L^2(\mathbb{R}^d))} \leq C \lambda^{-1}.$$

*Proof of Lemma 1.5.* We start with the expression defining the scattering amplitude (see Definition 1.2):

$$a_q(\theta, \omega, \lambda) = -\frac{1}{2i\lambda} \left( \frac{\lambda}{2\pi i} \right)^{\frac{d-1}{2}} \int q(y) \varphi_q(y, \omega, \lambda) e^{-i\lambda\theta \cdot y} dy$$

so by (1.5) we deduce that

$$\begin{aligned} a_q(\theta, \omega, \lambda) + \frac{1}{2i\lambda} \left( \frac{\lambda}{2\pi i} \right)^{\frac{d-1}{2}} \widehat{q}(\lambda(\theta - \omega)) \\ = -\frac{1}{2i\lambda} \left( \frac{\lambda}{2\pi i} \right)^{\frac{d-1}{2}} \int q(x) \varphi_q^{\text{scat}}(x, \omega, \lambda) e^{-i\lambda x \cdot \theta} dx. \end{aligned}$$

The resolvent identity reads (see for instance Formula (2.3) in [42])

$$(1.17) \quad R_q(\lambda) = R_0(\lambda) - R_0(\lambda) q R_q(\lambda)$$

which implies that if  $\chi \in \mathcal{C}_0^\infty(\mathbb{R}^d)$  is a cutoff function which equals 1 on the ball  $B(0, R)$ , then

$$\chi R_q(\lambda) \chi = \chi R_0(\lambda) \chi - (\chi R_0(\lambda) \chi) q (\chi R_q(\lambda) \chi).$$

If we apply this identity to  $-qe^{i\lambda x \cdot \omega}$ , we get in view of (1.6)

$$\chi \varphi_q^{\text{scat}} = -(\chi R_0(\lambda) \chi) (qe^{i\lambda x \cdot \omega}) + (\chi R_0(\lambda) \chi) (q \chi \varphi_q^{\text{scat}})$$

and using the estimate of Proposition 1.6, we obtain

$$\|\chi \varphi_q^{\text{scat}}\|_{L^2} \leq C\lambda^{-1} \|q\|_{L^2} + C\lambda^{-1} \|q\|_{L^\infty} \|\chi \varphi_q^{\text{scat}}\|_{L^2}.$$

Taking  $\lambda \geq 2CM$  we deduce

$$\|\chi \varphi_q^{\text{scat}}\|_{L^2} \leq 2C\lambda^{-1} \|q\|_{L^2}.$$

Using Cauchy-Schwarz's inequality, we get

$$\left| \int q \varphi_q^{\text{scat}} e^{-i\lambda x \cdot \omega} dx \right| \leq 2C\lambda^{-1} \|q\|_{L^2}^2$$

and this completes the proof of the lemma.  $\square$

Note that the main ingredient in the proof of Lemma 1.5 consists in combining the estimate for the non-perturbed resolvent (Proposition 1.6) with the resolvent identity (1.17), along with the fact that  $q$  is compactly supported.

This lemma is the basis of the reconstruction of the potential from the scattering amplitude at high frequencies, since one can choose  $\theta, \omega \in S^{d-1}$  such that  $\lambda(\theta - \omega) = \xi$  for any fixed frequency  $\xi$  with  $|\xi| \leq 2\lambda$  and let  $\lambda$  tend to infinity. We use a similar approach to prove a stability estimate at high frequencies.

*Proof of Theorem 2.* Again, we start with the expression defining the scattering amplitude and then express the potential in terms of the scattering amplitude: for  $j \in \{1, 2\}$  we write

$$\widehat{q}_j(\lambda(\theta - \omega)) = -2i(2\pi i)^{\frac{d-1}{2}} \lambda^{-\frac{d-3}{2}} a_{q_j}(\theta, \omega, \lambda) - \int q_j(x) \varphi_{q_j}^{\text{scat}}(x, \omega, \lambda) e^{-i\lambda x \cdot \theta} dx.$$

Taking the difference of the two expressions yields

$$(1.18) \quad \left| (\widehat{q_1 - q_2})(\lambda(\theta - \omega)) \right| \leq 2(2\pi)^{\frac{d-1}{2}} \lambda^{-\frac{d-3}{2}} |(a_{q_1} - a_{q_2})(\theta, \omega, \lambda)| \\ + \|q_1 - q_2\|_{L^2} \|\varphi_{q_1}^{\text{scat}}\|_{L^2(|x| \leq R)} + \|q_2\|_{L^2} \|\varphi_{q_1}^{\text{scat}} - \varphi_{q_2}^{\text{scat}}\|_{L^2(|x| \leq R)}$$

where we recall that the supports of  $q_1, q_2$  are contained in the ball of center 0 and radius  $R > 0$ . As in the proof of Lemma 1.5 we have

$$(1.19) \quad \|\varphi_{q_1}^{\text{scat}}\|_{L^2(|x| \leq R)} \leq C\lambda^{-1} \|q_1\|_{L^2}.$$

In light of (1.6), the difference of the two scattered waves satisfies

$$\varphi_{q_1}^{\text{scat}} - \varphi_{q_2}^{\text{scat}} = -(R_{q_1}(\lambda) - R_{q_2}(\lambda))(q_1 e^{i\lambda x \cdot \omega}) - R_{q_2}(\lambda)((q_1 - q_2) e^{i\lambda x \cdot \omega})$$

and by the resolvent identity (1.17), the first term on the right-hand side reads

$$R_{q_2}(\lambda)(q_1 - q_2)R_{q_1}(\lambda)(q_1 e^{i\lambda x \cdot \omega}) = -R_{q_2}(\lambda)(q_1 - q_2)\varphi_{q_1}^{\text{scat}}.$$

Let  $\chi \in C_0^\infty(\mathbb{R}^d)$  be a cutoff function which equals 1 on the ball  $B(0, R)$ , then

$$\begin{aligned} \chi(\varphi_{q_1}^{\text{scat}} - \varphi_{q_2}^{\text{scat}}) &= -(\chi R_{q_2}(\lambda)\chi)((q_1 - q_2)\varphi_{q_1}^{\text{scat}}) \\ &\quad - (\chi R_{q_2}(\lambda)\chi)((q_1 - q_2)e^{i\lambda x \cdot \omega}). \end{aligned}$$

Therefore along the same lines as in the proof of Lemma 1.5, the resolvent identity (1.17) and Proposition 1.6 give rise to

$$(1.20) \quad \|\varphi_{q_1}^{\text{scat}} - \varphi_{q_2}^{\text{scat}}\|_{L^2(|x| \leq R)} \lesssim \lambda^{-1} \|q_1 - q_2\|_{L^\infty} \|\varphi_{q_1}^{\text{scat}}\|_{L^2(|x| \leq R)} + \lambda^{-1} \|q_1 - q_2\|_{L^2}.$$

Indeed invoking the resolvent identity (1.17), we infer that

$$\chi R_{q_2}(\lambda)\chi = \chi R_0(\lambda)\chi - (\chi R_0(\lambda)\chi)q_2(\chi R_{q_2}(\lambda)\chi).$$

