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Asymptotic direction of random walks in Dirichlet environment

Laurent Tournier¹

Abstract In this short paper we generalize the result of directional transience from [SaTo10]. This enables us, by means of [Si07], [ZeMe01] and [Bo12] to conclude that, on \mathbb{Z}^d (for any dimension d), random walks in i.i.d. Dirichlet environment, or equivalently oriented-edge reinforced random walks, have almost-surely an asymptotic direction equal to the direction of the initial drift, i.e. $\frac{X_n}{\|X_n\|}$ converges to $\frac{E_o[X_1]}{\|E_o[X_1]\|}$ as $n \rightarrow \infty$, unless this drift is zero.

1 Introduction

Let $d \geq 1$, and denote by $\mathcal{V} = \{\vec{e}_1, -\vec{e}_1, \dots, \vec{e}_d, -\vec{e}_d\}$ the set of unit vectors in \mathbb{Z}^d , which will be used as possible steps. Assume we are given weights $\alpha_{\vec{e}} > 0$, $\vec{e} \in \mathcal{V}$.

Consider now the directed graph \mathbb{Z}^d whose oriented edges are the pairs $e = (x, y)$ such that $\vec{e} := y - x$ is an element of \mathcal{V} , endowed with (initial) weight

$$\alpha_e := \alpha_{\vec{e}}$$

and, for $x \in \mathbb{Z}^d$, define the law $P_x^{(\alpha)}$ of a random walk $(X_n)_{n \geq 0}$ on \mathbb{Z}^d in the following way: $P_x^{(\alpha)}$ -a.s., $X_0 = x$ and for every time $n \in \mathbb{N}$ and every edge e starting at X_n ,

$$P_x^{(\alpha)}((X_n, X_{n+1}) = e | X_0, \dots, X_n) = \frac{\alpha_e + N_n(e)}{\sum_{f: f=X_n} \alpha_f + N_n(f)} \quad (1)$$

where for every edge e we let $e =: (\underline{e}, \bar{e})$ and

$$N_n(e) := \#\{0 \leq i < n : (X_i, X_{i+1}) = e\}.$$

It is called the *oriented-edge reinforced random walk* (with initial weights $(\alpha_e)_e$, starting at x).

Due to the embedding of an independent Polya urn at each vertex and to de Finetti's theorem, this model admits an equivalent representation as a random walk in an i.i.d. random environment given by Dirichlet random variables. This representation (for which we refer for instance to [EnSa02] since it won't be explicitly used here besides its existence) makes the specificity of oriented-edge linear reinforcement and enabled several sharp results to be obtained, in contrast to the still very partial understanding of either random walks in random environment or reinforced random walks in dimension ≥ 2 .

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Define the *mean drift*

$$\vec{\Delta} := E_o^{(\alpha)}[X_1] = \frac{1}{\Sigma} \sum_{\vec{e} \in \mathcal{V}} \alpha_{\vec{e}} \vec{e}$$

where $\Sigma = \sum_{\vec{e} \in \mathcal{V}} \alpha_{\vec{e}}$.

In any dimension, Enriquez and Sabot [EnSa06] have given a sufficient ballisticity condition and bounds on the speed, later improved in [To09]. Non ballistic cases are known to occur when weights are sufficiently small (cf. [To09]), and Sabot and Tournier [SaTo10] proved that under the only assumption $\vec{\Delta} \neq \vec{0}$ (referred to as the *non symmetric case*), the random walk is transient with positive probability in the direction of a basis vector.

In dimension ≥ 3 , Sabot (in [Sa09]) proved that these random walks are transient (including in the symmetric case $\vec{\Delta} = \vec{0}$) and (in [Sa10]) characterized the ballistic regime (using in particular [SaTo10]).

The last three papers exploit a property of stability of Dirichlet distributions under time reversal. This technique proves again particularly efficient in the present paper to obtain the following theorem.

