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# LINEAR DIFFERENTIAL OPERATORS ON CONTACT MANIFOLDS

CHARLES H. CONLEY AND VALENTIN OVSIENKO

Abstract. We consider differential operators between sections of arbitrary powers of the determinant line bundle over a contact manifold. We extend the standard notions of the Heisenberg calculus: noncommutative symbolic calculus, the principal symbol, and the contact order to such differential operators. Our first main result is an intrinsically defined "subsymbol" of a differential operator, which is a differential invariant of degree one lower than that of the principal symbol. In particular, this subsymbol associates a contact vector field to an arbitrary second order linear differential operator. Our second main result is the construction of a filtration that strengthens the well-known contact order filtration of the Heisenberg calculus.

## 1. Introduction

The space $\mathcal{D}(M)$ of linear differential operators on a smooth manifold $M$ has a rich geometric structure. By the *geometry* of $\mathcal{D}(M)$, we understand its structure as a module over the group of all diffeomorphisms of $M$, and thereby also over its Lie algebra, the space $\mathrm{Vect}(M)$ of smooth vector fields on $M$. The most interesting geometric properties of $\mathcal{D}(M)$ are described by its invariants under the group of diffeomorphisms.

Additional structure on $M$ leads to a smaller group of diffeomorphisms, and therefore a richer set of invariants of $\mathcal{D}(M)$. Contact manifolds provide an important class of examples of geometric structures. In this paper we study the geometric properties of $\mathcal{D}(M)$ viewed as a module over the Lie algebra $\mathcal{K}(M)$ of all contact vector fields on $M$. This viewpoint fits into the general framework of *Heisenberg calculus*, see [BG88, EM98, vE10], where the geometric structure is a codimension-1 distribution in $T(M)$.

Our first main result is the association of a contact vector field to an arbitrary second order linear differential operator in a contact-invariant manner. Although we do not carry out the investigation here, this could provide a means to associate topological invariants to second order operators. We generalize the result to differential operators of arbitrary order, associating to each a certain *tensor density* on $M$. This tensor density is independent of the symbol of the operator and may be thought of as a partial "subsymbol".

By a tensor density, we mean a section of a power of the determinant line bundle. In fact, we state our results in the more general context of the spaces $\mathcal{D}_{\lambda,\mu}(M)$ of differential operators between such line bundles, rather than simply for differential operators on functions. An interesting feature is appearance of *contact resonances*, that is, of special powers of the determinant line bundle for which the geometric properties of differential operators are more complicated. These resonances were already observed in [FMP08]. Let us mention that the usual case where $\lambda = \mu$ is non-resonant.

Our second main result is the existence of a filtration refining the usual filtration given by the Heisenberg calculus. Recall that differential operators on a contact manifold have a *contact order*, in which vector fields tangent to the distribution are of order 1, and contact vector fields are of order 2. We introduce a contact-invariant filtration on $\mathcal{D}_{\lambda,\mu}(M)$ for which, roughly speaking,

*Key words and phrases.* Contact geometry, differential operators, Heisenberg calculus, infinitesimal characters, differential invariants.

tangential vector fields have order 1 and contact vector fields have order 3. However, this filtration is not compatible with composition. We prove its existence in the non-resonant case.

We remark that there is also an invariant double filtration on the space of differential forms on $M$ [Ru94]. In some ways, the situation for contact manifolds appears to be analogous to that for foliated manifolds. This may at first be surprising, as contact distributions are completely non-integrable, but such analogies have been observed before [ET98].

Although our results and their applications are essentially geometric, the proofs are algebraic. The considerations are local, so we may work in the Euclidean case, replacing $M$ by $\mathbb{R}^m$, where $m = 2\ell + 1$. Moreover, we need only consider the Lie algebra of polynomial contact vector fields on $\mathbb{R}^m$, which is the classical infinite dimensional Cartan algebra $\mathcal{K}_m$; see [Fu86].

Our main theorems are proven using certain underlying structural results concerning the cohomology of $\mathcal{K}_m$ with coefficients in spaces of differential operators between refined symbol modules. These results in turn are obtained using a quantization map which is equivariant with respect to the projective subalgebra $\mathfrak{s}_m$ of $\mathcal{K}_m$, a maximal subalgebra isomorphic to $\mathfrak{sp}_{2(\ell+1)}$. We prove the existence and uniqueness of this quantization map using the description of the infinitesimal characters of $\mathfrak{s}_m$ given by the Harish-Chandra homomorphism. We also calculate the map explicitly.

An $\mathfrak{s}_m$-module is said to have an infinitesimal character if the center $\mathfrak{Z}(\mathfrak{s}_m)$ of the universal enveloping algebra of $\mathfrak{s}_m$ acts on it by scalars. The infinitesimal character is then the resulting homomorphism from $\mathfrak{Z}(\mathfrak{s}_m)$ to $\mathbb{C}$. If an $\mathfrak{s}_m$-module has a finite Jordan-Hölder composition series of modules with distinct infinitesimal characters, then the module splits as the direct sum of its composition series modules.

Let us discuss at this point the role of infinitesimal characters in other forms of quantization. The *Casimir element* is the best known and simplest element of $\mathfrak{Z}(\mathfrak{s}_m)$. It turns out that for the contact projective quantization studied in this paper, it is not sufficient to consider the eigenvalues of the Casimir element alone, because there are fine symbol modules with distinct infinitesimal characters but identical Casimir eigenvalues. This is in contrast with the situation for projective quantization with respect to the full vector field Lie algebra $\mathrm{Vect}(\mathbb{R}^m)$, whose projective subalgebra is $\mathfrak{sl}_{m+1}$. The full principal symbol modules have infinitesimal characters under the action of $\mathfrak{sl}_{m+1}$, and these infinitesimal characters are distinct if and only if their Casimir eigenvalues are different [Le00]. Therefore in this setting there is no need to consider infinitesimal characters.

For conformal quantization, one replaces the projective subalgebra with the conformal subalgebra $\mathfrak{o}_{p+1,q+1}$, a maximal subalgebra of $\mathrm{Vect}(\mathbb{R}^{p+q})$. As was first observed in [DLO99], in this setting the Casimir element of $\mathfrak{o}_{p+1,q+1}$ is not sufficient to detect distinct infinitesimal characters among these submodules. Complete results concerning the existence and uniqueness of conformal quantization for differential operators between tensor density modules have recently been obtained in [Si09] and [Mi11]. It would be interesting to determine to what extent infinitesimal characters can be used to replicate them.

The crucial property that allows us to apply algebraic results in the geometric situation of an arbitrary contact manifold $M$ is the *uniqueness* of the $\mathfrak{sp}_{2(\ell+1)}$-equivariant quantization map. For example, the subsymbol is first defined locally in Darboux coordinates. Its uniqueness then implies that it is defined globally on $M$.

This paper is organized as follows. In Section 2 we define the modules of tensor densities, differential operators, and symbols, and formulate our main results. In Section 3 we fix local Darboux coordinates and review local properties of the contact Lie algebra. In Section 4 we study the modules of Section 2 under the action of the projective subalgebra, using infinitesimal characters to compute contact resonances. In Section 5 we prove the existence and uniqueness of the projective quantization, the natural projective equivalence from symbols to differential operators. Section 6 contains the proofs of two of our main results: the existence and uniqueness of the subsymbol and

of the fine filtration. Section 7 gives the explicit formula for the projective quantization and the subsymbol and proves our third main result.

## 2. Main results

Fix $m = 2\ell + 1$ odd, and let $M$ be a smooth $m$-dimensional manifold equipped with a contact distribution $\Xi$: a completely non-integrable distribution of codimension 1. As usual, locally we define the contact structure in terms of a contact form $\theta$ whose kernel is $\Xi$. The non-integrability of $\Xi$ is equivalent to the fact that $\theta \wedge (d\theta)^\ell$ is a local volume form. We define the subspace $\mathrm{Tan}(M)$ of $\mathrm{Vect}(M)$ to consist of the sections of $\Xi$, that is, the vector fields annihilated by $\theta$. We will refer to such vector fields as *tangential vector fields.*

2.1. **Definitions.** We will use the following notation throughout this paper. For $X \in \mathrm{Vect}(M)$, we write $L(X)$ for the associated *Lie derivative.* The non-negative integers will be denoted by $\mathbb{N}$, and the positive integers by $\mathbb{Z}^+$. For $x \in \mathbb{R}$, we use the floor notation for the greatest integer $\leq x$ and the ceiling notation for the least integer $\geq x$:

$$\lfloor x \rfloor := \sup\{n \in \mathbb{Z} : n \leq x\}, \qquad \lceil x \rceil := \inf\{n \in \mathbb{Z} : n \geq x\}.$$

Within $\mathrm{Vect}(M)$ we have the Lie subalgebra $\mathcal{K}(M)$ of *contact vector fields,* those which preserve $\Xi$. Contact vector fields are characterized locally as those whose Lie derivatives preserve the conformal class of $\theta$. More precisely, a vector field $X$ on $M$ is contact if

$$L(X)\theta = \tfrac{1}{\ell+1}\,\mathrm{Div}(X)\theta, \tag{1}$$

where Div is the divergence with respect to the volume form $\theta \wedge (d\theta)^\ell$. The complete non-integrability of $\Xi$ translates to

$$\mathrm{Vect}(M) = \mathcal{K}(M) \oplus \mathrm{Tan}(M).$$

This decomposition is invariant under the Lie action of $\mathcal{K}(M)$. Observe that $\mathcal{K}(M)$ is not invariant under multiplication by functions, and $\mathrm{Tan}(M)$ is not a Lie algebra. Let

$$\pi : \mathrm{Vect}(M) \to \mathcal{K}(M)$$

be the projection along $\mathrm{Tan}(M)$.

We now make several definitions valid for arbitrary (not necessarily contact) manifolds.

**Definition.**

(i) For $\lambda \in \mathbb{C}$, let $|\Lambda^m T^*(M)|^\lambda$ be the line bundle of homogeneous functions of degree $\lambda$ on the determinant bundle. The space $\mathcal{F}_\lambda(M)$ of *tensor densities of degree* $\lambda$ consists of the smooth sections of $|\Lambda^m T^*(M)|^\lambda$ with *complex coefficients.* It is a module for $\mathrm{Vect}(M)$, and we write $L_\lambda(X)$ for the action of a vector field $X$ on it.

(ii) Let $\mathcal{D}_{\lambda,\mu}(M)$ be the space of differential operators from $\mathcal{F}_\lambda(M)$ to $\mathcal{F}_\mu(M)$, and let $L_{\lambda,\mu}$ be the natural action of $\mathrm{Vect}(M)$ on it. For $k \in \mathbb{N}$, let $\mathcal{D}^k_{\lambda,\mu}(M)$ be the subspace of operators of order $\leq k$. The spaces $\mathcal{D}^k_{\lambda,\mu}(M)$ comprise the *order filtration* of $\mathcal{D}_{\lambda,\mu}(M)$ and are invariant under $\mathrm{Vect}(M)$.

(iii) We write $\delta$ for the difference between $\mu$ and $\lambda$:

$$\delta := \mu - \lambda.$$

(iv) The space of *principal symbols* of degree $k$ is the quotient

$$\mathcal{S}^k_\delta(M) \;:=\; \mathcal{D}^k_{\lambda,\mu}(M)/\mathcal{D}^{k-1}_{\lambda,\mu}(M).$$

It is well-known that its $\mathrm{Vect}(M)$-module structure depends only on $\delta$.

(v) The *principal symbol* is the natural projection

$$\sigma^k_{\lambda,\mu} : \mathcal{D}^k_{\lambda,\mu}(M) \to \mathcal{S}^k_\delta(M).$$

Let us give some natural examples of tensor density modules. The simplest is $C^\infty(M)$, which is $\mathcal{F}_0(M)$. In the contact setting, the following facts are well-known.

- The adjoint action of the Lie algebra $\mathcal{K}(M)$ of contact vector fields on itself is equivalent to $\mathcal{F}_{-\frac{1}{\ell+1}}(M)$. In other words, there is a $\mathcal{K}(M)$-equivalence

$$X : \mathcal{F}_{-\frac{1}{\ell+1}}(M) \to \mathcal{K}(M),$$

  associating a contact vector field $X_\varphi$ to each element $\varphi$ of $\mathcal{F}_{-\frac{1}{\ell+1}}(M)$. The tensor density $\varphi$ is called the *contact Hamiltonian* of $X_\varphi$. The notion of the contact Hamiltonian is independent of the choice of a contact form $\theta$. However, fixing $\theta$ one can (locally) identify tensor densities and functions and think of a contact Hamiltonian as of a function.
- The conformal class $C^\infty(M)\theta$ of the contact form $\theta$ is equivalent to $\mathcal{F}_{\frac{1}{\ell+1}}(M)$ as a $\mathcal{K}(M)$-module.

In fact, the second statement follows from (1), and the first follows from Lemma 3.1 below. Thus the $\mathcal{K}(M)$-modules of contact Hamiltonians and contact forms are dual over $C^\infty(M)$. We remark that the algebraic direct sum $\bigoplus_\lambda \mathcal{F}_\lambda(M)$ of all tensor density modules is a Poisson algebra under the Lagrange bracket.

The space $\mathcal{D}_{\lambda,\mu}(M)$ generalizes $\mathcal{D}_{0,0}(M)$, which is the usual space of differential operators acting on functions. Geometric properties of $\mathcal{D}_{\lambda,\mu}(M)$ vary with the parameters, and the structure of $\mathcal{D}_{\lambda,\mu}(M)$ viewed as a $\mathcal{K}(M)$-module can be special for certain values of $\lambda$ and $\mu$. Let us stress the fact that differential operators between tensor densities appear naturally in many geometric situations. We mention for example the classical notion of the conformally invariant Laplace operator, also known as the Yamabe Laplacian, which is an element of $\mathcal{D}_{\frac{1}{2}-\frac{1}{m},\frac{1}{2}+\frac{1}{m}}(M)$. The case $\lambda+\mu=1$ is particularly special. This is the case where the notions of *symmetric* and *skew-symmetric* operators are well-defined. More generally, if $A \in \mathcal{D}_{\lambda,\mu}(M)$, then the adjoint operator $A^*$ belongs to $\mathcal{D}_{1-\mu,1-\lambda}(M)$.

We now recall the classical notion of the Heisenberg order of a differential operator on a contact manifold; see for example [vE10] and references therein.

**Definition.**

(i) The space of differential operators of *Heisenberg order* $\leq d$ is

$$\mathcal{P}^d_{\lambda,\mu}(M) \;:=\; \mathrm{Span}\big\{T_c \circ L_\lambda(Y_1) \circ \cdots \circ L_\lambda(Y_t) \,:\, T_c \in \mathcal{D}^c_{\lambda,\mu}(M),\, Y_i \in \mathrm{Tan}(M),\, 2c+t \leq d\big\}.$$

The spaces $\mathcal{P}^d_{\lambda,\mu}(M)$, comprise the *Heisenberg filtration* of $\mathcal{D}_{\lambda,\mu}(M)$. They are invariant under $\mathcal{K}(M)$.

