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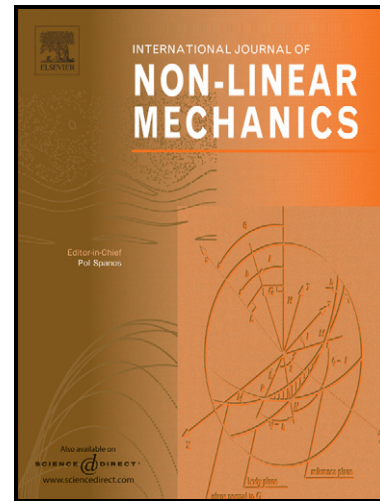
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Dual extremum principles for geometrically exact finite strain beams

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Abstract

Both necessary and sufficient conditions for the existence of two complementary-dual extremum principles for geometrically exact finite strain (one-dimensional) beam models are investigated by means of two different approaches. One is based on the results published by Gao and Strang, and the other relies on the approach proposed by Noble and Sewell. While the former is limited to beam models restricted to moderate large deformations, the latter is valid for arbitrarily large deformations (and strains). The numerical implementation of the complementary-dual extremum principles can lead to simple true global upper bounds of the error of the approximate solutions.

Key words: Geometrically exact finite strain beams, complementary-dual extremum principles, error estimation, global upper bounds.

1. Introduction

Complementary-dual extremum principles play fundamental roles in theoretical and computational mechanics. An *extremum principle* states that a solution of a given mathematical problem corresponds to a minimum or maximum value of a functional among a subset of solutions of the governing equations. *Dual principles*, also called complementary or reciprocal, form another class of variational principles. When a dual principle is available, the problem under consideration can be variationally formulated in two different, but interrelated, ways. In one formulation a solution is characterized by a maximum principle and in the other by a minimum principle. Furthermore, the maximum and minimum values of the respective functionals are equal.

Among the various stationary energy principles available for the analysis of fully (both geometrically and physically) linear problems, the two complementary-dual extremum principles well known in the literature as the minimum total potential energy (TPE) and maximum total complementary energy (TCE) principles are of great importance in practical applications. The difference between the values of both functionals for two different trial functions can be used to provide true global upper bounds of the error to the exact solution of a given boundary-value problem, *i.e.*, it can be used as a measure of the accuracy of an approximate

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solution. The extremum feature of these principles results from the absolute convexity of the strain energy density with respect to the configuration variables, which is also the necessary and sufficient condition for the existence of a unique solution. Furthermore, the Hellinger-Reissner energy (HRE) is a saddle-type functional, *i.e.*, a functional which is convex with respect to one group of variables and concave with respect to the other.

However, for geometrically nonlinear problems, the strain energy density is, in general, not convex with respect to the configuration variables and, therefore, neither global complementary-dual extremum principles nor uniqueness of solution exist in general. A dual energy-based *a posteriori* error estimation method, which is capable of providing true error bounds of the approximate solutions of a given boundary-value problem, requires however the existence of such dual extremum principles.

As shown by Noble and Sewell in the late sixties/early seventies [40, 28, 41, 42], complementary-dual extremum principles - minimum TPE under compatibility subsidiary conditions and maximum TCE under equilibrium conditions, can be generated from a single energy saddle-type functional.

Based on Noble-Sewell's theory, the extremum properties of the total potential and total complementary energies for the geometrically nonlinear analysis of pin joint networks were discussed by Bufler *et al.* (1980) [5]. The same author, in 1986, also following Noble and Sewell's concepts, established two complementary-dual extremum principles for the geometrically nonlinear (inextensional) Euler-Bernoulli (linear elastic planar) beam theory [4]. Positiveness of the stress-resultants normal to the cross-sections was shown to be the sufficient condition for the existence of the two dual extremum principles.

Dual extremum principles generated also from a single saddle-type functional in the framework of the nonlinear elasticity theory were derived and applied by Stumpf to the geometrically nonlinear analysis of plates and shells [47, 48, 49].

According to Gao and Strang (1989) [14], in the framework of the finite elasticity theory, characterized by a quadratic geometrical operator, both global and local extremality conditions depend on a so-called complementary gap functional, a non-zero quantity due to the nonconvexity of the strain energy with respect to the displacement variables. Furthermore, the HRE can only be regarded as a saddle-type functional if, and only if, the gap functional is positive for any statically admissible stress field. Under this condition, the minimum TPE principle is equivalent to a unique maximum dual variational principle. Otherwise, if the gap functional is negative, the system has two complementary-dual extremum principles.

Analytical applications obtained using Gao-Strang's theory to the geometrically nonlinear analysis of shells were given by Gao and Cheung (1990) [11]. Applications to the post-buckling analysis of a finite deformation (small strain) beam theory taking into account warping phenomena were presented by Gao (1997, 2000) [8, 10]. Recently, general analytical solutions have also been obtained for one-dimensional nonlinear elasticity and phase transition problems [12, 13].

Using a so-called sequential canonical dual transformation, Gao-Strang's theory has been extended by Gao (2000) (see Chapter 4 in [9]) to general geometrical nonlinear operators and the saddle complementary HRE principle was shown to hold also for higher-order geometrical nonlinear operators.

The objective of this paper is to present both necessary and sufficient conditions for the existence of two complementary-dual extremum principles developed for geometrically exact finite strain beams. Two

distinct approaches will be followed: one relies on Noble-Sewell's theory and the other has its basis on Gao-Strang's theory [14]. While the latter approach is restricted to quadratic geometrical (also called compatibility) operators, being valid therefore only for moderate large deflections, the former is valid for any magnitude of the displacements, rotations and strains.

2. The Geometrically Exact Finite Strain Beam Theory: Boundary-Value Problem

We consider the geometrically exact finite strain beam theory introduced by Reissner (1973, 1981) [34, 35] and Antman (1974) [1] and further extended by Simo and co-workers (1985, 1986, 1991, 1995) [43, 45, 46, 44]. Since these pioneering works, considerable progress has been made on the geometrically exact analysis of three-dimensional framed structures, from both theoretical and numerical points of view, see *e.g.* Cardona and Géradin (1988) [6], Ibrahimbegovic and co-workers (1995, 2000, 2003) [16, 19, 18, 17], Petrov and Géradin (1998) [31], Saje *et al.* (1998) [36], Crisfield and Jelenic (1999) [7, 22], Gruttmann *et al.* (2000) [15], Atluri and co-workers (1988, 1989, 1996, 1998, 2001) [20, 21, 33, 2], Betsch and Steinmann (2002) [3], Pimenta and Campello (2003) [32], Zupan and Saje (2003) [50], Kapania and Li (2003) [24, 23], Mata *et al.* (2007) [27], Makinen (2007) [26], Santos *et al.* (2009) [38, 37] and many others.

From a mechanical point of view, the most important feature of this model is that it does not make any (kinematical) assumptions regarding the magnitude of either the displacements (and rotations) or the strains, being therefore valid for arbitrarily large displacements, rotations and also strains.