Thus applying this identity to  $(q_1 - q_2)\varphi_{q_1}^{\text{scat}}$ , we obtain

$$\begin{aligned} \|(\chi R_{q_2}(\lambda)\chi)((q_1 - q_2)\varphi_{q_1}^{\text{scat}})\|_{L^2} &\leq C\lambda^{-1} \|q_1 - q_2\|_{L^\infty} \|\varphi_{q_1}^{\text{scat}}\|_{L^2(|x| \leq R)} \\ &\quad + C\lambda^{-1} \|q_2\|_{L^\infty} \|(\chi R_{q_2}(\lambda)\chi)((q_1 - q_2)\varphi_{q_1}^{\text{scat}})\|_{L^2}, \end{aligned}$$

which implies for  $\lambda > 2CM$

$$\|(\chi R_{q_2}(\lambda)\chi)((q_1 - q_2)\varphi_{q_1}^{\text{scat}})\|_{L^2} \leq 2C\lambda^{-1} \|q_1 - q_2\|_{L^\infty} \|\varphi_{q_1}^{\text{scat}}\|_{L^2(|x| \leq R)}.$$

We have also

$$\|(\chi R_{q_2}(\lambda)\chi)((q_1 - q_2)e^{i\lambda x \cdot \omega})\|_{L^2} \lesssim \lambda^{-1} \|q_1 - q_2\|_{L^2},$$

which achieves the proof of Estimate (1.20).

Taking into account those bounds, we obtain in view of (1.18) the estimate

$$(1.21) \quad |(\widehat{q_1 - q_2})(\lambda(\theta - \omega))| \lesssim \lambda^{-\frac{d-3}{2}} |(a_{q_1} - a_{q_2})(\theta, \omega, \lambda)| + \lambda^{-1} \|q_1 - q_2\|_{L^\infty}.$$

We denote  $r(\xi) = \sqrt{1 - |\xi|^2}$  when  $|\xi| \leq 1$ . For  $\xi \in \mathbb{R}^d$  with  $|\xi| \leq 2\lambda$ , choose  $\eta \in \xi^\perp$  with norm  $|\eta| = r(\xi/2\lambda)$ . The vectors

$$\theta = \eta + \frac{\xi}{2\lambda}, \quad \omega = \eta - \frac{\xi}{2\lambda},$$

have length one, taking the square of (1.21) and integrating the resulting estimate with respect to  $\eta$  yields

$$\begin{aligned} |(\widehat{q_1 - q_2})(\xi)|^2 &\lesssim \lambda^{-2} \|q_1 - q_2\|_{L^\infty}^2 \\ &\quad + \lambda^{-d+3} r(\xi/2\lambda)^{-d+2} \int_{\substack{|\eta|=r(\xi/2\lambda) \\ \eta \perp \xi}} \left| (a_{q_1} - a_{q_2})\left(\eta + \frac{\xi}{2\lambda}, \eta - \frac{\xi}{2\lambda}, \lambda\right) \right|^2 d\eta. \end{aligned}$$

When  $|\xi| \leq (2 - \varepsilon)\lambda$  we have  $r(\xi/2\lambda) \geq \sqrt{\varepsilon(4 - \varepsilon)}/2$  therefore multiplying the previous inequality by  $\langle \xi \rangle^{-\alpha} \leq 1$  and integrating with respect to  $\xi$  we get

$$\begin{aligned} \int_{|\xi| \leq (2-\varepsilon)\lambda} \langle \xi \rangle^{-\alpha} |\widehat{(q_1 - q_2)}(\xi)|^2 d\xi &\leq C_\varepsilon \lambda^{-2} \|q_1 - q_2\|_{L^\infty}^2 \\ &+ C_\varepsilon \lambda^3 \int_{|\xi| \leq 1} \int_{\substack{|\eta|=r(\xi) \\ \eta \perp \xi}} |(a_{q_1} - a_{q_2})(\eta + \xi, \eta - \xi, \lambda)|^2 d\eta d\xi. \end{aligned}$$

We consider the following  $2d - 2$  dimensional submanifold of  $S^{2d-1}$

$$\Sigma = \{(\xi, \eta) \in S^{2d-1} : \langle \xi, \eta \rangle = 0\}$$

and the following diffeomorphism

$$\begin{aligned} \varphi : \Sigma &\rightarrow S^{d-1} \times S^{d-1} \\ (\xi, \eta) &\mapsto (\xi + \eta, \eta - \xi), \end{aligned}$$

with inverse

$$\begin{aligned} \varphi^{-1} : S^{d-1} \times S^{d-1} &\rightarrow \Sigma \\ (\theta, \omega) &\mapsto \left( \frac{\theta - \omega}{2}, \frac{\theta + \omega}{2} \right) \end{aligned}$$

for which we have

$$\begin{aligned} \int_{\Sigma} F(\eta + \xi, \eta - \xi) d\eta \wedge d\xi &= \int_{S^{d-1} \times S^{d-1}} F(\theta, \omega) \varphi^{-1*}(d\eta \wedge d\xi) \\ &= 2^{-d} \int_{S^{d-1} \times S^{d-1}} F(\theta, \omega) d\omega \wedge d\theta. \end{aligned}$$

Then we finally get

$$\begin{aligned} \int_{|\xi| \leq (2-\varepsilon)\lambda} \langle \xi \rangle^{-\alpha} |\widehat{(q_1 - q_2)}(\xi)|^2 d\xi \\ \leq C_\varepsilon \lambda^3 \iint_{S^{d-1} \times S^{d-1}} |a_{q_1}(\theta, \omega, \lambda) - a_{q_2}(\theta, \omega, \lambda)|^2 d\theta d\omega \\ + C_\varepsilon \lambda^{-2} \|q_1 - q_2\|_{L^\infty}^2 \end{aligned}$$

and this completes the proof of our estimate.  $\square$

In particular, we recover the uniqueness of the potential from the scattering amplitude at high frequencies.

**Corollary 1.7.** *If for all  $(\theta, \omega, \lambda)$  belonging to  $S^{d-1} \times S^{d-1} \times \mathbb{R}_+^*$ , we have  $a_{q_1}(\theta, \omega, \lambda) = a_{q_2}(\theta, \omega, \lambda)$  then  $q_1 = q_2$ .*

## 2. NEAR FIELD PATTERN

**2.1. Definitions and notations.** Instead of considering the Helmholtz equation on the whole Euclidean space (with Sommerfeld's radiation condition) we focus on the Cauchy problem with Robin boundary condition on a bounded open set  $\Omega \subset \mathbb{R}^d$  with smooth boundary

$$(2.1) \quad \begin{cases} (D^2 - \lambda^2 + q)u = 0 & \text{in } \Omega, \\ (\partial_\nu - i\lambda)u|_{\partial\Omega} = f \in L^2(\partial\Omega). \end{cases}$$

This problem has a unique solution  $u \in H^1(\Omega)$  for all  $f \in L^2(\partial\Omega)$ . Writing a variational formulation of (2.1) and using a unique continuation argument shows the uniqueness of a solution to (2.1). The existence follows from Fredholm's alternative [41].

