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Stokes-Ramis matrices and connection constants for meromorphic linear differential systems with a single level: a perturbative approach

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Abstract

In the article *Matrices de Stokes-Ramis et constantes de connexion pour les systèmes différentiels linéaires de niveau unique* (P. Remy), we considered linear differential systems with a unique but arbitrary level and we stated formulae to express all the Stokes multipliers in terms of connection constants in the Borel plane generalizing thus the calculations made in the article *Resurgence, Stokes phenomenon and alien derivatives for level-one linear differential systems* (M. Loday-Richaud, P. Remy). In the present paper, we provide a new proof of these formulae. We perturb the given system in order that each Stokes value generate its own anti-Stokes direction. We state the connection-to-Stokes formulae for the perturbed system and we conclude by a limit process. We believe the method could provide an efficient tool for the numerical calculation of the Stokes multipliers. As an illustration, we develop an example. No assumption of genericity is made.

Keywords. Linear differential system, Stokes phenomenon, summability, resurgence, Stokes multipliers, connection constants

AMS subject classification. 34M03, 34M30, 34M35, 34M40

Introduction

All along the article, we are given a linear differential system (in short, a differential system or a system)

$$x^{r+1}\frac{dY}{dx} = A(x)Y \quad , \ A(x) \in M_n(\mathbb{C}\{x\}), \ A(0) \neq 0 \tag{0.1}$$

of dimension $n \geq 2$ with meromorphic coefficients of order $r+1 \geq 2$ at the origin 0 in $\mathbb{C}$. Such a system admits a formal fundamental solution of the form

$$\widetilde{Y}(x) = \widetilde{F}(x)x^L e^{Q(1/x)}$$

where $\widetilde{F}(x) \in M_n(\mathbb{C}[[x]])$ is a power series in x, the matrix $L \in M_n(\mathbb{C})$ of exponents of formal monodromy is a constant matrix and where the irregular part $Q(1/x) = \mathrm{diag}(q_1(1/x), ..., q_n(1/x))$ is a diagonal matrix with polynomial entries $q_j(1/x) \in x^{-1/\nu}\mathbb{C}[x^{-1/\nu}]$, $\nu \in \{1, ..., n!\}$, in a fractional power of $1/x$ ([1, 5]).

The finite algebraic extension $x \longmapsto x^\nu$ of the variable x and a suitable meromorphic gauge transformation $Y \longmapsto T(x)Y$ where $T(x)$ has explicit computable polynomial entries in x and $1/x$ allow to normalize $\widetilde{Y}(x)$ as follows ([1]):

- $\widetilde{F}(x) \in M_n(\mathbb{C}[[x]])$ is a power series in x with condition $\widetilde{F}(x) = I_n + O(x^r)$, where I_n is the identity matrix of size n,

- $L = \bigoplus_{j=1}^{J}(\lambda_j I_{n_j} + J_{n_j})$ where $J \geq 2$, $\mathrm{Re}(\lambda_j) \in [0, 1[$ and where

$$J_{n_j} = \begin{cases} 0 & \text{if } n_j = 1 \\ \begin{bmatrix} 0 & 1 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & 1 \\ 0 & \cdots & \cdots & 0 \end{bmatrix} & \text{if } n_j \geq 2 \end{cases}$$

  is an irreductible Jordan block of size n_j,

- $Q(1/x)$ is a diagonal matrix with polynomial entries in $1/x$ of the form

$$Q\left(\frac{1}{x}\right) = \bigoplus_{j=1}^{J} q_j\left(\frac{1}{x}\right) I_{n_j}$$

with

$$q_j\left(\frac{1}{x}\right) = -\frac{a_{j,r}}{x^r} - \frac{a_{j,r-1}}{x^{r-1}} - ... - \frac{a_{j,1}}{x} \in x^{-1}\mathbb{C}[x^{-1}]$$

Besides, we assume that

$$\lambda_1 = 0 \quad \text{and} \quad q_1 \equiv 0 \tag{0.2}$$

These conditions can always be fulfilled by means of the change of unknown vector $Y = x^{\lambda_1} e^{q_1(1/x)} Z$.

The assumption "*system (0.1) has the unique level r*" is equivalent to the conditions

(0.3)

1. $q_j - q_\ell \equiv 0$ or with degree r for all j, ℓ
2. there exists j such that $a_{j,r} \neq 0$

Note that these conditions imply

$$q_j \equiv q_\ell \Leftrightarrow a_{j,r} = a_{\ell,r}$$

The coefficients $a_{j,r}$ of the leading terms of the q_j's, $j = 1, ..., J$, are called *the Stokes values of system* (0.1). Recall that the r-th roots of the points $a_{j,r} - a_{\ell,r} \neq 0$ define the anti-Stokes directions of system (0.1).

Under the assumption (0.3), we are interested in the formulæ given the Stokes multipliers of $\widetilde{F}(x)$ in terms of connection constants in the Borel plane.

When $r = 1$, these constants are given by the singularities of the Borel transform $\widehat{F}(\xi)$ of $\widetilde{F}(x)$. Many proofs exist under sufficiently generic hypothesis (see [7] for instance). A complete proof without assumption of genericity was recently given by M. Loday-Richaud and the author ([6]). In this proof, we used Écalle's method by regular perturbation and majorant series quoted in [3]. We stated the summable-resurgence of $\widetilde{F}(x)$ and we displayed a precise description of the singularities of $\widehat{F}(\xi)$ in the Borel plane; the connection-to-Stokes formulæ are performed by interpreting the Stokes-Ramis matrices as Laplace integrals of these singularities.

Afterwards in [10], the author extended these results to the case $r \geq 2$ by using the classical method of rank reduction and by applying Écalle's method to the reduced system. In this case, instead of $\widetilde{F}(x)$, we considered its sub-series $\widetilde{\boldsymbol{F}}^{[u]}(t)$, $u = 0, ..., r-1$ and $t = x^r$, of terms r by r, also called *r-reduced series of* $\widetilde{F}(x)$. The connection-to-Stokes formulæ are performed by connecting the Stokes-Ramis matrices of system (0.1) to those of its reduced

system (*cf.* [5, prop. 4.2]) and the Stokes multipliers of the reduced system to the connection constants given by the singularities of the Borel transforms $\widehat{\boldsymbol{F}}^{[u]}(\tau)$, $u = 0, ..., r-1$.

In the present paper, we shall provide a new proof of connection-to-Stokes formulæ displayed in the case $r \geq 2$. This proof is quite different from the one in [10] since it is based on a perturbation of system (0.1) and a limit process.

In section 1, we recall the results from [10] on the r-reduced series of which we need.

In section 2, we state the connection-to-Stokes formulæ given in [10]. These formulæ make explicit the Stokes multipliers of $\widetilde{F}(x)$ in any anti-Stokes direction θ of system (0.1) in terms of the connection constants given by the singularities of the Borel transforms $\widehat{\boldsymbol{F}}^{[u]}(\tau)$'s at the various Stokes values generating θ.

Section 3 is devoted to the proof of the connection-to-Stokes formulæ. We first start by the generic situation where θ is generated by just one Stokes value (section 3.1). The case of several Stokes values is treated by means of a regular perturbation of system (0.1). We choose the perturbation so that the Stokes values fit the previous generic situation; then, we connect the Stokes-Ramis matrices of system (0.1) to the perturbed Stokes-Ramis matrices by means of a limit process (sections 3.3 and 3.4).

Throughout the article, we develop an example in order to illustrate our different results.

Acknowledgement I would like here to thank Professor M. Loday-Richaud for all her comments and advice which enabled me to finalize this article.

1 r-reduced series: summable-resurgence and singularities

For the convenience of the reader, we recall in this section some results from [10] concerning the summable-resurgence and the singularities in the Borel plane of the r-reduced series.

Since any of the J column-blocks of $\widetilde{F}(x)$ associated with the Jordan structure of L (matrix of exponents of formal monodromy) can be positionned at the first place by means of a permutation P on the columns of $\widetilde{Y}(x)$, we can restrict ourselves to the study of the column-block $\widetilde{f}(x)$ formed by the first n_1 (= the dimension of the first Jordan block of L) columns of $\widetilde{F}(x)$.

Recall that, after permutation, the new formal fundamental solution $\widetilde{Y}(x)P$ reads $\widetilde{Y}(x)P = \widetilde{F}(x)Px^{P^{-1}LP}e^{P^{-1}Q(1/x)P}$.

Recall that the r-reduced series of $\widetilde{f}(x) \in M_{n,n_1}(\mathbb{C}[[x]])$ are the the formal series $\widetilde{\boldsymbol{f}}^{[u]}(t) \in M_{n,n_1}(\mathbb{C}[[t]])$, $u = 0, ..., r-1$, defined by

$$\boxed{\widetilde{f}(x) := \widetilde{\boldsymbol{f}}^{[0]}(x^r) + x\widetilde{\boldsymbol{f}}^{[1]}(x^r) + ... + x^{r-1}\widetilde{\boldsymbol{f}}^{[r-1]}(x^r)} \tag{1.1}$$

In other words,

$$\widetilde{\boldsymbol{f}}^{[u]}(t) = \sum_{m\geq 0} f_{u+mr}t^m \quad \text{when} \quad \widetilde{f}(x) = \sum_{m\geq 0} f_m x^m$$

1.1 Summable-resurgence theorem

Recall that *a resurgent function* is an analytic function at $0 \in \mathbb{C}$ which can be analytically continued to an adequate Riemann surface $\mathcal{R}_\Omega$ associated with a so-called *singular support* $\Omega \subset \mathbb{C}$. For a more precise definition, we refer to [11] and [6, def. 2.1 and 2.2]. Recall that the difference between $\mathcal{R}_\Omega$ and the universal cover of $\mathbb{C}\backslash\Omega$ lies in the fact that $\mathcal{R}_\Omega$ has no branch point at 0 in the first sheet.

In the linear case, the singular support Ω is a finite set containing 0. In a more general framework, convolutions of singularities may occur what requires to consider for Ω a lattice, possibly dense in $\mathbb{C}$ (*cf.* [3, 8, 11] for instance).

To define the summable-resurgence, we extend the classical definition of sectorial regions of $\mathbb{C}$ used in summation theory into the one of sectorial regions of $\mathcal{R}_\Omega$. These regions are called *ν-sectorial regions* (*cf.* [6, def. 2.3]) and are defined for all $\nu > 0$ small enough by the data of

- an open disc D_ν centered at $0 \in \mathbb{C}$,
- an open sector Σ_ν with bounded opening at infinity,
- a tubular neighborhood $\mathcal{N}_\nu$ of a piecewise-$\mathcal{C}^1$ path γ connecting D_ν to Σ_ν after a finite number of turns around points of Ω,

such that the distance of D_ν to $\Omega^* = \Omega\backslash\{0\}$ and the distance of $\mathcal{N}_\nu \cup \Sigma_\nu$ to

Ω have to be greater than ν.

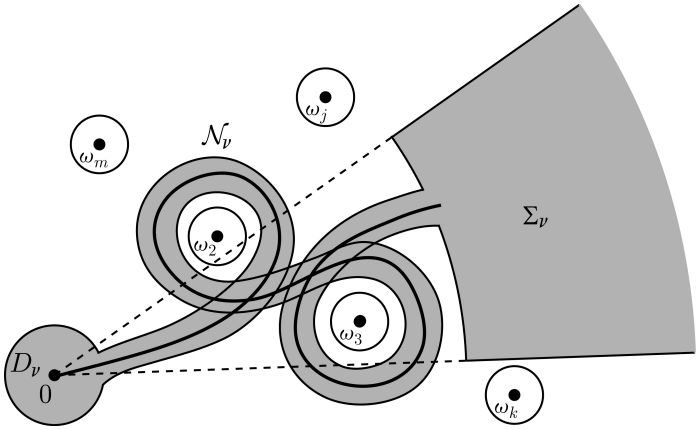

Figure 1.1 - A ν-sectorial region

Definition 1.1 (Summable-resurgent functions)
A resurgent function defined on $\mathcal{R}_\Omega$ is said to be summable-resurgent with singular support Ω *when it grows at most exponentially at infinity on any ν-sectorial region Δ_ν of $\mathcal{R}_\Omega$.*
We denote by $\widehat{\mathcal{R}es}_\Omega^{sum}$ the set of summable-resurgent functions with singular support Ω.

Definition 1.2 (Summable-resurgent series)
A formal series is said to be a summable-resurgent series with singular support Ω *when its formal Borel transform belongs to $\widehat{\mathcal{R}es}_\Omega^{sum}$.*
The set of summable-resurgent series with singular support Ω is denoted $\widetilde{\mathcal{R}es}_\Omega^{sum}$.

Recall that the formal Borel transformation $\widetilde{\mathcal{B}}$ is an isomorphism from the $\mathbb{C}$-differential algebra $(\mathbb{C}[[t]], +, \cdot, t^2\frac{d}{dt})$ to the $\mathbb{C}$-differential algebra $(\delta\mathbb{C} \oplus \mathbb{C}[[\tau]], +, *, \tau\cdot)$ that changes ordinary product $\cdot$ into convolution product $*$ and changes derivation $t^2\frac{d}{dt}$ into multiplication by τ. It also changes multiplication by $\frac{1}{t}$ into derivation $\frac{d}{d\tau}$ allowing thus to extend the isomorphism from the meromorphic series $\mathbb{C}[[t]][t^{-1}]$ to $\mathbb{C}[\delta^{(k)}, k \in \mathbb{N}] \oplus \mathbb{C}[[\tau]]$.

Under our hypothesis of "a single level equal to r" (*cf.* assumption (0.3)), we proved the following result in [10]:

Theorem 1.3 (Summable-resurgence theorem, [10, thm. 1.2])
Let $\Omega = \{a_{j,r}\ ,\ j = 1, ..., J\}$ denote the set of Stokes values of system (0.1). Then, for all $u = 0, ..., r-1$,

$$\boxed{\widetilde{\boldsymbol{f}}^{[u]}(t) \in \widetilde{\mathcal{R}es}_{\Omega}^{sum}}$$

1.2 Singularities in the Borel plane

For the convenience of the reader, we first recall some vocabulary used in resurgence theory (see [3, 8, 11] for instance).

Denote by $\mathcal{O}$ the space of holomorphic germs at 0 on $\mathbb{C}$ and $\widetilde{\mathcal{O}}$ the space of holomorphic germs at 0 on the Riemann surface $\widetilde{\mathbb{C}}$ of the logarithm. One calls *singularity at* 0 any element of the quotient space $\mathcal{C} := \widetilde{\mathcal{O}}/\mathcal{O}$ [1]. The canonical quotient map $\widetilde{\mathcal{O}} \longrightarrow \mathcal{C}$ is denoted by "can".

A singularity is usually denoted with a nabla. A representative of the singularity $\overset{\nabla}{\varphi}$ in $\widetilde{\mathcal{O}}$ is called *a major* of $\overset{\nabla}{\varphi}$ and is often denoted by $\breve{\varphi}$: $\mathrm{can}(\breve{\varphi}) = \overset{\nabla}{\varphi}$.

Given $\omega \neq 0$ in $\mathbb{C}$, we denote by $\mathcal{C}_\omega$ the space of the singularities at ω, *i.e.*, the space $\mathcal{C}$ translated from 0 to ω. Then, a function $\breve{\varphi}_\omega$ is a major of a singularity at ω if $\breve{\varphi}_\omega(\omega + \tau)$ is a major of a singularity at 0.

1.2.1 General structure of singularities

Theorem 1.3 tells us that, for all $u = 0, ..., r-1$, the Borel transform $\widehat{\boldsymbol{f}}^{[u]}(\tau)$ of $\widetilde{\boldsymbol{f}}^{[u]}(t)$ is analytic on the Riemann surface $\mathcal{R}_\Omega$, its possible singular points being the Stokes values of Ω including 0 out of the first sheet.

For any Stokes value $\omega \in \Omega$, we call *front of* ω the set

$$Fr(\omega) := \{q_j \text{ such that } a_{j,r} = \omega\}$$

of polynomials $q_j(1/x)$'s, the leading term of which is $-\omega/x^r$. Under our hypothesis of a single level (*cf.* assumption (0.3)), $Fr(\omega)$ is a singleton:

$$Fr(\omega) = \left\{ -\frac{\omega}{x^r} + \dot{q}_\omega\left(\frac{1}{x}\right) \right\}$$

where $\dot{q}_\omega \equiv 0$ or $\dot{q}_\omega(1/x)$ is a polynomial in $1/x$ of degree $\leq r-1$ and with no constant term.

[1] The elements of $\mathcal{C}$ are also called *micro-functions* by B. Malgrange ([8]) by analogy with hyper- and micro-functions defined by Sato, Kawai and Kashiwara in higher dimensions.

Definition 1.4 (Singularity with monomial front)
A Stokes value $\omega \in \Omega$ *is said to be* a singular point with monomial front *when* $\dot{q}_\omega \equiv 0$. *The corresponding singularity is called* singularity with monomial front.

In the case of level-one systems (case $r = 1$), all singularities are with monomial front and they are regular, *i.e.*, in the Nilsson class ([2]). A more precise description was displayed in [6, thm. 3.7].

For systems with single level $r \geq 2$, the situation is much more involved since $\dot{q}_\omega$ is not necessarily zero. This polynomial plays an essential role in the structure of the singularities. Indeed, one can show that the singularity at $\omega \in \Omega$ is irregular when $\dot{q}_\omega \not\equiv 0$ and keeps being regular when $\dot{q}_\omega \equiv 0$. For a general description of singularities, we refer to [10, thm. 2.13].

Actually, as we shall show in sections 2 and 3, it is sufficient to know the regular structure of the singularities with monomial front to make explicit and to prove the connection-to-Stokes formulæ in full generality. For such singularities, a more precise description than the one of [10, thm. 2.13] is displayed in [10, thm. 3.5]. We recall it in theorem 1.5 below.

1.2.2 Singularities with monomial front

For all $u = 0, ..., r-1$, the behavior of the function $\widehat{\boldsymbol{f}}^{[u]}$ at any Stokes value $\omega \in \Omega$ depends on the sheet of the Riemann surface $\mathcal{R}_\Omega$ where we are, *i.e.*, it depends on the "homotopic class of" the path γ of analytic continuation followed from 0 (first sheet) to a neighborhood of ω. We denote by $\mathrm{cont}_{\omega,\gamma}\widehat{\boldsymbol{f}}^{[u]}$ the analytic continuation of $\widehat{\boldsymbol{f}}^{[u]}$ along the path γ and by $\overset{\nabla}{\boldsymbol{f}}{}^{[u]}_{\omega,\gamma}$ the corresponding singularity.