The geometrically exact beam model has been developed using two different approaches. One is the approach followed by Simo, in which the beam theory is derived from a degenerating three-dimensional continuum theory restricted to some kinematical hypotheses. The other is the approach adopted by Reissner, in which the equilibrium differential equations of cross-sectional forces and moments defined for an infinitesimal element of the deformed beam axis are taken as basic and, finally, after stipulating an appropriate form of the principle of virtual work, a system of strain-displacement relations, involving the translational and rotational strains are obtained. Although formally both these approaches can be considered as equivalent, the former allows indeed to take into account finite strain constitutive relations in a more straightforward way than the latter. The latter benefits however from the advantages which are inherent to a true one-dimensional beam theory.

Only for the sake of simplicity, we will adopt here the approach followed by Reissner. Note however that all the results presented next should hold regardless of the approach one employs. The consequence of following this approach is that the constitutive equations should be assumed to be expressible in terms of generalized cross-sectional stress-resultants and (energy-conjugate) generalized strain measures, *i.e.* they must be regarded as true one-dimensional relations. Note however that, formally, this might not be possible in general. Notwithstanding, we highlight that such assumption should have no implications on the succeeding developments, as the constitutive equations can, without loss of generality, and once the (integral) relations between one-dimensional and corresponding tensorial quantities are known, be established in terms of consistent three-dimensional constitutive relations. For the derivation of a one-dimensional nonlinear elastic constitutive model consistent with the three-dimensional finite elasticity theory see, *e.g.*, the work by Lanzo (2004) [25] and the references therein.

We will employ the 'classical' geometrically exact finite strain beam theory, in which the configuration

space of a beam element includes not only the linear space of position vectors of the centroidal axis, but also the non-linear space of the rotations of the cross-sections, not necessarily normal to the line of centroids at the deformed state. The geometric shape of the cross-sections is assumed to be arbitrary and constant along the beam. The cross-sections are assumed to suffer only rigid body motions during deformation.

The centroidal axis is assumed to be parameterized by $S \in [0, L]$, with L representing the length of the beam in its reference configuration. The domain of the centroidal axis is decomposed into an internal part, Ω , and a boundary part, $\Gamma = \Gamma_N \cup \Gamma_D = \{0, L\}$, with Γ_N and Γ_D the Neumann and Dirichlet boundaries, respectively, such that $\Gamma_N \cap \Gamma_D = \emptyset$. The beam may be subject to generalized spatial distributed forces \mathbf{q} applied in Ω , generalized spatial concentrated forces $\bar{\mathbf{q}}$ applied on Γ_N , and generalized prescribed displacements $\bar{\mathbf{d}}$ defined on Γ_D . The loads are assumed to be conservative and to act at the centroidal axis of the beam.

The strong form of the boundary-value problem governing the mathematical (quasi-static) response of the present geometrically exact finite strain one-dimensional beam model with (homogeneous and isotropic) hyperelastic material behavior is given by the following equations

$$\mathcal{T}_e^r(\mathbf{d})\boldsymbol{\sigma}^r + \mathbf{q} = \mathbf{0}, \text{ in } \Omega \quad (1a)$$

$$\boldsymbol{\sigma}^r - \frac{\partial W(\boldsymbol{\varepsilon}^r)}{\partial \boldsymbol{\varepsilon}^r} = \mathbf{0}, \text{ in } \Omega \quad (1b)$$

$$\boldsymbol{\varepsilon}^r - \boldsymbol{\varepsilon}^r(\mathbf{d}) = \mathbf{0}, \text{ in } \Omega \quad (1c)$$

which represent the equilibrium, constitutive and compatibility (also called kinematical) differential equations, respectively, and also a set of prescribed boundary conditions subdivided into Neumann (also called static) and Dirichlet (also called compatibility) conditions, respectively, as follows

$$n\mathbf{H}(\mathbf{d})\boldsymbol{\sigma}^r - \bar{\mathbf{q}} = \mathbf{0}, \text{ on } \Gamma_N \quad (2a)$$

$$\mathbf{d} - \bar{\mathbf{d}} = \mathbf{0}, \text{ on } \Gamma_D \quad (2b)$$

with

$$n = \begin{cases} 1 & \text{if } S = L \\ -1 & \text{if } S = 0. \end{cases}$$

For details on the derivation of these equations and also the form of their corresponding operators, the reader is referred to [39, 38, 37]. It is only worth noting that \mathbf{d} represents a generalized displacement vector, including both the displacement vector of a point lying in the centroidal axis of the beam, \mathbf{u} , and also the rotation vector of the beam cross-section attached to that point, $\boldsymbol{\theta}$. Note also that, while $\boldsymbol{\varepsilon}^r$ denotes the generalized material strain vector, including both translational and rotational measures characterizing the strain state at that point, $\boldsymbol{\sigma}^r$ represents the generalized material (cross-sectional) stress-resultant vector, including both force and moment variables characterizing the stress state at that point.

Additionally, note that $\mathcal{T}_e^r(\mathbf{d})$ represents the material form of the equilibrium operator as follows

$$\mathcal{T}_e^r(\mathbf{d}) = \boldsymbol{\Psi}_e \Delta_e \boldsymbol{\Lambda}$$

with

$$\boldsymbol{\Psi}_e = \begin{bmatrix} \mathbf{I} & \mathbf{O} & \mathbf{O} \\ \mathbf{O} & \boldsymbol{\Gamma}^T \mathbf{Z} & \boldsymbol{\Gamma}^T \end{bmatrix}, \Delta_e = \begin{bmatrix} \mathbf{I} \frac{\partial}{\partial S} & \mathbf{O} \\ \mathbf{I} & \mathbf{O} \\ \mathbf{O} & \mathbf{I} \frac{\partial}{\partial S} \end{bmatrix}, \boldsymbol{\Lambda} = \begin{bmatrix} \boldsymbol{\mathcal{Q}} & \mathbf{O} \\ \mathbf{O} & \boldsymbol{\mathcal{Q}} \end{bmatrix}$$

where \mathbf{Z}' represents the skew-symmetric tensor associated with vector \mathbf{z}' , defined as

$$\mathbf{z}' = \mathbf{u}' + \mathbf{e}_3^r$$

with \mathbf{e}_3^r the unitary base vector taken normal to the beam cross sections. $(\cdot)'$ indicates differentiation of (\cdot) with respect to S . \mathbf{Q} and $\mathbf{\Gamma}$ represent the cross section rotation tensor and a transformation tensor, respectively, both defined as functions of the rotation vector $\boldsymbol{\theta}$. For details on their expressions the reader is referred to [39, 38]. \mathbf{I} denotes the standard second-order identity tensor.