**Remark 2.1.** Other classical boundary conditions are either Dirichlet or Neumann boundary conditions. However, one has to make the additional assumption that  $\lambda^2$  is not a Dirichlet (or Neumann) eigenvalue of  $D^2 + q$ , for the Dirichlet problem corresponding to (2.1) to have a unique solution for all  $f \in H^{\frac{1}{2}}(\partial\Omega)$ . Unfortunately, this does not make sense if one wants to take the high frequency limit  $\lambda \rightarrow \infty$ . To bypass this difficulty, we study the boundary problem (2.1). Indeed, this condition is natural. It approximates Sommerfeld's radiation condition at high frequencies [30, 38].

**Remark 2.2.** It follows from [41] that the unique solution  $u \in H^1(\Omega)$  of (2.1) satisfies

$$\|Du\|_{L^2(\Omega)} + \lambda\|u\|_{L^2(\Omega)} \leq C(\|qu\|_{L^2(\Omega)} + \|f\|_{L^2(\partial\Omega)})$$

for some constant  $C$  independent of  $\lambda$ . Therefore, as  $\lambda \rightarrow \infty$ , we have

$$\|Du\|_{L^2(\Omega)} + \frac{\lambda}{2}\|u\|_{L^2(\Omega)} \leq C\|f\|_{L^2(\partial\Omega)}.$$

**Definition 2.3.** *The near field pattern is the map*

$$\begin{aligned} \mathcal{N}_q(\lambda) : L^2(\partial\Omega) &\rightarrow L^2(\partial\Omega) \\ f &\mapsto u|_{\partial\Omega}. \end{aligned}$$

The typical inverse problem on the near field pattern is whether it uniquely determines the potential  $q$ . This was solved (for smooth potentials) by Sylvester and Uhlmann [57] in dimension  $d \geq 3$  in the case of the Dirichlet-to-Neumann map. Reconstruction methods were proposed by Nachman in [43] and stability issues were studied by Alessandrini [6]. It was shown by Mandache in [40] that the logarithmic stability result of Alessandrini in [6] is optimal.

**2.2. Stability estimates at high frequencies.** The following result is the counterpart of Theorem 2 in the near-field context.

**Theorem 3.** *For all  $M, R > 0$  and all  $\alpha > d$  there exist  $C, \lambda_0 > 0$  such that the following stability estimate holds true.*

*Let  $q_1, q_2 \in C_0^\infty(\mathbb{R}^d)$  be two potentials supported in the ball centered at 0 and of radius  $R$  such that  $\|q_1\|_{L^\infty}, \|q_2\|_{L^\infty} \leq M$ . Then for all  $\lambda \geq \lambda_0$ ,*

$$(2.2) \quad \int_{|\xi| \leq 2\lambda} \langle \xi \rangle^{-\alpha} |\widehat{q_1 - q_2}(\xi)|^2 d\xi \leq C \|\mathcal{N}_{q_1} - \mathcal{N}_{q_2}\|^2 + C\lambda^{-2} \|q_1 - q_2\|_{L^\infty}^2.$$

*Proof.* We start with two solutions  $u_1, u_2$  of the equations

$$(D^2 - \lambda^2 + q_j)u_j = 0$$

and computing

$$\begin{aligned} \int_{\Omega} (q_1 - q_2)u_1u_2 dx &= \int_{\Omega} \Delta u_1 u_2 dx - \int_{\Omega} u_1 \Delta u_2 dx \\ &= \int_{\partial\Omega} (\partial_\nu - i\lambda)u_1 u_2 d\sigma - \int_{\partial\Omega} u_1 (\partial_\nu - i\lambda)u_2 d\sigma \end{aligned}$$

yields the formula

$$\int_{\Omega} (q_1 - q_2)u_1u_2 dx = \int_{\partial\Omega} (u_2 \mathcal{N}_{q_1}(\lambda)u_1 - u_1 \mathcal{N}_{q_2}(\lambda)u_2) d\sigma.$$

Choosing  $q_1 = q_2$  shows that the application  $\mathcal{N}_{q_j}$  is symmetric, and using this additional information, we get the formula

$$\int_{\Omega} (q_1 - q_2)u_1u_2 dx = \int_{\partial\Omega} u_2 (\mathcal{N}_{q_1}(\lambda) - \mathcal{N}_{q_2}(\lambda))u_1 d\sigma.$$

Let us write  $q = q_1 - q_2$  and choose

$$\begin{aligned} u_1 &= \varphi_{q_1} = e^{i\lambda x \cdot \theta_1} + \varphi_{q_1}^{\text{scat}} \\ \text{and } u_2 &= \varphi_{q_2} = e^{i\lambda x \cdot \theta_2} + \varphi_{q_2}^{\text{scat}}, \end{aligned}$$

with  $\theta_1, \theta_2 \in S^{d-1}$ . Recall from the proof of Lemma 1.5 that the scattered waves satisfy the estimate

$$\|\varphi_{q_1}^{\text{scat}}\|_{L^2(B(0,R))} + \|\varphi_{q_2}^{\text{scat}}\|_{L^2(B(0,R))} \leq C\lambda^{-1}.$$

It follows that

$$\begin{aligned} \int_{\Omega} e^{i\lambda x \cdot (\theta_1 + \theta_2)} q(x) dx &= \int_{\partial\Omega} (\mathcal{N}_{q_1}(\lambda) - \mathcal{N}_{q_2}(\lambda))u_1u_2 d\sigma \\ &\quad - \int_{\Omega} q(x)(\varphi_{q_1}^{\text{scat}} + \varphi_{q_2}^{\text{scat}} + \varphi_{q_1}^{\text{scat}}\varphi_{q_2}^{\text{scat}}) dx \end{aligned}$$

and therefore

$$\begin{aligned} |\widehat{q}(\lambda(\theta_1 + \theta_2))| &\leq \|\mathcal{N}_{q_1}(\lambda) - \mathcal{N}_{q_2}(\lambda)\| \|u_1\|_{L^2(\partial\Omega)} \|u_2\|_{L^2(\partial\Omega)} + \frac{C}{\lambda} \|q\|_{L^\infty} \\ &\leq C \|\mathcal{N}_{q_1}(\lambda) - \mathcal{N}_{q_2}(\lambda)\| + \frac{C}{\lambda} \|q\|_{L^\infty} \end{aligned}$$

for all  $(\theta_1, \theta_2) \in S^{d-1} \times S^{d-1}$ .

