Classification of symmetric special biserial algebras with at most one non-uniserial indecomposable projective

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CLASSIFICATION OF SYMMETRIC SPECIAL BISERIAL ALGEBRAS WITH AT MOST ONE NON-UNISERIAL INDECOMPOSABLE PROJECTIVE

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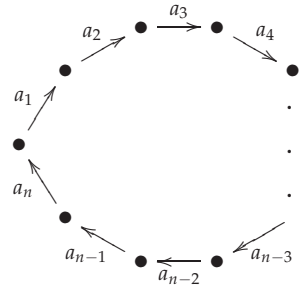
ABSTRACT. We consider a natural generalisation of symmetric Nakayama algebras, namely, symmetric special biserial algebras with at most one non-uniserial indecomposable projective module. We describe the basic algebras explicitly by quiver and relations, then classify them up to derived equivalence. This includes the algebras of [5], where they study the weakly symmetric algebras of Euclidean type.

INTRODUCTION

Let K be an algebraically closed field. We consider in this paper a generalisation of symmetric Nakayama K-algebras. A symmetric Nakayama K-algebra is a symmetric K-algebra A such that all indecomposable projective modules are uniserial. These algebras are well-known and have been classified up to Morita equivalence: every symmetric Nakayama algebra is Morita equivalent to exactly one algebra N_m^n defined by the quiver

$\Delta_n =$ 

and the ideal of relations L_m in $K\Delta_n$ generated by all paths of length $nm + 1$. Note that in particular, the basic algebra associated to A is special biserial.

Our aim is to describe the basic indecomposable finite-dimensional K-algebras A which are symmetric special biserial algebras with at most one non-uniserial indecomposable projective module. These algebras include the symmetric Nakayama algebras, as well as certain algebras in [5] that occur in the classification, up to derived equivalence, of all weakly symmetric algebras of Euclidean type. In this paper we also distinguish, up to derived equivalence, the basic indecomposable finite-dimensional symmetric special biserial algebras which have at most one non-uniserial indecomposable projective module. It is well-known that all special biserial algebras are tame. Morever, it was proved by Al-Nofayee in [1] (and by Rickard [12] for the symmetric case) that if A and B are derived equivalent algebras, then A is selfinjective if and only if B is selfinjective. The algebras given in [5] are Brauer graph algebras, and we recall that Brauer tree algebras play an important role in the Morita equivalence classification of blocks of group algebras of finite type (see [2, 4]). We use the theory of generalised Brauer tree algebras as part of the classification of our algebras up to derived equivalence. We refer the reader also to [13], where Skowroński discusses the extensive programme to determine the derived equivalence classes of all tame selfinjective algebras.

We begin this paper with some background and properties about basic symmetric algebras, so that, in Section 2, we can describe by quiver and relations all basic indecomposable finite-dimensional algebras which are symmetric special biserial algebras with at most one non-uniserial indecomposable projective module. In order to distinguish our algebras up to derived equivalence we use several invariants including Hochschild cohomology which we discuss in

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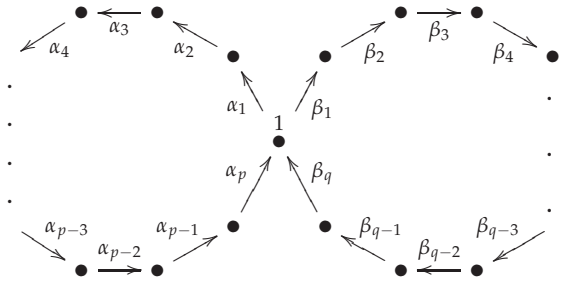
Section 3. The final section contains the full classification of our algebras up to derived equivalence, and in addition to the dimensions of the Hochschild cohomology groups, we use Cartan invariants (see [6, Proposition 1.5] for a proof of derived invariance) and Külshammer invariants (or generalised Reynolds ideals, whose derived invariance was proved in [15]).

We assume throughout that A is a basic indecomposable finite-dimensional algebra over the algebraically closed field K so that A is isomorphic to $K\mathcal{Q}/I$ for some unique connected quiver $\mathcal{Q}$ and admissible ideal I of $K\mathcal{Q}$. We let $\mathrm{rad}(A)$ denote the Jacobson radical of A.