Vector $\boldsymbol{\varepsilon}^r(\mathbf{d})$ represents the material compatibility operator, mapping generalized displacements onto generalized deformations as follows

$$\boldsymbol{\varepsilon}^r(\mathbf{d}) = \begin{bmatrix} \mathbf{Q}^T(\mathbf{u}' + \mathbf{e}_3^r) - \mathbf{e}_3^r \\ \mathbf{\Gamma}^T \boldsymbol{\theta}' \end{bmatrix}.$$

$\mathbf{H}(\mathbf{d})$ represents a geometrical nonlinear boundary operator given by

$$\mathbf{H}(\mathbf{d}) = \begin{bmatrix} \mathbf{Q} & \mathbf{0} \\ \mathbf{0} & \mathbf{\Gamma} \end{bmatrix}.$$

To conclude, it is emphasized that W represents the strain energy density, defined per unit reference length and regarded as function of the material strains in order to ensure material frame indifference. If W is convex with respect to $\boldsymbol{\varepsilon}^r$, the stiffness form of the constitutive relations (1b) can be uniquely established using the following alternative (also called inverse or flexibility) form

$$\boldsymbol{\varepsilon}^r - \frac{\partial W_c(\boldsymbol{\sigma}^r)}{\partial \boldsymbol{\sigma}^r} = \mathbf{0}, \text{ in } \Omega \quad (3)$$

where $W_c(\boldsymbol{\sigma}^r)$ represents the complementary strain energy density, regarded as the Legendre transform of $W(\boldsymbol{\varepsilon}^r)$.

In the present context, convexity means that the second derivative of W with respect to $\boldsymbol{\varepsilon}^r$ is positive definite. This fundamental physical requirement in hyperelasticity ensures that undesirable material instabilities are precluded (for a general discussion of convexity in hyperelasticity the reader is referred to [30], Sections 6 and 7). Note however that, in the theory of elasticity the notion of convexity of the strain energy function is dependent on the choice of deformation measure used and different deformation measures may lead to nonconvexity of W .

Conditions for the convexity of the strain energy density with respect to frame-indifferent deformation measures (material strain measures) and a full discussion of the existence of a uniquely defined inverse of the constitutive relations in the general framework of hyperelasticity were given by Ogden (1977) [29]. Ogden has concluded that, for all deformations of practical interest in rubberlike materials, it is reasonable to suppose the existence of a strictly convex function of the right stretch tensor.

3. Dual Variational Principles and the Gap Functional

3.1. Primal Variational Principle

The stationary principle of TPE is taken here as the primal variational principle.

Let \mathcal{U}_k and \mathcal{V}_k be the kinematically and the homogeneous kinematically admissible spaces, respectively, defined as

$$\mathcal{U}_k = \{\mathbf{d} \in \mathcal{H}^1(\Omega) \mid \mathbf{d} = \bar{\mathbf{d}} \text{ on } \Gamma_D\}$$

and

$$\mathcal{V}_k = \{\delta \mathbf{d} \in \mathcal{H}^1(\Omega) \mid \delta \mathbf{d} = \mathbf{0} \text{ on } \Gamma_D\}.$$

The TPE associated with vector \mathbf{d} is the one-field functional $\Pi_p : \mathcal{U}_k(\Omega) \rightarrow \mathcal{R}$ given by

$$\Pi_p(\mathbf{d}) = U(\boldsymbol{\varepsilon}^r(\mathbf{d})) + F(\mathbf{d})$$

where U represents the strain energy, or internal potential energy, given by

$$U(\boldsymbol{\varepsilon}^r(\mathbf{d})) = \int_0^L W(\boldsymbol{\varepsilon}^r(\mathbf{d})) d\Omega$$

and F represents the external potential energy defined as follows

$$F(\mathbf{d}) = - \int_0^L \mathbf{q} \cdot \mathbf{d} d\Omega - [\bar{\mathbf{q}} \cdot \mathbf{d}]_{\Gamma_N}.$$

Vector \mathbf{d} is called a stationary point of Π_p if the first-order Gâteaux derivative of $\Pi_p(\mathbf{d})$ at $\mathbf{d} \in \mathcal{U}_k$ in any direction $\delta \mathbf{d} \in \mathcal{V}_k$ vanishes, that is, if the following condition holds

$$\delta \Pi_p(\mathbf{d}; \delta \mathbf{d}) = 0, \quad \forall \delta \mathbf{d} \in \mathcal{V}_k$$

wherein $\delta \Pi_p(\mathbf{d}; \delta \mathbf{d})$ assumes the following form

$$\delta \Pi_p(\mathbf{d}; \delta \mathbf{d}) = \int_{\Omega} \left[\frac{\partial W}{\partial \boldsymbol{\varepsilon}^r} \cdot \delta \boldsymbol{\varepsilon}^r(\mathbf{d}; \delta \mathbf{d}) - \mathbf{q} \cdot \delta \mathbf{d} \right] d\Omega - [\bar{\mathbf{q}} \cdot \delta \mathbf{d}]_{\Gamma_N}.$$

This, upon substitution of the constitutive relation (1b), gives rise to

$$\delta \Pi_p(\mathbf{d}; \delta \mathbf{d}) = \int_{\Omega} [\boldsymbol{\sigma}^r \cdot \mathcal{T}_c^r(\mathbf{d}) \delta \mathbf{d} - \mathbf{q} \cdot \delta \mathbf{d}] d\Omega - [\bar{\mathbf{q}} \cdot \delta \mathbf{d}]_{\Gamma_N}$$

where $\delta \boldsymbol{\varepsilon}^r(\mathbf{d}; \delta \mathbf{d})$ has been replaced by $\mathcal{T}_c^r(\mathbf{d}) \delta \mathbf{d}$, with $\mathcal{T}_c^r(\mathbf{d})$ the material form of the tangent compatibility operator given by

$$\mathcal{T}_c^r(\mathbf{d}) = \mathbf{\Lambda}^T \boldsymbol{\Psi}_c \Delta_c$$

with

$$\boldsymbol{\Psi}_c = \begin{bmatrix} \mathbf{I} & \mathbf{O} & \mathbf{Z}'\boldsymbol{\Gamma} \\ \mathbf{O} & \boldsymbol{\Gamma} & \boldsymbol{\Gamma}' \end{bmatrix}, \quad \Delta_c = \begin{bmatrix} \mathbf{I} \frac{\partial}{\partial \mathbf{S}} & \mathbf{O} \\ \mathbf{O} & \mathbf{I} \frac{\partial}{\partial \mathbf{S}} \\ \mathbf{O} & \mathbf{I} \end{bmatrix}.$$

Finally, after integrating by parts, we obtain

$$\delta\Pi_p(\mathbf{d}; \delta\mathbf{d}) = - \int_{\Omega} [\mathcal{T}_e^r(\mathbf{d})\boldsymbol{\sigma}^r + \mathbf{q}] \cdot \delta\mathbf{d} \, d\Omega + [(n\mathbf{H}(\mathbf{d})\boldsymbol{\sigma}^r - \bar{\mathbf{q}}) \cdot \delta\mathbf{d}]_{\Gamma_N}$$

which shows that the Euler-Lagrange equations of $\delta\Pi_p$ are the equilibrium equations (1a) and (2a). It is worth noting that the operator \mathcal{T}_c^r can be regarded as the adjoint operator of the equilibrium operator \mathcal{T}_e^r , *i.e.*

$$\mathcal{T}_e^r(\mathbf{d})\boldsymbol{\sigma}^r \cdot \mathbf{d} = \mathcal{T}_c^r(\mathbf{d})\mathbf{d} \cdot \boldsymbol{\sigma}^r.$$

Hence, it can be concluded that the system is in equilibrium if, and only if, its TPE takes a stationary value for all displacement fields lying in the homogeneous kinematically admissible space \mathcal{V}^k . In other words, while the compatibility and constitutive equations are regarded as constraints assigned to the stationarity of the variational problem defined by the TPE, the equilibrium equations are derived as their Euler-Lagrange conditions.

