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► **To cite this version:**

| El Houcein El Abdalaoui. On the spectral type of some class of rank one flows. 2012. hal-00701070v2

HAL Id: hal-00701070

<https://hal.science/hal-00701070v2>

Preprint submitted on 10 Aug 2012

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ON THE SPECTRAL TYPE OF SOME CLASS OF RANK ONE FLOWS

E. H. EL ABDALAOUI

ABSTRACT. It is shown that a certain class of Riesz product type measure on \mathbb{R} is singular. This proves the singularity of the spectral types of some class of rank one flows. Our method is based on the extension of the Central Limit Theorem approach to the real line which gives a new extension of Salem-Zygmund Central Limit Theorem.

AMS Subject Classifications (2010): 37A15, 37A25, 37A30, 42A05, 42A55.

Key words and phrases: Rank one flows, spectral type, simple Lebesgue spectrum, singular spectrum, Salem-Zygmund Central Limit Theorem, Riesz products.

1. INTRODUCTION

The purpose of this paper is to study the spectral type of some class of rank one flows. Rank one flows have simple spectrum and using a random Ornstein procedure [22], A. Prikhod'ko in [26] produce a family of mixing rank one flows. It follows that the mixing rank one flows may possibly contain a candidate for the flow version of the Banach's well-known problem whether there exists a dynamical flow $(\Omega, \mathcal{A}, \mu, (T_t)_{t \in \mathbb{R}})$ with simple Lebesgue spectrum ¹. In [25], A. Prikhod'ko introduced a class of rank one flows called *exponential staircase rank one flows* and state that in this class the answer to the flow version of Banach problem is affirmative. Unfortunately, as we shall establish, this is not the case since the spectrum of a large class of exponential staircase rank one flow is singular and this class contain a subclass of Prikhod'ko examples.

Our main tools are on one hand an extension to \mathbb{R} of the CLT method (introduced in [1] for the torus) and on the other hand the generalized Bourgain methods [5] obtained in [2] (in the context of the Riesz products on \mathbb{R}).

This allows us to get a new extension of the Salem-Zygmund CLT Theorem [32] to the trigonometric sums with **real frequencies**.

Originally Salem-Zygmund CLT Theorem concerns the asymptotic stochastic behaviour of the lacunary trigonometric sums on the torus. Since Salem-Zygmund pioneering result, the central limit theorem for trigonometric sums has been intensively studied by many authors, Erdős [10], J.P. Kahane [13], J. Peyrière [24], Berkers [4], Murai [20], Takahashi [28], Fukuyama and Takahashi [11], and many others. The same method is used to study the asymptotic stochastic behaviour of Riesz-Raikov sums [23]. Nevertheless all these results concern only the trigonometric sums on the torus.

Here we obtain the same result on \mathbb{R} . The fundamental ingredient in our proof is based on the famous Hermite-Lindemann Lemma in the transcendental number theory [30].

Notice that the main argument used in the torus case [1] is based on the density of trigonometric polynomials. This argument cannot be applied here since the density of trigonometric polynomials in $L^1(\mathbb{R}, \omega(t)dt)$ (ω is a positive function in $L^1(\mathbb{R})$), is not verified unless ω satisfies some extra-condition. Nevertheless, using the density of the functions with compactly supported Fourier transforms, we are able to conclude.

The paper is organized as follows. In section 2, we review some standard facts from the spectral theory of dynamical flows. In section 3, we recall the basic construction of the rank one flows obtained by the cutting and stacking method and we state our main result. In section 4, we summarize and extend the relevant material on the Bourgain criterion concerning the singularity of the generalized Riesz products on \mathbb{R} . In section 5, we develop the CLT method for trigonometric sums with real frequencies and we prove our main result concerning the singularity of a exponential staircase rank one flows.

¹Ulam in his book [29, p.76] stated the Banach problem in the following form

Question 1.1 (Banach Problem). Does there exist a square integrable function $f(x)$ and a measure preserving transformation $T(x)$, $-\infty < x < \infty$, such that the sequence of functions $\{f(T^n(x)); n = 1, 2, 3, \dots\}$ forms a complete orthogonal set in Hilbert space?

2. BASIC FACTS FROM SPECTRAL THEORY OF DYNAMICAL FLOWS

A dynamical flow is a quadruplet $(X, \mathcal{A}, \mu, (T_t)_{t \in \mathbb{R}})$ where (X, \mathcal{A}, μ) is a Lebesgue probability space and $(T_t)_{t \in \mathbb{R}}$ is a measurable action of the group \mathbb{R} by measure preserving transformations. (It means that

- each T_t is a bimeasurable invertible transformation of the probability space such that, for any $A \in \mathcal{A}$, $\mu(T_t^{-1}A) = \mu(A)$,
- for all $s, t \in \mathbb{R}$, $T_s \circ T_t = T_{s+t}$,
- the map $(t, x) \mapsto T_t(x)$ is measurable from $\mathbb{R} \times X$ into X .)

Let us recall some classical definitions. A dynamical flow is *ergodic* if every measurable set which is invariant under all the maps T_t either has measure zero or one. A number λ is an *eigenfrequency* if there exists nonzero function $f \in L^2(X)$ such that, for all $t \in \mathbb{R}$, $f \circ T_t = e^{i\lambda t} f$. Such a function f is called an *eigenfunction*. An ergodic flow $(X, \mathcal{A}, \mu, (T_t)_{t \in \mathbb{R}})$ is *weakly mixing* if every eigenfunction is constant (a.e.). A flow $(X, \mathcal{A}, \mu, (T_t)_{t \in \mathbb{R}})$ is *mixing* if for all $f, g \in L^2(X)$,

$$\int f \circ T_t(x) \bar{g}(x) d\mu(x) \xrightarrow{|t| \rightarrow +\infty} \int f(x) d\mu(x) \int \bar{g}(x) d\mu(x).$$

Any dynamical flow $(T_t)_{t \in \mathbb{R}}$ induces an action of \mathbb{R} by unitary operators acting on $L^2(X)$ according to the formula $U_{T_t}(f) = f \circ T_{-t}$. When there will be no ambiguity on the choice of the flow, we will denote $U_t = U_{T_t}$.

The *spectral properties* of the flow are the property attached to the unitary representation associated to the flow. We recall below some classical facts; for details and references see [8] or [14].

Two dynamical flows $(X_1, \mathcal{A}_1, \mu_1, (T_t)_{t \in \mathbb{R}})$ and $(X_2, \mathcal{A}_2, \mu_2, (S_t)_{t \in \mathbb{R}})$ are *metrically isomorphic* if there exists a measurable map ϕ from $(X_1, \mathcal{A}_1, \mu_1)$ into $(X_2, \mathcal{A}_2, \mu_2)$, with the following properties:

- ϕ is one-to-one,
- For all $A \in \mathcal{A}_2$, $\mu_1(\phi^{-1}(A)) = \mu_2(A)$.
- $S_t \circ \phi = \phi \circ T_t$, $\forall t \in \mathbb{R}$.