We notice that

$$\begin{aligned} S^{d-1} \times S^{d-1} &\rightarrow B(0, 2) \\ (\theta_1, \theta_2) &\mapsto \theta_1 + \theta_2 \end{aligned}$$

is a submersion when  $\theta_1, \theta_2$  are not colinear. This implies that

$$(2.3) \quad |\widehat{q}(\xi)| \leq C \|\mathcal{N}_{q_1}(\lambda) - \mathcal{N}_{q_2}(\lambda)\| + \frac{C}{\lambda} \|q\|_{L^\infty}, \quad \xi \in B(0, 2\lambda).$$

Multiplying this estimate by  $\langle \xi \rangle^{-\alpha}/2$ , taking the square and integrating on the ball  $B(0, 2\lambda)$  completes the proof of the theorem.  $\square$

### 3. THE CASE OF A POTENTIAL LOCATED NEAR THE BOUNDARY

**3.1. Definitions and notations.** In this section we show, in the model case of the unit disk, that if the potential is supported close to the boundary of the disk, then a larger range of frequencies may be recovered by the near field  $\mathcal{N}_q(\lambda)$  than in the general case treated in the previous two sections. More precisely, introducing radial coordinates  $(x_1, x_2) = (r \cos \theta, r \sin \theta)$ , with  $(r, \theta) \in \mathbb{R}^+ \times [0, 2\pi[$ , we consider the following model problem in two space dimensions:

$$(3.1) \quad \begin{cases} (D^2 - \lambda^2 + q_\lambda)u_n = 0 & \text{in } B = \{x \in \mathbb{R}^2, |x| \leq 1\}, \\ (\partial_r - i\lambda)u_n|_{\partial B} = e^{in\theta}. \end{cases}$$

We suppose that  $q_\lambda$  is a smooth, radial function, with support included in  $D_\lambda^\kappa$  for some fixed constant  $\kappa > 0$ , where

$$(3.2) \quad D_\lambda^\kappa = \{r \in [0, 1], 1 - \kappa\lambda^{-1} < r < 1\}.$$

In the following for simplicity we shall drop the index  $\lambda$  in the notation of the potential.

The following result shows that in the monotone case, one can improve on the frequency band recovered in the general case (see Theorems 2 and 3). We recall that  $\|\mathcal{N}_q(\lambda)\|$  denotes the operator norm of  $\mathcal{N}_q(\lambda)$  in  $\mathcal{L}(L^2)$ .

**Theorem 4.** *Let  $q_1$  and  $q_2$  be two smooth radial potentials supported on  $D_\lambda^\kappa$  as defined in (3.2) and such that  $q_1 \geq q_2$ . There are positive constants  $\lambda_0$  and  $C$  such that the following holds.*

*Let  $\lambda \mapsto K(\lambda)$  be any function such that  $\lambda \leq K(\lambda)$ . Then the following stability estimate is valid for all  $\lambda \geq \lambda_0$ :*

$$\begin{aligned} \int_{|\xi| \leq K(\lambda)} |(\widehat{q_1 - q_2})(\xi)|^2 d\xi &\leq CK^2(\lambda) \left( \lambda^4 \|\mathcal{N}_{q_1}(\lambda) - \mathcal{N}_{q_2}(\lambda)\|^2 \right. \\ &\quad \left. + \frac{C_{q_1, q_2}^2}{\lambda^4} \|q_1 - q_2\|_{L^\infty}^2 \right), \end{aligned}$$

where  $C_{q_1, q_2} = \max(\|q_1\|_{L^\infty}, \|q_2\|_{L^\infty}, \|q_1\|_{L^\infty} \|q_2\|_{L^\infty})$ .



**Remark 3.1.** One has trivially that

$$\begin{aligned} \int_{|\xi| \leq K(\lambda)} |(\widehat{q_1 - q_2})(\xi)|^2 d\xi &\leq CK^2(\lambda) \|\widehat{q_1 - q_2}\|_{L^\infty}^2 \\ &\leq \frac{CK^2(\lambda)}{\lambda^2} \|q_1 - q_2\|_{L^\infty}^2, \end{aligned}$$

so the estimate provided in Theorem 4 is of a different nature.

**Remark 3.2.** The proof of Theorem 4 is presented in Section 3.2, as an immediate consequence of Lemma 3.3 proved in Section 3.3. That lemma relates the Laplace transform of a function to the near field operator. It holds in much more generality than Theorem 4, without the additional assumption that  $q_1 - q_2 \geq 0$ . However we are unable to relate the Fourier transform of a function to its Laplace transform in general (this fact is well-known to be difficult and in general very unstable); for nonnegative functions however the relation is very easy and enables us to conclude.

**3.2. Proof of Theorem 4.** We start by stating a lemma, proved in Section 3.3, which relates the Laplace transform of a function to the near field operator. It is stated in the framework of general, non radial functions.

We define the operator  $T : g \mapsto Tg$  by

$$Tg(r) = g(1 - r)$$

as well as the Laplace transform  $\mathcal{L}$ :

$$\mathcal{L}(g)(t) = \int_{\mathbb{R}} g(s) e^{-st} ds.$$

For any function  $\theta \mapsto f(\theta)$  we call  $c_k(f)$  its Fourier transform, for  $k \in \mathbb{Z}$ :

$$c_k(f)(r) = \int_0^{2\pi} f(r, \theta) e^{-ik\theta} d\theta.$$

**Lemma 3.3.** *Let  $q_1$  and  $q_2$  be two functions supported on  $D_\lambda^\kappa$ . There are three positive constants  $\lambda_0, K$  and  $C$  such that if  $t \geq 2K\lambda > 2K\lambda_0$ , then*

$$\begin{aligned} &\left| \mathcal{L}(c_k(T(r(q_1 - q_2)))(t) \right| \\ &\leq C \left( \lambda^2 \|\mathcal{N}_{q_1}(\lambda) - \mathcal{N}_{q_2}(\lambda)\| + \frac{C_{q_1, q_2}}{\lambda^2} \|q_1 - q_2\|_{L^\infty} \right) \end{aligned}$$

where  $C_{q_1, q_2} = \max(\|q_1\|_{L^\infty}, \|q_2\|_{L^\infty}, \|q_1\|_{L^\infty} \|q_2\|_{L^\infty})$ .

*Proof of Theorem 4.* Let  $q = q_1 - q_2$ . Since  $q$  is radial, we have  $q(x) = \mathcal{Q}(|x|)$  and

$$\widehat{q}(\xi) = \mathcal{Q}(|\xi|)$$

with for all  $\rho > 0$ ,

$$\mathcal{Q}(\rho) = \int_0^{2\pi} \int_0^\infty e^{-ir\rho \cos \theta} \mathcal{Q}(r) r dr d\theta.$$

In particular recalling that  $Q$  is nonnegative, we find that for all  $\xi \in \mathbb{R}^2$ ,

$$\begin{aligned} |\widehat{q}(\xi)| &\leq 2\pi \int_0^\infty Q(r) r dr \\ &\leq 2\pi \int_0^\infty e^{\zeta_0(1-r)} e^{-\zeta_0(1-r)} Q(r) r dr \end{aligned}$$

for any  $\zeta_0 \in \mathbb{R}$ . Then we can apply Lemma 3.3 to the particular case of a radial function, so choosing  $k = 0$  and  $\zeta_0 = 3K\lambda_0$  we infer that

$$\begin{aligned} (3.3) \quad |\widehat{q}(\xi)| &\leq C e^{\zeta_0 \kappa / \lambda} \mathcal{L}(T(rQ))(\zeta_0) \\ &\leq C e^{3K\kappa} \left( \lambda^2 \|\mathcal{N}_{q_1}(\lambda) - \mathcal{N}_{q_2}(\lambda)\| + \frac{C_{q_1, q_2}}{\lambda^2} \|q_1 - q_2\|_{L^\infty} \right). \end{aligned}$$

The end of the proof of Theorem 4 follows easily by taking the  $L^2$  norm in  $\xi$ .  $\square$

**3.3. Proof of Lemma 3.3.** The method of proof follows the ideas developed in the proofs of Theorems 2 and 3, adapting the estimates to our special situation where the potentials are located near the boundary. The heart of the matter consists to approximate the solutions to Helmholtz equation (3.1) by separable solutions in radial coordinates involving Bessel functions, which allows in light of Debye's formula to relate the Laplace transform of the potential to its near field operator.

