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# Auctions and Shareholdings

David Ettinger<sup>†</sup>

**ABSTRACT.** – This paper examines how shareholdings affect auctions' revenue and efficiency with independent private values. Two types of shareholdings are analyzed: Vertical (resp: horizontal) toeholds cover situations in which bidders own a fraction of the seller's profit (resp: a share of their competitor's profit). Expected revenue is an increasing (resp: decreasing) function of vertical (resp: horizontal) toeholds. With both types of toeholds, auction formats are not revenue equivalent. Expected revenue is affected to a greater extent by the presence of toeholds in the second-price auction than in the first-price auction.

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## Enchères et Participations Capitalistiques

**RÉSUMÉ :** Nous étudions l'impact des participations au capital sur les enchères à valeurs privées indépendantes. Le revenu de l'enchère décroît lorsque les acheteurs ont des participations croisées et croît lorsqu'ils détiennent une partie du capital du vendeur. Dans tous les cas de participations capitalistiques, les formats d'enchères ne sont plus équivalents. L'enchère au second prix est plus sensible, à la hausse ou à la baisse, que l'enchère au premier prix, à la présence de participations.

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# 1 Introduction

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This paper studies how shareholdings and cross-shareholdings affect firms' behavior in an auction context. But first, let us illustrate through two case studies why shareholdings can matter in an auction framework.

**The Global One case:** Global One was a joint-venture created in 1996 by Deutsche Telekom (40%), France Télécom (40%) and Sprint (20%). In 1999, in order to comply with anti-trust regulations, Sprint was forced to sell its Global One shares. In accordance with a previous agreement, the two remaining ex-partners were the only potential buyers for Sprint's shares. Neither of the two European companies was willing to own only 40% of Global One, while his competitor owned 60%. The remaining partners thus agreed to use a selling process that would result in one unique owner, possessing the totality of Global One. The three firms considered using an auction process.<sup>1</sup> However, the situation was slightly different from a standard auction setting because the bidders, France Télécom and Deutsche Telekom, were forced to assume contradictory roles, simultaneously potential buyers of remaining shares and potential sellers of their own shares.

In the Global One case, bidders owned, *ex ante*, a fraction of the good for sale. This is not the first time shareholdings have affected strategic incentives in an auction framework. We also observe situations in which bidders own a fraction of the capital of other bidders. Such a situation can be illustrated through the following case.

**RVI and Volvo Trucks** were two majors European truck manufacturers. Between 1991 and 1994, they had symmetric cross-shareholdings of 45%. At the same time, in some European countries, their joint market share exceeded 50%. Nevertheless, during that period, they still regularly competed in tenders organized by major haulage companies in these two countries. Once again, in the case of RVI and Volvo Trucks, one may speculate as to the impact that these toeholds had on strategic planning during bidding stages. As a matter of fact, each truck manufacturer benefited from a fraction of the other's profit. Thus, if RVI lost a market, it was in his interest that Volvo Trucks win the market and generate the highest possible profit.

In both of these cases, because of the presence of toeholds, losing bidders actually cared about the price paid by the winner. In this paper, we seek to understand how these toeholds affected their strategies. Generally speaking, do shareholdings have consequences on the efficiency or the expected revenue of the auction? Do they have the same impact on different auction formats? Which of the standard auction formats is preferable in these cases? What is the optimal auction format?

Two different types of shareholdings are considered. For the simplicity of this demonstration we use the following terminology. When a bidder owns shares of the seller or a fraction of the object for sale<sup>2</sup> (CF: the Global One case) we speak of a *vertical toehold*. When a bidder owns shares of another bidder, we speak of

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1. The author spoke with France Télécom's Finance Director before the choice of the auction process. Results of the current paper were evoked. The selected auction format remained secret.  
2. Both events are conceptually identical.

a *horizontal toehold*.<sup>3</sup> We examine how both types of toeholds affect the revenue and efficiency of auction formats in an independent private value setting in the first-price auction and the second-price auction.<sup>4</sup> We also put forward an optimal auction format for both types of toeholds.

The topic of toeholds in an auction context has been the object of several studies. More specifically, the work of ENGELBRECHT-WIGGANS [1994] can be reinterpreted as a study of an auction with vertical toeholds<sup>5</sup>. In this context, he showed that the expected price is higher with a second-price auction than with a first-price auction. This result is also used in GOEREE *et al.* [2005], MAASLAND and ONDERSTAL [2007] and ENGERS and McMANUS [2007]. GOEREE *et al.* [2005] also introduce an optimal auction format for charity auctions, the lowest-price all-pay auction, which can be applied to the vertical toehold case.

In the horizontal toehold case, DASGUPTA and TSUI [2004]<sup>6</sup> show that, in the independent and private value case, with two bidders and reserve prices, expected revenue is higher in a first-price auction than a second-price auction.

Compared to these papers, we intend to derive general results regarding the impact of toeholds on auctions through a joint analysis of the two types of toeholds. Moreover, we intend to provide a more systematic approach by considering the  $n$  bidders' case, revenue comparison, optimal auction formats and the bidders' point of view.

More specifically, in the vertical toehold case, we integrate ENGELBRECHT-WIGGANS' [1994] results but we also show that it is possible to use standard auction formats in an optimal mechanism and not only the lowest-price all-pay auction as suggested in GOEREE *et al.* [2005]. Once again, in the case of RVI and Volvo Trucks, one may speculate as to the impact that these toeholds had on strategic planning during bidding stages. As a matter of fact, each truck manufacturer benefited from a fraction of the others profit. Furthermore, we contribute to the existing literature regarding bidders preferences for one auction format over another. In the horizontal toehold case, we extend Dasgupta and Tsui's (2004) results to the  $n$  bidders' case and put forth an optimal auction format that was not included in their analysis. We also show that horizontal toeholds cannot be interpreted as *negative* vertical toeholds. As a matter of fact, a losing bidder with a vertical toehold only cares about the price paid by the winner. However, a losing bidder with a horizontal toehold cares not only about the price paid by the winner, but also about the winner's valuation for the good.

This joint analysis of both types of toeholds also allows us to observe that the second-price auction is more affected by the presence of any type of toehold.

Apart from the literature already cited, the impact of toeholds has also been discussed in several other studies. CRAMTON, GIBBONS and KLEMPERER [1987] and DE FRUTOS and KITTSTEINER [2006] study a case of vertical toeholds in which the whole target firm is owned, *ex ante*, by asymmetric competitors. They focus on the topic of

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3. A toehold is not consubstantially horizontal or vertical. The context determines whether it is horizontal or vertical.

4. The two other standard auction formats, the descending and the ascending auction are equivalent to the first-price auction and second-price auction, respectively.

5. His original motivation comes from Amish estate sales. After the death of a member of the community, the farm is auctioned off among heirs and the resulting revenue is divided equally amongst themselves.

6. This paper was developed simultaneously yet independently from DASGUPTA and TSUI [2004].

efficiency<sup>7</sup>. In their framework, the question of the auction revenue does not make much sense. Once the allocation of the good is determined, there are only transfers between bidders. There is no actual sale price. Alternatively, SINGH [1998] and BURKART [1995] analyze a contested takeover in which one bidder owns a vertical toehold in the target firm. He faces a bidder with no toehold. They observe overbidding and inefficiencies stemming from asymmetries among bidders. For technical reasons, they do not study the equilibrium of the first-price auction. Therefore, they do not suggest any revenue ranking. BULOW, HUANG and KLEMPERER [1999] also consider vertical toeholds but in a common value framework. They argue that the common value paradigm offers a more appropriate representation of *financial* bidders. To a certain extent, their approach is complementary to ours, which considers *strategic*<sup>8</sup> bidders. In their common value framework, they also observe that the revenue of the second-price auction is higher than the revenue of the first-price auction when toeholds are symmetric.