If two dynamical flows $(T_t)_{t \in \mathbb{R}}$ and $(S_t)_{t \in \mathbb{R}}$ are metrically isomorphic then the isomorphism ϕ induces an isomorphism V_ϕ between the Hilbert spaces $L^2(X_2)$ and $L^2(X_1)$ which acts according to the formula $V_\phi(f) = f \circ \phi$. In this case, since $V_\phi U_{S_t} = U_{T_t} V_\phi$, the adjoint groups (U_{T_t}) and (U_{S_t}) are unitary equivalent. Thus if two dynamical flows are metrically isomorphic then the corresponding adjoint groups of unitary operators are unitary equivalent. It is well known that the converse statement is false [8].

By Bochner theorem, for any $f \in L^2(X)$, there exists a unique finite Borel measure σ_f on \mathbb{R} such that

$$\widehat{\sigma}_f(t) = \int_{\mathbb{R}} e^{-it\xi} d\sigma_f(\xi) = \langle U_t f, f \rangle = \int_X f \circ T_t(x) \cdot \bar{f}(x) d\mu(x).$$

σ_f is called the *spectral measure* of f . If f is eigenfunction with eigenfrequency λ then the spectral measure is the Dirac measure at λ .

The following fact derives directly from the definition of the spectral measure: let $(a_k)_{1 \leq k \leq n}$ be complex numbers and $(t_k)_{1 \leq k \leq n}$ be real numbers; consider $f \in L^2(X)$ and denote $F = \sum_{k=1}^n a_k \cdot f \circ T_{t_k}$. Then the spectral measure σ_F is absolutely

continuous with respect to the spectral measure σ_f and

$$(1) \quad \frac{d\sigma_F}{d\sigma_f}(\xi) = \left| \sum_{k=1}^n a_k e^{it_k \xi} \right|^2.$$

Here is another classical result concerning spectral measures : let (g_n) be a sequence in $L^2(X)$, converging to $f \in L^2(X)$; then the sequence of real measures $(\sigma_{g_n} - \sigma_f)$ converges to zero in total variation norm.

The *maximal spectral type* of $(T_t)_{t \in \mathbb{R}}$ is the equivalence class of Borel measures σ on \mathbb{R} (under the equivalence relation $\mu_1 \sim \mu_2$ if and only if $\mu_1 \ll \mu_2$ and $\mu_2 \ll \mu_1$), such that $\sigma_f \ll \sigma$ for all $f \in L^2(X)$ and if ν is another measure for which $\sigma_f \ll \nu$ for all $f \in L^2(X)$ then $\sigma \ll \nu$.

The maximal spectral type is realized as the spectral measure of one function: there exists $h_1 \in L^2(X)$ such that σ_{h_1} is in the equivalence class defining the maximal spectral type of $(T_t)_{t \in \mathbb{R}}$. By abuse of notation, we will call this measure the maximal spectral type measure.

The reduced maximal type σ_0 is the maximal spectral type of $(U_t)_{t \in \mathbb{R}}$ on $L_0^2(X) \stackrel{\text{def}}{=} \left\{ f \in L^2(X) : \int f d\mu = 0 \right\}$. The spectrum of $(T_t)_{t \in \mathbb{R}}$ is said to be discrete (resp. continuous, resp. singular, resp. absolutely continuous, resp. Lebesgue) if σ_0 is discrete (resp. continuous, resp. singular with respect to Lebesgue measure, resp. absolutely continuous with respect to Lebesgue measure).

The cyclic space of $h \in L^2(X)$ is

$$Z(h) \stackrel{\text{def}}{=} \overline{\text{span}\{U_t h : t \in \mathbb{R}\}}.$$

There exists an orthogonal decomposition of $L^2(X)$ into cyclic spaces

$$(2) \quad L^2(X) = \bigoplus_{i=1}^{\infty} Z(h_i), \quad \sigma_{h_1} \gg \sigma_{h_2} \gg \dots$$

Each decomposition (2) is called a *spectral decomposition* of $L^2(X)$ (while the sequence of measures is called a *spectral sequence*). A spectral decomposition is unique up to equivalent class of the spectral sequence. The spectral decomposition is determined by the maximal spectral type and the *multiplicity function* $M : \mathbb{R} \rightarrow \{1, 2, \dots\} \cup \{+\infty\}$, which is defined σ_{h_1} -a.e. by $M(s) = \sum_{i=1}^{\infty} 1_{Y_i}(s)$, where $Y_1 = \mathbb{R}$ and $Y_i = \text{supp} \frac{d\sigma_{x_i}}{d\sigma_{x_1}}$ for $i \geq 2$.

The flow has *simple spectrum* if 1 is the only essential value of M . The multiplicity is *homogeneous* if there is only one essential value of M . The essential supremum of M is called the *maximal spectral multiplicity*.

Von Neumann showed that the flow $(T_t)_{t \in \mathbb{R}}$ has homogeneous Lebesgue spectrum if and only if the associated group of unitary operators $(U_t)_{t \in \mathbb{R}}$ satisfy the Weyl commutation relations for some one-parameter group $(V_t)_{t \in \mathbb{R}}$ i.e.

$$U_t V_s = e^{-ist} V_s U_t, \quad s, t \in \mathbb{R},$$

where e^{-ist} denotes the operator of multiplication by e^{-ist} .

It is easy to show that the Weyl commutation relations implies that the maximal spectral type is invariant with respect to the translations. The proof of von Neumann homogeneous Lebesgue spectrum theorem can be found in [8].

3. RANK ONE FLOWS BY CUTTING AND STACKING METHOD

Several approach of the notion of rank one flow have been proposed in the literature. The notion of *approximation of a flow by periodic transformations* has been introduced by Katok and Stepin in [15] (see Chapter 15 of [8]). This was the first attempt of a definition of a rank one flow.

In [9], del Junco and Park adapted the classical Chacon construction [6] to produce similar construction for a flow. The flow obtain by this method is called the Chacon flow.

This cutting and stacking construction has been extended by Zeitz ([31]) in order to give a general definition of a rank one flow. In the present paper we follow this cutting and stacking (CS) approach and we recall it now. We assume that the reader is familiar with the CS construction of a rank one map acting on certain measure space which may be finite or σ -finite. A nice account may be founded in [12].

Let us fix a sequence $(p_n)_{n \in \mathbb{N}}$ of integers ≥ 2 and a sequence of finite sequences of non-negative real numbers $\left((s_{n,j})_{j=1}^{p_n-1} \right)_{n > 0}$.

Let \overline{B}_0 be a rectangle of height 1 with horizontal base B_0 . At stage one divide B_0 into p_0 equal parts $(A_{1,j})_{j=1}^{p_0}$. Let $(\overline{A}_{1,j})_{j=1}^{p_0}$ denotes the flow towers over $(A_{1,j})_{j=1}^{p_0}$. In order to construct the second flow tower, put over each tower $\overline{A}_{1,j}$ a rectangle spacer of height $s_{1,j}$ (and base of same measure as $A_{1,j}$) and form a stack of height $h_1 = p_0 + \sum_{j=1}^{p_0} s_{1,j}$ in the usual fashion. Call this second tower \overline{B}_1 , with $B_1 = A_{1,1}$.

At the k^{th} stage, divide the base B_{k-1} of the tower \overline{B}_{k-1} into p_{k-1} subsets $(A_{k,j})_{j=1}^{p_{k-1}}$ of equal measure. Let $(\overline{A}_{k,j})_{j=1}^{p_{k-1}}$ be the towers over $(A_{k,j})_{j=1}^{p_{k-1}}$ respectively. Above each tower $\overline{A}_{k,j}$, put a rectangle spacer of height $s_{k,j}$ (and base of same measure as $A_{k,j}$). Then form a stack of height $h_k = p_{k-1}h_{k-1} + \sum_{j=1}^{p_{k-1}} s_{k,j}$ in the usual fashion. The new base is $B_k = A_{k,1}$ and the new tower is \overline{B}_k .

