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► **To cite this version:**

Léo Glangetas, Mohamed Najeme. Analytical regularizing effect for the radial and spatially homogeneous Boltzmann equation. *Kinetic and Related Models*, 2013, 10.3934/krm.2013.6.407. hal-00701857v3

HAL Id: hal-00701857

<https://hal.science/hal-00701857v3>

Submitted on 5 Jun 2012

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ANALYTICAL REGULARIZING EFFECT FOR THE RADIAL AND SPATIALLY HOMOGENEOUS BOLTZMANN EQUATION

LEO GLANGETAS AND MOHAMED NAJEME

ABSTRACT. In this paper, we consider a class of spatially homogeneous Boltzmann equation without angular cutoff. We prove that any radial symmetric weak solution of the Cauchy problem become analytic for positive time.

1. INTRODUCTION

This paper deals with the analytic regularity of the radially symmetric solutions of the following Cauchy problem for the spatially homogeneous Boltzmann equation :

$$(1.1) \quad \frac{\partial f}{\partial t} = Q(f, f), \quad v \in \mathbb{R}^3, t > 0; \quad f|_{t=0} = f_0,$$

where $f(t, v) : \mathbb{R}^+ \times \mathbb{R}^3 \rightarrow \mathbb{R}$ is the probability density of a gas, $v \in \mathbb{R}^3$ the velocity and $t \geq 0$ the time. The Boltzmann collision operator $Q(g, f)$ is a bi-linear functional given by

$$Q(f, g) = \int_{\mathbb{R}^3} \int_{S^2} B(v - v_*, \sigma) \{f(v'_*)g(v') - f(v_*)g(v)\} d\sigma dv_*,$$

where, for $\sigma \in S^2$,

$$v' = \frac{v + v_*}{2} + \frac{|v - v_*|}{2} \sigma, \quad v'_* = \frac{v + v_*}{2} - \frac{|v - v_*|}{2} \sigma.$$

Theses relations between the post and pre-collisional velocities follow from the conservation of momentum and kinetic energy. The non-negative function $B(z, \sigma)$ is called the Boltzmann collision kernel, depends only on $|z|$ and on the cosine of the deviation angle θ

$$\cos \theta = \left\langle \frac{v - v_*}{|v - v_*|}, \sigma \right\rangle$$

and is defined by

$$B(v - v_*, \cos \theta) = \Psi(|v - v_*|) b(\cos \theta), \quad 0 \leq \theta \leq \frac{\pi}{2}.$$

We will consider the Maxwellian case $\Psi \equiv 1$ and we suppose that the cross-section kernel b has a singularity at $\theta = 0$ (the so-called non-cutoff problem) and satisfies :

$$(1.2) \quad B(v - v_*, \cos \theta) = b(\cos \theta) \sim |\theta|^{-2-2s} \quad \text{when } \theta \rightarrow 0, \quad 0 < s < 1.$$

2010 *Mathematics Subject Classification.* 35A20, 35B65, 35D10, 35H20, 35Q20, 76P05, 82C40.

Key words and phrases. Boltzmann equation, Non-cutoff Kac's equation, smoothing effect of Cauchy problem, analytical regularizing, Gevrey regularizing.

We put $\langle v \rangle = (1 + v^2)^{\frac{1}{2}}$ for $v \in \mathbb{R}^n$ and we shall use the following standard weighted Sobolev spaces, for $k, \ell \in \mathbb{R}$, as

$$\begin{aligned} L_\ell^p(\mathbb{R}^n) &= \left\{ f ; \langle v \rangle^\ell f \in L^p(\mathbb{R}^n) \right\}, \\ H_\ell^k(\mathbb{R}^n) &= \left\{ f \in \mathcal{S}'(\mathbb{R}^n) ; \langle v \rangle^\ell f \in H^k(\mathbb{R}^n) \right\}, \\ L \log L(\mathbb{R}^n) &= \left\{ f \in \mathcal{S}'(\mathbb{R}^n) ; \|f\|_{L \log L} = \int_{\mathbb{R}^n} |f(v)| \log(1 + |f(v)|) dv < \infty \right\}. \end{aligned}$$

The Gevrey space is given for $\alpha > 0$ by:

$$G^{\frac{1}{\alpha}}(\mathbb{R}^n) = \left\{ f ; e^{c_0 \langle D \rangle^\alpha} \in L^2(\mathbb{R}^n) \right\},$$

where $\langle D \rangle = (1 + |D_v|^2)^{\frac{1}{2}}$. Remark that $G^1(\mathbb{R}^n)$ is the usual analytical functions space.

A solution of Boltzmann equation is known to satisfy the conservation of mass, kinetic energy and the entropy inequality:

$$\begin{aligned} \int_{\mathbb{R}^3} f(t, v) dv &= \int_{\mathbb{R}^3} f_0(v) dv, \\ \int_{\mathbb{R}^3} f(t, v) |v|^2 dv &= \int_{\mathbb{R}^3} f_0(v) |v|^2 dv, \\ \int_{\mathbb{R}^3} f(t, v) \log(f(t, v)) dv &\leq \int_{\mathbb{R}^3} f_0(v) \log(f_0(v)) dv. \end{aligned}$$

We say that a function $f(v)$ is spatially radially symmetric with respect to $v \in \mathbb{R}^3$ if for any rotation A in \mathbb{R}^3

$$f(v) = f(Av).$$

A lot of progress has been made on the study of the non cut-off problems. For the existence of weak solutions, see [17] and the references therein.

In [13], Lions proved that strong compactness is available at the level of renormalized solutions. Then Desvillettes proved in [6] that there is a regularizing effect in the case for radially symmetric solutions of the Cauchy problem for a 2D Boltzmann equation with Maxwellian molecules. And this is definitively different from the cutoff case, for which there is no smoothing effect. The Sobolev smoothing effect for solutions of the Cauchy problem was then studied in other works (see [1, 9, 2, 11, 15]).

Some gain of regularity is also obtained for a solution to the Cauchy problem of a modified 1D model of the Boltzmann equation involving a kinetic transport term (see [8]). For recent works on the non-homogeneous Boltzmann equation, see [3, 4, 5].

In [16], Ukai showed that the Cauchy problem for the Boltzmann equation has a unique local solution in Gevrey classes. Then Desvillettes, Furioli and Terraneo proved in [7] the propagation of Gevrey regularity for solutions of Boltzmann equation for Maxwellian molecules. For the non-Maxwellian case, Morimoto and Ukai considered in [14] the Gevrey regularity of C^∞ solutions in the case with a modified kinetic factor $\Psi(|v - v_*|) = (1 + |v - v_*|^2)^{\frac{\gamma}{2}}$ and recently Zhang and Yin in [18] the case with the general kinetic factor $\Psi(|v - v_*|) = |v - v_*|^\gamma$. In [15], it was proved that the solutions of the linearized Cauchy problem are in the Gevrey space $G^{\frac{1}{s}}(\mathbb{R}^3)$ for any $0 < s < 1$.

Recently, Lekrine and Xu have proved in [12] that, in the case $0 < s < \frac{1}{2}$, any symmetric weak solution of the Boltzmann equation belongs to the Gevrey space $G^{\frac{1}{2s'}}(\mathbb{R}^3)$ for any $0 < s' < s$ and time $t > 0$.

In this work, we consider the case $\frac{1}{2} \leq s < 1$ and we get the following result.

Theorem 1.1. *Assume that the cross-section kernel B satisfies (1.2) with $\frac{1}{2} < s < 1$ and the initial datum $f_0 \in L^1_{2+2s} \cap L \log L(\mathbb{R}^3)$, $f_0 \geq 0$ is radially symmetric. If f is a nonnegative radially symmetric weak solution of the Cauchy problem for the Boltzmann equation (1.1) such that $f \in L^\infty([0, \infty[; L^1_{2+2s} \cap L \log L(\mathbb{R}^3))$, then $f(t, \cdot) \in G^1(\mathbb{R}^3)$ for any $t > 0$.*

However, for $s = \frac{1}{2}$, we have $f(t, \cdot) \in G^{1/\alpha}(\mathbb{R}^3)$ for any $0 < \alpha < 1$ and $t > 0$.

It is well-known that the study of radially symmetric solutions of the Boltzmann equation can be reduced to the study of the solutions of the following Kac equation (see [6] and also section 5)

$$(1.3) \quad \begin{cases} \frac{\partial f}{\partial t} = K(f, f), \\ f|_{t=0} = f_0, \end{cases}$$

where $f = f(t, v)$ is the density distribution function with velocity $v \in \mathbb{R}$ and the Kac's bilinear collisional operator K is given by

$$K(f, g) = \int_{\mathbb{R}} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \beta(\theta) \{f(v'_*)g(v') - f(v_*)g(v)\} d\theta dv_*,$$

where

$$v' = v \cos \theta - v_* \sin \theta, \quad v'_* = v \sin \theta + v_* \cos \theta.$$

The non-negative cross-section β satisfies

$$(1.4) \quad \beta(\theta) = b_0 \frac{|\cos \theta|}{|\sin \theta|^{1+2s}} \quad \text{when } \theta \rightarrow 0$$

for $0 < s < 1$ and $b_0 > 0$. Remark that

$$(1.5) \quad \int_{-\pi/2}^{\pi/2} \beta(\theta) |\theta|^2 d\theta < \infty.$$

There is also conservation of the mass, the kinetic energy and the entropy inequality for the solutions of the Kac's equation. We will prove the following result:

Theorem 1.2. *Assume that the cross-section kernel β satisfies (1.4) with $\frac{1}{2} < s < 1$, the initial datum $f_0 \in L^1_{2+2s} \cap L \log L(\mathbb{R})$. For $T_0 > 0$, if $f \in L^\infty([0, \infty[; L^1_{2+2s} \cap L \log L(\mathbb{R}))$ is a nonnegative weak solution of the Cauchy problem of the Kac's equation (1.3), then $f(t, \cdot) \in G^1(\mathbb{R})$ for any $t > 0$.*

However, for $s = \frac{1}{2}$, we have $f(t, \cdot) \in G^{1/\alpha}(\mathbb{R})$ for any $0 < \alpha < 1$ and $t > 0$.

Same as in the paper of [12], the Theorem 1.1 is a direct consequence of the Theorem 1.2. We are reduced to study the Cauchy problem for spatially homogeneous Kac's equation.

This paper is organized as follows: In the next section, we prove some estimates which will be used in section 4. In section 3, we study the regularity in weighted Sobolev spaces for the weak solutions of the Cauchy problem of the Kac's equation. The section 4 is devoted to the proof of the Theorem 1.2 and in section 5 we conclude the proof the Theorem 1.1.

2. ESTIMATES OF THE COMMUTATORS

In this section, we will get the estimates of some terms that we call ‘‘commutators’’ and we will see in section 4 that they are the main point to get the regularity of weak solutions for the Cauchy problem of the Kac's equation. We recall the following coercivity inequality deduced from the non cut-off of collision kernel.