From now on, given a matrix M split into blocks fitting to the Jordan structure of L (matrix of exponents of formal monodromy), we denote by $M^{j;\bullet}$ the j-th row-block of M. So, $M^{j;\bullet}$ is a $n_j \times p$-matrix when M is a $n \times p$-matrix (recall that n_j is the size of the j-th Jordan block of L).

Theorem 1.5 (Singularities with monomial front, [10, thm. 3.5])
Fix $u \in \{0, ..., r-1\}$ *and* $\omega \in \Omega\backslash\{0\}$ *a singular point of* $\widehat{\boldsymbol{f}}^{[u]}$ *with monomial front.*

For any path γ *on* $\mathbb{C}\backslash\Omega$ *from* 0 *to a neighborhood of* ω, *the singularity* $\overset{\nabla}{\boldsymbol{f}}{}^{[u]}_{\omega,\gamma}$ *admits a major* $\breve{\boldsymbol{f}}^{[u]}_{\omega,\gamma}$ *of the form*

$$\boxed{\breve{\boldsymbol{f}}^{[u]j;\bullet}_{\omega,\gamma}(\omega+\tau) = \tau^{\frac{\lambda_j - u}{r} - 1}\tau^{\frac{J_{n_j}}{r}}\boldsymbol{K}^{[u]j;\bullet}_{\omega,\gamma}\tau^{-\frac{J_{n_1}}{r}} + rem^{[u]j;\bullet}_{\omega,\gamma}(\tau)}$$

for all $j = 1, ..., J$ *with a remainder*

$$rem^{[u]j;\bullet}_{\omega,\gamma}(\tau) = \sum_{\lambda_\ell; a_{\ell,r}=\omega} \sum_{v=0}^{r-1} \tau^{\frac{\lambda_\ell - v}{r}} \boldsymbol{R}^{[u]j;\bullet}_{\lambda_\ell,v;\omega,\gamma}(\ln \tau)$$

where

- $\boldsymbol{K}^{[u]j;\bullet}_{\omega,\gamma}$ *denotes a constant* $n_j \times n_1$*-matrix such that* $\boldsymbol{K}^{[u]j;\bullet}_{\omega,\gamma} = 0$ *when* $a_{j,r} \neq \omega$*,*

- $\boldsymbol{R}^{[u]j;\bullet}_{\lambda_\ell,v;\omega,\gamma}(X)$ *denotes a polynomial matrix with summable-resurgent coefficients in* $\widehat{\mathcal{R}es}^{sum}_{\Omega-\omega}$*, the columns of which are of log-degree*

$$N[\ell] = \begin{cases} \begin{bmatrix} (n_\ell - 1) & (n_\ell - 1) + 1 & \cdots & (n_\ell - 1) + (n_1 - 1) \end{bmatrix} & \text{if } \lambda_\ell \neq 0 \\ \begin{bmatrix} n_\ell & n_\ell + 1 & \cdots & n_\ell + (n_1 - 1) \end{bmatrix} & \text{if } \lambda_\ell = 0 \end{cases}$$

The constants $\boldsymbol{K}^{[u]j;\bullet}_{\omega,\gamma}$ and the remainders $rem^{[u]j;\bullet}_{\omega,\gamma}$ depend on the path of analytic continuation γ and on the chosen determination of the argument around ω. From now on,

- we consider a path γ^+ going along the straight line $[0, \omega]$ from 0 to a point τ close to ω and avoiding all singular points of $\Omega \cap]0, \omega]$ to the right (see figure 1.2 below),

- we choose the principal determination of the variable τ around ω, say $\arg(\tau) \in]-2\pi, 0]$.

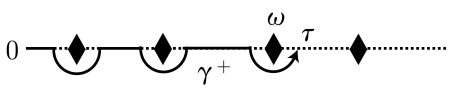

Figure 1.2

For such choices, we respectively denote $\overset{\nabla}{\boldsymbol{f}}^{[u]}_{\omega^\star,+}$, $\boldsymbol{K}^{[u]j;\bullet}_{\omega^\star,+}$ and $rem^{[u]j;\bullet}_{\omega^\star,+}$ for $\overset{\nabla}{\boldsymbol{f}}^{[u]}_{\omega,\gamma}$, $\boldsymbol{K}^{[u]j;\bullet}_{\omega,\gamma}$ and $rem^{[u]j;\bullet}_{\omega,\gamma}$.

Definition 1.6 (Principal major and connection constants)

- *Given $u \in \{0, ..., r-1\}$ and a singular point $\omega \in \Omega \backslash \{0\}$ of $\widehat{\boldsymbol{f}}^{[u]}$ with monomial front, we call* principal major of $\widehat{\boldsymbol{f}}^{[u]}$ at ω *the major $\check{\boldsymbol{f}}^{[u]}_{\omega^\star,+}$ of $\overset{\nabla}{\boldsymbol{f}}^{[u]}_{\omega^\star,+}$ given in theorem 1.5. Recall that, for all $j = 1, ..., J$,*

$$\boxed{\check{\boldsymbol{f}}^{[u]j;\bullet}_{\omega^\star,+}(\omega+\tau) = \tau^{\frac{\lambda_j - u}{r} - 1} \tau^{\frac{J_{n_j}}{r}} \boldsymbol{K}^{[u]j;\bullet}_{\omega^\star,+} \tau^{-\frac{J_{n_1}}{r}} + rem^{[u]j;\bullet}_{\omega^\star,+}(\tau)} \tag{1.2}$$

- *The entries of the matrices $\boldsymbol{K}^{[u]j;\bullet}_{\omega^\star,+}$ when $a_{j,r} = \omega$ are called* the connection constants of $\widehat{\boldsymbol{f}}^{[u]}$ at ω. Recall that $\boldsymbol{K}^{[u]j;\bullet}_{\omega^\star,+} = 0$ when $a_{j,r} \neq \omega$.

Note that, in practice, the matrix $\boldsymbol{K}^{[u]j;\bullet}_{\omega^\star,+}$ can be determined as the coefficient of the monomial $\tau^{(\lambda_j - u)/r - 1}$.

Let us end this section with an example which will be resumed throughout the article in order to illustrate our different results.

Example 1.7 We consider the system

$$x^3 \frac{dY}{dx} = \begin{bmatrix} 0 & 0 & 0 \\ x^4 - x^5 & 2 & 0 \\ x^4 + x^5 & 0 & 4 + \frac{x^2}{2} \end{bmatrix} Y \tag{1.3}$$

and its formal fundamental solution $\widetilde{Y}(x) = \widetilde{F}(x) x^L e^{Q(1/x)}$ where

- $Q\left(\frac{1}{x}\right) = \operatorname{diag}\left(0, -\frac{1}{x^2}, -\frac{2}{x^2}\right)$,
- $L = \operatorname{diag}\left(0, 0, \frac{1}{2}\right)$,
- $\widetilde{F}(x) = \begin{bmatrix} 1 & 0 & 0 \\ \widetilde{f}_2(x) & 1 & 0 \\ \widetilde{f}_3(x) & 0 & 1 \end{bmatrix}$ is a power series such that $\widetilde{F}(x) = I_3 + O(x^4)$.

System (1.3) has the unique level 2 and the set of its Stokes values is $\Omega = \{0, 1, 2\}$.

Although system (1.3) may seem a little bit involved since it admits two aligned non-zero Stokes values, the fact that its matrix is triangular makes it simple enough to allow the *exact* calculation of the connection constants and the Stokes multipliers. For a more general system, *.i.e.*, the matrix of which

is no longer triangular, such exact calculations no longer hold in general.

The 2-reduced series of the first column of $\widetilde{F}(x)$ are of the form

$$\widetilde{\boldsymbol{f}}^{[0]}(t) = \begin{bmatrix} 1 \\ \widetilde{\boldsymbol{f}}_2(t) \\ \widetilde{\boldsymbol{f}}_3(t) \end{bmatrix} \quad \text{and} \quad \widetilde{\boldsymbol{f}}^{[1]}(t) = \begin{bmatrix} 0 \\ \widetilde{\boldsymbol{f}}_5(t) \\ \widetilde{\boldsymbol{f}}_6(t) \end{bmatrix}$$

where the $\widetilde{\boldsymbol{f}}_j$'s are power series in t satisfying $\widetilde{\boldsymbol{f}}_j(t) = O(t^2)$ (*cf.* (1.1)). Our aim is the calculation of the connection constants of the $\widehat{\boldsymbol{f}}^{[u]}$'s, $u = 0, 1$, at the Stokes values $\tau = 1$ and $\tau = 2$.

By using rank reduction ([5]), we can check that the matrix

$$\widetilde{\boldsymbol{f}}(t) := \begin{bmatrix} \widetilde{\boldsymbol{f}}^{[0]}(t) \\ \widetilde{\boldsymbol{f}}^{[1]}(t) \end{bmatrix} \in M_{6,1}(\mathbb{C}[[t]])$$

is uniquely determined by the system

$$2t^2 \frac{d\widetilde{\boldsymbol{f}}}{dt} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ t^2 & 2 & 0 & -t^3 & 0 & 0 \\ t^2 & 0 & 4+\frac{t}{2} & t^3 & 0 & 0 \\ 0 & 0 & 0 & -t & 0 & 0 \\ -t^2 & 0 & 0 & t^2 & 2-t & 0 \\ t^2 & 0 & 0 & t^2 & 0 & 4-\frac{t}{2} \end{bmatrix} \widetilde{\boldsymbol{f}}$$

jointly with the initial condition $\widetilde{\boldsymbol{f}}(0) = I_{6,1}$ (first column of the identity matrix I_6 of size 6). Therefore, the formal series $\widetilde{\boldsymbol{f}}_j$'s are the unique solutions of the equations

$$2t^2 \frac{d\widetilde{\boldsymbol{f}}_2}{dt} - 2\widetilde{\boldsymbol{f}}_2 = t^2 \qquad 2t^2 \frac{d\widetilde{\boldsymbol{f}}_5}{dt} - (2-t)\widetilde{\boldsymbol{f}}_5 = -t^2$$

$$2t^2 \frac{d\widetilde{\boldsymbol{f}}_3}{dt} - \left(4 + \frac{t}{2}\right)\widetilde{\boldsymbol{f}}_3 = t^2 \qquad 2t^2 \frac{d\widetilde{\boldsymbol{f}}_6}{dt} - \left(4 - \frac{t}{2}\right)\widetilde{\boldsymbol{f}}_6 = t^2$$

satisfying the condition $\widetilde{\boldsymbol{f}}_j(t) = O(t^2)$. As a result, their Borel transforms $\widehat{\boldsymbol{f}}_j$'s are defined, for all $|\tau| < 1$, by

$$\boxed{\begin{array}{ll} \widehat{\boldsymbol{f}}_2(\tau) = -\dfrac{\tau}{2(1-\tau)} & \widehat{\boldsymbol{f}}_5(\tau) = \dfrac{1}{3}(1-\tau)^{-3/2} - \dfrac{1}{3} \\[2ex] \widehat{\boldsymbol{f}}_3(\tau) = -\dfrac{2^{7/4}}{3}(2-\tau)^{-3/4} + \dfrac{2}{3} & \widehat{\boldsymbol{f}}_6(\tau) = -\dfrac{2^{9/4}}{5}(2-\tau)^{-5/4} + \dfrac{2}{5} \end{array}}$$

In particular, $\widehat{\boldsymbol{f}}_2$ and $\widehat{\boldsymbol{f}}_5$ (resp. $\widehat{\boldsymbol{f}}_3$ and $\widehat{\boldsymbol{f}}_6$) have just one singular point, located at the Stokes value $\tau = 1$ (resp. $\tau = 2$). More precisely, their analytic continuation $\widehat{\boldsymbol{f}}_j^+$ to the right of these points are defined by

$$\boxed{\begin{array}{ll} \widehat{\boldsymbol{f}}_2^+(1+\tau) = \dfrac{\tau+1}{2\tau} & \widehat{\boldsymbol{f}}_5^+(1+\tau) = \dfrac{i}{3}\tau^{-3/2} - \dfrac{1}{3} \\[2ex] \widehat{\boldsymbol{f}}_3^+(2+\tau) = -\dfrac{2^{7/4}e^{-\frac{3i\pi}{4}}}{3}\tau^{-3/4} + \dfrac{2}{3} & \widehat{\boldsymbol{f}}_6^+(2+\tau) = -\dfrac{2^{9/4}e^{-\frac{5i\pi}{4}}}{5}\tau^{-5/4} + \dfrac{2}{5} \end{array}}$$

Consequently, the connection matrices $\boldsymbol{K}_{1,+}^{[u]}$ and $\boldsymbol{K}_{2,+}^{[u]}$ are given by

$$\boxed{\begin{array}{ll} \boldsymbol{K}_{1,+}^{[0]} = \begin{bmatrix} 0 \\ k_{1,+}^{[0]2} = \dfrac{1}{2} \\ 0 \end{bmatrix} & \boldsymbol{K}_{2,+}^{[0]} = \begin{bmatrix} 0 \\ 0 \\ k_{2,+}^{[0]3} = -\dfrac{2^{7/4}}{3}e^{-3i\pi/4} \end{bmatrix} \\[4ex] \boldsymbol{K}_{1,+}^{[1]} = \begin{bmatrix} 0 \\ k_{1,+}^{[1]2} = \dfrac{i}{3} \\ 0 \end{bmatrix} & \boldsymbol{K}_{2,+}^{[1]} = \begin{bmatrix} 0 \\ 0 \\ k_{2,+}^{[1]3} = -\dfrac{2^{9/4}}{5}e^{-5i\pi/4} \end{bmatrix} \end{array}}$$

We end the study of system (1.3) with the calculation of its Stokes multipliers in section 2.2.1 (see example 2.3).

2 Stokes-Ramis matrices and connection constants

2.1 Stokes-Ramis automorphisms

Given a non anti-Stokes direction $\theta \in \mathbb{R}/2\pi\mathbb{Z}$ of system (0.1) and a choice of an argument of θ, say its principal determination $\theta^\star \in]-2\pi, 0]$ as previously[2], we consider the sum of $\widetilde{Y}$ in the direction θ given by

$$Y_\theta(x) = s_{r;\theta}(\widetilde{F})(x)Y_{0,\theta^\star}(x)$$

[2] Any choice is convenient. However, to be compatible, on the Riemann sphere, with the usual choice $0 \leq \arg(z = 1/x) < 2\pi$ of the principal determination at infinity, we suggest to choose $-2\pi < \arg(x) \leq 0$ as principal determination about 0 as well as about any ω at finite distance.

where $s_{r;\theta}(\widetilde{F})(x)$ is the uniquely determined r-sum of $\widetilde{F}$ at θ and where $Y_{0,\theta^\star}(x)$ is the actual analytic function $Y_{0,\theta^\star}(x) := x^L e^{Q(1/x)}$ defined by the choice $\arg(x)$ close to $\theta^\star$ (denoted below by $\arg(x) \simeq \theta^\star$). Recall that $s_{r;\theta}(\widetilde{F})$ is an analytic function defined and $\frac{1}{r}$-Gevrey asymptotic to $\widetilde{F}$ on a sector bisected by θ with opening larger than π/r.

When $\theta \in \mathbb{R}/2\pi\mathbb{Z}$ is an anti-Stokes direction of system (0.1), we consider the two lateral sums $s_{r;\theta^-}(\widetilde{F})$ and $s_{r;\theta^+}(\widetilde{F})$ respectively obtained as analytic continuations of $s_{r;\theta-\eta}(\widetilde{F})$ and $s_{r;\theta+\eta}(\widetilde{F})$ to a sector with vertex 0, bisected by θ and opening π/r. Notice that such analytic continuations exist without ambiguity when $\eta > 0$ is small enough. We denote by Y_{θ^-} and Y_{θ^+} the two sums of $\widetilde{Y}$ respectively defined for $\arg(x) \simeq \theta^\star$ by $Y_{\theta^-}(x) := s_{r;\theta^-}(\widetilde{F})(x)Y_{0,\theta^\star}(x)$ and $Y_{\theta^+}(x) := s_{r;\theta^+}(\widetilde{F})(x)Y_{0,\theta^\star}(x)$.

The two lateral sums $s_{r;\theta^-}(\widetilde{F})$ and $s_{r;\theta^+}(\widetilde{F})$ of $\widetilde{F}$ are not analytic continuations from each other in general. This fact is the *Stokes phenomenon* of system (0.1). It is characterized by the collection, for all anti-Stokes directions $\theta \in \mathbb{R}/2\pi\mathbb{Z}$ of system (0.1), of the automorphisms

$$St_{\theta^\star} : Y_{\theta^+} \longmapsto Y_{\theta^-}$$

that one calls *Stokes-Ramis automorphisms relative to* $\widetilde{Y}$.

Definition 2.1 (Stokes-Ramis matrices)
One calls Stokes-Ramis matrix associated with $\widetilde{Y}$ in the direction θ *the matrix of* $St_{\theta^\star}$ *in the basis* Y_{θ^+} [3]. *We denote it by* $I_n + C_{\theta^\star}$.

Note that the matrix $I_n + C_{\theta^\star}$ is uniquely determined by the relation

$$\boxed{Y_{\theta^-}(x) = Y_{\theta^+}(x)(I_n + C_{\theta^\star}) \qquad \text{for } \arg(x) \simeq \theta^\star}$$

2.2 Relations between Stokes multipliers and connection constants

Recall that the set Ω denotes the set of Stokes values $a_{j,r}$ of system (0.1). Given a direction $\theta \in \mathbb{R}/2\pi\mathbb{Z}$, we denote by

- d_θ the half line issuing from 0 with argument θ,

[3] In the literature, a Stokes matrix has a more general meaning where one allows to compare any two asymptotic solutions whose domains of definition overlap. According to the custom initiated by J.-P. Ramis ([9]) in the spirit of Stokes' work, we exclude this case here. We consider only matrices providing the transition between the sums on each side of a same anti-Stokes direction.