For any two positive integers p and q with $p \leqslant q$, we define the quiver $\mathcal{Q}_{(p,q)}$ to be the quiver formed of two oriented cycles, of lengths p and q respectively, joined at one vertex labelled 1, as follows:

$\mathcal{Q}_{(p,q)}$:
$1 \leqslant p \leqslant q$



We denote the trivial path at the vertex i by e_i. Paths are written from left to right. We write $\mathfrak{o}(\alpha)$ for the trivial path corresponding to the origin of the arrow α and write $\mathfrak{t}(\alpha)$ for the trivial path corresponding to the terminus of the arrow α. The vertices of the quiver $\mathcal{Q}_{(p,q)}$ are labelled by $1, \ldots, p+q-1$, in such a way that $\mathfrak{o}(\alpha_i) = i$ for $i = 1, \ldots, p$, and $\mathfrak{t}(\beta_j) = p + j$ for $j = 1, \ldots, q-1$. Thus $\mathfrak{t}(\alpha_i) = i+1$ for $i = 1, \ldots, p-1$, $\mathfrak{t}(\alpha_p) = 1$, $\mathfrak{o}(\beta_1) = 1$ and $\mathfrak{o}(\beta_j) = p + j - 1$ for $j = 2, \ldots, q$.

Set $\gamma = \alpha_1\alpha_2\cdots\alpha_p$ and $\delta = \beta_1\beta_2\cdots\beta_q$. We define the following two admissible ideals in $K\mathcal{Q}_{(p,q)}$:

(a) for a positive integer r, let I_r be the ideal generated by

$$\begin{array}{l} \alpha_p\alpha_1, \qquad \beta_q\beta_1, \qquad (\gamma\delta)^r - (\delta\gamma)^r, \\ \alpha_i\cdots\alpha_p(\delta\gamma)^{r-1}\delta\alpha_1\cdots\alpha_i \text{ for all } 2 \leqslant i \leqslant p-1, \\ \beta_j\cdots\beta_q(\gamma\delta)^{r-1}\gamma\beta_1\cdots\beta_j \text{ for all } 2 \leqslant j \leqslant q-1; \end{array}$$

(b) for a pair of positive integers (s,t), let $J_{(s,t)}$ be the ideal generated by

$$\begin{array}{l} \alpha_p\beta_1, \qquad \beta_q\alpha_1, \qquad \gamma^s - \delta^t, \\ \alpha_i\cdots\alpha_p\gamma^{s-1}\alpha_1\cdots\alpha_i \text{ for all } 2 \leqslant i \leqslant p-1, \\ \beta_j\cdots\beta_q\delta^{t-1}\beta_1\cdots\beta_j \text{ for all } 2 \leqslant j \leqslant q-1, \end{array}$$

where, if $p = 1$ then $s \geqslant 2$, and, if $q = 1$ then $p = 1, s \geqslant 2$ and $t \geqslant 2$.

The algebras considered in [5] are special cases of these algebras. Specifically $A(p,q) = K\mathcal{Q}_{(p,q)}/I_1$ so that $r = 1$, and $\Lambda(n) = K\mathcal{Q}_{(1,n)}/J_{(2,2)}$ so that $p = 1, q = n$ and $s = 2 = t$.

1. Background

The following result and especially its consequences will be used repeatedly. They are given in [3], but we include the proofs here for completeness.

Proposition 1.1. *Let $\mathcal{Q}$ be a quiver and let I be an admissible ideal in $K\mathcal{Q}$ such that $A = K\mathcal{Q}/I$ is a symmetric algebra. Let ρ be a path in $\mathcal{Q}$ with $\rho \neq 0$ in A. Then there exists a cycle $\rho\rho_1$ in $\mathcal{Q}$ with $\rho\rho_1$ and $\rho_1\rho$ non-zero in A.*