3.2. Dual Variational Principle

The stationary principle of TCE is taken here as the dual variational principle.

Let \mathcal{U}_s and \mathcal{V}_s be the statically and the homogeneous statically admissible spaces, respectively, defined as

$$\mathcal{U}_s = \{(\boldsymbol{\sigma}^r, \mathbf{d}) \in \mathcal{H}^1(\Omega) \times \mathcal{H}^1(\Omega) \mid \mathcal{T}_e^r(\mathbf{d})\boldsymbol{\sigma}^r + \mathbf{q} = \mathbf{0} \text{ in } \Omega \text{ and } n\mathbf{H}(\mathbf{d})\boldsymbol{\sigma}^r = \bar{\mathbf{q}} \text{ on } \Gamma_N\}$$

and

$$\mathcal{V}_s = \{(\delta\boldsymbol{\sigma}^r, \mathbf{d}) \in \mathcal{H}^1(\Omega) \times \mathcal{H}^1(\Omega) \mid \mathcal{T}_e^r(\mathbf{d})\delta\boldsymbol{\sigma}^r = \mathbf{0} \text{ in } \Omega \text{ and } n\mathbf{H}(\mathbf{d})\delta\boldsymbol{\sigma}^r = \mathbf{0} \text{ on } \Gamma_N\}.$$

As shown recently by Santos *et al.* [39], the TCE for a geometrically exact finite strain beam is the two-field functional $\Pi_c : \mathcal{U}_s(\Omega) \rightarrow \mathcal{R}$ given by

$$\Pi_c(\boldsymbol{\sigma}^r, \mathbf{d}) = \int_{\Omega} [\mathcal{W}_c(\boldsymbol{\sigma}^r) - \boldsymbol{\sigma}^r \cdot \boldsymbol{\varepsilon}^r(\mathbf{d}) + \boldsymbol{\sigma}^r \cdot \mathcal{T}_c^r(\mathbf{d})\mathbf{d}] \, d\Omega - [\bar{\mathbf{d}} \cdot n\mathbf{H}(\mathbf{d})\boldsymbol{\sigma}^r]_{\Gamma_D}. \quad (4)$$

It can be easily shown that the pair $(\boldsymbol{\sigma}^r, \mathbf{d}) \in \mathcal{U}_s$ is a solution of the boundary-value problem if, and only if, the following condition holds

$$\delta\Pi_c(\boldsymbol{\sigma}^r, \mathbf{d}; \delta\boldsymbol{\sigma}^r) = 0, \quad \forall \delta\boldsymbol{\sigma}^r \in \mathcal{V}_s.$$

In fact, taking the first-order Gâteaux derivative of Π_c defined at $(\boldsymbol{\sigma}^r, \mathbf{d}) \in \mathcal{U}_s$ in the direction of $\delta\boldsymbol{\sigma}^r \in \mathcal{V}_s$ yields

$$\delta\Pi_c(\boldsymbol{\sigma}^r, \mathbf{d}; \delta\boldsymbol{\sigma}^r) = \int_{\Omega} [\boldsymbol{\varepsilon}^r - \boldsymbol{\varepsilon}^r(\mathbf{d})] \cdot \delta\boldsymbol{\sigma}^r \, d\Omega + \int_{\Omega} \mathcal{T}_c^r(\mathbf{d})\mathbf{d} \cdot \delta\boldsymbol{\sigma}^r \, d\Omega - [\bar{\mathbf{d}} \cdot n\mathbf{H}(\mathbf{d})\delta\boldsymbol{\sigma}^r]_{\Gamma_D}.$$

Thus, after integrating by parts and making use of the equilibrium equations given by

$$\begin{aligned} \mathcal{T}_e^r(\mathbf{d})\delta\boldsymbol{\sigma}^r &= \mathbf{0}, \text{ in } \Omega \\ n\mathbf{H}(\mathbf{d})\delta\boldsymbol{\sigma}^r &= \mathbf{0}, \text{ on } \Gamma_N \end{aligned}$$

it is finally obtained that

$$\delta\Pi_c(\boldsymbol{\sigma}^r, \mathbf{d}; \delta\boldsymbol{\sigma}^r) = \int_{\Omega} [\boldsymbol{\varepsilon}^r - \boldsymbol{\varepsilon}^r(\mathbf{d})] \cdot \delta\boldsymbol{\sigma}^r d\Omega + [(\mathbf{d} - \bar{\mathbf{d}}) \cdot n\mathbf{H}(\mathbf{d})\delta\boldsymbol{\sigma}^r]_{\Gamma_D}.$$

Hence, requiring $\delta\Pi_c$ to vanish for independent variations $\delta\boldsymbol{\sigma}^r \in \mathcal{V}_s$ leads to the compatibility equations of the boundary-value problem (1c) and (2b).

Therefore, a generalized displacement vector of a beam, \mathbf{d} , is said to be a kinematically admissible field if, and only if, the TCE of the beam takes a stationary value for all stress-resultant fields lying in the homogeneous statically admissible space \mathcal{V}_s .

3.3. Conjugate Energy and the Gap Functional

For linear elastic material behavior, the strain energy density W is a quadratic and, therefore, convex functional of the material strain vector $\boldsymbol{\varepsilon}^r$. However, if one considers a general hyperelastic material, W might not be convex with respect to the material strain measures. Nevertheless, as noted above, for the deformations of practical interest in rubberlike materials, it is reasonable to suppose the existence of a strictly convex strain energy density function.

Note however that, either for linear or nonlinear elastic responses, as long as the strain vector $\boldsymbol{\varepsilon}^r$ is regarded as a nonlinear function of \mathbf{d} , then, W may be nonconvex with respect to \mathbf{d} and, therefore, the TPE may be a nonconvex functional of the displacement vector as well.

Accordingly, on the basis of the physical behavior of rubberlike materials, we postulate the existence of a convex strain energy density W taken with respect to $\boldsymbol{\varepsilon}^r$. Note that this is true in general only for fully (both physically and geometrically) linear responses. This ensures that the constitutive relation has a unique inversion, *i.e.*, a unique deformation $\boldsymbol{\varepsilon}^r$ can be found for a given stress $\boldsymbol{\sigma}^r$ and, furthermore, the Legendre transformation of $W(\boldsymbol{\varepsilon}^r)$ is unique and leads to a strictly convex complementary strain energy density. For simplicity, let us also assume homogeneous Dirichlet boundary conditions. Under these assumptions, the Legendre transforms of $W(\boldsymbol{\varepsilon}^r)$ and $F(\mathbf{d})$ read

$$\begin{aligned} W_c(\boldsymbol{\sigma}^r) &= \boldsymbol{\varepsilon}^r \cdot \boldsymbol{\sigma}^r - W(\boldsymbol{\varepsilon}^r) \\ F_c(\mathbf{r}) &= -F(\mathbf{d}) - \int_0^L \mathbf{q} \cdot \mathbf{d} d\Omega - [\bar{\mathbf{q}} \cdot \mathbf{d}]_{\Gamma_N} \end{aligned}$$

where \mathbf{r} represents the energy-conjugate variable associated to the vector of prescribed displacements $\bar{\mathbf{d}}$. W_c and F_c are referred to as the complementary strain energy and the external complementary energy, respectively.