(ii) The bifiltration $\mathcal{D}^{k,d}_{\lambda,\mu}(M) := \mathcal{D}^k_{\lambda,\mu}(M) \cap \mathcal{P}^d_{\lambda,\mu}(M)$ gives rise to the *fine symbol* modules:

$$\Sigma^{k,d}_\delta(M) \;:=\; \mathcal{D}^{k,d}_{\lambda,\mu}(M)/\big(\mathcal{D}^{k-1,d}_{\lambda,\mu}(M) + \mathcal{D}^{k,d-1}_{\lambda,\mu}(M)\big).$$

(iii) The *fine symbol* is the corresponding projection

$$\mathrm{f}\sigma^{k,d}_{\lambda,\mu} : \mathcal{D}^{k,d}_{\lambda,\mu}(M) \to \Sigma^{k,d}_\delta(M).$$

In the simplest case $k=1$ and $\delta=0$, $\Sigma^{1,1}_0(M)$ is $\mathrm{Tan}(M)$ and $\Sigma^{1,2}_0(M)$ is $\mathcal{K}(M)$. Here $\mathrm{f}\sigma^{1,2}_{\lambda,\lambda}$ is nothing but the projection $\pi$ defined above. More generally, it follows from Proposition 3.3 below that

$$\Sigma^{k,2k}_\delta(M) \cong \mathcal{F}_{\delta-\frac{k}{\ell+1}}(M).$$

Therefore $\mathrm{f}\sigma^{k,2k}_{\lambda,\mu}$ may be regarded as a $\mathcal{K}(M)$-equivariant linear projection from $\mathcal{D}^k_{\lambda,\mu}(M)$ to $\mathcal{F}_{\delta-\frac{k}{\ell+1}}(M)$. This type of equivariant assignment of a tensor density to a differential operator is known as a *differential invariant.*

**Definition.** We say that $\delta$ is *contact-resonant* if it lies in the set

$$\left\{\frac{1}{\ell+1}+\frac{n}{2(\ell+1)}\,\Big|\,n\in\mathbb{N}\right\}.$$

We will see in Section 4.5 that contact resonances arise from the representation theory of $\mathfrak{sp}_{2(\ell+1)}$.

2.2. **The subsymbol.** Our first main theorem gives a new contact differential invariant.

**Theorem A.** *If $\delta$ is not contact-resonant, then there exists a unique $\mathcal{K}(M)$-equivariant linear map*

$$\mathrm{s}\sigma^k_{\lambda,\mu}:\mathcal{D}^k_{\lambda,\mu}(M)\to\Sigma^{k-1,\,2(k-1)}_{\delta}(M)$$

*whose restriction to $\mathcal{D}^{k-1}_{\lambda,\mu}(M)$ is $\mathrm{f}\sigma^{k-1,\,2(k-1)}_{\lambda,\mu}$.*

We refer to $\mathrm{s}\sigma^k_{\lambda,\mu}$ as the *contact subsymbol.* We will give an explicit formula for it in Proposition 7.4. It may be regarded as a $\mathcal{K}(M)$-equivariant projection from $\mathcal{D}^k_{\lambda,\mu}(M)$ to $\mathcal{F}_{\delta-\frac{k-1}{\ell+1}}(M)$.

We remark that in the general self-adjoint case, where $\lambda+\mu=1$ and $k$ is arbitrary, the existence of such a differential invariant is obvious. Indeed, for $T$ in $\mathcal{D}^k_{\lambda,\mu}(M)$, the operator $T-(-1)^kT^*$ is in $\mathcal{D}^{k-1}_{\lambda,\mu}(M)$, and so can be projected to $\Sigma^{k-1,\,2(k-1)}_{\delta}(M)$.

Since $\mathcal{F}_{-\frac{1}{\ell+1}}(M)$ is equivalent to $\mathcal{K}(M)$, the case that $k=2$ and $\mu=\lambda$ is of particular interest, as there the differential invariant given by the contact subsymbol may be viewed as a contact vector field. In other words, for all $\lambda\in\mathbb{C}$, the subsymbol $\mathrm{s}\sigma^2_{\lambda,\lambda}$ defines a $\mathcal{K}(M)$-equivariant projection from $\mathcal{D}^2_{\lambda,\lambda}(M)$ to $\mathcal{K}(M)$.

In order to give an intrinsically defined and manifestly contact-invariant formula for $\mathrm{s}\sigma^2_{\lambda,\lambda}$, observe that any second order differential operator can be represented as a linear combination of compositions of vector fields. On contact manifolds, contact vector fields and tangential vector fields are intrinsically distinguished. Thus we are led to express an arbitrary second order operator on $\mathcal{F}_\lambda(M)$ as a linear combination of operators of the form

$$\begin{aligned}T &= L_\lambda(X_{\varphi_1})\circ L_\lambda(X_{\varphi_2})+L_\lambda(X_{\varphi_3})\circ L_\lambda(Y_1)+L_\lambda(Y_2)\circ L_\lambda(Y_3)\\ &\quad +L_\lambda(X_{\varphi_4})+L_\lambda(Y_4)+f,\end{aligned}\tag{2}$$

where the $\varphi_i$ are arbitrary contact Hamiltonians, the $Y_i$ are tangential vector fields, and $f$ is a function.

**Theorem B.** *The subsymbol $\mathrm{s}\sigma^2_{\lambda,\lambda}(T)$ is the contact vector field*

$$\tfrac{1}{2}\big[X_{\varphi_1},X_{\varphi_2}\big]-\big(\tfrac{\ell+1}{\ell+2}\big)\big(\lambda-\tfrac{1}{2}\big)X_{L(Y_1)\varphi_3}+\tfrac{1}{2}\pi\big[Y_2,Y_3\big]+X_{\varphi_4},$$

*where $L(Y_1)\varphi_3$ denotes the natural action of $Y_1$ on the $-\frac{1}{\ell+1}$-density $\varphi_3$.*

Let us comment on this formula. It only contains natural operations, so it is clearly contact-invariant. Conversely, equivariance with respect to $\mathcal{K}(M)$ (in fact the affine subalgebra suffices) implies that $\mathrm{s}\sigma^2_{\lambda,\lambda}(T)$ has to be of the form

$$c_{12}\big[X_{\varphi_1},X_{\varphi_2}\big]+c_{13}X_{L(Y_1)\varphi_3}+c_{23}\pi\big[Y_2,Y_3\big]+c_4X_{\varphi_4},$$

where the $c$'s are constants. The normalization condition on $\mathcal{D}^1_{\lambda,\lambda}$ gives $c_4=1$. Skew-symmetrizing the expression then yields $c_{12}=c_{23}=\frac{1}{2}$. Symmetrizing the expression implies that $c_{13}$ vanishes in the self-adjoint case $\lambda=\frac{1}{2}$, but its exact form must be deduced by computation.

The main content of the theorem is that the formula is actually well-defined. Indeed, the choice of the $\varphi_i$ and $Y_i$ in (2) is not unique: one can write an operator as a linear combination of such expressions in many different ways. However, the formula is independent of the choice. Moreover, the uniqueness statement of Theorem A implies that, up to a scalar, this is not true for any other choice of the $c$'s.

2.3. **The fine filtration.** In order to explain the significance of our next theorem, consider the following arrangement of the fine symbol modules (we have omitted $M$ and $\delta$ for clarity):

$$\begin{array}{ccccccccc}
 & & & & & & \Sigma^{6,12} & & \\
 & & & & & \Sigma^{5,10} & \Sigma^{6,11} & & \\
 & & & & \Sigma^{4,8} & \Sigma^{5,9} & \Sigma^{6,10} & & \\
 & & & \Sigma^{3,6} & \Sigma^{4,7} & \Sigma^{5,8} & \Sigma^{6,9} & \cdots & \\
 & & \Sigma^{2,4} & \Sigma^{3,5} & \Sigma^{4,6} & \Sigma^{5,7} & \Sigma^{6,8} & & \\
 & \Sigma^{1,2} & \Sigma^{2,3} & \Sigma^{3,4} & \Sigma^{4,5} & \Sigma^{5,6} & \Sigma^{6,7} & & \\
\Sigma^{0,0} & \Sigma^{1,1} & \Sigma^{2,2} & \Sigma^{3,3} & \Sigma^{4,4} & \Sigma^{5,5} & \Sigma^{6,6} & &
\end{array}$$

Observe that the graded module of $\mathcal{S}^k_\delta(M)$ defined by the bifiltration $\mathcal{D}^{k,d}_{\lambda,\mu}(M)$ is the "vertical" sum

$$\operatorname{gr}\mathcal{S}^k_\delta(M) = \bigoplus_{k\le d\le 2k} \Sigma^{k,d}_\delta(M).$$

The graded module of $\mathcal{P}^d_{\lambda,\mu}(M)/\mathcal{P}^{d-1}_{\lambda,\mu}(M)$ is the "slope $-1$" sum

$$\operatorname{gr}\big(\mathcal{P}^d_{\lambda,\mu}(M)/\mathcal{P}^{d-1}_{\lambda,\mu}(M)\big) = \bigoplus_{\lceil\frac{d}{2}\rceil\le k\le d} \Sigma^{k,d}_\delta(M).$$

The content of our next theorem is that there exists a $\mathcal{K}(M)$-invariant filtration that strengthens the filtration $\mathcal{P}^d_{\lambda,\mu}(M)$. The graded modules of its subquotients are the "slope $-\frac{1}{2}$" sums.

**Theorem C.** *Assume that $\delta$ is not contact-resonant. Then there is a unique $\mathcal{K}(M)$-invariant filtration of $\mathcal{D}_{\lambda,\mu}(M)$,*

$$\mathcal{D}^{(0)}_{\lambda,\mu}(M) \subset \cdots \subset \mathcal{D}^{(b)}_{\lambda,\mu}(M) \subset \mathcal{D}^{(b+1)}_{\lambda,\mu}(M) \subset \cdots,$$

*such that the graded module of $\mathcal{D}^{(b)}_{\lambda,\mu}(M)$ is given by*

$$\operatorname{gr}\mathcal{D}^{(b)}_{\lambda,\mu}(M) = \bigoplus_{2d-k\le b} \Sigma^{k,d}_\delta(M).$$

For example, $\operatorname{gr}\big(\mathcal{D}^{(6)}_{\lambda,\mu}(M)/\mathcal{D}^{(5)}_{\lambda,\mu}(M)\big) = \Sigma^{6,6}_\delta(M) \oplus \Sigma^{4,5}_\delta(M) \oplus \Sigma^{2,4}_\delta(M)$, as indicated by the boundaries in the diagram above. We will define $\mathcal{D}^{(b)}_{\lambda,\mu}(M)$ via the projective quantization: see Section 6.3.

2.4. **Comments on non-existence and conjectures.** In this paper we are concerned with existence results rather than non-existence results, but we remark that in the contact-resonant case our theorems are false for most values of $\lambda$. For example, if $k = \delta(\ell+1) - \ell$ in Theorem A, then there exists no subsymbol except in the self-adjoint case $\lambda + \mu = 1$, where there exists a 1-parameter family of such maps. Non-existence results can be interpreted in terms of cohomological obstructions. We conjecture that the filtration of Theorem C does not exist in the contact-resonant case. These questions will be addressed elsewhere.

A module is called *uniserial,* or *completely indecomposable,* if it has a unique maximal invariant filtration. Such modules may be thought of as opposite to completely reducible modules. We conjecture that for generic values of $\lambda$ and $\mu$, $\mathcal{D}^{(b)}_{\lambda,\mu}(M)/\mathcal{D}^{(b-1)}_{\lambda,\mu}(M)$ is uniserial, because we expect that in the Euclidean case, there is a non-trivial projectively relative 1-cohomology class of $\mathcal{K}_m$ linking $\Sigma^{k,d}_\delta$ to $\Sigma^{k-2,d+1}_\delta$: an analog of the Schwarzian derivative. This conjecture says essentially that the bifiltration $\mathcal{D}^k_{\lambda,\mu}(M) \cap \mathcal{D}^{(b)}_{\lambda,\mu}(M)$ is the best possible. In other words, if the conjecture is true then there is no $\mathcal{K}(M)$-invariant filtration of $\mathcal{D}_{\lambda,\mu}(M)$ whose elements are composed of all those fine symbol modules on or below the lines of any fixed slope shallower than $-\frac{1}{2}$ passing through the above diagram.

## 3. The Euclidean contact Lie algebra

Since all of the theorems in Section 2 are local, their proofs essentially reduce to the case $M = \mathbb{R}^m$. Therefore in this section we establish notation and state some well-known results for Euclidean contact manifolds: the proofs are straightforward and are usually omitted. All $C^\infty(\mathbb{R}^m)$-modules of finite rank are equipped with their usual topologies as Frechet spaces, and by definition all Hom spaces between such modules include only continuous linear maps.

3.1. **Darboux coordinates.** Fix coordinates $x_i$, $y_i$, and $z$ on $\mathbb{R}^m$, where $m = 2\ell+1$ and $1 \le i \le \ell$. Henceforth we use *Einstein's summation convention:* unless stated otherwise, *repeated indices are summed over from* 1 *to* $\ell$. Let $\theta$ and $\omega$ be the standard contact and volume forms on $\mathbb{R}^m$:

$$\theta := dz + \tfrac{1}{2}(x_i dy_i - y_i dx_i), \qquad \omega := \tfrac{1}{\ell!}\,\theta \wedge (d\theta)^\ell = dz \wedge \textstyle\bigwedge_1^\ell (dx_i \wedge dy_i).$$

The standard divergence operator $\mathrm{Div} : \mathrm{Vect}(\mathbb{R}^m) \to C^\infty(\mathbb{R}^m)$ is defined by

$$L(X)\,\omega = \mathrm{Div}(X)\omega.$$

Recall from Section 2 the definitions of the Lie algebra of contact vector fields and the space of tangent vector fields:

$$\begin{aligned} \mathcal{K}(\mathbb{R}^m) &:= \big\{X \in \mathrm{Vect}(\mathbb{R}^m) : L(X)\,\theta = \tfrac{1}{\ell+1}\,\mathrm{Div}(X)\theta\big\}, \\ \mathrm{Tan}(\mathbb{R}^m) &:= \big\{X \in \mathrm{Vect}(\mathbb{R}^m) : \langle \theta, X\rangle = 0\big\}. \end{aligned}$$

The space $\mathrm{Tan}(\mathbb{R}^m)$ is a module over $\mathcal{K}(\mathbb{R}^m)$, and so one has the $\mathcal{K}(\mathbb{R}^m)$-invariant decomposition

$$\mathrm{Vect}(\mathbb{R}^m) = \mathcal{K}(\mathbb{R}^m) \oplus \mathrm{Tan}(\mathbb{R}^m).$$

It is important to keep in mind that while $\mathrm{Tan}(\mathbb{R}^m)$ is closed under multiplication by smooth functions, $\mathcal{K}(\mathbb{R}^m)$ is not. We now give explicit descriptions of both spaces.

Define the following *Euler operators:*

$$\mathcal{E}_z := z\partial_z, \qquad \mathcal{E}_{xy} := x_i\partial_{x_i} + y_i\partial_{y_i}.$$

For $1 \le i \le \ell$, define the vector fields

$$A_i := \partial_{x_i} + \tfrac{1}{2}y_i\partial_z, \qquad B_i := -\partial_{y_i} + \tfrac{1}{2}x_i\partial_z.$$

The following statements are classical.