Proposition 2.1. (see [1]) Assume that the cross-section and satisfies the assumption (1.4). Let $f \geq 0$, $f \neq 0$, $f \in L_1^1(\mathbb{R}) \cap L \log L(\mathbb{R})$, then there exists a constant $c_f > 0$, depending only of, β , $\|f\|_{L_1^1}$ and $\|f\|_{L \log L}$, such that

$$-(K(f, g), g)_{L^2} \geq c_f \|g\|_{H^s}^2 - C \|f\|_{L^1} \|g\|_{L^2}^2$$

for any smooth function $g \in H^s(\mathbb{R})$.

Remark. From [11, 15], if $m, \ell \in \mathbb{R}$, $0 < s < 1$ and f and g are suitable functions, the Kac collision kernel has the following regularity ($\ell^+ = \max(0, \ell)$)

$$\|K(f, g)\|_{H_\ell^m} \leq C \|f\|_{L_{\ell^+ + 2s}^1} \|g\|_{H_{(\ell^+ + 2s)^+}^{m+2s}}.$$

As in [15], we introduce the following mollifier

$$(2.1) \quad G_\delta(t, \xi) = \frac{e^{c_0 t \langle \xi \rangle^\alpha}}{1 + \delta e^{c_0 t \langle \xi \rangle^\alpha}}$$

where $\langle \xi \rangle = (1 + |\xi|^2)^{\frac{1}{2}}$, $\xi \in \mathbb{R}$, $c_0 > 0$ and $0 < \delta < 1$ will be chosen small enough and $\alpha \in]0, 2[$ are fixed. It is easy to check that, for any $0 < \delta < 1$,

$$G_\delta(t, \xi) \in L^\infty(]0, T[\times \mathbb{R}).$$

We denote by \hat{f} the Fourier transform of f

$$\hat{f}(\xi) = \mathcal{F}(f)(\xi) = \int_{\mathbb{R}} e^{-iv\xi} f(v) dv$$

and by $G_\delta(t, D_v)$ the Fourier multiplier of symbol $G_\delta(t, \xi)$ (see [10])

$$G_\delta g(t, v) = G_\delta(t, D_v)g(t, v) = \mathcal{F}^{-1}(G_\delta(t, \cdot) \hat{g}(t, \cdot))(v).$$

The proof of Theorem 1.2 will be based on the uniform estimate with respect to $0 < \delta < 1$ of $\|G_\delta(t, D_v)f(t, \cdot)\|_{L_1^2}$ where $f(t, \cdot)$ is a weak solution of the Cauchy problem of the Kac's equation (1.3).

In the following, C will represent a generic constant independent of δ and $t \in [0, T]$ (but it will depend on the kernel β and the norms $\|f(t, \cdot)\|_{L_1^1}$, $\|f(t, \cdot)\|_{L \log L}$ used for the coercivity).

Lemma 2.2. Let $T > 0$. We have that for any $0 < \delta < 1$ and $0 \leq t \leq T$, $\xi \in \mathbb{R}$,

$$\begin{aligned} |\partial_t G_\delta(t, \xi)| &\leq c_0 \langle \xi \rangle^\alpha G_\delta(t, \xi), \\ |\partial_\xi G_\delta(t, \xi)| &\leq \alpha c_0 t \langle \xi \rangle^{\alpha-1} G_\delta(t, \xi), \\ |\partial_\xi^2 G_\delta(t, \xi)| &\leq C \langle \xi \rangle^{2\alpha-2} G_\delta(t, \xi) \end{aligned}$$

with $C > 0$ independent of δ and $t \in [0, T]$.

Proof. We compute

$$\begin{aligned} \partial_t G_\delta(t, \xi) &= c_0 \frac{\langle \xi \rangle^\alpha}{1 + \delta e^{c_0 t \langle \xi \rangle^\alpha}} G_\delta(t, \xi), \\ \partial_\xi G_\delta(t, \xi) &= \alpha c_0 t \xi (1 + |\xi|^2)^{\frac{\alpha}{2}-1} G_\delta(t, \xi) \frac{1}{1 + \delta e^{c_0 t \langle \xi \rangle^\alpha}}, \\ \partial_\xi^2 G_\delta(t, \xi) &= \left(\alpha c_0 t \xi (1 + |\xi|^2)^{\frac{\alpha}{2}-1} \right)^2 G_\delta(t, \xi) \frac{1 - \delta e^{c_0 t \langle \xi \rangle^\alpha}}{(1 + \delta e^{c_0 t \langle \xi \rangle^\alpha})^2} \\ &\quad + \alpha c_0 t \left((1 + |\xi|^2)^{\frac{\alpha}{2}-1} + (\alpha - 2) \xi^2 (1 + |\xi|^2)^{\frac{\alpha}{2}-2} \right) G_\delta(t, \xi) \frac{1}{1 + \delta e^{c_0 t \langle \xi \rangle^\alpha}}, \end{aligned}$$

and the estimates of the lemma follow easily. \square

Lemma 2.3. *There exists $C > 0$ such that for all $0 < \delta < 1$ and $\xi \in \mathbb{R}$*

$$\begin{aligned} |G_\delta(t, \xi) - G_\delta(t, \xi \cos \theta)| &\leq C \sin^2 \frac{\theta}{2} \langle \xi \rangle^\alpha G_\delta(t, \xi \cos \theta) G_\delta(t, \xi \sin \theta), \\ |(\partial_\xi G_\delta)(t, \xi) - (\partial_\xi G_\delta)(t, \xi \cos \theta)| &\leq C \sin^2 \frac{\theta}{2} \langle \xi \rangle^{2\alpha-1} G_\delta(t, \xi \cos \theta) G_\delta(t, \xi \sin \theta). \end{aligned}$$

Proof. This lemma 2.3 is proved by Taylor formula, the estimates from lemma 2.2 and the following inequality :

$$(2.2) \quad G_\delta(t, \xi) \leq 3G_\delta(t, \xi \cos \theta) G_\delta(t, \xi \sin \theta). \quad \square$$

We now estimate the commutator of the Kac's operator with the mollifier:

Proposition 2.4. *Assume that $0 < \alpha < 2$. Let $f, g \in L_1^2$ and $h \in H^{\alpha/2}(\mathbb{R})$, then we have*

$$|(G_\delta K(f, g), h)_{L^2} - (K(f, G_\delta g), h)_{L^2}| \leq C \|G_\delta f\|_{L_1^2} \|G_\delta g\|_{H^{\alpha/2}} \|h\|_{H^{\alpha/2}}.$$

Proof. By definition, of G_δ we have for a regular f ,

$$\mathcal{F}(G_\delta f)(\xi) = G_\delta \hat{f}(\xi),$$

and

$$\mathcal{F}(vG_\delta f)(\xi) = i\partial_\xi (G_\delta(t, \xi) \hat{f}(t, \xi)).$$

We recall the Bobilev formula

$$(2.3) \quad \mathcal{F}(K(f, g))(\xi) = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \beta(\theta) \{ \hat{f}(\xi \sin \theta) \hat{g}(\xi \cos \theta) - \hat{f}(0) \hat{g}(\xi) \} d\theta.$$

From the Bobilev and Plancherel formulas

$$\begin{aligned} &(G_\delta K(f, g), h)_{L^2} - (K(f, G_\delta g), h)_{L^2} \\ &= \int_{\mathbb{R}_\xi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \beta(\theta) G_\delta(t, \xi) \{ \hat{f}(\xi \sin \theta) \hat{g}(\xi \cos \theta) - \hat{f}(0) \hat{g}(\xi) \} d\theta \overline{h} d\xi \\ &- \int_{\mathbb{R}_\xi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \beta(\theta) \{ \hat{f}(\xi \sin \theta) \mathcal{F}(G_\delta g)(\xi \cos \theta) - \hat{f}(0) \mathcal{F}(G_\delta g)(\xi) \} \overline{\hat{h}(\xi)} d\theta d\xi \\ &= \int_{\mathbb{R}_\xi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \beta(\theta) \hat{f}(\xi \sin \theta) \{ G_\delta(\xi) - G_\delta(\xi \cos \theta) \} \hat{g}(\xi \cos \theta) \overline{\hat{h}(\xi)} d\theta d\xi. \end{aligned}$$

By the previous formula, lemma 2.3 and the Cauchy-Schwarz inequality we have

$$\begin{aligned} &\left| (G_\delta K(f, g), h)_{L^2} - (K(f, G_\delta g), h)_{L^2} \right| \\ &\leq \int_{\mathbb{R}_\xi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \beta(\theta) \sin^2 \frac{\theta}{2} |G_\delta(\xi \sin \theta) \hat{f}(\xi \sin \theta)| \\ &\quad \times |G_\delta(\xi \cos \theta) \hat{g}(\xi \cos \theta)| \langle \xi \rangle^\alpha |\hat{h}(\xi)| d\theta d\xi \\ &\leq C \|G_\delta \hat{f}\|_{L^\infty} \|G_\delta g\|_{H^{\alpha/2}} \|h\|_{H^{\alpha/2}} \\ &\leq C \|G_\delta f\|_{L_1^2} \|G_\delta g\|_{H^{\alpha/2}} \|h\|_{H^{\alpha/2}} \end{aligned}$$

where we have used the following continuous embedding

$$L_1^2(\mathbb{R}) \subset L^1(\mathbb{R})$$

and the assumption (1.5) on the kernel β . \square

We again estimate the commutator of the Kac's operator with the mollifier weighted as in [12]. We will need to use a property of symmetry for the Kac's operator.

Proposition 2.5. *Assume that $\frac{1}{2} < s < 1$ and let $f, g \in L_2^1(\mathbb{R})$ and $h \in H^{\frac{1}{2}}(\mathbb{R})$. Then we have*

$$\begin{aligned} & |((vG_\delta)K(f, g), h)_{L^2} - (K(f, (vG_\delta)g), h)_{L^2}| \\ & \leq C \left(\|f\|_{L_2^1} + C \|G_\delta f\|_{L_1^2} \right) \|G_\delta g\|_{H^{\frac{1}{2}}} \|h\|_{H^{\frac{1}{2}}}. \end{aligned}$$

Remark. For $s = \frac{1}{2}$, the previous estimate is not enough accurate. In order to use some interpolation argument, we will need the following estimate.