Our approach is also reminiscent of some aspects of the study of auctions with externalities initiated by JEHIEL and MOLDOVANU. In JEHIEL and MOLDOVANU [2000], for instance, they consider an asymmetric information setting in which a losing bidder derives a positive or negative fixed externality from the allocation of the good to another bidder. A priori, our vertical (resp: horizontal) toeholds could be considered as a negative (resp: positive) externality. However, what really matters here is that the externality term depends on the price. This specific issue is absent from most auction with externalities literature, but it is crucial to our comparison of the first-price auction and the second-price auction. This also explains why the revenue equivalence of the main standard auction formats is preserved in their framework and not in ours.

More recently, and independently from our work, LU [2007] proposes a general approach of optimal auctions with both types of externalities. Both approaches have much in common except that LU [2007] focuses more specifically on optimal allocation mechanisms with a weaker interest in standard auction format.

The remainder of this paper is divided into four sections. In section 2, we present the model. In sections 3 and 4, we analyze respectively the impact of vertical and horizontal toeholds, and in section 5, we present our concluding remarks.

## 2 The model

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A good is sold through an auction process with  $n > 2$  risk-neutral bidders.  $\forall i \in \{1, \dots, n\}$ , bidder  $i$  perfectly reflects the interests of firm  $i$ .<sup>9</sup> Firm  $i$ 's valuation

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7. In our framework with symmetric toeholds, the allocation is always efficient.

8. For a discussion on the difference between a strategic and a financial bidder, see the introduction of BULOW, HUANG and KLEMPERER [1999]. In short, a *financial* bidder buys shares of companies in order to sell them later; as such he cares about the common value component of the target firm. A *strategic* bidder buys firms in order to merge with this firm and to realize synergies. A *strategic* bidder cares more about the private value component of the target firm.

9. Throughout the remainder of the paper, we will identify bidders with the firms they represent.

for the good is  $v_i$  which is bidder  $i$ 's private information<sup>10</sup>. It is common knowledge that  $v_i$ 's are independently distributed according to an identical cumulative distribution  $F$  with density  $f$  on the interval  $[0, 1]$ . Furthermore, we assume that  $F$  is continuous, differentiable, strictly increasing on  $[0, 1]$  and satisfies the monotone hazard rate condition<sup>11</sup>  $\left( \frac{d}{dt} \frac{1-F(t)}{f(t)} < 0 \right)$  on all the interval.

We define two categories of toeholds. A bidder has a *horizontal toehold* when he owns a fraction of the capital of the other bidders. A bidder has a *vertical toehold* when he owns a fraction of the capital of the seller. For the sake of simplicity, we assume that the seller's only asset is the good for sale. For a bidder, owning a vertical toehold is equivalent to possessing a fraction of the good for sale.

We focus on two polar cases regarding the distribution of toeholds.<sup>12</sup>

- Vertical toeholds: All the firms own an identical fraction  $\alpha \in \left( 0, \frac{1}{n} \right)$  of the capital of the seller and no horizontal toehold.
- Horizontal toeholds: Each firm owns an identical fraction  $\theta \in \left( 0, \frac{1}{n-1} \right)$  of the capital of all the other firms and no vertical toehold.

In both cases, toeholds are common knowledge.

We consider two different auction formats: The second-price auction and the first-price auction. In both auctions, each bidder simultaneously submits a bid  $b \geq 0$  and the bidder who submits the highest bid obtains the good. In the first-price auction, the winner pays the amount of his bid. In the second-price auction, he pays the second highest bid. In both auctions, if more than one bidder submits the highest bid, the seller randomly selects the winner; all have an equal chance of winning. The winner obtains the good and pays the common bid.

In the vertical toehold case, we assume that losers always agree to sell their fraction of the good at the price defined through the auction.<sup>13</sup> Bidder  $i$  owns a fraction  $\alpha$  of the good. Then, if he wins the auction, he buys the remaining  $(1 - \alpha)$  shares.

10. We consider the independent private value paradigm in order to identify effects that are specifically due to the presence of toeholds. Our results are independent from the effect of affiliation and the explosive impact of asymmetry in a common value framework as identified in BULOW *et al.* [1999]. Moreover, in the two introductory examples, the private value hypothesis seems justified. As a matter of fact, in the RVI/Volvo Trucks case, players' private information concerns their production costs whom they almost perfectly know and which does not give them more information about the specific production costs of their opponents. In the Global One case, of course, an important fraction of the value of the good is common among the two buyers. However, we claim that the residual private information of the bidders concerns the independent and private part of their valuations: private synergies, long-run industrial strategies... The common value part of Global One is known by both bidders who have access to all the relevant information concerning the firm and all the studies on the perspectives of the telecom sector.

11. In fact, this condition is only required for propositions 4, ?? and 9.

12. In both cases, we assume that bidders are symmetric. It is a necessary condition to solve equilibria.

13. Consider, for instance, a contested takeover. In that case, a losing competitor usually prefers not to keep his toehold. If another bidder takes control of the target firm, he will probably divert the extra profits he can create. The loser is better off selling his shares before this dilution. Besides, the winning bidder, in most legislations, cannot refuse to buy his adversaries' toehold at the price of the winning tender. If we interpret toeholds as a fraction of the capital of the seller owned by bidders, through his shares, a bidder always gets a fraction  $\alpha$  of the extra profit of the selling company:  $\alpha p$ . A losing bidder gets  $\alpha p$  and the winning bidder gets  $v - p + \alpha p = v - (1 - \alpha)p$ .

On the other hand, if he loses the auction, bidder  $i$  sells his toehold. Utilities are then defined as follows:

If  $i$  obtains the good and pays a price  $p$ :

$$U_i = v_i - p + \alpha p$$

If  $i$  loses the auction and the selling price is  $p$ :

$$U_i = \alpha p$$

More generally, denoting by  $q_k$  the probability that bidder  $k$  obtains the good and  $p_k$  the expected payment of bidder  $k$ , the expected utility of bidder  $i$  can be expressed as follows:

$$U_i = q_i v_i - p_i + \alpha \sum_{k=1}^n p_k$$

The horizontal toehold case is slightly more complex. We assume that through dividends or the rise of share value, any additional profit from a firm is distributed to its shareholders in proportion to their stakes. If bidder  $i$  wins the auction and pays a price  $p$ , firm  $i$  derives a direct profit from this purchase:  $v_i - p$ . Consequently, the value of a fraction  $\theta$  of firm  $i$  increases by  $\theta(v_i - p)$ . Since, for any  $j \neq i$ , firm  $j$  owns a fraction  $\theta$  of firm  $i$ , whenever firm  $i$  wins at the price  $p$ , firm  $j$ 's value increases by  $\theta(v_i - p)$ . But, firm  $i$  also owns a fraction  $\theta$  of the capital of these other firms. As they all own a fraction of each other's capital, they all receive a fraction of this extra profit through dividends. This mechanism reproduces itself *ad infinitum*.

We suggest the following method to solve this issue. Let  $h(\theta, n)$  be the fraction of an extra profit of firm  $k \neq i$  that firm  $i$  gets depending on  $\theta$  and  $n$ . If firm  $k$  receives 1 Euro, all the other firms get  $h(\theta, n)$  Euro and firm  $k$  gets  $1 + \theta(n - 1)h(\theta, n)$  Euro. The value of all the firms raises by:  $1 + (1 + \theta)(n - 1)h(\theta, n)$ . Since we assume that any extra profit is distributed to shareholders through dividends, accordingly none of the firms earn any money from this operation. This means that the amount of dividends given to the *other shareholders*, those who do not have horizontal toeholds, is equal to 1. This can be written as follows:  $(1 - (n - 1)\theta)(1 + (1 + \theta)(n - 1)h(\theta, n)) = 1$  which gives  $h(\theta, n) = \frac{\theta}{(1 + \theta)(1 - (n - 1)\theta)}$  and  $1 + \theta(n - 1)h(\theta, n) = \frac{1 + (2 - n)\theta}{(1 + \theta)(1 - (n - 1)\theta)}$ .