Theorem 1. *Assume $\vec{\Delta} \neq \vec{0}$. For any $\vec{u} \in \mathbb{R}^d$ with rational slopes such that $\vec{u} \cdot \vec{\Delta} > 0$,*

$$P_o^{(\alpha)} \left(X_n \cdot \vec{u} \xrightarrow{n} +\infty \right) > 0.$$

In [SaTo10], this theorem was proved in the case when $\vec{u} = \vec{e}_i$. The interest in the present refinement lies in the corollary below, obtained by combining the theorem with the 0-1 laws of [ZeMe01] ($d = 2$) and of the recent [Bo12] ($d \geq 3$) together with the main result of [Si07] for the conclusion (details follow).

Corollary 1. *Assume $\vec{\Delta} \neq \vec{0}$. Then*

$$\frac{X_n}{\|X_n\|} \xrightarrow[n]{} \frac{\vec{\Delta}}{\|\vec{\Delta}\|} \quad P_o^{(\alpha)}\text{-a.s.}$$

Remarks.

- In [EnSa06], Enriquez and Sabot gave an expansion of the speed as $\gamma \rightarrow \infty$ when the parameters are $\alpha_i^{(\gamma)} := \gamma \alpha_i$, and noticed that the second order was surprisingly colinear to the first one, i.e. to $\vec{\Delta}$. This is not anymore a surprise given the above corollary; but this highlights the fact that the simplicity of the corollary comes as a surprise itself. Correlations between the transition probabilities at one site indeed affect the speed (cf. for instance [Sa04]), and thus the speed of a random walk in random environment is typically not expected to be colinear with the mean drift, if not for symmetry reasons.
- The theorem does actually not depend on the graph structure of \mathbb{Z}^d besides translation invariance, meaning that the result also holds for non nearest neighbour models: we may enable \mathcal{V} to be any finite subset of \mathbb{Z}^d , and the proof is written in such a way that it covers this case. The same is true for the main results of [Bo12] and [Si07] with little modification, hence the corollary also generalizes in this way in dimension ≥ 3 . The

intersection property for planar walks used in [ZeMe01] may however fail if jumps are allowed in such a way that the graph is not anymore planar. But if it is planar, then the proof carries closely. This includes in particular the case of the triangular lattice (by taking $\mathcal{V} = \{\pm\vec{e}_1, \pm\vec{e}_2, \pm(\vec{e}_1 + \vec{e}_2)\}$).

- Using the above-mentioned 0-1 laws, the probability in the theorem equals 1 and the rationality assumption is readily waived; the theorem was stated this way in order to keep its proof essentially contained in the present paper, in contrast to its corollary.
- Statement (d) of Theorem 1.8 of [DrRa10] (and Lemma 4 of [ZeMe01]) implies that in any dimension, when $\vec{u} \cdot \vec{\Delta} = 0$, $P_o^{(\alpha)}$ -a.s., $\limsup_n X_n \cdot \vec{u} = +\infty$ and $\liminf_n X_n \cdot \vec{u} = -\infty$. In dimension at least 3, this is showed in [Bo12] as well.
- Theorem 2 of [Bo12] also implies the existence of a deterministic yet unspecified asymptotic direction in dimension at least 3. Further remarks regarding the derivation of the corollary from the theorem are deferred to the end of the proof.

2 Proof of the theorem

The proof, like [SaTo10], uses a time reversal property from [Sa09] (re-proved in a more probabilistic way in [SaTo10]). To keep the present proof more self-contained, the following very elementary lemma sums up the only aspect of this property that will be used later (this is Lemma 1 of [SaTo10]).