- $\Omega_\theta := \Omega^* \cap d_\theta$ with $\Omega^* = \Omega\backslash\{0\}$ the set of non-zero Stokes values of system (0.1) with argument θ.

The *anti-Stokes directions of system (0.1) associated with* $\widetilde{f}$ are the directions of maximal decay of the exponentials $e^{q_j(1/x)}$, $q_j \not\equiv 0$. Therefore, to each non-zero polynomial q_j $-i.e.$, such that $a_{j,r} \neq 0$ under our hypothesis of a single level (*cf.* assumption (0.3))$-$ correspond r anti-Stokes directions $\theta_0, \theta_1, ..., \theta_{r-1} \in \mathbb{R}/2\pi\mathbb{Z}$ regularly distributed around $x = 0$. They are given by the arguments of the r-th roots of $a_{j,r}$; then, we say that $a_{j,r}$ *generates the collection* $(\theta_k)_{k=0,...,r-1}$[4].

Such a collection being chosen, we assume, to fix ideas, that the principal determinations $\theta_k^\star$ satisfy

$$-2\pi < \theta_{r-1}^\star < ... < \theta_1^\star < \theta_0^\star \leq 0$$

A Stokes value $\omega \in \Omega^*$ generates the collection $(\theta_k)_{k=0,...,r-1}$ if and only if $\omega \in \Omega_{r\theta_0}$.

Let $\rho := e^{-2i\pi/r}$. For all $k = 0, ..., r-1$, the Stokes-Ramis matrix $I_n + C_{\theta_k^\star}$ of $\widetilde{Y}$ in the direction θ_k is uniquely determined by the relation

$$\boxed{Y_{\theta_k^-}(\rho^k x) = Y_{\theta_k^+}(\rho^k x)(I_n + C_{\theta_k^\star}) \qquad \text{for } \arg(x) \simeq \theta_0^\star} \tag{2.1}$$

We denote by $c_{\theta_k^\star}$ the first n_1 columns of $C_{\theta_k^\star}$ and we split $c_{\theta_k^\star}$ into row-blocks $c_{\theta_k^\star}^{j;\bullet}$ accordingly to the Jordan structure of L (we refer to page 8 for the notations). The $n_j \times n_1$-matrix $c_{\theta_k^\star}^{j;\bullet}$ is zero for all $k = 0, ..., r-1$ as soon as $a_{j,r} \notin \Omega_{r\theta_0}$. When $a_{j,r} = \omega \in \Omega_{r\theta_0}$, the entries of $c_{\theta_k^\star}^{j;\bullet}$ are called *Stokes multipliers of* $\widetilde{f}$ *associated with* ω *in the direction* θ_k.

2.2.1 Case of singularities with monomial front

Recall that

- $\rho = e^{-2i\pi/r}$,
- the $\widetilde{\boldsymbol{f}}^{[u]}$'s, $u = 0, ..., r-1$, denote the r-reduced series of $\widetilde{f}$ (*cf.* (1.1)),
- the singularities of the Borel transforms $\widehat{\boldsymbol{f}}^{[u]}$'s of the $\widetilde{\boldsymbol{f}}^{[u]}$'s are located at the Stokes values of system (0.1) (*cf.* thm. 1.3).

[4]From now on, we say that a point $\omega \neq 0$ generates a collection of r directions $\alpha_0, \alpha_1, ..., \alpha_{r-1} \in \mathbb{R}/2\pi\mathbb{Z}$ regularly distribued around 0 when $\omega \in d_{r\alpha_0}$.

We denote moreover by $L_j := \lambda_j I_{n_j} + J_{n_j}$ the j-th Jordan block of L (matrix of exponents of formal monodromy).

Let $\omega \in \Omega_{r\theta_0}$ be a non-zero Stokes value of system (0.1) generating the collection $(\theta_k)_{k=0,\dots,r-1}$. We assume besides, in this section, that the front of ω is *monomial*.

Under this hypothesis, theorem 2.2 below tells us that the Stokes multipliers of $\widetilde{f}$ associated with ω are expressed in terms of the connection constants at ω of the $\widehat{\boldsymbol{f}}^{[u]}$'s, $u = 0, ..., r-1$.

Theorem 2.2 (Connection-to-Stokes formulæ, [10, thm. 4.4])
For all j such that $a_{j,r} = \omega$, the data of $(c^{j;\bullet}_{\theta^\star_k})_{k=0,\dots,r-1}$ and of $(\boldsymbol{K}^{[u]j;\bullet}_{\omega^\star,+})_{u=0,\dots,r-1}$ are equivalent and are related, for all $k = 0, ..., r-1$, by the relations

$$\boxed{c^{j;\bullet}_{\theta^\star_k} = \sum_{u=0}^{r-1} \rho^{k(uI_{n_j} - L_j)} \boldsymbol{I}^{[u]j;\bullet}_{\omega^\star} \rho^{kJ_{n_1}}} \tag{2.2}$$

where

$$\boxed{\boldsymbol{I}^{[u]j;\bullet}_{\omega^\star} := \int_{\gamma_0} \tau^{\frac{\lambda_j - u}{r} - 1} \tau^{\frac{J_{n_j}}{r}} \boldsymbol{K}^{[u]j;\bullet}_{\omega^\star,+} \tau^{-\frac{J_{n_1}}{r}} e^{-\tau} d\tau} \tag{2.3}$$

and where γ_0 is a Hankel type path around the non-negative real axis $\mathbb{R}^+$ with argument from -2π to 0.

The proof given in [10, § 4.3] of theorem 2.2 is, on the one hand, based on the relations between the Stokes-Ramis matrices of system (0.1) and those of its reduced system ([5, prop. 4.2]) and requires, on the other hand, to know explicitly the structure of *all* singularities in the Borel plane (*i.e.*, with monomial front or not). The proof that we shall give in section 3 is quite different since it is based on a regular perturbation of the matrix $A(x)$ of system (0.1) and a limit process. As a consequence, we show in particular that it is sufficient to know the structure of singularities with monomial front.

An expanded form providing each entry of the connection-to-Stokes formulæ (2.2) is given in [10, cor. 4.6]. This can be useful for effective numerical calculations. We recall this expanded form below in the particular case where the matrix L of exponents of formal monodromy is diagonal: $L = \bigoplus_{j=1}^n \lambda_j$.

In this case, the matrices $c^{j;\bullet}_{\theta^\star_k}$ and $\boldsymbol{K}^{[u]j;\bullet}_{\omega^\star,+}$ are reduced to just one entry which we respectively denote by $c^{j}_{\theta^\star_k}$ and $\boldsymbol{K}^{[u]j}_{\omega^\star,+}$.

Since the Jordan blocks J_{n_j} are zero for all j, identity (2.3) becomes

$$\int_{\gamma_0} \tau^{\frac{\lambda_j-u}{r}-1} \boldsymbol{K}^{[u]j}_{\omega^\star,+} e^{-\tau} d\tau = 2i\pi \frac{e^{-i\pi\frac{\lambda_j-u}{r}}}{\Gamma\left(1-\frac{\lambda_j-u}{r}\right)} \boldsymbol{K}^{[u]j}_{\omega^\star,+}$$

Therefore, the Stokes multipliers $c^j_{\theta^\star_k}$ are related to the connection constants $\boldsymbol{K}^{[u]j}_{\omega^\star,+}$ by the formulæ

$$\boxed{c^j_{\theta^\star_k} = 2i\pi \sum_{u=0}^{r-1} \rho^{k(u-\lambda_j)} \frac{e^{-i\pi\frac{\lambda_j-u}{r}}}{\Gamma\left(1-\frac{\lambda_j-u}{r}\right)} \boldsymbol{K}^{[u]j}_{\omega^\star,+} \quad \text{for all } k=0,...,r-1} \tag{2.4}$$

Let us end this section with a numerical application of theorem 2.2.

Example 2.3 We calculate here below the Stokes multipliers of system (1.3) by using theorem 2.2 and formula (2.4) above.

With notations as example 1.7, the anti-Stokes directions of system (1.3) associated with the first column $\widetilde{f}(x)$ of $\widetilde{F}(x)$ are $\theta_0 = 0$ and $\theta_1 = -\pi$ (the directions of maximal decay of the exponentials e^{-1/x^2} and e^{-2/x^2}). Obviously, the Stokes-Ramis matrices $I_3 + C_0$ and $I_3 + C_{-\pi}$ are of the form

$$C_0 = \begin{bmatrix} 0 & 0 & 0 \\ c^2_0 & 0 & 0 \\ c^3_0 & 0 & 0 \end{bmatrix} \qquad \text{and} \qquad C_{-\pi} = \begin{bmatrix} 0 & 0 & 0 \\ c^2_{-\pi} & 0 & 0 \\ c^3_{-\pi} & 0 & 0 \end{bmatrix}$$

Indeed, $\widetilde{f}(x)$ is the unique column of $\widetilde{F}(x)$ which is divergent.

The collection $(0, -\pi)$ is generated by the two Stokes values 1 and 2 ($\Omega_0 = \{1, 2\}$) which are both with monomial front. Therefore, we deduce, from theorem 2.2, that the Stokes multipliers c^2_0 and $c^2_{-\pi}$ (resp. c^3_0 and $c^3_{-\pi}$) are expressed in terms of the connection constants at $\tau = 1$ (resp. $\tau = 2$) of

$\widehat{\boldsymbol{f}}^{[0]}(\tau)$ and $\widehat{\boldsymbol{f}}^{[1]}(\tau)$. Precisely, since L is diagonal, it results from (2.4) that

$$
\boxed{
\begin{aligned}
c_0^2 &= 2i\pi k_{1,+}^{[0]2} + 2i\pi \frac{e^{\frac{i\pi}{2}}}{\Gamma\left(\frac{3}{2}\right)} k_{1,+}^{[1]2} \\
c_0^3 &= 2i\pi \frac{e^{-\frac{i\pi}{4}}}{\Gamma\left(\frac{3}{4}\right)} k_{2,+}^{[0]3} + 2i\pi \frac{e^{\frac{i\pi}{4}}}{\Gamma\left(\frac{5}{4}\right)} k_{2,+}^{[1]3} \\
c_{-\pi}^2 &= 2i\pi k_{1,+}^{[0]2} + 2i\pi e^{-i\pi} \frac{e^{\frac{i\pi}{2}}}{\Gamma\left(\frac{3}{2}\right)} k_{1,+}^{[1]2} \\
c_{-\pi}^3 &= 2i\pi e^{\frac{i\pi}{2}} \frac{e^{-\frac{i\pi}{4}}}{\Gamma\left(\frac{3}{4}\right)} k_{2,+}^{[0]3} + 2i\pi e^{-\frac{i\pi}{2}} \frac{e^{\frac{i\pi}{4}}}{\Gamma\left(\frac{5}{4}\right)} k_{2,+}^{[1]3}
\end{aligned}
} \tag{2.5}
$$

(recall that $\rho = e^{-i\pi}$ since system (1.3) has the unique level 2). Hence,

$$
\boxed{
\begin{aligned}
c_0^2 &= \left(\pi - \frac{4\sqrt{\pi}}{3}\right) i & c_0^3 &= 2^{3/4}\left(\frac{4\pi}{3\Gamma(\frac{3}{4})} + \frac{16}{5}\Gamma(\tfrac{3}{4})\right) i \\
c_{-\pi}^2 &= \left(\pi + \frac{4\sqrt{\pi}}{3}\right) i & c_{-\pi}^3 &= -2^{3/4}\left(\frac{4\pi}{3\Gamma(\frac{3}{4})} - \frac{16}{5}\Gamma(\tfrac{3}{4})\right)
\end{aligned}
} \tag{2.6}
$$

2.2.2 General case

Let $\omega \in \Omega_{r\theta_0}$ be a non-zero Stokes value of system (0.1) generating the collection $(\theta_k)_{k=0,\dots,r-1}$. Recall that the front of ω reads

$$
Fr(\omega) = \left\{ q_\omega\left(\frac{1}{x}\right) = -\frac{\omega}{x^r} + \dot{q}_\omega\left(\frac{1}{x}\right) \right\}
$$

where $\dot{q}_\omega \equiv 0$ or $\dot{q}_\omega(1/x)$ is a polynomial in $1/x$ of degree $\leq r-1$ and with no constant term (*cf.* section 1.2.1).

When ω is with monomial front (*i.e.*, $\dot{q}_\omega \equiv 0$), theorem 2.2 above allows us to express the Stokes multipliers of $\widetilde{f}$ associated with ω in terms of connection constants in the Borel plane.

When ω is with non-monomial front (*i.e.*, $\dot{q}_\omega \not\equiv 0$), a result of the same type exists but requires to reduce ω into a Stokes value with monomial front by means of lemma 2.4 below.

Lemma 2.4 (M. Loday-Richaud, [4])

1. *There exists, in the x-plane (Laplace plane), a change of the variable x of the form*

$$x = \frac{y}{1 + \alpha_1 y + ... + \alpha_{r-1} y^{r-1}} \qquad , \alpha_1, ..., \alpha_{r-1} \in \mathbb{C} \tag{2.7}$$

such that the polar part $p_\omega(1/y)$ of $q_\omega(1/x(y))$ reads

$$p_\omega\left(\frac{1}{y}\right) = -\frac{\omega}{y^r}$$

2. *The Stokes-Ramis matrices of system (0.1) are preserved by the change of variable (2.7).*

Indeed, lemma 2.4 allows us to construct a *new* system (S) such that

- (S) has the unique level r and satisfies normalizations as system (0.1) (*cf.* page 2),
- ω is a Stokes value of (S) and is with monomial front,
- systems (S) and (0.1) have the same Stokes-Ramis matrices.

Hence, applying theorem 2.2 to (S), we can again express the Stokes multipliers of $\widetilde{f}$ associated with ω in terms of connection constants in the Borel plane. Note however that these constants are calculated from system (S) and not from system (0.1). A numerical example was treated in detail in [10, § 5.3].

3 Proof of theorem 2.2

3.1 Case of a unique Stokes value

In this section, we assume that

$(A1)$ the collection $(\theta_k)_k$ is generated by a unique Stokes value $\omega \neq 0$ of system (0.1): $\Omega_{r\theta_0} = \{\omega\}$,

$(A2)$ the front of ω is monomial.

Recall that condition $(A2)$ can always be fulfilled by means of a change of the variable x (*cf.* lemma 2.4).

Given $k \in \{0, ..., r-1\}$, identity (2.1) reads

$$s_{r;\theta_k^-}(\widetilde{F})(\rho^k x) - s_{r;\theta_k^+}(\widetilde{F})(\rho^k x) = s_{r;\theta_k^+}(\widetilde{F})(\rho^k x) Y_{0,\theta_k^\star}(\rho^k x) C_{\theta_k^\star} Y_{0,\theta_k^\star}^{-1}(\rho^k x)$$

for $\arg(x) \simeq \theta_0^\star$. Recall that $Y_{0,\theta_k^\star}(X)$ is the actual analytic function $Y_{0,\theta_k^\star}(X) := X^L e^{Q(1/X)}$ defined by the choice of $\arg(X) \simeq \theta_k^\star$.

Since the matrices $c_{\theta_k^\star}^{j;\bullet}$ are zero as soon as $a_{j,r} \neq \omega$, we obtain, in restriction to the first n_1 columns,

$$\boxed{s_{r;\theta_k^-}(\widetilde{f})(\rho^k x) - s_{r;\theta_k^+}(\widetilde{f})(\rho^k x) = s_{r;\theta_k^+}(\widetilde{F})(\rho^k x) M_{k,\omega^\star}(x) e^{-\omega/x^r} \quad \text{for } \arg(x) \simeq \theta_0^\star} \tag{3.1}$$

where

$$\boxed{M_{k,\omega^\star}^{j;\bullet} = \begin{cases} 0_{n_j \times n_1} & \text{if } a_{j,r} \neq \omega \\ (\rho^k x)^{L_j} c_{\theta_k^\star}^{j;\bullet} (\rho^k x)^{-J_{n_1}} & \text{if } a_{j,r} = \omega \end{cases}} \tag{3.2}$$

for all $j = 1, ..., J$. Recall that n_j is the size of the j-th Jordan block $L_j = \lambda_j I_{n_j} + J_{n_j}$ of L. The matrix $0_{n_j \times n_1}$ denotes the $n_j \times n_1$-null matrix.

By definition (*cf.* (1.1)), the series $\widetilde{f}$ is related to its r-reduced series $\widetilde{\boldsymbol{f}}^{[u]}$'s by the formula

$$\widetilde{f}(x) = \sum_{u=0}^{r-1} x^u \widetilde{\boldsymbol{f}}^{[u]}(x^r)$$

Therefore, the r-sum $s_{r;\theta}(\widetilde{f})$ of $\widetilde{f}$ in a direction θ is related to the 1-sums (or Borel-Laplace sums) $s_{1;r\theta}(\widetilde{\boldsymbol{f}}^{[u]})$'s of the $\widetilde{\boldsymbol{f}}^{[u]}$'s in the direction $r\theta$. More precisely,

$$s_{r;\theta}(\widetilde{f})(x) = \sum_{u=0}^{r-1} x^u s_{1;r\theta}(\widetilde{\boldsymbol{f}}^{[u]})(x^r)$$

Consequently, the left hand side of (3.1) reads, for $\arg(x) \simeq \theta_0^\star$,

$$s_{r;\theta_k^-}(\widetilde{f})(\rho^k x) - s_{r;\theta_k^+}(\widetilde{f})(\rho^k x) = \sum_{u=0}^{r-1} (\rho^k x)^u \left(s_{1;r\theta_0^-}(\widetilde{\boldsymbol{f}}^{[u]})(x^r) - s_{1;r\theta_0^+}(\widetilde{\boldsymbol{f}}^{[u]})(x^r) \right) \tag{3.3}$$

Recall that, when θ is not an anti-Stokes direction for $\widetilde{\boldsymbol{f}}^{[u]}$ (*i.e.*, $\Omega_\theta = \varnothing$), the 1-sum $s_{1;\theta}(\widetilde{\boldsymbol{f}}^{[u]})(t)$ is given by the Borel-Laplace integral $\displaystyle\int_{d_\theta} \widehat{\boldsymbol{f}}^{[u]}(\tau) e^{-\tau/t} d\tau$ in the direction θ (*cf.* thm. 1.3 for instance).

