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On digit patterns in expansions of rational numbers with prime denominator

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Abstract

We show that, for any fixed $\varepsilon > 0$ and almost all primes p , the g -ary expansion of any fraction m/p with $\gcd(m, p) = 1$ contains almost all g -ary strings of length $k < (5/24 - \varepsilon) \log_g p$. This complements a result of J. Bourgain, S. V. Konyagin, and I. E. Shparlinski that asserts that, for almost all primes, all g -ary strings of length $k < (41/504 - \varepsilon) \log_g p$ occur in the g -ary expansion of m/p .

1 Introduction

Let us fix some integer $g \geq 2$. It is well-known that if $\gcd(n, gm) = 1$ then the g -ary expansion of the rational fractions m/n is purely periodic with period t_n , which is independent of m and equals the multiplicative order of g modulo n , see [9]. In the series of works [3, 8, 9], the distribution of digit patterns in such expansions has been studied. In particular, for positive integers k and $m < n$ with $\gcd(n, gm) = 1$, we denote by $T_{m,n}(k)$ the number

of distinct g -ary strings $(d_1, \dots, d_k) \in \{0, 1, \dots, g-1\}^k$ that occur among the first t_n strings $(\delta_r, \dots, \delta_{r+k-1})$, $r = 1, \dots, t_n$, from the g -ary expansion

$$\frac{m}{n} = \sum_{r=1}^{\infty} \delta_r g^{-r}, \quad \delta_r \in \{0, 1, \dots, g-1\}. \quad (1)$$

Motivated by applications to pseudorandom number generators, see [1], we are interested in describing the conditions under which $T_{m,n}(k)$ is close to its trivial upper bound

$$T_{m,n}(k) \leq \min\{t_n, g^k\}.$$

Since $t_n \leq n$, it is clear that only values $k \leq \lceil \log_g n \rceil$ are of interest. It has been shown in [8, Theorem 11.1] that, for any fixed $\varepsilon > 0$ and for almost all primes p (that is, for all but $o(x/\log x)$ primes $p \leq x$), we have $T_{m,p}(k) = g^k$, provided that $k \leq (3/37 - \varepsilon) \log_g p$. The coefficient $3/37$ has been increased up to $41/504$ in [3, Corollary 8]. Here we show that, for almost all primes p , we have $T_{m,p}(k) = (1 + o(1))g^k$ for much larger string lengths k .

Theorem 1. *For any fixed $\varepsilon > 0$, for almost all primes p , we have*

$$T_{m,p}(k) = (1 + o(1))g^k$$

as $p \rightarrow \infty$, provided that $k \leq (5/24 - \varepsilon) \log_g p$.

Our arguments depend on the reduction of the problem to the study of intersections of intervals and multiplicative groups modulo p generated by g , that has been established in [8]. In turn, the question about the intersections of intervals and subgroups in residue rings has been studied in a number of works [3, 4, 8]. In particular, the results of [3, Corollary 8] and [8, Theorem 11.1] are based on estimates of the length of the longest interval that is not hit by a subgroup of the multiplicative group \mathbb{F}_p^* of the field \mathbb{F}_p of p elements. To prove Theorem 1, we use the results and ideas of [3] to estimate the total number of intervals of a given length that do not intersect a given subgroup of \mathbb{F}_p^* .

2 Multiplicative Orders

We recall the following well-known implication of the classical result of [5].

Lemma 2. *For almost all primes p , the multiplicative order t of g modulo p satisfies $t > p^{1/2}$.*

3 Bounds of Some Exponential Sums

Let p be prime and let $\mathcal{G} \subseteq \mathbb{F}_p^*$ be a subgroup of order t , where \mathbb{F}_p is a finite field of p elements.