The conjugate energy of the TPE comes out therefore as the following convex functional

$$\Pi^*(\boldsymbol{\sigma}^r) = U_c(\boldsymbol{\sigma}^r) + F_c(\mathbf{r}(\boldsymbol{\sigma}^r)) = \int_0^L W_c(\boldsymbol{\sigma}^r) d\Omega. \quad (7)$$

This, in turn, under subsidiary equilibrium equations gives rise to the following augmented Lagrangian

$$L_c(\boldsymbol{\sigma}^r, \mathbf{d}) = \int_0^L W_c(\boldsymbol{\sigma}^r) d\Omega + \int_0^L (\mathcal{T}_e^r(\mathbf{d})\boldsymbol{\sigma}^r + \mathbf{q}) \cdot \mathbf{d} d\Omega - [(n\mathbf{H}(\mathbf{d})\boldsymbol{\sigma}^r - \bar{\mathbf{q}}) \cdot \mathbf{d}]_{\Gamma_N}.$$

Its first-order Gâteaux derivative in the direction of $\delta\boldsymbol{\sigma}^r$ yields

$$\delta L_c(\boldsymbol{\sigma}^r, \mathbf{d}; \delta\boldsymbol{\sigma}^r) = \int_0^L \frac{\partial W_c(\boldsymbol{\sigma}^r)}{\partial \boldsymbol{\sigma}^r} \cdot \delta\boldsymbol{\sigma}^r d\Omega + \int_0^L \mathcal{T}_c^r(\mathbf{d}) \delta\boldsymbol{\sigma}^r \cdot \mathbf{d} d\Omega - [n\mathbf{H}(\mathbf{d})\delta\boldsymbol{\sigma}^r \cdot \mathbf{d}]_{\Gamma_N}$$

which, by means of (3) and after integrating by parts, leads to

$$\delta L_c(\boldsymbol{\sigma}^r, \mathbf{d}; \delta\boldsymbol{\sigma}^r) = \int_0^L \boldsymbol{\varepsilon}^r \cdot \delta\boldsymbol{\sigma}^r d\Omega - \int_0^L \mathcal{T}_c^r(\mathbf{d}) \mathbf{d} \cdot \delta\boldsymbol{\sigma}^r d\Omega + [n\mathbf{H}(\mathbf{d})\delta\boldsymbol{\sigma}^r \cdot \mathbf{d}]_{\Gamma_D}.$$

This shows that the conjugate energy Π^* is stationary at an equilibrium state if the following equations hold

$$\boldsymbol{\varepsilon}^r = \mathcal{T}_c^r(\mathbf{d})\mathbf{d} \text{ in } \Omega \quad (8a)$$

$$\mathbf{d} = \mathbf{0} \text{ on } \Gamma_D. \quad (8b)$$

As it can be seen, while the right-hand side term of (8a) represents the generalized vector of elastic material strain measures (arising from the constitutive relations), its left-hand side term corresponds only to the tangent part of the generalized material strain vector, rather than the actual strain vector. Therefore, the one-field functional $\Pi^*(\boldsymbol{\sigma}^r)$ obtained from the Legendre transformation of $\Pi_p(\mathbf{d})$ cannot be regarded as the true complementary energy.

By comparison, the true complementary energy (4) differs from the conjugate energy (7) on the following integral term

$$\Pi_c(\boldsymbol{\sigma}^r, \mathbf{d}) - \Pi^*(\boldsymbol{\sigma}^r) = \text{Gap}(\boldsymbol{\sigma}^r, \mathbf{d}) = - \int_0^L (\boldsymbol{\varepsilon}^r(\mathbf{d}) - \mathcal{T}_c^r(\mathbf{d})\mathbf{d}) \cdot \boldsymbol{\sigma}^r d\Omega \quad (9)$$

called the *gap functional*. This nomenclature was introduced first by Gao and Strang in the general context of geometric nonlinearity [14].

3.4. Complementary Hellinger-Reissner Variational Principle

According to Santos *et al.* (2010) [39], the complementary HRE for a geometrically exact finite strain beam is the two-field functional $\Pi_{HR} : \mathcal{H}^0(\Omega) \times \mathcal{H}^1(\Omega) \rightarrow \mathcal{R}$ given by

$$\Pi_{HR}(\boldsymbol{\sigma}^r, \mathbf{d}) = \int_{\Omega} [W_c(\boldsymbol{\sigma}^r) - \boldsymbol{\sigma}^r \cdot \boldsymbol{\varepsilon}^r(\mathbf{d}) + \mathbf{q} \cdot \mathbf{d}] d\Omega + [\bar{\mathbf{q}} \cdot \mathbf{d}]_{\Gamma_N} + [n\mathbf{H}(\mathbf{d})\boldsymbol{\sigma}^r \cdot (\mathbf{d} - \bar{\mathbf{d}})]_{\Gamma_D}. \quad (10)$$

Its first-order Gâteaux derivative at $(\boldsymbol{\sigma}^r, \mathbf{d})$ in the direction of $(\delta\boldsymbol{\sigma}^r, \delta\mathbf{d}) \in \mathcal{H}^0(\Omega) \times \mathcal{H}^1(\Omega)$ can be written as

$$\begin{aligned} \delta\Pi_{HR} = \int_{\Omega} \left[\frac{\partial W_c(\boldsymbol{\sigma}^r)}{\partial \boldsymbol{\sigma}^r} \cdot \delta\boldsymbol{\sigma}^r - \boldsymbol{\varepsilon}^r(\mathbf{d}) \cdot \delta\boldsymbol{\sigma}^r - \boldsymbol{\sigma}^r \cdot \mathcal{T}_c^r(\mathbf{d})\delta\mathbf{d} + \mathbf{q} \cdot \delta\mathbf{d} \right] d\Omega \\ + [\bar{\mathbf{q}} \cdot \delta\mathbf{d}]_{\Gamma_N} + [(\mathbf{d} - \bar{\mathbf{d}}) \cdot n\delta(\mathbf{H}(\mathbf{d})\boldsymbol{\sigma}^r)]_{\Gamma_D} + [n\mathbf{H}(\mathbf{d})\boldsymbol{\sigma}^r \cdot \delta\mathbf{d}]_{\Gamma_D}. \end{aligned}$$