All the rectangles are equipped with Lebesgue two-dimensional measure that will be denoted by ν . Proceeding this way we construct what we call a rank one flow $(T_t)_{t \in \mathbb{R}}$ acting on a certain measure space (X, \mathcal{B}, ν) which may be finite or σ -finite depending on the number of spacers added at each stage.

This rank one flow will be denoted by

$$(T^t)_{t \in \mathbb{R}} \stackrel{\text{def}}{=} \left(T_{(p_n, (s_{n+1,j})_{j=1}^{p_n-1})_{n \geq 0}}^t \right)_{t \in \mathbb{R}}$$

The invariant measure ν will be finite if and only if

$$\sum_{k=0}^{+\infty} \frac{\sum_{j=1}^{p_k} s_{k+1,j}}{p_k h_k} < +\infty.$$

In that case, the measure will be normalized in order to have a probability.

Remarks 3.1. The only thing we use from [31] is the definition of rank one flows. Actually a careful reading of Zeitz paper [31] shows that the author assumes that for any rank one flow there exists always at least one time t_0 such that T_{t_0} has rank one property. But, it turns out that this is not the case in general as proved by Ryzhikhov in [27]. Furthermore, if this property was satisfied then the weak closure theorem for flows would hold as a direct consequence of the King weak

closure theorem ($C(T_t) \subset C(t_0) \stackrel{\text{J.King}}{=} WCT(T_{t_0}) \subset WCT(T_t) \subset C(T_t)$, where $C(t_0)$ is the centralizer of T_{t_0} and $WCT(T_{t_0})$ is the weak closure of T_{t_0}).

3.1. Exponential staircase rank one flows of type I. The main issue of this note is to study the spectrum of a subclass of a rank one flows called *exponential staircase rank one flows of type I* which are defined as follows.

Let $(m_n, p_n)_{n \in \mathbb{N}}$ be a sequence of positive integers such that m_n and p_n goes to infinity as n goes to infinity. Let ε_n be a sequence of rational numbers which converge to 0. Put

$$\omega_n(p) = \frac{m_n}{\varepsilon_n^2} p_n \left(\exp\left(\frac{\varepsilon_n p}{p_n}\right) - 1 \right) \text{ for any } p \in \{0, \dots, p_n - 1\},$$

and define the sequence of the spacers $((s_{n+1,p})_{p=0, \dots, p_n-1})_{n \geq 0}$ by

$$h_n + s_{n+1,p+1} = \omega_n(p+1) - \omega_n(p), \quad p = 0, \dots, p_n - 1, n \in \mathbb{N}.$$

In this definition we assume that

- (1) $m_n \geq \varepsilon_n \cdot h_n$, for any $n \in \mathbb{N}$.
- (2) $\frac{\log(p_n)}{m_n} \xrightarrow{n \rightarrow \infty} 0$ if $p_n \geq \frac{m_n}{\varepsilon_n}$
- (3) $\frac{\log(p_n)}{m_n} \xrightarrow{n \rightarrow \infty} 0$ and $\frac{\log(p_n)}{p_n} \leq \varepsilon_n$ if $p_n < \frac{m_n}{\varepsilon_n}$

We will denote this class of rank one flow by

$$(T^t)_{t \in \mathbb{R}} \stackrel{\text{def}}{=} \left(T_{(p_n, \omega_n)_{n \geq 0}}^t \right)_{t \in \mathbb{R}}.$$

It is easy to see that this class of flows contain a large class of examples introduced by Prikhodko [25]. Indeed, assume that $h_n^\beta \geq p_n \geq h_n^{1+\alpha}$, $\beta \geq 2$ and $\alpha \in]0, \frac{1}{4}[$. Then, by assumption (1), we have

$$\frac{\log(p_n)}{m_n} \leq \beta \frac{\log(h_n)}{\varepsilon_n h_n},$$

Taking $\beta = \varepsilon_n (\lfloor h_n^\delta \rfloor + 1)$, $0 < \delta < 1$, we get

$$\frac{\log(p_n)}{m_n} \xrightarrow{n \rightarrow \infty} 0.$$

In [2] it is proved that the spectral type of of any rank one flow $(T_{(p_n, (s_{n+1,j})_{j=1}^{p_n})_{n \geq 0}}^t)_{t \in \mathbb{R}}$ is given by some kind of Riesz-product measure on \mathbb{R} . To be more precise, the authors in [2] proved the following theorem

Theorem 3.2 (Maximal spectral type of rank one flows). For any $s \in (0, 1]$, the spectral measure $\sigma_{0,s}$ is the weak limit of the sequence of probability measures

$$\prod_{k=0}^n |P_k(\theta)|^2 K_s(\theta) d\theta,$$

where

$$P_k(\theta) = \frac{1}{\sqrt{p_k}} \left(\sum_{j=0}^{p_k-1} e^{i\theta(jh_k + \bar{s}_k(j))} \right), \quad \bar{s}_k(j) = \sum_{i=1}^j s_{k+1,i}, \quad \bar{s}_k(0) = 0.$$

and

$$K_s(\theta) = \frac{s}{2\pi} \cdot \left(\frac{\sin(\frac{s\theta}{2})}{\frac{s\theta}{2}} \right)^2.$$

In addition the continuous part of spectral type of the rank one flow is equivalent to the continuous part of $\sum_{k \geq 1} 2^{-k} \sigma_{0, \frac{1}{k}}$.

The theorem above gives a new generalization of Choksi-Nadkarni Theorem [7], [21]. We point out that in [3], the author generalized the Choksi-Nadkarni Theorem to the case of funny rank one group actions for which the group is compact and Abelian.

We end this section by stating our main result.

Theorem 3.3. Let $(T^t)_{t \in \mathbb{R}} = \left(T^t_{(p_n, \omega_n)_{n \geq 0}} \right)_{t \in \mathbb{R}}$ be a exponential staircase rank one flow of type I associated to

$$\omega_n(p) = \frac{m_n}{\varepsilon_n^2} p_n \exp\left(\frac{\varepsilon_n \cdot p}{p_n}\right), \quad p = 0, \dots, p_n - 1.$$

Then the spectrum of $(T_t)_{\mathbb{R}}$ is singular.

4. ON THE BOURGAIN SINGULARITY CRITERIUM OF GENERALIZED RIESZ PRODUCTS ON \mathbb{R}

In this section, for the convenience of the reader we repeat the relevant material from [2] without proofs, thus making our exposition self-contained. Let us fix $s \in (0, 1)$ and denote by μ_s the probability measure of density K_s on \mathbb{R} , that is,

$$d\mu_s(\theta) = K_s(\theta) d\theta$$

We denote by σ the spectral measure of $\mathbb{1}_{\overline{B_{0,s}}}$ given as the weak limit of the following generalized Riesz products

$$(3) \quad d\sigma = W\text{-}\lim_{N \rightarrow +\infty} \prod_{k=1}^N |P_k|^2 d\mu_s,$$

where

$$P_k(\theta) = \frac{1}{\sqrt{p_k}} \left(\sum_{j=0}^{p_k-1} e^{i\theta(jh_k + \bar{s}_k(j))} \right), \quad \bar{s}_k(j) = \sum_{i=1}^j s_{k+1,i}, \quad \bar{s}_k(0) = 0.$$

Let us recall the following Bourgain criterion established in [2].