Lemma 1. *Let $G = (V, E)$ be a directed graph, endowed with positive weights $(\alpha_e)_{e \in E}$. We denote by $\check{G} = (V, \check{E})$ its reversed graph, i.e. $\check{E} := \{\check{e} := (\bar{e}, \underline{e}) : e = (\underline{e}, \bar{e}) \in E\}$, endowed with the weights $\check{\alpha}_{\check{e}} := \alpha_e$. Assume that $\text{div}(\alpha) = 0$, i.e., for every $x \in V$,*

$$\alpha_x := \sum_{e: \underline{e}=x} \alpha_e = \sum_{e: \bar{e}=x} \alpha_e =: \check{\alpha}_x.$$

Then, for any closed path $\sigma = (x_0, x_1, \dots, x_{n-1}, x_0)$ in G , letting $\check{\sigma} := (x_0, x_{n-1}, \dots, x_1, x_0)$ denote its reverse (in \check{G}), we have

$$P_{x_0}^{(\alpha)}((X_0, \dots, X_n) = \sigma) = P_{x_0}^{(\check{\alpha})}((X_0, \dots, X_n) = \check{\sigma}),$$

where the laws of oriented-edge reinforced random walks on G or \check{G} are defined as in (1).

Proof. From the definition of $P_{x_0}^{(\alpha)}$ we get

$$P_{x_0}^{(\alpha)}((X_0, \dots, X_n) = \sigma) = \frac{\prod_{e \in E} \alpha_e (\alpha_e + 1) \cdots (\alpha_e + n_e(\sigma) - 1)}{\prod_{x \in V} \alpha_x (\alpha_x + 1) \cdots (\alpha_x + n_x(\sigma) - 1)},$$

where $n_e(\sigma)$ (resp. $n_x(\sigma)$) is the number of crossings of the oriented edge e (resp. the number of visits of the vertex x) in the path σ . Cyclicity gives $n_e(\sigma) = n_{\check{e}}(\check{\sigma})$ and $n_x(\sigma) = n_x(\check{\sigma})$ for all $e \in E, x \in V$. Furthermore we have by assumption $\check{\alpha}_x = \alpha_x$ for every vertex x , and by definition $\alpha_e = \check{\alpha}_{\check{e}}$ for every edge e . This shows that the previous product matches the similar product with \check{E} , $\check{\alpha}$ and $\check{\sigma}$ instead of E , α and σ , hence the lemma. \square

Let us turn to the proof of Theorem 1. Assume $\vec{\Delta} \neq \vec{0}$, and let \vec{u} be a vector with rational slopes such that $\vec{\Delta} \cdot \vec{u} > 0$.

Up to multiplication by a constant, we may assume that $\vec{u} \in \mathbb{Z}^d$, and that $\|\vec{u}\| \geq \|\vec{e}\|$, $\forall \vec{e} \in \mathcal{V}$ (cf. second remark after the corollary). We complete \vec{u} into an orthogonal basis $(\vec{u}, \vec{u}_2, \dots, \vec{u}_d)$ chosen in such a way that $\vec{u}_i \in \mathbb{Z}^d$ for all i .

Let us consider the event $D := \{\forall n \geq 0, X_n \cdot \vec{u} \geq 0\}$. Because the weights are translation invariant, the probabilities $P_x^{(\alpha)}(D)$ are invariant by the translations $x \mapsto x \pm \vec{u}_i$ for all $i \geq 2$, and thus take only finitely many different values when x is in the “discrete hyperplane”

$$\overline{\mathcal{H}} := \{x \in \mathbb{Z}^d : \exists \vec{e} \in \mathcal{V}, (x - \vec{e}) \cdot \vec{u} < 0 \leq x \cdot \vec{u}\},$$

namely for instance each of the values obtained when x belongs to the finite set

$$\overline{\mathcal{H}}_0 := \overline{\mathcal{H}} \cap (\mathbb{R}_+ \vec{u} + [0, \vec{u}_2) + \dots + [0, \vec{u}_d)).$$

We introduce a probability measure μ on $\overline{\mathcal{H}}_0$ as follows: for all $x \in \overline{\mathcal{H}}_0$,

$$\mu(x) := \frac{1}{Z} \sum_{\substack{\vec{e} \in \mathcal{V}: \\ (x - \vec{e}) \cdot \vec{u} < 0}} \alpha_{\vec{e}} \quad (2)$$

where Z is a normalizing constant. Let us define a finite graph that will enable us to bound $P_\mu^{(\alpha)}(D) := \sum_{x \in \overline{\mathcal{H}}_0} \mu(x) P_x^{(\alpha)}(D)$ from below.