Proposition 2.6. *Assume that $s = \frac{1}{2}$ and let $0 < \alpha, \alpha' < 1$, $f, g \in L_2^1(\mathbb{R})$, and $h \in H^{\frac{\alpha}{2}}(\mathbb{R})$. Then we have*

$$\begin{aligned} & |((vG_\delta)K(f, g), h)_{L^2} - (K(f, (vG_\delta)g), h)_{L^2}| \\ & \leq C \|f\|_{L_{1+\alpha'}^1} \|G_\delta g\|_{H^{\frac{\alpha'}{2}}} \|h\|_{H^{\frac{\alpha'}{2}}} + C \|G_\delta f\|_{L_1^2} \|G_\delta g\|_{H^{\frac{\alpha}{2}}} \|h\|_{H^{\frac{\alpha}{2}}}. \end{aligned}$$

We will prove these Propositions by using the Bobylev formula (2.3) and the Plancherel formula. We can write

$$((vG_\delta)K(f, g), h)_{L^2} - (K(f, (vG_\delta)g), h)_{L^2} = i \int_{\mathbb{R}_\xi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \beta(\theta) A(\xi, \theta) \overline{\hat{h}(\xi)} d\theta d\xi$$

where

$$A(\xi, \theta) = \partial_\xi \{ \hat{f}(\xi \sin \theta) G_\delta(\xi) \hat{g}(\xi \cos \theta) \} - \hat{f}(\xi \sin \theta) \partial_\xi \{ G_\delta \hat{g} \}(\xi \cos \theta).$$

We decompose $A = A_1 + A_2 + A_3$ where

$$\begin{aligned} A_1 &= \sin \theta (\partial_\xi \hat{f})(\xi \sin \theta) G_\delta(\xi) \hat{g}(\xi \cos \theta), \\ A_2 &= \hat{f}(\xi \sin \theta) \{ G_\delta(\xi) \cos \theta - G_\delta(\xi \cos \theta) \} (\partial_\xi \hat{g})(\xi \cos \theta), \\ A_3 &= \hat{f}(\xi \sin \theta) \{ \partial_\xi G_\delta(\xi) - (\partial_\xi G_\delta)(\xi \cos \theta) \} \hat{g}(\xi \cos \theta), \end{aligned}$$

and we put for $k = 1, 2, 3$

$$I_k = i \int_{\mathbb{R}_\xi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \beta(\theta) A_k(\xi, \theta) \overline{\hat{h}(\xi)} d\theta d\xi.$$

Therefore we have

$$(2.4) \quad |((vG_\delta)K(f, g), h)_{L^2} - (K(f, (vG_\delta)g), h)_{L^2}| \leq |I_1| + |I_2| + |I_3|.$$

In the following, we will estimate the three terms I_1 , I_2 and I_3 .

Estimate of I_1 . We decompose $I_1 = I_{1a} + I_{1b}$ where

$$\begin{aligned} I_{1a} &= i \int_{\mathbb{R}_\xi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \beta(\theta) \sin \theta (\partial_\xi \hat{f})(\xi \sin \theta) G_\delta(\xi \cos \theta) \hat{g}(\xi \cos \theta) \overline{\hat{h}(\xi)} d\theta d\xi, \\ I_{1b} &= i \int_{\mathbb{R}_\xi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \beta(\theta) \sin \theta (\partial_\xi \hat{f})(\xi \sin \theta) (G_\delta(\xi) - G_\delta(\xi \cos \theta)) \hat{g}(\xi \cos \theta) \overline{\hat{h}(\xi)} d\theta d\xi. \end{aligned}$$

Lemma 2.7. *Suppose that $\frac{1}{2} < s < 1$. Then there exists a constant C such that*

$$|I_{1a}| \leq C \|f\|_{L_2^1} \|G_\delta g\|_{H^{\frac{1}{2}}} \|h\|_{H^{\frac{1}{2}}}.$$

Proof. We use some symmetry property of the Kac's equation. We write the first term $I_{1a} = \frac{1}{2}I_{1a} + \frac{1}{2}I_{1a}$ and we use the change of variables $\theta \rightarrow -\theta$. We then have

$$(2.5) \quad I_{1a} = \int_{\mathbb{R}_\xi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \beta(\theta) \sin \theta \tilde{A}(\xi, \theta) G_\delta(\xi \cos \theta) \hat{g}(\xi \cos \theta) \overline{\hat{h}(\xi)} d\theta d\xi$$

where

$$\tilde{A}(\xi, \theta) = \frac{1}{2} (\partial_\xi \hat{f}(\xi \sin \theta) - \partial_\xi \hat{f}(-\xi \sin \theta)).$$

We compute

$$\tilde{A}(\xi, \theta) = \int_{\mathbb{R}} v \sin(\xi v \sin \theta) f(v) dv$$

and we estimate

$$|\tilde{A}(\xi, \theta)| \leq |\xi| |\sin \theta| \|f\|_{L^1_2} \leq \langle \xi \rangle |\sin \theta| \|f\|_{L^1_2}.$$

Finally we obtain

$$|I_{1a}| \leq C \|f\|_{L^1_2} \|G_\delta g\|_{H^{\frac{1}{2}}} \|h\|_{H^{\frac{1}{2}}}. \quad \square$$

Lemma 2.8. *Suppose that $s = \frac{1}{2}$. Then for any $0 < \alpha' < 1$, there exists a constant C such that*

$$|I_{1a}| \leq C \|f\|_{L^1_{1+\alpha'}} \|G_\delta g\|_{H^{\frac{\alpha'}{2}}} \|h\|_{H^{\frac{\alpha'}{2}}}.$$

Proof. Following the proof of the previous lemma, we consider again the identity (2.5) where

$$\tilde{A}(\xi, \theta) = \int_{\mathbb{R}} v \sin(\xi v \sin \theta) f(v) dv.$$

We then estimate

$$|\tilde{A}(\xi, \theta)| \leq |\xi|^{\alpha'} |\sin \theta|^{\alpha'} \|f\|_{L^1_{1+\alpha'}} \leq \langle \xi \rangle^{\alpha'} |\sin \theta|^{\alpha'} \|f\|_{L^1_{1+\alpha'}}.$$

Finally we obtain

$$|I_{1a}| \leq C \|f\|_{L^1_{1+\alpha'}} \|G_\delta g\|_{H^{\frac{\alpha'}{2}}} \|h\|_{H^{\frac{\alpha'}{2}}}. \quad \square$$

Lemma 2.9. *There exists a constant C such that*

$$|I_{1b}| \leq C \left(\|G_\delta f\|_{L^2_1} + \|G_\delta f\|_{H^{(\alpha-1)^+}} \right) \|G_\delta g\|_{H^{\frac{\alpha}{2}}} \|h\|_{H^{\frac{\alpha}{2}}}.$$

Proof. We estimate

$$I_{1b} = i \int_{\mathbb{R}_\xi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \beta(\theta) \sin \theta (\partial_\xi \hat{f})(\xi \sin \theta) (G_\delta(\xi) - G_\delta(\xi \cos \theta)) \hat{g}(\xi \cos \theta) \overline{\hat{h}(\xi)} d\theta d\xi.$$

By using lemma 2.3,

$$\begin{aligned} |I_{1b}| &\leq \int_{\mathbb{R}_\xi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \beta(\theta) \sin^2 \frac{\theta}{2} |\sin \theta| G_\delta(\xi \sin \theta) |(\partial_\xi \hat{f})(\xi \sin \theta)| \\ &\quad \langle \xi \rangle^{\frac{\alpha}{2}} G_\delta(\xi \cos \theta) |\hat{g}(\xi \cos \theta)| \langle \xi \rangle^{\frac{\alpha}{2}} \overline{\hat{h}(\xi)} d\theta d\xi. \end{aligned}$$

From

$$\|\langle \cdot \rangle^{\frac{\alpha}{2}} G_\delta \hat{g}\|_{L^\infty} \leq \|G_\delta g\|_{H^{\frac{\alpha}{2}}}$$

and the Cauchy-Schwarz inequality, we get

$$\begin{aligned}
& \int_{\mathbb{R}_\xi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \beta(\theta) \sin^2 \frac{\theta}{2} |\sin \theta| G_\delta(\xi \sin \theta) |(\partial_\xi \hat{f})(\xi \sin \theta)| \langle \xi \rangle^{\frac{\alpha}{2}} \overline{\hat{h}(\xi)} d\theta d\xi \\
& \leq \left(\int_{\mathbb{R}_\xi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \beta(\theta) \sin^2 \frac{\theta}{2} |\sin \theta| G_\delta(\xi \sin \theta)^2 |(\partial_\xi \hat{f})(\xi \sin \theta)|^2 d\theta d\xi \right)^{1/2} \\
& \quad \left(\int_{\mathbb{R}_\xi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \beta(\theta) \sin^2 \frac{\theta}{2} |\sin \theta| \langle \xi \rangle^{\frac{\alpha}{2}} |\hat{h}(\xi)|^2 d\theta d\xi \right)^{1/2} \\
& \leq \|G_\delta \partial_\xi \hat{f}\|_{L^2} \times \|\langle \cdot \rangle^{\frac{\alpha}{2}} \hat{h}\|_{L^2}.
\end{aligned}$$

We then observe that from lemma 2.2

$$\begin{aligned}
\|G_\delta(\partial_\xi \hat{f})\|_{L^2} & \leq \|\partial_\xi(G_\delta \hat{f})\|_{L^2} + \|(\partial_\xi G_\delta)\hat{f}\|_{L^2} \\
& \leq C \|G_\delta f\|_{L_1^2} \|\langle \cdot \rangle^{\alpha-1} \hat{f}\|_{L^2}
\end{aligned}$$

and we conclude

$$|I_{1b}| \leq C \left(\|G_\delta f\|_{L_1^2} + \|G_\delta f\|_{H^{(\alpha-1)+}} \right) \|G_\delta g\|_{H_1^{\frac{\alpha}{2}}} \|h\|_{H^{\frac{\alpha}{2}}}.$$

□

Estimate of I_2 . We decompose $I_2 = I_{2a} + I_{2b}$ where

$$\begin{aligned}
I_{2a} & = i \int_{\mathbb{R}_\xi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \beta(\theta) \hat{f}(\xi \sin \theta) (\cos \theta - 1) G_\delta(\xi) (\partial_\xi \hat{g})(\xi \cos \theta) \overline{\hat{h}(\xi)} d\theta d\xi, \\
I_{2b} & = i \int_{\mathbb{R}_\xi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \beta(\theta) \hat{f}(\xi \sin \theta) (G_\delta(\xi) - G_\delta(\xi \cos \theta)) (\partial_\xi \hat{g})(\xi \cos \theta) \overline{\hat{h}(\xi)} d\theta d\xi.
\end{aligned}$$

Lemma 2.10. *There exists a constant C such that*

$$|I_{2a}| \leq C \|G_\delta f\|_{L_1^2} \|G_\delta g\|_{H^{(\alpha-1)+}} \|h\|_{L^2}.$$

Proof. For I_{2a} we use (2.2) and $\cos \theta - 1 = -2 \sin^2 \frac{\theta}{2}$:

$$\begin{aligned}
I_{2a} & \leq C \|G_\delta f\|_{L_1^2} \|G_\delta(vg)\|_{L^2} \|h\|_{L^2} \\
& \leq C \|G_\delta f\|_{L_1^2} (\|G_\delta g\|_{L^2} + \|G_\delta g\|_{H^{(\alpha-1)+}}) \|h\|_{L^2}.
\end{aligned}$$

