A MULTI-MODELS APPROACH OF SAINT-VENANT'S EQUATIONS: A STABILITY STUDY BY LMIs

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A MULTI-MODELS APPROACH OF SAINT-VENANT'S EQUATIONS: A STABILITY STUDY BY LMI

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This paper deals with the stability study of Partial Differential nonlinear Equation (PDE) of Saint-Venant. The proposed approach is based on the Multi-Models concept which takes into account some Linear Time Invariant (LTI) models defined around a set of operating points. This method allows to describe the dynamic of this nonlinear system in infinite dimension over a wide operating range. A stability analysis of the nonlinear PDE of Saint-Venant is proposed both by the use of Linear Matrix Inequality (LMI) and an Internal Model Boundary Control (IMBC) structures. The method is applied both in simulations and real experimentations through a micro channel, illustrating thus the theoretical results developed in the paper.

Keywords: EDP, Saint-Venant Equations, Multi-Models, LMI, Finite/Infinite dimension, Exponential Stability, C_0-Semigroup, IMBC.

1. Introduction

Irrigation channels regulation problem presents an economic and environment interest and many research have been done in this area. Indeed, the water is a precious resource which has to be efficiently managed and protected. However, water losses in irrigation channels are substantial while it is the biggest consumer of fresh water ($\simeq 80\%$). The automation of irrigation channels have improved the output of such process, but for irrigation channels the loss are still around 30%, due to inefficient management and control. In order to avoid overflows and to satisfy specific water requirements, the level of instrumentation (e.g. water level measurements and motor-driven gates) and automation in open channel networks increase (see (Mareels *et al.*, 2005) for an overview). On the other hand, the water request, the ecological constraints and the necessary limitations have become more and more important these last years. The instrumentation and the control (by a state feedback for example) allow to improve the management of such systems; nevertheless it is necessary to improve them so as to take into account in a more precise way every event that can occur. In order to deliver water, it is important to ensure that the water level and the flow rate in the open channel remain at given values. The difficulty of this control system is that only the gates positions are able to meet performance specifications: that's why the use of boundary control laws satisfying the control objectives are required. Open surface channels possess a nonlinear complex dynamics because they couple phenomena of transport and phenomena of delay. Those distributed parameters systems have a dynamic represented by hyperbolic Partial Differential Equations (PDE), which depend on time and space: the equations of Saint-Venant. This problem has been previously considered in the literature using a wide variety of technics. See the use of classical linear control theory in (Malaterre *et al.*, 1998; Papageorgiou and Messmer, 1989; Weyer, 2002). Some of them take into account the uncertainties and apply some robust control approaches (see (Litrico and Georges, 1999) e.g.). Others researchers have studied directly the nonlinear dynamics as in (Zaccarian *et al.*, 2007; Litrico *et al.*, 2005; Dulhoste *et al.*, 2001; Dos Santos and Prieur, 2008). Recent approaches have considered the distributed feature of the system. Using the Riemann's coordinates approach on the Saint-Venant equations, stability results are given in (Greenberg and Li, 1984) for a system of two conservation laws and in (Li, 1994) for system of larger dimension. Lyapunov technics have been used in (Coron *et al.*, 2007; Dos Santos *et al.*, 2008; Dos Santos and Prieur, 2008).
In practice, process industries as mining, chemical, water treatment processes are characterized by complex pro-

cesses which often operate in multiple operating regimes (Blesa *et al*., 2010). It is often difficult to obtain nonlinear models that accurately describe plants in all regimes. Also, considerable effort is required for development of nonlinear models. Comparatively, different techniques for linear systems identification, control and monitoring are available. An attractive alternative to nonlinear technique is to use a multiple linear models strategy. The concept of the multiple models or 'Multi-Models' are based on the partitioning of the operating range of a system into separated regions by applying local linear control to each region (Murray-Smith and Johansen, 1997). The Multi-Models structure is well adapted for nonlinear systems because this structure allows to determine a set of linear models defined around some predefined operating points. Each local model (called sub-model) is defined as a Linear Time Invariant (LTI) model dedicated to a specific operating point. The Multi-Models philosophy is based on weighting functions which ensure the transition between the different locals models. These functions represent the degree of validity of each local model. This degree of validity is a function of the system inputs, outputs and time. The Multi-Models approach has often been used for modelling and control of nonlinear systems (Porfirio *et al*., 2003; Athans *et al*., 2005) and for fault diagnosis (Bhagwat *et al*., 2003; Gatzke and Doyle, 2002; Rodrigues *et al*., 2008). Some authors speak about gain scheduling strategy like in (Leith and Leithead, 2000), or Linear Parameter Varying (LPV) systems with the same formalism (Hamdi *et al*., 2011) or interpolated controllers (Banerjee *et al*., 1995) or switching controllers (Narendra *et al*., 1995).

The use of Multi-Models representation for stability study of systems described by nonlinear PDE is something new in the literature for such systems; but some researchers like (Wang *et al*., 2011) have recently developed a control strategy by Takagi-Sugeno models for PDE nonlinear systems with a stability study. More generally, common approaches are based on a finite dimensional approximation of the nonlinear PDE and adaptive control. The stability and the control of such systems in infinite dimension is still an open problem.

In this paper, an analysis of the stability of the nonlinear PDE of Saint-Venant is proposed by the use of the Multi-Models and Internal Model Boundary Control (IMBC) structures. The stability study is performed by Linear Matrix Inequality (LMI) due to the effectiveness for calculating a unique gain solution for multiple models (Lopez-Toribio *et al*., 1999; Rodrigues *et al*., 2007; Dos Santos Martins and Rodrigues, 2011).

The paper is organized as follows: firstly, the Saint-Venant equations are presented as well as the control problem. The Internal Model Boundary Control is explained and the physical constraints are given. Secondly, the linearized systems are developed around a set of equilibrium which depend on the space variable. Their insertion into the LMI formalism are also described into this second part. The third part of the paper is dedicated to the design of an integral feedback gain by LMI which ensures the stability of the system: an integral controller is designed and implemented using a "Lyapunov-LMI" approach. The last section is dedicated to the simulations and the experimentations. The data used are the one of the water channel of Valence. Comparisons between initial experimental results using a PI-controller (done some years ago) and simulations with the presented integral controller using theoretically tuned gain, are realized. New experimentations have been implemented too with these theoretical gains found by the LMI synthesis.