We denote

$$\mathbf{e}_p(z) = \exp(2\pi iz/p)$$

and define exponential sums

$$S_\lambda(p; \mathcal{G}) = \sum_{v \in \mathcal{G}} \mathbf{e}_p(\lambda v).$$

Using [6, Lemma 3] (see also [8, Lemma 3.3]) if $t < p^{2/3}$, and the well known bounds

$$|S_\lambda(p; \mathcal{G})| \leq p^{1/2} \quad \text{and} \quad \sum_{\lambda \in \mathbb{F}_p^*} |S_\lambda(p; \mathcal{G})|^2 \leq pt$$

(see [8, Equations (3.4) and (3.15)]) if $t \geq p^{2/3}$, we derive:

Lemma 3. *For any prime p and a subgroup $\mathcal{G} \subseteq \mathbb{F}_p^*$ of order t , we have*

$$\sum_{\lambda \in \mathbb{F}_p^*} |S_\lambda(p; \mathcal{G})|^4 \ll pt^{5/2}.$$

4 Intervals Avoiding Subgroups

As before, let p be prime and let $\mathcal{G} \subseteq \mathbb{F}_p^*$ be a subgroup of order t .

Let $\mathcal{U}(p; \mathcal{G}, H)$ be the set of $u \in \mathbb{F}_p$ such the congruence

$$v \equiv u + x \pmod{p}, \quad v \in \mathcal{G}, \quad 0 \leq x < H,$$

has no solution.

Lemma 4. *Assume that \mathcal{G} is of order $t > p^{1/2}$. Then, for any fixed integer $\nu \geq 1$, we have*

$$\begin{aligned} \#\mathcal{U}(p; \mathcal{G}, H) \leq & p^{2-1/4(\nu+1)+o(1)} H^{-1/2} t^{-5/4+(2\nu+1)/4\nu(\nu+1)} \\ & + p^{5/2-1/2\nu+o(1)} H^{-1} t^{-5/4+1/2\nu}. \end{aligned}$$

Proof. Let us fix some $\varepsilon > 0$. We put

$$s = \left\lceil \frac{3}{2}(1 + \varepsilon^{-1}) \right\rceil, \quad h = \lceil p^{1+\varepsilon}/H \rceil, \quad Z = \lceil H/s \rceil.$$

We can assume that $h < p/2$, as otherwise the bound is trivial (for example, it follows immediately from the bound of Heath-Brown and Konyagin [6, Theorem 1]). Obviously

$$\mathcal{U}(p; \mathcal{G}, H) \subseteq \mathcal{W}_s(p; \mathcal{G}, Z), \quad (2)$$

where $\mathcal{W}_s(p; \mathcal{G}, Z)$ is the set of $u \in \mathbb{F}_p$ such the congruence

$$v \equiv u + x_1 + \dots + x_s \pmod{p}, \quad v \in \mathcal{G}, \quad 0 \leq x_1, \dots, x_s < Z, \quad (3)$$

has no solution.

For the number $Q_s(p; \mathcal{G}, Z, u)$ of solutions to the congruence (3), exactly as in the proof of [8, Lemma 7.1], we obtain

$$Q_s(p; \mathcal{G}, Z, u) = \frac{1}{p} \sum_{|a| < p/2} \mathbf{e}_p(-au) \left(\sum_{0 \leq x < Z} \mathbf{e}_p(ax) \right)^s S_a(p; \mathcal{G}).$$

where the sums $S_a(p; \mathcal{G})$ are defined in Section 3.

Separating the term $tZ^s p^{-1}$ corresponding to $a = 0$ and summing over all $u \in \mathcal{W}_s(p; \mathcal{G}, Z)$ yields

$$0 = \sum_{u \in \mathcal{W}_s(p; \mathcal{G}, Z)} Q_s(p; \mathcal{G}, Z, u) \geq \frac{tWZ^s}{p} - \frac{\sigma}{p},$$

where

$$W = \#\mathcal{W}_s(p; \mathcal{G}, Z)$$

and

$$\sigma = \sum_{1 \leq |a| < p/2} \left| \sum_{u \in \mathcal{W}_s(p; \mathcal{G}, Z)} \mathbf{e}_p(au) \right| \left| \sum_{0 \leq x < Z} \mathbf{e}_p(ax) \right|^s |S_a(p; \mathcal{G})|.$$

Using the Cauchy inequality, and then the orthogonality relation for exponential functions, we obtain

$$\begin{aligned} \sigma^2 &\leq \sum_{1 \leq |a| < p/2} \left| \sum_{u \in \mathcal{W}_s(p; \mathcal{G}, Z)} \mathbf{e}_p(au) \right|^2 \sum_{1 \leq |a| < p/2} \left| \sum_{0 \leq x < Z} \mathbf{e}_p(ax) \right|^{2s} |S_a(p; \mathcal{G})|^2 \\ &\leq pW \sum_{1 \leq |a| < p/2} \left| \sum_{0 \leq x < Z} \mathbf{e}_p(ax) \right|^{2s} |S_a(p; \mathcal{G})|^2. \end{aligned}$$