This, in turn, after integrating by parts, leads to

$$\delta\Pi_{HR} = \int_{\Omega} \left[\left(\frac{\partial W_c(\boldsymbol{\sigma}^r)}{\partial \boldsymbol{\sigma}^r} - \boldsymbol{\varepsilon}^r(\mathbf{d}) \right) \cdot \delta\boldsymbol{\sigma}^r + (\mathcal{T}_c^r(\mathbf{d})\boldsymbol{\sigma}^r + \mathbf{q}) \cdot \delta\mathbf{d} \right] d\Omega + [(\bar{\mathbf{q}} - n\mathbf{H}(\mathbf{d})\boldsymbol{\sigma}^r) \cdot \delta\mathbf{d}]_{\Gamma_N} + [(\mathbf{d} - \bar{\mathbf{d}}) \cdot n\delta(\mathbf{H}(\mathbf{d})\boldsymbol{\sigma}^r)]_{\Gamma_D}.$$

Accordingly, requiring $\delta\Pi_{HR}$ to vanish for independent variations of $\delta\boldsymbol{\sigma}^r$ and $\delta\mathbf{d}$ gives rise to the following Euler-Lagrange equations

$$\begin{aligned}\frac{\partial W_c(\boldsymbol{\sigma}^r)}{\partial \boldsymbol{\sigma}^r} - \boldsymbol{\varepsilon}^r(\mathbf{d}) &= \mathbf{0}, \text{ in } \Omega \\ \mathcal{T}_e^r(\mathbf{d})\boldsymbol{\sigma}^r + \mathbf{q} &= \mathbf{0}, \text{ in } \Omega \\ \bar{\mathbf{q}} - n\mathbf{H}(\mathbf{d})\boldsymbol{\sigma}^r &= \mathbf{0}, \text{ on } \Gamma_N \\ \mathbf{d} - \bar{\mathbf{d}} &= \mathbf{0}, \text{ on } \Gamma_D\end{aligned}$$

which represent all the equations governing the boundary-value problem.

4. Application of Noble and Sewell's Approach to the Derivation of Dual Extremum Principles

The variational statements previously presented are stationary principles without extremum properties. Hence, it is the purpose of the present and also the succeeding sections to establish the necessary and sufficient conditions for the existence of complementary-dual extremum principles in the framework of the geometrically exact finite strain beam theory.

Following Noble and Sewell's approach [28, 41, 42], one can derive a pair of complementary-dual extremum principles from the complementary HRE given by (10), regarded as a generating saddle-shaped functional.

The saddle-point property of the complementary HRE requires that

$$\begin{aligned}\Pi_{HR}(\boldsymbol{\sigma}_2^r, \mathbf{d}_2) - \Pi_{HR}(\boldsymbol{\sigma}_1^r, \mathbf{d}_1) &\leq \int_0^L (\mathcal{T}_e^r(\mathbf{d}_1)\boldsymbol{\sigma}_1^r + \mathbf{q}) \cdot \delta\mathbf{d} \, d\Omega + \int_0^L \left(\frac{\partial W_c}{\partial \boldsymbol{\sigma}^r} \Big|_{\sigma_2^r} - \boldsymbol{\varepsilon}^r(\mathbf{d}_2) \right) \cdot \delta\boldsymbol{\sigma}^r \, d\Omega \\ &\quad + [(\bar{\mathbf{q}} - n\mathbf{H}(\mathbf{d}_1)\boldsymbol{\sigma}_1^r) \cdot \delta\mathbf{d}]_{\Gamma_N} + [(\mathbf{d}_2 - \bar{\mathbf{d}}) \cdot n\delta(\mathbf{H}(\mathbf{d}_1)\boldsymbol{\sigma}^r)]_{\Gamma_D}.\end{aligned}$$

Since $(\boldsymbol{\sigma}_1^r, \mathbf{d}_1)$ has been assumed as a solution of the problem, this inequality can be reduced to

$$\Pi_{HR}(\boldsymbol{\sigma}_2^r, \mathbf{d}_2) - \Pi_{HR}(\boldsymbol{\sigma}_1^r, \mathbf{d}_1) \leq \int_0^L \left(\frac{\partial W_c}{\partial \boldsymbol{\sigma}^r} \Big|_{\sigma_2^r} - \boldsymbol{\varepsilon}^r(\mathbf{d}_2) \right) \cdot \delta\boldsymbol{\sigma}^r \, d\Omega + [(\mathbf{d}_2 - \bar{\mathbf{d}}) \cdot n\delta(\mathbf{H}(\mathbf{d}_1)\boldsymbol{\sigma}^r)]_{\Gamma_D}. \quad (12)$$

Using Taylor's formula in the HRE defined at $(\boldsymbol{\sigma}_2^r, \mathbf{d}_2)$ gives

$$\begin{aligned}\Pi_{HR}(\boldsymbol{\sigma}_2^r, \mathbf{d}_2) &= \Pi_{HR}(\boldsymbol{\sigma}_1^r, \mathbf{d}_1) + \frac{1}{2} \int_0^L \delta\boldsymbol{\sigma}^r \frac{\partial^2 W_c(\boldsymbol{\sigma}^r)}{\partial \boldsymbol{\sigma}^r \partial \boldsymbol{\sigma}^r} \Big|_{\sigma_1^r} \cdot \delta\boldsymbol{\sigma}^r \, d\Omega - \\ &\quad \int_0^L \delta\boldsymbol{\sigma}^r \cdot \delta\boldsymbol{\varepsilon}^r(\mathbf{d}) \Big|_{d_1} \, d\Omega - \frac{1}{2} \int_0^L \boldsymbol{\sigma}^r \cdot \delta^2(\boldsymbol{\varepsilon}^r(\mathbf{d})) \Big|_{(\sigma_1^r, d_1)} \, d\Omega.\end{aligned}$$

Applying also Taylor's formula in the first term of the right-hand side term of inequality (12) yields

$$\frac{\partial W_c}{\partial \boldsymbol{\sigma}^r} \Big|_{\sigma_2^r} - \boldsymbol{\varepsilon}^r(\mathbf{d}_2) = \frac{\partial W_c}{\partial \boldsymbol{\sigma}^r} \Big|_{\sigma_1^r} + \frac{\partial^2 W_c}{\partial \boldsymbol{\sigma}^r \partial \boldsymbol{\sigma}^r} \Big|_{\sigma_1^r} \delta\boldsymbol{\sigma}^r - \boldsymbol{\varepsilon}^r(\mathbf{d}_1) - \frac{\partial \boldsymbol{\varepsilon}^r}{\partial \mathbf{d}} \Big|_{d_1} \delta\mathbf{d}.$$

Finally, on insertion of these results into (12), the following local saddle-point condition is obtained

$$\int_0^L \delta \sigma^r \frac{\partial^2 W_c(\sigma^r)}{\partial \sigma^r \partial \sigma^r} \Big|_{\sigma_1^r} \cdot \delta \sigma^r d\Omega + \int_0^L \sigma^r \cdot \delta^2(\boldsymbol{\varepsilon}^r(\mathbf{d})) \Big|_{(\sigma_1^r, \mathbf{d}_1)} d\Omega \geq 0, \quad \forall (\delta \sigma^r, \delta \mathbf{d}) \in \mathcal{H}^0(\Omega) \times \mathcal{H}^1(\Omega). \quad (13)$$

Under this saddle-point property, a pair of complementary-dual extremum principles can be obtained: one is the *principle of minimum TPE* whereas the other is the *principle of maximum TCE*.