□

Lemma 2.11. *There exists a constant C such that*

$$|I_{2b}| \leq C \|G_\delta f\|_{L_1^2} \left(\|G_\delta g\|_{H_1^{\frac{\alpha}{2}}} + \|G_\delta g\|_{H^{(\frac{3\alpha}{2}-1)+}} \right) \|h\|_{H^{\frac{\alpha}{2}}}.$$

Proof. Using lemma 2.3 we get

$$I_{2b} \leq C \|G_\delta f\|_{L^\infty} \|\langle \cdot \rangle^{\frac{\alpha}{2}} G_\delta(\partial_\xi \hat{g})\|_{L^2} \|\langle \cdot \rangle^{\frac{\alpha}{2}} \hat{h}\|_{L^2}$$

and

$$\|\langle \cdot \rangle^{\frac{\alpha}{2}} G_\delta(\partial_\xi \hat{g})\|_{L^2} \leq \|\langle \cdot \rangle^{\frac{\alpha}{2}} \partial_\xi(G_\delta \hat{g})\|_{L^2} + \|\langle \cdot \rangle^{\frac{\alpha}{2}} (\partial_\xi G_\delta) \hat{g}\|_{L^2}.$$

□

Estimate of I_3 We recall

$$I_3 = i \int_{\mathbb{R}_\xi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \beta(\theta) \hat{f}(\xi \sin \theta) \{ \partial_\xi G_\delta(\xi) - (\partial_\xi G_\delta)(\xi \cos \theta) \} \hat{g}(\xi \cos \theta) \overline{\hat{h}(\xi)} d\theta d\xi.$$

Lemma 2.12. *There exists a constant C such that*

$$I_3 \leq C \|G_\delta f\|_{L_1^2} \|G_\delta \hat{g}\|_{H^{(\alpha-\frac{1}{2})}} \|h\|_{H^{(\alpha-\frac{1}{2})}}.$$

Proof.

$$I_3 \leq C \|G_\delta \hat{f}\|_{L^\infty} \|\langle \cdot \rangle^{(\alpha-\frac{1}{2})} G_\delta \hat{g}\|_{L^2} \|\langle \cdot \rangle^{(\alpha-\frac{1}{2})} \hat{h}\|_{L^2}.$$

□

Proof of Proposition 2.5. We use the previous lemmas 2.7 and 2.9-2.12. By summing the above estimates, we deduce from (2.4)

$$\begin{aligned} & |((vG_\delta)K(f, g), h)_{L^2} - (K(f, (vG_\delta)g), h)_{L^2}| \\ & \leq C \|f\|_{L_1^2} \|G_\delta g\|_{H^{\frac{1}{2}}} \|h\|_{H^{\frac{1}{2}}} + C \|G_\delta f\|_{H_1^{(\alpha-1)+}} \|G_\delta g\|_{H^{\frac{\alpha}{2}}} \|h\|_{H^{\frac{\alpha}{2}}} \\ & + C \|G_\delta f\|_{L_1^2} \|G_\delta g\|_{H_1^{(\frac{3\alpha}{2}-1)+}} \|h\|_{H^{\frac{\alpha}{2}}} + C \|G_\delta f\|_{L_1^2} \|G_\delta g\|_{H^{(\alpha-\frac{1}{2})+}} \|h\|_{H^{(\alpha-\frac{1}{2})+}}. \end{aligned}$$

Taking $\alpha = 1$, this finishes the proof of Proposition 2.5. □

Proof of Proposition 2.6. We recall $s = \frac{1}{2}$. We have from (2.4)

$$|((vG_\delta)K(f, g), h)_{L^2} - (K(f, (vG_\delta)g), h)_{L^2}| \leq |I_{1a}| + |I_{1b}| + |I_{2a}| + |I_{2b}| + |I_3|.$$

We use the lemma 2.8 and the lemmas 2.9-2.12 taking $0 < \alpha < 1$, and this concludes the proof. □

We now estimate some scalar product terms which involve the derivative of the mollifier with respect to time:

Lemma 2.13. *There exists $C > 0$ such that*

$$(2.6) \quad |((\partial_t G_\delta)(t, D_v)f(t, \cdot), G_\delta(t, D_v)f(t, \cdot))_{L^2}| \leq C \|G_\delta f\|_{H^{\alpha/2}}^2,$$

and

$$(2.7) \quad |(v(\partial_t G_\delta)(t, D_v)f(t, \cdot), vG_\delta(t, D_v)f(t, \cdot))_{L^2}| \leq C \left(\|G_\delta f\|_{H_1^{\alpha/2}}^2 + \|G_\delta f\|_{H_1^{\alpha-\frac{1}{2}}}^2 \right).$$

Proof. We have by the Plancherel formula

$$((\partial_t G_\delta)(t, D_v)f(t, \cdot), G_\delta(t, D_v)f(t, \cdot))_{L^2} = \int (\partial_t G_\delta) \hat{f} \overline{G_\delta \hat{f}} d\xi.$$

The estimate (2.6) can be deduced directly from lemma 2.2. For (2.7), we compute

$$\begin{aligned} & (v(\partial_t G_\delta)(t, D_v)f(t, \cdot), vG_\delta(t, D_v)f(t, \cdot))_{L^2} \\ & = \int \left\{ \partial_\xi \left(\frac{c_0 \langle \xi \rangle^\alpha}{1 + \delta e^{c_0 t \langle \xi \rangle^\alpha}} \right) (G_\delta \hat{f}) + \left(\frac{c_0 \langle \xi \rangle^\alpha}{1 + \delta e^{c_0 t \langle \xi \rangle^\alpha}} \right) \partial_\xi (G_\delta \hat{f}) \right\} \overline{\partial_\xi (G_\delta \hat{f})} d\xi \end{aligned}$$

and we use the following estimate

$$\left| \partial_\xi \left(\frac{\langle \xi \rangle^\alpha}{1 + \delta e^{c_0 t \langle \xi \rangle^\alpha}} \right) \right| \leq C \langle \xi \rangle^{2\alpha-1}.$$

□

3. SOBOLEV REGULARIZING EFFECT FOR KAC'S EQUATION

In this section, we prove the regularity in weighted Sobolev spaces of the weak solutions for the Cauchy problem of the Kac's equation.

Theorem 3.1. *Assume that the initial datum $f_0 \in L^1_{2+2s} \cap L \log L(\mathbb{R})$, and the cross-section weak β satisfies (1.4) with $\frac{1}{2} \leq s < 1$. If $f \in L^\infty([0, +\infty[; L^1_{2+2s} \cap L \log L(\mathbb{R}))$ is a nonnegative weak solution of the Cauchy problem (1.3), then $f(t, \cdot) \in H_2^{+\infty}(\mathbb{R})$ for any $t > 0$.*

Remark. This Theorem has been proved in [12] in the case $0 < s < \frac{1}{2}$.

We also obtain the following propagation of Sobolev regularity:

Corollary 3.2. *Under the assumptions of Theorem 3.1, for any $T_0 > 0$, there exists a constant C which depends only on β and $\|f\|_{L^\infty([0, +\infty[; L^1_{2+2s} \cap L \log L(\mathbb{R}))}$ such that*

$$\forall t \geq T_0, \quad \|f(t, \cdot)\|_{H_2^2} \leq e^{C(t-T_0)} \|f(T_0, \cdot)\|_{H_2^2}.$$

Throughout this section, we will distinguish the case $\frac{1}{2} < s < 1$ and the limit case $s = \frac{1}{2}$. We introduce as in [15] the mollifier of polynomial type

$$M_\delta(t, \xi) = \frac{\langle \xi \rangle^{Nt-1}}{(1 + \delta |\xi|^2)^{N_0}}$$

for $0 < \delta < 1$, $t \in [0, T_0]$ and $2N_0 = T_0N + 4$.

Lemma 3.3. *We have that for any $0 < \delta < 1$ and $0 \leq t \leq T_0$, $\xi \in \mathbb{R}$,*

$$|\partial_t M_\delta(t, \xi)| \leq N \log(\langle \xi \rangle) M_\delta(t, \xi).$$

For $-\frac{\pi}{4} \leq \theta \leq \frac{\pi}{4}$,

$$\begin{aligned} |M_\delta(t, \xi) - M_\delta(t, \xi \cos \theta)| &\leq C \sin^2 \frac{\theta}{2} M_\delta(t, \xi \cos \theta), \\ |(\partial_\xi M_\delta)(t, \xi) - (\partial_\xi M_\delta)(t, \xi \cos \theta)| &\leq C \sin^2 \frac{\theta}{2} \langle \xi \rangle^{-1} M_\delta(t, \xi \cos \theta), \\ |(\partial_\xi^2 M_\delta)(t, \xi) - (\partial_\xi^2 M_\delta)(t, \xi \cos \theta)| &\leq C \sin^2 \frac{\theta}{2} \langle \xi \rangle^{-2} M_\delta(t, \xi \cos \theta). \end{aligned}$$

Proof. We compute

$$\begin{aligned} \log M_\delta(t, \xi) &= \frac{Nt-1}{2} \log(1 + \xi^2) - N_0 \log(1 + \delta \xi^2), \\ \partial_t M_\delta(t, \xi) &= \frac{N}{2} \log(1 + \xi^2) M_\delta(t, \xi). \end{aligned}$$

Using the estimates

$$\begin{aligned} \left| \partial_\xi^k (M_\delta(t, \xi)) \right| &\leq C_k \langle \xi \rangle^{-k} M_\delta(t, \xi), \\ |M_\delta(t, \xi)| &\leq C M_\delta(t, \xi \cos \theta) \end{aligned}$$

and the Taylor formula, we obtain the proof of the lemma. \square

We estimate the first commutator:

Proposition 3.4. *Let $f, g \in L^2_1$ and $h \in L^2(\mathbb{R})$, then we have that*

$$\left| (M_\delta K(f, g), h)_{L^2} - (K(f, M_\delta g), h)_{L^2} \right| \leq C \|f\|_{L^1} \|M_\delta g\|_{L^2} \|h\|_{L^2}.$$

Proof. By the definition of M_δ , we have

$$\begin{aligned}\mathcal{F}(M_\delta f)(\xi) &= M_\delta \hat{f}(\xi), \\ \mathcal{F}(vM_\delta f)(\xi) &= i\partial_\xi (M_\delta(t, \xi)\hat{f}(\xi)).\end{aligned}$$

We now use the Bobilev formula (2.3) and the Plancherel formula

$$\begin{aligned}& (M_\delta K(f, g), h)_{L^2} - (K(f, M_\delta g), h)_{L^2} \\ &= \int_{\mathbb{R}_\xi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \beta(\theta) \hat{f}(\xi \sin \theta) \{M_\delta(\xi) - M_\delta(\xi \cos \theta)\} \hat{g}(\xi \cos \theta) \overline{\hat{h}(\xi)} d\theta d\xi.\end{aligned}$$

By the previous formula, lemma 3.3, the Cauchy-Schwarz inequality and (1.5) we have

$$\begin{aligned}& |(M_\delta K(f, g), h)_{L^2} - (K(f, M_\delta g), h)_{L^2}| \\ & \leq \int_{\mathbb{R}_\xi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \beta(\theta) \sin^2 \frac{\theta}{2} |\hat{f}(\xi \sin \theta)| \times |M_\delta(\xi \cos \theta) \hat{g}(\xi \cos \theta) \hat{h}(\xi)| d\theta d\xi \\ & \leq C \|\hat{f}\|_{L^\infty} \|M_\delta g\|_{L^2} \|h\|_{L^2} \\ & \leq C \|f\|_{L^1} \|M_\delta g\|_{L^2} \|h\|_{L^2}.\end{aligned}$$

□

In the same spirit of Proposition 2.5, we will use some symmetry property of the Kac's equation to estimate the weighted commutator.