Utility functions can be defined as follows:

If bidder  $i$  obtains the good for a price  $p$ :

$$U_i = \frac{1 + (2 - n)\theta}{(1 + \theta)(1 - (n - 1)\theta)} (v_i - p)$$

If bidder  $j$  obtains the good for a price  $p$ :

$$U_i = \frac{\theta}{(1+\theta)(1-(n-1)\theta)} (v_j - p)$$

Using the same notations as in the vertical toehold case, the expected utility of bidder  $i$  can be defined as follows:

$$U_i = \frac{1}{(1+\theta)(1-(n-1)\theta)} \left( (1+(2-n)\theta)(q_i v_i - p_i) + \theta \sum_{k \neq i} (q_k v_j - p_k) \right)$$

Notice that a horizontal toehold cannot be modeled as a negative vertical toehold. There is a specific element that does not appear in the vertical toehold case. Here, a losing bidder not only cares about the price paid by the loser, as in the vertical toehold case but he also cares about the winning bidder's valuation. That is why these two types of toeholds cannot be represented by a unique coefficient whose sign would be positive for a vertical toehold and negative for a horizontal one.

In both the first-price and the second-price auction and for vertical and horizontal toeholds, we limit our attention to symmetric equilibria. In order to avoid disruptive equilibrium multiplicity, we also assume that bidders have lexicographic preferences. Firstly, they care about their utility as we define it. Secondly, they prefer that the good be sold<sup>14</sup>.

## 3 Auctions with vertical toeholds

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In this section, we consider the case of vertical toeholds (*i.e.*, when bidders own a fraction of the seller or a fraction of the good for sale). In the first subsection, we review some of the existing results in this domain, the equilibria of the two auction formats and the expected revenue comparison. We then introduce some new results and put forth a new and simpler optimal auction format. We consider the buyers point of view and discuss the possible uses of toeholds to raise the sellers revenue.

### 3.1 Equilibria and revenue comparison

All the propositions stated in this subsection are corollary of ENGELBRECHT-WIGGANS' [1994] results, which considers an affiliated values environment.

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14. This assumption is only required for propositions 4 and 9.

PROPOSITION 1. *There is a unique symmetric equilibrium of the second-price auction. For  $i = \{1, 2, \dots, n\}$  bidder  $i$  with valuation  $v_i$  bids  $b_V^H(v_i)$  where:*

$$b_V^H(v_i) = v_i + \int_{v_i}^1 \left( \frac{1-F(t)}{1-F(v_i)} \right)^{\frac{1}{\alpha}} dt$$

PROPOSITION 2. *There is a unique symmetric equilibrium of the first-price auction. For  $i = \{1, 2, \dots, n\}$  bidder  $i$  with valuation  $v_i$  bids  $b_V^L(v_i)$  where:*

$$b_V^L(v_i) = v_i - \int_0^{v_i} \left( \frac{F(t)}{F(v_i)} \right)^{\frac{n-1}{1-\alpha}} dt$$

COROLLARY 1. *With both auction formats, the allocation is efficient and the expected revenue is increasing in  $\alpha$ .*

PROPOSITION 3. *For any  $\alpha \in \left(0, \frac{1}{n}\right)$ , the expected price is strictly higher with a second-price auction than with a first-price auction. If  $\alpha = 0$  or  $\alpha = \frac{1}{n}$ , the expected price is the same with both auction formats.*

With both formats, the auction is efficient and expected revenue increases in accordance with the size of the toehold. However, vertical toeholds affect bidding behavior in the two auction formats through two different channels.

In the second-price auction, bidders are interested in the price paid,  $p$ , because they receive residual benefits of  $\alpha p$  regardless of whether or not they obtain the goods in the end. Therefore, contrary to the standard case without toeholds, bidding one's own valuation is not a dominant strategy. Bidders tend to bid more than their valuations to raise the price, should they lose the auction. At the same time, bidding too high can be dangerous, for in doing so, a bidder might actually end up winning and pay a price that exceeds his valuation for the good. The equilibrium bid is the result of this trade-off.

In the first-price auction, losing bids have no effect on the price paid by the winner. There is no direct strategic way for the loser to raise the price paid by the winner. However, toeholds have an impact on bidders' incentives, even in a first-price auction. Each bidder has two roles, that of buyer and seller. On the one hand, as a potential buyer, a bidder, if he bids  $\varepsilon$  more and wins the auction, does not pay  $\varepsilon$  more but rather  $(1 - \alpha)\varepsilon$  more. On the other hand, as a potential seller, by increasing his bid, a bidder reduces the probability of selling his toehold for a low price. High bids are less costly and more profitable than in the standard case.

Even though vertical toeholds have the same qualitative effects on both auction formats, they do not affect the expected price of both auction formats to the same degree (as stated in Proposition 3). We propose the following intuitive explanation for this revenue ranking. In the second-price auction, bidders have a more obvious reason to raise their bids, for it is a direct way of increasing the price should they lose the auction. By the very definition of auction formats, this motivation cannot exist in a first-price auction. That is why vertical toeholds have a more direct impact on bidding strategies in the second-price auction than in the first-price auction.

### 3.2 A simple optimal auction format

GOEREE *et al.* [2005] showed that a lowest-price all-pay auction with an adequate entry fee is an optimal auction format in this context. They therefore advocate for the use of the lowest-price all-pay auction format by claiming that standard auction formats are non-optimal in this context. I intend to show that the way they present their results is partially biased since it is possible to build an optimal auction format based on any standard auction format. For the sake of simplicity, we will show this result with the first-price auction.

Since, we are in a standard Myersonian environment, the properties of an optimal auction format are well-known. The bidder with the highest marginal revenue obtains the good provided that his marginal revenue is positive<sup>15</sup> and the bidders' reservation utility is equal to zero. We introduce a modified first-price auction,  $\Gamma_V$ , which contains these properties. It is defined as follows:

Step 1. The seller asks the bidders to pay an entry fee  $c$  equal to:

$$\alpha(n-1) \int_{R^*}^1 \left( t - \int_{R^*}^t \left( \frac{F(u)}{F(t)} \right)^{\frac{n-1}{1-\alpha}} du \right) f(t) F^{n-2}(t) dt$$

with  $R^* = \frac{1 - F(R^*)}{f(R^*)}$ .

If one of the bidders refuses to pay the entry fee, the auction process is closed and the good remains unsold. Otherwise, the auction process continues.

Between steps 1 and 2, if all the bidders accept to pay, they pay  $c$  to the seller.

Step 2: If all the bidders have paid the entry fee, the seller organizes a first-price auction with a reserve price  $R^*$ .

PROPOSITION 4. *There is a unique symmetric equilibrium of  $\Gamma_V$  in which bidders participate with a strictly positive probability. In step 1, all the bidders accept to pay the entry fee. In step 2,  $\forall i \in \{1, \dots, n\}$ , bidder  $i$  bids according to the bidding function defined as follows:  $\forall v_i < R^*$ , bidder  $i$  does not participate in the auc-*

*tion and  $\forall v_i \geq R^*$ ,  $b(v_i) = v_i - \int_{R^*}^{v_i} \left( \frac{F(t)}{F(v_i)} \right)^{\frac{n-1}{1-\alpha}} dt$ .*

$\Gamma_V$  is an optimal mechanism.