Let $N, L \in \mathbb{N}^*$. We first consider the cylinder

$$C_{N,L} := \left\{ x \in \mathbb{Z}^d : 0 \leq x \cdot \vec{u} \leq L \|\vec{u}\|^2 \right\} / (N\mathbb{Z}\vec{u}_2 + \dots + N\mathbb{Z}\vec{u}_d),$$

i.e. the slab $\{0 \leq x \cdot \vec{u} \leq L \|\vec{u}\|^2\} \cap \mathbb{Z}^d$ where vertices that differ by $N\vec{u}_i$ for some $i \in \{2, \dots, d\}$ are identified. Let \mathcal{R} denote its “right” end, i.e.

$$\mathcal{R} := \left\{ x \in \mathbb{Z}^d : \exists \vec{e} \in \mathcal{V}, x \cdot \vec{u} \leq L \|\vec{u}\|^2 < (x + \vec{e}) \cdot \vec{u} \right\} / (N\mathbb{Z}\vec{u}_2 + \dots + N\mathbb{Z}\vec{u}_d) \subset C_{N,L}$$

(note that the inclusion is due to the constraint $\|\vec{u}\| \geq \|\vec{e}\|$ for small L) and similarly $\mathcal{L} \subset C_{N,L}$ for the “left” end. We may now define the finite graph $G_{N,L}$ (refer to Figure 1 for an example in \mathbb{Z}^2). Its vertex set is

$$V_{N,L} := C_{N,L} \cup \{R, \partial\},$$

where R and ∂ are new vertices, and the edges of $G_{N,L}$ are of the following types:

- a) edges induced by those of \mathbb{Z}^d inside $C_{N,L}$;
- b) edges from (resp. to) the vertices of \mathcal{L} to (resp. from) ∂ , corresponding to the edges of \mathbb{Z}^d exiting (resp. entering) the cylinder “through the left end”;
- c) edges from (resp. to) the vertices of \mathcal{R} to (resp. from) R , corresponding to the edges of \mathbb{Z}^d exiting (resp. entering) the cylinder “through the right end”;
- d) a new edge from R to ∂ .

Note that in b) and c) several edges may connect two vertices, and that in d) no edge goes from ∂ to R . We also introduce weights $\alpha_e^{N,L}$ on the edges of $G_{N,L}$ as follows (invoking the translation invariance of the weights in \mathbb{Z}^d):

- edges defined in a), b) and c) have the weight of the corresponding edge in \mathbb{Z}^d ;
- the edge from R to ∂ has weight

$$\alpha_{(R,\partial)}^{N,L} := \left(\sum_{\substack{x \in \mathcal{R}, \vec{e} \in \mathcal{V}: \\ x + \vec{e} \notin C_{N,L}}} \alpha_{\vec{e}} \right) - \left(\sum_{\substack{x \in \mathcal{L}, \vec{e} \in \mathcal{V}: \\ x - \vec{e} \notin C_{N,L}}} \alpha_{\vec{e}} \right).$$

By construction, we have $\operatorname{div} \alpha^{N,L} = 0$. The main point to check however is that $\alpha_{(R,\partial)}^{N,L}$ is positive.