When θ is an anti-Stokes direction for $\widetilde{\boldsymbol{f}}^{[u]}$, the 1-sums $s_{1;\theta^\pm}(\widetilde{\boldsymbol{f}}^{[u]})(t)$ to the left and right of θ are defined as the analytic continuations to a germ of half-plane bisected by θ of $s_{1;\theta\pm\eta}(\widetilde{\boldsymbol{f}}^{[u]})(t)$ as η tends to 0.

Thus, for $\eta > 0$ small enough, each term of the right hand side of (3.3) can be seen as the Laplace integral

$$s_{1;r\theta_0^-}(\widetilde{\boldsymbol{f}}^{[u]})(x^r) - s_{1;r\theta_0^+}(\widetilde{\boldsymbol{f}}^{[u]})(x^r) = \int_{\gamma_{r\theta_0}} \widehat{\boldsymbol{f}}^{[u]}(\tau)e^{-\tau/x^r}d\tau \tag{3.4}$$

where $\gamma_{r\theta_0}$ is the path going along the straight line $d_{r\theta_0+\eta}$ from infinity to 0 and going back to infinity along the straight line $d_{r\theta_0-\eta}$.

Due to the summable-resurgence of the $\widetilde{\boldsymbol{f}}^{[u]}$'s (*cf.* thm. 1.3) and hypothesis $(A1)$, the value of integral (3.4) is preserved by deforming the path $\gamma_{r\theta_0}$ into a Hankel type path $\gamma^+_{r\theta_0}(\omega)$ with asymptotic direction $r\theta_0$ around ω as shown on figure 3.1 below[5].

Hence, by means of a translation T from ω to 0 and using the fact that holomorphic functions at ω contributes 0 to the integral around ω, we can replace $\widehat{\boldsymbol{f}}^{[u]}$ by its principal major $\breve{\boldsymbol{f}}^{[u]}_{\omega^\star,+}(\omega+\tau)$ at ω obtaining so, for all $u = 0, ..., r-1$,

$$s_{1;r\theta_0^-}(\widetilde{\boldsymbol{f}}^{[u]})(x^r) - s_{1;r\theta_0^+}(\widetilde{\boldsymbol{f}}^{[u]})(x^r) = e^{-\omega/x^r}\int_{\gamma^+_{r\theta_0}} \breve{\boldsymbol{f}}^{[u]}_{\omega^\star,+}(\omega+\tau)e^{-\tau/x^r}d\tau \tag{3.5}$$

where, as shown on figure 3.2 below, $\gamma^+_{r\theta_0} := T(\gamma^+_{r\theta_0}(\omega))$ is the image of $\gamma^+_{r\theta_0}(\omega)$ by T. Recall that, since the front of ω is monomial (*cf.* assumption $(A2)$), the major $\breve{\boldsymbol{f}}^{[u]}_{\omega^\star,+}(\omega+\tau)$ is given by (1.2).

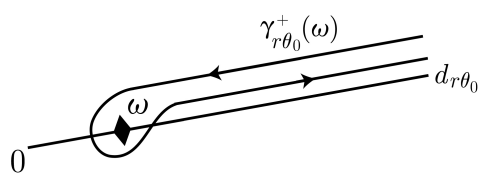

Figure 3.1

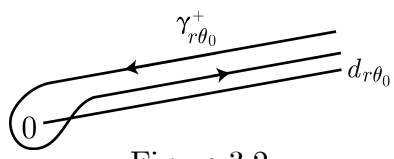

Figure 3.2

Then, we deduce from (3.1), (3.3) and (3.5) the new i Figure 3.1dentity

[5]Contrarily to formula (3.4) which only requires the 1-summability of the series $\widetilde{\boldsymbol{f}}^{[u]}$'s, the individual resurgence and 1-summability are not sufficient here. We do need summable-resurgence.

$$\boxed{\sum_{u=0}^{r-1}(\rho^k x)^u \int_{\gamma^+_{r\theta_0}} \breve{\boldsymbol{f}}^{[u]}_{\omega^\star,+}(\omega+\tau)e^{-\tau/x^r}d\tau = s_{r;\theta_k^+}(\widetilde{F})(\rho^k x)M_{k,\omega^\star}(x) \quad \text{for } \arg(x)\simeq\theta_0^\star} \tag{3.6}$$

Lemma 3.1 ([6, prop. 4.1 and thm. 4.3])
Let $x=t^{1/r}$ be the r-th root of t corresponding to the choice $\arg(x=t^{1/r})\simeq\theta_0^\star$.
Then, for all $u=0,...,r-1$ and $j=1,...,J$ such that $a_{j,r}=\omega$,

$$\int_{\gamma^+_{r\theta_0}} \breve{\boldsymbol{f}}^{[u]j;\bullet}_{\omega^\star,+}(\omega+\tau)e^{-\tau/t}d\tau = t^{\frac{\lambda_j-u}{r}}t^{\frac{J_{n_j}}{r}}\boldsymbol{I}^{[u]j;\bullet}_{\omega^\star}t^{-\frac{J_{n_1}}{r}}+ \sum_{\lambda_\ell;a_{\ell,r}=\omega}\sum_{v=0}^{r-1}t^{\frac{\lambda_\ell-v}{r}+1}\boldsymbol{P}^{[u]j;\bullet}_{\lambda_\ell,v;\omega^\star,+}(\ln t)$$

where

- $\boldsymbol{I}^{[u]j;\bullet}_{\omega^\star}$ *is the integral given in (2.3),*
- $\boldsymbol{P}^{[u]j;\bullet}_{\lambda_\ell,v;\omega^\star,+}(\ln t)$ *is a polynomial in $\ln t$, the coefficients of which are 1-sums of 1-summable series in direction $r\theta_0+\varepsilon$ ($\varepsilon>0$ small enough).*

Consequently, identity (3.6) becomes, for all $j=1,...,J$ such that $a_{j,r}=\omega$ and $\arg(x)\simeq\theta_0^\star$,

$$\sum_{u=0}^{r-1}\left(\rho^{ku}x^{L_j}\boldsymbol{I}^{[u]j;\bullet}_{\omega^\star}x^{-J_{n_1}}+\sum_{\lambda_\ell;a_{\ell,r}=\omega}\sum_{v=0}^{r-1}x^{\lambda_\ell-v+r}\boldsymbol{Q}^{[u]j;\bullet}_{\lambda_\ell,v}(\ln x)\right) = s_{r;\theta_k^+}(\widetilde{F})(\rho^k x)M_{k,\omega^\star}(x)$$

where $\boldsymbol{Q}^{[u]j;\bullet}_{\lambda_\ell,v}(\ln x)$ is a polynomial in $\ln x$, the coefficients of which are r-sums of r-summable series in direction θ_k^+. Then, equating the dominant terms, we obtain

$$\sum_{u=0}^{r-1}\rho^{ku}\boldsymbol{I}^{[u]j;\bullet}_{\omega^\star} = \rho^{kL_j}c^{j;\bullet}_{\theta_k^\star}\rho^{-kJ_{n_1}}$$

for all j such that $a_{j,r}=\omega$. This ends the proof of theorem 2.2 in the case of a unique Stokes value. ■

3.2 An example

Before to start the calculations in the case when the collection (θ_k) is generated by $p \geq 2$ Stokes values, let us study again the system

$$x^3 \frac{dY}{dx} = A(x)Y \ , \ A(x) = \begin{bmatrix} 0 & 0 & 0 \\ x^4 - x^5 & 2 & 0 \\ x^4 + x^5 & 0 & 4 + \frac{x^2}{2} \end{bmatrix} \tag{1.3}$$

together with its formal fundamental solution $\widetilde{Y}(x) = \widetilde{F}(x) x^L e^{Q(1/x)}$ in another way. We refer to examples 1.7 and 2.3 for the notations.

Recall that

- the set of Stokes values of system (1.3) is $\Omega = \{0, 1, 2\}$,
- the non-zero Stokes values 1 and 2 generate the collection $(\theta_0 = 0, \theta_1 = -\pi)$ of the anti-Stokes directions associated with the first column $\widetilde{f}(x)$ of $\widetilde{F}(x)$.

Recall also that the corresponding Stokes multipliers are given by (2.6) and have been calculated by means of theorem 2.2 (*cf.* example 2.3).

The method of calculation which we shall present below is based on a regular perturbation of system (1.3) and on the result of section 3.1. This method will be generalized in sections 3.3 and 3.4 in order to end the proof of theorem 2.2.

3.2.1 A perturbed system

We consider, for all $\varepsilon \geq 0$, the regularly perturbed system

$$x^3 \frac{dY}{dx} = A^\varepsilon(x)Y \ , \ A^\varepsilon(x) = \begin{bmatrix} 0 & 0 & 0 \\ x^4 - x^5 & 2 & 0 \\ x^4 + x^5 & 0 & 4e^{-2i\varepsilon} + \frac{x^2}{2} \end{bmatrix} \tag{3.7}$$

together with its formal fundamental solution $\widetilde{Y}^\varepsilon(x) = \widetilde{F}^\varepsilon(x) x^L e^{Q^\varepsilon(1/x)}$ at 0 where

- $Q^\varepsilon(\frac{1}{x}) = \mathrm{diag}(0, -\frac{1}{x^2}, -\frac{2e^{-2i\varepsilon}}{x^2})$ (hence, system (3.7) has the unique level 2 and the set of its Stokes values is $\Omega^\varepsilon := \{0, 1, 2e^{-2i\varepsilon}\}$),
- $L = \mathrm{diag}(0, 0, \frac{1}{2})$,
- $\widetilde{F}^\varepsilon(x) = \begin{bmatrix} 1 & 0 & 0 \\ \widetilde{f}_2^\varepsilon(x) & 1 & 0 \\ \widetilde{f}_3^\varepsilon(x) & 0 & 1 \end{bmatrix}$ is a power series in x verifying $\widetilde{f}_j^\varepsilon(x) \in x^4\mathbb{C}[[x]]$.

Note that, for $\varepsilon = 0$, we get $A^0 \equiv A$, $\widetilde{Y}^0 \equiv \widetilde{Y}$ and $\Omega^0 \equiv \Omega$.

3.2.2 Action of the perturbation

The perturbation in ε acts on the anti-Stokes directions of system (1.3) as follows: for $\varepsilon > 0$ small enough, the collection (θ_0, θ_1) splits into the distinct three collections $(0, -\pi)$, $(-\varepsilon, -\varepsilon - \pi)$ and $(\alpha_\varepsilon, \beta_\varepsilon := \alpha_\varepsilon - \pi)$ respectively generated by the Stokes values 1, $2e^{-2i\varepsilon}$ and $2e^{-2i\varepsilon} - 1$ so that

$$0 > -\varepsilon > \alpha_\varepsilon^\star \quad \text{and} \quad -\pi > -\varepsilon - \pi > \beta_\varepsilon^\star$$

Note that the first two collections are the anti-Stokes directions associated with the first column $\widetilde{f}^\varepsilon$ of $\widetilde{F}^\varepsilon$ and are generated by *just one* Stokes value of system (3.7).

For any previous six directions $*$, we denote by $I_3 + \mathfrak{S}^\varepsilon_{*^\star}$ the corresponding Stokes-Ramis matrix associated with $\widetilde{Y}^\varepsilon$. Clearly,

$$\mathfrak{S}^\varepsilon_0 = \begin{bmatrix} 0 & 0 & 0 \\ \sigma^\varepsilon_0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad \mathfrak{S}^\varepsilon_{-\pi} = \begin{bmatrix} 0 & 0 & 0 \\ \sigma^\varepsilon_{-\pi} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\mathfrak{S}^\varepsilon_{-\varepsilon} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \sigma^\varepsilon_{-\varepsilon} & 0 & 0 \end{bmatrix} \qquad \mathfrak{S}^\varepsilon_{-\varepsilon-\pi} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \sigma^\varepsilon_{-\varepsilon-\pi} & 0 & 0 \end{bmatrix}$$

$$\mathfrak{S}^\varepsilon_{\alpha^\star_\varepsilon} = 0 \qquad \mathfrak{S}^\varepsilon_{\beta^\star_\varepsilon} = 0$$

Indeed, $\widetilde{f}^\varepsilon(x)$ is the unique column of $\widetilde{F}^\varepsilon(x)$ which is divergent.

Proposition 3.2 *The Stokes-Ramis matrices $I_3 + C_0$ and $I_3 + C_{-\pi}$ of system (1.3) are related to the perturbed Stokes-Ramis matrices above by the formulæ*

$$\boxed{\begin{aligned} I_3 + C_0 &= \lim_{\varepsilon \to 0} (I_3 + \mathfrak{S}^\varepsilon_0)(I_3 + \mathfrak{S}^\varepsilon_{-\varepsilon})(I_3 + \mathfrak{S}^\varepsilon_{\alpha^\star_\varepsilon}) \\ I_3 + C_{-\pi} &= \lim_{\varepsilon \to 0} (I_3 + \mathfrak{S}^\varepsilon_{-\pi})(I_3 + \mathfrak{S}^\varepsilon_{-\varepsilon-\pi})(I_3 + \mathfrak{S}^\varepsilon_{\beta^\star_\varepsilon}) \end{aligned}} \tag{3.8}$$

In particular,

$$\boxed{\begin{aligned} &\lim_{\varepsilon \to 0} \sigma^\varepsilon_0 = c_0^2 && \lim_{\varepsilon \to 0} \sigma^\varepsilon_{-\pi} = c_{-\pi}^2 \\ &\lim_{\varepsilon \to 0} \sigma^\varepsilon_{-\varepsilon} = c_0^3 && \lim_{\varepsilon \to 0} \sigma^\varepsilon_{-\varepsilon-\pi} = c_{-\pi}^3 \end{aligned}} \tag{3.9}$$

Proof. Fix $\delta > 0$ small enough such that

1. for all $\varepsilon \in]0,\delta]$, the directions 0, $-\varepsilon$ and α_ε (resp. $-\pi$, $-\varepsilon-\pi$ and β_ε) are the only anti-Stokes directions of system (3.7) with principal determinations in $[\alpha_\varepsilon^\star, 0]$ (resp. $[\beta_\varepsilon^\star, -\pi]$),

2. the direction $\alpha_\delta^\star$ (resp. $\beta_\delta^\star$) is close enough to 0 (resp. $-\pi$) in order that the two sectors with vertex 0, opening $\pi/2$ and respectively bisected by 0 and α_δ (resp. $-\pi$ and β_δ) overlap.

Then, the sums $Y^\varepsilon_{\alpha_\varepsilon^-}$ and $Y^\varepsilon_{0^+}$, on the one hand, and the sums $Y^\varepsilon_{\beta_\varepsilon^-}$ and $Y^\varepsilon_{-\pi^+}$, on the other hand, are related, for $\arg(x) \simeq 0$, by the formulæ

$$\boxed{\begin{aligned} Y^\varepsilon_{\alpha_\varepsilon^-}(x) &= Y^\varepsilon_{0^+}(x)(I_3+\mathfrak{S}^\varepsilon_0)(I_3+\mathfrak{S}^\varepsilon_{-\varepsilon})(I_3+\mathfrak{S}^\varepsilon_{\alpha_\varepsilon^\star}) \\ Y^\varepsilon_{\beta_\varepsilon^-}(-x) &= Y^\varepsilon_{-\pi^+}(-x)(I_3+\mathfrak{S}^\varepsilon_{-\pi})(I_3+\mathfrak{S}^\varepsilon_{-\varepsilon-\pi})(I_3+\mathfrak{S}^\varepsilon_{\beta_\varepsilon^\star}) \end{aligned}} \tag{3.10}$$

Otherwise, let us denote by $\widetilde{\boldsymbol{f}}^{\,\varepsilon[u]}(t)$, $u=0,1$, the 2-reduced series of $\widetilde{f}^{\,\varepsilon}(x)$. Similar calculations to those of example 1.7 show that their Borel transforms $\widehat{\boldsymbol{f}}^{\,\varepsilon[u]}(\tau)$'s are defined, for all $|\tau|<1$, by

$$\widehat{\boldsymbol{f}}^{\,\varepsilon[0]}(\tau) = \begin{bmatrix} 0 \\ -\dfrac{\tau}{2(1-\tau)} \\ -\dfrac{2^{7/4}}{3}e^{-3i\varepsilon/2}(2e^{-2i\varepsilon}-\tau)^{-3/4}+\dfrac{2}{3} \end{bmatrix}$$

and

$$\widehat{\boldsymbol{f}}^{\,\varepsilon[1]}(\tau) = \begin{bmatrix} 0 \\ \dfrac{1}{3}(1-\tau)^{-3/2}-\dfrac{1}{3} \\ -\dfrac{2^{9/4}}{5}e^{-5i\varepsilon/2}(2e^{-2i\varepsilon}-\tau)^{-5/4}+\dfrac{2}{5} \end{bmatrix}$$

In particular,

$$\lim_{\varepsilon\to 0}\widehat{\boldsymbol{f}}^{\,\varepsilon[u]}(\tau) = \widehat{\boldsymbol{f}}^{\,[u]}(\tau)$$

for all $u=0,1$ and $\tau \notin \bigcup_{\varepsilon\in[0,\delta]}\Omega^\varepsilon$. Recall that the $\widehat{\boldsymbol{f}}^{\,[u]}$'s denote the Borel transforms of the 2-reduced series of the first column $\widetilde{f}(x)$ of $\widetilde{F}(x)$ (*cf.* example 1.7 for more precisions). The last two columns of $\widetilde{F}^{\,\varepsilon}(x)$ and $\widetilde{F}(x)$

being trivial, we can then check, by using Lebesgue dominated convergence theorem, that, for $\arg(x) \simeq 0$,

$$\lim_{\varepsilon\to 0} s_{r;0^+}(\widetilde{F}^\varepsilon)(x) = s_{r;0^+}(\widetilde{F})(x) \qquad \lim_{\varepsilon\to 0} s_{r;-\pi^+}(\widetilde{F}^\varepsilon)(-x) = s_{r;-\pi^+}(\widetilde{F})(-x)$$

$$\lim_{\varepsilon\to 0} s_{r;\alpha_\varepsilon^-}(\widetilde{F}^\varepsilon)(x) = s_{r;0^-}(\widetilde{F})(x) \qquad \lim_{\varepsilon\to 0} s_{r;\beta_\varepsilon^-}(\widetilde{F}^\varepsilon)(-x) = s_{r;-\pi^-}(\widetilde{F})(-x)$$

obtaining so, for $\arg(x) \simeq 0$,

$$\lim_{\varepsilon\to 0} Y^\varepsilon_{0^+}(x) = Y_{0^+}(x) \qquad \lim_{\varepsilon\to 0} Y^\varepsilon_{-\pi^+}(-x) = Y_{-\pi^+}(-x)$$

$$\lim_{\varepsilon\to 0} Y^\varepsilon_{\alpha_\varepsilon^-}(x) = Y_{0^-}(x) \qquad \lim_{\varepsilon\to 0} Y^\varepsilon_{\beta_\varepsilon^-}(-x) = Y_{-\pi^-}(-x)$$