15. The marginal revenue of a bidder with valuation  $v$  and distribution function  $F$  is equal to:  $v - \frac{1 - F(v)}{f(v)}$ .

We see that it is possible to design an optimal mechanism based on a standard auction format. A standard auction can select the bidder with the highest marginal revenue whereas the nullification of bidders' reservation utilities through the entry fee can be done with any auction format. Therefore, as long as it is possible to use this very specific type of entry fee, there is no need to introduce a non-standard auction format to maximize the seller's expected revenue<sup>16</sup>.

Let us also remark that we chose to model an optimal auction based on a first-price auction for simplicity reasons, but it could have also been done with a second-price auction and a lower value for the entry fee. The reserve price would then be  $\hat{R}$  equal to:

$$(1) \quad \hat{R} = \frac{F(R^*)R^* - \alpha(n-1)(1-F(R^*))b_V^H(R^*)}{(1-\alpha)F(R^*) + \alpha(n-1)(1-F(R^*))}$$

And the entry fee would be equal to:

$$(2) \quad \alpha \left[ (n-1)\hat{R}F^{n-2}(R^*)(1-F(R^*)) + (n-1)(n-2) \int_{R^*}^1 B_V^H(t)(1-F(t))f(t)F^{n-3}(t)dt \right]$$

With  $F^{-1}(t) = 0$  for any  $t \in [0, 1]$ . At the equilibrium, all the bidders pay the entry fee. They participate when their valuations are higher or equal to  $R^*$  and submit their bids according to  $b_V^H$ <sup>17</sup>.

### 3.3 The bidders' point of view

After having studied the sellers' interests, we consider bidders' preferences regarding auction formats.

PROPOSITION 5. For any  $\alpha \in \left(0, \frac{1}{n}\right)$ ,  $i = \{1, 2, \dots, n\}$  and  $v_i \in [0, 1]$ , the expected utility of bidder  $i$  with a first-price auction is strictly higher than his expected utility with a second price auction by an amount  $D_V$  which is independent of his valuation  $v_i$ . If,  $\alpha = 0$  or  $\alpha = \frac{1}{n}$ , bidder  $i$  is indifferent between the two auction formats.

**Proof:** The expected utility of a bidder, whatever his valuation, is a function of the allocation plus his reservation utility (Revenue Equivalence Theorem). Here,

16. Introducing a new auction format can come with a cost. Bidders are often reluctant. Besides, it takes time to learn how to play the equilibrium of a new auction format.

17. This result can be proved the same way as Proposition 4.

the allocation rule is the same with both auction formats. Therefore, whatever  $v_i$ , for bidder  $i$ , the difference between his expected utility in the two auction formats is the same. Q.E.D.

The resulting uniformity, although it relies on standard regularity properties, is not intuitive. It means that the bidder's preference for one auction format over another depends on  $\alpha$ , the size of his toehold but not on his valuation. If a bidder has a high valuation and is almost sure to win, the extra utility he derives from the choice of a first-price auction rather than a second-price auction is the same as when he has a low valuation and is almost sure to lose the auction. In the first case, he gets an extra utility because he pays a lower price. In the second case, he gets an extra utility because his opponent pays a higher price. Proposition 5 tells us that the two effects compensate for one another.

For  $\alpha = \frac{1}{n}$ , bidders' expected utilities are identical with both auction formats since this case corresponds to the partnership dissolution studied by Cramton *et al.* [1987] which shows that the first-price and the second-price auction are equivalent. Besides, it is a well-known result that in the standard case, for  $\alpha = 0$ , bidders' expected utilities are the same with both auction formats. From these results, we derive the following corollary.

COROLLARY 2. For any  $i \in \{1, 2, \dots, n\}$  and  $v_i \in [0, 1]$ , the difference between bidder  $i$ 's expected utility with a first-price auction and his expected utility with a second-price auction is a non-monotonic function of  $\alpha$ .

For low values of  $\alpha$ , toeholds do not have much of an effect on bidding strategies. Thus, the differences between the two auction formats are minor. For high values of  $\alpha$ , close to  $\frac{1}{n}$ , the auction tends to be equivalent to the allocation of a good between his exclusive owners. In that case, expected utilities uniquely depend on the allocation rule. Since, with both auction formats, the allocation is the same, expected utilities of bidders are also identical with both auction formats. Thus, the choice of one auction format over another really only matters for bidders when  $\alpha$  has an intermediary value.

## 4 Auctions with horizontal toeholds

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In this section, we study the case of horizontal toeholds. We first characterize the symmetric equilibria, then we compare generated revenues, introduce an optimal auction format and consider the bidders' point of view.

With horizontal toeholds, incentives are diametrically opposed to what we have observed with vertical toeholds. If bidder  $i$  loses the auction, he has nothing to sell to the winner. By contrast, as he owns a fraction  $\theta$  of the winning firm, he prefers this firm make the highest possible profit and that the price be low.

Thus, at equilibrium, bidders submit lower bids than in the standard case. The following propositions illustrate exactly how horizontal toeholds affect the equilibrium bidding functions with the two auction formats.

## 4.1 Equilibria Characterization

PROPOSITION 6. *There is a unique symmetric equilibrium of the second-price auction. For any  $i \in \{1, 2, \dots, n\}$ , bidder  $i$  bids  $b_H^{\text{II}}(v_i, n)$  where:*

$$b_H^{\text{II}}(v_i, n) = v_i - \int_0^{v_i} \left( \frac{1 - F(v_i)}{1 - F(t)} \right)^{\frac{1+(1-n)\theta}{\theta}} dt$$

PROPOSITION 7. *There is a unique symmetric equilibrium of the first-price auction. For  $i \in \{1, 2, \dots, n\}$ , bidder  $i$  bids  $b_H^{\text{I}}(v_i, n)$  where:*

$$b_H^{\text{I}}(v_i, n) = v_i - \int_0^{v_i} \left( \frac{F(t)}{F(v_i)} \right)^{\frac{(1+(1-n)\theta)(n-1)}{1+(2-n)\theta}} dt$$

COROLLARY 3. *Both auction formats are efficient. Except if  $v_i = 0$ , bidding functions are strictly decreasing in  $\theta$  and so are the actual and expected revenues.*

**Proof (of corollary 3):** Since in both auction formats, bidders have identical and strictly increasing bidding functions, the allocation is efficient. The other properties are direct consequences of the shapes of  $b_H^{\text{II}}$  and  $b_H^{\text{I}}$ .

Let us be more explicit in describing how horizontal toeholds affect the two auction formats.

In the second-price auction, bidding its own valuation is not a dominant strategy. A losing bidder's utility is a decreasing function of his bid because his bid may determine the price paid by the winner. Nevertheless, an extremely low bid cannot be part of an equilibrium bidding strategy. By bidding that way, bidders would lose opportunities to obtain the good at a low price. Bidding strategies, in the second-price auction, are the result of this trade-off. As  $\theta$  grows, bidders receive a larger fraction of winner's profit. Thus, it becomes increasingly important for a bidder, should he lose, to submit a low bid. That is why the equilibrium bidding function is decreasing in  $\theta$ .

In the first-price auction, losing bids do not determine the price. Nevertheless, toeholds still affect bidders' strategies. As a matter of fact, if a bidder does

not obtain the good, his utility is the profit of the winning bidder multiplied by a strictly positive coefficient. The expected utility upon losing the auction becomes positive. The equilibrium bidding function, which is the result of a tension between the fear to lose and the will to make a higher profit when obtaining the good, is consequently affected. *Ceteris paribus*, the expected utility of a bidder, in the extent of a loss, is increasing in  $\theta$ . For larger values of  $\theta$ , bidders are less eager to win the auction with a minimal difference between their valuations and their bids. As a result, equilibrium bidding functions are also decreasing in  $\theta$ .