Proposition 3.5. *Suppose that $\frac{1}{2} < s < 1$. We then have :*

$$|((v^2 M_\delta)K(f, g), h)_{L^2} - (K(f, (v^2 M_\delta)g), h)_{L^2}| \leq C \|f\|_{L^1} \|M_\delta g\|_{H^{\frac{1}{2}}} \|h\|_{H^{\frac{1}{2}}}.$$

Proof. We have

$$\begin{aligned}& ((v^2 M_\delta)K(f, g), h)_{L^2} - (K(f, (v^2 M_\delta)g), h)_{L^2} \\ &= - \int \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \beta(\theta) (\sin^2 \theta) (\partial_\xi^2 \hat{f})(\xi \sin \theta) M_\delta(\xi) \hat{g}(\xi \cos \theta) \overline{\hat{h}(\xi)} d\theta d\xi \\ & - 2 \int \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \beta(\theta) \sin \theta (\partial_\xi \hat{f})(\xi \sin \theta) \partial_\xi (M_\delta(\xi) \hat{g}(\xi \cos \theta)) \overline{\hat{h}(\xi)} d\theta d\xi \\ & - \int \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \beta(\theta) \hat{f}(\xi \sin \theta) \left\{ \partial_\xi^2 (M_\delta(t, \xi) \hat{g}(\xi \cos \theta)) - \partial_\xi^2 (M_\delta \hat{g})(\xi \cos \theta) \right\} \overline{\hat{h}(\xi)} d\theta d\xi \\ &= B_1 + B_2 + B_3.\end{aligned}$$

Then

$$|B_1| \leq C \|f\|_{L^1} \|M_\delta g\|_{L^2} \|h\|_{L^2}.$$

For B_2 , we will use the symmetry and the change of variables $\theta \rightarrow -\theta$ (see proof of lemma 2.7). We write $B_2 = B_{2a} + B_{2b}$ where

$$\begin{aligned}B_{2a} &= -2 \int_{\mathbb{R}_\xi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \beta(\theta) \sin \theta (\partial_\xi \hat{f})(\xi \sin \theta) (\partial_\xi M_\delta)(\xi) \hat{g}(\xi \cos \theta) \overline{\hat{h}(\xi)} d\theta d\xi, \\ B_{2b} &= -2 \int_{\mathbb{R}_\xi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \beta(\theta) \sin \theta (\partial_\xi \hat{f})(\xi \sin \theta) (\partial_\xi \hat{g})(\xi \cos \theta) \cos \theta M_\delta(\xi) \overline{\hat{h}(\xi)} d\theta d\xi.\end{aligned}$$

The symmetry and the estimate of lemma 3.3 implies

$$\begin{aligned} |B_{2a}| &\leq C \int_{\mathbb{R}_\xi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \beta(\theta) (\sin^2 \theta) |\xi| \|\partial_\xi^2 \hat{f}\|_{L^\infty} \langle \xi \rangle^{-1} M_\delta(\xi \cos \theta) |\hat{g}(\xi \cos \theta)| \times |\hat{h}(\xi)| d\theta d\xi \\ &\leq C \|f\|_{L^1_2} \|M_\delta g\|_{L^2} \|h\|_{L^2}. \end{aligned}$$

We note that

$$\|\langle \xi \rangle^{\frac{1}{2}} M_\delta \partial_\xi \hat{g}\|_{L^2} = \|M_\delta(vg)\|_{H^{\frac{1}{2}}} \leq C \|M_\delta g\|_{H^{\frac{1}{2}}}.$$

Using again the symmetry and the previous estimate, we get

$$\begin{aligned} |B_{2b}| &\leq C \int_{\mathbb{R}_\xi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \beta(\theta) \sin^2 \theta \langle \xi \rangle \|\partial_\xi^2 \hat{f}\|_{L^\infty} |M_\delta(\xi \cos \theta) \partial_\xi \hat{g}(\xi \cos \theta)| \times |\hat{h}(\xi)| d\theta d\xi \\ &\leq C \|f\|_{L^1_2} \|M_\delta g\|_{H^{\frac{1}{2}}} \|h\|_{H^{\frac{1}{2}}}. \end{aligned}$$

For B_3 we have

$$\begin{aligned} B_3 &= - \int_{\mathbb{R}_\xi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \beta(\theta) \hat{f}(\xi \sin \theta) \\ &\quad \left\{ \partial_\xi^2 \left(M_\delta(t, \xi) \hat{g}(\xi \cos \theta) \right) - \partial_\xi^2 \left(M_\delta \hat{g} \right) (\xi \cos \theta) \right\} \overline{\hat{h}(\xi)} d\theta d\xi \end{aligned}$$

and we compute

$$\partial_\xi^2 \left(M_\delta(t, \xi) \hat{g}(\xi \cos \theta) \right) - \partial_\xi^2 \left(M_\delta \hat{g} \right) (\xi \cos \theta) = D_1 + D_2 + D_3 + D_4 + D_5$$

where

$$\begin{aligned} D_1 &= \left(\partial_\xi^2 M_\delta(\xi) - \partial_\xi^2 M_\delta(\xi \cos \theta) \right) \hat{g}(\xi \cos \theta), \\ D_2 &= (M_\delta(\xi) - M_\delta(\xi \cos \theta)) \partial_\xi^2 \hat{g}(\xi \cos \theta), \\ D_3 &= (\cos^2 \theta - 1) M_\delta(\xi \cos \theta) \partial_\xi^2 \hat{g}(\xi \cos \theta), \\ D_4 &= 2 (\partial_\xi M_\delta(\xi) - \partial_\xi M_\delta(\xi \cos \theta)) \partial_\xi \hat{g}(\xi \cos \theta), \\ D_5 &= 2(\cos \theta - 1) \partial_\xi M_\delta(\xi) \partial_\xi \hat{g}(\xi \cos \theta). \end{aligned}$$

For $1 \leq i \leq 5$, we note $J_i = - \int_{\mathbb{R}_\xi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \beta(\theta) \hat{f}(\xi \sin \theta) D_i(\xi, \theta) \overline{\hat{h}(\xi)} d\theta d\xi$.

We successively estimate :

$$\begin{aligned} |J_1| &\leq C \|\hat{f}\|_{L^\infty} \int_{\mathbb{R}_\xi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \beta(\theta) \sin^2 \frac{\theta}{2} M_\delta(\xi \cos \theta) |\hat{g}(\xi \cos \theta) \hat{h}(\xi)| d\theta d\xi \\ &\leq C \|f\|_{L^1} \|M_\delta g\|_{L^2} \|h\|_{L^2}, \\ |J_2| + |J_3| &\leq C \|\hat{f}\|_{L^\infty} \int_{\mathbb{R}_\xi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \beta(\theta) (\sin^2 \frac{\theta}{2} + \sin^2 \theta) M_\delta(\xi \cos \theta) |\partial_\xi^2 \hat{g}(\xi \cos \theta) \hat{h}(\xi)| d\theta d\xi \\ &\leq C \|f\|_{L^1} \|M_\delta g\|_{L^2} \|h\|_{L^2}, \\ |J_4| + |J_5| &\leq C \|\hat{f}\|_{L^\infty} \int_{\mathbb{R}_\xi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \beta(\theta) \sin^2 \frac{\theta}{2} M_\delta(\xi \cos \theta) |\partial_\xi \hat{g}(\xi \cos \theta) \hat{h}(\xi)| d\theta d\xi \\ &\leq C \|f\|_{L^1} \|M_\delta g\|_{L^1_2} \|h\|_{L^2}. \end{aligned}$$

From the previous inequalities we deduce

$$|B_3| \leq C \|f\|_{L^1} \|M_\delta g\|_{L^2} \|h\|_{L^2}$$

and this finishes the proof of the Proposition 3.5. \square

For the case $s = \frac{1}{2}$, we will need a different estimate of the weighted commutator.

Proposition 3.6. *Assume that $s = \frac{1}{2}$. Then for any $0 < \alpha' < 1$ we have*

$$(3.1) \quad \left| \left((v^2 M_\delta) K(f, g), h \right)_{L^2} - \left(K(f, (v^2 M_\delta) g), h \right)_{L^2} \right| \leq C \|f\|_{L^1} \|M_\delta g\|_{H^{\frac{\alpha'}{2}}} \|h\|_{H^{\frac{\alpha'}{2}}}.$$

The proof of this Proposition use the same arguments of Proposition 3.5 and lemma 2.8.

Proof of the Theorem 3.1.

- **Case :** $\frac{1}{2} < s < 1$.

We consider $f \in L^1_{2+2s} \cap L \log L$ a weak solution of the Cauchy problem (1.3) and we multiply the equation with the test function

$$\varphi(t, v) = M_\delta(t, D_v)(1 + v^4)M_\delta(t, D_v)f(t, v).$$

Therefore we obtain the equality

$$(3.2) \quad (\partial_t f, \varphi)_{L^2} = (K(f, f), \varphi)_{L^2}.$$

Using some similar arguments in [15], we can suppose that $\varphi \in C^1([0, T_0]; H^5_{-2+2s}(\mathbb{R}))$.