Theorem 4.1 (Bourgain criterion). The following are equivalent

- (i) σ is singular with respect to Lebesgue measure.
- (ii) $\inf \left\{ \int_{\mathbb{R}} \prod_{\ell=1}^L |P_{n_\ell}| d\mu_s : L \in \mathbb{N}, n_1 < n_2 < \dots < n_L \right\} = 0.$

As noted in [2] to prove the singularity of the spectrum of the rank one flow it is sufficient to prove that a weak limit point of the sequence $(\|P_m\|^2 - 1)$ is bounded by below by a positive constant. More precisely, the authors in [2] established the following proposition.

Proposition 4.2. Let E be an infinite set of positive integers. Suppose that there exists a constant $c > 0$ such that, for any positive function $\phi \in L^2(\mathbb{R}, \mu_s)$,

$$\liminf_{\substack{m \rightarrow +\infty \\ m \in E}} \int_{\mathbb{R}} \phi ||P_m|^2 - 1| d\mu_s \geq c \int_{\mathbb{R}} \phi d\mu_s.$$

Then σ is singular.

The proof of the proposition 4.2 is based on the following lemma.

Lemma 4.3. Let E be an infinite set of positive integers. Let L be a positive integer and $0 \leq n_1 < n_2 < \dots < n_L$ be integers. Denote $Q = \prod_{\ell=1}^L |P_{n_\ell}|$. Then

$$\limsup_{\substack{m \rightarrow +\infty \\ m \in E}} \int Q |P_m| d\mu_s \leq \int Q d\mu_s - \frac{1}{8} \left(\liminf_{\substack{m \rightarrow +\infty \\ m \in E}} \int Q ||P_m|^2 - 1| d\mu_s \right)^2.$$

For sake of completeness we recall from [2] the proof of the proposition 4.2.

Proof of Proposition 4.2.

Let $\beta = \inf \left\{ \int Q d\mu_s : Q = \prod_{\ell=1}^L |P_{n_\ell}|, L \in \mathbb{N}, 0 \leq n_1 < n_2 < \dots < n_L \right\}$. Then, for any such Q , we have

$$\int Q d\mu_s \geq \beta \quad \text{and} \quad \liminf \int Q |P_m| d\mu_s \geq \beta.$$

Thus by Lemma 4.3 and by taking the infimum over all Q we get

$$\beta \leq \beta - \frac{1}{8}(c\beta)^2$$

It follows that

$$\beta = 0,$$

and the proposition follows from Theorem 4.1. \square

5. THE CLT METHOD FOR TRIGONOMETRIC SUMS AND THE SINGULARITY OF THE SPECTRUM OF EXPONENTIAL STAIRCASE RANK ONE FLOWS OF TYPE I

The main goal of this section is to prove the following proposition

Proposition 5.1. Let $(T^t)_{t \in \mathbb{R}} = \left(T_{(p_n, \omega_n)_{n \geq 0}}^t \right)_{t \in \mathbb{R}}$ be a exponential staircase rank one flow of type I associated to

$$\omega_n(p) = \frac{m_n}{\varepsilon_n^2} p_n \exp\left(\frac{\varepsilon_n p}{p_n}\right), \quad p = 0, \dots, p_n - 1.$$

Then, there exists a constant $c > 0$ such that, for any positive function f in $L^2(\mathbb{R}, \mu_s)$, we have

$$\liminf_{m \rightarrow +\infty} \int_{\mathbb{R}} f(t) ||P_m(t)|^2 - 1| d\mu_s(t) \geq c \int_{\mathbb{R}} f(t) d\mu_s(t).$$

The proof of the proposition 5.1 is based on the study of the stochastic behaviour of the sequence $|P_m|$. For that, we follow the strategy introduced in [1] based on the method of the Central Limit Theorem for trigonometric sums.

This methods takes advantage of the following classical expansion

$$\overline{\exp(ix)} = (1 + ix) \exp\left(-\frac{x^2}{2} + r(x)\right),$$

where $|r(x)| \leq |x|^3$, for all real number x ², combined with some ideas developed in the proof of martingale central limit theorem due to McLeish [19]. Precisely, the main ingredient is the following theorem proved in [1].

Theorem 5.2. Let $\{X_{nj} : 1 \leq j \leq k_n, n \geq 1\}$ be a triangular array of random variables and t a real number. Let

$$S_n = \sum_{j=1}^{k_n} X_{nj}, \quad T_n = \prod_{j=1}^{k_n} (1 + itX_{nj}),$$

and

$$U_n = \exp\left(-\frac{t^2}{2} \sum_{j=1}^{k_n} X_{nj}^2 + \sum_{j=1}^{k_n} r(tX_{nj})\right).$$

Suppose that

- (1) $\{T_n\}$ is uniformly integrable.
- (2) $\mathbb{E}(T_n) \xrightarrow{n \rightarrow \infty} 1$.
- (3) $\sum_{j=1}^{k_n} X_{nj}^2 \xrightarrow{n \rightarrow \infty} 1$ in probability.
- (4) $\max_{1 \leq j \leq k_n} |X_{nj}| \xrightarrow{n \rightarrow \infty} 0$ in probability.

Then $\mathbb{E}(\exp(itS_n)) \rightarrow \exp(-\frac{t^2}{2})$.

We remind that the sequence $\{X_n, n \geq 1\}$ of random variables is said to be uniformly integrable if and only if

$$\lim_{c \rightarrow +\infty} \int_{\{|X_n| > c\}} |X_n| d\mathbb{P} = 0 \quad \text{uniformly in } n.$$

and it is well-known that if

$$(4) \quad \sup_{n \in \mathbb{N}} \left(\mathbb{E}(|X_n|^{1+\varepsilon}) \right) < +\infty,$$

for some ε positive, then $\{X_n\}$ is uniformly integrable.

Using Theorem 5.2 we shall prove the following extension to \mathbb{R} of Salem-Zygmund CLT theorem, which seems to be of independent interest.

Theorem 5.3. Let A be a Borel subset of \mathbb{R} with $\mu_s(A) > 0$ and let $(m_n, p_n)_{n \in \mathbb{N}}$ be a sequence of positive integers such that m_n and p_n goes to infinity as n goes to infinity. Let ε_n be a sequence of rationals numbers which converge to 0 and

$$\omega_n(j) = \frac{m_n}{\varepsilon_n^2} p_n \exp\left(\frac{\varepsilon_n \cdot j}{p_n}\right) \quad \text{for any } j \in \{0, \dots, p_n - 1\}.$$

²this is a direct consequence of Taylor formula with integral remainder.

Assume that

- (1) $m_n \geq \varepsilon_n \cdot h_n$, for any $n \in \mathbb{N}$.
- (2) $\frac{\log(p_n)}{m_n} \xrightarrow{n \rightarrow \infty} 0$ if $p_n \geq \frac{m_n}{\varepsilon_n}$
- (3) $\frac{\log(p_n)}{m_n} \xrightarrow{n \rightarrow \infty} 0$ and $\frac{\log(p_n)}{p_n} \leq \varepsilon_n$ if $p_n < \frac{m_n}{\varepsilon_n}$.