2. Problem statement about channel regulation

Let us consider the following class of water channels represented on the figures (1) and (2), i.e. a reach of an open channel delimited by underflow and/or overflow gates where:

- $Q(x,t)$ is the water flow rate,
- $Z(x,t)$ is the height of water channel,
- L is the length of the reach taken between the upstream $x_{up} = 0$ and the downstream $x_{do} = L$,
- $U_{up} = U_0(t)$, $U_{do} = U_L(t)$ are the opening of the gates at upstream and downstream.

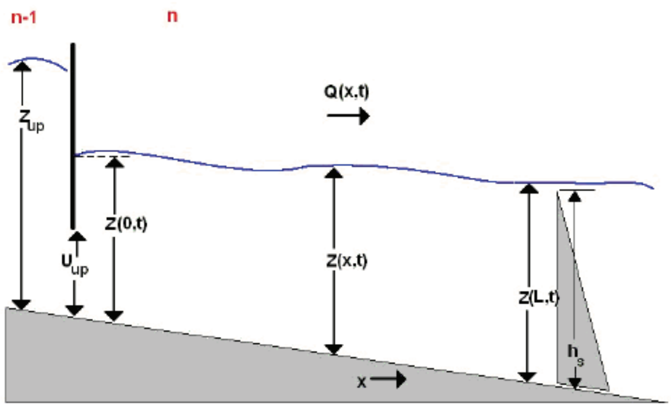

Fig. 1. Channel scheme: upstream underflow and downstream overflow gates

The regulation problem concerns the stabilization of the water flow rate and/or the height of the water around an equilibrium for a reach denoted by $(Z_e(x), Q_e(x))$. A linear model with varying coefficients can be deduced from the nonlinear PDE, in order to describe the variation

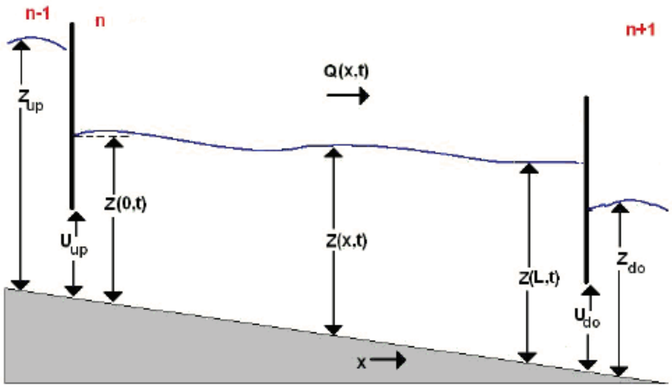

Fig. 2. Channel scheme: two underflow gates

of the water level and flow for an open channel. Let recall these models.

2.1. A model of a reach.

The channel is supposed to have a sufficient length L such that one can consider that the lateral movement is uniform. Nonlinear PDE of Saint-Venant, which describe the flow on the channel, are the following (Georges, 2002):

$$\partial_t Z = -\partial_x \frac{Q}{b}, \quad (1)$$

$$\partial_t Q = -\partial_x \left(\frac{Q^2}{bZ} + \frac{1}{2} gbZ^2\right) + gbZ(I - J), \quad (2)$$

$$Z_0(x) = Z(x, 0), \quad Q_0(x) = Q(x, 0), \quad (3)$$

$\forall x \in \Omega = (x_{up}, x_{do}) = (0, L)$, $t > 0$, where I is the slope, b is the channel width, g is the constant gravity.
J is the friction slope from the formula of Manning-Strickler and R is the hydraulic radius. J and R are defined such that:

$$J = \frac{n^2 Q^2}{(bZ)^2 R^{4/3}}, \quad R = \frac{bZ}{b + 2Z}. \quad (4)$$

The different limits conditions bring us to consider two control cases, $\forall\ x \in \Gamma = \partial\Omega$:

Case a *Single variable control, spillway case*:
The equation of the upstream condition of the reach ($x = x_{up}$) is given by

$$Q(x_{up}, t) = U_{up}(t)\Psi_1(Z(x_{up}, t)), \quad (5)$$

with $\Psi_1(Z) = K_1\sqrt{2g(Z_{up} - Z)}$. The downstream condition of the reach ($x = x_{do}$) is given by the spillway equation (figure (1)):

$$Z(x_{do}, t) = \Psi_2(Q(x_{do}, t)), \quad (6)$$

where

$$\Psi_2(Q) = \left(\frac{Q^2}{2gK_2}\right)^{1/3} + h_s.$$

Z_{up} is the water height at upstream of the gate, K_i is the product of the channel width with the water flow rate coefficient of the gate $n^0 i$, $U_{up}(t)$ is the upstream control. h_s is the height of the spillway considered constant in this case. Let denote that the variable to control is the height $Z(x_{do})$. So, in this case $x_{up} = 0$, $x_{do} = L$, $U_{up} = U_0$ (cf. figure (1)).

Case b *Multi-variable control* :
The upstream condition equation is still the equation (5). Another control can appear at the downstream of the reach, i.e. in $x = x_{do}$ (figure (2)):

$$Q(x_{do}, t) = U_{do}(t)\Psi_3(Z(x_{do}, t)),$$

where $\Psi_3(Z) = K_2\sqrt{2g(Z - Z_{do})}$ and $U_{do}(t)$ is the downstream control of the reach, Z_{do} is the water height downstream of the gate (cf. figure (2)).

- Upstream and downstream depend of the considered reach, it is the same thing for abscissa and gates.
- The case b is considered, i.e. the multi-variable control case.