- There is a linear bijection $X : C^\infty(\mathbb{R}^m) \to \mathcal{K}(\mathbb{R}^m)$ mapping $f$ to $X_f$, the unique contact vector field such that $\langle \theta, X_f \rangle = f$. It has the following explicit formulas:

$$\begin{aligned} X_f &= f\partial_z + B_i(f)A_i - A_i(f)B_i \\ &= \big(1 - \tfrac{1}{2}\mathcal{E}_{xy}\big)(f)\partial_z + \tfrac{1}{2}\partial_z(f)\mathcal{E}_{xy} + \big(\partial_{x_i}(f)\partial_{y_i} - \partial_{y_i}(f)\partial_{x_i}\big). \end{aligned} \tag{3}$$

- $\{A_i,\, B_i : 1 \le i \le \ell\}$ is a basis of $\mathrm{Tan}(\mathbb{R}^m)$ over $C^\infty(\mathbb{R}^m)$, and

$$[A_i, B_j] = \delta_{ij}\partial_z, \qquad [\partial_z, A_i] = 0, \qquad [\partial_z, B_i] = 0.$$

We remark that one can verify (1) directly in this setting by checking that

$$L(X_f)\,\theta = \partial_z(f)\,\theta, \qquad \mathrm{Div}(X_f) = (\ell+1)\partial_z(f). \tag{4}$$

**Definition.** The *Lagrange bracket* $\{f, g\}$ on $C^\infty(\mathbb{R}^m)$ is defined by

$$X_{\{f,g\}} := [X_f, X_g].$$

The following formulas for $\{f, g\}$ may be deduced from (3).

**Lemma 3.1.** *The Lagrange bracket is given by*

$$\begin{aligned} \{f,g\} &= X_f(g) - g\partial_z(f) \;=\; f\partial_z(g) - X_g(f) \\ &= f\partial_z(g) - \partial_z(f)g + B_i(f)A_i(g) - A_i(f)B_i(g) \\ &= \big(1 - \tfrac{1}{2}\mathcal{E}_{xy}\big)(f)\partial_z(g) - \big(1 - \tfrac{1}{2}\mathcal{E}_{xy}\big)(g)\partial_z(f) + \big(\partial_{x_i}(f)\partial_{y_i}(g) - \partial_{y_i}(f)\partial_{x_i}(g)\big). \end{aligned}$$

3.2. **Tensor density modules.** Recall from Section 2 the tensor density module $\mathcal{F}_\lambda(\mathbb{R}^m)$ of $\mathrm{Vect}(\mathbb{R}^m)$. As a vector space, $\mathcal{F}_\lambda(\mathbb{R}^m)$ is nothing but $C^\infty(\mathbb{R}^m)$. However, the action of $\mathrm{Vect}(\mathbb{R}^m)$ depends on $\lambda$ and is given by

$$L_\lambda(X)(g) := X(g) + \lambda\,\mathrm{Div}(X)\,g,$$

where $g \in C^\infty(\mathbb{R}^m)$ and $\lambda \in \mathbb{R}$. Since the volume form $\omega$ is global, we may regard $\mathcal{F}_\lambda(\mathbb{R}^m)$ as $\omega^\lambda C^\infty(\mathbb{R}^m)$, so that the action of $\mathrm{Vect}(\mathbb{R}^m)$ is identified with the usual Lie derivative:

$$L(X)(\omega^\lambda g) = \omega^\lambda\, L_\lambda(X)(g).$$

We remark that the full family $\big\{\mathcal{F}_\lambda(\mathbb{R}^m) : \lambda \in \mathbb{C}\big\}$ of $\mathrm{Vect}(\mathbb{R}^m)$-modules can be understood algebraically as a non-trivial deformation of the module $C^\infty(\mathbb{R}^m)$.

We will consider $\mathcal{F}_\lambda(\mathbb{R}^m)$ as a module over the subalgebra of contact vector fields $\mathcal{K}(\mathbb{R}^m)$. In light of (1) and the global contact form $\theta$, we may regard $\mathcal{F}_\lambda(\mathbb{R}^m)$ in this context as either $\omega^\lambda C^\infty(\mathbb{R}^m)$ or $\theta^{\lambda(\ell+1)}C^\infty(\mathbb{R}^m)$. In particular, (4) gives

$$L(X_f)(\theta^{\lambda(\ell+1)}g) = \theta^{\lambda(\ell+1)}L_\lambda(X_f)(g) = \theta^{\lambda(\ell+1)}\big(X_f(g) + \lambda(\ell+1)\partial_z(f)g\big).$$

As mentioned in Section 2, the adjoint action of $\mathcal{K}(\mathbb{R}^m)$ on itself is equivalent to the module of $-\frac{1}{\ell+1}$-densities. The following definition and lemma state this formally.

**Definition.** Henceforth we regard $X$ as the map

$$X : \mathcal{F}_{-\frac{1}{\ell+1}}(\mathbb{R}^m) \to \mathcal{K}(\mathbb{R}^m), \qquad X(\theta^{-1}f) := X_f.$$

**Lemma 3.2.** *The map $X$ is a linear bijection and a $\mathcal{K}(\mathbb{R}^m)$-equivalence.*

*Proof.* By Lemma 3.1, $\{f, g\} = L_{-\frac{1}{\ell+1}}(X_f)(g)$. Therefore $X$ intertwines the $\mathcal{K}(\mathbb{R}^m)$-action on $\mathcal{F}_{-\frac{1}{\ell+1}}(\mathbb{R}^m)$ and the adjoint action. □

3.3. **Differential operator modules.** We now turn to the focus of the paper, the modules $\mathcal{D}_{\lambda,\mu}(\mathbb{R}^m)$ of differential operators from $\mathcal{F}_\lambda(\mathbb{R}^m)$ to $\mathcal{F}_\mu(\mathbb{R}^m)$. We can write the action $L_{\lambda,\mu}$ of $\mathrm{Vect}(\mathbb{R}^m)$ on these modules concretely as follows:

$$L_{\lambda,\mu}(X)(T) := L_\mu(X)\circ T - T\circ L_\lambda(X).$$

The structure of the spaces $\mathcal{D}_{\lambda,\mu}(\mathbb{R}^m)$ viewed as $\mathrm{Vect}(\mathbb{R}^m)$-modules has been thoroughly studied; see for example [LMT96, DO97, GMO05, Co09] and the references therein. We will be interested in these spaces viewed as $\mathcal{K}(\mathbb{R}^m)$-modules, as which they are less rigid, because $\mathcal{K}(\mathbb{R}^m)$ is smaller than $\mathrm{Vect}(\mathbb{R}^m)$. In particular, the $\mathcal{D}_{\lambda,\mu}(\mathbb{R}^m)$ admit more $\mathcal{K}(\mathbb{R}^m)$-invariant operations than they do $\mathrm{Vect}(\mathbb{R}^m)$-invariant operations, such as projections to tensor fields. We will understand such operations as differential invariants.

3.4. **Symbol modules.** In Section 2 we defined the principal symbol modules $\mathcal{S}^k_\delta(\mathbb{R}^m)$ and the fine symbol modules $\Sigma^{k,d}_\delta(\mathbb{R}^m)$, $k\le d\le 2k$. We will write $\mathcal{S}_\delta(\mathbb{R}^m)$ and $\Sigma_\delta(\mathbb{R}^m)$ for the *total symbol module* and the *total fine symbol module,* respectively; the graded modules associated to the order filtration and the bifiltration of $\mathcal{D}_{\lambda,\mu}(\mathbb{R}^m)$:

$$\mathcal{S}_\delta(\mathbb{R}^m) \;:=\; \bigoplus_{k=0}^{\infty}\mathcal{S}^k_\delta(\mathbb{R}^m), \qquad \Sigma_\delta(\mathbb{R}^m) \;:=\; \bigoplus_{k=0}^{\infty}\bigoplus_{d=k}^{2k}\Sigma^{k,d}_\delta(\mathbb{R}^m).$$

Let $L^{\mathcal{S}}_\delta$ be the natural action of $\mathrm{Vect}(\mathbb{R}^m)$ on $\mathcal{S}_\delta(\mathbb{R}^m)$, and let $L^{\Sigma}_\delta$ be the natural action of $\mathcal{K}(\mathbb{R}^m)$ on $\Sigma_\delta(\mathbb{R}^m)$. Our next proposition gives formulas for $L^{\mathcal{S}}_\delta(X_f)$ and $L^{\Sigma}_\delta(X_f)$. In order to state it we must develop a variation of the usual symbol calculus which is adapted to the fine filtration. Let $\alpha_i$, $\beta_i$, and $\zeta$ be the symbols associated to the vector fields $A_i$, $B_i$, and $\partial_z$. More explicitly,

$$\alpha_i = \xi_{x_i} + \tfrac{1}{2}\,y_i\xi_z, \qquad \beta_i = -\xi_{y_i} + \tfrac{1}{2}\,x_i\xi_z, \qquad \zeta = \xi_z, \tag{5}$$

where $\xi_{x_1},\xi_{y_1},\dots,\xi_{x_\ell},\xi_{y_\ell},\xi_z$ are the coordinates on $T^*\mathbb{R}^m$ dual to $x_1,y_1,\dots,x_\ell,y_\ell,z$.

We shall abuse notation and use $\alpha_i$ and $\beta_i$ also to denote fine symbols. Thus

$$\begin{aligned}\mathcal{S}^k_\delta(\mathbb{R}^m) &= \mathrm{Span}_{C^\infty(\mathbb{R}^m)}\big\{\zeta^c\alpha^I\beta^J : |I|+|J|+c=k\big\},\\ \Sigma^{k,d}_\delta(\mathbb{R}^m) &= \mathrm{Span}_{C^\infty(\mathbb{R}^m)}\big\{\zeta^{d-k}\alpha^I\beta^J : |I|+|J|=2k-d\big\},\end{aligned} \tag{6}$$

where $I$ and $J$ are multi-indices: $I=(I_1,\dots,I_\ell)$ and $J=(J_1,\dots,J_\ell)$. Note that in local coordinates we do not write the shift $\omega^\delta$ in the tensor density degree explicitly.

Any differential operator from one symbol module to another, or from one fine symbol module to another, may be written as a linear combination over $C^\infty(\mathbb{R}^m)$ of products of the operators

$$A_i,\ B_i,\ \partial_z,\ \alpha_i,\ \beta_i,\ \zeta,\ \partial_{\alpha_i},\ \partial_{\beta_i},\ \partial_\zeta.$$

Such a combination is to be interpreted as follows. The operators $A_i$, $B_i$, and $\partial_z$ act solely on the $C^\infty(\mathbb{R}^m)$ coefficients of the basis elements in (6), while the remaining operators act solely on the basis elements themselves.

Imitating the definitions of $\mathcal{E}_z$ and $\mathcal{E}_{xy}$, we set

$$\mathcal{E}_\zeta := \zeta\partial_\zeta, \qquad \mathcal{E}_{\alpha\beta} := \alpha_i\partial_{\alpha_i} + \beta_i\partial_{\beta_i}.$$

**Proposition 3.3.** (i) *The action $L^{\Sigma}_\delta$ of $\mathcal{K}(\mathbb{R}^m)$ on $\Sigma_\delta(\mathbb{R}^m)$ is*

$$\begin{aligned}L^{\Sigma}_\delta(X_f) \;=\;& f\partial_z + B_i(f)A_i - A_i(f)B_i + \partial_z(f)\big(\delta(\ell+1) - \mathcal{E}_\zeta - \tfrac{1}{2}\mathcal{E}_{\alpha\beta}\big)\\ &+ \tfrac{1}{2}(A_iB_i + B_iA_i)(f)(\beta_i\partial_{\beta_j} - \alpha_i\partial_{\alpha_j}) + A_iA_j(f)\beta_i\partial_{\alpha_j} - B_iB_j(f)\alpha_i\partial_{\beta_j}.\end{aligned}$$

(ii) *The action $L^{\Sigma}_\delta$ of $\mathcal{K}(\mathbb{R}^m)$ on $\mathcal{S}_\delta(\mathbb{R}^m)$ is*

$$L^{\mathcal{S}}_\delta(X_f) = L^{\Sigma}_\delta(X_f) + \big(\partial_zA_i(f)\beta_i - \partial_zB_i(f)\alpha_i\big)\partial_\zeta.$$

*Proof.* Taking in to account the shift $\omega^\delta$ in tensor density degree, $L^{\mathcal{S}}$ and $L^{\Sigma}$ are derivations in an obvious sense. Therefore it is only necessary to check the formulas on the generators $\alpha_i$, $\beta_i$, and $\zeta$ and on functions $g$. Keep in mind that $A_i$ and $B_i$ do not commute, although $\alpha_i$ and $\beta_i$, being symbols, do. □

The difference between $L^{\mathcal{S}}_\delta$ and $L^{\Sigma}_\delta$ is due to the fact that $\Sigma_\delta(\mathbb{R}^m)$ is the graded module of $\mathcal{S}_\delta(\mathbb{R}^m)$. Observe that $L^{\mathcal{S}}_\delta - L^{\Sigma}_\delta$ maps $\Sigma^{k,d}_\delta(\mathbb{R}^m)$ to $\Sigma^{k,d-1}_\delta(\mathbb{R}^m)$.

Henceforth we will frequently drop the argument $\mathbb{R}^m$ of the various tensor density, differential operator, and symbol modules, writing simply $\mathcal{F}_\lambda$, $\mathcal{D}_{\lambda,\mu}$, $\mathcal{S}^k_\delta$, $\Sigma^{k,d}_\delta$, and so on.

## 4. The projective subalgebra

Here we recall the *projective subalgebra* $\mathfrak{s}_m$ of $\mathcal{K}(\mathbb{R}^m)$, which is isomorphic to $\mathfrak{sp}_{2(\ell+1)}$. Restriction of $\mathcal{K}(\mathbb{R}^m)$-modules to $\mathfrak{s}_m$ will be central to our strategy throughout this paper, for two reasons. First, $\mathfrak{s}_m$ is a maximal polynomial subalgebra of $\mathcal{K}(\mathbb{R}^m)$, and it turns out that for most values of their parameters, the tensor density modules and the fine symbol modules are not only algebraically irreducible under $\mathcal{K}(\mathbb{R}^m)$, they remain so under restriction to $\mathfrak{s}_m$. (By "algebraically irreducible", we mean irreducible in the polynomial category.) Second, $\mathfrak{s}_m$ is a finite dimensional semisimple Lie algebra, and so we can bring the representation theory of such algebras to bear on the restricted modules.

In fact, the restrictions to $\mathfrak{s}_m$ of the tensor density modules and fine symbol modules are duals of $\mathfrak{sp}_{2\ell}$-relative Verma modules. Our approach to the construction of the projective quantization referred to in the introduction will be to observe that for generic $\delta$, the fine symbol modules composing $\mathcal{D}_{\lambda,\mu}$ have distinct infinitesimal characters under the action of $\mathfrak{s}_m$. This implies that there is a unique $\mathfrak{s}_m$-equivariant splitting of $\mathcal{D}_{\lambda,\mu}$ into the sum of its fine symbol modules. This splitting may be regarded as a projectively invariant total symbol. The projective quantization is, by definition, its inverse.

In addition to the projective subalgebra $\mathfrak{s}_m$, two other subalgebras of $\mathcal{K}(\mathbb{R}^m)$ will be important to us: the *affine subalgebra* $\mathfrak{t}_m$ and its nilradical $\mathfrak{u}_m$.