The former of these principles states that, among all kinematically admissible solutions satisfying condition (13), the solution of the boundary-value problem minimizes the primal functional $P : \mathcal{U}_k(\Omega) \rightarrow \mathcal{R}$ given by

$$\begin{aligned} P &= -\Pi_{HR} + \int_{\Omega} \left(\frac{\partial W_c(\sigma^r)}{\partial \sigma^r} - \boldsymbol{\varepsilon}^r(\mathbf{d}) \right) \cdot \sigma^r d\Omega - [(\mathbf{d} - \bar{\mathbf{d}}) \cdot n\mathbf{H}(\mathbf{d})\sigma^r]_{\Gamma_D} \\ &= \Pi_p(\mathbf{d}). \end{aligned}$$

representing indeed the one-field TPE. Conversely, the latter states that, among all statically admissible solutions satisfying condition (13), the solution point of the boundary-value problem maximizes the dual functional $D : \mathcal{U}_s(\Omega) \rightarrow \mathcal{R}$ defined by

$$\begin{aligned} D &= -\Pi_{HR} + \int_{\Omega} (\mathcal{T}_e^r(\mathbf{d})\sigma^r + \mathbf{q}) \cdot \mathbf{d} d\Omega + [(\bar{\mathbf{q}} - n\mathbf{H}(\mathbf{d})\sigma^r) \cdot \mathbf{d}]_{\Gamma_N} \\ &= -\Pi_c(\sigma^r, \mathbf{d}). \end{aligned}$$

identified as the two-field TCE with a minus sign. Also under condition (13), the following relations hold at the solution point of the boundary-value problem

$$\Pi_p(\mathbf{d}_1) = -\Pi_c(\sigma_1^r, \mathbf{d}_1) = -\Pi_{HR}(\sigma_1^r, \mathbf{d}_1).$$

In other words, it can be stated that, under the local saddle-point condition given by (13), the following inequalities hold

$$\Pi_c(\sigma_s^r, \mathbf{d}_s) \geq \Pi_c(\sigma_1^r, \mathbf{d}_1) = -\Pi_p(\mathbf{d}_1) \geq -\Pi_p(\mathbf{d}_k), \quad \forall (\sigma_s^r, \mathbf{d}_s) \in \mathcal{U}_s \text{ and } \forall \mathbf{d}_k \in \mathcal{U}_k$$

which are indeed the necessary and sufficient conditions for the existence of global upper bounds of the exact errors in energy.

5. Application of Gao and Strang's Approach to the Derivation of Dual Extremum Principles

The purpose of this section is to establish the saddle-point condition of the complementary HRE given by (10), assumed to be specialized to a beam theory in which the compatibility operator is taken as follows

$$\boldsymbol{\varepsilon}^r(\mathbf{d}) = \mathcal{L}_c^r(\mathbf{d})\mathbf{d}$$

with \mathcal{L}_c^r a quadratic operator of vector \mathbf{d} . It will be shown that, under this assumption, two dual extremum principles can be formulated following Gao and Strang's approach [14].

In the following, $(\sigma_1^r, \mathbf{d}_1)$ denotes a solution of the boundary-value problem, whereas $(\sigma_2^r, \mathbf{d}_2)$ represents an infinitesimally adjacent solution.

Since a quadratic compatibility operator has been assumed, then, according to the general theory of convex analysis, the HRE can be regarded as a convex functional with respect to \mathbf{d} , *i.e.*, we have

$$\Pi_{HR}(\sigma_2^r, \mathbf{d}_1) \geq \Pi_{HR}(\sigma_1^r, \mathbf{d}_1), \quad \forall \sigma_2^r \in \mathcal{H}^0(\Omega). \quad (14)$$

However, as it will be shown next, the convexity of $\Pi_{HR}^1(\sigma^r, \mathbf{d})$ with respect to σ^r is not so obvious. It depends on the sign of the gap functional introduced in (9). To see this, let us introduce the first-order Gâteaux variation of $\varepsilon^r(\mathbf{d})$ defined at \mathbf{d}_1 in the direction of \mathbf{d}_2 as follows

$$\delta \varepsilon^r(\mathbf{d}_1; \mathbf{d}_2) = \mathcal{T}_c^r(\mathbf{d}_1) \mathbf{d}_2$$

where $\mathcal{T}_c^r(\mathbf{d}_1)$ is the Gâteaux derivative of ε^r at \mathbf{d}_1 . Let us also consider the following operator decomposition

$$\mathcal{L}_c^r(\mathbf{d}) = \mathcal{T}_c^r(\mathbf{d}) + \mathcal{N}_c^r(\mathbf{d})$$

where $\mathcal{N}_c^r(\mathbf{d})$ represents a complementary geometrical operator. Applying Taylor's formula and the lemma for quadratic operators [14] to the compatibility operator $\varepsilon^r(\mathbf{d})$ gives

$$\varepsilon^r(\mathbf{d}_2) = \varepsilon^r(\mathbf{d}_1) + \left. \frac{\partial \varepsilon^r}{\partial \mathbf{d}} \right|_{\mathbf{d}_1} \delta \mathbf{d} + \frac{1}{2} \delta \mathbf{d} \left. \frac{\partial^2 \varepsilon^r}{\partial \mathbf{d} \partial \mathbf{d}} \right|_{\mathbf{d}_1} \delta \mathbf{d} = \varepsilon^r(\mathbf{d}_1) + \mathcal{T}_c^r(\mathbf{d}_1) \delta \mathbf{d} - \mathcal{N}_c^r(\delta \mathbf{d}) \delta \mathbf{d}, \quad \forall \mathbf{d}_2 = \mathbf{d}_1 + \delta \mathbf{d} \in \mathcal{H}^1(\Omega). \quad (15)$$

Since the external part of the HRE, defined by

$$F(\mathbf{d}) = \int_{\Omega} \mathbf{q} \cdot \mathbf{d} \, d\Omega + [\bar{\mathbf{q}} \cdot \mathbf{d}]_{\Gamma_N}$$

is linear with respect to its argument, then, using Taylor's formula, it can be stated that

$$F(\mathbf{d}_2) = F(\mathbf{d}_1) + \int_{\Omega} \mathbf{q} \cdot \delta \mathbf{d} \, d\Omega + [\bar{\mathbf{q}} \cdot \delta \mathbf{d}]_{\Gamma_N}. \quad (16)$$

Accordingly, we obtain

$$\begin{aligned} \Pi_{HR}(\sigma_1^r, \mathbf{d}_2) - \Pi_{HR}(\sigma_1^r, \mathbf{d}_1) &= \int_{\Omega} [W_c(\sigma_1^r) - \sigma_1^r \cdot \varepsilon^r(\mathbf{d}_2)] \, d\Omega + F(\mathbf{d}_2) \\ &\quad - \int_{\Omega} [W_c(\sigma_1^r) - \sigma_1^r \cdot \varepsilon^r(\mathbf{d}_1)] \, d\Omega - F(\mathbf{d}_1). \end{aligned}$$