Proof. Since A is a symmetric algebra there exists a symmetric form $f : A \to K$ on A whose kernel contains no non-zero left or right ideals of A. Then ρA is not contained in $\mathrm{Ker}\, f$ so there exists a path ρ_1 such that $f(\rho\rho_1) \neq 0$. In particular, $\rho\rho_1 \neq 0$ and $\mathfrak{t}(\rho) = \mathfrak{o}(\rho_1)$. Moreover, since f is symmetric, $f(\rho_1\rho) = f(\rho\rho_1) \neq 0$ so $\rho_1\rho \neq 0$. Therefore $\mathfrak{t}(\rho_1) = \mathfrak{o}(\rho)$. Hence $\rho\rho_1$ and $\rho_1\rho$ are cycles in $\mathcal{Q}$ which are non-zero in A. □

Corollary 1.2. *Let $A = K\mathcal{Q}/I$ be an indecomposable finite-dimensional symmetric special biserial algebra that is not isomorphic to $N_1^1 \cong K[X]/(X^2)$. Then:*

(1) For any arrow α, there is a unique arrow α' such that $\alpha'\alpha \neq 0$ and a unique arrow α'' such that $\alpha\alpha'' \neq 0$.
(2) For any vertex v in $\mathcal{Q}$, the number of arrows that start at v is equal to the number of arrows that end at v, and this number is either 1 or 2.

Proof. The second statement follows easily from the first and the definition of a special biserial algebra. Here we prove the first statement.

First suppose that α is an arrow which is not a loop. Then α is a non-zero path, so, by Proposition 1.1, there exists a path $\rho \neq \alpha$ such that $\alpha\rho$ and $\rho\alpha$ are non-zero cycles. Therefore we can take α' to be the last arrow in ρ and α'' the first arrow in ρ. The uniqueness of these arrows follows from the definition of a special biserial algebra.

Now suppose that α is a loop. Assume for contradiction that $\alpha\beta = 0$ for every arrow β in $\mathcal{Q}$. Then α is in the socle of the indecomposable projective module $\mathfrak{o}(\alpha)A$. If no other arrow starts at $\mathfrak{o}(\alpha)$ then, since A is indecomposable and $\alpha^2 = 0$, we get $A \cong K[X]/(X^2)$, a contradiction. Therefore there is another arrow ρ with $\mathfrak{o}(\alpha) = \mathfrak{o}(\rho)$. Choose a path σ which is maximal with the property $\rho\sigma \neq 0$ so that $\rho\sigma$ is in the socle of $\mathfrak{o}(\alpha)A$. Since A is a self-injective algebra, $\mathrm{soc}(\mathfrak{o}(\alpha)A)$ is one-dimensional so that there exists a non-zero $c \in K$ such that $\alpha = c\rho\sigma$, which is a contradiction since the ideal I is admissible. Therefore there exists an arrow α' with $\alpha'\alpha \neq 0$. The proof of the existence of the other arrow α'' is similar. □

2. Classification theorem

Our main theorem is Theorem 2.2 where we classify, by quiver and relations, all basic indecomposable finite-dimensional symmetric special biserial algebras with at most one non-uniserial indecomposable projective module.

Proposition 2.1. *The algebras $K\mathcal{Q}_{(p,q)}/I_r$ and $K\mathcal{Q}_{(p,q)}/J_{(s,t)}$ are symmetric special biserial algebras with at most one non-uniserial indecomposable projective module.*

Proof. It is easy to see that these algebras are special biserial and that all but one of each of their indecomposable projective modules are uniserial. Moreover, the algebras are weakly symmetric, that is, the top and the socle of each indecomposable projective module are isomorphic. It remains to prove that the algebras are symmetric.

For $A = K\mathcal{Q}_{(p,q)}/I_r$, the socle of A has a K-basis consisting of $(\delta\gamma)^r$, $\alpha_i \cdots \alpha_p(\delta\gamma)^{r-1}\alpha_1 \cdots \alpha_{i-1}$ and $\beta_j \cdots \beta_q(\gamma\delta)^{r-1}\beta_1 \cdots \beta_{j-1}$ for $2 \leqslant i \leqslant p$ and $2 \leqslant j \leqslant q$, that is, all the paths obtained from cyclic permutations of $(\gamma\delta)^r$, where we recall that $\gamma = \alpha_1\alpha_2 \cdots \alpha_p$ and $\delta = \beta_1\beta_2 \cdots \beta_q$. Complete this K-basis of $\mathrm{soc}(A)$ with paths in $K\mathcal{Q}_{(p,q)}$ to obtain a basis of A, and define $f : A \to K$ on this basis by sending the elements in the socle to 1 and the others to 0. Then it follows from [10, Proposition 3.1], that $\mathrm{Ker}\, f$ contains no non-zero left or right ideals of A. Moreover, f is clearly symmetric, since the paths on which it is non-zero are all the cyclic permutations of a single path. Thus A is a symmetric algebra.