4.1. **The projective subalgebra of the full vector field Lie algebra.** We first recall the definitions of the analogous subalgebras of the full vector field Lie algebra. Let $u_1, \dots, u_m$ be any coordinates on $\mathbb{R}^m$. The full Euler operator is

$$\mathcal{E}_u := \sum_{i=1}^m u_i \partial_{u_i}.$$

Within $\mathrm{Vect}(\mathbb{R}^m)$ we have the projective subalgebra $\mathfrak{a}_m$, the affine subalgebra $\mathfrak{b}_m$, and the constant coefficient subalgebra $\mathfrak{c}_m$:

$$\begin{aligned}
\mathfrak{a}_m &:= \mathrm{Span}_{\mathbb{C}}\{\partial_{u_i}, u_j\partial_{u_i}, u_j\mathcal{E}_u : 1 \le i,j \le m\},\\
\mathfrak{b}_m &:= \mathrm{Span}_{\mathbb{C}}\{\partial_{u_i}, u_j\partial_{u_i} : 1 \le i,j \le m\},\\
\mathfrak{c}_m &:= \mathrm{Span}_{\mathbb{C}}\{\partial_{u_i} : 1 \le i \le m\}.
\end{aligned}$$

Clearly $\mathfrak{a}_m \supset \mathfrak{b}_m \supset \mathfrak{c}_m$. In fact, $\mathfrak{a}_m$ is isomorphic to $\mathfrak{sl}_{m+1}$, and $\mathfrak{b}_m$ is a maximal parabolic subalgebra of $\mathfrak{a}_m$ with Levi factor $\mathfrak{gl}_m$ and nilradical $\mathfrak{c}_m$. The center of the Levi factor is $\mathbb{C}\mathcal{E}_u$.

There is a standard conceptual proof of $\mathfrak{a}_m \cong \mathfrak{sl}_{m+1}$ which we briefly sketch. Let $u_0, \dots, u_m$ be coordinates on $\mathbb{R}^{m+1}$, and note that $\mathrm{Span}_{\mathbb{C}}\{u_j\partial_{u_i} : 0 \le i,j \le m\}$ is a subalgebra of $\mathrm{Vect}(\mathbb{R}^{m+1})$ isomorphic to $\mathfrak{gl}_{m+1}$. Restricting the action of $\mathrm{Vect}(\mathbb{R}^{m+1})$ to functions on $\mathbb{P}^m$ defines the canonical projection from this copy of $\mathfrak{gl}_{m+1}$ to $\mathfrak{sl}_{m+1}$. Regarding $\mathbb{P}^m$ locally as the hyperplane defined by $u_0 = 1$ and identifying this hyperplane with $\mathbb{R}^m$ yields the isomorphism.

4.2. **The projective subalgebra of the contact Lie algebra.** We now define the projective and affine subalgebras of $\mathcal{K}(\mathbb{R}^m)$: they are simply the intersections of the projective and affine subalgebras of $\mathrm{Vect}(\mathbb{R}^m)$ with $\mathcal{K}(\mathbb{R}^m)$.

**Definition.** The *projective subalgebra* $\mathfrak{s}_m$ and the *affine subalgebra* $\mathfrak{t}_m$ of $\mathcal{K}(\mathbb{R}^m)$ are

$$\mathfrak{s}_m := \mathfrak{a}_m \cap \mathcal{K}(\mathbb{R}^m), \qquad \mathfrak{t}_m := \mathfrak{b}_m \cap \mathcal{K}(\mathbb{R}^m).$$

We define also the following subspaces of $\mathcal{K}(\mathbb{R}^m)$:

$$\begin{aligned} \mathfrak{u}_m &:= \mathrm{Span}_{\mathbb{C}}\{X_1, X_{x_i}, X_{y_i} : 1 \le i \le \ell\}, \\ \mathfrak{l}^{\mathrm{s}}_m &:= \mathrm{Span}_{\mathbb{C}}\{X_{x_iy_j}, X_{x_ix_j}, X_{y_iy_j} : 1 \le i, j \le \ell\}, \\ \mathfrak{l}_m &:= \mathfrak{l}^{\mathrm{s}}_m \oplus \mathbb{C}X_z. \end{aligned}$$

Observe that $X_z = \mathcal{E}_z + \frac{1}{2}\mathcal{E}_{xy}$. This is the natural Euler operator in $\mathcal{K}(\mathbb{R}^m)$. The space $\mathfrak{l}_m$ is the 0-weight space of its adjoint action, and $\mathfrak{u}_m$ is the sum of its $-\frac{1}{2}$-eigenspace $\mathrm{Span}_{\mathbb{C}}\{X_{x_i}, X_{y_i} : 1 \le i \le \ell\}$ and its $-1$-eigenspace $\mathbb{C}X_1$.

Next we give explicit descriptions of $\mathfrak{s}_m$ and $\mathfrak{t}_m$ and prove that $\mathfrak{s}_m$ is symplectic. We also show that $\mathfrak{u}_m$, $\mathfrak{l}^{\mathrm{s}}_m$, and $\mathfrak{l}_m$ are subalgebras of $\mathfrak{t}_m$, and that $\mathfrak{u}_m$ is the contact-analog of the constant coefficient algebra $\mathfrak{c}_m$.

**Lemma 4.1.** (i) *The space $\mathfrak{l}^{\mathrm{s}}_m$ is a subalgebra of $\mathfrak{s}_m$ isomorphic to $\mathfrak{sp}_{2\ell}$.*

(ii) *The space $\mathfrak{l}_m$ is a Levi subalgebra of $\mathfrak{s}_m$, with semisimple part $\mathfrak{l}^{\mathrm{s}}_m$ and center $\mathbb{C}X_z$.*

(iii) *The space $\mathfrak{u}_m$ is a Heisenberg Lie algebra with center $\mathbb{C}X_1$.*

(iv) *The affine subalgebra $\mathfrak{t}_m$ is the semidirect sum $\mathfrak{l}_m \oplus_{\mathrm{s}} \mathfrak{u}_m$. It is a maximal parabolic subalgebra of $\mathfrak{s}_m$ with nilradical $\mathfrak{u}_m$.*

(v) *The projective subalgebra $\mathfrak{s}_m$ is isomorphic to $\mathfrak{sp}_{2(\ell+1)}$, and*

$$\mathfrak{s}_m = X\big(\theta^{-1}\{\text{Polynomials of degree} \le 2 \text{ on } \mathbb{R}^m\}\big).$$

*Proof.* Most of this can be left to the reader. To prove (i), use (3) to obtain

$$X_{x_iy_j} = y_j\partial_{y_i} - x_i\partial_{x_j}, \quad X_{x_ix_j} = x_i\partial_{y_j} + x_j\partial_{y_i}, \quad X_{y_iy_j} = -y_i\partial_{x_j} - y_j\partial_{x_i}.$$

Then note that the natural action of $\mathfrak{l}^{\mathrm{s}}_m$ on $\mathrm{Span}_{\mathbb{C}}\{x_i, y_i : 1 \le i \le \ell\}$ preserves the skew-symmetric form defined by $\langle x_i, x_j\rangle = 0$, $\langle y_i, y_j\rangle = 0$, and $\langle x_i, y_j\rangle = \delta_{ij}$. (Alternately, observe that the Lagrange bracket defines a non-degenerate $\mathfrak{l}_m$-invariant skew-symmetric form on the $-\frac{1}{2}$-eigenspace of $X_z$.) Parts (ii), (iii), and (iv) now follow by computation.

For a direct proof of the displayed equation in (v), first check that $\mathcal{E}_{xy} = x_iA_i - y_iB_i$. Then verify that the vector fields

$$A_i, \quad B_i, \quad x_jA_i - y_iB_j, \quad x_jB_i - x_iB_j, \quad y_jA_i - y_iA_j, \quad zA_i + \tfrac{1}{2}y_i\mathcal{E}_{xy}, \quad zB_i + \tfrac{1}{2}x_i\mathcal{E}_{xy},$$

$1 \le i, j \le \ell$, span a $(2\ell+3)\ell$-dimensional subspace of $\mathfrak{a}_m \cap \mathrm{Tan}(\mathbb{R}^m)$.

Next, use Lemma 3.2 to verify that the right side of the display in (v) is a $(2\ell+3)(\ell+1)$-dimensional subspace of $\mathfrak{a}_m \cap \mathcal{K}(\mathbb{R}^m)$. Since $\mathfrak{a}_m$ is $(2\ell+3)(2\ell+1)$-dimensional, (v) is proven.

The fact that $\mathfrak{s}_m$ is a copy of $\mathfrak{sp}_{2(\ell+1)}$ can now be proven using (ii) and an adaptation of the above argument proving that $\mathfrak{a}_m$ is a copy of $\mathfrak{sl}_{m+1}$. □

Let us establish some notation for later use: we will write $\mathbb{C}^{2\ell}$ for the basic module of $\mathfrak{l}^{\mathrm{s}}_m \cong \mathfrak{sp}_{2\ell}$, and $\mathrm{Sym}^r\,\mathbb{C}^{2\ell}$ for its $r^{\mathrm{th}}$ symmetric power. It is well-known that $\mathrm{Sym}^r\,\mathbb{C}^{2\ell}$ is self-dual and irreducible for all $r$.

The following lemma defines a Cartan subalgebra $\mathfrak{h}_m$ of $\mathfrak{s}_m$ and gives the associated root system and Weyl group. Its proof may be found in any text on Lie theory; see *e.g.,* [Va84].

**Lemma 4.2.** (i) *The set* $\{2X_z, X_{x_1y_1}, \dots, X_{x_\ell y_\ell}\}$ *is a basis of a Cartan subalgebra* $\mathfrak{h}_m$ *of* $\mathfrak{s}_m$*. Under the Cartan-Killing form this basis is orthogonal and all of its elements are of length* $2\sqrt{\ell+2}$*. Let* $\{e_0, e_1, \dots, e_\ell\}$ *be the dual basis of* $\mathfrak{h}_m^*$*.*

(ii) *The adjoint action of* $\mathfrak{h}_m$ *on the polynomial contact vector fields is semisimple:* $X(x^I y^J z^c)$ *is of* $\mathfrak{h}_m$*-weight* $(2c+|I|+|J|-2)e_0 - (I_i - J_i)e_i$*.*

(iii) *The roots of* $\mathfrak{s}_m$ *are all of the non-zero* $\pm e_i \pm e_j$*. The Weyl group* $W(\mathfrak{s}_m)$ *is* $S_{\ell+1} \ltimes \mathbb{Z}_2^{\ell+1}$*, acting in the usual way on the basis* $\{e_0, \dots, e_\ell\}$*.*

(iv) *The order* $0 < e_\ell < \dots < e_0$ *gives the following simple root system* $\Pi^+(\mathfrak{s}_m)$ *and positive root system* $\Delta^+(\mathfrak{s}_m)$ *of* $\mathfrak{s}_m$*:*

$$\Pi^+(\mathfrak{s}_m) := \big\{e_{i-1} - e_i,\, 2e_\ell\big\}_{i=1}^{\ell}, \quad \Delta^+(\mathfrak{s}_m) := \big\{e_i - e_j\big\}_{i<j} \cup \big\{e_i + e_j\big\}_{i,j}.$$

*The half-sum of the positive roots is* $\rho(\mathfrak{s}_m) = \sum_0^\ell (\ell+1-i)e_i$*. The dominant weights are those* $\sum_0^\ell \gamma_i e_i$ *with* $0 \le \gamma_\ell \le \dots \le \gamma_0$*. The negative root vectors are* $X_1$*,* $X_{x_i}$*,* $X_{y_i}$*,* $X_{x_ix_j}$*, and those* $X_{x_iy_j}$ *with* $i<j$*.*

(v) *The algebra* $\mathfrak{h}_m$ *is also a Cartan subalgebra of the Levi subalgebra* $\mathfrak{l}_m$ *of* $\mathfrak{s}_m$*. The roots of* $\mathfrak{l}_m$ *are the non-zero* $\pm e_i \pm e_j$ *with* $i,j>0$*. It inherits the simple root system*

$$\Pi^+(\mathfrak{l}_m) = \big\{e_{i-1} - e_i,\, 2e_\ell\big\}_{i=2}^{\ell}.$$

(vi) *The subalgebra* $\mathfrak{h}_m^{\mathrm{s}} := \mathrm{Span}\{X_{x_1y_1}, \dots, X_{x_\ell y_\ell}\}$ *of* $\mathfrak{h}_m$ *is a Cartan subalgebra of* $\mathfrak{l}_m^{\mathrm{s}}$*. The roots of* $\mathfrak{l}_m^{\mathrm{s}}$ *are the same as those of* $\mathfrak{l}_m$*.*

4.3. **The $\mathfrak{s}_m$-structure of the symbol modules.** In this subsection we analyze the action of $\mathfrak{s}_m$ on the principal and fine symbol modules. Our first lemma gives the restriction to $\mathfrak{s}_m$ of the actions $L_\delta^{\mathcal{S}}$ and $L_\delta^{\Sigma}$ on $\mathcal{S}_\delta$ and $\Sigma_\delta$. It is a corollary of Proposition 3.3. In order to state it concisely, we define the *total weight operator* $W$ to act on both $\mathcal{S}_\delta$ and $\Sigma_\delta$ by

$$W|_{\mathcal{S}_\delta} = W|_{\Sigma_\delta} := (\mathcal{E}_z + \tfrac{1}{2}\mathcal{E}_{xy}) - (\mathcal{E}_\zeta + \tfrac{1}{2}\mathcal{E}_{\alpha\beta}) + \delta(\ell+1).$$

**Lemma 4.3.** (i) *The restrictions of* $L_\delta^{\mathcal{S}}$ *and* $L_\delta^{\Sigma}$ *to the affine subalgebra* $\mathfrak{t}_m$ *coincide.*

(ii) *Their restriction to the nilradical* $\mathfrak{u}_m$ *of* $\mathfrak{t}_m$ *is the identity map* $X_f \mapsto X_f$*:*

$$X_1 \;\mapsto\; \partial_z, \quad X_{x_i} \;\mapsto\; x_i\partial_z - B_i, \quad X_{y_i} \;\mapsto\; y_i\partial_z - A_i.$$

(iii) *Their restriction to the Levi factor* $\mathfrak{l}_m$ *of* $\mathfrak{t}_m$ *is given by*

$$\begin{aligned}
X_z &\mapsto W,\\
X_{x_iy_j} &\mapsto x_iy_j\partial_z - (x_iA_j + y_jB_i) + (\alpha_j\partial_{\alpha_i} - \beta_i\partial_{\beta_j}),\\
X_{x_ix_j} &\mapsto x_ix_j\partial_z - (x_iB_j + x_jB_i) + (\beta_i\partial_{\alpha_j} + \beta_j\partial_{\alpha_i}),\\
X_{y_iy_j} &\mapsto y_iy_j\partial_z - (y_iA_j + y_jA_i) - (\alpha_i\partial_{\beta_j} + \alpha_j\partial_{\beta_i}).
\end{aligned}$$

(iv) *The action of the rest of* $\mathfrak{s}_m$ *under* $L_\delta^{\Sigma}$ *is given by*

$$\begin{aligned}
L_\delta^{\Sigma}(X_{x_iz}) &= x_iW - zB_i - \tfrac{1}{2}(x_r\alpha_r - y_r\beta_r)\partial_{\alpha_i} + \tfrac{1}{2}\beta_i(x_s\partial_{\beta_s} + y_s\partial_{\alpha_s}),\\
L_\delta^{\Sigma}(X_{y_iz}) &= y_iW - zA_i + \tfrac{1}{2}(x_r\alpha_r - y_r\beta_r)\partial_{\beta_i} + \tfrac{1}{2}\alpha_i(x_s\partial_{\beta_s} + y_s\partial_{\alpha_s}),\\
L_\delta^{\Sigma}(X_{z^2}) &= 2zW - z^2\partial_z - \tfrac{1}{2}(x_r\alpha_r - y_r\beta_r)(x_s\partial_{\beta_s} + y_s\partial_{\alpha_s}).
\end{aligned}$$

(v) *The action of the rest of* $\mathfrak{s}_m$ *under* $L^{\mathcal{S}}_\delta$ *is given by*

$$\begin{aligned} L^{\mathcal{S}}_\delta(X_{x_iz}) &= L^{\Sigma}_\delta(X_{x_iz}) + \beta_i\partial_\zeta, \\ L^{\mathcal{S}}_\delta(X_{y_iz}) &= L^{\Sigma}_\delta(X_{y_iz}) + \alpha_i\partial_\zeta, \\ L^{\mathcal{S}}_\delta(X_{z^2}) &= L^{\Sigma}_\delta(X_{z^2}) - (x_r\alpha_r - y_r\beta_r)\partial_\zeta. \end{aligned}$$

Note that the operators $x_r\alpha_r - y_r\beta_r$ and $x_s\partial_{\beta_s} + y_s\partial_{\alpha_s}$ occurring in (iv) are $\mathfrak{l}^{\mathrm{s}}_m$-invariants of total weights 0 and 1, respectively.