Indeed, $\lim_{\varepsilon\to 0}\alpha_\varepsilon = 0$ and $\lim_{\varepsilon\to 0}\beta_\varepsilon = -\pi$. Therefore, when ε goes to 0, identities (3.10) become, for $\arg(x) \simeq 0$,

$$\begin{aligned}\lim_{\varepsilon\to 0}(I_3 + \mathfrak{S}^\varepsilon_0)(I_3 + \mathfrak{S}^\varepsilon_{-\varepsilon})(I_3 + \mathfrak{S}^\varepsilon_{\alpha^\star_\varepsilon}) &= \lim_{\varepsilon\to 0} Y^\varepsilon_{0^+}(x)^{-1} Y^\varepsilon_{\alpha_\varepsilon^-}(x)\\ &= Y_{0^+}(x)^{-1} Y_{0^-}(x)\\ &= I_3 + C_0\end{aligned}$$

and

$$\begin{aligned}\lim_{\varepsilon\to 0}(I_3 + \mathfrak{S}^\varepsilon_{-\pi})(I_3 + \mathfrak{S}^\varepsilon_{-\varepsilon-\pi})(I_3 + \mathfrak{S}^\varepsilon_{\beta^\star_\varepsilon}) &= \lim_{\varepsilon\to 0} Y^\varepsilon_{-\pi^+}(-x)^{-1} Y^\varepsilon_{\beta_\varepsilon^-}(-x)\\ &= Y_{-\pi^+}(-x)^{-1} Y_{-\pi^-}(-x)\\ &= I_3 + C_{-\pi}\end{aligned}$$

Hence the result. ■

3.2.3 Perturbed Stokes multipliers vs Stokes multipliers

We are left to calculate the perturbed Stokes multipliers defined in (3.9). As we previously said, for $\varepsilon > 0$ small enough, the two collections of anti-Stokes directions $(0, -\pi)$ and $(-\varepsilon, -\varepsilon - \pi)$ associated with $\widetilde{f}^{\,\varepsilon}$ are generated by *just one* Stokes value of system (3.7), respectively 1 and $2e^{-2i\varepsilon}$. Since these two points are both *with monomial front*, section 3.1 applies: the perturbed Stokes multipliers σ^ε_0 and $\sigma^\varepsilon_{-\pi}$ (resp. $\sigma^\varepsilon_{-\varepsilon}$ and $\sigma^\varepsilon_{-\varepsilon-\pi}$) are expressed in terms of the connection constants at $\tau = 1$ (resp. $\tau = 2e^{-2i\varepsilon}$) of the $\widehat{\boldsymbol{f}}^{\,\varepsilon[u]}(\tau)$'s, $u = 0, 1$. Similar calculations to those detailed in examples 1.7 and 2.3 give

us

$$
\boxed{
\begin{aligned}
\sigma_0^\varepsilon &= 2i\pi k_{1,+}^{\varepsilon[0]2} + 2i\pi \frac{e^{\frac{i\pi}{2}}}{\Gamma\left(\frac{3}{2}\right)} k_{1,+}^{\varepsilon[1]2} \\
\sigma_{-\varepsilon}^\varepsilon &= 2i\pi \frac{e^{-\frac{i\pi}{4}}}{\Gamma\left(\frac{3}{4}\right)} k_{2e^{-2i\varepsilon},+}^{\varepsilon[0]3} + 2i\pi \frac{e^{\frac{i\pi}{4}}}{\Gamma\left(\frac{5}{4}\right)} k_{2e^{-2i\varepsilon},+}^{\varepsilon[1]3} \\
\sigma_{-\pi}^\varepsilon &= 2i\pi k_{1,+}^{\varepsilon[0]2} + 2i\pi e^{-i\pi} \frac{e^{\frac{i\pi}{2}}}{\Gamma\left(\frac{3}{2}\right)} k_{1,+}^{\varepsilon[1]2} \\
\sigma_{-\varepsilon-\pi}^\varepsilon &= 2i\pi e^{\frac{i\pi}{2}} \frac{e^{-\frac{i\pi}{4}}}{\Gamma\left(\frac{3}{4}\right)} k_{2e^{-2i\varepsilon},+}^{\varepsilon[0]3} + 2i\pi e^{-\frac{i\pi}{2}} \frac{e^{\frac{i\pi}{4}}}{\Gamma\left(\frac{5}{4}\right)} k_{2e^{-2i\varepsilon},+}^{\varepsilon[1]3}
\end{aligned}
}
\tag{3.11}
$$

where

$$
\boxed{
\begin{aligned}
k_{1,+}^{\varepsilon[0]2} &= \frac{1}{2} & k_{2e^{-2i\varepsilon},+}^{\varepsilon[0]3} &= -\frac{2^{7/4}}{3} e^{-3i\varepsilon/2} e^{-3i\pi/4} \\
k_{1,+}^{\varepsilon[1]2} &= \frac{i}{3} & k_{2e^{-2i\varepsilon},+}^{\varepsilon[1]3} &= -\frac{2^{9/4}}{5} e^{-5i\varepsilon/2} e^{-5i\pi/4}
\end{aligned}
}
\tag{3.12}
$$

We obtain

$$
\boxed{
\begin{aligned}
\sigma_0^\varepsilon &= \left(\pi - \frac{4\sqrt{\pi}}{3}\right) i & \sigma_{-\varepsilon}^\varepsilon &= 2^{3/4} \left(\frac{4\pi e^{-3i\varepsilon/2}}{3\Gamma(\frac{3}{4})} + \frac{16}{5}\Gamma(\tfrac{3}{4}) e^{-5i\varepsilon/2}\right) i \\
\sigma_{-\pi}^\varepsilon &= \left(\pi + \frac{4\sqrt{\pi}}{3}\right) i & \sigma_{-\varepsilon-\pi}^\varepsilon &= -2^{3/4} \left(\frac{4\pi e^{-3i\varepsilon/2}}{3\Gamma(\frac{3}{4})} - \frac{16}{5}\Gamma(\tfrac{3}{4}) e^{-5i\varepsilon/2}\right)
\end{aligned}
}
$$

Note that, letting ε tend to 0, these relations reduce to relations (3.9).

Remark 3.3 When ε goes to 0, identities (3.12) become

$$
\boxed{
\begin{aligned}
\lim_{\varepsilon\to 0} k_{1,+}^{\varepsilon[0]2} &= k_{1,+}^{[0]2} & \lim_{\varepsilon\to 0} k_{2e^{-2i\varepsilon},+}^{\varepsilon[0]3} &= k_{2,+}^{[0]3} \\
\lim_{\varepsilon\to 0} k_{1,+}^{\varepsilon[1]2} &= k_{1,+}^{[1]2} & \lim_{\varepsilon\to 0} k_{2e^{-2i\varepsilon},+}^{\varepsilon[1]3} &= k_{2,+}^{[1]3}
\end{aligned}
}
\tag{3.13}
$$

where $k_{1,+}^{[u]2}$ and $k_{2,+}^{[u]3}$, $u = 0, 1$, respectively denote the connection constants of the $\widehat{\boldsymbol{f}}^{[u]}$'s at $\tau = 1$ and $\tau = 2$ (*cf.* example 1.7).

Remark 3.4 Obviously, identities (3.9), (3.11) and (3.13) imply identities (2.5), *i.e.*, theorem 2.2 in restriction to system (1.3). Actually, as we shall show below, this kind of identities is the core of the proof of theorem 2.2 when the collection (θ_k) is generated by $p \geq 2$ Stokes values.

3.3 Case of two Stokes values

We assume in this section that the collection $(\theta_k)_k$ of anti-Stokes directions associated with $\widetilde{f}(x)$ is generated by just two Stokes values ω_1 and ω_2 satisfying $|\omega_1| < |\omega_2|$.

Recall that we must prove identity (2.2) when the front of ω_1 (resp. ω_2) is monomial. We proceed similarly as in section 3.2 by considering a regular perturbation of system (0.1).

3.3.1 A perturbed system

According to the normalization of $\widetilde{Y}(x)$ (*cf.* page 2), the matrix $A(x)$ of system (0.1) reads

$$A(x) = \bigoplus_{j=1}^{J} \left[\left(\sum_{k=1}^{r} k a_{j,k} x^{r-k} \right) I_{n_j} + x^r L_j \right] + B(x)$$

where $L_j := \lambda_j I_{n_j} + J_{n_j}$ denotes the j-th Jordan block of the matrix L of exponents of formal monodromy and where $B(x)$ is analytic at the origin $0 \in \mathbb{C}$. More precisely, split $B(x) = [B^{j;\ell}(x)]$ into blocks fitting to the Jordan structure of L. Then,

$$B^{j;\ell}(x) = \begin{cases} O(x^r) & \text{if } a_{j,r} \neq a_{\ell,r} \\ O(x^{2r}) & \text{if } a_{j,r} = a_{\ell,r} \end{cases} \tag{3.14}$$

For all $\varepsilon \geq 0$, we define the regularly perturbed system

$$x^{r+1} \frac{dY}{dx} = A^{\varepsilon}(x) Y \tag{3.15}$$

with

$$A^{\varepsilon}(x) = \bigoplus_{j=1}^{J} \left[\left(r a_{j,r}^{\varepsilon} + \sum_{k=1}^{r-1} k a_{j,k} x^{r-k} \right) I_{n_j} + x^r L_j \right] + B(x)$$

where, for all $j = 1, ..., J$,

$$a_{j,r}^{\varepsilon} = \begin{cases} a_{j,r} & \text{if } a_{j,r} \neq \omega_2 \\ \omega_2 e^{-ri\varepsilon} & \text{if } a_{j,r} = \omega_2 \end{cases}$$

For all $\varepsilon \geq 0$, system (3.15) has for formal fundamental solution the matrix $\widetilde{Y}^{\varepsilon}(x) = \widetilde{F}^{\varepsilon}(x) x^L e^{Q^{\varepsilon}(1/x)}$ where

- $\widetilde{F}^{\varepsilon}(x) \in M_n(\mathbb{C}[[x]])$ is a power series in x verifying same normalization as $\widetilde{F}(x)$: $\widetilde{F}^{\varepsilon}(x) = I_n + O(x^r)$,
- L is the matrix of exponents of formal monodromy of system (0.1),
- $Q^{\varepsilon}(1/x) = \bigoplus_{j=1}^{J} q_j^{\varepsilon}(1/x) I_{n_j}$ is the diagonal matrix, the polynomial entries q_j^{ε} of which are defined, for all j, by

$$\begin{cases} q_j^{\varepsilon} \equiv q_j & \text{if } a_{j,r} \neq \omega_2 \\ q_j^{\varepsilon}\left(\dfrac{1}{x}\right) = -\dfrac{\omega_2 e^{-ri\varepsilon}}{x^r} + \dot{q}_j\left(\dfrac{1}{x}\right) & \text{if } a_{j,r} = \omega_2 \end{cases}$$

Note that the following condition holds for $\varepsilon > 0$ small enough:

$(C1)$ System (3.15) has the unique level r and satisfies normalizations as system (0.1); the set Ω^{ε} of its Stokes values is deduced from the set Ω of Stokes values of system (0.1) by replacing ω_2 by $\omega_2 e^{-ri\varepsilon}$.

Note also that, for $\varepsilon = 0$, we get $A^0 \equiv A$, $\widetilde{Y}^0 \equiv \widetilde{Y}$ and $\Omega^0 \equiv \Omega$.

Let us now fix $\delta > 0$ such that condition $(C1)$ above be verified for all $\varepsilon \in [0, \delta]$.

For all $\varepsilon \in [0, \delta]$, we denote as previously by

- $\widetilde{f}^{\varepsilon}(x)$ the first n_1 columns of $\widetilde{F}^{\varepsilon}(x)$,
- $\widetilde{\boldsymbol{f}}^{\,\varepsilon[u]}(t)$, $u = 0, ..., r-1$, the r-reduced series of $\widetilde{f}^{\varepsilon}(x)$ (*cf.* (1.1)).

Proposition 3.5 *Let Ω^{ε} denote the set of Stokes values of system (3.15). Then, for all $\varepsilon \in [0, \delta]$ and $u = 0, ..., r-1$,*

$$\boxed{\widetilde{\boldsymbol{f}}^{\,\varepsilon[u]}(t) \in \widetilde{\mathcal{R}es}^{\,sum}_{\Omega^{\varepsilon}}}$$

Moreover, the singularity $\overset{\nabla}{\boldsymbol{f}}{}^{\varepsilon[u]}_{\omega^{\star},+}$ at any Stokes value $\omega \in \Omega^{\varepsilon} \backslash \{0\}$ with monomial front is given by (1.2).

Proposition 3.5 is straightforward from condition $(C1)$ and theorem 1.3. However, in view to study the dependence of the Borel transforms $\widehat{\boldsymbol{f}}^{\,\varepsilon[u]}(\tau)$ of $\widetilde{\boldsymbol{f}}^{\,\varepsilon[u]}(t)$ in the parameter ε, we need to give again a complete proof of the summable-resurgence of the $\widetilde{\boldsymbol{f}}^{\,\varepsilon[u]}(t)$'s. We proceed as in [10] by following Écalle's method by regular perturbation and majorant series quoted in [3].

3.3.2 Proof of the summable-resurgence for the perturbed system

Fix $\varepsilon \in [0, \delta]$. By definition of rank reduction, the vector

$$\widetilde{\boldsymbol{f}}^{\,\varepsilon}(t) := \begin{bmatrix} \widetilde{\boldsymbol{f}}^{\,\varepsilon[0]}(t) \\ \vdots \\ \widetilde{\boldsymbol{f}}^{\,\varepsilon[r-1]}(t) \end{bmatrix} \in M_{rn,n_1}(\mathbb{C}[[t]])$$

is a column-block of a formal fundamental solution of the reduced system of (3.15) ([5]). Hence, its homological system provides a system characterizing the $\widetilde{\boldsymbol{f}}^{\,\varepsilon[u]}(t)$'s.

Denote respectively by $\boldsymbol{A}^{\varepsilon[u]}(t)$ and $\boldsymbol{B}^{[u]}(t)$ the r-reduced series of $A^{\varepsilon}(x)$ and $B(x)$. Then ([5]), $\widetilde{\boldsymbol{f}}^{\,\varepsilon}(t)$ is uniquely determined by the system

$$rt^2 \frac{d\boldsymbol{f}}{dt} = \boldsymbol{A}^{\varepsilon}(t)\boldsymbol{f} - t\boldsymbol{f}\, J_{n_1} \tag{3.16}$$

jointly with the initial condition $\widetilde{\boldsymbol{f}}^{\,\varepsilon}(0) = I_{rn,n_1}$ (the first n_1 columns of the identity matrix of size rn). The matrix $\boldsymbol{A}^{\varepsilon}(t)$ is the matrix of size $rn \times rn$ defined by

$$\boldsymbol{A}^{\varepsilon}(t) = \begin{bmatrix} \boldsymbol{A}^{\varepsilon[0]}(t) & t\boldsymbol{A}^{\varepsilon[r-1]}(t) & \cdots & \cdots & t\boldsymbol{A}^{\varepsilon[1]}(t) \\ \boldsymbol{A}^{\varepsilon[1]}(t) & \boldsymbol{A}^{\varepsilon[0]}(t) & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \boldsymbol{A}^{\varepsilon[0]}(t) & t\boldsymbol{A}^{\varepsilon[r-1]}(t) \\ \boldsymbol{A}^{\varepsilon[r-1]}(t) & \cdots & \cdots & \boldsymbol{A}^{\varepsilon[1]}(t) & \boldsymbol{A}^{\varepsilon[0]}(t) \end{bmatrix} - \bigoplus_{u=0}^{r-1} utI_n$$

where

$$\boldsymbol{A}^{\varepsilon[0]}(t) = \bigoplus_{j=1}^{J} \left(ra^{\varepsilon}_{j,r} I_{n_j} + tL_j \right) + \boldsymbol{B}^{[0]}(t)$$

and, for all $u = 1, ..., r-1$,

$$\boldsymbol{A}^{\varepsilon[u]}(t) = \bigoplus_{j=1}^{J} (r-u) a_{j,r-u} I_{n_j} + \boldsymbol{B}^{[u]}(t)$$

Moreover, according to (3.14), the series $\boldsymbol{B}^{[u]}(t) = [\boldsymbol{B}^{[u]j;\ell}(t)] \in M_n(\mathbb{C}\{t\})$ satisfy, for all $u = 0, ..., r-1$, the condition

$$\boldsymbol{B}^{[u]j;\ell}(t) = \begin{cases} O(t) & \text{if } a_{j,r} \neq a_{\ell,r} \\ O(t^2) & \text{if } a_{j,r} = a_{\ell,r} \end{cases} \tag{3.17}$$

Note that the definition of the matrix $A^\varepsilon(x)$ of system (3.15) implies that only the matrix $\boldsymbol{A}^{\varepsilon[0]}(t)$ depends on the parameter ε.

Following J. Écalle ([3]), we consider, instead of system (3.16), the regularly perturbed system

$$rt^2\frac{d\boldsymbol{f}}{dt} = \boldsymbol{A}^\varepsilon(t,\alpha)\boldsymbol{f} - t\boldsymbol{f}\,J_{n_1} \tag{3.18}$$

obtained by substituting $\alpha\boldsymbol{B}^{[u]}$ for $\boldsymbol{B}^{[u]}$ for all $u = 0, ..., r-1$ in the matrix $\boldsymbol{A}^\varepsilon(t)$ of system (3.16).