Due to periodicity, we have

$$\alpha_{(R,\partial)}^{N,L} = N^{d-1} \|\vec{u}_2\| \cdots \|\vec{u}_d\| \sum_{\vec{e} \in \mathcal{V}} (\Phi_{\vec{u}}(\vec{e}) - \Phi_{-\vec{u}}(\vec{e})) \alpha_{\vec{e}}$$

where $\Phi_{\vec{u}}(\vec{e})$ is the flux of \vec{e} through the oriented hyperplane \vec{u}^\perp :

$$\begin{aligned} \Phi_{\vec{u}}(\vec{e}) &:= \frac{1}{N^{d-1} \|\vec{u}_2\| \cdots \|\vec{u}_d\|} \#\{x \in \mathcal{R} : x + \vec{e} \notin C_{N,L}\} \\ &= \frac{1}{\|\vec{u}_2\| \cdots \|\vec{u}_d\|} \#\left(\left\{ x \in \mathbb{Z}^d : x \cdot \vec{u} \leq 0 < (x + \vec{e}) \cdot \vec{u} \right\} / (\mathbb{Z}\vec{u}_2 + \cdots + \mathbb{Z}\vec{u}_d) \right). \end{aligned}$$

Clearly $\Phi_{\vec{u}}(\vec{e})$ is zero if $\vec{u} \cdot \vec{e} \leq 0$ and otherwise it is a simple geometric fact that the above cardinality equals the volume of the parallelotope on the vectors $\vec{e}, \vec{u}_2, \dots, \vec{u}_d$. Indeed, this is also the number of lattice points in the torus $\mathbb{R}^d / (\mathbb{Z}\vec{e} + \mathbb{Z}\vec{u}_2 + \cdots + \mathbb{Z}\vec{u}_d)$, and this torus is partitioned into the unit cubes $x + [0, 1)^d$ indexed by the lattice points x . Hence in any case

$$\Phi_{\vec{u}}(\vec{e}) = \left(\frac{\vec{u}}{\|\vec{u}\|} \cdot \vec{e} \right)_+$$

which gives

$$\begin{aligned} \alpha_{(R,\partial)}^{N,L} &= \frac{N^{d-1} \|\vec{u}_2\| \cdots \|\vec{u}_d\|}{\|\vec{u}\|} \sum_{\vec{e} \in \mathcal{V}} ((\vec{u} \cdot \vec{e})_+ - (-\vec{u} \cdot \vec{e})_+) \alpha_{\vec{e}} = \frac{N^{d-1} \|\vec{u}_2\| \cdots \|\vec{u}_d\|}{\|\vec{u}\|} \sum_{\vec{e} \in \mathcal{V}} (\vec{u} \cdot \vec{e}) \alpha_{\vec{e}} \\ &= N^{d-1} \|\vec{u}_2\| \cdots \|\vec{u}_d\| \frac{\vec{u}}{\|\vec{u}\|} \cdot \Sigma \vec{\Delta} \end{aligned}$$

therefore finally $\alpha_{(R,\partial)}^{N,L} > 0$ since $\vec{u} \cdot \vec{\Delta} > 0$, as expected.

NB. The above computation also shows that, introducing a new notation,

$$\alpha_{(\mathcal{L},\partial)}^{N,L} := \sum_{x \in \mathcal{L}} \alpha_{(x,\partial)}^{N,L} = \sum_{\substack{x \in \mathcal{L}, \vec{e} \in \mathcal{V}: \\ x - \vec{e} \notin C_{N,L}}} \alpha_{\vec{e}} = N^{d-1} \|\vec{u}_2\| \cdots \|\vec{u}_d\| \sum_{\vec{e} \in \mathcal{V}} \left(- \frac{\vec{u}}{\|\vec{u}\|} \cdot \vec{e} \right)_+ \alpha_{\vec{e}},$$

hence in particular

$$\frac{\alpha_{(\mathcal{L},\partial)}^{N,L}}{\alpha_{(R,\partial)}^{N,L}} = \frac{\sum_{\vec{e} \in \mathcal{V}} (-\vec{u} \cdot \vec{e})_+ \alpha_{\vec{e}}}{\sum_{\vec{e} \in \mathcal{V}} (\vec{u} \cdot \vec{e}) \alpha_{\vec{e}}} = \frac{E_o^{(\alpha)} [(X_1 \cdot \vec{u})_-]}{E_o^{(\alpha)} [X_1 \cdot \vec{u}]}. \quad (3)$$