We now define a space which will regularly play an important role in our arguments:

$$(\Sigma^{k,d}_\delta)^{\mathfrak{u}_m} := \big\{P \in \Sigma^{k,d}_\delta : L^{\Sigma}_\delta(\mathfrak{u}_m)P = 0\big\},$$

the subspace of $\Sigma^{k,d}_\delta$ invariant under the Heisenberg algebra $\mathfrak{u}_m$. Since $\mathfrak{t}_m$ normalizes $\mathfrak{u}_m$, $(\Sigma^{k,d}_\delta)^{\mathfrak{u}_m}$ is a $\mathfrak{t}_m$-module on which $\mathfrak{u}_m$ acts trivially.

**Lemma 4.4.** (i) $(\Sigma^{k,d}_\delta)^{\mathfrak{u}_m} = \mathrm{Span}_{\mathbb{C}}\{\zeta^{d-k}\alpha^I\beta^J : |I| + |J| = 2k - d\}$.

(ii) *Under* $W$, $(\Sigma^{k,d}_\delta)^{\mathfrak{u}_m}$ *has total weight* $\delta(\ell+1) - \frac{1}{2}d$.

(iii) *Under* $\mathfrak{l}^{\mathrm{s}}_m \cong \mathfrak{sp}_{2\ell}$, $(\Sigma^{k,d}_\delta)^{\mathfrak{u}_m}$ *is equivalent to* $\mathrm{Sym}^{2k-d}\,\mathbb{C}^{2\ell}$.

*Proof.* Part (i) is clear from the action of $\mathfrak{u}_m$ given in Lemma 4.3. Part (ii) is straightforward, and Part (iii) follows from the fact that as a module of $\mathfrak{l}^{\mathrm{s}}_m$, $\mathrm{Span}_{\mathbb{C}}\{\alpha_i, \beta_i\}_i$ is equivalent to the irreducible module $\mathbb{C}^{2\ell}$ of $\mathfrak{sp}_{2\ell}$. □

Thus the $\mathfrak{u}_m$-invariant fine symbols are precisely the constant fine symbols. In addition to the spaces of constant symbols, we will encounter the spaces of *polynomial fine symbols,* which are modules of the Lie algebra of the *polynomial contact vector fields.* Under the action of $\mathfrak{s}_m$, these modules turn out to be the restricted duals of $\mathfrak{l}_m$-relative Verma modules.

To make this precise, let us write $\mathrm{Poly}(\mathbb{R}^m)$ for the polynomials in $C^\infty(\mathbb{R}^m)$. We will denote the polynomial subspaces of $\mathcal{K}(\mathbb{R}^m)$, $\mathcal{F}_\lambda(\mathbb{R}^m)$, $\mathcal{D}_{\lambda,\mu}(\mathbb{R}^m)$, $\mathcal{S}_\delta(\mathbb{R}^m)$, and $\Sigma_\delta(\mathbb{R}^m)$ by writing $\mathbb{R}^m_{\mathrm{poly}}$ in place of $\mathbb{R}^m$. Thus for example $\mathcal{K}(\mathbb{R}^m_{\mathrm{poly}})$ is the classical Cartan algebra $\mathcal{K}_m$ discussed in the introduction, and

$$\Sigma^{k,d}_\delta(\mathbb{R}^m_{\mathrm{poly}}) := \mathrm{Span}_{\mathrm{Poly}(\mathbb{R}^m)}\big\{\zeta^{d-k}\alpha^I\beta^J : |I| + |J| = 2k - d\big\}.$$

Note that $\mathcal{K}_m$ is a dense subalgebra of $\mathcal{K}(\mathbb{R}^m)$ containing $\mathfrak{s}_m$, and all of the above polynomial subspaces are dense $\mathcal{K}_m$-submodules of their smooth counterparts.

We will need the fact that smooth globally defined eigenfunctions of Euler operators are polynomials.

**Lemma 4.5.** (i) *Let* $f \in C^\infty(\mathbb{R}^m)$ *be an eigenfunction of the full Euler operator* $\mathcal{E}_u = \sum_{i=1}^m u_i\partial_i$ *with eigenvalue* $\lambda$. *Then* $\lambda \in \mathbb{N}$ *and* $f$ *is a homogeneous polynomial in* $u$ *of degree* $\lambda$.

(ii) *Let* $f \in C^\infty(\mathbb{R}^m)$ *be an eigenfunction of the contact Euler operator* $\mathcal{E}_z + \frac{1}{2}\mathcal{E}_{xy}$ *with eigenvalue* $\lambda$. *Then* $\lambda \in \frac{1}{2}\mathbb{N}$ *and* $f$ *is a homogeneous polynomial in* $(x, y, z)$ *of degree* $\lambda$, *in the sense that* $x_i$ *and* $y_i$ *have degree* $\frac{1}{2}$ *and* $z$ *has degree* 1.

*Proof.* The first statement is classical, and the second follows from the first by the change of coordinates $(x, y, z) \mapsto (x, y, z^2)$. □

Recall from Lemma 4.2 the Cartan subalgebra $\mathfrak{h}_m$ of $\mathfrak{s}_m$. Given any $\mathfrak{h}_m$-module $V$ and any $\nu \in \mathfrak{h}^*_m$, we use the standard notation $V_\nu$ for the $\nu$-weight space of $V$. By the *restricted dual* of $V$, we mean the direct sum of the duals of its weight spaces. A *lowest weight vector* in an $\mathfrak{s}_m$-module is a weight vector annihilated by the negative root vectors; see Lemma 4.2(iv). Recall also that total weights are eigenvalues of the operator $W$.

**Lemma 4.6.** (i) $\Sigma_\delta^{k,d}(\mathbb{R}^m_{\rm poly})$ *is the span of those elements of* $\Sigma_\delta^{k,d}$ *with well-defined total weight.*

(ii) $(\Sigma_\delta^{k,d})^{\mathfrak{u}_m}$ *is the lowest total weight space of* $\Sigma_\delta^{k,d}(\mathbb{R}^m_{\rm poly})$.

(iii) $\Sigma_\delta^{k,d}(\mathbb{R}^m_{\rm poly})$ *has lowest* $\mathfrak{h}_m$*-weight*

$$\nu_\delta^{k,d} = \big[2\delta(\ell+1)-d\big]e_0 - \big[2k-d\big]e_1.$$

*Its lowest* $\mathfrak{h}_m$*-weight space is the line spanned by* $\zeta^{d-k}\beta_1^{2k-d}$.

*Proof.* Lemma 4.5 implies (i), and (ii) is due to the fact that polynomials have non-negative total weights. For (iii), recall from Lemma 4.2 the basis $\{e_0,\dots,e_\ell\}$ of the weight space $\mathfrak{h}_m^*$ and its order $0<e_\ell<\dots<e_0$. The lowest $\mathfrak{h}_m$-weight space is contained in the lowest total weight space $(\Sigma_\delta^{k,d})^{\mathfrak{u}_m}$. Since the $\mathfrak{h}_m$-weights of $\zeta$, $\alpha_i$, and $\beta_i$ are, respectively, $-2e_0$, $-e_0+e_i$, and $-e_0-e_i$, (iii) follows from Lemma 4.4. □

We now recall the basic notions of Verma modules. Let us denote by $\mathfrak{u}_m^+$ and $\mathfrak{t}_m^+$ the subalgebras of $\mathfrak{s}_m$ opposite to $\mathfrak{u}_m$ and $\mathfrak{t}_m$, respectively:

$$\mathfrak{u}_m^+ = \mathrm{Span}_{\mathbb{C}}\big\{X_{x_iz},\, X_{y_iz},\, X_{z^2}\,:\,1\le i\le\ell\big\},\qquad \mathfrak{t}_m^+ = \mathfrak{l}_m\oplus_{\rm s}\mathfrak{u}_m^+.$$

Given an irreducible module $V$ of $\mathfrak{l}_m$, the associated relative Verma module of $\mathfrak{s}_m$ is $\mathfrak{U}(\mathfrak{s}_m)\otimes_{\mathfrak{t}_m^+}V$, where $\mathfrak{u}_m^+$ acts trivially on $V$.

**Lemma 4.7.** *As an* $\mathfrak{s}_m$*-module,* $\Sigma_\delta^{k,d}(\mathbb{R}^m_{\rm poly})$ *is equivalent to the restricted dual of the relative Verma module* $\mathfrak{U}(\mathfrak{s}_m)\otimes_{\mathfrak{t}_m^+}\mathrm{Sym}^{2k-d}\,\mathbb{C}^{2\ell}$.

*Proof.* By Lemmas 4.3 and 4.4, any non-zero $\mathfrak{t}_m$-invariant subspace of $\Sigma_\delta^{k,d}(\mathbb{R}^m_{\rm poly})$ contains $(\Sigma_\delta^{k,d})^{\mathfrak{u}_m}$. Taking duals, we see that the dual of $(\Sigma_\delta^{k,d})^{\mathfrak{u}_m}$ generates the restricted dual $\Sigma_\delta^{k,d}(\mathbb{R}^m_{\rm poly})^*$ under the action of $\mathfrak{t}_m$. It follows that $\Sigma_\delta^{k,d}(\mathbb{R}^m_{\rm poly})^*$ is a quotient of $\mathfrak{U}(\mathfrak{s}_m)\otimes_{\mathfrak{t}_m^+}\big((\Sigma_\delta^{k,d})^{\mathfrak{u}_m}\big)^*$. The reader may easily check that the two have the same total weight space dimensions, so they are equivalent. The result now follows from Lemma 4.4(iii) and the fact that $\mathbb{C}^{2\ell}$ is self-dual. □

4.4. **The $\mathfrak{s}_m$-structure of $\mathcal{K}_m$.** As a particular case of the $\mathfrak{s}_m$-module structures of spaces of fine symbols, we investigate the $\mathfrak{s}_m$-structure of the algebra $\mathcal{K}_m$ itself, which is by definition $\Sigma_0^{1,2}$.

**Proposition 4.8.** *The quotient* $\mathcal{K}_m/\mathfrak{s}_m$ *is irreducible under* $\mathfrak{s}_m$*. Its lowest weight vector is* $X_{x_1^3}$*, which has weight* $e_0-3e_1$.

*Proof.* The reader may use Lemma 4.3 to check that the lowest weight vectors of $\mathcal{K}_m/\mathfrak{s}_m$ under $\mathfrak{l}_m$ are precisely all the elements of the form $X_{x_1^iz^c}$ with $i+c\ge 3$. The same lemma shows that it is possible to move from any one of these lowest weight vectors to any other using the elements $X_1$, $X_{x_1}$, $X_{x_1z}$, and $X_{z^2}$ of $\mathfrak{s}_m$. The weight of $X_{x_1^3}$ is given by Lemma 4.2(ii). □

It will be important to understand the space of $\mathfrak{u}_m$-invariants in $\mathcal{K}_m/\mathfrak{s}_m$. The following result is immediate from Proposition 4.8.

**Corollary 4.9.** (i) $(\mathcal{K}_m/\mathfrak{s}_m)^{\mathfrak{u}_m}=\mathrm{Span}\{X_{x^Iy^J}:|I|+|J|=3\}$.

(ii) $(\mathcal{K}_m/\mathfrak{s}_m)^{\mathfrak{u}_m}$ *has total weight* $\frac12$.

(iii) $(\mathcal{K}_m/\mathfrak{s}_m)^{\mathfrak{u}_m}$ *is equivalent to* $\mathrm{Sym}^3\,\mathbb{C}^{2\ell}$ *under the action of* $\mathfrak{l}_m^{\rm s}$.

4.5. **Infinitesimal characters.** We now turn to the infinitesimal characters of the fine symbol modules. A module of a complex semisimple Lie algebra $\mathfrak{g}$ is said to have an *infinitesimal character* if the center $\mathfrak{Z}(\mathfrak{g})$ of the universal enveloping algebra $\mathfrak{U}(\mathfrak{g})$ acts on it by scalars. In this case, the infinitesimal character is the resulting homomorphism from $\mathfrak{Z}(\mathfrak{g})$ to $\mathbb{C}$. By Schur's lemma, all irreducible modules have infinitesimal characters.

Suppose that a Cartan subalgebra and a positive root system of $\mathfrak{g}$ are fixed. Let $\rho$ be the half-sum of the positive roots. If $V$ and $V'$ are two lowest weight modules of $\mathfrak{g}$ with lowest weights $\nu$ and $\nu'$, respectively, and both modules have infinitesimal characters, then it is a consequence of the Harish-Chandra homomorphism that their infinitesimal characters are the same if and only if $\nu-\rho$ and $\nu'-\rho$ lie in the same orbit of the Weyl group of $\mathfrak{g}$.

**Proposition 4.10.** *Under the action of $\mathfrak{s}_m$, the fine symbol modules have infinitesimal characters. Fix two fine symbol modules $\Sigma_\delta^{k,d}$ and $\Sigma_\delta^{k',d'}$ such that either $k'<k$, or $k'=k$ and $d'<d$. They have the same infinitesimal characters if and only if at least one of the following four conditions holds:*

$$\begin{array}{llllll}
\text{(i)} & k' & = & k, & d' & = & 2\delta(\ell+1)-1+2k-d. \\
\text{(ii)} & k' & = & 2(\delta-1)(\ell+1)+1-k, & d' & = & 2(\delta-1)(\ell+1)+1-2k+d. \\
\text{(iii)} & k' & = & (2\delta-1)(\ell+1)+k-d, & d' & = & 2(2\delta-1)(\ell+1)-d. \\
\text{(iv)} & k' & = & d-k-\ell, & d' & = & 2(\delta-1)(\ell+1)+1-2k+d.
\end{array}$$

*Cases (i), (ii), (iii), and (iv) cannot occur unless $2\delta(\ell+1)$ is in*

$$2+\mathbb{N},\quad 2(\ell+1)+\mathbb{N},\quad \ell+2+\mathbb{N},\quad 2\ell+1+\mathbb{N},$$

*respectively. Moreover, if $\ell=0$, then Case (iv) cannot occur unless $2\delta\in 2+\mathbb{N}$. Therefore if $\delta$ is not contact-resonant (see Section 2.1), then all of the fine symbol modules of $\mathcal{D}_{\lambda,\mu}$ have distinct infinitesimal characters.*

*Proof.* It is well-known that Verma modules have infinitesimal characters, and so their restricted duals do also. Therefore by Lemma 4.7, $\mathfrak{Z}(\mathfrak{s}_m)$ acts by scalars on $\Sigma_\delta^{k,d}(\mathbb{R}^m_{\text{poly}})$. By a density argument, it acts by the same scalars on $\Sigma_\delta^{k,d}(\mathbb{R}^m)$. Thus the fine symbol modules have infinitesimal characters.