An identification of equal powers in α allows us to state that system (3.18) admits a unique formal solution of the form

$$\widetilde{\boldsymbol{f}}^{\,\varepsilon}(t,\alpha) = \sum_{m\geq 0}\widetilde{\boldsymbol{f}}^{\,\varepsilon}_m(t)\alpha^m$$

satisfying $\widetilde{\boldsymbol{f}}^{\,\varepsilon}_0(t) = I_{rn,n_1}$ and $\widetilde{\boldsymbol{f}}^{\,\varepsilon}_m(t) \in M_{rn,n_1}(\mathbb{C}[[t]])$ for all $m \geq 1$. More precisely, split $\widetilde{\boldsymbol{f}}^{\,\varepsilon}_m(t) = \left[\widetilde{\boldsymbol{f}}^{\,\varepsilon[0]}_m(t), ..., \widetilde{\boldsymbol{f}}^{\,\varepsilon[r-1]}_m(t)\right]$ into r blocks of size $n \times n_1$ like $\widetilde{\boldsymbol{f}}^{\,\varepsilon}(t)$ and denote by

$$\widetilde{\boldsymbol{f}}^{\,\varepsilon}_{m,j}(t) := \begin{bmatrix} \widetilde{\boldsymbol{f}}^{\,\varepsilon[0]j;\bullet}_m(t) \\ \vdots \\ \widetilde{\boldsymbol{f}}^{\,\varepsilon[r-1]j;\bullet}_m(t) \end{bmatrix} \quad \text{for all } j = 1, ..., J$$

the matrix of size $rn_j \times n_1$ formed by the j-th row-blocks of the $\widetilde{\boldsymbol{f}}^{\,\varepsilon[u]}_m(t)$'s (we refer to page 8 for the notations).

Lemma 3.6 *For all $m \geq 1$, the components $\widetilde{\boldsymbol{f}}^{\,\varepsilon}_{m,j}(t) \in M_{rn_j,n_1}(\mathbb{C}[[t]])$ of $\widetilde{\boldsymbol{f}}^{\,\varepsilon}_m(t)$ are uniquely determined, for all $j = 1, ..., J$, as formal solutions of systems*

$$rt^2\frac{d\widetilde{\boldsymbol{f}}^{\,\varepsilon}_{m,j}}{dt} - \boldsymbol{A}^\varepsilon_j\widetilde{\boldsymbol{f}}^{\,\varepsilon}_{m,j} - t\boldsymbol{A}_j\widetilde{\boldsymbol{f}}^{\,\varepsilon}_{m,j} = \boldsymbol{B}_j(t)\widetilde{\boldsymbol{f}}^{\,\varepsilon}_{m-1} - t\widetilde{\boldsymbol{f}}^{\,\varepsilon}_{m,j}J_{n_1} \tag{3.19}$$

where

$$\boldsymbol{B}_j(t) := \begin{bmatrix} \boldsymbol{B}^{[0]j;\bullet}(t) & t\boldsymbol{B}^{[r-1]j;\bullet}(t) & \cdots & \cdots & t\boldsymbol{B}^{[1]j;\bullet}(t) \\ \boldsymbol{B}^{[1]j;\bullet}(t) & \boldsymbol{B}^{[0]j;\bullet}(t) & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \boldsymbol{B}^{[0]j;\bullet}(t) & t\boldsymbol{B}^{[r-1]j;\bullet}(t) \\ \boldsymbol{B}^{[r-1]j;\bullet}(t) & \cdots & \cdots & \boldsymbol{B}^{[1]j;\bullet}(t) & \boldsymbol{B}^{[0]j;\bullet}(t) \end{bmatrix}$$

is a matrix of size $rn_j \times rn$ with analytic entries at $0 \in \mathbb{C}$ and where the matrices $\boldsymbol{A}^\varepsilon_j$ and $\boldsymbol{A}_j$ are the constant $rn_j \times rn_j$-matrices defined by

- $\boldsymbol{A}_j^\varepsilon := \begin{bmatrix} ra_{j,r}^\varepsilon & 0 & \cdots & 0 \\ (r-1)a_{j,r-1} & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ a_{j,1} & \cdots & (r-1)a_{j,r-1} & ra_{j,r}^\varepsilon \end{bmatrix} \otimes I_{n_j}$

- $\boldsymbol{A}_j := \begin{bmatrix} 0 & a_{j,1} & \cdots & (r-1)a_{j,r-1} \\ \vdots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & a_{j,1} \\ 0 & \cdots & \cdots & 0 \end{bmatrix} \otimes I_{n_j} - \bigoplus_{u=0}^{r-1} (L_j - uI_{n_j})$

Remark 3.7 When $a_{j,r}^\varepsilon \neq 0$, the matrix $\boldsymbol{A}_j^\varepsilon$ is invertible. Moreover, since system (3.15) has the unique level r, assumption (0.3) implies that $\boldsymbol{A}_j^\varepsilon = 0$ and

$$\boldsymbol{A}_j = \bigoplus_{u=0}^{r-1} \left(L_j - uI_{n_j}\right)$$

as soon as $a_{j,r}^\varepsilon = 0$. Note also that only the matrix $\boldsymbol{A}_j^\varepsilon$ depends on ε.

Relations (3.19) and normalizations (3.17) show in particular that

$$\widetilde{\boldsymbol{f}}^{\,\varepsilon}_{2m-1,j}(t) = O(t^m) \quad \text{and} \quad \widetilde{\boldsymbol{f}}^{\,\varepsilon}_{2m,j}(t) = \begin{cases} O(t^m) & \text{if } a_{j,r} = 0 \\ O(t^{m+1}) & \text{if } a_{j,r} \neq 0 \end{cases}$$

for all $m \geq 1$ and $j = 1, ..., J$. As a result, the series $\widetilde{\boldsymbol{f}}^{\,\varepsilon}(t,\alpha)$ can be rewritten as a series in t with polynomial coefficients in α. Consequently, $\widetilde{\boldsymbol{f}}^{\,\varepsilon}(t) = \widetilde{\boldsymbol{f}}^{\,\varepsilon}(t,1)$ (by unicity of $\widetilde{\boldsymbol{f}}^{\,\varepsilon}(t)$ and $\widetilde{\boldsymbol{f}}^{\,\varepsilon}(t,1)$) and, for all α, the series $\widetilde{\boldsymbol{f}}^{\,\varepsilon}(t,\alpha)$ admits a formal Borel transform $\boldsymbol{\varphi}^\varepsilon(\tau,\alpha)$ with respect to t of the form

$$\boldsymbol{\varphi}^\varepsilon(\tau,\alpha) = \delta I_{rn,n_1} + \sum_{m\geq 1} \boldsymbol{\varphi}_m^\varepsilon(\tau)\alpha^m$$

where $\boldsymbol{\varphi}_m^\varepsilon(\tau)$ denotes, for all $m \geq 1$, the Borel transform of $\widetilde{\boldsymbol{f}}^{\,\varepsilon}_m(t)$. In particular, for $\alpha = 1$, lemma 3.8 below tells us that the Borel transform $\widehat{\boldsymbol{f}}^{\,\varepsilon}(\tau) = \boldsymbol{\varphi}^\varepsilon(\tau,1)$ of $\widetilde{\boldsymbol{f}}^{\,\varepsilon}(t)$ can be interpreting as a series of resurgent functions on $\mathcal{R}_{\Omega^\varepsilon}$.

As previously, we split $\boldsymbol{\varphi}_m^\varepsilon(\tau) = \left[\boldsymbol{\varphi}_m^{\varepsilon[0]}(\tau), ..., \boldsymbol{\varphi}_m^{\varepsilon[r-1]}(\tau)\right]$ into r blocks of size $n \times n_1$ and we denote by

$$\boldsymbol{\varphi}_{m,j}^\varepsilon(\tau) := \begin{bmatrix} \boldsymbol{\varphi}_m^{\varepsilon[0]j;\bullet}(\tau) \\ \vdots \\ \boldsymbol{\varphi}_m^{\varepsilon[r-1]j;\bullet}(\tau) \end{bmatrix} \quad \text{for all } j = 1, ..., J$$

the matrix of size $rn_j \times n_1$ formed by the j-th row-blocks of the $\boldsymbol{\varphi}_m^{\varepsilon[u]}(\tau)$'s. Since $\boldsymbol{\varphi}_{m,j}^{\varepsilon}(\tau)$ is the Borel transform of $\widetilde{\boldsymbol{f}}_{m,j}^{\varepsilon}(t)$, identities (3.19) provide identities satisfied by the $\boldsymbol{\varphi}_{m,j}^{\varepsilon}(\tau)$'s. More precisely, we can check the following result:

Lemma 3.8 (Decomposition of the Borel transform $\widehat{\boldsymbol{f}}^{\,\varepsilon}(\tau)$)
The Borel transform $\widehat{\boldsymbol{f}}^{\,\varepsilon}(\tau)$ of $\widetilde{\boldsymbol{f}}^{\,\varepsilon}(t)$ reads in the form

$$\widehat{\boldsymbol{f}}^{\,\varepsilon}(\tau) = \sum_{m\geq 1} \boldsymbol{\varphi}_m^{\varepsilon}(\tau) \quad \textit{with } \boldsymbol{\varphi}_m^{\varepsilon}(\tau) \in M_{rn,n_1}(\mathbb{C}[[\tau]]) \textit{ for all } m \geq 1$$

Moreover, for all $m \geq 1$, the components $\boldsymbol{\varphi}_{m,j}^{\varepsilon}(\tau) \in M_{rn_j,n_1}(\mathbb{C}[[\tau]])$ of $\boldsymbol{\varphi}_m^{\varepsilon}(\tau)$ are iteratively determined, for all $j = 1, ..., J$, as solutions of systems

$$\boxed{R_j^{\varepsilon} \frac{d\boldsymbol{\varphi}_{m,j}^{\varepsilon}}{d\tau} = S_j \boldsymbol{\varphi}_{m,j}^{\varepsilon} + \frac{d}{d\tau}\left(\widehat{\boldsymbol{B}}_j * \boldsymbol{\varphi}_{m-1}^{\varepsilon}\right) - \boldsymbol{\varphi}_{m,j}^{\varepsilon} J_{n_1}} \tag{3.20}$$

where $\boldsymbol{\varphi}_0^{\varepsilon} = \delta I_{rn,n_1}$ and where

- $R_j^{\varepsilon} = \begin{bmatrix} r(\tau - a_{j,r}^{\varepsilon}) & 0 & \cdots & 0 \\ (r-1)a_{j,r-1} & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ a_{j,1} & \cdots & (r-1)a_{j,r-1} & r(\tau - a_{j,r}^{\varepsilon}) \end{bmatrix} \otimes I_{n_j},$

- $S_j = \begin{bmatrix} 0 & a_{j,1} & \cdots & (r-1)a_{j,r-1} \\ \vdots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & a_{j,1} \\ 0 & \cdots & \cdots & 0 \end{bmatrix} \otimes I_{n_j} - \bigoplus_{u=0}^{r-1} (L_j - (u+r)I_{n_j}).$

In particular, the $\boldsymbol{\varphi}_m^{\varepsilon}(\tau)$, $m \geq 1$, are resurgent functions defined on $\mathcal{R}_{\Omega^{\varepsilon}}$.

The resurgence of the $\boldsymbol{\varphi}_m^{\varepsilon}$'s is due to the fact that the only singularities of systems (3.20) are the Stokes values $a_{j,r}^{\varepsilon} \in \Omega^{\varepsilon}$. Indeed, the homogeneous equation associated with (3.20) is singular at $a_{j,r}^{\varepsilon}$. Moreover, since $\boldsymbol{B}_j(t)$ is analytic at 0, its Borel transform $\widehat{\boldsymbol{B}}_j(\tau)$ is an entire function on $\mathbb{C}$ and, consequently, the singularities of the convolution product $\widehat{\boldsymbol{B}}_j * \boldsymbol{\varphi}_{m-1}^{\varepsilon}$ are those of $\boldsymbol{\varphi}_{m-1}^{\varepsilon}$.

We are left to prove (*cf.* def 1.2) that

(a) $\widehat{\boldsymbol{f}}^{\,\varepsilon}(\tau)$ is analytic at 0 and can be analytically continued to $\mathcal{R}_{\Omega^{\varepsilon}}$ (we keep denoting by $\widehat{\boldsymbol{f}}^{\,\varepsilon}(\tau)$ the analytic continuation),

(b) $\widehat{\boldsymbol{f}}^{\varepsilon}(\tau)$ grows at most exponentially on any ν-sectorial region $\Delta_{\nu}^{\varepsilon}$ of $\mathcal{R}_{\Omega^{\varepsilon}}$.

These properties are proved below by using a technique of majorant series satisfying a convenient system. There exists, of course, many possible majorant system. Here, we make explicit a possible one.

Let $\boldsymbol{g} = \left[\boldsymbol{g}^{[0]}, ..., \boldsymbol{g}^{[r-1]}\right]$ be a matrix of size $rn \times n_1$ split as previously into r blocks of size $n \times n_1$ and, for all $j = 1, ..., J$, let

$$\boldsymbol{g}_j := \begin{bmatrix} \boldsymbol{g}^{[0]j;\bullet} \\ \vdots \\ \boldsymbol{g}^{[r-1]j;\bullet} \end{bmatrix}$$

be the matrix of size $rn_j \times n_1$ formed by the j-th row-blocks of the $\boldsymbol{g}^{[u]}$'s. When $\boldsymbol{g} = I_{rn,n_1}$, we denote by I^j_{rn,n_1} the matrix $\boldsymbol{g}_j$.

Fix $\nu > 0$ and consider, for $j = 1, ..., J$, the perturbed linear system

(3.21)

$$\begin{cases} C_j(\boldsymbol{g}_j - I^j_{rn,n_1}) = (I_r \otimes J_{n_j})\boldsymbol{g}_j + \boldsymbol{g}_j J_{n_1} - 2I^j_{rn,n_1} J_{n_1} + \alpha \dfrac{|\boldsymbol{B}_j|(t)}{t}\boldsymbol{g} \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{if } a^{\varepsilon}_{j,r} = 0 \\ (\mathcal{R}_j - t\mathcal{S}_j)\boldsymbol{g}_j = t\boldsymbol{g}_j J_{n_1} + \alpha\,|\boldsymbol{B}_j|(t)\boldsymbol{g} \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{if } a^{\varepsilon}_{j,r} \neq 0 \end{cases}$$

where

- $|\boldsymbol{B}_j|(t)$ denotes the series $\boldsymbol{B}_j(t)$ in which the coefficients of the powers of t are replaced by their module,
- C_j is an invertible constant $rn_j \times rn_j$-diagonal matrix with positive entries,
- $\mathcal{R}_j$ and $\mathcal{S}_j$ are the constant $rn_j \times rn_j$-matrices defined by

$$\mathcal{R}_j := \begin{bmatrix} \nu & 0 & \cdots & 0 \\ -|a_{j,r-1}| & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ -|a_{j,1}| & \cdots & -|a_{j,r-1}| & \nu \end{bmatrix} \otimes I_{n_j}$$

and

$$\mathcal{S}_j := \begin{bmatrix} 0 & |a_{j,1}| & \cdots & |a_{j,r-1}| \\ \vdots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & |a_{j,1}| \\ 0 & \cdots & \cdots & 0 \end{bmatrix} \otimes I_{n_j} + \bigoplus_{u=0}^{r-1} \left(\left| \frac{\lambda_j}{r} - \frac{u}{r} - 1 \right| I_{n_j} + J_{n_j} \right)$$

Note that, for now, the constants C_j are just arbitrary. They are to be adequatly chosen below (*cf.* lemma 3.8).

Note also that system (3.21) does not depend on the parameter ε. Actually, system (3.21) is the majorant system that we used in [10] to prove summable-resurgent theorem for $\widetilde{\boldsymbol{f}}(t)$ (thm. 1.3). In particular, we showed that the Borel transformed system of system (3.21) admits, for $\alpha = 1$, a solution of the form

$$\widehat{\boldsymbol{g}}(\tau) = \delta I_{rn,n_1} + \sum_{m\geq 1} \boldsymbol{\Phi}_m(\tau) \in M_{rn,n_1}(\mathbb{C}[[\tau]])$$

which is entire on all $\mathbb{C}$ with exponential growth at infinity. Moreover, for all $m \geq 1$, the $\boldsymbol{\Phi}_m(\tau)$'s are also entire functions on all $\mathbb{C}$ with exponential growth at infinity. More precisely, using notations as above, the components $\boldsymbol{\Phi}_{m,j}(\tau)$ of $\boldsymbol{\Phi}_m(\tau)$, $m \geq 1$ and $j = 1, ..., J$, are iteratively determined from $\boldsymbol{\Phi}_0 = \delta I_{rn,n_1}$ as solutions of the following systems:

- Case $a^{\varepsilon}_{j,r} = 0$:

$$C_j \boldsymbol{\Phi}_{m,j} = (I_r \otimes J_{n_j})\boldsymbol{\Phi}_{m,j} + \boldsymbol{\Phi}_{m,j} J_{n_1} + \frac{d}{d\tau}\left(\widehat{|\boldsymbol{B}_j|} * \boldsymbol{\Phi}_{m-1}\right)$$

- Case $a^{\varepsilon}_{j,r} \neq 0$:

$$\mathcal{R}_j \frac{d\boldsymbol{\Phi}_{m,j}}{d\tau} = \mathcal{S}_j \boldsymbol{\Phi}_{m,j} + \boldsymbol{\Phi}_{m,j} J_{n_1} + \frac{d}{d\tau}\left(\widehat{|\boldsymbol{B}_j|} * \boldsymbol{\Phi}_{m-1}\right)$$

In particular, the series $\boldsymbol{\Phi}_m(\tau)$ have non-negative coefficients.

Since only the Stokes values $a^{\varepsilon}_{j,r} \in \Omega^{\varepsilon}$ depend on the parameter ε in system (3.15), the calculations detailed in [10, § 3.2.2] apply allowing us to make explicit a convenient majorant series of $\widehat{\boldsymbol{f}}^{\,\varepsilon}(\tau)$ (*cf.* lemma 3.9 below).