2.2. A regulation model.

An equilibrium state ($\partial_t(.) \equiv 0$) of the system verifies the following equations:

$$\partial_x Q_e = 0 \quad (7)$$

$$\partial_x Z_e = gbZ_e \frac{I - J_e}{gbZ_e - Q_e^2/bZ_e^2}, \quad (8)$$

Remark 1. The fluvial case is considered and it follows that:

$$Z_e > \sqrt[3]{Q_e^2/(gb^2)}. \quad (9)$$

Let denote that Q_e is constant but that z_e depends of space variable. The linearized model around an equilibrium point $(Z_e(x) \;\; Q_e(x))^t$ is, with

$$\xi(t) = (z(t) \;\; q(t))^t$$

The linearized state variables are:

$$\begin{aligned}
\partial_t \xi(x,t) &= A_1(x)\partial_x \xi(x,t) + A_2(x)\xi(x,t) \quad (10)\\
&= \mathcal{A}(x)\xi(x,t)\\
\xi(x,0) &= \xi_0(x)\\
q(x_{up},t) &= U_{up,e}\partial_z \Psi_1(Z_e(x_{up},t))z(x_{up},t)\\
&\quad + u_{up}(t)\Psi_1(Z_e(x_{up},t)) \quad (11)\\
q(x_{do},t) &= U_{do,e}\partial_z \Psi_3(Z_e(x_{do},t))z(x_{do},t)\\
&\quad + u_{do}(t)\Psi_3(Z_e(x_{do},t)) \quad (12)
\end{aligned}$$

where $U_{up,e}$, $U_{do,e}$ are the openings gates for the upstream and downstream at the equilibrium and $u_{up}(t)$, $u_{do}(t)$ are the variations of these openings gates to be controlled. The matrices $A_1(x)$, $A_2(x)$ are given by:

$$A_1(x) = \begin{pmatrix} 0 & -a_1 \\ -a_2(x) & -a_3(x) \end{pmatrix}, \quad (13)$$

$$A_2(x) = \begin{pmatrix} 0 & 0 \\ a_4(x) & -a_5(x) \end{pmatrix} \quad (14)$$

with $a_1 = 1/b$, $\quad a_2(x) = gbZ_e(x) - \frac{Q_e^2}{bZ_e^2(x)}$, $a_3(x) = \frac{2Q_e}{bZ_e(x)}$, $a_4(x) = gb(I + J_e(x) + \frac{\frac{4}{3}J_e(x)}{1+2Z_e(x)/b})$, $a_5(x) = \frac{2gbJ_e(x)Z_e(x)}{Q_e}$.

The control problem is to find the variations of $u_{up}(t)$ at extremity $x = x_{up}$ and $u_{do}(t)$ at the extremity $x = x_{do}$ of the reach such that downstream water level, $z(x_{do}, t) = z(L, t)$ (measured variables), tracks a reference signal $r(t)$.

The reference signal $r(t)$ is chosen for all cases: constant or non-persistent (a stable step answer of a non-oscillatory system).

In this paper, the control scheme based on the Internal Model Boundary Control (IMBC) (Dos Santos, 2004; Dos Santos *et al.*, 2005) is adopted as illustrated on figure (3). This control strategy integrates the process model in real time and allows to regulate the water height in all the points of the channel by taking into account the error between the linearized model and the real system (or the nonlinear model for the simulations).

- M_f is the linear filtering model of finite dimension which aims at filtering the error signal $e(t) = y_s(t) - y(t)$.

- M_r is the pursuit model which allows to put a dynamic by regards of the fixed reference $r(t)$.

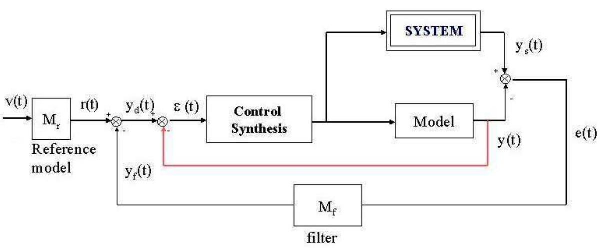

Fig. 3. IMBC structure: Internal Model Boundary Control

2.3. Stability of the system.

The equation (10) describes the dynamic of the system in open loop. In this representation, the state vector $\xi(x,t)$ is not explicitly linked with the boundary control. In order to design an output feedback and to study the closed-loop stability, an operator D of distribution of the boundary control is introduced. It is a bounded operator such that $Im(D) \subset Ker(A)$ and $Du \in D(\mathcal{A})$ and (Dos Santos, 2004; Touré and Rudolph, 2002; Sakawa and Matsushita, 1975):

$$\xi(x,t) \quad = \quad \varphi(x,t) + Du(t). \quad (15)$$

This operator is naturally null in the domain of $A(x)$ as it is active only on the boundary of the domain. This change of variables allows to get a Kalman representation of the system (Touré and Rudolph, 2002; Sakawa and Matsushita, 1975; Alizadeh Moghadam *et al.*, 2011):

$$\partial_t \varphi(x,t) \quad = \quad \mathcal{A}(x)\varphi(x,t) - D\dot{u} \quad (16)$$

$$\varphi(x,0) \quad = \quad \varphi_0(x) = \xi_0(x) - Du(0). \quad (17)$$

It has been proved that the open-loop system described below is exponentially stable (Dos Santos, 2004; Dos Santos and Toure, 2005), as the operator of the linearized system in infinite dimension generates an exponentially stable C_0-semigroup. Moreover, under a PI control $u(t) = \alpha_i\kappa_i \int \varepsilon(s)ds + \alpha_p\kappa_p\varepsilon(t) \in U = \mathbb{R}^n$, $u \in C^\alpha([0,\infty], U)$ [1], conditions on the tuning parameters are also given to ensure the stability of the closed-loop nonlinear system using the IMBC structure and the properties of stability of the closed-loop linearized system, figure (3). For example, some of those conditions are taken for the tuned parameters of the PI-control:

$$0 \le \alpha_i < \alpha_{i,max} = \min_{\lambda\in\Gamma}(a\|R(\lambda;\mathcal{A}_e)\| + 1)^{-1} \quad (18)$$

$$0 \le \alpha_p < \alpha_{p,max} = (\sup_{\lambda\in\Gamma} a\|R(\lambda;\mathcal{A})\|)^{-1} \quad (19)$$

where $\mathcal{A}_e$ is a part of the series development of the closed loop operator (Dos Santos and Toure, 2005), and $R(\lambda; K)$ is the resolvent operator of K, a a constant which depend on $\mathcal{A}_e$.

Those theoretical results have been coupled with simulations and experimentations which have been confirmed this approach (Dos Santos, 2004; Dos Santos and Toure, 2005).

Those experimentations have risen up the limitations due to the linearization around an equilibrium state and a first attempt of a multi-models experimentations have been successfully realized (figure (10)) but it was not optimal and no theoretical proof has been given. The aim of this paper is to develop a first step for this proof.