For $A = K\mathcal{Q}_{(p,q)}/I_{(s,t)}$ the argument is similar, but this time the socle of A is generated by all the cyclic permutations of the two paths γ^s and δ^t. □

We now have the following result.

Theorem 2.2. *Let A be a basic indecomposable finite-dimensional symmetric special biserial algebra with at most one non-uniserial indecomposable projective module. Then A is isomorphic to a Nakayama algebra N_m^n for some positive integers m and n, or to $K\mathcal{Q}_{(p,q)}/I_r$ for some positive integers p, q and r, or to $K\mathcal{Q}_{(p,q)}/J_{(s,t)}$ for some positive integers p, q, s and t.*

Proof. Set $A = K\mathcal{Q}/I$ for some quiver $\mathcal{Q}$ and some admissible ideal I. It is already known that a symmetric (special biserial) algebra with no non-uniserial indecomposable projective module is isomorphic to a Nakayama algebra N_m^n. We may therefore assume that all except one indecomposable projective A-modules are uniserial. Consequently, using Corollary 1.2(2), we must have one vertex that is the end point of exactly two arrows and the starting point of

exactly two arrows, which we label 1, and the other vertices are the end point of exactly one arrow and the starting point of exactly one arrow. Therefore the quiver of A must be $\mathcal{Q}_{(p,q)}$ for some positive integers p, q. Without loss of generality we may assume that $p \leqslant q$.

Now consider the composition $\alpha_p\alpha_1$. There are two cases: $\alpha_p\alpha_1 = 0$ and $\alpha_p\alpha_1 \neq 0$.

First assume that $\alpha_p\alpha_1 = 0$. Then it follows from Corollary 1.2(1) that $\alpha_p\beta_1 \neq 0$, $\beta_q\alpha_1 \neq 0$ and $\beta_q\beta_1 = 0$. Now, for each vertex k with $k \neq 1$, e_kA is uniserial and, since A is symmetric, $\operatorname{top}(e_kA) \cong \operatorname{soc}(e_kA) \cong S_k$, the simple module at k. Therefore, for $i \neq 1$, we get a relation $\alpha_i \cdots \alpha_p(\delta\gamma)^{u_i}\delta\alpha_1 \cdots \alpha_i = 0$ with $\alpha_i \cdots \alpha_p(\delta\gamma)^{u_i}\delta\alpha_1 \cdots \alpha_{i-1} \neq 0$ for some integer u_i, and, for $j \neq 1$, we get a relation $\beta_j \cdots \beta_p(\gamma\delta)^{v_j}\gamma\beta_1 \cdots \beta_j = 0$ with $\beta_j \cdots \beta_p(\gamma\delta)^{v_j}\gamma\beta_1 \cdots \beta_{j-1} \neq 0$ for some integer v_j. Now consider e_1A. Since $\operatorname{rad}(e_1A) = \alpha_1A + \beta_1A$ and $\operatorname{soc}(e_1A) \cong S_1$, there is an element in $\operatorname{soc}(e_1A)$ of the form $(\gamma\delta)^r$ or $(\gamma\delta)^r\gamma$ for some integer r, and there is an element of $\operatorname{soc}(e_1A)$ of the form $(\delta\gamma)^s$ or $(\delta\gamma)^s\delta$ for some integer s. But, $\operatorname{soc}(e_1A)$ is simple so we must have a relation of one of the following forms:

(i) $(\gamma\delta)^r\gamma = c(\delta\gamma)^s \neq 0$ for some non-zero $c \in K$. Note that since I is admissible we must have $s \geqslant 1$. But then, if $r \geqslant s$, we would have $(\gamma\delta)^r\gamma = \gamma(\delta\gamma)^s(\delta\gamma)^{r-s} = c^{-1}\gamma(\gamma\delta)^r\gamma(\delta\gamma)^{r-s} = 0$, which is a contradiction, and if $r < s$ we would have $(\delta\gamma)^s = \delta(\gamma\delta)^{s-1-r}(\gamma\delta)^r\gamma = c\delta(\gamma\delta)^{s-1-r}(\delta\gamma)^s = 0$, which is also a contradiction. Therefore we cannot have this type of relation.