More precisely, we shall look for solutions to (3.1) under the following form, for  $n \in \mathbb{Z}$ :

$$\begin{aligned} u_n(r, \theta) &= \frac{J_{|n|}(\lambda r)}{\lambda(J'_{|n|}(\lambda) - iJ_{|n|}(\lambda))} e^{in\theta} + v_n \\ &= z_n(r, \lambda) e^{in\theta} + v_n, \end{aligned}$$

where  $J_n$  is a Bessel function of the first kind (see Appendix A), solution to

$$\left( \partial_r^2 + \frac{1}{r} \partial_r - \frac{n^2}{r^2} + 1 \right) J_n(r) = 0.$$

Suppose that for  $\ell \in \{1, 2\}$ ,  $u_n^\ell$  solves (3.1) with potential  $q_\ell$ . Then writing

$$u_n^\ell(r, \theta) = z_n(r, \lambda) e^{in\theta} + v_n^\ell(r, \theta),$$

we find that

$$\begin{cases} (D^2 - \lambda^2 + q_\ell) v_n^\ell = -q_\ell z_n e^{in\theta} & \text{in } B, \\ (\partial_r - i\lambda) v_n^\ell|_{\partial B} = 0. \end{cases}$$

Due to Property (A.6) we have if  $1 - \lambda^{-1} \leq r \leq 1$  and  $|n| \geq K\lambda$  for  $K$  large, that  $|\lambda z_n(r, \lambda)|$  is bounded, for  $\lambda \geq \lambda_0$ , by a constant depending only on  $K$  and  $\lambda_0$ .

For now on we shall denote by  $C(\lambda_0, K)$  such a constant, which may change from line to line.

Therefore, arguing as in the proof of Lemma 1.5 and taking advantage of the fact that on the support of  $q_\ell$ ,  $r$  varies in an interval of size  $\lambda^{-1}$ , we find as soon as  $\lambda$  is large enough compared to  $\|q_\ell\|_{L^\infty}$

$$(3.4) \quad \|v_n^\ell\|_{L^2} \leq \frac{C(\lambda_0, K)}{\lambda} \|q_\ell(r, \theta) z_n(r, \lambda)\|_{L^2} \leq \frac{C(\lambda_0, K)}{\lambda^{\frac{5}{2}}} \|q_\ell\|_{L^\infty}.$$

Now going back to the computations of Section 2 we consider two positive integers  $n$  and  $m$  such that  $n, m \geq K\lambda$ , and we write

$$\int_B (q_1 - q_2) u_n^1 u_{-m}^2 dx = \int_{\partial B} (\mathcal{N}_{q_1}(\lambda) - \mathcal{N}_{q_2}(\lambda)) e^{in\theta} e^{-im\theta} d\theta.$$

Therefore decomposing

$$\begin{aligned} u_n^1(r, \theta) &= z_n(r, \lambda) e^{in\theta} + v_n^1(r, \theta) \quad \text{and} \\ u_{-m}^2(r, \theta) &= z_m(r, \lambda) e^{-im\theta} + v_{-m}^2(r, \theta), \end{aligned}$$

we get

$$(3.5) \quad \begin{aligned} & \int_0^{2\pi} (\mathcal{N}_{q_1}(\lambda) - \mathcal{N}_{q_2}(\lambda)) e^{in\theta} e^{-im\theta} d\theta \\ &= \int_B (q_1 - q_2) e^{i(n-m)\theta} z_n(r, \lambda) z_m(r, \lambda) r dr d\theta \\ & \quad + \int_B (q_1 - q_2) e^{in\theta} z_n(r, \lambda) v_{-m}^2 r dr d\theta \\ & \quad + \int_B (q_1 - q_2) e^{-im\theta} z_m(r, \lambda) v_n^1 r dr d\theta \\ & \quad + \int_B (q_1 - q_2) v_n^1 v_{-m}^2 r dr d\theta. \end{aligned}$$

As  $|\lambda z_n(r, \lambda)|$  is bounded by a constant  $C(\lambda_0, K)$ , we have (recalling that  $q_1$  and  $q_2$  are compactly supported in an interval in  $r$  of size  $\lambda^{-1}$ )

$$\left| \int_B (q_1 - q_2) e^{in\theta} z_n(r, \lambda) v_{-m}^2 r dr d\theta \right| \leq \frac{C(\lambda_0, K)}{\lambda^{\frac{3}{2}}} \|q_1 - q_2\|_{L^\infty} \|v_{-m}^2\|_{L^2}$$

and similarly

$$\left| \int_B (q_1 - q_2) e^{-im\theta} z_m(r, \lambda) v_n^1 r dr d\theta \right| \leq \frac{C(\lambda_0, K)}{\lambda^{\frac{3}{2}}} \|q_1 - q_2\|_{L^\infty} \|v_n^1\|_{L^2}.$$

Finally

$$\left| \int_B (q_1 - q_2) v_n^1 v_{-m}^2 r dr d\theta \right| \leq \|q_1 - q_2\|_{L^\infty} \|v_n^1\|_{L^2} \|v_{-m}^2\|_{L^2}.$$

Thus by (3.4), we get from (3.5), for  $n, m \geq K\lambda > K\lambda_0$ ,

$$(3.6) \quad \left| \int_B (q_1 - q_2) e^{i(n-m)\theta} z_n(r, \lambda) z_m(r, \lambda) r dr d\theta \right| \\ \leq \left| \int_0^{2\pi} (\mathcal{N}_{q_1}(\lambda) - \mathcal{N}_{q_2}(\lambda)) e^{in\theta} e^{-im\theta} d\theta \right| + \frac{C_{q_1, q_2}}{\lambda^4} \|q\|_{L^\infty},$$

where  $q = q_1 - q_2$  and

$$C_{q_1, q_2} = C(\lambda_0, K) \max(\|q_1\|_{L^\infty}, \|q_2\|_{L^\infty}, \|q_1\|_{L^\infty} \|q_2\|_{L^\infty}).$$