Then, the distribution of the sequence of random variables $\frac{\sqrt{2}}{\sqrt{p_n}} \sum_{j=0}^{p_n-1} \cos(\omega_n(j)t)$ converges to the Gauss distribution. That is, for any real number x , we have

$$(5) \quad \frac{1}{\mu_s(A)} \mu_s \left\{ t \in A : \frac{\sqrt{2}}{\sqrt{p_n}} \sum_{j=0}^{p_n-1} \cos(\omega_n(j)t) \leq x \right\} \\ \xrightarrow{n \rightarrow \infty} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^x e^{-\frac{1}{2}t^2} dt \stackrel{\text{def}}{=} \mathcal{N}(-\infty, x).$$

We start by proving the following proposition.

Proposition 5.4. There exists a subsequence of the sequence $\left(\left| |P_n(\theta)|^2 - 1 \right| \right)_{n \geq \mathbb{N}}$ which converges weakly in $L^2(\mathbb{R}, \mu_s)$ to some non-negative function ϕ bounded above by 2.

In the proof of Proposition 5.4 we shall need the following lemma proved in [2].

Lemma 5.5. The sequence of probability measures $|P_n(\theta)|^2 K_s(\theta) d\theta$ converges weakly to $K_s(\theta) d\theta$.

Proof of Proposition 5.4. Since for all n , we have $\|P_n\|_{L^2(\mu_s)}^2 = 1$. Therefore, the sequence $\left(\left| |P_n|^2 - 1 \right| \right)_{n \in \mathbb{N}}$ is bounded in $L^2(\mathbb{R}, \mu_s)$, thus admits a weakly convergent subsequence. Let us denote by ϕ one such weak limit function. Let f be a bounded continuous function on \mathbb{R} . We have

$$\int f \cdot \left| |P_n|^2 - 1 \right| d\mu_s \leq \int f \cdot |P_n|^2 d\mu_s + \int f d\mu_s.$$

By Lemma 5.5 we get

$$\lim_{n \rightarrow +\infty} \int f \cdot \left| |P_n|^2 - 1 \right| d\mu_s \leq 2 \int f d\mu_s.$$

Hence, for any bounded continuous function f on \mathbb{R} , we have

$$\int f \cdot \phi d\mu_s \leq 2 \int f d\mu_s.$$

which proves that the weak limit ϕ is bounded above by 2. \square

Let us prove now that the function ϕ is bounded by below by a universal positive constant. For that we need to prove Proposition 5.1. Let n be a positive integer and put

$$\mathbb{W}_n \stackrel{\text{def}}{=} \left\{ \sum_{j \in I} \eta_j \omega_n(j) : \eta_j \in \{-1, 1\}, I \subset \{0, \dots, p_n - 1\} \right\}.$$

The element $w = \sum_{i \in I} \eta_j \omega_n(j)$ is called a word.

We shall need the following two combinatorial lemmas. The first one is a classical result in the transcendental number theory and it is due to Hermite-Lindemann.

Lemma 5.6 (Hermite-Lindemann, 1882). Let α be a non-zero algebraic number. Then, the number $\exp(\alpha)$ is transcendental.

We state the second lemma as follows.

Lemma 5.7. For any $n \in \mathbb{N}^*$. All the words of \mathbb{W}_n are distinct.

Proof. Let $w, w' \in \mathbb{W}_n$, write

$$\begin{aligned} w &= \sum_{j \in I} \eta_j \omega_n(j), \\ w' &= \sum_{j \in I'} \eta'_j \omega_n(j). \end{aligned}$$

Then $w = w'$ implies

$$\sum_{j \in I} \eta_j \omega_n(j) - \sum_{j \in I'} \eta'_j \omega_n(j) = 0$$

Hence

$$\sum_{j \in I} \eta_j \exp\left(\frac{\varepsilon_n}{p_n} j\right) - \sum_{j \in I'} \eta'_j \exp\left(\frac{\varepsilon_n}{p_n} j\right) = 0$$

But Lemma 5.6 tell us that e^{ε_n/p_n} is a transcendental number. This clearly forces $I = I'$ and the proof of the lemma is complete. \square

Proof of Theorem 5.3. Let A be a Borel set with $\mu_s(A) > 0$ and notice that for any positive integer n , we have

$$\int_{\mathbb{R}} \left| \frac{\sqrt{2}}{\sqrt{p_n}} \sum_{j=0}^{p_n-1} \cos(\omega_n(j)t) \right|^2 d\mu_s(t) \leq 1.$$

Therefore, applying the Helly theorem we may assume that the sequence

$$\left(\frac{\sqrt{2}}{\sqrt{p_n}} \sum_{j=0}^{p_n-1} \cos(\omega_n(j)t) \right)_{n \geq 0}$$

converge in distribution. As is well-known, it is sufficient to show that for every real number x ,

$$\frac{1}{\mu_s(A)} \int_A \exp \left\{ -ix \frac{\sqrt{2}}{\sqrt{p_n}} \sum_{j=0}^{p_n-1} \cos(\omega_n(j)t) \right\} d\mu_s(t) \xrightarrow{n \rightarrow \infty} \exp\left(-\frac{x^2}{2}\right).$$

To this end we apply theorem 5.2 in the following context. The measure space is the given Borel set A of positive measure with respect to the probability measure μ_s on \mathbb{R} equipped with the normalised measure $\frac{\mu_s}{\mu_s(A)}$ and the random variables are given by

$$X_{nj} = \frac{\sqrt{2}}{\sqrt{p_n}} \cos(\omega_n(j)t), \quad \text{where } 0 \leq j \leq p_n - 1, \quad n \in \mathbb{N}.$$

It is easy to check that the variables $\{X_{nj}\}$ satisfy condition (4). Further, condition (3) follows from the fact that

$$\int_{\mathbb{R}} \left| \sum_{j=0}^{p_n-1} X_{nj}^2 - 1 \right|^2 d\mu_s(t) \xrightarrow{n \rightarrow \infty} 0.$$

It remains to verify conditions (1) and (2) of Theorem 5.2. For this purpose, we set

$$\begin{aligned} \Theta_n(x, t) &= \prod_{j=0}^{p_n-1} \left(1 - ix \frac{\sqrt{2}}{\sqrt{p_n}} \cos(\omega_n(j)t) \right) \\ &= 1 + \sum_{w \in \mathbb{W}_n} \rho_w^{(n)}(x) \cos(wt), \end{aligned}$$

and

$$\mathbb{W}_n = \bigcup_r \mathbb{W}_n^{(r)},$$

where $\mathbb{W}_n^{(r)}$ is the set of words of length r . Hence

$$|\Theta_n(x, t)| \leq \left\{ \prod_{j=0}^{p_n-1} \left(1 + \frac{2x^2}{p_n} \right) \right\}^{\frac{1}{2}}.$$

But, since $1 + u \leq e^u$, we get

$$(6) \quad |\Theta_n(x, t)| \leq e^{x^2}.$$

This shows that the condition (1) is satisfied. It still remains to prove that the variables $\{X_{nj}\}$ satisfy condition (2). For that, it is sufficient to show that

$$(7) \quad \int_A \prod_{j=0}^{p_n-1} \left(1 - ix \frac{\sqrt{2}}{\sqrt{p_n}} \cos(\omega_n(j)t) \right) d\mu_s(t) \xrightarrow{n \rightarrow \infty} \mu_s(A).$$