With both auction formats, equilibrium bids are decreasing functions of  $\theta$ . This result is not surprising, but it raises an issue concerning anti-trust regulations. Let us illustrate our point quoting the Commission of the European Communities. In a 1990 report (Case N IV/M.0004 (1990)), it explained that a shareholdings exchange of 25% between two competitors need not be controlled by the regulatory authorities provided that the exchange “does not in itself either give sole control of one party over the other or create a situation of common control” (in application of Council Regulation N 4064/89, article 3). In such a case, even if no common decision is made, horizontal toeholds distort bidders’ behaviors and affect price. Therefore, such an exchange, because of its possible consequences, should also be controlled by the authorities in charge of market regulation<sup>18</sup>. The possible consequences are from being negligible. In our framework, with uniform distribution functions, 2 bidders and horizontal toeholds of 25%, the expected revenue for the seller is  $\frac{1}{5}$ , compared to the expected revenue without horizontal toeholds:  $\frac{1}{3}$ , which represents a loss of 40%.

## 4.2 Revenue Comparison

We have seen that, through two different channels, in both auction formats, vertical toeholds have a decreasing effect on bids. We may also ask whether or not there is a general ranking in terms of expected revenue, as in the vertical toehold case.

PROPOSITION 8. For any  $\theta \in \left(0, \frac{1}{n-1}\right)$ , the expected price is strictly higher with a first-price auction than with a second-price auction.

Even though the revenue ranking is opposite, the intuition of this result is similar to the intuition in the vertical toehold case. As a matter of fact, as in the vertical toehold case, a losing bidder can more directly influence the price in the second-price auction. Thus, the downward variations of expected revenue due to horizontal toeholds are exacerbated in the second-price auction.

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18. This point was already established in the context of a Cournot model (see REYNOLDS and SNAPP [1986]). We have demonstrated that it remains true with both the first-price auction and the second-price auction.

### 4.3 The optimal auction format

Thus far, we have compared expected revenue in standard auction formats with the presence of horizontal toeholds. A complementary approach would consist of defining the expected revenue maximizing (optimal) auction format.

The reasoning is the same as in subsection 3.2.

Since we are in a standard Myersonian environment, we know the properties of an optimal auction format. The bidder with the highest marginal revenue obtains the good provided that this marginal revenue is positive and that the bidders' reservation utility is equal to zero.

In the standard case, with symmetric distribution functions and without toeholds, an ascending auction with a reserve price  $R$  such that  $R = \frac{1-F(R)}{f(R)}$  has these two properties. When bidders have horizontal toeholds, this is no longer the case. A bidder reservation is not zero, it is equal to the share of the winning firm's profit that he will obtain through the toeholds. Therefore, to maximize the seller's revenue, we must find a way to reduce this reservation utility. Again, this can be done by setting a pre-auction round in which bidders would be asked to pay their reservation utilities. We introduce a modified first-price auction,  $\Gamma_H$ , which contains these properties and can be defined as follows.

Step 1. The seller asks the bidders to pay an entry fee  $c$  equal to

$$\frac{\theta}{1+\theta} \int_{R^*}^1 \int_{R^*}^u \left( \left( \frac{F(t)}{F(u)} \right)^{\frac{(1+(1-n)\theta)(n-1)}{1+(2-n)\theta}} dt \right) f(u) F^{n-2}(u) du$$

With  $R^* = \frac{1-F(R^*)}{f(R^*)}$ .

If one of the bidders refuses to pay the entry fee, the auction process comes to a close and the good remains unsold. Otherwise, the auction process goes on.

Between step 1 and 2, if all bidders accept to pay, they pay  $c$  to the seller.

Step 2: If all bidders pay the entry fee, the seller organizes a first-price auction with a reserve price  $R^*$ .

**PROPOSITION 9.** *There is a unique symmetric equilibrium of  $\Gamma_H$  in which bidders participate with a strictly positive probability. In step 1, all the bidders accept to pay the entry fee,  $c$ . In step 2,  $\forall i \in \{1, \dots, n\}$ , bidder  $i$  bids according to the bidding function defined as follows:  $\forall v_i < R^*$ , bidder  $i$  does not participate in the*

*auction and  $\forall v_i \geq R^*$ ,  $b(v_i) = v_i - \int_{R^*}^{v_i} \left( \frac{1-F(v_i)}{1-F(t)} \right)^{\frac{(1+(1-n)\theta)(n-1)}{1+(2-n)\theta}} dt$ .*

$\Gamma_H$  is an optimal mechanism.

We have seen that what distinguishes the auction with horizontal toeholds from a standard auction is the strictly positive bidders' reservation utility. With the entry fee, the seller can fix the reservation utility to any positive level, with zero being the most advantageous choice for him. Therefore, it is always possible to build an optimal auction with the help of this entry fee.

We chose to model an optimal auction based on a first-price auction, but it could have also been done with a second-price auction and a higher value for the entry fee. The reserve price would remain the same and the entry fee would be equal to:

$$\frac{\theta}{1+\theta} \left\{ (n-1)(n-2) \int_{R^*}^1 \int_{R^*}^u \left[ u - \left( t - \int_{R^*}^t \left( \frac{1-F(t)}{1-F(v)} \right)^{\frac{(1+(1-n)\theta)}{\theta}} dv \right) \right] f(t) F^{n-3}(t) f(u) du + (n-1) \int_{R^*}^1 (u - R^*) f(u) F^{n-2}(R^*) du \right\}$$

With  $F^{-1}(t) = 0$  for any  $t \in [0, 1]$ .

At the equilibrium, all the bidders pay the entry fee. They participate when their valuations are higher or equal to  $R^*$  and submit

$$b_H^H(v_i, n, R^*) = v_i - \int_{R^*}^{v_i} \left( \frac{1-F(v_i)}{1-F(t)} \right)^{\frac{(1+(1-n)\theta)}{\theta}} dt^{19}.$$

In fact, as long as the seller can credibly commit to not selling the good if one of the bidders refuses to pay the entry fee, he can construct a revenue-maximizing auction based on any standard auction format.

### 4.4 The bidder's point of view

After having studied the sellers' interest, we consider bidders' preferences regarding auction formats.

**COROLLARY 4.** For any  $\theta \in \left( 0, \frac{1}{n-1} \right)$ ,  $i = 1, 2$  and  $v_i \in [0, 1]$ , the expected utility of bidder  $i$  with a second-price auction is higher than his expected utility with a first-price auction by an amount  $D_H$  that is independent of  $v_i$ .

**Proof:** From the Equivalence Revenue Theorem we derive that in any auction mechanism, the expected utility of a bidder is a function of the allocation rule and his valuation plus his reservation utility. Here, the allocation is the same with both auction formats. Therefore, whatever  $v_i$ , for bidder  $i$ , the difference between his expected utility in the two auction formats is the same, equal to the difference in reservation utility. Q.E.D.

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19. This result can be proved the same way as Proposition 9.

Bidders' preferences for one auction format over another depends on  $\theta$ , the size of the toehold but not on their valuations. The use of a second-price auction rather than a first-price auction is worth a fixed amount to a bidder. This amount does not change with the valuation. For low valuations, bidders prefer the second-price auction mainly because it reduces the price paid in the event of a loss. For high values, they prefer the second-price auction because it reduces the price he pays, should he win. Corollary 4 tells us that the two effects perfectly compensate for one another.