Recall that, according to the definition of $\Delta^{\varepsilon}_{\nu}$ (*cf.* page 5), there exists a constant $K > 0$ so that, for all $\tau \in \Delta^{\varepsilon}_{\nu}$, there is a piecewise-$\mathcal{C}^1$-path γ_τ contained in $\Delta^{\varepsilon}_{\nu}$ and parametrized by arc length from 0 to τ such that the arc length s_η of all $\eta \in \gamma_\tau$ satisfies $|\eta| \leq s_\eta \leq K\,|\eta|$; $|\eta|$ denotes the modulus of the projection of η in $\mathbb{C}$ ([6, lem. 2.4]). Besides, since points in $\Delta^{\varepsilon}_{\nu}$ have bounded arguments, there also exists a constant $a > 0$ such that $|\arg(\tau)| \leq a$ for all $\tau \in \Delta^{\varepsilon}_{\nu}$.

Lemma 3.9 (Majorant series, [10])
Let K and a be two constants as above.
Let

$$C_j = \frac{1}{K \max\limits_{1\le j\le J} \exp(2a\,|\mathrm{Im}\,\lambda_j|)} \bigoplus_{u=0}^{r-1} \left(1 - \mathrm{Re}\left(\frac{\lambda_j}{r} - \frac{u}{r}\right)\right) I_{n_j}$$

for all $j = 1, ..., J$.
Then, for all $m \ge 1$, $\tau \in \Delta_\nu^\varepsilon$ and $j = 1, ..., J$, the following inequalities hold:

$$\left|\boldsymbol{\varphi}_{m,j}^\varepsilon(\tau)\right| \le \boldsymbol{\Phi}_{m,j}(s_\tau) \le \boldsymbol{\Phi}_{m,j}(K\,|\tau|) \tag{3.22}$$

In particular, the series $\widehat{\boldsymbol{g}}(K\,|\tau|) = \sum\limits_{m\ge1} \boldsymbol{\Phi}_m(K\,|\tau|)$ is a majorant series of $\widehat{\boldsymbol{f}}^{\,\varepsilon}(\tau)$.

Recall that the second inequality of (3.22) is due to the fact that the series $\boldsymbol{\Phi}_m(\tau)$ has non-negative coefficients; the first inequality is proved by using Grönwall lemma.

Since $\widehat{\boldsymbol{g}}$ is well defined on Δ_ν^ε with exponential growth at infinity, the same property holds for $\widehat{\boldsymbol{f}}^{\,\varepsilon}(\tau)$ which achieves the proof of the summable-resurgence of $\widehat{\boldsymbol{f}}^{\,\varepsilon}(\tau)$.

Remark 3.10 Although this proof is similar to the one in [10], it has its own interest. Indeed, it allows to deal with a particular case of regularly perturbed systems with a single level.

Remark 3.11 The majorant series $\widehat{\boldsymbol{g}}(K|\tau|)$ does not depend on the parameter ε. This point, which is crucial in the proof of the connection-to-Stokes formulæ that we present in this article, was obtained because *the parameter ε is only related to the Stokes values of Ω^ε*. Of course, for a more general perturbation, this fact no longer holds in general.

3.3.3 Dependence of the Borel transforms $\widehat{\boldsymbol{f}}^{\,\varepsilon[u]}(\tau)$ in ε

Let $W_\delta := \bigcup_{\varepsilon\in[0,\delta]} \Omega^\varepsilon$ denote the set of Stokes values of systems (3.15) for all $\varepsilon \in [0, \delta]$. By definition, W_δ is the union of the set $\Omega\backslash\{\omega_2\}$ of Stokes values $a_{j,r} \ne \omega_2$ and the circle arc $A(\omega_2, \delta) := \{\omega_2 e^{-ri\varepsilon}\ ,\ \varepsilon \in [0, \delta]\}$ of length δ. Note that, according to condition $(C1)$, the sets $\Omega\backslash\{\omega_2\}$ and $A(\omega_2, \delta)$ are disjoints.

Definition 3.12 (ν-generalized sectors of $\mathbb{C}$)
Let $\nu > 0$ be a positive number smaller than half the minimal distance between the elements of $\Omega\backslash\{\omega_2\}$ and smaller than the distance between the closed sets $\Omega\backslash\{\omega_2\}$ and $A(\omega_2, \delta)$. We call ν-generalized sectors of $\mathbb{C}$ associated with W_δ *any open domain $\Sigma_\nu \subset \mathbb{C}$ satisfying*

1. *Σ_ν is of the form*

$$\{x \in \mathbb{C}\ ;\ \alpha < \arg(x) < \beta \text{ and } |x-w| \geq \nu \text{ for all } w \in W_\delta \backslash \{0\}\}$$

with $\alpha, \beta \in]-2\pi, 0]$,

2. *Σ_ν is simply connected.*

We are now able to state the result in view in this section.

Proposition 3.13 (Dependence of the Borel transforms $\widehat{\boldsymbol{f}}^{\,\varepsilon[u]}(\tau)$)
Fix $u \in \{0, .., r-1\}$.

1. *Given $\nu > 0$ as in definition 3.12, there exists an entire function $\boldsymbol{\psi}^{[u]}$ on all $\mathbb{C}$ growing at most exponentially at infinity such that, for any ν-generalized sectors Σ_ν of $\mathbb{C}$ associated with W_δ, there is a constant $c \geq 1$ such that*

$$\boxed{\left|\widehat{\boldsymbol{f}}^{\,\varepsilon[u]}(\tau)\right| \leq \boldsymbol{\psi}^{[u]}(c\,|\tau|) \quad \text{for all } \tau \in \Sigma_\nu \text{ and } \varepsilon \in [0, \delta]}$$

2. *For all $\tau \notin W_\delta$, the function $\varepsilon \longmapsto \widehat{\boldsymbol{f}}^{\,\varepsilon[u]}(\tau)$ is continuous on $[0, \delta]$. In particular,*

$$\boxed{\lim_{\varepsilon \to 0} \widehat{\boldsymbol{f}}^{\,\varepsilon[u]}(\tau) = \widehat{\boldsymbol{f}}^{\,[u]}(\tau)}$$

Recall that the $\widehat{\boldsymbol{f}}^{\,\varepsilon[u]}(\tau)$'s, $u = 0, ..., r-1$, are the components of the Borel transform $\widehat{\boldsymbol{f}}^{\,\varepsilon}(\tau)$ of $\widetilde{\boldsymbol{f}}^{\,\varepsilon}(t)$:

$$\widehat{\boldsymbol{f}}^{\,\varepsilon}(\tau) = \begin{bmatrix} \widehat{\boldsymbol{f}}^{\,\varepsilon[0]}(\tau) \\ \vdots \\ \widehat{\boldsymbol{f}}^{\,\varepsilon[r-1]}(\tau) \end{bmatrix}$$

Proof. The proof of proposition 3.13 is derived from the proof of summable-resurgence of the $\widetilde{\boldsymbol{f}}^{\,\varepsilon[u]}(t)$'s given above.
Point (1) is straightforward from lemma 3.9. Indeed, Σ_ν can be seen as a ν-sectorial region Δ_ν^ε of $\mathcal{R}_{\Omega^\varepsilon}$ for any $\varepsilon \in [0, \delta]$ and the majorant series of $\widehat{\boldsymbol{f}}^{\,\varepsilon}(\tau)$ does not depend on ε.
Otherwise, lemma 3.8 shows that $\widehat{\boldsymbol{f}}^{\varepsilon}(\tau)$ reads as a series $\widehat{\boldsymbol{f}}^{\varepsilon}(\tau) = \sum_{m \geq 1} \boldsymbol{\varphi}_m^\varepsilon(\tau)$ where, for all $m \geq 1$, the terms $\boldsymbol{\varphi}_m^\varepsilon(\tau)$ are analytic functions on all $\mathcal{R}_{\Omega^\varepsilon}$ and are solutions of systems (3.20) which continuously depend on ε. In particular, for all $\tau \notin W_\delta$ and $m \geq 1$, the function $\varepsilon \longmapsto \boldsymbol{\varphi}_m^\varepsilon(\tau)$ is continuous on

$[0,\delta]$. Thus, by applying again lemma 3.9, the series $\sum_{m\geq 1}\boldsymbol{\varphi}_m^{\varepsilon}(\tau)$ is, for all $\tau\notin W_\delta$, a series of continuous functions in ε which normally converges on $[0,\delta]$. Hence, point (2) and proposition 3.13. ■

Note that the fact that the majorant series given in lemma 3.9 does not depend on the parameter ε is crucial.

Proposition 3.13 above can be extended to the other columns of $\widetilde{F}^{\varepsilon}$. Let us denote by

- $\widetilde{\boldsymbol{F}}^{\varepsilon[u]}$, $u=0,...,r-1$, the r-reduced series of $\widetilde{F}^{\varepsilon}$,
- $\boldsymbol{\Omega}^{\varepsilon}:=\{\alpha-\beta,\ \alpha,\beta\in\Omega^{\varepsilon}\}$ the set of Stokes values of the homological system associated with (3.15),
- $\boldsymbol{W}_\delta:=\bigcup_{\varepsilon\in[0,\delta]}\boldsymbol{\Omega}^{\varepsilon}$ the set of Stokes values of the homological systems of (3.15) for all $\varepsilon\in[0,\delta]$.

Since $\boldsymbol{W}_\delta$ is, like W_δ, the union of a finite set of points and a finite number of circle arcs of length δ [6], we can extend definition 3.12 into the one of *ν-generalized sectors of $\mathbb{C}$ associated with $\boldsymbol{W}_\delta$* allowing so to state the following result:

Corollary 3.14 (Dependence of the Borel transforms $\widehat{\boldsymbol{F}}^{\varepsilon[u]}(\tau)$)
Fix $u=0,...,r-1$.

1. *Given $\nu>0$ small enough, there exists an entire function $\boldsymbol{\Psi}^{[u]}$ on all $\mathbb{C}$ growing at most exponentially at infinity such that, for any ν-generalized sectors Σ_ν of $\mathbb{C}$ associated with $\boldsymbol{W}_\delta$, there is a constant $C\geq 1$ such that*

$$\boxed{\left|\widehat{\boldsymbol{F}}^{\varepsilon[u]}(\tau)\right|\leq\boldsymbol{\Psi}^{[u]}(C\,|\tau|)\quad \textit{for all } \tau\in\Sigma_\nu \textit{ and } \varepsilon\in[0,\delta]}$$

2. *For all $\tau\notin\boldsymbol{W}_\delta$, the function $\varepsilon\longmapsto\widehat{\boldsymbol{F}}^{\varepsilon[u]}(\tau)$ is continuous on $[0,\delta]$. In particular,*

$$\boxed{\lim_{\varepsilon\to 0}\widehat{\boldsymbol{F}}^{\varepsilon[u]}(\tau)=\widehat{\boldsymbol{F}}^{[u]}(\tau)}$$

[6] More precisely, $\boldsymbol{W}_\delta$ is the union of the set of Stokes values $a_{j,r}-a_{\ell,r}$ with $a_{*,r}\neq\omega_2$ and the circle arcs $\{\pm(\omega_2 e^{-ri\varepsilon}-a_{\ell,r})\ ,\ \varepsilon\in[0,\delta]$ and $a_{\ell,r}\neq\omega_2\}$.

3.3.4 Perturbed Stokes-Ramis matrices vs Stokes-Ramis matrices

Recall that

- Ω denotes the set of Stokes values of system (0.1),
- $\Omega_{r\theta_0} := \Omega^* \cap d_{r\theta_0}$ denotes the set of non-zero Stokes values of system (0.1) generating the collection $(\theta_k)_{k=0,\dots,r-1}$ of anti-Stokes direction associated with $\widetilde{f}(x)$.

Recall also that we suppose in section 3.3 that $\Omega_{r\theta_0} = \{\omega_1, \omega_2\}$ with $|\omega_1| < |\omega_2|$.

In addition to these notations, we denote by

- $\mathbf{\Omega} := \{\alpha - \beta \ ; \ \alpha, \beta \in \Omega\}$ the set of Stokes values of the homological system of (0.1),
- $\mathbf{\Omega}_{r\theta_0} := \mathbf{\Omega}^* \cap d_{r\theta_0}$ the set of non-zero Stokes values of the homological system of (0.1) generating the collection $(\theta_k)_{k=0,\dots,r-1}$.

Note that the Stokes values of $\mathbf{\Omega}_{r\theta_0}$ are the three Stokes values ω_1, ω_2 and $\omega_2 - \omega_1$ and possibly the Stokes values of the form

- $\omega_j - a_{k,r}$ with $j = 1, 2$ and $\arg(a_{k,r}) = r\theta_0 - \pi$,
- $\alpha - \beta$ with $\alpha, \beta \notin \Omega_{r\theta_0} = \{\omega_1, \omega_2\}$.

We can now study the action of the perturbation in ε on the collection (θ_k).

Lemma 3.15 (Action of the perturbation on (θ_k))
For $\varepsilon > 0$ small enough, the collection $(\theta_k)_{k=0,\dots,r-1}$ splits into the following collections of anti-Stokes directions of system (3.15):

1. *the three collections $(\theta_k)_{k=0,\dots,r-1}$, $(\theta_k - \varepsilon)_{k=0,\dots,r-1}$ and $(\theta_{k,\varepsilon})_{k=0,\dots,r-1}$ respectively generated by the Stokes values ω_1 [7], ω_2 and $\omega_2 e^{-2i\varepsilon} - \omega_1$,*
2. *the possibly collections which are generated by the Stokes values of the form $\omega_2 e^{-2i\varepsilon} - a_{k,r}$ with $\arg(a_{k,r}) = r\theta_0 - \pi$.*

[7] Note the collection $(\theta_k)_{k=0,\dots,r-1}$ can be also generated by the Stokes values of the form $\omega_1 - a_{k,r}$ with $\arg(a_{k,r}) = r\theta_0 - \pi$ or of the form $\alpha - \beta$ with $\alpha, \beta \notin \Omega_{r\theta_0}$ if they exist.

When they exist, we denote the collections of (2) by $(\theta_{k,s,\varepsilon})_{k=0,...,r-1}$, $s = 1, ..., p$, so that

$$\theta_k^\star > \theta_{k,1,\varepsilon}^\star > ... > \theta_{k,p,\varepsilon}^\star > (\theta_k - \varepsilon)^\star > \theta_{k,\varepsilon}^\star \tag{3.23}$$

for all $k = 0, ..., r-1$. Note that only (θ_k) and $(\theta_k - \varepsilon)$ are collections of anti-Stokes directions associated with $\widetilde{f}^\varepsilon(x)$.

For any directions $*$ above, we denote by $I_n + \mathfrak{S}_{*\star}^\varepsilon$ the corresponding Stokes-Ramis matrix associated with $\widetilde{Y}^\varepsilon$ (the formal fundamental solution of system (3.15) given page 28). As usually, we split $\mathfrak{S}_{*\star}^\varepsilon := [\mathfrak{S}_{*\star}^{\varepsilon\ j;\ell}]$ into blocks $\mathfrak{S}_{*\star}^{\varepsilon\ j;\ell}$ of size $n_j \times n_\ell$ fitting to the Jordan structure of L (matrix of exponents of formal monodromy). Recall that n_j is the size of the j-th Jordan block $L_j = \lambda_j I_{n_j} + J_{n_j}$ of L.

Lemma 3.16 (Perturbed Stokes-Ramis matrices)
For all $k = 0, ..., r-1$,

- *the block $\mathfrak{S}_{\theta_k^\star}^{\varepsilon\ j;1}$ is zero as soon as $a_{j,r} \neq \omega_1$,*
- *the ℓ-th column-block $\mathfrak{S}_{\theta_k^\star}^{\varepsilon\ \bullet;\ell}$ of $\mathfrak{S}_{\theta_k^\star}^\varepsilon$ is zero as soon as $a_{\ell,r} = \omega_2$,*
- *the block $\mathfrak{S}_{(\theta_k-\varepsilon)^\star}^{\varepsilon\ j;\ell}$ is zero as soon as $a_{j,r} \neq \omega_2$ or $\ell \neq 1$,*
- *the block $\mathfrak{S}_{\theta_{k,\varepsilon}^\star}^{\varepsilon\ j;\ell}$ is zero as soon as $a_{j,r} \neq \omega_2$ or $a_{\ell,r} \neq \omega_1$*
- *when the direction $\theta_{k,s,\varepsilon}$ exists, the ℓ-th column-block $\mathfrak{S}_{\theta_{k,s,\varepsilon}^\star}^{\varepsilon\ \bullet;\ell}$ of $\mathfrak{S}_{\theta_{k,s,\varepsilon}^\star}^\varepsilon$ is zero as soon as $\ell = 1$ or $a_{\ell,r} = \omega_2$.*

Proposition 3.17 below makes explicit the Stokes-Ramis matrices $I_n + C_{\theta_k^\star}$ of system (0.1) in terms of the perturbed Stokes-Ramis matrices above.

Proposition 3.17 (Perturbed vs initial Stokes-Ramis matrices)
Fix $k \in \{0, ..., r-1\}$.
Let M_k^ε be the matrix defined by

$$M_k^\varepsilon := (I_n + \mathfrak{S}_{\theta_{k,1,\varepsilon}^\star}^\varepsilon)(I_n + \mathfrak{S}_{\theta_{k,2,\varepsilon}^\star}^\varepsilon)...(I_n + \mathfrak{S}_{\theta_{k,p,\varepsilon}^\star}^\varepsilon)$$

when the collections $(\theta_{k,s,\varepsilon})$, $s = 1, ..., p$, exist and $M_k^\varepsilon := I_n$ otherwise. Then,

$$\boxed{I_n + C_{\theta_k^\star} = \lim_{\varepsilon \to 0} (I_n + \mathfrak{S}_{\theta_k^\star}^\varepsilon) M_k^\varepsilon (I_n + \mathfrak{S}_{(\theta_k-\varepsilon)^\star}^\varepsilon)(I_n + \mathfrak{S}_{\theta_{k,\varepsilon}^\star}^\varepsilon)} \tag{3.24}$$

In particular, for $a_{j,r} \in \{\omega_1, \omega_2\}$, the Stokes multipliers $c^{j;\bullet}_{\theta^\star_k}$ of $\widetilde{f}$ satisfy

$$\boxed{\begin{array}{ll} c^{j;\bullet}_{\theta^\star_k} = \lim\limits_{\varepsilon\to 0} \mathfrak{S}^{\varepsilon\ j;1}_{\theta^\star_k} & \textit{if } a_{j,r} = \omega_1 \\ \\ c^{j;\bullet}_{\theta^\star_k} = \lim\limits_{\varepsilon\to 0} \mathfrak{S}^{\varepsilon\ j;1}_{(\theta_k-\varepsilon)^\star} & \textit{if } a_{j,r} = \omega_2 \end{array}} \tag{3.25}$$

Proof. $\star$ Relations (3.25) are straightforward from relations (3.24) and lemma 3.16.