In order to control the water level over a wide operating range, a set of local models are considered around judicious operating points: each model is an approximation

[1] Regularity coefficient is generally taken as $\alpha = 2$.

of the process in a small interval of the operating range and a control is synthesized and activated on this interval when the system goes trough it. The idea here, is to define necessary conditions to preserve the stability of this system all along the operating range.

2.4. A Multi-Models representation of Saint-Venant's Equations.

The Multi-Models structure like (Rodrigues *et al.*, 2008; Hamdi *et al.*, 2011), allows to control the system over a wide operating range because it takes into account the different sub-models which can be activated under different operating regimes (Murray-Smith and Johansen, 1997). The representation of Saint-Venant's PDE around N operating points by the Multi-Models approach is defined by the following equations:

$$\begin{aligned} \partial_t \xi(x,t) &= \sum_{i=1}^{N} \mu_i(\zeta(t))\mathcal{A}_i(x)\xi(x,t) && (20) \\ \mathcal{A}_i(x) &= A_{1,i}(x)\partial_x + A_{2,i}(x) && (21) \\ \xi_0(x) &= \xi(x,0) \end{aligned}$$

- $\mathcal{A}_i(x)$ is the operator which corresponds to the i^{th} equilibrium state.
- $\zeta(t)$ is a function depending of some decision variables directly linked with the measurable states variables and eventually to the input.
- $\mu_i(\zeta(t))$ is the weighting functions that are based on the output height of the process z_L and determine which sub-model is used for the control law.

A Multi-Models approach can be developed and made possible the study of the stability by the Lyapunov second method.
In the following paragraph, the synthesis of a control law by LMI technics is developed. An output feedback is considered under an integral control and the synthesis of the gain by LMI technics ensures the stability of the system.

3. Stability study by LMI

In this part, the closed-loop structure (figure (3)) is studied under an integral feedback. The pursuit model (M_r) and filtering model (M_f) are not considered. The choice of an integrator can be justified by the fact that the derivative of the control $\dot{u}$ appears in the state equation (16).

3.1. Closed-loop structure for an integral feedback.

For a control with an output feedback, K is defined as the gain, $u(t) = K \int \varepsilon(t)dt$, it follows that (Dos Santos, 2004):

$$\begin{aligned} \varepsilon(t) &= r(t) - y(t) && (22) \\ u(t) &= K \int [r(\tau) - y(\tau)]d\tau && (23) \end{aligned}$$

with $y(t) = C(\xi(x,t) + Eq(x,t))$ where $Eq(x,t) = \sum_{i=1}^{N} \mu_i(\zeta(t))(z_{e,i}(x) \;\; q_{e,i})^t$ the equilibrium state and for example $CEq(x,t) = \sum_{i=1}^{N} \mu_i(\zeta(t))z_{e,i}(L)$ if the aim is to regulate the water level at $x = L$. From equation (15), one deduces:

$$y(t) = C\varphi(x,t) + CEq(x,t) + CDu(t) \qquad (24)$$

and by replacing $y(t)$ into the control equation:

$$\begin{aligned} u(t) = K \int & [r(\tau) - CEq(x,\tau) \\ & -C\varphi(x,\tau) - CDu(\tau)]\, d\tau \\ \Rightarrow \dot{u}(t) = K & [r(t) - CEq(x,t) - C\varphi(x,t) - CDu(t)] \end{aligned}$$

and $\dot{u}$ into the equation (16), the closed-loop expression is then

$$\begin{aligned} \partial_t \varphi(x,t) &= \sum_{i=1}^{N} \mu_i(\zeta(t)) \big[\mathcal{A}_i(x)\varphi(x,t) \\ & -DK\left(r(t) - CEq(x,t) - C\varphi(x,t) - CDu(t)\right) \big] \\ &= \sum_{i=1}^{N} \mu_i(\zeta(t)) \big[\left(\mathcal{A}_i(x) + DKC\right)\varphi(x,t) \\ & +DK\left(CDu(t) + CEq(x,t) - r(t)\right) \big] && (25) \end{aligned}$$

Let define:

$$\tilde{K} = DK \qquad (26)$$

The equation (25) can be written as

$$\begin{aligned} \partial_t \varphi(x,t) &= \sum_{i=1}^{N} \mu_i(\zeta(t))[(\mathcal{A}_i(x) + \tilde{K}C)\varphi(x,t) \\ & +\tilde{K}(CDu(t) + CEq(x,t) - r(t))] = \mathcal{M}_i(x,t) && (27) \end{aligned}$$

The stability conditions are ensured by using a quadratic Lyapunov function as in (Rodrigues *et al.*, 2007; Hamdi *et al.*, 2011) in order to guarantee the convergence of the water height to the reference $r(t)$ over the widest operating range.

3.2. Stability study with a quadratic Lyapunov function.

Let us consider:

$$V(\varphi(x,t),t) = < \varphi(x,t), P\varphi(x,t) > \qquad (28)$$

where $< .,. >$ is the inner product considered. The Multi-Models representation of the linearized PDE of Saint-Venant defined by equation (27) is asymptotically stable if there exists a matrix/operator $P > 0$ such that[1]:

For finite dimension systems:

$$\dot{V}(\varphi(x,t),t) < 0, \tag{29}$$

$$\Leftrightarrow < \dot{\varphi}, P\varphi > + < \varphi, P\dot{\varphi} >< 0, \tag{30}$$

For infinite dimension systems:

$$< \dot{\varphi}, P\varphi > + < P\varphi, \dot{\varphi} >= - < \varphi, \varphi > \tag{31}$$

Remark 2. As told previously, the authors want to give first elements of a proof of the stability using LMI technics usually defined for finite dimension systems but applied to infinite dimension systems in this paper. The authors in (Hante and Sigalotti, 2010) have worked on the stability of switching systems in infinite dimension and they are still working to link those mathematical results to the LMI approach. The tools for LMI technics developed in infinite dimension have not been found by the authors and we try to developed them making a parallel with known technics. The authors (Wang *et al.*, 2011) have also used LMI technics for the stability study.
The previous results, like in the example part, can be put in parallel with the following theorem (Curtain and Zwart, 1995):