## 5 Conclusions

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We have demonstrated that, in the presence of a horizontal toehold, an expected revenue-maximizing seller is always better off choosing a first-price auction. We can apply this finding to the RVI/Volvo Trucks case presented in the introduction. The clients of RVI and Volvo Trucks should have chosen the first-price auction rather than the second-price auction in order to strengthen the competition among the two suppliers and to steer clear of the negative effects of horizontal toeholds as much as possible.<sup>20</sup>

We also observed that regulators choose not to control cross-shareholdings unless they create a situation of common control or give sole control of one party over the other. Our results show that, as in case of mergers, Competition Authorities should control cross-shareholdings since with or without common control, cross-shareholdings may affect competition<sup>21</sup>.

When bidders own a share of the seller or a fraction of the good for sale, the expected revenue is higher with the second-price auction than with the first-price auction. If we apply this result to the Global One case, we can deduce that a second-price auction was more favorable to Sprint (the seller), whereas a first-price auction was more favorable to France Télécom and Deutsche Telekom (the buyers).

We may also note in the Global One case another complexity deriving from the fact that the seller was the one who chose the auction format. As a matter of fact, the two possible *buyers* owned 80% of Global One while, the *seller* owned only 20% of Global One. Therefore, it is also unclear who was really controlling the agenda.

A natural extension to this work would consist in modeling a pre-auction bargaining about the choice of the auction procedure. This may be treated in future work. For the time being, we can make the following remark. We established in corollary 5 that bidders' preferences for an auction format over another one do not depend on their valuations. Then, without getting into further details of this pre-auction bargaining, we can say that it will not make it possible to directly extract information about bidders' valuations<sup>22</sup>.

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20. As far as we know, they used a format that is closed to the first-price auction. However, we do not know what motivated them to do so.

21. In practice, a 10% threshold may be appropriate.

22. Contrary to what de FRUTOS and KITTSTEINER [2006] derive in a partnership dissolution environment, here, the choice of the auction format does not affect the allocation, it only affects the surplus distribution.

A broader interpretation of our results is also possible. As a matter of fact, losing bidders may care about the final price in many other situations. We could analyze such situations with our model:  $\alpha$  and  $\theta$  would represent, respectively, the coefficient of mutual *malevolence* and of mutual *benevolence* among bidders. Using this interpretation, we could extend the application field of our results, deriving the following: In order to benefit more from the effects of mutual *malevolence* among bidders, a seller should choose the second-price auction. And to protect himself from mutual *benevolence* among bidders, he should choose a first-price auction.

This interpretation recalls some insights found in existing auction theory literature, although there exists no general study of this issue. Here are two examples. Firstly, in the collusion situation in which we can assume that bidders are mutually *benevolent*, it has been shown that collusion is much easier to sustain with a second-price auction than with a first-price auction (see Robinson (1985) on this issue). Secondly, in the case of budget-constrained bidders in two sequential auctions, in the first item auction, the losing bidder prefers that the winning bidder pays a high price. If he spends more in the first auction, this winning bidder will be a less formidable opponent in the following auction. Thus, we can talk of mutual *malevolence* in the first auction. PITCHIK and SCHOTTER [1988] studied that case, focusing on the revenue earned of the first auction. They showed that the first-price auction generates a lower revenue in the first item auction than the second-price auction, both in theory and in practice.

In these two examples, the revenue ranking follows the same general logic observed in our model. In the *benevolent* (resp: *malevolent*) case, a first-price auction (resp: second-price auction) generates more revenue.<sup>23</sup> This tends to indicate that the our findings should be applicable, more generally, to situations in which a seller has to deal with mutual *benevolence* or mutual *malevolence* among bidders ■

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23. We would not obtain such results if we assumed that losing bidders derive a fixed externality that only depends on the identity of the loser. JEHL and MOLDOVANU [2000] showed that, in that case, there is no such revenue ranking. The revenue ranking relies on bidders caring about the price paid by the winner.

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## A Proofs

### A.1 Proof of proposition 4

#### The equilibrium

First, for any  $\underline{v}$  and  $\bar{v}$  such that  $0 \leq \underline{v} < \bar{v} \leq 1$ , if at the equilibrium, a bidder with valuation  $\underline{v}$  accepts to pay the entry fee, then he also accepts to pay it when his valuation is  $\bar{v}$  (because, by participating he can get at least what he would get if he were to participate with a valuation  $\underline{v}$ ). Then, the strategy of bidder  $i$  in the first step can be represented by a threshold (for a valuation below this threshold, a bidder does not accept to pay the entry fee and, for a valuation above this threshold he accepts to pay the entry fee). Let us call this threshold  $v^*$ .

Suppose that  $v^* \in (0, R^*)$ , then in the second step, a bidder with valuation  $v^*$  does not participate since he prefers losing the auction than winning it for a price higher or equal to  $R^*$ . Therefore, if a bidder has a valuation in the interval  $[0, v^*)$ , he could also accept to pay the entry fee in step 1 and obtain exactly the same pay-off without bidding in step 2. Because of the lexicographic preferences, he cannot be indifferent to participating or not participating, so he strictly prefers participating when his valuation lies in the interval  $[0, v^*)$ . Therefore,  $v^* \in (0, R^*)$  cannot be part of an equilibrium.

Suppose that  $v^* \in [R^*, 1)^{24}$ , then we can show that, in the second step, bidders would submit bids increasing in valuations. Now, a bidder with valuation  $v^*$  is sure to lose the auction in step 2. Therefore, if a bidder has a valuation lower than  $v^*$ , he can accept to pay the entry fee in step 1 and not participate in step 2, he will get the same expected utility as a bidder with valuation  $v^*$ . Because of the lexicographic preferences, he cannot be indifferent to participating or not participating so he strictly prefers participating when his valuation lies in the interval  $[0, v^*)$ . Therefore,  $v^* \in (R^*, 1)$  cannot be part of an equilibrium.

Only  $v^* = 0$  can be part of an equilibrium. So, let us suppose that  $v^* = 0$  and verify whether or not there exists an equilibrium.

In the auction itself, a bidder with valuation strictly lower than  $R^*$  has a dominant strategy, which is not participating in the auction. If there exists a bidding function,  $b$ , such that, at the symmetric equilibrium, a bidder with valuation  $\hat{v} \geq R^*$  submits  $b(\hat{v}) \geq R^*$  with  $b$  strictly increasing, it must be such that:

$$v_i f(v_i) - b(v_i) f(v_i) - \frac{1-\alpha}{n-1} F(v_i) b'(v_i) = 0$$

As  $b(R^*) = R^*$ , the solution of the differential equation is

$$b(v_i) = v_i - \int_{R^*}^{v_i} \left( \frac{F(t)}{F(v_i)} \right)^{\frac{n-1}{1-\alpha}} dt$$

24. Note that if  $v^* = 1$ , bidders participate in the auction with zero probability. We excluded this kind of equilibrium.

We need to check that with such a bidding strategy in step 2, it is an equilibrium for the bidders always to pay the entry fee in step 1. To do so, we only need to check that a bidder with a valuation  $v = 0$  is better off paying the entry fee. If he pays, his expected utility is equal to:

$$-\alpha(n-1) \int_{R^*}^1 \left( t - \int_{R^*}^t \left( \frac{F(u)}{F(t)} \right)^{\frac{n-1}{1-\sigma}} du \right) f(t) F^{n-2}(t) dt + \alpha(n-1) \int_{R^*}^1 \left( t - \int_{R^*}^t \left( \frac{F(u)}{F(t)} \right)^{\frac{n-1}{1-\sigma}} du \right) f(t) F^{n-2}(t) dt = 0$$

The expected utility is equal to zero and a bidder with valuation zero prefers paying the entry fee because of his lexicographic preferences.