This gives rise to

$$(3.7) \quad \left| \int_B (q_1 - q_2) e^{i(n-m)\theta} z_n(r, \lambda) z_m(r, \lambda) r dr d\theta \right| \\ \leq \|\mathcal{N}_{q_1}(\lambda) - \mathcal{N}_{q_2}(\lambda)\| + \frac{C_{q_1, q_2}}{\lambda^4} \|q\|_{L^\infty},$$

which leads in light of (A.6) to

$$\left| \int_B (q_1 - q_2) e^{i(n-m)\theta} e^{-(n+m)(1-r)} r dr d\theta \right| \\ \leq C(\lambda_0, K) \left( \lambda^2 \|\mathcal{N}_{q_1}(\lambda) - \mathcal{N}_{q_2}(\lambda)\| + \frac{C_{q_1, q_2}}{\lambda^2} \|q\|_{L^\infty} \right),$$

for all  $n, m \geq K\lambda$ . In conclusion we have for any  $k \in \mathbb{Z}$ , any  $\ell \in \mathbb{N}$  and any  $j \geq 2K\lambda > 2K\lambda_0$ ,

$$(3.8) \quad \left| \int_B (q_1 - q_2) e^{-ik\theta} e^{-j(1-r)} r dr d\theta \right| \\ \leq C(\lambda_0, K) \left( \lambda^2 \|\mathcal{N}_{q_1}(\lambda) - \mathcal{N}_{q_2}(\lambda)\| + \frac{C_{q_1, q_2}}{\lambda^2} \|q\|_{L^\infty} \right).$$

The conclusion follows from (3.3). Lemma 3.3 is proved.  $\square$

#### 4. CONCLUDING REMARKS

In this paper we have shown that the low-frequency component of the potential can be determined in a stable way from the scattering measurements and justified the resolution limit. We have also proved that in the near-field we have in the monotone case infinite resolution in reconstructing the potential near the boundary. We think that the result holds in the general case. However, its proof seems to be out of reach. In fact, even though a sampling (or interpolation) formula for the Laplace transform does exist [20, 54], making norm-estimates similar to those in Theorem 4 is very challenging. Our results can be extended in many directions. It would be very interesting to study the limited-view case and show, as in [11], that we recover infinite resolution from near-field measurements on the overlap of the source and receiver apertures. Another challenging problem is to understand how probe interaction can improve local resolution by converting

evanescent modes of the potential to propagating ones [23]. These problems will be the subject of forthcoming works.

#### APPENDIX A. BESSEL FUNCTIONS

Bessel's equation arises when finding separable solutions to the Helmholtz equation in spherical coordinates, and writes as follows:

$$(A.1) \quad \left( \partial_z^2 + \frac{1}{z} \partial_z + \left( 1 - \frac{n^2}{z^2} \right) \right) u = 0, \quad n \in \mathbb{Z}.$$

It is well known (see for instance [39, 49] and the references therein) that one of the solutions of Bessel's equation is the entire function  $J_n(z)$  known as the Bessel function of the first kind of order  $n$ , and defined for arbitrary  $z \in \mathbb{C}$  by the convergent series

$$J_n(z) = \sum_{k=0}^{k=\infty} \frac{(-1)^k}{k! (n+k)!} (z/2)^{n+2k},$$

in the case where  $n \in \mathbb{N}$  and by  $J_{-n}(z) = (-1)^n J_n(z)$ .

To find a general solution of Bessel's equation (A.1), we need a second solution of (A.1) which is linearly independent of  $J_n(z)$ . For such a solution, we usually choose  $Y_n(z)$  the Bessel function of the second kind which is entire in the complex plane cut along the segment  $]-\infty, 0]$  and defined for  $n \in \mathbb{N}$  by

$$\begin{aligned} Y_n(z) &= \frac{2}{\pi} J_n(z) \log \frac{z}{2} - \frac{1}{\pi} \sum_{k=0}^{k=n-1} \frac{(n-k-1)!}{k!} \left( \frac{z}{2} \right)^{2k-n} \\ &\quad - \frac{1}{\pi} \sum_{k=0}^{k=\infty} \frac{(-1)^k (z/2)^{n+2k}}{k! (n+k)!} [\psi(k+1) + \psi(k+n+1)], \end{aligned}$$

where  $\psi(m+1) = -\gamma + 1 + \frac{1}{2} + \dots + \frac{1}{m}$ ,  $\gamma$  being the Euler constant. We also define  $Y_{-n}(z) = (-1)^n Y_n(z)$ .

Since  $J_n$  and  $Y_n$  are linearly independent, the general expression for solutions of (A.1) is a linear combination of Bessel functions of the first and second kinds, i.e.,

$$u(z) = A J_n(z) + B Y_n(z),$$

where  $A$  and  $B$  are constants.

Another basis of solutions to the differential equation (A.1) is given by the Bessel functions of the third kind or Hankel functions, denoted by  $H_n^{(1)}$  and  $H_n^{(2)}$ . These functions are defined by the formulas

$$(A.2) \quad H_n^{(1)}(z) = J_n(z) + iY_n(z) \quad \text{and} \quad H_n^{(2)}(z) = J_n(z) - iY_n(z),$$

where  $z$  is any point of the complex plane cut along the segment  $]-\infty, 0]$ . The motivation for introducing the Hankel functions is that the linear combination of  $J_n(z)$  and  $Y_n(z)$  have very simple asymptotic expansions for large  $|z|$ : it is thus well-known that

$$(A.3) \quad H_n^{(1)}(z) = \left(\frac{2}{\pi z}\right)^{\frac{1}{2}} e^{i(z - \frac{n\pi}{2} - \frac{\pi}{4})} \left(1 + \mathcal{O}\left(\frac{1}{|z|}\right)\right) \text{ and}$$

$$(A.4) \quad H_n^{(2)}(z) = \left(\frac{2}{\pi z}\right)^{\frac{1}{2}} e^{-i(z - \frac{n\pi}{2} - \frac{\pi}{4})} \left(1 + \mathcal{O}\left(\frac{1}{|z|}\right)\right)$$

as  $|z| \rightarrow \infty$ .

Furthermore, we have the following Debye formulas whose proof can be found for instance in [49, Chapter 9.4] and [1, Chapter 9]:

$$J_n(n \operatorname{sech} \alpha) \sim \frac{e^{-n(\alpha - \tanh \alpha)}}{(2\pi n \tanh \alpha)^{\frac{1}{2}}} \left(1 + \left(\frac{1}{8} \coth \alpha - \frac{5}{24} (\coth \alpha)^3\right) \frac{1}{n} + \dots\right)$$

and particularly

$$(A.5) \quad \begin{aligned} J_n(n \operatorname{sech} \alpha) &= \frac{e^{-n(\alpha - \tanh \alpha)}}{(2\pi n \tanh \alpha)^{\frac{1}{2}}} \left(1 + \mathcal{O}\left(\frac{1}{n}\right)\right), \\ J'_n(n \operatorname{sech} \alpha) &= \frac{(\tanh \alpha)^{\frac{1}{2}} e^{-n(\alpha - \tanh \alpha)}}{(4\pi n)^{\frac{1}{2}}} \left(1 + \mathcal{O}\left(\frac{1}{n}\right)\right) \end{aligned}$$

as  $n \rightarrow \infty$ , where  $\operatorname{sech} z$  denotes the hyperbolic secant of  $z$  defined by

$$\operatorname{sech} z = \frac{1}{\cosh z}$$

with  $\cosh z$  the hyperbolic cosine.