$\star$ Prove identity (3.24). In addition to condition $(C1)$, we can suppose, after possibly replacing δ by a smaller value $\delta' \in]0, \delta]$, that the two following conditions are satisfied:

$(C2)$ for all $\varepsilon \in]0, \delta]$ and $k = 0, ..., r-1$, the directions θ_k, $\theta_k - \varepsilon$, $\theta_{k,\varepsilon}$ and $\theta_{k,s,\varepsilon}$, $s = 1, ..., p$, are the only anti-Stokes directions of system (3.15) with principal determinations in $[\theta^\star_{k,\varepsilon}, \theta^\star_k]$,

$(C3)$ for all $k = 0, ..., r-1$, the direction $\theta^\star_{k,\delta}$ is close enough to $\theta^\star_k$ in order that the two sectors with vertex 0, opening π/r and respectively bisected by θ_k and $\theta_{k,\delta}$ overlap.

Then, according to (3.23), the sums $Y^\varepsilon_{\theta^-_{k,\varepsilon}}$ and $Y^\varepsilon_{\theta^+_k}$ are related, for all $k = 0, ..., r-1$ and $\arg(x) \simeq \theta^\star_0$, by the relation

$$\boxed{Y^\varepsilon_{\theta^-_{k,\varepsilon}}(\rho^k x) = Y^\varepsilon_{\theta^+_k}(\rho^k x)(I_n + \mathfrak{S}^\varepsilon_{\theta^\star_k})M^\varepsilon_k(I_n + \mathfrak{S}^\varepsilon_{(\theta_k-\varepsilon)^\star})(I_n + \mathfrak{S}^\varepsilon_{\theta^\star_{k,\varepsilon}})} \tag{3.26}$$

Otherwise, it results from the definition of the perturbation in ε that $\lim\limits_{\varepsilon\to 0} Q^\varepsilon \equiv Q$. Thus, lemma 3.18, point (2), below implies

$$\boxed{\lim_{\varepsilon\to 0} Y^\varepsilon_{\theta^-_{k,\varepsilon}}(\rho^k x) = Y_{\theta^-_k}(\rho^k x) \quad \text{and} \quad \lim_{\varepsilon\to 0} Y^\varepsilon_{\theta^+_k}(\rho^k x) = Y_{\theta^+_k}(\rho^k x)}$$

for $\arg(x) \simeq \theta^\star_0$. Consequently, we deduce from (3.26) that

$$\begin{aligned} \lim_{\varepsilon\to 0}(I_n + \mathfrak{S}^\varepsilon_{\theta^\star_k})M^\varepsilon_k(I_n + \mathfrak{S}^\varepsilon_{(\theta_k-\varepsilon)^\star})(I_n + \mathfrak{S}^\varepsilon_{\theta^\star_{k,\varepsilon}}) &= \lim_{\varepsilon\to 0} Y^\varepsilon_{\theta^+_k}(\rho^k x)^{-1} Y^\varepsilon_{\theta^-_{k,\varepsilon}}(\rho^k x) \\ &= Y_{\theta^+_k}(\rho^k x)^{-1} Y_{\theta^-_k}(\rho^k x) \\ &= I_n + C_{\theta^\star_k} \end{aligned}$$

for all $k = 0, ..., r-1$. Hence, proposition 3.17. ■

Lemma 3.18 *With notations as above:*

1. *For all* $u = 0, ..., r-1$ *and* $\arg(t) \simeq r\theta_0^\star$,

$$\boxed{\begin{array}{l} \lim\limits_{\varepsilon\to 0} s_{1;r\theta_0^+}(\widetilde{\boldsymbol{F}}^{\,\varepsilon[u]})(t) = s_{1;r\theta_0^+}(\widetilde{\boldsymbol{F}}^{\,[u]})(t) \\ \lim\limits_{\varepsilon\to 0} s_{1;r\theta_{0,\varepsilon}^-}(\widetilde{\boldsymbol{F}}^{\,\varepsilon[u]})(t) = s_{1;r\theta_0^-}(\widetilde{\boldsymbol{F}}^{\,[u]})(t) \end{array}}$$

2. *For all* $k = 0, ..., r-1$ *and* $\arg(x) \simeq \theta_0^\star$,

$$\boxed{\begin{array}{l} \lim\limits_{\varepsilon\to 0} s_{r;\theta_k^+}(\widetilde{F}^{\varepsilon})(\rho^k x) = s_{r;\theta_k^+}(\widetilde{F})(\rho^k x) \\ \lim\limits_{\varepsilon\to 0} s_{r;\theta_{k,\varepsilon}^-}(\widetilde{F}^{\varepsilon})(\rho^k x) = s_{r;\theta_k^-}(\widetilde{F})(\rho^k x) \end{array}}$$

Proof. $\star$ Point (2) is straightforward from point (1) and the two relations

$$s_{r;*_k}(\widetilde{F}_\varepsilon)(\rho^k x) = \sum_{u=0}^{r-1} (\rho^k x)^u s_{1;r*_0}(\widetilde{\boldsymbol{F}}^{\,\varepsilon[u]})(x^r)$$

$$s_{r;\theta_k^\pm}(\widetilde{F})(\rho^k x) = \sum_{u=0}^{r-1} (\rho^k x)^u s_{1;r\theta_0^\pm}(\widetilde{\boldsymbol{F}}^{\,[u]})(x^r)$$

connecting the r-sum of $\widetilde{F}^\varepsilon$ (resp. $\widetilde{F}$) in direction $*_k \in \{\theta_k^+, \theta_{k,\varepsilon}^-\}$ (resp. $\theta_k^\pm$) to the 1-sums of its r-reduced series $\widetilde{\boldsymbol{F}}^{\,\varepsilon[u]}$ (resp. $\widetilde{\boldsymbol{F}}^{\,[u]}$) in direction $r*_0$ (resp. $r\theta_0^\pm$).

$\star$ Prove the first identity of point (1). Fix $u \in \{0, ..., r-1\}$ and $\arg(t) \simeq r\theta_0^\star$. For $\eta > 0$ small enough, the sums $s_{1;r\theta_0^+}(\widetilde{\boldsymbol{F}}^{\,\varepsilon[u]})(t)$ and $s_{1;r\theta_0^+}(\widetilde{\boldsymbol{F}}^{\,[u]})(t)$ are respectively given by the Borel-Laplace integrals

$$s_{1;r\theta_0^+}(\widetilde{\boldsymbol{F}}^{\,\varepsilon[u]})(t) = \int_{d_{r\theta_0+\eta}} \widehat{\boldsymbol{F}}^{\,\varepsilon[u]}(\tau) e^{-\tau/t} d\tau = \int_0^{+\infty} \widehat{\boldsymbol{G}}_1^{\varepsilon[u]}(\tau) d\tau$$

where

$$\widehat{\boldsymbol{G}}_1^{\varepsilon[u]}(\tau) := \widehat{\boldsymbol{F}}^{\,\varepsilon[u]}(\tau e^{i(r\theta_0+\eta)}) e^{-\tau \exp(i(r\theta_0+\eta))/t}$$

and

$$s_{1;r\theta_0^+}(\widetilde{\boldsymbol{F}}^{\,[u]})(t) = \int_{d_{r\theta_0+\eta}} \widehat{\boldsymbol{F}}^{\,[u]}(\tau) e^{-\tau/t} d\tau = \int_0^{+\infty} \widehat{\boldsymbol{G}}_1^{[u]}(\tau) d\tau$$

where

$$\widehat{\boldsymbol{G}}_1^{[u]}(\tau) := \widehat{\boldsymbol{F}}^{\,[u]}(\tau e^{i(r\theta_0+\eta)}) e^{-\tau \exp(i(r\theta_0+\eta))/t}$$

Apply corollary 3.14:

– the function $\varepsilon \longmapsto \widehat{\boldsymbol{G}}_1^{\varepsilon[u]}(\tau)$ is *continuous on* $[0,\delta]$ for all $\tau \geq 0$ and

$$\boxed{\lim_{\varepsilon\to 0} \widehat{\boldsymbol{G}}_1^{\varepsilon[u]}(\tau) = \widehat{\boldsymbol{G}}_1^{[u]}(\tau)}$$

– there exist *an entire function* $\boldsymbol{\Psi}^{[u]}$ *on all* $\mathbb{C}$ *growing at most exponentially at infinity* and a constant $C \geq 1$ such that, for all $\varepsilon \in [0,\delta]$ and $\tau \geq 0$,

$$\boxed{\left|\widehat{\boldsymbol{G}}_1^{\varepsilon[u]}(\tau)\right| \leq \boldsymbol{\Psi}^{[u]}(C\tau)e^{-\tau\,\mathrm{Re}(\exp(i(r\theta_0+\eta))/t)} := \boldsymbol{M}_1^{[u]}(\tau)} \tag{3.27}$$

Then, since $\boldsymbol{M}_1^{[u]}(\tau)$ is integrable on $[0;+\infty[$, we deduce from Lebesgue dominated convergence theorem that the function $\varepsilon \longmapsto s_{1;r\theta_0^+}(\widetilde{\boldsymbol{F}}^{\varepsilon[u]})(t)$ is continuous on $[0,\delta]$ and

$$\lim_{\varepsilon\to 0} s_{1;r\theta_0^+}(\widetilde{\boldsymbol{F}}^{\varepsilon[u]})(t) = \int_0^{+\infty} \lim_{\varepsilon\to 0} \widehat{\boldsymbol{G}}_1^{\varepsilon[u]}(\tau)d\tau = \int_0^{+\infty} \widehat{\boldsymbol{G}}_1^{[u]}(\tau)d\tau = s_{1;r\theta_0^+}(\widetilde{\boldsymbol{F}}^{[u]})(t)$$

Hence, the first identity of point (1).

$\star$ The second identity of point (1) is obtained similarly. Note that, in addition to conditions $(C1)$, $(C2)$ and $(C3)$ above, we can suppose, after possibly replacing δ by a smaller value in $]0,\delta]$, that the following condition is satisfied:

$(C4)$ for any $\varepsilon \in]0,\delta[$, there is no anti-Stokes direction of system (3.15) with a principal determination in $[\theta^\star_{k,\delta}, \theta^\star_{k,\varepsilon}[$.

As before, there exists $\eta > 0$ such that the sums $s_{1;r\theta_{0,\varepsilon}^-}(\widetilde{\boldsymbol{F}}^{\varepsilon[u]})(t)$ and $s_{1;r\theta_0^-}(\widetilde{\boldsymbol{F}}^{[u]})(t)$ are respectively given by the Borel-Laplace integrals

$$s_{1;r\theta_{0,\varepsilon}^-}(\widetilde{\boldsymbol{F}}^{\varepsilon[u]})(t) = \int_{d_{r\theta_{0,\delta}-\eta}} \widehat{\boldsymbol{F}}^{\varepsilon[u]}(\tau)e^{-\tau/t}d\tau = \int_0^{+\infty} \widehat{\boldsymbol{G}}_2^{\varepsilon[u]}(\tau)d\tau$$

where

$$\widehat{\boldsymbol{G}}_2^{\varepsilon[u]}(\tau) := \widehat{\boldsymbol{F}}^{\varepsilon[u]}(\tau e^{i(r\theta_{0,\delta}-\eta)})e^{-\tau\exp(i(r\theta_{0,\delta}-\eta))/t}$$

and

$$s_{1;r\theta_0^-}(\widetilde{\boldsymbol{F}}^{[u]})(t) = \int_{d_{r\theta_{0,\delta}-\eta}} \widehat{\boldsymbol{F}}^{[u]}(\tau)e^{-\tau/t}d\tau = \int_0^{+\infty} \widehat{\boldsymbol{G}}_2^{[u]}(\tau)d\tau$$

where

$$\widehat{\boldsymbol{G}}_2^{[u]}(\tau) := \widehat{\boldsymbol{F}}^{[u]}(\tau e^{i(r\theta_{0,\delta}-\eta)})e^{-\tau\exp(i(r\theta_{0,\delta}-\eta))/t}$$

We conclude as above by using corollary 3.14. ■

Remark 3.19 Inequality (3.27) allowing to apply Lebesgue dominated convergence theorem is the key point of the proof of lemma 3.18. Recall (*cf.* corollary 3.14) that such an inequality was obtained because *the parameter ε is only related to the non-zero Stokes values of* Ω. Of course, for a more general perturbation, *i.e.*, the parameter of which is no longer related only to Ω^*, inequality (3.27) and hence lemma 3.18 no longer hold in general.

We are now able to prove theorem 2.2 in the case when the collection $(\theta_k)_{k=0,\dots,r-1}$ is generated by just two Stokes values.

3.3.5 *Proof of theorem 2.2*

Recall that we must prove identity (2.2) when the front of ω_1 (resp. ω_2) is monomial. Recall also that such a condition can always be fulfilled by means of a change of the variable x in system (0.1) (*cf.* lemma 2.4).

First case: the front of ω_1 is monomial. For all $\varepsilon \in]0,\delta]$, the collection $(\theta_k)_{k=0,\dots,r-1}$ is a collection of anti-Stokes directions of system (3.15) associated with $\widetilde{f}^{\,\varepsilon}$ and is generated by *just one* Stokes value of Ω^ε (here, ω_1). Then, the hypothesis "ω_1 *is with monomial front*" allows us to apply section 3.1: the perturbed Stokes multipliers $\mathfrak{S}^{\varepsilon\ j;1}_{\theta_k^\star}$, j such that $a_{j,r}=\omega_1$, of $\widetilde{f}^{\varepsilon}$ at ω_1 are expressed in terms of the connection constants of the $\widehat{\boldsymbol{f}}^{\,\varepsilon[u]}$'s, $u=0,\dots,r-1$, at ω_1. More precisely, denoting by $\boldsymbol{K}^{\varepsilon[u]}_{\omega_1^\star,+}$ the connection matrix of $\widehat{\boldsymbol{f}}^{\,\varepsilon[u]}$ at ω_1 and $\boldsymbol{K}^{\varepsilon[u]j;\bullet}_{\omega_1^\star,+}$ its j-th row-block accordingly to the Jordan structure of L (matrix of exponents of formal monodromy), we obtain

$$\boxed{\mathfrak{S}^{\varepsilon\ j;1}_{\theta_k^\star}=\sum_{u=0}^{r-1}\rho^{k(uI_{n_j}-L_j)}\boldsymbol{I}^{\varepsilon[u]j;\bullet}_{\omega_1^\star}\rho^{kJ_{n_1}}} \tag{3.28}$$

for all $k=0,\dots,r-1$, where

$$\boxed{\boldsymbol{I}^{\varepsilon[u]j;\bullet}_{\omega_1^\star}:=\int_{\gamma_0}\tau^{\frac{\lambda_j-u}{r}-1}\tau^{\frac{J_{n_j}}{r}}\boldsymbol{K}^{\varepsilon[u]j;\bullet}_{\omega_1^\star,+}\tau^{-\frac{J_{n_1}}{r}}e^{-\tau}d\tau}$$

We refer to theorem 2.2 for the definition of path γ_0.

Otherwise, proposition 3.13, point (2), implies that the principal majors $\breve{\boldsymbol{f}}^{\varepsilon[u]}_{\omega_1^\star,+}$ and $\breve{\boldsymbol{f}}^{[u]}_{\omega_1^\star,+}$ of $\widehat{\boldsymbol{f}}^{\varepsilon[u]}$ and $\widehat{\boldsymbol{f}}^{[u]}$ respectively are related, for all $u=0,\dots,r-1$, by the relation

$$\lim_{\varepsilon\to 0}\breve{\boldsymbol{f}}^{\,\varepsilon[u]}_{\omega_1^\star,+}(\tau)=\breve{\boldsymbol{f}}^{\,[u]}_{\omega_1^\star,+}(\tau)\quad\text{for all }\tau\notin\bigcup_{\varepsilon\in[0,\delta]}\Omega^\varepsilon$$

Then, we obtain

$$\boxed{\lim_{\varepsilon\to 0}\boldsymbol{K}^{\varepsilon[u]j;\bullet}_{\omega_1^\star,+} = \boldsymbol{K}^{[u]j;\bullet}_{\omega_1^\star,+}}$$

and, consequently, proposition 3.17 applied to identity (3.28) implies identity (2.2).

Second case: the front of ω_2 ***is monomial.*** Identity (2.2) is obtained in the same way. Indeed, for all $\varepsilon \in]0,\delta]$, the collection $(\theta_k - \varepsilon)_{k=0,\dots,r-1}$ still is a collection of anti-Stokes directions of system (3.15) associated with $\widetilde{f}^{\varepsilon}$ and still is generated by *just one* Stokes value of Ω^ε, here $\omega_2 e^{-ri\varepsilon}$, the front of which is *monomial*. Therefore, we can apply again section 3.1 and we can conclude as above by using propositions 3.13 and 3.17. ■

3.4 General case

We are left to prove theorem 2.2 when the collection $(\theta_k)_{k=0,\dots,r-1}$ is generated by $p \geq 3$ Stokes values $(\omega_k)_{k=1,\dots,p}$ satisfying $|\omega_1| < |\omega_2| < \dots < |\omega_p|$.

We proceed similarly as in section 3.3 by considering, for $\varepsilon \geq 0$, the regularly perturbed system

$$x^{r+1}\frac{dY}{dx} = A^\varepsilon(x)Y \tag{3.29}$$

with

$$A^\varepsilon(x) = \bigoplus_{j=1}^{J}\left[\left(ra^\varepsilon_{j,r} + \sum_{k=1}^{r-1} k a_{j,k} x^{r-k}\right) I_{n_j} + x^r L_j\right] + B(x)$$

where, for all $j = 1, \dots, J$,

$$a^\varepsilon_{j,r} = \begin{cases} a_{j,r} & \text{if } a_{j,r} \notin \{\omega_1, \omega_2, \dots, \omega_p\} \\ \omega_s \exp\left(-ri\frac{(s-1)\varepsilon}{p-1}\right) & \text{if } a_{j,r} = \omega_s \end{cases}$$

Doing so, we can check that the results of section 3.3 can be extended to system (3.29). This ends the proof of theorem 2.2. ■

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