We compute

$$\begin{aligned} & (M_\delta \partial_t f, M_\delta f)_{L^2} + (v^2 M_\delta \partial_t f, v^2 M_\delta f)_{L^2} \\ &= (M_\delta K(f, f), M_\delta f)_{L^2} + (v^2 M_\delta K(f, f), v^2 M_\delta f)_{L^2}. \end{aligned}$$

We will use the following notations

$$\begin{aligned} \text{time}_0 &= ((\partial_t M_\delta) f, M_\delta f)_{L^2}, \\ \text{time}_2 &= (v^2 (\partial_t M_\delta) f, v^2 M_\delta f)_{L^2}, \end{aligned}$$

and, concerning the commutators of the Kac operator and the weighted mollifier,

$$\begin{aligned} \text{com}_0 &= (M_\delta K(f, f), M_\delta f)_{L^2} - (K(f, M_\delta f), M_\delta f)_{L^2}, \\ \text{com}_2 &= (v^2 M_\delta K(f, f), v^2 M_\delta f)_{L^2} - (K(f, v^2 M_\delta f), v^2 M_\delta f)_{L^2}. \end{aligned}$$

Therefore the relation (3.2) become

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \left(\|Mf\|_{L^2}^2 + \|v^2 Mf\|_{L^2}^2 \right) - (K(f, M_\delta f), M_\delta f)_{L^2} - (K(f, v^2 M_\delta f), v^2 M_\delta f)_{L^2} \\ &= \text{time}_0 + \text{time}_2 + \text{com}_0 + \text{com}_2. \end{aligned}$$

From the coercivity inequality of Proposition 2.1, we derive the following differential in-equation

$$(3.3) \quad \begin{aligned} & \frac{1}{2} \frac{d}{dt} \left(\|Mf\|_{L^2}^2 + \|v^2 Mf\|_{L^2}^2 \right) + c_f \|Mf\|_{H^s}^2 \\ & \leq \text{time}_0 + \text{time}_2 + \text{com}_0 + \text{com}_2 + C \|f\|_{L^1} \|M_\delta f\|_{L^2}. \end{aligned}$$

Lemma 3.7. *Assume that $0 < s < 1$ and $\varepsilon > 0$. Then there exists a constant C_ε such that :*

$$\begin{aligned} & |((\partial_t M_\delta) f, M_\delta f)_{L^2}| \leq \varepsilon \|M_\delta f\|_{H^s}^2 + C_\varepsilon \|M_\delta f\|_{L^2}^2, \\ & |(v^2 (\partial_t M_\delta) f, v^2 M_\delta f)_{L^2}| \leq \varepsilon \|M_\delta f\|_{H^s}^2 + C_\varepsilon \|M_\delta f\|_{L^2}^2. \end{aligned}$$

Proof. We compute

$$\text{time}_0 = \left(\frac{N}{2} \log(1 + \xi^2) M_\delta \hat{f}, M_\delta \hat{f} \right)_{L^2}.$$

For $\varepsilon > 0$, there exists a constant C_ε such that :

$$(3.4) \quad \frac{N}{2} \log(1 + \xi^2) \leq \varepsilon(1 + \xi^2)^s + C_\varepsilon.$$

Therefore

$$|\text{time}_0| \leq \varepsilon \left((1 + \xi^2)^s M_\delta \hat{f}, M_\delta \hat{f} \right)_{L^2} + C_\varepsilon \left(M_\delta \hat{f}, M_\delta \hat{f} \right)_{L^2}.$$

We estimate the term $\text{time}_2 = \left(\partial_\xi^2 (\partial_t M_\delta \hat{f}), \mathcal{F}(v^2 M_\delta f) \right)_{L^2}$. We compute

$$\begin{aligned} \partial_\xi^2 (\partial_t M_\delta \hat{f}) &= \underbrace{\partial_\xi^2 \left(\frac{N}{2} \log(1 + \xi^2) \right)}_{\leq C} M_\delta \hat{f} + 2 \underbrace{\partial_\xi \left(\frac{N}{2} \log(1 + \xi^2) \right)}_{\leq C} \partial_\xi (M_\delta \hat{f}) \\ &\quad + \frac{N}{2} \log(1 + \xi^2) \partial_\xi^2 (M_\delta \hat{f}). \end{aligned}$$

Using again the inequality (3.4),

$$\begin{aligned} |\text{time}_2| &\leq C \left| (M_\delta \hat{f}, \mathcal{F}(v^2 M_\delta f))_{L^2} \right| + C \left| (\partial_\xi (M_\delta \hat{f}), \mathcal{F}(v^2 M_\delta f))_{L^2} \right| \\ &\quad + \varepsilon \left| \left(\partial_\xi^2 (M_\delta \hat{f}), \mathcal{F}(v^2 M_\delta f) \right)_{L^2} \right| + C_\varepsilon \left| \left((1 + \xi^2)^s \partial_\xi^2 (M_\delta \hat{f}), \mathcal{F}(v^2 M_\delta f) \right)_{L^2} \right| \end{aligned}$$

where

$$\begin{aligned} \left| (M_\delta \hat{f}, \mathcal{F}(v^2 M_\delta f))_{L^2} \right| &= \|v M_\delta f\|_{L^2}^2 \leq \|M_\delta f\|_{L^2}^2, \\ \left| (\partial_\xi (M_\delta \hat{f}), \mathcal{F}(v^2 M_\delta f))_{L^2} \right| &= |(v M_\delta f, v^2 M_\delta f)_{L^2}| \leq \|M_\delta f\|_{L^2}, \\ \left| \left(\partial_\xi^2 (M_\delta \hat{f}), \mathcal{F}(v^2 M_\delta f) \right)_{L^2} \right| &= |(v^2 M_\delta f, v^2 M_\delta f)_{L^2}| \leq \|M_\delta f\|_{L^2}, \\ \left| \left((1 + \xi^2)^s \partial_\xi^2 (M_\delta \hat{f}), \mathcal{F}(v^2 M_\delta f) \right)_{L^2} \right| &= \|M_\delta f\|_{H^s}^2. \end{aligned}$$

This concludes the proof of lemma 3.7. \square

Plugging the estimates of Propositions 3.4, 3.5 and lemma 3.7 into (3.3), we get

$$\begin{aligned} &\frac{1}{2} \frac{d}{dt} \left(\|M_\delta f\|_{L^2}^2 + \|v^2 M_\delta f\|_{L^2}^2 \right) + c_f \|M_\delta f\|_{H^s}^2 \\ &\leq \varepsilon \|M_\delta f\|_{H^s}^2 + C_\varepsilon \|f\|_{L^2}^2 + C \|f\|_{L^1} \|M_\delta f\|_{H^s}^2. \end{aligned}$$

From the interpolation estimate

$$(3.5) \quad \|g\|_{H^{\frac{1}{2}}}^2 \leq \lambda \|g\|_{H^s}^2 + \lambda^{\frac{-1}{2s-1}} \|g\|_{L^2}^2,$$

we deduce

$$\begin{aligned} &\frac{1}{2} \frac{d}{dt} \left(\|M_\delta f\|_{L^2}^2 + \|v^2 M_\delta f\|_{L^2}^2 \right) + c_f \|M_\delta f\|_{H^s}^2 \\ &\leq \left(\varepsilon + C\lambda \|f\|_{L^1} \right) \|M_\delta f\|_{H^s}^2 + \left(C_\varepsilon + C\lambda^{\frac{-1}{2s-1}} \|f\|_{L^1} \right) \|M_\delta f\|_{L^2}^2. \end{aligned}$$

Choosing ε and λ small enough, we get

$$(3.6) \quad \frac{1}{2} \frac{d}{dt} \left(\|M_\delta f\|_{L^2}^2 + \|v^2 M_\delta f\|_{L^2}^2 \right) \leq C \|M_\delta f\|_{L^2}^2 \leq C \left(\|M_\delta f\|_{L^2}^2 + \|v^2 M_\delta f\|_{L^2}^2 \right).$$

From Gronwall's lemma we have

$$\|M_\delta f\|_{L^2}^2 + \|v^2 M_\delta f\|_{L^2}^2 \leq e^{2Ct} (\|M_\delta(0)f_0\|_{L^2}^2 + \|v^2 M_\delta(0)f_0\|_{L^2}^2),$$

that is

$$\|M_\delta f\|_{L^2}^2 \leq e^{2Ct} \|M_\delta(0)f_0\|_{L^2}^2.$$

We write :

$$\|(1 - \delta\Delta)^{-N_0} f\|_{H_2^{N_0-1}}^2 \leq e^{2Ct} \|(1 - \delta\Delta)^{-N_0} f_0\|_{H_2^{-1}}^2.$$

By Fatou's lemmas, letting $\delta \rightarrow 0$,

$$\|f\|_{H_2^{N_0-1}}^2 \leq e^{2Ct} \|f_0\|_{H_2^{-1}}^2 \leq C' e^{2Ct} \|f_0\|_{L^1}^2.$$

For $t \in [0, T_0]$ we have proved

$$(1 + |D_v|^2)^m f(t, \cdot) \in L^2_2(\mathbb{R})$$

for all $T_0 > 0$ and $m = Nt - 1 > 0$. Therefore we have obtained that $f(t, \cdot) \in H_2^m(\mathbb{R})$ and that concludes the proof of Theorem 3.1 in the case $\frac{1}{2} < s < 1$.

- **Case $s = \frac{1}{2}$.** The proof is similar to the case $\frac{1}{2} < s < 1$. We choose $\alpha' = \frac{1}{2}$ in Proposition 3.6 and we plug the estimate of Proposition 3.1 in the differential inequation (3.3). We get the same estimate (3.6) and from Gronwall's lemma we conclude the proof of Theorem 3.1. \square

Proof of Corollary 3.2.

We consider the case $\frac{1}{2} < s < 1$. We first note that, from Theorem 3.1, $f(t, \cdot) \in H_2^{+\infty}(\mathbb{R})$ for all $t > 0$. We introduce the mollifier

$$M(\xi) = 1 + \xi^2$$

which corresponds to the differential operator $M = 1 - \Delta_v$.

By a proof similar to that of propositions 3.4 and 3.5, since M satisfies obviously the estimates of lemma 3.3, we can prove the following estimates of the commutators: For $f, g \in L^1_1$ and $h \in L^2(\mathbb{R})$, we have

$$\begin{aligned} |(MK(f, g), h)_{L^2} - (K(f, Mg), h)_{L^2}| &\leq C \|f\|_{L^1} \|Mg\|_{L^2} \|h\|_{L^2}, \\ |((v^2 M)K(f, g), h)_{L^2} - (K(f, (v^2 M)g), h)_{L^2}| &\leq C \|f\|_{L^1} \|Mg\|_{H_2^{\frac{1}{2}}} \|h\|_{H^{\frac{1}{2}}} \end{aligned}$$

where C depends only on β and $\|f\|_{L^\infty([0, +\infty[; L^1_{2+2s} \cap L \log L(\mathbb{R}))}$.