Theorem 1. (S)*uppose that A is the infinitesimal generator of the C_0-semigroup $T(t)$ on the Hilbert space Z. Then $T(t)$ is exponentially stable if and only if there exists a positive operator $P \in \mathcal{L}$ such that (with $\dot{z} = Az$)*

$$< Az, Pz > + < Pz, Az >= - < z, z >, \forall z \in Z.$$

□

The main difference here between this stability result in finite and infinite dimension, is located in the inequality of the Lyapunov function for finite dimension system and equality for infinite one. This equality complexity can be removed in some cases: for example for operators with compact resolvent (Triggiani, 1975; Dos Santos, 2004; Curtain and Zwart, 1995) and in this case the same inequality from finite dimension is a sufficient and necessary condition for the infinite dimension case. Indeed, the inequality from finite dimension can not be transposed directly in infinite dimension because the spectral growth assumption is not satisfied in general (it is, in finite dimension) i.e. an operator A generator of a C_0-semigroup $T_A(t)$ satisfies the spectral growth assumption if:

$$\sup\{\mathcal{R}(\lambda); \lambda \in \sigma(A)\} = w_0(A) := \inf_{t>0} \frac{\|T_A(t)\|}{t}$$

So, if the spectral growth assumption is satisfied, and if there exists a positive operator $P \in \mathcal{L}$ such that (with $\dot{z} = Az$)

$$< Az, Pz > + < Pz, Az > \quad < 0, \forall z \in Z \tag{32}$$

[1]We suppose that $\partial_t \psi = \dot{\psi}$ whatever the function ψ.

then $T(t)$ is exponentially stable.
For the Saint-Venant equations, it has been shown that the operator has a compact resolvent (Dos Santos and Toure, 2005; Dos Santos, 2004) so it satisfies the spectral growth assumption.

Then, taking account of (27)-(32), it follows that one has to prove this inequality:

$$< \mathcal{M}_i, P\varphi > + < \varphi, P\mathcal{M}_i > \quad < 0 \tag{33}$$

The development of this inequality leads us to consider an inequality for each sub-system of index i such that:

$$\begin{aligned} & < [\mathcal{A}_i(x) + \tilde{K}C]\varphi(x,t), P\varphi(x,t) > \\ & + < \tilde{K}(CDu(t) - r(t) + CEq(x,t)), P\varphi(x,t) > \\ & + < \varphi(x,t), P\tilde{K}(CDu(t) - r(t) + CEq(x,t)) > \\ & + < \varphi(x,t), P[\mathcal{A}_i(x) + \tilde{K}C]\varphi(x,t) >< 0 \end{aligned} \tag{34}$$

In the inequality (34), which defines the stability condition of the system, the control parameter u appears in this inequality and it is a difficulty for the design of the gain $\tilde{K}$. Let us consider the following equality deduced from (15):

$$CDu(t) - r(t) = C\xi(x,t) - r(t) - C\varphi(x,t) \tag{35}$$

Proposition 1. (I) *f there exists a matrix P positive definite, a matrix W and a scalar α such that the following statements hold true:*

$$a) < \varphi, P\widetilde{K}\,(CDu(t) + CEq(x,t) - r(t)) > \tag{36}$$
$$\leq \alpha\varphi^T P\widetilde{K}C\varphi,$$

$$b)\ \mathcal{A}_i^T P + P\mathcal{A}_i + WC + C^T W^T < 0, \tag{37}$$

with $\tilde{K} = \frac{1}{1+\alpha} P^{-1}W$. Then, the system (16) with an integral control input (23) is stable. ■

Proof. Let consider the quadratic Lyapunov function

$$V(\varphi(x,t),t) =< \varphi, P\varphi >= \varphi^T P\varphi$$

then, one can wrote $\dot{V}(t) < 0$ such that (34) can be upper bounded. Indeed inequality (36) implies that

$$\begin{aligned} & \varphi^T P\left[(\mathcal{A}_i + \widetilde{K}C)\varphi + \widetilde{K}(CDu - r)\right] \\ & \quad \leq \varphi^T P\left[(\mathcal{A}_i + \widetilde{K}C)\varphi + \alpha\widetilde{K}C\varphi\right] \\ & \quad \leq \varphi^T P\left[\mathcal{A}_i\varphi + \widetilde{K}C\varphi(1+\alpha)\right] \end{aligned} \tag{38}$$

So with the last consideration, the inequality (34) can be then upper bounded by:

$$\begin{aligned}&\varphi^T(x,t)[\mathcal{A}_i(x)+\tilde{K}C]^T P\varphi(x,t)\\&\quad+\varphi^T(x,t)P[\mathcal{A}_i(x)+\tilde{K}C]\varphi(x,t)\\&\quad+[\tilde{K}(CDu(t)-r(t)+CEq(x,t))]^T P\varphi(x,t)\\&\quad+\varphi^T(x,t)P[\tilde{K}(CDu(t)-r(t)+CEq(x,t))]\\&\quad\le\varphi^T(x,t)\Big\{[\mathcal{A}_i(x)+(1+\alpha)\tilde{K}C]^T P\\&\quad+P[\mathcal{A}_i(x)+(1+\alpha)\tilde{K}C]\Big\}\varphi(x,t)\qquad(39)\\&=\varphi^T\left[\mathcal{A}_i^T P+P\mathcal{A}_i+WC+C^TW^T\right]\varphi<0\end{aligned}$$

with $\tilde{K}=\frac{1}{1+\alpha}P^{-1}W$. Now, let discuss about the inequality (36);

$$\varphi^T P\tilde{K}\left(CDu(t)+CEq(x,t)-r(t)\right)\le\alpha\varphi^T P\tilde{K}C\varphi.$$

Let remember that the system is exponentially stable in open-loop and for a PI-controller in closed-loop, with gains correctly tuned (Dos Santos, 2004; Dos Santos and Toure, 2005) for a time t well chosen, so one can assume that $\exists\, k>0$ such that

$$\mid C\xi(x,t)+CEq(x,t)-r(t)\mid\le k\mid C\varphi(x,t)\mid.\quad(40)$$

Let pose $\varepsilon_{f(t)}=sign(f(t))$, then

$$\begin{aligned}&CDu(t)-(r(t)-CEq(x,t))\\&\quad=C\xi(x,t)-(r(t)-CEq(x,t))-C\varphi(x,t)\\&\mid CDu-r+CEq(x,t)\mid\le\mid C\xi-(r-CEq)\mid+\mid C\varphi\mid\end{aligned}$$