Recall from Lemma 4.6 the lowest weight $\nu_\delta^{k,d}$ of $\Sigma_\delta^{k,d}(\mathbb{R}^m_{\text{poly}})$, and from Lemma 4.2 the half-sum $\rho(\mathfrak{s}_m)$ of the positive roots:

$$\rho(\mathfrak{s}_m)=\sum_{i=0}^{\ell}(\ell+1-i)e_i.$$

Recall also that the Weyl group $W(\mathfrak{s}_m)=S_{\ell+1}\ltimes\mathbb{Z}_2^{\ell+1}$ acts by permutations and sign changes on the $e_i$.

As stated above, the Harish-Chandra homomorphism shows that the infinitesimal characters of $\Sigma_\delta^{k,d}$ and $\Sigma_\delta^{k',d'}$ are the same if and only if there is an element $w$ of $W(\mathfrak{s}_m)$ such that

$$w\big(\nu_\delta^{k,d}-\rho(\mathfrak{s}_m)\big)=\nu_\delta^{k',d'}-\rho(\mathfrak{s}_m).$$

By Lemma 4.6, such a $w$ exists if and only if the two sets

$$\big\{|(2\delta-1)(\ell+1)-d|,\ \ell+2k-d,\ \ell-1,\ \ell-2,\ \dots,2,\ 1\big\},$$

$$\big\{|(2\delta-1)(\ell+1)-d'|,\ \ell+2k'-d',\ \ell-1,\ \ell-2,\ \dots,2,\ 1\big\}$$

are equal. Since $\ell+2k-d>\ell-1>\cdots>1>0$, this can occur only in the following ways.

First, $|(2\delta-1)(\ell+1)-d|$ and $|(2\delta-1)(\ell+1)-d')|$ can be equal. In this case $\ell+2k-d$ and $\ell+2k'-d'$ must also be equal. Since we are assuming that $(k,d)$ and $(k',d')$ are distinct, this leads to Case (iii).

Second, we could have

$$|(2\delta-1)(\ell+1)-d| = \ell+2k'-d', \qquad |(2\delta-1)(\ell+1)-d'| = \ell+2k-d.$$

This can occur in three ways, depending on the signs of the arguments of the absolute values. If both are negative we arrive at Case (i), if both are positive we are in Case (ii), and if they are different we obtain Case (iv).

Finally, if $|(2\delta-1)(\ell+1)-d|$ is equal to one of $\ell-1,\dots,1$, say $i$, then again $\ell+2k-d$ and $\ell+2k'-d'$ must be equal and $|(2\delta-1)(\ell+1)-d|$ must also be $i$. Therefore here we are still in Case (iii), albeit with a different $w$. □

## 5. Projective quantization

Recall from Section 3.4 the total symbol modules $\mathcal{S}_\delta$ and the total fine symbol modules $\Sigma_\delta$. In this section we study quantizations of $\Sigma_\delta$ invariant under the projective subalgebra $\mathfrak{s}_m$ of $\mathcal{K}(\mathbb{R}^m)$. We begin in Section 5.1 with a review of quantizations of $\mathcal{S}_\delta$ invariant under the projective subalgebra $\mathfrak{a}_m$ of $\mathrm{Vect}(\mathbb{R}^m)$, as these quantizations are a component of the fine projective quantizations.

5.1. **Projective quantization of symbols.** A *quantization of* $\mathcal{S}_\delta$ is defined to be a linear bijection $Q$ from $\mathcal{S}_\delta$ to $\mathcal{D}_{\lambda,\mu}$ which *preserves degree* and is the *identity on symbols.* By this we mean that it carries $\mathcal{S}_\delta^k$ into $\mathcal{D}_{\lambda,\mu}^k$, and its restriction to $\mathcal{S}_\delta^k$ is a right-inverse of the principal symbol map $\sigma_{\lambda,\mu}^k$:

$$\sigma_{\lambda,\mu}^k \circ Q|_{\mathcal{S}_\delta^k} : \mathcal{S}_\delta^k \to \mathcal{S}_\delta^k \ \text{ is the identity map.}$$

Suppose that $\mathfrak{g}$ is any Lie subalgebra of $\mathrm{Vect}(\mathbb{R}^m)$. A quantization is said to be a $\mathfrak{g}$-*equivariant quantization* (or simply a $\mathfrak{g}$-*quantization*) if it intertwines the two $\mathfrak{g}$-actions $L_\delta^{\mathcal{S}}|_{\mathfrak{g}}$ and $L_{\lambda,\mu}|_{\mathfrak{g}}$.

The more vector fields a quantization is invariant with respect to, the more useful it is. Recall from Section 4.1 the projective subalgebra $\mathfrak{a}_m$ and the affine subalgebra $\mathfrak{b}_m$ of $\mathrm{Vect}(\mathbb{R}^m)$. At one extreme, one might ask for a $\mathrm{Vect}(\mathbb{R}^m)$-quantization of $\mathcal{S}_\delta$. However, there is no such map: $\mathcal{S}_\delta$ and $\mathcal{D}_{\lambda,\mu}$ are not $\mathrm{Vect}(\mathbb{R}^m)$-equivalent for any $(\lambda,\mu)$. At the opposite extreme, $\mathfrak{b}_m$-quantizations are easy to find, not unique, and not very useful. The critical intermediate case is afforded by $\mathfrak{a}_m$, because it is a simple finite dimensional maximal subalgebra of $\mathrm{Vect}(\mathbb{R}^m)$.

**Definition.** We say that $\delta$ is *projectively resonant* if it lies in the set

$$\left\{1+\frac{n}{m+1} \;\middle|\; n\in\mathbb{N}\right\}.$$

The following theorem was proven in [CMZ97] for $m=1$, in [LO99] for arbitrary $m$ at $p=0$, and in general in [Le00].

**Theorem 5.1.** *For $\delta$ not projectively resonant, there exists a unique $\mathfrak{a}_m$-quantization*

$$Q_{\lambda,\mu}^{\mathfrak{a}_m} : \mathcal{S}_\delta \to \mathcal{D}_{\lambda,\mu}.$$

Theorem 5.1 may be proven using only the eigenvalues of the Casimir operator; the full infinitesimal characters are not needed. This is the approach taken in [Le00]. The explicit formula for the projective quantization $Q_{\lambda,\mu}^{\mathfrak{a}_m}$ was given in [CMZ97] for $m=1$, in [LO99] for arbitrary $m$ at $p=0$, and in general in [DO01]; see Section 7.2 below.

5.2. **Projective quantization of fine symbols.** By analogy with quantizations of $\mathcal{S}_\delta$, we define a *quantization of* $\Sigma_\delta$, sometimes called a *fine quantization,* to be a linear bijection $Q$ from $\Sigma_\delta$ to $\mathcal{D}_{\lambda,\mu}$, carrying $\Sigma_\delta^{k,d}$ into $\mathcal{D}_{\lambda,\mu}^{k,d}$, which is the identity on fine symbols in the sense that its restriction to $\Sigma_\delta^{k,d}$ is a right-inverse for the fine symbol map $\mathrm{f}\sigma_{\lambda,\mu}^{k,d}$:

$$\mathrm{f}\sigma_{\lambda,\mu}^{k,d} \circ Q|_{\Sigma_\delta^{k,d}} : \Sigma_\delta^{k,d} \to \Sigma_\delta^{k,d} \text{ is the identity map.}$$

For any Lie subalgebra $\mathfrak{g}$ of $\mathcal{K}(\mathbb{R}^m)$, we say that a fine quantization is a *fine* $\mathfrak{g}$*-equivariant quantization* if it intertwines $L_\delta^\Sigma|_\mathfrak{g}$ and $L_{\lambda,\mu}|_\mathfrak{g}$.

The picture for quantizations of $\Sigma_\delta$ is similar to that for quantizations of $\mathcal{S}_\delta$: there is no fine $\mathcal{K}(\mathbb{R}^m)$-quantization, there are many fine $\mathfrak{t}_m$-quantizations, and for most $\delta$ there is a unique fine $\mathfrak{s}_m$-quantization. The following theorem makes this precise. It may be proven by combining the results of [DO01] and [FMP08]. We understand it as a corollary of Proposition 4.10. As remarked before that proposition, it cannot be proven using the Casimir operator of $\mathfrak{s}_m$ alone; the full infinitesimal characters of the fine symbol modules are required.

**Theorem 5.2.** *For $\delta$ not contact-resonant (see Section 2.1), there exists a unique fine $\mathfrak{s}_m$-quantization*

$$Q_{\lambda,\mu}^{\mathfrak{s}_m} : \Sigma_\delta \to \mathcal{D}_{\lambda,\mu}.$$

*Proof.* Write $\chi_\delta^{k,d}$ for the $\mathfrak{s}_m$-infinitesimal character of $\Sigma_\delta^{k,d}$, and $(\mathcal{D}_{\lambda,\mu})_{\chi_\delta^{k,d}}$ for the subspace of $\mathcal{D}_{\lambda,\mu}$ on which the center $\mathfrak{Z}(\mathfrak{s}_m)$ of $\mathfrak{U}(\mathfrak{s}_m)$ acts by $\chi_\delta^{k,d}$. The $\chi_\delta^{k,d}$ are distinct by Proposition 4.10, so we have the $\mathfrak{s}_m$-decomposition

$$\mathcal{D}_{\lambda,\mu} = \bigoplus_{k,d} (\mathcal{D}_{\lambda,\mu})_{\chi_\delta^{k,d}}.$$

The fine symbol map $\mathrm{f}\sigma_{\lambda,\mu}^{k,d}$ restricts to the unique fine symbol-preserving $\mathfrak{s}_m$-equivalence from $(\mathcal{D}_{\lambda,\mu})_{\chi_\delta^{k,d}}$ to $\Sigma_\delta^{k,d}$. The fine $\mathfrak{s}_m$-quantization $Q_{\lambda,\mu}^{\mathfrak{s}_m}$ is the direct sum of the inverses of these restrictions. □

The explicit formula for $Q_{\lambda,\mu}^{\mathfrak{s}_m}$ will be given in Section 7.3.

## 6. Lowest weight calculations

In this section we prove Theorem A and Theorem C. The proofs rely on lowest weight calculations in modules of homomorphisms between the fine symbol spaces.

6.1. **The structure of** $\mathrm{Hom}_{\mathfrak{u}_m}(\Sigma_\delta^{k,d}, \Sigma_\delta^{k',d'})$**.** We begin with a description of the total weight spaces of $\mathrm{Hom}_{\mathfrak{u}_m}(\Sigma_\delta^{k,d}, \Sigma_\delta^{k',d'})$ which will be needed in both proofs. Recall that for $\nu \in \mathfrak{h}_m^*$, $V_\nu$ denotes the $\nu$-weight space of any $\mathfrak{h}_m$-module $V$. We will also use the following notation: for $w \in \mathbb{C}$, $V_{(w)}$ denotes the $w$-total weight space of $V$. Note that if $\nu = \sum_0^\ell \nu_i e_i$, then $V_\nu \subseteq V_{(\frac{\nu_0}{2})}$.

We will abbreviate $\mathbb{C}[\partial_z, A_1, B_1 \ldots, A_\ell, B_\ell]$ by $\mathbb{C}[\partial_z, A, B]$.

**Lemma 6.1.** (i) *The total weight space* $\mathrm{End}_{\mathfrak{u}_m}\big(C^\infty(\mathbb{R}^m)\big)_{(w)}$ *is zero unless* $w \in -\frac{1}{2}\mathbb{N}$*, when it is* $\mathbb{C}[\partial_z, A, B]_{(w)}$.

(ii) *The total weight space* $\mathrm{Hom}_{\mathfrak{u}_m}(\Sigma_\delta^{k,d}, \Sigma_\delta^{k',d'})_{(w)}$ *is zero unless* $w \in \frac{1}{2}(d-d') - \frac{1}{2}\mathbb{N}$*, when it is*

$$\begin{aligned} \mathrm{Span}_\mathbb{C}\big\{\zeta^{d'-k'}\alpha^{I'}\beta^{J'}\partial_\zeta^{d-k}\,\partial_\alpha^I\,\partial_\beta^J \quad &: \quad |I|+|J| = 2k-d,\ |I'|+|J'| = 2k'-d'\big\} \\ &\otimes \quad \mathbb{C}[\partial_z, A, B]_{\left(w-\frac{1}{2}(d-d')\right)}. \end{aligned}$$

*Proof.* Let $T : C^\infty(\mathbb{R}^m) \to C^\infty(\mathbb{R}^m)$ be of total weight $w$. Use Lemma 4.5 to see that $T$ maps polynomials to polynomials, and verify that $\mathrm{End}\big(\mathrm{Poly}(\mathbb{R}^m)\big)$ is $\mathbb{C}[x,y,z]\,[[\partial_z, A, B]]$. Hence

$$\mathrm{End}\big(\mathrm{Poly}(\mathbb{R}^m)\big)_{(w)} = \mathbb{C}[x,y,z]\,[\partial_z, A, B]_{(w)}.$$

Since $\partial_z$, $A$, and $B$ commute with $\mathfrak{u}_m$, the $\mathfrak{u}_m$-equivariant maps in this space are precisely those independent of $x$, $y$, and $z$. This proves (i).

Under $\mathfrak{u}_m$, $\Sigma_\delta^{k,d}$ is equivalent to $(\Sigma_\delta^{k,d})^{\mathfrak{u}_m} \otimes C^\infty(\mathbb{R}^m)$, so $\mathrm{Hom}_{\mathfrak{u}_m}\big(\Sigma_\delta^{k,d}, \Sigma_\delta^{k',d'}\big)$ is

$$\mathrm{Hom}_{\mathbb{C}}\big((\Sigma_\delta^{k,d})^{\mathfrak{u}_m}, (\Sigma_\delta^{k',d'})^{\mathfrak{u}_m}\big) \otimes \mathrm{End}_{\mathfrak{u}_m}\big(C^\infty(\mathbb{R}^m)\big).$$

Thus (ii) follows from (i) and the fact that $\mathrm{Hom}_{\mathbb{C}}\big((\Sigma_\delta^{k,d})^{\mathfrak{u}_m}, (\Sigma_\delta^{k',d'})^{\mathfrak{u}_m}\big)$ is

$$\mathrm{Span}_{\mathbb{C}}\big\{\zeta^{d'-k'}\alpha^{I'}\beta^{J'}\partial_\zeta^{d-k}\,\partial_\alpha^I\,\partial_\beta^J \;:\; |I|+|J| = 2k-d,\ |I'|+|J'| = 2k'-d'\big\}. \qquad \square$$

We will also need the structure of $\mathrm{Hom}_{\mathfrak{u}_m}\big(\Sigma_\delta^{k,d}, \Sigma_\delta^{k',d'}\big)_{(w)}$ as a module of the subalgebra $\mathfrak{l}_m^{\mathrm{s}}$, which recall is isomorphic to $\mathfrak{sp}_{2\ell}$.