Hence

$$W \leq \frac{p}{t^2 Z^{2s}} \Sigma, \quad (4)$$

where

$$\Sigma = \sum_{1 \leq |a| < p/2} \left| \sum_{0 \leq x < Z} \mathbf{e}_p(ax) \right|^{2s} |S_a(p; \mathcal{G})|^2.$$

Following the idea of the proof of [8, Lemma 7.1], we write

$$\Sigma = \Sigma_1 + \Sigma_2, \quad (5)$$

where

$$\begin{aligned} \Sigma_1 &= \sum_{1 \leq |a| \leq h} \left| \sum_{0 \leq x < Z} \mathbf{e}_p(ax) \right|^{2s} |S_a(p; \mathcal{G})|^2, \\ \Sigma_2 &= \sum_{h < |a| < p/2} \left| \sum_{0 \leq x < Z} \mathbf{e}_p(ax) \right|^{2s} |S_a(p; \mathcal{G})|^2. \end{aligned}$$

For $1 \leq |a| \leq h$, we use the trivial estimate

$$\left| \sum_{0 \leq x < Z} \mathbf{e}_p(ax) \right| \leq Z$$

and derive

$$\begin{aligned} \Sigma_1 &\leq Z^{2s} \sum_{1 \leq |a| \leq h} |S_a(p; \mathcal{G})|^2 = \frac{Z^{2s}}{t} \sum_{1 \leq |a| \leq h} \sum_{w \in \mathcal{G}} |S_{aw}(p; \mathcal{G})|^2 \\ &= \frac{Z^{2s}}{t} \sum_{\lambda \in \mathbb{F}_p^*} M_\lambda(p; \mathcal{G}, h) |S_\lambda(p; \mathcal{G})|^2, \end{aligned}$$

where $M_\lambda(p; \mathcal{G}, h)$ denotes the number of solutions to the congruence

$$\lambda \equiv aw \pmod{p}, \quad 1 \leq |a| \leq h, \quad w \in \mathcal{G}.$$

Hence, by the Cauchy inequality

$$\Sigma_1 \leq \frac{Z^{2s}}{t} \left(\sum_{\lambda \in \mathbb{F}_p^*} M_\lambda(p; \mathcal{G}, h)^2 \right)^{1/2} \left(\sum_{\lambda \in \mathbb{F}_p^*} |S_\lambda(p; \mathcal{G})|^4 \right)^{1/2}.$$

As in [3, Section 3.3], we have

$$\sum_{\lambda \in \mathbb{F}_p^*} M_\lambda(p; \mathcal{G}, h)^2 \leq tN(p; \mathcal{G}, h),$$

where $N(p; \mathcal{G}, h)$ is the number of solutions of the congruence

$$ux \equiv y \pmod{p}, \quad 0 < |x|, |y| \leq h, \quad u \in \mathcal{G}.$$

Therefore,

$$\Sigma_1 \leq \frac{Z^{2s}}{t^{1/2}} N(p; \mathcal{G}, h)^{1/2} \left(\sum_{\lambda \in \mathbb{F}_p^*} |S_\lambda(p; \mathcal{G})|^4 \right)^{1/2}. \quad (6)$$

It is shown in [3, Theorem 1] that if $t \geq p^{1/2}$ then for any fixed integer ν and any positive number h , we have

$$N(p; \mathcal{G}, h) \leq ht^{(2\nu+1)/2\nu(\nu+1)} p^{-1/2(\nu+1)+o(1)} + h^2 t^{1/\nu} p^{-1/\nu+o(1)}. \quad (7)$$

Therefore, using Lemma 3 and the bound (7) we derive from (6) that

$$\Sigma_1 \leq p^{1/2} t^{3/4} Z^{2s} \left(h^{1/2} t^{(2\nu+1)/4\nu(\nu+1)} p^{-1/4(\nu+1)+o(1)} + ht^{1/2\nu} p^{-1/2\nu+o(1)} \right). \quad (8)$$