That is to say that one can bound $(CDu-r)$ by:

$$\begin{aligned}&-(k+1)\varepsilon_{(C\varphi)}C\varphi\le CDu-r\le(k+1)\varepsilon_{(C\varphi)}C\varphi\\&-(k+1)\varepsilon_{(C\varphi)}\varepsilon_{(\varphi^TP\tilde{K})}\varphi^TP\tilde{K}C\varphi\le\varphi^TP\tilde{K}\left(CDu-r\right)\\&\quad\le(k+1)\varepsilon_{(C\varphi)}\varepsilon_{(\varphi^TP\tilde{K})}\varphi^TP\tilde{K}C\varphi\end{aligned}$$

and

$$\Rightarrow\varphi^TP\tilde{K}\left(CDu-r\right)\le(k+1)\varepsilon_{(\varphi^TP\tilde{K}C\varphi)}\varphi^TP\tilde{K}C\varphi$$

So, the inequality (36) is proved and

$$\begin{aligned}&\varphi^TP[\mathcal{A}_i+\tilde{K}C]\varphi+\varphi^TP[\tilde{K}(CDu-r)\varphi\qquad(41)\\&\quad\le\varphi^TP\left[\mathcal{A}_i+(1+(k+1)\varepsilon_{(\varphi^TP\tilde{K}C\varphi)})\tilde{K}C\right]\varphi\end{aligned}$$

then, one get the gain $\tilde{K}=\frac{1}{1+\alpha}P^{-1}W$ as

$$\varphi^TP[\mathcal{A}_i+\tilde{K}C]\varphi+\varphi^TP[\tilde{K}(CDu-r)\varphi$$
$$\begin{cases}\le\varphi^TP\left[\mathcal{A}_i+(k+2)\tilde{K}C\right]\varphi\\\quad\text{if }\varepsilon_{(\varphi^TP\tilde{K}C\varphi)}=1,\ \text{with }\alpha=k+1\\\le\varphi^TP\left[\mathcal{A}_i+k\tilde{K}C\right]\varphi\\\quad\text{if }\varepsilon_{(\varphi^TP\tilde{K}C\varphi)}=-1,\ \text{with }\alpha=-k.\end{cases}\qquad(42)$$

■

Remark 3. The solution of Proposition (1) may lead to a conservatism due to the fact that a unique gain has been determined for all the models. However, it is here a first attempt into infinite dimension and the use of LMI applied to such PDE systems is very recent.
Now, the gain $\tilde{K}$ has been implemented into the discretized model of simulations so as to verify the stability of the system. The results have been obtained for a single reach with two underflow gates. The aim is to compare the simulations and experimentations curves obtained with this method and the ones obtained experimentally by Dos Santos Martins in previous works (Dos Santos, 2004; Dos Santos *et al.*, 2005).

4. Simulations and experimentations results

Firstly, let's describe the benchmark used for the simulations and the experimentations which are presented in the second and third subsections respectively.

4.1. Configuration and data of the channel.

An experimental validation has been performed on the Valence micro-channel, figures (4) and (5), Tab.1. This pilot channel is located at ESISAR[2] /INPG[3] engineering school in Valence (France). It is operated under the responsibility of the LCIS[4] laboratory. This experimental channel (total length=8 meters) has an adjustable slope and a rectangular cross-section (width=0.1 meter). The channel is ended at its downstream by a variable overflow spillway and equipped with three underflow control gates (figures (4) and (5)). Ultrasound sensors provide water level measurements at different locations of the channel (figure (6)). Note that water flow is deduced from the gate equations and has not been measured directly.

parameters	$B(m)$	$L\ (m)$	$K\ (m^{1/3}.s^{-1})$
values	0.1	7	97
parameters	μ_0	μ_L	slope $(m.m^{-1})$
values	0.6	0.73	1.6 ‰

Table 1. Parameters of the channel of Valence

For all numerical simulations, the Chang and Cooper theta-scheme of order 2 is used (Cordier *et al.*, 2004). To validate this numerical discretization, comparisons between the numerical simulations with real data and with numerical simulations using the Preissmann scheme (which is used in other works dealing with the control of flows (Litrico and Georges, 1999; Ouarit *et al.*, 2003)) were done in (Dos Santos and Prieur, 2008). They validated the numerical discretization and the identification

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Fig. 4. Pilot channel of Valence

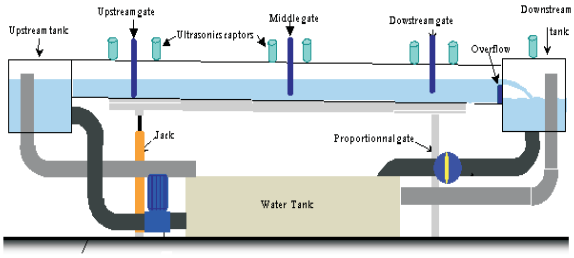

Fig. 5. Pilot channel of Valence

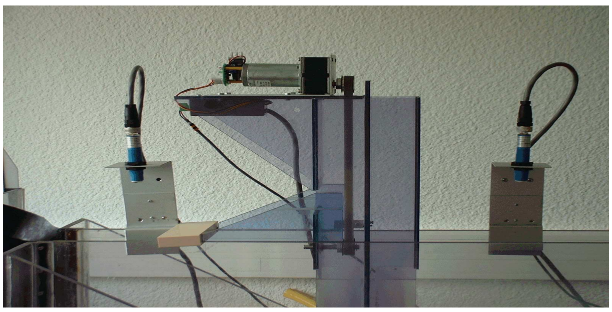

Fig. 6. Pilot channel of Valence: gate and ultrasound sensors

of the parameters. It also has been done for the micro-channel (figure (7)).

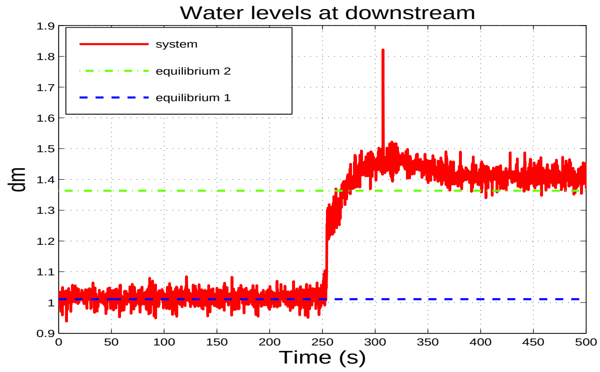

Fig. 7. Pilot channel of Valence: open loop identification

The experimental data depicted below have been filtered to get a better idea of the experimentation results. One reach has a length of $0.7m$, the water level at upstream of the first gate is $Z_{up} = 0.172m$ and at downstream of the second gate $Z_{do} = 0.085m$ (theoretical values).
For this study, the following set of parameters from the practical Valence's channel (figure (5)) is considered where the data are defined such that:

- $n = 20$ is the number of the discretized points,
- Z_L is the water height to regulate.