**Lemma 6.2.** *Suppose that $w$ is in $\frac{1}{2}(d-d') - \frac{1}{2}\mathbb{N}$. Then under the action of $\mathfrak{l}_m^{\mathrm{s}}$,*

$$\mathrm{Hom}_{\mathfrak{u}_m}\big(\Sigma_\delta^{k,d}, \Sigma_\delta^{k',d'}\big)_{(w)} \;\cong\; \mathrm{Sym}^{2k-d}\,\mathbb{C}^{2\ell} \otimes \mathrm{Sym}^{2k'-d'}\,\mathbb{C}^{2\ell} \otimes \Big(\bigoplus_{r\geq 0} \mathrm{Sym}^{d-d'-2r-2w}\,\mathbb{C}^{2\ell}\Big),$$

*where the direct sum is only over non-negative exponents.*

*Proof.* Consider the explicit basis of $\mathrm{Hom}_{\mathfrak{u}_m}\big(\Sigma_\delta^{k,d}, \Sigma_\delta^{k',d'}\big)_{(w)}$ given in Lemma 6.1(ii). We have the following $\mathfrak{l}_m^{\mathrm{s}}$-equivalences:

$$\begin{aligned}
\mathrm{Span}_{\mathbb{C}}\big\{\alpha^{I'}\beta^{J'} \;:\; |I'|+|J'| = 2k'-d'\big\} \quad &\cong \quad \mathrm{Sym}^{2k'-d'}\,\mathbb{C}^{2\ell},\\
\mathrm{Span}_{\mathbb{C}}\big\{\partial_\alpha^I\,\partial_\beta^J \;:\; |I|+|J| = 2k-d\big\} \quad &\cong \quad \mathrm{Sym}^{2k-d}\,\mathbb{C}^{2\ell},
\end{aligned}$$

and for $v \in -\frac{1}{2}\mathbb{N}$, $\mathbb{C}[\partial_z, A, B]_{(v)} \cong \bigoplus_{r\geq 0} \mathrm{Sym}^{-2v-2r}\,\mathbb{C}^{2\ell}$. Since $\zeta$, $\partial_\zeta$, and $\partial_z$ are $\mathfrak{l}_m^{\mathrm{s}}$-invariant, the result follows. $\square$

6.2. **Proof of Theorem A.** Assume temporarily that $M$ is $\mathbb{R}^m$, equipped with the standard contact structure. Since $\delta$ is not contact-resonant, Theorem 5.2 shows that there is a unique $\mathfrak{s}_m$-equivariant map from $\mathcal{D}_{\lambda,\mu}^k$ to $\Sigma_\delta^{k-1,\,2(k-1)}$, namely

$$\mathrm{s}\sigma_{\lambda,\mu}^k := \pi_{k-1,\,2(k-1)} \circ (Q_{\lambda,\mu}^{\mathfrak{s}_m})^{-1}, \tag{7}$$

where $\pi_{j,d}$ denotes the canonical projection from $\Sigma_\delta$ to $\Sigma_\delta^{j,d}$. Our task is to prove that this map is in fact $\mathcal{K}(\mathbb{R}^m)$-equivariant.

We use the projective quantization to pull the action $L_{\lambda,\mu}$ of $\mathcal{K}(\mathbb{R}^m)$ on $\mathcal{D}_{\lambda,\mu}$ back to an action $\mathcal{L}_{\lambda,\mu}$ of $\mathcal{K}(\mathbb{R}^m)$ on $\Sigma_\delta$:

$$\mathcal{L}_{\lambda,\mu}(X_f) \;:=\; (Q_{\lambda,\mu}^{\mathfrak{s}_m})^{-1} \circ L_{\lambda,\mu}(X_f) \circ Q_{\lambda,\mu}^{\mathfrak{s}_m}.$$

The statement that $\mathrm{s}\sigma_{\lambda,\mu}^k$ is $\mathcal{K}(\mathbb{R}^m)$-equivariant is equivalent to the statement that

$$\pi_{k-1,\,2(k-1)} : \bigoplus_{0\leq j\leq k} \bigoplus_{j\leq d\leq 2j} \Sigma_\delta^{j,d} \to \Sigma_\delta^{k-1,\,2(k-1)}$$

intertwines the $\mathcal{K}(\mathbb{R}^m)$-actions $\mathcal{L}_{\lambda,\mu}$ and $L_\delta^\Sigma$.

We may regard $\mathcal{L}_{\lambda,\mu}$ as a block matrix with entries

$$\mathcal{L}_{\lambda,\mu}^{(j,d),(j',d')}(X_f) : \Sigma_\delta^{j,d} \to \Sigma_\delta^{j',d'}.$$

This matrix is triangular with respect to the dictionary order on $(j,d)$, and the diagonal entries are simply the usual actions on fine symbols: $\mathcal{L}_{\lambda,\mu}^{(j,d),(j,d)}$ is $L_\delta^\Sigma$ restricted to $\Sigma_\delta^{j,d}$. As a result of the $\mathfrak{s}_m$-equivariance of $Q_{\lambda,\mu}^{\mathfrak{s}_m}$, the off-diagonal entries $\mathcal{L}_{\lambda,\mu}^{(j,d),(j',d')}$ are $\mathfrak{s}_m$-relative 1-cochains of $\mathcal{K}(\mathbb{R}^m)$: they vanish on $\mathfrak{s}_m$ and are $\mathfrak{s}_m$-equivariant maps from $\mathcal{K}(\mathbb{R}^m)/\mathfrak{s}_m$ to $\mathrm{Hom}(\Sigma_\delta^{j,d},\Sigma_\delta^{j',d'})$.

One finds that $\pi_{k-1,\,2(k-1)}\circ\mathcal{L}_{\lambda,\mu}$ restricted to $\bigoplus_{0\le j\le k}\bigoplus_{j\le d\le 2j}\Sigma_\delta^{j,d}$ is

$$L_\delta^\Sigma\circ\pi_{k-1,\,2(k-1)}+\sum_{d=k}^{2k}\mathcal{L}_{\lambda,\mu}^{(k,d),(k-1,\,2(k-1))}\circ\pi_{k,d}.$$

Therefore it suffices to prove that entries $\mathcal{L}_{\lambda,\mu}^{(k,d),(k-1,\,2(k-1))}$ are zero for all $d$. The following lemma is the key to the situation.

**Lemma 6.3.** *Let $V$ be a module of $\mathcal{K}(\mathbb{R}^m)$ such that the space $V_{(\frac12)}^{\mathfrak{u}_m}$ of $\mathfrak{u}_m$-invariants in $V$ of total weight $\frac12$ contains no copies of $\mathrm{Sym}^3\mathbb{C}^{2\ell}$ under the action of $\mathfrak{l}_m^{\mathrm{s}}$. Then the space $C^1\big(\mathcal{K}(\mathbb{R}^m),\mathfrak{s}_m;V\big)$ of $\mathfrak{s}_m$-relative 1-cochains of $\mathcal{K}(\mathbb{R}^m)$ with values in $V$ is zero.*

*Proof.* Apply Corollary 4.9. □

In order to apply this lemma, we must prove that $\mathrm{Hom}_{\mathfrak{u}_m}\big(\Sigma_\delta^{k,d},\Sigma_\delta^{k-1,\,2(k-1)}\big)_{(\frac12)}$ contains no copies of $\mathrm{Sym}^3\mathbb{C}^{2\ell}$ under $\mathfrak{l}_m^{\mathrm{s}}$ for $k\le d\le 2k$. When $d=2k$, by Lemma 6.1 we obtain

$$\mathrm{Hom}_{\mathfrak{u}_m}\big(\Sigma_\delta^{k,2k},\Sigma_\delta^{k-1,\,2(k-1)}\big)_{(\frac12)}=\zeta^{k-1}\partial_\zeta^k\ \mathrm{Span}\{A,B\},$$

which is a copy of $\mathbb{C}^{2\ell}$. When $d=2k-1$, we obtain

$$\mathrm{Hom}_{\mathfrak{u}_m}\big(\Sigma_\delta^{k,2k-1},\Sigma_\delta^{k-1,\,2(k-1)}\big)_{(\frac12)}=\zeta^{k-1}\partial_\zeta^{k-1}\ \mathrm{Span}\{\partial_\alpha,\partial_\beta\},$$

which is again a copy of $\mathbb{C}^{2\ell}$. Finally, when $d<2k-1$, we obtain zero.

This completes the proof of Theorem A when $M=\mathbb{R}^m$. If $M$ is an arbitrary contact manifold, Darboux's theorem implies the existence of an atlas of local charts on $M$ that are diffeomorphic to the standard contact structure on $\mathbb{R}^m$. We have just proved that for every chart $U$, there is a unique (locally defined) map $\mathrm{s}\sigma_{\lambda,\mu}^k(U)$ equivariant with respect to $\mathcal{K}(U)$. If $U'$ is another chart, uniqueness implies that the maps $\mathrm{s}\sigma_{\lambda,\mu}^k(U)$ and $\mathrm{s}\sigma_{\lambda,\mu}^k(U')$ coincide on $U\cap U'$. Therefore $\mathrm{s}\sigma_{\lambda,\mu}^k(M)$ is well-defined on all of $M$ and obviously commutes with $\mathcal{K}(M)$.

6.3. **Proof of Theorem C.** As in the proof of Theorem A, take $M=\mathbb{R}^m$ and consider the action $\mathcal{L}_{\lambda,\mu}$ of $\mathcal{K}(\mathbb{R}^m)$ on $\Sigma_\delta$. We first prove that certain of its matrix elements $\mathcal{L}_{\lambda,\mu}^{(k,d),(k',d')}$ vanish.

**Lemma 6.4.** *In both of the following cases, the space of $\mathfrak{s}_m$-relative 1-cochains of $\mathcal{K}(\mathbb{R}^m)$ with coefficients in $\mathrm{Hom}\big(\Sigma_\delta^{k,d},\Sigma_\delta^{k',d'}\big)$ is zero:*

(i) *$k'=k-1$ and $d'\ge d$.*
(ii) *$k'<k-1$ and $d'\ge d-(k-k')+2$.*

*Proof.* By Lemma 6.3, we must prove that under $\mathfrak{l}_m^{\mathrm{s}}$ the space $\mathrm{Hom}_{\mathfrak{u}_m}(\Sigma_\delta^{k,d},\Sigma_\delta^{k',d'})_{(\frac12)}$ contains no copies of $\mathrm{Sym}^3\mathbb{C}^{2\ell}$ in either Case (i) or Case (ii). By Lemma 6.1, this space is zero unless $d-d'\in\mathbb{Z}^+$. In particular, it is zero in Case (i). In Case (ii) with $d-d'\in\mathbb{Z}^+$, Lemma 6.2 shows that under $\mathfrak{l}_m^{\mathrm{s}}$ it is equivalent to

$$\mathrm{Sym}^{2k-d}\mathbb{C}^{2\ell}\otimes\mathrm{Sym}^{2k'-d'}\mathbb{C}^{2\ell}\otimes\Big(\bigoplus_{r\ge0}\mathrm{Sym}^{d-d'-2r-1}\mathbb{C}^{2\ell}\Big).\tag{8}$$

It is well-known (see, *e.g.*, [Va84]) that the largest irreducible component of $\mathrm{Sym}^r\,\mathbb{C}^{2\ell}\otimes\mathrm{Sym}^s\,\mathbb{C}^{2\ell}$ is $\mathrm{Sym}^{r+s}\,\mathbb{C}^{2\ell}$, and for $r\geq s$, its smallest irreducible component is its Parthasarathy - Ranga Rao - Varadarajan submodule, a copy of $\mathrm{Sym}^{r-s}\,\mathbb{C}^{2\ell}$. Therefore the maximal component of

$$\mathrm{Sym}^{2k'-d'}\,\mathbb{C}^{2\ell}\otimes\left(\bigoplus_{r\geq 0}\mathrm{Sym}^{d-d'-2r-1}\,\mathbb{C}^{2\ell}\right)$$

is $\mathrm{Sym}^{d-2d'+2k'-1}\,\mathbb{C}^{2\ell}$. In Case (ii) we have $k'-d'<k-d$, so the smallest irreducible component of (8) is $\mathrm{Sym}^{2(d'-d+k-k')+1}\,\mathbb{C}^{2\ell}$, which is larger than $\mathrm{Sym}^3\,\mathbb{C}^{2\ell}$. □

This lemma implies that $\mathcal{L}^{(k,d),(k',d')}_{\lambda,\mu}=0$ under the conditions of Cases (i) and (ii). Therefore the space $\mathcal{D}^{(b)}_{\lambda,\mu}(\mathbb{R}^m)$ defined by

$$\mathcal{D}^{(b)}_{\lambda,\mu}(\mathbb{R}^m)\;:=\;Q^{\mathfrak{s}_m}_{\lambda,\mu}\left(\bigoplus_{2d-k\leq b}\Sigma^{k,d}_\delta\right)$$

is invariant under the action $L_{\lambda,\mu}$ of $\mathcal{K}(\mathbb{R}^m)$. Since under $\mathfrak{s}_m$ there is unique copy of $\Sigma^{k,d}_\delta$ in $\mathcal{D}_{\lambda,\mu}$ for all $(k,d)$, $\mathcal{D}^{(b)}_{\lambda,\mu}$ is the unique subspace of $\mathcal{D}_{\lambda,\mu}$ whose graded module is as in Theorem C. This completes the proof for $M=\mathbb{R}^m$. For $M$ arbitrary, local existence and uniqueness allows us to conclude global existence and uniqueness of $\mathcal{D}^{(b)}_{\lambda,\mu}(M)$ as in the proof of Theorem A.

## 7. Explicit formulas

7.1. **The affine invariants.** We begin by proving that the following maps between fine symbol modules are equivariant with respect to the affine subalgebra $\mathfrak{t}_m$:

**Definition.** The *contact divergence* is the map

$$\mathrm{Div}_{\mathrm{C}}:=\partial_z\partial_\zeta:\Sigma^{k,d}_\delta\to\Sigma^{k-1,d-2}_\delta.$$

The *tangential divergence* is the map

$$\mathrm{Div}_{\mathrm{T}}:=A_r\partial_{\alpha_r}+B_r\partial_{\beta_r}:\Sigma^{k,d}_\delta\to\Sigma^{k-1,d-1}_\delta.$$

Finally, define

$$\Delta:=(\alpha_rB_r-\beta_rA_r)\partial_\zeta:\Sigma^{k,d}_\delta\to\Sigma^{k,d-1}_\delta.$$

**Lemma 7.1.** (i) $\mathrm{Div}_{\mathrm{C}}$, $\mathrm{Div}_{\mathrm{T}}$, *and* $\Delta$ *are all* $\mathfrak{t}_m$*-equivariant.*

(ii) *The contact divergence* $\mathrm{Div}_{\mathrm{C}}$ *commutes with both* $\mathrm{Div}_{\mathrm{T}}$ *and* $\Delta$, *and*

$$[\mathrm{Div}_{\mathrm{T}},\Delta]=(\ell+\mathcal{E}_{\alpha\beta})\,\mathrm{Div}_{\mathrm{C}}\,.$$

(iii) *Regarded as a map from* $\mathcal{S}^k_\delta$ *to* $\mathcal{S}^{k-1}_\delta$, $\mathrm{Div}_{\mathrm{T}}+\mathrm{Div}_{\mathrm{C}}$ *is the full divergence* $\mathrm{Div}$.