Observe that

$$\int_A \Theta_n(x, t) d\mu_s(t) = \mu_s(A) + \sum_{w \in \mathbb{W}_n} \rho_w^{(n)}(x) \int_A \cos(wt) d\mu_s(t)$$

and for $w = \sum_{j=1}^r \omega_n(q_j) \in \mathbb{W}_n$, we have

$$(8) \quad |\rho_w^{(n)}(x)| \leq \frac{2^{1-r} |x|^r}{p_n^{\frac{r}{2}}},$$

hence

$$\max_{w \in \mathbb{W}_n} |\rho_w^{(n)}(x)| \leq \frac{|x|}{p_n^{\frac{1}{2}}} \xrightarrow{n \rightarrow \infty} 0.$$

We claim that it is sufficient to prove the following

$$(9) \quad \int_{\mathbb{R}} \phi \prod_{j=0}^{p_n-1} \left(1 - ix \frac{\sqrt{2}}{\sqrt{p_n}} \cos(\omega_n(j)t) \right) dt \xrightarrow{n \rightarrow \infty} \int_{\mathbb{R}} \phi dt,$$

for any function ϕ with compactly supported Fourier transforms. Indeed, assume that (9) holds and let $\epsilon > 0$. Then, by the density of the functions with compactly supported Fourier transforms [16, p.126], one can find a function ϕ_ϵ with compactly supported Fourier transforms such that

$$\left\| \chi_A \cdot K_s - \phi_\epsilon \right\|_{L^1(\mathbb{R})} < \epsilon,$$

where χ_A is indicator function of A . Hence, according to (9) combined with (7), for n sufficiently large, we have

$$(10) \quad \left| \int_A \Theta_n(x, t) d\mu_s(t) - \mu_s(A) \right| = \left| \int_A \Theta_n(x, t) d\mu_s(t) - \int_{\mathbb{R}} \Theta_n(x, t) \phi_\epsilon(t) dt + \right. \\ (11) \quad \left. \int_{\mathbb{R}} \Theta_n(x, t) \phi_\epsilon(t) dt - \int_{\mathbb{R}} \phi_\epsilon(t) dt + \int_{\mathbb{R}} \phi_\epsilon(t) dt - \mu_s(A) \right| < e^{x^2} \epsilon + 2\epsilon.$$

The proof of the claim is complete. It still remains to prove (9). For that, let us compute the cardinality of words of length r which can belong to the support of ϕ .

By the well-known sampling theorem, we can assume that the support of ϕ is $[-\Omega, \Omega]$, $\Omega > 0$. First, it is easy to check that for all odd r , $|w_n^{(r)}| \xrightarrow{n \rightarrow \infty} +\infty$. It suffices to consider the words with even length. The case $r = 2$ is easy, since it is obvious to obtain the same conclusion. Moreover, as we will see later, it is sufficient to consider the case $r = 2^k$, $k \geq 2$. We argue that the cardinality of words of length 2^k , $k \geq 2$ which can belong to $[-\Omega, \Omega]$ is less than $\Omega \cdot \frac{p_n^k (\log(p_n))^{k-1}}{m_n \epsilon_n^{k-2}}$. Indeed, for $k = 2$. Write

$$w_n^{(4)} = \eta_1 \omega_n(k_1) + \eta_2 \omega_n(k_2) + \eta_3 \omega_n(k_3) + \eta_4 \omega_n(k_4),$$

with $\eta_i \in \{-1, 1\}$ and $k_i \in \{0, \dots, p_n - 1\}$, $i = 1, \dots, 4$, and put

$$e_n(p) = \exp\left(\frac{\epsilon_n}{p_n} \cdot p\right), \quad p \in \{0, \dots, p_n - 1\}.$$

If $\sum_{i=1}^4 \eta_i \neq 0$ then there is nothing to prove since $|w_n^{(4)}| \xrightarrow{n \rightarrow \infty} +\infty$. Therefore, let us assume that $\sum_{i=1}^4 \eta_i = 0$. In this case, without loss of generality (WLOG), we will assume that $k_1 < k_2 < k_3 < k_4$. Hence

$$w_n^{(4)} = \eta_1 \omega_n(k_1) + \eta_2 \omega_n(k_2) + \eta_3 \omega_n(k_3) + \eta_4 \omega_n(k_4) \\ = \frac{m_n}{\epsilon_n^2} p_n e_n(k_1) \left(\eta_1 + \eta_2 e_n(\alpha_1) + \eta_3 e_n(\alpha_2) + \eta_4 e_n(\alpha_3) \right)$$

where, $\alpha_i = k_{i+1} - k_1$, $i = 1, \dots, 3$. At this stage, we may assume again WLOG that $\eta_1 + \eta_2 = 0$ and $\eta_3 + \eta_4 = 0$. It follows that

$$w_n^{(4)} = \frac{m_n}{\epsilon_n^2} p_n e_n(k_1) \left(\eta_2 (e_n(\alpha_1) - 1) + \eta_4 e_n(\alpha_2) (e_n(\alpha'_1) - 1) \right) \quad \text{with } \alpha'_1 = \alpha_3 - \alpha_2.$$

Consequently, we have two cases to deal with.

- **Case 1:** $\eta_2 = \eta_4$. Then

$$\begin{aligned} |w_n^{(4)}| &\geq \frac{m_n}{\epsilon_n^2} p_n e_n(k_1) (e_n(\alpha_1) - 1) \\ &\geq \frac{m_n}{\epsilon_n^2} p_n \frac{\epsilon_n}{p_n} \xrightarrow{n \rightarrow \infty} +\infty. \end{aligned}$$

- **Case 2:** $\eta_2 = -\eta_4$. We thus get

$$w_n^{(4)} = \frac{m_n}{\epsilon_n^2} p_n \cdot e_n(k_1) \eta_4 (e_n(\alpha_2)(e_n(\alpha'_1) - 1) - (e_n(\alpha_1) - 1))$$

Hence

$$|w_n^{(4)}| = \frac{m_n}{\epsilon_n^2} p_n e_n(k_1) |e_n(\alpha_2)(e_n(\alpha'_1) - 1) - (e_n(\alpha_1) - 1)|$$

Therefore, we have three cases to deal with.