Q.E.D.

### The optimality

It is a well known result that an auction is optimal if and only if it allocates the good to the bidder with the highest marginal revenue (provided that this marginal revenue is positive) and that bidders' reservation utility is equal to zero.

Since we assumed that the distribution function satisfies the monotone hazard rate condition, the marginal revenue of a bidder is strictly increasing with his valuation and positive when  $v \geq v^*$  with  $v^*$  such that  $v^* = \frac{1 - F(v^*)}{f(v^*)}$ .

The reserve price of the auction is precisely equal to  $v^*$  and the equilibrium of the auction is such that the bidder with the highest valuation above  $v^*$  wins the auction. Then, the allocation of the auction coincides with the optimal allocation rule.

Now, we need to check bidders' reservation valuation, but we already showed in the first part of the proof (the equilibrium) that the reservation utility is equal to zero.

Q.E.D.

## A.2 Proof of proposition 6

To start off with, we must show that any symmetric equilibrium bidding function of the second-price auction,  $b$  must satisfy the following conditions:  $b$  is continuous in the interval  $[0, 1]$ , strictly increasing on the interval  $[0, 1]$  and  $b(0) = 0$ .

First, let us prove that  $b$  is nondecreasing: If  $\underline{v} < \bar{v}$ , then  $b(\bar{v}) < b(\underline{v})$  is impossible. As a matter of fact, as  $b(\underline{v})$  is a best response for a bidder with valuation  $\underline{v}$ , a bidder with valuation  $\bar{v}$  can profitably deviate by submitting  $b(\underline{v})$  rather than  $b(\bar{v}) < b(\underline{v})$ . Thus,  $b$  must be nondecreasing. We can also exclude the possibility that  $b$  has an atom (an interval of valuations for which bidder  $i$  submits the same bid). As a matter of fact, it is impossible that, an interval of types, bidder 2 prefers

to quit simultaneously with bidder 1's atom rather than leave just before or just after.

Now, let us show that  $b$  must be continuous on  $[0, 1)$ . Suppose that  $b$  has a gap in  $v^* \in [0, 1)$ . Since bidders strictly prefer to sell their shares for the lowest possible price and  $b$  is strictly increasing, there always exist an  $\varepsilon > 0$  such that a bidder with valuation  $v^* - \varepsilon$  is strictly better off submitting  $\frac{\lim b(v^*) + \lim b(v^*)}{2}$  rather than  $b(v^* + \varepsilon)$  which means that  $b$  is not constitutive of a symmetric equilibrium. Thus,  $b$  must be continuous on  $(0, 1)$ . The continuity in 0 can be proved the same way.

Finally,  $b(0) = 0$  is a dominant strategy.

Now, consider a bidding function  $b$  respecting these conditions. If bidders bid according to  $b$ ,  $\forall i \in \{1, 2, \dots, n\}$ , it is a dominated strategy for bidder  $i$  to bid less than  $b(0)$  and he cannot be better off bidding more than  $b(1)$  than he would be bidding  $b(1)$ . Thus, we can restrict bidder  $i$ 's strategy to the choice of a  $g: [0, 1] \rightarrow [0, 1]$ , such that he bids  $b(g(v_i))$ . Let us define  $U_i^H(v_i, \hat{v}_i)$  as the expected utility of bidder  $i$  with valuation  $v_i$  bidding  $b(\hat{v}_i)$ . As we can limit our study to the case  $\hat{v}_i \in [0, 1]$ , we obtain the following expression:

$$U_i^H(v_i, \hat{v}_i) = \frac{(1 + (2 - n)\theta)(n - 1)}{(1 + \theta)(1 - (n - 1)\theta)} \int_0^{\hat{v}_i} (v_i - b(t))f(t)F^{n-2} dt + \frac{\theta(n - 1)}{(1 + \theta)(1 - (n - 1)\theta)} \int_{\hat{v}_i}^1 (t - b(\hat{v}_i))f(t)F^{n-2} dt + \frac{\theta(n - 2)(n - 1)}{(1 + \theta)(1 - (n - 1)\theta)} \int_{\hat{v}_i}^1 \left( \int_t^1 (x - b(t))f(x)dx \right) f(t)F^{n-3}(t) dt$$

We obtain the following necessary and sufficient condition<sup>25</sup> for  $b$  to be a symmetric equilibrium strategy:

$$\frac{\partial U_i^H(v_i, \hat{v}_i)}{\partial \hat{v}_i} = 0 \text{ for } \hat{v}_i = v_i, \text{ for } v_i \in [0, 1]$$

This can be written:<sup>26</sup>

$$(v_i - b(v_i))f(v_i) - \frac{\theta}{1 + (1 - n)\theta} (1 - F(v_i))b'(v_i) = 0$$

As  $b(0) = 0$ , the solution of the differential equation is:

$$b(v_i) = v_i - \int_0^{v_i} \left( \frac{1 - F(v_i)}{1 - F(t)} \right)^{\frac{1 + (1 - n)\theta}{\theta}} dt$$

Q.E.D.

25. We can exclude corner solutions.

26. We assume that  $b'$  is well defined on the interval in question, a condition that is verified at the equilibrium.

### A.3 Proof of proposition 7

We apply the same arguments as in the proof of proposition 6 and study the expression:

$$(1 + (2 - n)\theta)(F^{n-1}(\hat{v}_i)(v_i - b(\hat{v}_i))) + \theta(n-1) \int_{\hat{v}_i}^1 (t - b(t))F^{n-2}f(t)dt$$

The first-order condition is

$$v_i f(v_i) - b(v_i) f(v_i) - \frac{1 + (2 - n)\theta}{(n-1)(1 + (1 - n)\theta)} F(v_i) b'(v_i) = 0$$

As  $b(0) = 0$ , the solution of the differential equation is

$$b(v_i) = v_i - \int_0^{v_i} \left( \frac{F(t)}{F(v_i)} \right)^{\frac{(n-1)(1+(1-n)\theta)}{1+(2-n)\theta}} dt$$

Q.E.D.

### A.4 Proof of proposition 8

The Revenue Equivalence Theorem says that the revenue of an auction is a function of the allocation rule minus bidders' reservation utilities. In the present case, the allocation is identical with both auction formats. Thus, in order to compare the expected revenues of these two auction formats, we can focus on the comparison of expected utility of lowest type,  $v = 0$ .

We prove the proposition by induction.

First, suppose that  $n = 2$ .

In the first-price auction the reservation utility of both bidders is:

$$\frac{\theta}{1+\theta} E[v_j - b_H^I(v_j)] = \frac{\theta}{1+\theta} \int_0^1 (t - b_H^I(t)) f(t) dt$$

In the second-price auction, the reservation utility is:

$$\frac{\theta}{1+\theta} E[v_j - b_H^{II}(0)] = \frac{\theta}{1+\theta} \int_0^1 t f(t) dt$$

$\theta > 0$  and  $\int_0^1 b_H^I(t) f(t) dt > 0$ , then the reservation utility is strictly higher with the second-price auction than with the first-price auction. Consequently, the

expected revenue is higher with the first-price auction than with the second-price auction.

Now, suppose that the proposition is verified for  $n = \tilde{n} \geq 2$  for any value of  $\theta \in \left(0; \frac{1}{\tilde{n}-1}\right)$ . Let us consider the case with  $\tilde{n}+1$  bidders and any  $\hat{\theta} \in \left(0, \frac{1}{\tilde{n}}\right)$ .