Debye's formula gives rise to the following asymptotic behavior for the function introduced in Section 3:

$$z_n(\lambda, r) = \frac{J_{|n|}(\lambda r)}{\lambda(J'_{|n|}(\lambda) - iJ_{|n|}(\lambda))}$$

defined for  $n \in \mathbb{Z}$ . Let us prove that for  $|n| \gg \lambda$  and  $r \sim 1$ ,

$$(A.6) \quad \begin{aligned} \operatorname{Re} z_n(\lambda, r) &\sim \frac{C}{\lambda} e^{-n(1-r)} \left(1 + \mathcal{O}\left(\frac{1}{n}\right)\right), \\ \text{and } \operatorname{Im} z_n(\lambda, r) &\sim \frac{C}{\lambda} e^{-n(1-r)} \left(1 + \mathcal{O}\left(\frac{1}{n}\right)\right). \end{aligned}$$

Without loss of generality we may assume that  $n \in \mathbb{N}$ , then defining

$$\cosh \alpha_1 = \frac{n}{\lambda r} \text{ and } \cosh \alpha_2 = \frac{n}{\lambda},$$

it is easy to see that under the above assumptions ( $\lambda \ll n$  and  $r \sim 1$ ), we have necessarily  $\cosh \alpha_i \gg 1$  for  $i \in \{1, 2\}$ , hence  $\alpha_i \gg 1$ . This implies  $\cosh \alpha_i \sim e^{\alpha_i}$ ,  $\sinh \alpha_i \sim e^{\alpha_i}$ , and  $\tanh \alpha_i \sim 1$ , which gives rise to

$$\operatorname{Re} z_n(\lambda, r) = \frac{J_n(n \operatorname{sech} \alpha_1) J'_n(n \operatorname{sech} \alpha_2)}{\lambda(J_n'^2(n \operatorname{sech} \alpha_2) + J_n^2(n \operatorname{sech} \alpha_2))}$$

and

$$\operatorname{Im} z_n(\lambda, r) = \frac{J_n(n \operatorname{sech} \alpha_1) J_n(n \operatorname{sech} \alpha_2)}{\lambda(J_n'^2(n \operatorname{sech} \alpha_2) + J_n^2(n \operatorname{sech} \alpha_2))}.$$

Finally, taking advantage of (A.5) we get

$$\operatorname{Re} z_n(\lambda, r) \sim \frac{e^{-n(\alpha_1 - \alpha_2)}}{\lambda} \sim \frac{e^{-n \log(\frac{1}{r})}}{\lambda}.$$

The computation is identical for  $\operatorname{Im} z_n(\lambda, r)$ , so using the fact that  $r$  is near to 1, we obtain the desired conclusion.

#### REFERENCES

- [1] M. Abramowitz, I. Stegun, *Handbook of mathematical functions with formulas, graphs, and mathematical tables*, National Bureau of Standards Applied Mathematics Series, **55**, 1964, xiv+1046 pp.
- [2] S. Agmon, *Spectral properties of Schrödinger operators and scattering theory*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. **2** (1975), 151–218.
- [3] S. Agmon, L. Hörmander, *Asymptotic properties of solutions of differential equations with simple characteristics*, J. Anal. Math. **30** (1976), 1–38.
- [4] G. Alessandrini, *Stable determination of conductivity by boundary measurements*, Appl. Anal. **27** (1988), 153–172.
- [5] G. Alessandrini, *Examples of instability in inverse boundary-value problems*, Inverse Problems **13** (1997), 887–897.
- [6] G. Alessandrini, *Open issues of stability for the inverse conductivity problem*, J. Inverse Ill-Posed Probl. **15** (2007), 451–460.
- [7] H. Ammari, *An Introduction to Mathematics of Emerging Biomedical Imaging*, Math. & Appl., Volume 62, Springer-Verlag, Berlin, 2008.
- [8] H. Ammari, J. Garnier, V. Jugnon, H. Kang, *Direct reconstruction methods in ultrasound imaging of small anomalies*, Lecture Notes in Mathematics, Vol. **2035**, 31–55, Springer-Verlag, Berlin, 2011.
- [9] H. Ammari, J. Garnier, H. Kang, M. Lim, K. Sølna, *Multistatic imaging of extended targets*, SIAM J. Imaging Sci. **5** (2012), 564–600.
- [10] H. Ammari, J. Garnier, K. Sølna, *Resolution and stability analysis in full aperture, linearized conductivity and wave imaging*, Proc. Amer. Math. Soc., to appear.
- [11] H. Ammari, J. Garnier, K. Sølna, *Limited view resolving power of conductivity imaging from boundary measurements*, submitted.
- [12] H. Ammari, H. Kang, *Reconstruction of small inhomogeneities from boundary measurements*, Lecture Notes in Mathematics, Vol. **1846**, Springer, Berlin, 2004.
- [13] H. Ammari, H. Kang, E. Kim, J.-Y. Lee, *The generalized polarization tensors for resolved imaging. Part II: Shape and electromagnetic parameters reconstruction of an electromagnetic inclusion from multistatic measurements*, Math. Comp. **81** (2012), 839–860.
- [14] G. Bao, Y. Chen, F. Ma, *Regularity and stability for the scattering map of a linearized inverse medium problem*, J. Math. Anal. Appl. **247** (2000), 255–271.