On insertion of (15) and (16) into the previous equation gives

$$\Pi_{HR}(\sigma_1^r, \mathbf{d}_2) - \Pi_{HR}(\sigma_1^r, \mathbf{d}_1) = \int_{\Omega} \sigma_1^r \cdot [\mathcal{N}_c^r(\delta \mathbf{d}) \delta \mathbf{d} - \mathcal{T}_c^r(\mathbf{d}_1) \delta \mathbf{d}] \, d\Omega + \int_{\Omega} \mathbf{q} \cdot \delta \mathbf{d} \, d\Omega + [\bar{\mathbf{q}} \cdot \delta \mathbf{d}]_{\Gamma_N}$$

which, after integrating by parts, leads to

$$\begin{aligned} \Pi_{HR}(\sigma_1^r, \mathbf{d}_2) - \Pi_{HR}(\sigma_1^r, \mathbf{d}_1) &= \int_{\Omega} [\mathcal{T}_e^r(\mathbf{d}_1) \sigma_1^r + \mathbf{q}] \cdot \delta \mathbf{d} \, d\Omega + \int_{\Omega} \sigma_1^r \cdot \mathcal{N}_c^r(\delta \mathbf{d}) \delta \mathbf{d} \, d\Omega \\ &\quad + [(\bar{\mathbf{q}} - n\mathbf{H}(\mathbf{d})\sigma^r) \cdot \delta \mathbf{d}]_{\Gamma_N} - [n\mathbf{H}(\mathbf{d})\sigma^r \cdot \delta \mathbf{d}]_{\Gamma_D}. \end{aligned}$$

Additionally, since $(\boldsymbol{\sigma}'_1, \mathbf{d}_1)$ is a solution of the boundary-value problem, it follows that

$$\Pi_{HR}(\boldsymbol{\sigma}'_1, \mathbf{d}_2) - \Pi_{HR}(\boldsymbol{\sigma}'_1, \mathbf{d}_1) = -\text{Gap}(\boldsymbol{\sigma}'_1, \delta \mathbf{d}) \quad (17)$$

with

$$\text{Gap}(\boldsymbol{\sigma}'_1, \delta \mathbf{d}) = - \int_{\Omega} \boldsymbol{\sigma}'_1 \cdot \mathcal{N}'_c(\delta \mathbf{d}) \delta \mathbf{d} \, d\Omega \quad (18)$$

representing the complementary gap functional. Since the compatibility operator $\boldsymbol{\varepsilon}'(\mathbf{d})$ has been assumed as a quadratic function of \mathbf{d} , the gap functional comes out also as a pure quadratic function of \mathbf{d} . Therefore, if $\text{Gap}(\boldsymbol{\sigma}'_1, \mathbf{d}_1) \geq 0$, we have also $\text{Gap}(\boldsymbol{\sigma}'_1, \mathbf{d}_2) \geq 0$. Hence, from (17) and (18) we can conclude that

$$\Pi_{HR}(\boldsymbol{\sigma}'_1, \mathbf{d}_2) \leq \Pi_{HR}(\boldsymbol{\sigma}'_1, \mathbf{d}_1), \text{ if } \text{Gap}(\boldsymbol{\sigma}'_1, \mathbf{d}_2) \geq 0, \quad \forall \mathbf{d}_2 \in \mathcal{H}^1(\Omega). \quad (19)$$

From this condition we can also state that

$$-\Pi_p(\mathbf{d}_2) \leq -\Pi_p(\mathbf{d}_1) \Leftrightarrow \Pi_p(\mathbf{d}_1) = \inf_{\mathbf{d}_2} \Pi_p(\mathbf{d}_2), \text{ if } \text{Gap}(\boldsymbol{\sigma}'_1, \mathbf{d}_2) \geq 0, \quad \forall \mathbf{d}_2 \in \mathcal{U}_k$$

which shows that, if the gap functional is positive then the system is stable. Note however that the equilibrium may be stable even when the gap functional is negative.

Combining results (14) and (19), it follows that $(\boldsymbol{\sigma}'_1, \mathbf{d}_1)$ is a saddle-point of Π_{HR} , *i.e.*

$$\Pi_{HR}(\boldsymbol{\sigma}'_2, \mathbf{d}_1) \geq \Pi_{HR}(\boldsymbol{\sigma}'_1, \mathbf{d}_1) \geq \Pi_{HR}(\boldsymbol{\sigma}'_1, \mathbf{d}_2), \quad \forall (\boldsymbol{\sigma}'_2, \mathbf{d}_2) \in \mathcal{H}^0(\Omega) \times \mathcal{H}^1(\Omega)$$

if $\text{Gap}(\boldsymbol{\sigma}'_1, \mathbf{d}_2) \geq 0$ holds for all $\mathbf{d}_2 \in \mathcal{H}^1(\Omega)$.

Thus, we can finally state that, if the gap functional is positive for any statically admissible stress-resultant $\boldsymbol{\sigma}'_s$, the following inequalities hold

$$\Pi_c(\boldsymbol{\sigma}'_s, \mathbf{d}_s) \geq \Pi_c(\boldsymbol{\sigma}'_1, \mathbf{d}_1) = -\Pi_p(\mathbf{d}_1) \geq -\Pi_p(\mathbf{d}_k), \quad \forall (\boldsymbol{\sigma}'_s, \mathbf{d}_s) \in \mathcal{U}_s \text{ and } \forall \mathbf{d}_k \in \mathcal{U}_k$$

which are indeed the necessary and sufficient conditions for the existence of global upper bounds of the exact errors in energy.

6. Conclusions

Two complementary-dual extremum principles, namely the principle of minimum TPE and the principle of maximum TCE, have been derived for the geometrically exact finite strain beam theory using two distinct approaches: one relying on the general results published by Noble and Sewell within the context of applied mathematics, and the other having its conceptual basis on the duality theory proposed by Gao and Strang for geometrically nonlinear solid mechanics problems. While the former is valid for the geometrically exact finite strain beam theory, which may involve arbitrarily large displacements, rotations and strains, the latter is restricted to geometrically nonlinear beam theories involving moderate large deflections. In both cases, the necessary and sufficient condition for the existence of the complementary-dual extremum principles have been shown to be the saddle-point property of the stationary HRE principle.

Under the saddle-point condition, upper bounds of the energy error can be directly obtained from the absolute value of the sum of the functionals at dual solution points, obtained for instance from numerical techniques relying on these variational principles. These upper bounds can be used to assess the quality of the approximate solutions provided by such numerical techniques. In the context of finite element formulations, as these bounds are expressed as a sum of elemental contributions, they can be used as error indicators for adaptive refinement procedures. Further research is necessary to characterize, from the viewpoint of error estimation and bounding, the case in which the saddle-point condition is not verified, as well as the use of these approaches to obtain bounds of the outputs for local quantities of interest, such as generalized displacements and stress-resultants. Extension of Gao-Strang's theory should also be carried out in the framework of the geometrically exact finite strain beam theory using the so-called sequential canonical dual transformation proposed by Gao.

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