In this single reach with two gates, the regulation of the water height Z_L at $x = L$, is done by controlling the openings $U_0(t)$ and $U_L(t)$ of the gates at upstream and downstream respectively: it is a multi-variable control (Cf figure (2)).
The equilibria profiles have been chosen such that the calculated control law from the local models can be efficient over all the operating range of the water height (Dos Santos, 2004). Let notice that it has been experimentally verified that a local model is valid around $\pm 20\%$ of a water level equilibrium profile, i.e. the model and the data have the same behavior and values. In order to assign references which are included between $0.06m$ and $0.2m$, the operating points at $x = 0$ are the following:

Table 2. Initial set points for the simulations and the experimentations

Simulations		Experimentations	
$z_{e1}(x=0)$	$0.062m$		
$z_{e2}(x=0)$	$0.077m$	$z_{e1}(x=0)$	$0.062m$
$z_{e3}(x=0)$	$0.099m$	$z_{e2}(x=0)$	$0.094m$
$z_{e4}(x=0)$	$0.135m$	$z_{e3}(x=0)$	$0.141m$
$z_{e5}(x=0)$	$0.18m$		

The efficiency of the computer managing the D-Space card can not bear more than three equilibrium states as it is working on a Windows 95 version. So the numbers of them towards the simulations had to be reduced.
In this application, the weighting function $\mu_i(\zeta(t))$ is equal to 1 if the output's height is included into the validity domain of the model, otherwise it is equal to 0. The parameter $\zeta(t)$ exclusively depends on the output which is the only one variable of decision in this precise case.

4.2. Simulations.

These results are obtained from an IMBC Control and a Multi-Models approach with a LMI gain previously calculated in the previous section. The figure (8) shows that the output $Z(L)$ converges to the reference even if this one strongly varies (variations $> 100\%$). The reference tracks a slow dynamic and one can see that the convergence of the output is good.

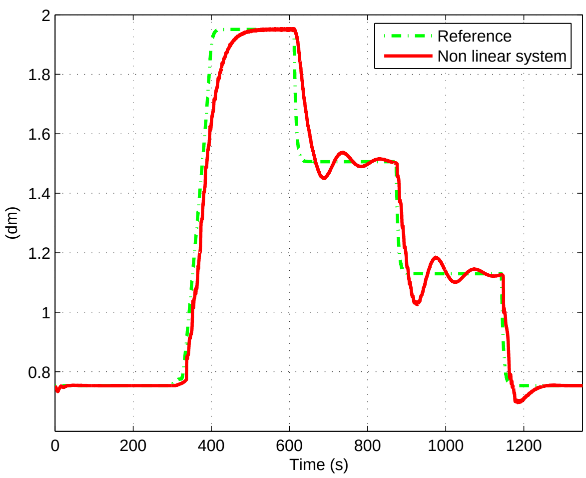

Fig. 8. Variations of the reference greater than 100%

The curves that describe the upstream and downstream gates openings of the reach are given by the figure (9). The convergence of the output to the reference is ensured even when the reference is decreasing or increasing.

Next simulations are a first comparison between simulations using the theoretical gain obtains through LMI approach with the first tests realized some years ago by (Dos Santos *et al.*, 2005), using an experimental Multi-Models gain, without any theoretical study. The figure (10) represents the dynamic evolution of the simulated system and the experimental data. Note that the references curves are equals, the experimental one stands for the signal $v(t)$, the simulation one stands for the signal $r(t)$ after the reference model M_r (cf. figure (3)). The figure (11) compares the dynamic of the gate openings. The dynamic of the gates as of the water level

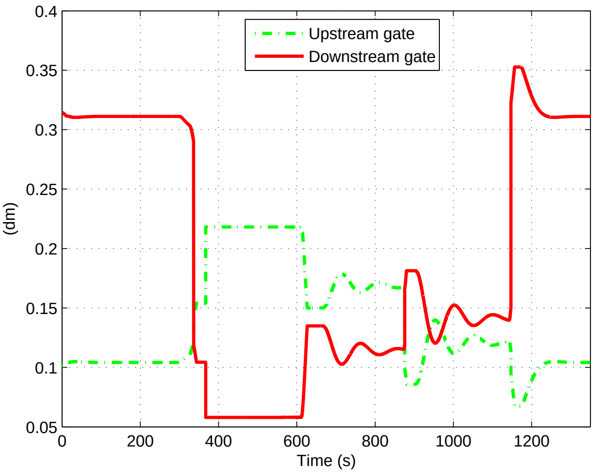

Fig. 9. Gates opening

are similar and so are promising.

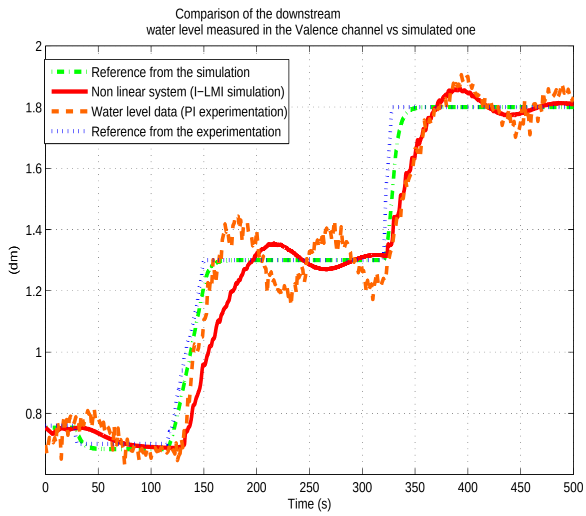

Fig. 10. Comparison of the downstream water level measured in the Valence channel with the first Multi-Models approach in 2004 versus the simulated one with the LMI approach

One can observe on figure (10), that the convergence are better than the one obtained experimentally and the overshoot is less too. But the rising time is obviously slowly (at time $t = 120 - 200s$ example given) with the integral controller (simulation) versus the PI controller (experimentation). The next step is to design a new PI controller using LMI, to compare it to the experimental PI. It is actually under study.