- [15] G. Bao, J. Junshan, F. Triki, *A multi-frequency inverse source problem*, J. Diff. Equat. **249** (2010), 3443–3465.
- [16] G. Bao, J. Junshan, F. Triki, *Numerical solution of the inverse source problem for the Helmholtz equation with multiple frequency data*, Contemp. Math. **548** (2011), 45–60.
- [17] G. Bao, K. Yun, *On the stability of an inverse problem for the wave equation*, Inverse Problems **25** (2009), 045003 (7pp).
- [18] M. Bertero, P. Boccacci, M. Piana, *Resolution and super-resolution in inverse diffraction*, in *Inverse Problems of Wave Propagation and Diffraction*, Lecture Notes in Physics, Vol. **486**, 1–17, Springer, Berlin, 1997.
- [19] M. Born, E. Wolf, *Principles of Optics*, Cambridge University Press, Cambridge, 1999.
- [20] A. Boumenir, V. K. Tuan, *Sampling eigenvalues in Hardy spaces*, SIAM J. Numer. Anal. **45** (2007), 473–483.
- [21] N. Burq, *Semi-classical estimates for the resolvent in nontrapping geometries*, Int. Math. Res. Not. **5** (2002), 221–241.
- [22] A. Calderón, *On an inverse boundary value problem*, Seminar on Numerical Analysis and its Applications to Continuum Physics, Rio de Janeiro, Editors W. H. Meyer and M. A. Raupp, Sociedade Brasileira de Matematica (1980), 65–73.
- [23] P. S. Carney, J. C. Schotland, *Near-field tomography*, in *Inside out: inverse problems and applications*, Math. Sci. Res. Inst. Publ. **47**, 133–168, Cambridge Press, Cambridge, 2003.
- [24] Y. Chen, *Inverse scattering via Heisenberg’s uncertainty principle*, Inverse Problems **13** (1997), 253–282.
- [25] Y. Chen, V. Rokhlin, *On the inverse scattering problem for the Helmholtz equation in one dimension*, Inverse Problems **8** (1992), 365–391.
- [26] R. Coifman, M. Goldberg, T. Hrycak, M. Israeli, V. Rokhlin, *An improved operator expansion algorithm for direct and inverse scattering computations*, Waves Random Media **9** (1999), 441–457.
- [27] G. Derveaux, G. Papanicolaou, C. Tsogka, *Resolution and denoising in near-field imaging*, Inverse Problems **22** (2006), 1437–1456,
- [28] D. L. Donoho, P. B. Stark, *Uncertainty principles and signal recovery*, SIAM J. Appl. Math. **49** (1989), 906–931.
- [29] C. L. Epstein, J. Schotland, *The bad truth about Laplace’s transform*, SIAM Rev. **50** (2008), 504–520.
- [30] B. Engquist, A. Majda, *Absorbing boundary conditions for the numerical simulation of waves*, Math. Comp. **31** (1977), 629–651.
- [31] C. Gérard, *A proof of the abstract limiting absorption principle by energy estimates*, J. Funct. Anal. **254** (2008), 2707–2724.
- [32] I. M. Gelfand, G. E. Shilov, *Generalized Functions I: Properties and Operations*, Academic Press, 1964.
- [33] M. V. de Hoop, L. Qiu, O. Scherzer, *Local analysis of inverse problems: Hölder stability and iterative reconstruction*, Inverse Problems **28** (2012), 045001.
- [34] V. Isakov, *Increased stability in the Cauchy problem for some elliptic equations*, in *Instability in Models Connected with Fluid Flows. I*, C. Bardos and A. Fursikov, eds., International Math. Series, Vol. **6**, 339–362, Springer, 2008.
- [35] V. Isakov, S. Kindermann, *Subspaces of stability in the Cauchy problem for the Helmholtz equation*, Meth. Appl. Anal. **18** (2011), 1–29.
- [36] F. John, *Continuous dependence on data for solutions of partial differential equations with a prescribed bound*, Comm. Pure Appl. Math. **13** (1960), 551–585.
- [37] S. N. Karp, *A convergent ‘farfield’ expansion for two-dimensional radiation functions*, Comm. Pure Appl. Math. **14** (1961), 427–434.
- [38] J. B. Keller, D. Givoli, *Exact nonreflecting boundary conditions*, J. Comput. Phys. **82**, 172–192.



- [39] N. N. Lebedev, *Special functions and their applications*, Prentice Hall, 1965.
- [40] N. Mandache, *Exponential instability in an inverse problem for the Schrödinger equation*, *Inverse Problems* **17** (2001), 1435–1444.
- [41] J. M. Melenk, *On generalized finite element methods*, PhD Thesis, University of Maryland, 1995.
- [42] R. B. Melrose, *Geometric scattering theory*, Cambridge University Press, 1995.
- [43] A. I. Nachman, *Reconstruction from boundary measurements*, *Ann. of Math.* **128** (1988), 531–576.
- [44] A. I. Nachman, *Global uniqueness for a two-dimensional inverse boundary value problem*, *Ann. of Math.* **142** (1995), 71–96.
- [45] S. Nagayasu, G. Uhlmann, J.-N. Wang, *Depth dependent stability estimates in electrical impedance tomography*, *Inverse Problems* **25** (2009), 075001.
- [46] S. Nagayasu, G. Uhlmann, J.-N. Wang, *Increasing stability in an inverse problem for the acoustic equation*, preprint.
- [47] R. G. Novikov, *A multidimensional inverse spectral problem for the equation  $-\Delta\psi + (v(x) - Eu(x))\psi = 0$* , *Funct. Anal. Appl.* **22** (1988), 263–272.
- [48] R. G. Novikov, *Formulae and equations for finding scattering data from the Dirichlet-to-Neumann map with nonzero background potential*, *Inverse Problems* **21** (2005), 257–270.
- [49] F. W. J. Olver, *Asymptotics and special functions*, Academic Press, 1974.
- [50] B. Perthame, L. Vega, *Morrey-Campanato estimates for Helmholtz equations*, *J. Func. Anal.* **164** (1999), 340–355.
- [51] A. G. Ramm, *Recovery of the potential from fixed energy scattering data*, *Inverse Problems* **4** (1988), 877–886.
- [52] A. G. Ramm, *Finding potential from the fixed-energy scattering data via DN map*, *J. Inverse Ill-Posed Probl.* **4** (1996), 145–151.
- [53] J. Rauch, *Fourier Analysis from Complex Analysis*, homepage.
- [54] A. Rybkin, V. K. Tuan, *A new interpolation formula for the Titchmarsh-Weyl  $m$ -function*, *Proc. Amer. Math. Soc.* **137** (2009), 4177–4185.
- [55] D. Slepian, *Some comments on Fourier analysis, uncertainty and modeling*, *SIAM Review* **25** (1983), 379–393.
- [56] P. Stefanov, *Stability of the inverse problem in potential scattering at fixed energy*, *Ann. Inst. Fourier* **40** (1990), 867–884.
- [57] J. Sylvester, G. Uhlmann, *A global uniqueness theorem for an inverse boundary value problem*, *Ann. of Math* **125** (1987), 153–169.
- [58] M. E. Taylor, *Partial differential equations II*, Springer Verlag.
- [59] M. E. Taylor, *Estimates for approximate solutions to acoustic inverse scattering problems*, in *Inverse problems in wave propagation* (Minneapolis, MN, 1995), 463–499, IMA Vol. Math. Appl., **90**, Springer, New York, 1997.
- [60] G. Uhlmann, A. Vasy, *Fixed energy inverse problem for exponentially decreasing potentials*, *Methods Appl. Anal.* **9** (2002), 239–247.
- [61] R. Weder, *Global uniqueness at a fixed energy in multidimensional inverse scattering theory*, *Inverse Problems* **7** (1991), 927–938.
- [62] D. Yafaev, *High-energy and smoothness asymptotic expansion of the scattering amplitude*, *J. Funct. Anal.* **202** (2003), 526–570.
- [63] H. Zhao, *Analysis of the response matrix for an extended target*, *SIAM J. Appl. Math.* **64** (2004), 725–745.

DEPARTMENT OF MATHEMATICS AND APPLICATIONS, ÉCOLE NORMALE SUPÉRIEURE,  
45 RUE D'ULM, F-75005 PARIS, FRANCE  
*E-mail address:* `habib.ammari@ens.fr`

UNIVERSITÉ PARIS EST-CRÉTEIL, UMR 8050 LAMA, 61 AVENUE DU GÉNÉRAL DE  
GAULLE, F-94010 CRÉTEIL CEDEX, FRANCE  
*E-mail address:* `hbahouri@math.cnrs.fr`

UNIVERSITÉ PARIS 13, CNRS, UMR 7539 LAGA, 99 AVENUE JEAN-BAPTISTE CLÉMENT,  
F-93430 VILLETANEUSE, FRANCE  
*E-mail address:* `dds@math.univ-paris13.fr`

INSTITUT DE MATHÉMATIQUES UMR 7586, UNIVERSITÉ PARIS VII, 175 RUE DU CHEVALERET,  
F-75013 PARIS, FRANCE  
*E-mail address:* `Isabelle.Gallagher@math.jussieu.fr`