*Proof.* The statement follows from Lemma 4.3 and short computations. □

**Remark.** It follows from the first fundamental theorem of invariant theory for $\mathfrak{sp}_{2\ell}$ that the associative algebra generated by the operators

$$\mathrm{Div}_{\mathrm{C}},\qquad\mathrm{Div}_{\mathrm{T}},\qquad\Delta,\qquad\mathcal{E}_{\alpha\beta},\qquad\mathcal{E}_\zeta$$

coincides with the algebra $\mathrm{End}_{\mathfrak{t}_m}(\Sigma_\delta)$ of all affine invariants: see [FMP07].

7.2. **The $\mathfrak{sl}_{m+1}$-equivariant quantization of $\mathcal{S}_\delta$.** Continuing the discussion of Section 5.1, we now give an explicit formula for the quantization map $Q^{\mathfrak{a}_m}_{\lambda,\mu}$ of Theorem 5.1. This formula is part of the explicit formula for the fine projective quantization in the contact setting.

We begin with the standard symbol calculus on $\mathcal{S}_\delta$. As in Section 4.1, fix any coordinates $u_1,\dots,u_m$ on $\mathbb{R}^m$. Write $\xi_i$ for the symbol of the vector field $\partial_{u_i}$. Then

$$\mathcal{S}_\delta = \mathrm{Span}_{C^\infty(\mathbb{R}^m)}\{\xi^I : I\in\mathbb{N}^m\},\qquad \mathcal{D}_{\lambda,\mu} = \mathrm{Span}_{C^\infty(\mathbb{R}^m)}\{\partial_u^I : I\in\mathbb{N}^m\}.$$

As noted in Section 3.4, these spaces carry the $\mathrm{Vect}(\mathbb{R}^m)$-actions $L^{\mathcal{S}}_\delta$ and $L_{\lambda,\mu}$, respectively.

One of the simplest quantizations is the *normal order quantization:*

$$\mathrm{N}:\mathcal{S}_\delta\to\mathcal{D}_{\lambda,\mu},\qquad \mathrm{N}\big(f_I(u)\xi^I\big) := f_I(u)\partial_u^I.$$

In the literature, $\mathcal{S}_\delta$ and $\mathcal{D}_{\lambda,\mu}$ are frequently identified via N, which then does not appear explicitly in the formulas. Although N is not an $\mathfrak{a}_m$-equivariant quantization, it does turn out to be a $\mathfrak{b}_m$-quantization. The first step in computing $Q^{\mathfrak{a}_m}_{\lambda,\mu}$ is to find explicit formulas for $L^{\mathcal{S}}_\delta$ and $L_{\lambda,\mu}$. In fact, one computes the pull-back of $L_{\lambda,\mu}$ to $\mathcal{S}_\delta$ via N, that is,

$$L^{\mathrm{N}^{-1}}_{\lambda,\mu}(X) \;:=\; \mathrm{N}^{-1}\circ L_{\lambda,\mu}(X)\circ\mathrm{N}.$$

Towards this end, note that any differential operator on $\mathcal{S}_\delta$ may be written as a $C^\infty(\mathbb{R}^m)$-linear combination of monomials $\partial_u^I\xi^J\partial_\xi^K$.

Given any vector field $X=\sum_1^m X_i\partial_{u_i}$, it is straightforward to obtain

$$L^{\mathrm{N}^{-1}}_{\lambda,\mu}(X) = X+\delta\,\mathrm{Div}(X) - \sum_{|I|>0}\tfrac{1}{I!}\Big(\partial_u^I\big[\lambda\,\mathrm{Div}(X)+\textstyle\sum_j X_j\xi_j\big]\Big)\partial_\xi^I.$$

The action $L^{\mathcal{S}}_\delta$ is simply the part of $L^{\mathrm{N}^{-1}}_{\lambda,\mu}$ which preserves $\xi$-degree:

$$L^{\mathcal{S}}_\delta(X) = X+\delta\,\mathrm{Div}(X) - \sum_{i,j=1}^m(\partial_{u_i}X_j)\xi_j\partial_{\xi_i}.$$

Observe that these two formulas are the same if and only if $X\in\mathfrak{b}_m$. Thus as claimed, N is a $\mathfrak{b}_m$-quantization.

As usual, let $\mathcal{E}_\xi$ denote the $\xi$-Euler operator. The full divergence operator is

$$\mathrm{Div} = \sum_{1\le i\le m}\partial_{u_i}\partial_{\xi_i} : \mathcal{S}^k_\delta\to\mathcal{S}^{k-1}_\delta.$$

Bear in mind that $\mathcal{E}_\xi$ and Div do not commute: $[\mathcal{E}_\xi,\mathrm{Div}] = -\,\mathrm{Div}$. The theorem is as follows.

**Theorem 7.2.** [LO99, DO01] *For $\delta$ non-resonant,*

$$Q^{\mathfrak{a}_m}_{\lambda,\mu} = \mathrm{N}\circ\sum_{s=0}^\infty\tfrac{1}{s!}\,\mathrm{Div}^s\circ\binom{\mathcal{E}_\xi+\lambda(m+1)-1}{s}\binom{2\mathcal{E}_\xi-\delta(m+1)+m-1}{s}^{-1}.$$

It is worth mentioning that this formula can be understood as a (non-commutative) hypergeometric function: see [DO01].

7.3. **The $\mathfrak{sp}_{2(\ell+1)}$-equivariant quantization of $\Sigma_\delta$.** We now proceed to derive an explicit formula for $Q^{\mathfrak{s}_m}_{\lambda,\mu}$. We begin by defining a map $\mathcal{S}Q^{\mathfrak{s}_m}_\delta$ from $\Sigma_\delta$ to $\mathcal{S}_\delta$:

$$\mathcal{S}Q^{\mathfrak{s}_m}_\delta := \sum_{s=0}^\infty\frac{2^s}{(s!)^2}\Delta^s\circ\binom{2\mathcal{E}_\zeta-2\delta(\ell+1)}{s}^{-1}. \tag{9}$$

This formula is well-defined provided that $\delta$ is not contact-resonant. Recall from (6) that we are abusing notation and using the same bases for $\Sigma_\delta$ and $\mathcal{S}_\delta$ (this is analogous to regarding the normal

order quantization N as the identity). Therefore we may and do regard $\mathcal{S}Q^{\mathfrak{s}_m}_\delta$ as a map from $\Sigma_\delta$ to $\mathcal{S}_\delta$.

**Theorem 7.3.** *The fine $\mathfrak{s}_m$-equivariant quantization of Theorem 5.2 is*

$$Q^{\mathfrak{s}_m}_{\lambda,\mu} = Q^{\mathfrak{a}_m}_{\lambda,\mu} \circ \mathcal{S}Q^{\mathfrak{s}_m}_\delta.$$

*Proof.* Observe that it suffices to prove that $\mathcal{S}Q^{\mathfrak{s}_m}_\delta$ intertwines the restrictions of the actions $L^\Sigma_\delta$ and $L^{\mathcal{S}}_\delta$ to $\mathfrak{s}_m$. By Lemma 7.1, $\mathcal{S}Q^{\mathfrak{s}_m}_\delta$ intertwines the $\mathfrak{t}_m$-actions. Restricted to $\Sigma^{k,d}_\delta$, it is of the form $\sum_0^\infty \Delta^s C_s$ for some constants $C_s$. One obtains (9) by deriving a recursion relation for these constants. Since $X_{x_i z}$ generates $\mathfrak{s}_m$ under $\mathfrak{t}_m$ for any $i$, we need only impose the condition

$$L^{\mathcal{S}}_\delta(X_{x_i z}) \circ \mathcal{S}Q^{\mathfrak{s}_m}_\delta = \mathcal{S}Q^{\mathfrak{s}_m}_\delta \circ L^\Sigma_\delta(X_{x_i z}).$$

By Lemma 4.3, $L^{\mathcal{S}}_\delta(X_{x_i z}) - L^\Sigma_\delta(X_{x_i z})$ is $\beta_i\partial_\zeta$, which commutes with $\Delta$. Therefore we find that $\mathcal{S}Q^{\mathfrak{s}_m}_\delta \circ \beta_i\partial_\zeta$ and $\big[\mathcal{S}Q^{\mathfrak{s}_m}_\delta, L^\Sigma_\delta(X_{x_i z})\big]$ must be equal, *i.e.,*

$$0 = \textstyle\sum_s \big(\Delta^s \circ \beta_i\partial_\zeta - \big[\Delta^s, L^\Sigma_\delta(X_{x_i z})\big]\big)C_s.$$

Using the same lemma, deduce the following commutator:

$$[L^\Sigma_\delta(X_{x_i z}), \Delta^s] = -s\Delta^{s-1}\beta_i\partial_\zeta\big(\mathcal{E}_\zeta - \delta(\ell+1) - \tfrac{1}{2}(s-1)\big),$$

This gives the recursion relation

$$C_{s-1} = \tfrac{1}{2}s\big(2c - 2\delta(\ell+1) - (s-1)\big)C_s.$$

Since $C_0 = 1$, the theorem follows. □

7.4. **The subsymbol.** In order to give an explicit local formula for $\mathrm{s}\sigma^k_{\lambda,\mu}$, let us fix a system of Darboux coordinates as in Section 3.1. An arbitrary differential operator $T$ of order $\le k$ may be expressed as

$$T = \sum_{c+|I|+|J|\le k} T_{c,I,J}\,\partial_z^c\partial_x^I\partial_y^J,$$

where the $T_{c,I,J}$ are smooth functions. As usual, we replace $\partial_z, \partial_x, \partial_y$ by their symbols $\xi_z, \xi_x, \xi_y$, respectively. This amounts to replacing $T$ by

$$\mathrm{N}^{-1}(T) = \sum_{c+|I|+|J|\le k} T_{c,I,J}\,\xi_z^c\xi_x^I\xi_y^J.$$

The formula has two ingredients: the full divergence Div, and the projection $\pi_{k-1,2(k-1)} : \mathcal{S}^{k-1}_\delta \to \Sigma^{k-1,2(k-1)}_\delta$. The full divergence $\mathrm{Div}(\mathrm{N}^{-1}(T))$ is

$$\sum_{c+|I|+|J|\le k}\Bigg(c\partial_z(T_{c,I,J})\,\xi_z^{c-1}\xi_x^I\xi_y^J + \sum_{s=1}^{\ell}\Big(I_s\partial_{x_s}(T_{c,I,J})\,\xi_z^c\xi_x^{I-e_s}\xi_y^J + J_s\partial_{y_s}(T_{c,I,J})\,\xi_z^c\xi_x^I\xi_y^{J-e_s}\Big)\Bigg).$$

In the $(\alpha,\beta,\zeta)$-coordinates on $\mathcal{S}^{k-1}_\delta$ given in Section 3.4, the projection $\pi_{k-1,2(k-1)}$ simply gives the $\zeta^{k-1}$ term. In Darboux coordinates,

$$\pi_{k-1,2(k-1)}\big(T_{c,I,J}\,\xi_z^{k-1-|I|-|J|}\xi_x^I\xi_y^J\big) = (-1)^{|I|}\big(\tfrac{1}{2}\big)^{|I|+|J|}y^Ix^JT_{c,I,J}\zeta^{k-1},$$

because by (5), $\xi_{x_i} = \alpha_i - \frac{1}{2}y_i\zeta$, $\xi_{y_i} = -\beta_i + \frac{1}{2}x_i\zeta$, and $\xi_z = \zeta$.

**Proposition 7.4.** *In Darboux coordinates, the subsymbol $\mathrm{s}\sigma^k_{\lambda,\mu}(T)$ is given by*

$$\mathrm{s}\sigma^k_{\lambda,\mu}(T) = \pi_{k-1,2(k-1)} \circ \Big(1 - \frac{(k-1)+2\lambda(\ell+1)}{2(k-1)-2(\delta-1)(\ell+1)}\,\mathrm{Div}\Big) \circ \mathrm{N}^{-1}(T).$$

*Proof.* Using (7) and Theorem 7.3, we obtain $\mathrm{s}\sigma^k_{\lambda,\mu}(T) = \pi_{k-1,2(k-1)} \circ \left(\mathcal{S}Q^{\mathfrak{s}_m}_\delta\right)^{-1} \circ \left(Q^{\mathfrak{a}_m}_{\lambda,\mu}\right)^{-1}$. Since $\Sigma^{k-1,2(k-1)}_\delta$ is the fine symbol module in $\mathcal{S}^{k-1}_\delta$ of the highest contact order, (9) gives

$$\pi_{k-1,2(k-1)} \circ \left(\mathcal{S}Q^{\mathfrak{s}_m}_\delta\right)^{-1} = \pi_{k-1,2(k-1)}.$$

To calculate the $\mathcal{S}^{k-1}_\delta$-component of $\left(Q^{\mathfrak{a}_m}_{\lambda,\mu}\right)^{-1}$, we use the formula of Theorem 7.2. □

Note that the formula for $\mathrm{s}\sigma^k_{\lambda,\mu}$ is well-defined for all but one contact-resonant value of $\delta$, the value $\delta = \frac{\ell+k}{\ell+1}$. By continuity, it retains $\mathcal{K}_m$-equivariance whenever it is well defined.

7.5. **Proof of Theorem B.** Let $T$ be a second order operator from $\mathcal{F}_\lambda$ to $\mathcal{F}_\lambda$. Since $\Sigma^{1,2}_0$ is equivalent to $\mathcal{F}_{-\frac{1}{\ell+1}}$, the subsymbol $\mathrm{s}\sigma^2_{\lambda,\lambda}(T)$ may be written as a contact Hamiltonian. If the operator $T$ is of the form

$$\begin{aligned} T \;=\; & T_{2,0,0}\,\partial_z^2 + T_{1,i,0}\,\partial_z\partial_{x_i} + T_{1,0,i}\,\partial_z\partial_{y_i} + T_{0,ij,0}\,\partial_{x_i}\partial_{x_j} + T_{0,i,j}\,\partial_{x_i}\partial_{y_j} + T_{0,0,ij}\,\partial_{y_i}\partial_{y_j} \\ & + T_{1,0,0}\,\partial_z + T_{0,i,0}\,\partial_{x_i} + T_{0,0,i}\,\partial_{y_i} + T_{0,0,0}, \end{aligned}$$

then the formula of Proposition 7.4 reads

$$\begin{aligned} \mathrm{s}\sigma^2_{\lambda,\lambda}(T) \;=\; & -\tfrac{1+2\lambda(\ell+1)}{\ell+2}\Big(\partial_z(T_{2,0,0} - \tfrac12 y_i T_{1,i,0} + \tfrac12 x_i T_{1,0,i}) \\ & \qquad + \partial_{x_i}(T_{1,i,0} - \tfrac12 y_j T_{0,ij,0} + \tfrac12 x_j T_{0,i,j}) \\ & \qquad + \partial_{y_i}(T_{1,0,i} + \tfrac12 x_j T_{0,0,ij} - \tfrac12 y_j T_{0,j,i})\Big) \\ & + T_{1,0,0} - \tfrac12 y_i T_{0,i,0} + \tfrac12 x_i T_{0,0,i}. \end{aligned}$$

One can verify directly that this formula coincides with that of Theorem B. The most efficient approach is to prove first the comments following the statement of the theorem, after which it suffices to carry out the verification for the single operator $T = L_\lambda(X_z) \circ L_\lambda(A_1)$, because that determines $c_{13}$.

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