Following the proof of Theorem 3.1 and the same notations, we get a differential inequation similar to (3.3) (remark that the mollifier M is independent of the time)

$$\frac{1}{2} \frac{d}{dt} (\|Mf\|_{L^2}^2 + \|v^2 Mf\|_{L^2}^2) + c_f \|Mf\|_{H_2^s}^2 \leq \text{com}_0 + \text{com}_2 + C \|f\|_{L^1} \|Mf\|_{L^2}^2.$$

The previous estimates of the commutators imply

$$\frac{1}{2} \frac{d}{dt} (\|Mf\|_{L^2}^2 + \|v^2 Mf\|_{L^2}^2) + c_f \|Mf\|_{H_2^s}^2 \leq C \|f\|_{L^1} \|Mf\|_{H_2^{\frac{1}{2}}}^2.$$

Using the interpolation estimate (3.5) we deduce the following differential inequation

$$\frac{1}{2} \frac{d}{dt} (\|Mf\|_{L^2}^2 + \|v^2 Mf\|_{L^2}^2) \leq C (\|Mf\|_{L^2}^2 + \|v^2 Mf\|_{L^2}^2)$$

and from Gronwall's lemma we derive

$$\|Mf\|_{L^2}^2 \leq e^{2Ct} \|Mf_0\|_{L^2}^2.$$

That concludes the proof of Corollary 3.2 in the case $\frac{1}{2} < s < 1$.

The proof in the case $s = \frac{1}{2}$ is similar. \square

4. ANALYTICITY PROPERTY FOR KAC'S EQUATION

From the Theorem 3.1, we know that the weak solution of the Cauchy problem of the Kac's equation (1.3) has the following regularity: for any $t_0 > 0$, $f \in L^\infty([t_0, T_0]; H_2^2(\mathbb{R}))$. Therefore f is a solution of the following Cauchy problem :

$$\begin{cases} \frac{\partial f}{\partial t} = K(f, f), \\ f|_{t=0} = f_0 \in H_2^2(\mathbb{R}). \end{cases}$$

and we can suppose that the initial datum is $f_0 \in H_2^2(\mathbb{R}) \cap L_2^1(\mathbb{R})$.

We have the local analytic regularizing effect of Cauchy problem.

Theorem 4.1. *Assume that the cross-section kernel β satisfies (1.4), the initial datum $f_0 \in L_{2+2s}^1 \cap H_2^2(\mathbb{R})$ and $f \in L^\infty([0, T_0]; H_2^2 \cap L_2^1(\mathbb{R}))$ is a nonnegative weak solution of the Cauchy problem of the Kac's equation (1.3) for some $T_0 > 0$.*

- **Case** $\frac{1}{2} < s < 1$.

There exist $0 < T_* \leq T_0$ and $c_0 > 0$ such that

$$e^{c_0 t \langle D_v \rangle} f \in L^\infty([0, T_*]; L_1^2(\mathbb{R})).$$

Therefore we have $f(t, \cdot) \in G^1(\mathbb{R})$ for any $0 < t \leq T_*$.

- **Case** $s = \frac{1}{2}$.

For any $0 < \alpha < 1$, there exist $0 < T_* \leq T_0$ and $c_0 > 0$ such that

$$e^{c_0 t \langle D_v \rangle^\alpha} f \in L^\infty([0, T_*]; L_1^2(\mathbb{R})).$$

Therefore for any $0 < \alpha < 1$ and $0 < t \leq T_*$ we have $f(t, \cdot) \in G^{1/\alpha}(\mathbb{R})$.

Proof of the Theorem 4.1.

We choose the test function

$$\tilde{\varphi}(t, \cdot) = (G_\delta(t, D_v) \langle v \rangle^2 G_\delta(t, D_v) f)(t, \cdot) \in L^\infty([0, T_0]; H^2(\mathbb{R}))$$

where the mollifier G_δ is given in section 2 by (2.1).

We have

$$(\partial_t f, \tilde{\varphi})_{L^2} = (K(f, f), \tilde{\varphi})_{L^2}.$$

First, the left-hand side term is

$$(\partial_t f, \tilde{\varphi})_{L^2} = \frac{1}{2} \frac{d}{dt} \|G_\delta f\|_{L_1^2}^2 - ((\partial_t G_\delta) f, G_\delta f)_{L^2} - (v(\partial_t G_\delta) f, v G_\delta f)_{L^2}.$$

The rights-hand side is

$$\begin{aligned} (K(f, f), \tilde{\varphi})_{L^2} &= (G_\delta(K(f, f)), (1 + v^2)G_\delta f)_{L^2} \\ &= (K(f, G_\delta f), G_\delta f)_{L^2} + (K(f, vG_\delta f), vG_\delta f)_{L^2} \\ &\quad + (G_\delta K(f, f) - K(f, G_\delta f), G_\delta f)_{L^2} \\ &\quad + (vG_\delta K(f, f) - K(f, vG_\delta f), vG_\delta f)_{L^2}. \end{aligned}$$

Therefore we can write :

$$(4.1) \quad \begin{aligned} \frac{1}{2} \frac{d}{dt} \|G_\delta f\|_{L_1^2}^2 - (K(f, G_\delta f), G_\delta f)_{L^2} - (K(f, vG_\delta f), vG_\delta f)_{L^2} \\ = (\text{time term}) + (\text{commutator}) \end{aligned}$$

where

$$\begin{aligned} \text{(time term)} &= -((\partial_t G_\delta)(\cdot, D_v)f, G_\delta(t, D_v)f(t, \cdot))_{L^2} \\ &\quad - (v(\partial_t G_\delta)(\cdot, D_v)f, vG_\delta(t, D_v)f)_{L^2} \end{aligned}$$

and

$$\begin{aligned} \text{(commutator)} &= (G_\delta K(f, f) - K(f, G_\delta f), G_\delta f)_{L^2} \\ &\quad + (vG_\delta K(f, f) - K(f, vG_\delta f), vG_\delta f)_{L^2}. \end{aligned}$$

Furthermore,

$$\|G_\delta f\|_{H_1^s}^2 \leq \|G_\delta f\|_{H^s}^2 + \|vG_\delta f\|_{H^s}^2 + \|G_\delta f\|_{L^2}^2$$

and by the Proposition 2.1,

$$-(K(f, G_\delta f), (G_\delta f))_{L^2} \geq c_f \|G_\delta f\|_{H^s}^2 - C\|f\|_{L^1} \|G_\delta f\|_{L^2}^2$$

and

$$-(K(f, vG_\delta f), v(G_\delta f))_{L^2} \geq c_f \|vG_\delta f\|_{H^s}^2 - C\|f\|_{L^1} \|vG_\delta f\|_{L^2}^2.$$

Then the equality (4.1) became

$$(4.2) \quad \begin{aligned} \frac{1}{2} \frac{d}{dt} \|G_\delta f\|_{L_1^2}^2 + c_f \|G_\delta f\|_{H_1^s}^2 \\ \leq C\|f\|_{L^1} \|G_\delta f\|_{L_1^2}^2 + \text{(time term)} + \text{(commutator)}. \end{aligned}$$

- **Case :** $\frac{1}{2} < s < 1$.

We consider the mollifier G_δ defined in (2.1) and chosen with $\alpha = 1$

$$G_\delta(t, \xi) = \frac{e^{c_0 t \langle \xi \rangle}}{1 + \delta e^{c_0 t \langle \xi \rangle}}.$$

Remark 4.2. This is the optimal choice for $\alpha \in]0, 2[$ as it can be seen in the estimates of section 2 : for example, from lemma 2.11, the term $\|G_\delta f\|_{H^{(\frac{3\alpha}{2}-1)^+}} \|G_\delta f\|_{H^{\frac{\alpha}{2}}}$ can be controlled by the coercivity only if $\alpha \leq 1$.

Using the Propositions 2.4, 2.5 and the lemma 2.13 of section 2, we get :

- Estimate of commutators terms:

$$(4.3) \quad |(G_\delta K(f, f) - K(f, G_\delta f), G_\delta f)_{L^2}| \leq C \|G_\delta f\|_{L_1^2} \|G_\delta f\|_{H^{1/2}}^2$$

and

$$(4.4) \quad |(vG_\delta)K(f, f) - K(f, (vG_\delta)f), vG_\delta f)_{L^2}| \leq C(\|f\|_{L_1^2} + \|G_\delta f\|_{L_1^2}) \|G_\delta f\|_{H_1^{\frac{1}{2}}}^2.$$

- Estimate of the terms involving the derivative with respect to time:

$$(4.5) \quad |((\partial_t G_\delta)(t, D_v)f(t, \cdot), G_\delta(t, D_v)f(t, \cdot))_{L^2}| \leq C \|G_\delta f\|_{H^{1/2}}^2,$$

and

$$(4.6) \quad |(v(\partial_t G_\delta)(t, D_v)f(t, \cdot), vG_\delta(t, D_v)f(t, \cdot))_{L^2}| \leq C \|G_\delta f\|_{H_1^{1/2}}^2.$$

Therefore, plugging the estimates (4.3)-(4.6) into (4.2), we obtain

$$\frac{1}{2} \frac{d}{dt} \|G_\delta f\|_{L_1^2}^2 + c_f \|G_\delta f\|_{H_1^s}^2 \leq C \|G_\delta f\|_{H_1^{\frac{1}{2}}}^2 + C \|G_\delta f\|_{L_1^2} \|G_\delta f\|_{H_1^{\frac{1}{2}}}^2.$$

From the interpolation inequality (3.5) we have

$$C \|G_\delta f\|_{H_1^{\frac{1}{2}}}^2 \leq C\lambda_1 \|G_\delta f\|_{H_1^s}^2 + C\lambda_1^{\frac{-1}{2s-1}} \|G_\delta f\|_{L_1^2}^2$$

and

$$C\|G_\delta f\|_{L_1^2}\|G_\delta f\|_{H_1^{\frac{1}{2}}}^2 \leq C\lambda_2\|G_\delta f\|_{L_1^2}\|G_\delta f\|_{H_1^2}^2 + C\lambda_2^{\frac{-1}{2s-1}}\|G_\delta f\|_{L_1^2}^3.$$

Choosing λ_1 and λ_2 such that $C\lambda_1 = c_f/4$ and $C\lambda_2\|G_\delta f\|_{L_1^2} = c_f/4$ we get

$$\frac{1}{2}\frac{d}{dt}\|G_\delta f\|_{L_1^2}^2 \leq C_1\|G_\delta f\|_{L_1^2}^2 + C_2\|G_\delta f\|_{L_1^2}^{\frac{1}{2s-1}+3}$$

for $t \in [0, T_0]$, with $C_1, C_2 > 0$ are independent of t and $\delta > 0$. Then

$$\frac{d}{dt}\|G_\delta f\|_{L_1^2} \leq C_1\|G_\delta f\|_{L_1^2} + C_2\|G_\delta f\|_{L_1^2}^\gamma$$

where $\gamma = \frac{1}{2s-1} + 2$. We set $\psi(t) = \|G_\delta f(t, \cdot)\|_{L_1^2}$. Therefore

$$\frac{d}{dt}\psi(t) \leq C_1\psi(t) + C_2\psi(t)^\gamma.$$

Solving the previous differential inequation, we easily get

$$\psi(t) \leq \frac{e^{C_1 t} \psi(0)}{\left(1 - \frac{C_2}{C_1} (e^{(\gamma-1)C_1 t} - 1) \psi(0)^{\gamma-1}\right)^{\frac{1}{\gamma-1}}},$$

that is

$$\|G_\delta f(t, \cdot)\|_{L_1^2} \leq \frac{e^{C_1 t} \|f_0\|_{L_1^2}}{\left(1 - \frac{C_2}{C_1} \left(e^{\frac{s}{s-\frac{1}{2}} t} - 1\right) \|f_0\|_{L_1^2}^{\frac{s}{s-\frac{1}{2}}}\right)^{\frac{s-\frac{1}{2}}{s}}}.$$

We now choose $0 < T_* \leq T_0$ small enough so that for $t \in [0, T_*]$

$$(4.7) \quad 1 - \frac{C_2}{C_1} \left(e^{\frac{s}{s-\frac{1}{2}} t} - 1\right) \|f_0\|_{L_1^2}^{\frac{s}{s-\frac{1}{2}}} \geq \left(\frac{1}{2}\right)^{\frac{s}{s-\frac{1}{2}}},$$

and taking $\delta \rightarrow 0$, we have for $t \in]0, T_*]$,

$$\|e^{c_0 t \langle D_v \rangle} f(t, \cdot)\|_{L_1^2} \leq 2e^{C_1 t} \|f_0\|_{L_1^2}.$$

This concludes the proof of Theorem 4.1 in the case $\frac{1}{2} < s < 1$.