- **Case 1:** $\alpha'_1 > \alpha_1$. In this case,

$$\begin{aligned} |w_n^{(4)}| &= \frac{m_n}{\epsilon_n^2} p_n e_n(k_1) (e_n(\alpha_2)(e_n(\alpha'_1) - 1) - (e_n(\alpha_1) - 1)) \\ &\geq \frac{m_n}{\epsilon_n^2} p_n (e_n(\alpha'_1) - e_n(\alpha_1)) \\ &\geq \frac{m_n}{\epsilon_n^2} p_n (e_n(\alpha'_1 - \alpha_1) - 1) \xrightarrow{n \rightarrow \infty} +\infty. \end{aligned}$$

- **Case 2:** $\alpha'_1 < \alpha_1$. Write $\alpha_1 = \alpha'_1 + \beta$. Thus

$$\begin{aligned} |w_n^{(4)}| &= \frac{m_n}{\epsilon_n^2} p_n e_n(k_1) |e_n(\alpha_2)(e_n(\alpha'_1) - 1) - (e_n(\alpha'_1 + \beta) - 1)| \\ &\geq \frac{m_n}{\epsilon_n^2} p_n |e_n(\alpha'_1 + \alpha_2) - e_n(\alpha_2) - e_n(\alpha'_1 + \beta) + 1| \\ &\geq \frac{m_n}{\epsilon_n^2} p_n |(e_n(\alpha_2) - 1)(e_n(\alpha'_1) - 1) - e_n(\alpha'_1)(e_n(\beta) - 1)| \\ &\geq \frac{m_n}{\epsilon_n^2} p_n \frac{\epsilon_n \beta \alpha'_1}{p_n} \left(\frac{e_n(\alpha'_1)(e_n(\beta) - 1)}{\frac{\epsilon_n \beta \alpha'_1}{p_n}} - \frac{(e_n(\alpha_2) - 1)(e_n(\alpha'_1) - 1)}{\frac{\epsilon_n \beta \alpha'_1}{p_n}} \right) \\ &\geq \frac{m_n}{\epsilon_n} \beta \alpha'_1 \left(\frac{e_n(\alpha'_1)(e_n(\beta) - 1)}{\frac{\epsilon_n \beta \alpha'_1}{p_n}} - \frac{(e_n(\alpha_2) - 1)(e_n(\alpha'_1) - 1)}{\frac{\epsilon_n \beta \alpha'_1}{p_n}} \right) \end{aligned}$$

But for any $x \in [0, \log(2)]$, we have $x \leq e^x - 1 \leq 2x$. Therefore

$$\frac{(e_n(\alpha_2) - 1)(e_n(\alpha'_1) - 1)}{\frac{\epsilon_n \beta \alpha'_1}{p_n}} \leq 4\epsilon_n.$$

Hence

$$|w_n^{(4)}| \xrightarrow{n \rightarrow \infty} +\infty$$

- **Case 3:** $\alpha'_1 = \alpha_1$. In this case, we get

$$|w_n^{(4)}| \geq \frac{m_n}{\epsilon_n^2} p_n (e_n(\alpha_2) - 1) (e_n(\alpha_1) - 1).$$

From this, we have

$$\begin{aligned} |w_n^{(4)}| &\geq \frac{m_n}{\epsilon_n^2} p_n \cdot \left(\frac{\epsilon_n}{p_n}\right)^2 \cdot \alpha_2 \cdot \alpha_1. \\ &\geq \frac{m_n}{p_n} \alpha_2 \cdot \alpha_1 \end{aligned}$$

It follows that

$$|w_n^{(4)}| \leq \Omega \implies \alpha_2 \cdot \alpha_1 \leq \frac{p_n}{m_n} \cdot \Omega.$$

But the cardinality of (α_1, α_2) such that $\alpha_2 \cdot \alpha_1 \leq \Omega \frac{p_n}{m_n}$ is less than $\Omega \frac{p_n}{m_n} \log(p_n)$.

This gives that the cardinality of words of length 4 which can belong to $[-\Omega, \Omega]$ is less than $\Omega \frac{p_n^2}{m_n} \log(p_n)$.

Repeated the same argument as above we deduce that the only words to take into account at the stage $k = 3$ are the form

$$w_n^{(8)} = \frac{m_n}{\epsilon_n^2} p_n e_n(\alpha) \left((e_n(\alpha_1) - 1)(e_n(\alpha_2) - 1) + \eta \cdot e_n(\alpha_3)(e_n(\alpha_4) - 1)(e_n(\alpha_5) - 1) \right),$$

where $\eta = \pm 1$. In the case $\eta = 1$, it is easy to see that

$$|w_n^{(8)}| \geq \frac{m_n \epsilon_n}{p_n^2} \alpha_3 \tilde{\alpha}_1 \tilde{\alpha}_2, \quad \text{where } \tilde{\alpha}_i = \inf(\alpha_i, \alpha_{3+i}).$$

For $\eta = -1$, write

$$\begin{aligned} |w_n^{(8)}| &= \frac{m_n}{\epsilon_n^2} p_n e_n(\alpha) \left| (e_n(\alpha_1) - 1)(e_n(\alpha_2) - 1) - \right. \\ &\quad \left. (e_n(\alpha_3) - 1)(e_n(\alpha_4) - 1)(e_n(\alpha_5) - 1) - (e_n(\alpha_4) - 1)(e_n(\alpha_5) - 1) \right|. \end{aligned}$$

Using the following expansion

$$(12) \quad e_n(x) = 1 + \frac{\epsilon_n \cdot x}{p_n} + o(1),$$

we obtain, for a large n ,

$$|w_n^{(8)}| \geq \frac{m_n \epsilon_n}{p_n^2} \alpha_3 \tilde{\alpha}_1 \tilde{\alpha}_2.$$

We deduce that the cardinality of words of length 8 which can belong to $[-\Omega, \Omega]$ is less than

$$\Omega \cdot p_n \cdot \frac{p_n^2}{m_n \epsilon_n} \sum_{\tilde{\alpha}_1, \tilde{\alpha}_2} \frac{1}{\tilde{\alpha}_1 \tilde{\alpha}_2} \leq \Omega \cdot \frac{p_n^3}{m_n \epsilon_n} (\log(p_n))^2.$$

In the same manner as before consider the words of length k in the following form

$$w_n^{(2^k)} = \frac{m_n}{\epsilon_n^2} p_n e_n(\alpha) \left((e_n(\alpha_1) - 1) \cdots (e_n(\alpha_k) - 1) \right).$$

Therefore

$$\begin{aligned} |w_n^{(2^k)}| &\geq \frac{m_n}{\epsilon_n^2} p_n \cdot \left(\frac{\epsilon_n}{p_n}\right)^k \cdot \alpha_1 \cdot \alpha_2 \cdots \alpha_k. \\ &\geq \frac{m_n}{p_n^{k-1}} \epsilon_n^{k-2} \alpha_1 \cdot \alpha_2 \cdots \alpha_k \end{aligned}$$

which yields as above that the cardinality of words of length 2^k which can belong to $[-\Omega, \Omega]$ is less than

$$\Omega \cdot p_n \cdot \frac{p_n^{k-1}}{m_n \varepsilon_n^{k-2}} \sum_{\alpha_2, \dots, \alpha_k} \frac{1}{\alpha_2 \cdots \alpha_k} \leq \Omega \cdot p_n \cdot \frac{p_n^{k-1}}{m_n \varepsilon_n^{k-2}} (\log(p_n))^{k-1}.$$

Now, if r is any arbitrary even number. Write r in base 2 as

$$r = 2^{l_s} + \cdots + 2^{l_1} \quad \text{with } l_s > \cdots > l_1 \geq 1.$$

and write

$$w_n^{(r)} = w_n^{(2^{l_s})} + \cdots + w_n^{(2^{l_1})},$$

with

$$w_n^{(2^{l_j})} = \eta_1^{(j)} \omega_n(k_1^{(j)}) + \cdots + \eta_{2^{l_j}}^{(j)} \omega_n(k_{l_j}^{(j)}), \quad j = 1, \dots, s,$$

and

$$\sum_{i=1}^{2^{l_j}} \eta_i^{(j)} = 0, \quad j = 1, \dots, s.$$

Observe that the important case to consider is the case

$$w_n^{(2^{l_j})} = \pm \frac{m_n}{\varepsilon_n} p_n \prod_{i=1}^{l_j} (e(\alpha_i^{(j)}) - 1), \quad j = 1, \dots, s.$$