Remark 4. The reference level of the channel is limited by physical constraints: the minimum is obtained with the maximum between the water height of the downstream reach and the fluvial condition, obtained from the initial model (9). The maximum from the size of the channel itself. In these simulations, the critical water height from fluvial constraint is $z_{ec} = 0.0369m$.

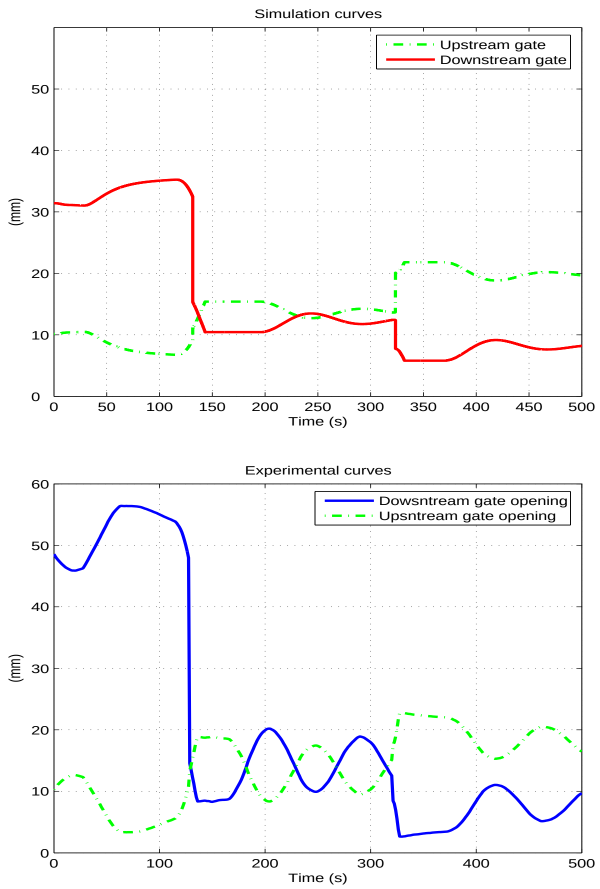

Fig. 11. Gates opening

4.3. Experimentations.

These experimentations have been realized into the Valence channel (figure (5)) with a Multi-Models approach and a gain calculated via the LMI approach. In the experimentations figures (12)-(13), the wide range of the accessible water level is attempt. Let remark that some problems appear with the gates as sometimes they stay jammed because of the friction (e.g. here at times $t = 50s$, $t = 120s$, $t = 550s$ and $t = 625s$). Those problems act like perturbations and the integral controller try to compensate them.

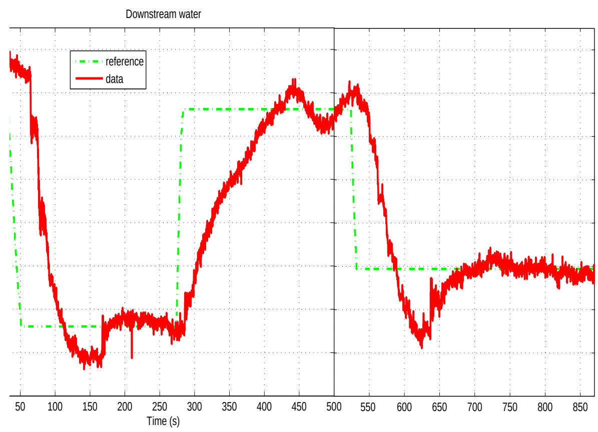

Fig. 12. Valence micro channel: Downstream water level

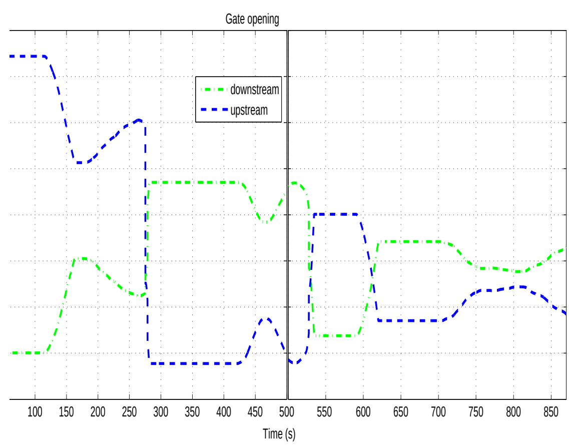

Fig. 13. Valence micro channel: Gates openings

The convergence of the downstream water level is ensured in spite of the perturbations, but it necessary to improve the controller to take them into account.

Those experimental results are relevant and promising towards the applicability of our approach. Nevertheless, some improvements have to be done and a first step is to extend first results to a PI controller, then to get the robustness by this way.

5. Acknowledgments

Thanks to professor E. Mendes and the LCIS for allowing us to realize our experimentations on the micro-channel.

6. Conclusion

First attempts of a multi-models approach on irrigation channels control, through an IMBC structure, have been realized some years ago (Dos Santos, 2004; Dos Santos *et al.*, 2005). Good experimental results, but without theoretical approach, were obtained and have shown promising results. In this paper, the authors have formalized a LMI approach of the problem and given first theoretical results in order to tune the new feedback gain trough LMI in the case of an integral controller. Simulations have shown the improvements realized towards the initial multi-models approach and new experimentations have confirmed the new theoretical tuning of the gain. This paper has allowed to find some stability results of infinite dimension systems with LMI tools for finite dimension systems.

Extensions to a PI-controller are actually in study and preliminary results have been published (Dos Santos Martins and Rodrigues, 2011). The complexity is here located in the fact that it is a boundary control and by its distribution on the state, that the control appears in a derivative form. Further experimentations are planned for a new PI-controller and thus comparisons with the same controller have been and are already performed.

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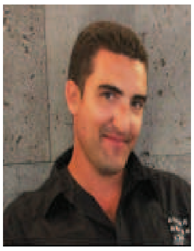

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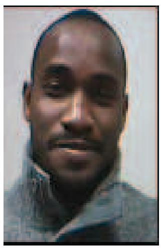

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