(ii) $(\gamma\delta)^r = c(\delta\gamma)^s\delta$ for some non-zero $c \in K$. As in the previous case, this relation cannot occur.

(iii) $(\gamma\delta)^r\gamma = c(\delta\gamma)^s\delta \neq 0$ for some non-zero $c \in K$. Then, if $r > s$, we have $(\gamma\delta)^r\gamma = \gamma(\delta\gamma)^r = \gamma(\delta\gamma)^s\delta(\gamma\delta)^{r-1-s}\gamma = c^{-1}\gamma(\gamma\delta)^r\gamma(\gamma\delta)^{r-1-s}\gamma = 0$, which is a contradiction, and if $s > r$ we have a similar contradiction. Therefore $s = r$. Now we also have $\alpha_2 \cdots \alpha_p(\delta\gamma)^{u_2}\delta\alpha_1\alpha_2 = 0$ so, multiplying on the left by α_1 and on the right by $\alpha_3 \cdots \alpha_p$, we get $\gamma(\delta\gamma)^{u_2+1} = 0$ so that $u_2 + 1 > r$. But $\alpha_2 \cdots \alpha_p(\delta\gamma)^r\delta\alpha_1 = c^{-1}\alpha_2 \cdots \alpha_p(\gamma\delta)^r\gamma\alpha_1 = 0$ so $u_2 < r$ which is impossible. Therefore a relation of this form cannot occur either.

(iv) $(\gamma\delta)^r = c(\delta\gamma)^s$ for some non-zero $c \in K$, and this is the only possible type of relation. Here again, if $r > s$, then $(\gamma\delta)^r = \gamma(\delta\gamma)^s(\delta\gamma)^{r-s-1}\delta = c^{-1}\gamma(\gamma\delta)^r(\delta\gamma)^{r-s-1}\delta = 0$ which is a contradiction and if $s > r$ we get a similar contradiction. Therefore $r = s$ so that the relation is $(\gamma\delta)^r = c(\delta\gamma)^r$ for some $r \geqslant 1$ and $c \in K^*$.

Given this relation, we are now able to determine the u_i and the v_j. Since $\alpha_i \cdots \alpha_p(\delta\gamma)^r\delta\alpha_1 \cdots \alpha_{i-1} = c^{-1}\alpha_i \cdots \alpha_p(\gamma\delta)^r\delta\alpha_1 \cdots \alpha_{i-1} = 0$, we must have $r > u_i$. Moreover,

$$(\gamma\delta)^{u_i+2} = \alpha_1 \cdots \alpha_{i-1}(\alpha_i \cdots \alpha_p(\delta\gamma)^{u_i}\delta\alpha_1 \cdots \alpha_i)\alpha_{i+1} \cdots \alpha_p\delta = 0$$

so that $u_i + 2 > r$. Hence $u_i = r - 1$ for all $i = 2, \ldots, p$. Similarly, $v_j = r - 1$ for all $j = 2, \ldots, q$. Moreover, we note that the relations $\alpha_p(\delta\gamma)^{r-1}\delta\alpha_1 \cdots \alpha_p = 0$ (when $i = p$) and $\beta_p(\gamma\delta)^{r-1}\gamma\beta_1 \cdots \beta_p = 0$ (when $j = q$) are superfluous, so are not required to give a minimal generating set of the ideal I_r.

Finally, we show that c must be equal to 1. Since A is symmetric, there exists a symmetric linear map $f : A \to K$ whose kernel does not contain any non-zero left or right ideal. In particular, the socle of A is not contained in $\operatorname{Ker} f$. But, from the relations obtained above, we see that the socle is generated as a K-vector space by all the paths obtained by cyclic permutations of $(\gamma\delta)^r$. Since f is symmetric it follows that $f((\gamma\delta)^r) \neq 0$. But we have

$$f((\gamma\delta)^r) = f((\delta\gamma)^r) = f(c(\gamma\delta)^r) = cf((\gamma\delta)^r)$$

so that $c = 1$.