Letting N and then L go to infinity as in [SaTo10] (applied to each of the finitely many possible values of X_0 in $\overline{\mathcal{H}}_0$), we get

$$P_\mu^{(\alpha)}(\forall n \geq 0, X_n \cdot \vec{u} \geq 0) \geq 1 - \frac{E_o^{(\alpha)}[(X_1 \cdot \vec{u})_-]}{E_o^{(\alpha)}[(X_1 \cdot \vec{u})_+]}$$

hence, by translation invariance of $P_o^{(\alpha)}$ and Kalikow's 0-1 law (and Lemma 4 of [ZeMe01], showing that the walk cannot stay in a slab),

$$P_o^{(\alpha)}(X_n \cdot \vec{u} \rightarrow_n +\infty) = P_\mu^{(\alpha)}(X_n \cdot \vec{u} \rightarrow_n +\infty) \geq 1 - \frac{E_o^{(\alpha)}[(X_1 \cdot \vec{u})_-]}{E_o^{(\alpha)}[(X_1 \cdot \vec{u})_+]} > 0.$$

This is the content of Theorem 1. One may note that the above lower bound is a continuous function of $\frac{\vec{u}}{\|\vec{u}\|}$.

3 Proof of the corollary

Recall that oriented-edge reinforced random walks are also random walks in Dirichlet environment. Due to the 0-1 law of Zerner and Merkl [ZeMe01] (cf. also [Ze07]) in dimension 2 (for random walks in elliptic random environment), and of Bouchet [Bo12] in dimension at least 3 (for random walks in Dirichlet environment), the result of Theorem 1 turns into: for any $\vec{u} \in \mathbb{R}^d$ with rational slopes and such that $\vec{u} \cdot \vec{\Delta} > 0$,

$$X_n \cdot \vec{u} \xrightarrow[n]{} +\infty, \quad P_o^{(\alpha)} - a.s. \quad (4)$$

Note that the set of directions $\vec{u} \in \mathbb{R}^d$ such that the previous statement holds also has to be convex, therefore it contains the half-space $\{\vec{u} \in \mathbb{R}^d : \vec{u} \cdot \vec{\Delta} > 0\}$.

By Theorem 1 of [Si07], there exists a direction $\vec{v} \in \mathbb{S}^{d-1}$ such that

$$\frac{X_n}{\|X_n\|} \xrightarrow[n]{} \vec{v}, \quad P_o^{(\alpha)} - a.s.$$

On the other hand, this direction satisfies $\vec{v} \cdot \vec{u} \geq 0$ for every \vec{u} that satisfies (4), in particular for every \vec{u} such that $\vec{\Delta} \cdot \vec{u} > 0$. This fully characterizes \vec{v} , which therefore has to be

$$\vec{v} = \frac{\vec{\Delta}}{\|\vec{\Delta}\|}.$$

Remarks

- Before learning about the article [Bo12], a former version of the present paper gave a weaker result in dimension at least 3, namely that an asymptotic direction exists, although it remained unidentified, and possibly random (two-valued). Indeed, by the 0-1 law of Kalikow (in its elliptic version proved in [ZeMe01]) and Theorem 1.8 of [DrRa10], there exists $\vec{v} \in \mathbb{S}^{d-1}$ and an event A such that

$$\frac{X_n}{\|X_n\|} \xrightarrow[n]{} (\mathbf{1}_A - \mathbf{1}_{A^c})\vec{v}$$

but identifying \vec{v} from Theorem 1 is hindered by the restriction to rational slopes due to the possible non-convexity of the set of directions \vec{u} of transience (i.e. satisfying the theorem).

- In dimension at least 3, since [Bo12] already proves the existence of an asymptotic direction, an alternative derivation of the corollary without [Si07] would consist in using Theorem 1 in the proof of Theorem 2 of [Bo12] instead of referring to [SaTo10].

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