If $h < |a| < p/2$, then we use the bound

$$\sum_{0 \leq x < Z} \mathbf{e}_p(ax) \ll \frac{p}{|a|},$$

see [7, Bound (8.6)]. From the trivial bound

$$|S_a(p; \mathcal{G})| \leq t,$$

recalling the choice of h , we obtain

$$\Sigma_2 \ll \sum_{h < |a| < p/2} \left(\frac{p}{|a|} \right)^{2s} t^2 \ll t^2 \frac{p^{2s}}{h^{2s-1}} \ll t^2 \frac{Z^{2s} h}{p^{2s\varepsilon}} \leq \frac{Z^{2s} p^3}{p^{2s\varepsilon}} \ll Z^{2s},$$

as $2s\varepsilon > 3$ for our choice of s . Thus the bound on Σ_2 is dominated by the bound (8) on Σ_1 . Using (4) and (5), we obtain

$$W \leq p^{3/2} t^{-5/4} \left(h^{1/2} t^{(2\nu+1)/4\nu(\nu+1)} p^{-1/4(\nu+1)+o(1)} + h t^{1/2\nu} p^{-1/2\nu+o(1)} \right).$$

Recalling (2), the choice of h and that ε is arbitrary, after simple calculations, we obtain the result. \square

Corollary 5. *Assume that \mathcal{G} is of order $t > p^{1/2}$. Then for any $\varepsilon > 0$ and*

$$H \geq p^{19/24+\varepsilon}$$

we have

$$\#\mathcal{U}(p; \mathcal{G}, H) = o(p).$$

Proof. Since $t > p^{1/2}$, we have, for any fixed integer $\nu \geq 1$,

$$\#\mathcal{U}(p; \mathcal{G}, H) \leq p^{11/8+1/8\nu(\nu+1)+o(1)} H^{-1/2} + p^{15/8-1/4\nu+o(1)} H^{-1}.$$

Taking $\nu = 2$ or $\nu = 3$, we conclude the proof. \square

5 Proof of Theorem 1

By Lemma 2 it is enough to consider prime p for which the multiplicative order t of g modulo p satisfies $t > p^{1/2}$.

We now take a positive integer $k \leq (5/24 - \varepsilon) \log_g p$ and consider the intervals $\left[\frac{D}{g^k}, \frac{D+1}{g^k} \right)$. As in the proof of [8, Theorem 11.1], we observe that, for any integer $\ell \geq 0$ and any g -ary string (d_1, \dots, d_k) , we have $\delta_{\ell+i} = d_i$, $i = 1, \dots, k$, if and only if

$$\frac{mg^\ell}{p} - \left\lfloor \frac{mg^\ell}{p} \right\rfloor \in \left[\frac{D}{g^k}, \frac{D+1}{g^k} \right),$$

where $D = d_1g^{k-1} + d_2g^{k-2} + \dots + d_k$ and the δ_r , $r = 1, 2, \dots$, are defined by (1) with $n = p$. Thus, if a string (d_1, \dots, d_k) is not present in the g -ary expansion of m/p , then each interval $[u, u + H)$ with

$$u = \left[\frac{D}{g^k} p \right], \dots, \left[\frac{D + 1/2}{g^k} p \right] \quad \text{and} \quad H = \left[\frac{1}{2g^k} p \right]$$

contains no element of the conjugacy class $m\mathcal{G}_p$ of the group \mathcal{G}_p generated by g modulo p . Clearly, different strings (d_1, \dots, d_k) correspond to different intervals of the values of u , and each of them contains

$$\left[\frac{D + 1/2}{g^k} p \right] - \left[\frac{D}{g^k} p \right] \gg \frac{p}{g^k}$$

values of u . Therefore, the number of missing strings (d_1, \dots, d_k) satisfies

$$g^k - T_{m,p}(k) \ll \frac{g^k}{p} \#\mathcal{U}(p; \mathcal{G}_p, H).$$

Since $g^k \leq p^{5/24-\varepsilon}$, we infer from Corollary 5 that $\#\mathcal{U}(p; \mathcal{G}_p, H) = o(p)$, which proves Theorem 1.

6 Composite Denominators

It is quite likely that one can also study $T_{m,n}(k)$ for almost all composite n by supplementing the ideas of this work with those of [2] (to get an analogue of Lemma 3) and also using the result of [10] that gives an analogue of Lemma 2.

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