First, let us observe that since  $\hat{\theta} < \frac{1}{\tilde{n}}$ , there always exists a  $\theta' < \frac{1}{\tilde{n}-1}$  such that  $\frac{\hat{\theta}}{1+(1-\tilde{n})\hat{\theta}} = \frac{\theta'}{1+(2-\tilde{n})\theta'}$ . In a second-price auction, bidders make the same equilibrium submissions when there are  $\tilde{n}$  bidders and a toehold of size  $\theta'$  or when there are  $\tilde{n}+1$  bidders and a toehold of size  $\hat{\theta}$ . In a first-price auction, bidders make higher equilibrium submissions when there are  $\tilde{n}+1$  bidders and a toehold of size  $\hat{\theta}$  or when there are  $\tilde{n}$  bidders and a toehold of size  $\theta'$ .

Since, with both auctions, the allocation is the same (*i.e.* the bidder with the highest valuation wins the auction), we just need to compare reservation utility. The reservation utility is equal to  $\hat{\theta}$  times the expected utility of the bidder with the highest valuation provided that the bidder has a valuation equal to zero. In a second-price auction, this is equal to the expected utility of the bidder with the highest valuation, with  $\tilde{n}$  bidders and a toehold of size  $\theta'$  (we will denote it:  $E\bar{U}^H(\theta', \tilde{n})$ ). In a first-price auction, this is less than the expected utility of the bidder with the highest valuation, with  $\tilde{n}$  bidders and a toehold of size  $\theta'$  (we will denote it:  $E\bar{U}^I(\theta', \tilde{n})$ ).

With  $\tilde{n}$  bidders, whatever the size of the toehold (as long as it is strictly positive), the expected price paid is strictly higher with a first-price auction than with a second-price auction and the allocation is the same. Therefore, since the allocation is the same with both auction formats, regardless of the valuation, the expected utility of a bidder is strictly higher with a second-price than with a first-price auction. Then  $E\bar{U}^H(\theta', \tilde{n}) > E\bar{U}^I(\theta', \tilde{n})$ . This means that, with  $\tilde{n}+1$  bidders and a toehold of size  $\hat{\theta}$ , the reservation utility is strictly higher in a second-price than in a first-price auction and the expected revenue of the seller is strictly higher in a first-price auction.

Q.E.D.

## A.5 Proof of proposition

### The equilibrium

First, for any  $\underline{v}$  and  $\bar{v}$  such that  $0 \leq \underline{v} < \bar{v} \leq 1$ , if at the equilibrium, a bidder with valuation  $\underline{v}$  accepts to pay the entry fee, then he also accepts to pay it when his valuation is  $\bar{v}$  (because, by participating he can get at least what he would get if he were to participate with a valuation  $\underline{v}$ ). Then, the strategy of bidder  $i$  in the first step can be represented by a threshold (for a valuation below this threshold, a bidder does not accept to pay the entry fee and, for a valuation above this threshold, he accepts to pay the entry fee). Let us call this threshold  $v^*$ .

Suppose that  $v^* \in (0, R^*)$ , then in the second step, a bidder with valuation  $v^*$  does not participate since he prefers losing the auction than winning it for a price higher or equal to  $R^*$ . Therefore, if a bidder has a valuation in the interval  $[0, v^*)$ ,

he could also accept to pay the entry fee in step 1 and obtain exactly the same payoff without bidding in step 2. Because of the lexicographic preferences, he cannot be indifferent to participating or not participating so he strictly prefers participating when his valuation lies in the interval  $[0, v^*)$ . Therefore,  $v^* \in (0, R^*)$  cannot be part of an equilibrium.

Suppose that  $v^* \in (R^*, 1)$ <sup>27</sup>, then we can show that, in the second step, bidders would submit bids increasing in their valuations. Now, a bidder with valuation  $v^*$  is sure to lose the auction in step 2. Therefore, if a bidder has a valuation lower than  $v^*$ , he can accept to pay the entry fee in step 1 and not participate in step 2, he will get the same expected utility as a bidder with valuation  $v^*$ . Because of the lexicographic preferences, he cannot be indifferent to participating or not participating so he strictly prefers participating when his valuation lies in the interval  $[0, v^*)$ . Therefore,  $v^* \in (R^*, 1)$  cannot be part of an equilibrium.

Only  $v^* = 0$  can be part of an equilibrium. So, let us suppose that  $v^* = 0$  and check whether there exists an equilibrium.

In the auction itself, a bidder with valuation strictly lower than  $R^*$  has a dominant strategy: not to participate in the auction. If there exists a bidding function,  $b$ , such that, at the symmetric equilibrium, a bidder with valuation  $\hat{v} \geq R^*$  submits  $b(\hat{v}) \geq R^*$  with  $b$  strictly increasing, it must be such that (following the proof of proposition 7):

$$v_i f(v_i) - b(v_i) f(v_i) - \frac{1 + (2 - n)\theta}{(n - 1)(1 + (1 - n)\theta)} F(v_i) b'(v_i) = 0$$

As  $b(R^*) = R^*$ , the solution of the differential equation is

$$b(v_i) = v_i - \int_{R^*}^{v_i} \left( \frac{F(t)}{F(v_i)} \right)^{\frac{(n-1)(1+(1-n)\theta)}{1+(2-n)\theta}} dt$$

We need to check that with such a bidding strategy in step 2, it is an equilibrium for the bidders always to pay the entry fee in step 1. To do so, we only need to check that a bidder with a valuation  $v = 0$  is better off paying the entry fee. If he pays, his expected utility is equal to:

27. Note that if  $v^* = 1$ , bidders participate in the auction with a zero probability. We excluded this kind of equilibrium.

$$\begin{aligned}
& -\frac{1+(2-n)\theta+(n-1)\theta}{(1+\theta)(1-(n-1)\theta)} \frac{\theta}{1+\theta} \int_{R^*}^1 \int_{R^*}^u \left( \left( \frac{F(t)}{F(u)} \right)^{\frac{(1+(1-n)\theta)(n-1)}{1+(2-n)\theta}} dt \right) f(u) F^{n-2}(u) du + \\
& \frac{\theta}{(1+\theta)(1-(n-1)\theta)} \int_{R^*}^1 \int_{R^*}^u \left( \left( \frac{F(t)}{F(u)} \right)^{\frac{(1+(1-n)\theta)(n-1)}{1+(2-n)\theta}} dt \right) f(u) F^{n-2}(u) du \\
& = \left( \frac{1+(2-n)\theta+(n-1)\theta}{1+\theta} - 1 \right) \frac{\theta}{(1+\theta)(1-(n-1)\theta)} \int_{R^*}^1 \int_{R^*}^u \left( \left( \frac{F(t)}{F(u)} \right)^{\frac{(1+(1-n)\theta)(n-1)}{1+(2-n)\theta}} dt \right) \\
& \qquad \qquad \qquad f(u) F^{n-2}(u) du = 0
\end{aligned}$$

The expected utility is equal to zero and a bidder with valuation zero prefers paying the entry fee because of his lexicographic preferences.

Q.E.D.

### The optimality

It is a well known result that an auction is optimal if and only if it allocates the good to the bidder with the highest marginal revenue (provided that this marginal revenue is positive) and that bidders' reservation utility is equal to zero.

Since we assumed that the distribution function satisfies the monotone hazard rate condition, the marginal revenue of a bidder is strictly increasing with his valuation and positive when  $v \geq v^*$  with  $v^*$  such that  $v^* = \frac{1-F(v^*)}{f(v^*)}$ .

The reserve price of the auction is precisely equal to  $v^*$  and the equilibrium of the auction is such that the bidder with the highest valuation above  $v^*$  wins the auction. Then, the allocation of the auction coincides with the optimal allocation rule.

Now, we need to verify bidders' reservation valuation but we have already showed in the first part of the proof (the equilibrium) that the reservation utility is equal to zero.

Q.E.D.