Using again (12) we obtain for a large n that

$$|w_n^{(r)}| \geq \frac{m_n}{p_n^{l_s-1}} \varepsilon_n^{l_s-2} \prod_{i=1}^{l_s} \alpha_i^{(s)}.$$

Hence, the cardinality of words of length r which can belong to $[-\Omega, \Omega]$ is less than

$$\Omega \cdot \frac{1}{m_n} \cdot \frac{p_n^{|\log_2(r)|}}{\varepsilon_n^{|\log_2(r)|-2}} (\log(p_n))^{|\log_2(r)|-1},$$

and, in consequence, by (8), we deduce that

$$\sum_{w \in \mathbb{W}_n} \left| \rho_w^{(n)}(x) \int_{\mathbb{R}} e^{-iwt} \phi(t) dt \right| \leq \sum_{\substack{w \in \mathbb{W}_n^{(r)}, r \text{ even} \\ 4 \leq r \leq p_n}} \left| \rho_w^{(n)}(x) \int_{\mathbb{R}} \phi(t) e^{-iwt} dt \right|.$$

Therefore, under the assumption (2), we get

$$\begin{aligned} \sum_{w \in \mathbb{W}_n} \left| \rho_w^{(n)}(x) \int_{\mathbb{R}} e^{-iwt} \phi(t) dt \right| &\leq \Omega \frac{|x|}{m_n} \sum_{\substack{r \text{ even} \\ 4 \leq r \leq p_n}} \frac{(\log(p_n))^{|\log_2(r)|-1}}{p_n^{\left(\frac{r}{2}-2|\log_2(r)|+2\right)}} \cdot \frac{1}{(p_n \cdot \varepsilon_n)^{|\log_2(r)|-2}} \\ &\leq \Omega \cdot |x| \cdot \frac{\log(p_n)}{m_n} \sum_{\substack{r \text{ even} \\ 4 \leq r \leq p_n}} \frac{1}{p_n^{\left(\frac{r}{2}-2|\log_2(r)|+2\right)}}. \end{aligned}$$

The last inequality is due to the fact that for a large n we may assume that $\frac{\log(p_n)}{m_n}$ is

strictly less than 1. In addition, since $p_n \geq 2$, $\sum_{\substack{r \text{ even} \\ 4 \leq r \leq p_n}} \frac{1}{p_n^{\left(\frac{r}{2}-2|\log_2(r)|+2\right)}}$ is convergent.

We conclude that

$$\left| \sum_{w \in W_n} \rho_w^{(n)}(x) \int_{\mathbb{R}} \phi(t) \cos(wt) dt \right| \leq \Omega \cdot |x| \cdot \frac{\log(p_n)}{m_n} \sum_{\substack{r \text{ even} \\ 4 \leq r \leq p_n}} \frac{1}{p_n^{(\frac{r}{2} - 2 \lfloor \log_2(r) \rfloor + 2)}} \xrightarrow{n \rightarrow \infty} 0.$$

This finishes the proof, the other case is left to the reader. \square

Remark. Let us point out that if

$$(13) \quad \frac{h_n \cdot \varepsilon_n^k}{p_n^k} \xrightarrow{n \rightarrow \infty} +\infty, \quad \text{for any } k \geq 1.$$

Then the spectrum of the associated exponential staircase flow is singular. Indeed,

Let $W_n^{(r)} = \frac{m_n \cdot p_n}{\varepsilon_n^2} \sum_{j=1}^r \eta_j e_n(k_j)$ a word of length r . If $W_n^{(r)} \neq 0$. Then, by Lemma 5.6, there exists $\alpha \geq 1$ such that

$$\sum_{j=1}^r \frac{\eta_j k_j^\alpha}{\alpha!} \neq 0.$$

Therefore, using a Taylor expansion of e_n and assuming that all the terms of degree less than α are 0, we have

$$\begin{aligned} |W_n^{(r)}| &= \frac{m_n \cdot p_n}{\varepsilon_n} \left| \left(\frac{\varepsilon_n}{p_n} \right)^\alpha \sum_{j=1}^r \frac{\eta_j k_j^\alpha}{\alpha!} + o\left(\left(\frac{\varepsilon_n}{p_n} \right)^\alpha \sum_{j=1}^r \frac{\eta_j k_j^\alpha}{\alpha!} \right) \right| \\ &\geq \left| \frac{h_n \cdot \varepsilon_n^{\alpha-1}}{p_n^{\alpha-1}} \sum_{j=1}^r \frac{\eta_j k_j^\alpha}{\alpha!} + o\left(\frac{h_n \cdot \varepsilon_n^{\alpha-1}}{p_n^{\alpha-1}} \sum_{j=1}^r \frac{\eta_j k_j^\alpha}{\alpha!} \right) \right| \xrightarrow{n \rightarrow \infty} +\infty. \end{aligned}$$

One can apply Lagrangian method to show that α is strictly less than two. Furthermore, to ensure that the assumption (13) holds, it suffices to take $p_n = n$, $\varepsilon_n = \frac{1}{n^2}$ and $m_n = \frac{h_n}{n^2}$.

Proof of Proposition 5.3. Let A be a Borel subset of \mathbb{R} , and $x \in]1, +\infty[$, then, for any positive integer m , we have

$$\begin{aligned} \int_A \left| |P_m(\theta)|^2 - 1 \right| d\mu_s(\theta) &\geq \int_{\{\theta \in A : |P_m(\theta)| > x\}} \left| |P_m(\theta)|^2 - 1 \right| d\mu_s(\theta) \\ &\geq (x^2 - 1) \mu_s \left\{ \theta \in A : |P_m(\theta)| > x \right\} \\ &\geq (x^2 - 1) \mu_s \left\{ \theta \in A : |\Re(P_m(\theta))| > x \right\} \end{aligned}$$

Let m goes to infinity and use Theorem 5.3 and Proposition 5.4 to get

$$\int_A \phi d\mu_s \geq (x - 1) \{1 - \mathcal{N}([-\sqrt{2}x, \sqrt{2}x])\} \mu_s(A).$$

Put $K = (x - 1) \{1 - \mathcal{N}([-\sqrt{2}x, \sqrt{2}x])\}$. Hence

$$\int_A \phi d\mu_s \geq K \mu_s(A),$$

for any Borel subset A of \mathbb{R} . This end the proof of the proposition. \square

Now, we give the proof of our main result.

Proof of Theorem 3.3. Follows easily from the proposition 5.1 combined with proposition 4.2. \square

Remark. It is shown in [2] that the spectrum of the Ornstein rank one flows is singular and in the forthcoming papers we shall extended the classical results of Klemes [17] and Klemes-Reinhold [18]. This allow us to ask the following question

Question 5.8. Does any rank one flow have singular spectrum?

Acknowledgment 1. The author wishes to express his thanks to J-P. Thouvenot, B. Host, Mariusz Lemanczyk, Emmanuel Lesigne, T. de la Rue, A. A. Prikhod'ko and Margherita Disertori for fruitful discussions on the subject. He would like to express his special thanks to Bassam Fayad for pointing out a gap in the previous version of this paper.

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