- **Case :** $s = \frac{1}{2}$.

We consider the mollifier G_δ defined in (2.1) with $0 < \alpha < 1$. Taking $\alpha' = \frac{1}{2}$ in the estimate of the commutator in Proposition 2.6 we obtain

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\|G_\delta f\|_{L_1^2}^2 + c_f\|G_\delta f\|_{H_1^{\frac{1}{2}}}^2 &\leq \\ C\|f\|_{L_1^1}\|G_\delta f\|_{H_1^{\frac{1}{4}}}^2 + C\|G_\delta f\|_{L_1^2}\|G_\delta f\|_{H_1^{\frac{3}{2}}}^2 + C\|G_\delta f\|_{H_1^{\frac{3}{2}}}^2 &\cdot \end{aligned}$$

From an interpolation estimate similar as (3.5), we get the following differential inequation

$$\frac{1}{2}\frac{d}{dt}\|G_\delta f\|_{L_1^2}^2 \leq C'_1\|G_\delta f\|_{L_1^2}^2 + C'_2\|G_\delta f\|_{L_1^2}^{\frac{\alpha}{1-\alpha}+3}.$$

where $C'_1, C'_2 > 0$ are independent of $\delta > 0$. This concludes the proof of the Theorem 4.1. \square

Proof of the propagation of analyticity and end of the proof of Theorem 1.2.

We could use the Theorem 2.6 of [7] (propagation of Gevrey regularity in the case of an even initial datum f_0). We present below a direct proof.

We consider the case $\frac{1}{2} < s < 1$. Let f a nonnegative weak solution of the Cauchy problem (1.3) which fulfills the assumptions of Theorem 1.2: From Theorem 3.1, we have $f(t, \cdot) \in H^2_2(\mathbb{R})$ for all $t > 0$.

Let us consider some arbitrary and fixed $0 < T_0 < T_1$. From Corollary 3.2, there exists a constants C_0 which depends only on β , $\|f\|_{L^\infty([0, +\infty[; L^1_{2+2s} \cap L \log L(\mathbb{R}))}$, T_0 , T_1 and $\|f(T_0, \cdot)\|_{H^2_2}$ such that

$$(4.8) \quad \forall t \in [T_0, T_1 + 1], \quad \|f(t, \cdot)\|_{H^2_2} \leq C_0.$$

We then consider $t_0 \in [T_0, T_1]$ and we apply Theorem 4.1 for the initial value $\tilde{f}_0 = f(t_0, \cdot)$ and for $t \in [0, 1]$. There exist $0 < T_* \leq 1$ and some constants $c_0 > 0$ and $C_1 > 0$ such that

$$\forall t \in]t_0, t_0 + T_*], \quad \|e^{c_0(t-t_0) \langle Dv \rangle} f(t, \cdot)\|_{L^2_1} \leq 2e^{C_1(t-t_0)} \|f(t_0, \cdot)\|_{L^2_1}.$$

In addition, we remark from the proof of Theorem 4.1 and the inequality (4.7) that the time T_* depends only on T_0 , T_1 and $\|f\|_{L^\infty([T_0, T_1+1]; H^2_2 \cap L^1_2(\mathbb{R}))}$, which is controlled by $\|f_0\|_{L^1_2}$ and by the constant C_0 from (4.8),

Hence $T_* > 0$ can be chosen independent of $t_0 \in [T_0, T_1]$. Therefore $f(t, \cdot) \in G^1(\mathbb{R})$ for all $t \in]t_0, t_0 + T_*]$, and this will remain true for $t \in]T_0, T_1]$. Since $T_0 < T_1$ are arbitrary, the solution of the Cauchy problem (1.3) satisfies $f(t, \cdot) \in G^1(\mathbb{R})$ for any $t > 0$.

The proof for the case $s = \frac{1}{2}$ is similar. This concludes the proof of Theorem 1.2. \square

5. ANALYTICITY PROPERTY FOR BOLTZMANN EQUATION

In this section, we will prove the analyticity of the radially symmetric solutions of the Boltzmann equation (Theorem 1.1)

$$\frac{\partial f}{\partial t} = Q(f, f), \quad v \in \mathbb{R}^3, t > 0; \quad f|_{t=0} = f_0.$$

Using the Bobylev's formula, we have for $\xi \in \mathbb{R}^3$

$$\mathcal{F}(Q(f, g))(\xi) = \int_{S^2} b\left(\frac{\xi}{|\xi|} \cdot \sigma\right) \{\hat{f}(\xi^-) \hat{g}(\xi^+) - \hat{f}(0) \hat{g}(\xi)\} d\sigma$$

where

$$\xi^+ = \frac{\xi + |\xi| \sigma}{2}, \quad \xi^- = \frac{\xi - |\xi| \sigma}{2}.$$

We define θ by

$$\cos \theta = \left\langle \frac{\xi}{|\xi|}, \sigma \right\rangle.$$

We then have

$$|\xi^+| = |\xi| |\cos \frac{\theta}{2}|, \quad |\xi^-| = |\xi| |\sin \frac{\theta}{2}|.$$

Let f be a radially symmetric function. That is $f(Av) = f(v)$ for any orthogonal 3×3 matrix A . Therefore $f(v) = f(0, 0, |v|)$. We compute the Fourier transform in \mathbb{R}^3 :

$$\mathcal{F}_{\mathbb{R}^3}(f)(\xi) = \int_{\mathbb{R}^3} e^{-i\xi \cdot v} f(v) dv = \int_{\mathbb{R}^3} e^{-i|\xi| |v|} f(v) dv = \int_{\mathbb{R}} e^{-i|\xi| |u|} F(u) du$$

where

$$(5.1) \quad F(u) = \int_{\mathbb{R}^2} f(v_1, v_2, u) dv_1 dv_2.$$

Then

$$(5.2) \quad \hat{f}(\xi) = \hat{F}(|\xi|)$$

is also radially symmetric. Let us consider two radially symmetric functions f and g and let us denote F and G the associated functions defined by (5.1). We compute

$$\begin{aligned}\mathcal{F}(Q(f, g))(\xi) &= \int_{S^2} b\left(\frac{\xi}{|\xi|} \cdot \sigma\right) \{\hat{F}(|\xi^-|)\hat{G}(|\xi^+|) - \hat{F}(0)\hat{G}(|\xi|)\} d\sigma \\ &= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \beta(\theta) \{\hat{F}(|\xi^-|)\hat{G}(|\xi^+|) - \hat{F}(0)\hat{G}(|\xi|)\} d\theta\end{aligned}$$

where

$$\beta(\theta) = 2\pi \sin \theta b(\cos \theta).$$

Let $f(t, \cdot)$ be a solution of the Boltzmann equation. We put for $t \geq 0$

$$F(t, u) = \int_{\mathbb{R}^2} f(t, v_1, v_2, u) dv_1 dv_2.$$

Therefore $\hat{f}(t, \cdot)$ is solution of the equation

$$\partial_t \hat{f}(t, \xi) = \mathcal{F}(Q(f(t, \cdot), f(t, \cdot))) (\xi)$$

and from (5.2) we prove that $F(t, \cdot)$ is a solution of the equation

$$\partial_t \hat{F}(t, |\xi|) = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \beta(\theta) \{\hat{F}(t, |\xi \sin \frac{\theta}{2}|)\hat{F}(t, |\xi \cos \frac{\theta}{2}|) - \hat{F}(t, 0)\hat{F}(t, |\xi|)\} d\theta.$$

We use the following lemma (see [12]):

Lemma 5.1. *Let $f \in L_k^1(\mathbb{R}^3)$ radially symmetric, $f \geq 0$ and define F by (5.1). Then $F \in L_k^1(\mathbb{R})$. Assume that f is also uniformly integrable $f \geq 0$. Then F is also uniformly integrable.*

From lemma 5.1, $F(t, \cdot) \in L_{2+2s}^1(\mathbb{R})$, but we do not have $F(t, \cdot) \in L \log L(\mathbb{R})$. However F is uniformly integrable, and it is enough to get the coercivity property of proposition 2.1.

Proof of Theorem 1.1.

Case $\frac{1}{2} < s < 1$ (the proof for the case $s = \frac{1}{2}$ is similar). We apply the Theorem 1.2 : for a fixed $t > 0$, there exists a constant c_0 such that

$$\|e^{c_0 \langle \cdot \rangle} \hat{F}(t, |\cdot|)\|_{L^2(\mathbb{R})} < \infty$$

that is $F(t, \cdot) \in G^1(\mathbb{R})$. Since

$$e^{c_0 \langle \xi \rangle} \hat{F}(t, |\xi|) = e^{c_0 \langle \xi \rangle} \hat{f}(t, \xi)$$

we have

$$\|e^{\frac{c_0}{2} \langle \cdot \rangle} \hat{f}(t, \cdot)\|_{L^2(\mathbb{R}^3)} \leq C \|e^{c_0 \langle \cdot \rangle} \hat{f}(t, |\cdot|)\|_{L^2(\mathbb{R})} < \infty$$

and therefore $f(t, \cdot) \in G^1(\mathbb{R}^3)$.

The proof for the case $s = \frac{1}{2}$ is similar. This concludes the proof of Theorem 1.1. \square

Acknowledgements : The authors wish to thank Chao-Jiang Xu for interesting conversations.

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