Hence we have shown that $A \cong K\mathcal{Q}_{(p,q)}/I_r$.

Now assume that $\alpha_p\alpha_1 \neq 0$. Then it follows from Corollary 1.2(1) that $\alpha_p\beta_1 = 0$, $\beta_q\alpha_1 = 0$ and $\beta_q\beta_1 \neq 0$. The same methods as in the previous case show that we must have a relation of the form $\gamma^s = c\delta^t$ for some non-zero $c \in K$ and some positive integers s and t (by considering at the structure of e_1A) and relations $\alpha_i \cdots \alpha_p\gamma^{s-1}\alpha_1 \cdots \alpha_i$ for all $2 \leqslant i \leqslant p$ and $\beta_j \cdots \beta_q\delta^{t-1}\beta_1 \cdots \beta_j$ for all $2 \leqslant j \leqslant q$ (from the structure of the other indecomposable projectives and using the relation $\gamma^s = c\delta^t$). Moreover, since K is algebraically closed, we may replace α_1 by $c'\alpha_1$, where c' is a t-th root of c, and thus we may replace the relation $\gamma^s = c\delta^t$ by $\gamma^s = \delta^t$. Again we may find a minimal set of relations, and so conclude that $A \cong K\mathcal{Q}_{(p,q)}/J_{(s,t)}$. □

3. Hochschild cohomology groups

Our aim is now to investigate the derived equivalences among these algebras. It is well-known that Hochschild cohomology is an invariant under derived equivalence, and this section determines some of the Hochschild cohomology groups of the algebras $K\mathcal{Q}_{(p,q)}/I_r$ and $K\mathcal{Q}_{(p,q)}/J_{(s,t)}$, so that a full classification up to derived equivalence can be given in Section 4.

Let $\Gamma(p,q;r) = K\mathcal{Q}_{(p,q)}/I_r$ and $\Lambda(p,q;s,t) = K\mathcal{Q}_{(p,q)}/J_{(s,t)}$. The special cases of the algebras $\Gamma(p,q;1)$ and $\Lambda(1,n;2,2)$ were considered in [5], where, in their notation, we have $A(p,q) = K\mathcal{Q}_{(p,q)}/I_1 = \Gamma(p,q;1)$ and $\Lambda(n) = K\mathcal{Q}_{(1,n)}/J_{(2,2)} = \Lambda(1,n;2,2)$.

We begin by describing $\mathrm{HH}^0(A)$ and $\mathrm{HH}^1(A)$ for the algebras $A = \Lambda(p,q;s,t)$ and $A = \Gamma(p,q;r)$. Recall that $\mathrm{HH}^0(A) = Z(A)$, the centre of the algebra A.

3.1. $\mathrm{HH}^0(A)$ **and** $\mathrm{HH}^1(A)$ **for the algebra** $A = \Lambda(p,q;s,t)$**.** We begin with the algebra $\Lambda(p,q;s,t) = K\mathcal{Q}_{(p,q)}/J_{(s,t)}$ where $1 \leqslant p \leqslant q$. Recall that $\gamma = \alpha_1\alpha_2\cdots\alpha_p$ and $\delta = \beta_1\beta_2\cdots\beta_q$. Let $\gamma_i = \alpha_i\cdots\alpha_p\alpha_1\cdots\alpha_{i-1}$ for $i = 1,\ldots,p$, and let $\delta_j = \beta_j\cdots\beta_q\beta_1\cdots\beta_{j-1}$ for $j = 1,\ldots,q$, so that $\gamma = \gamma_1$ and $\delta = \delta_1$.

Proposition 3.1. *Consider the algebra $\Lambda(p,q;s,t)$ and let $1 \leqslant p \leqslant q$. Let $x = \sum_{i=1}^{p}\gamma_i$ and $y = \sum_{j=1}^{q}\delta_j$. Then*

$$\dim \mathrm{HH}^0(\Lambda(p,q;s,t)) = p+q+s+t-2$$

and the set

$$\left\{1, x, \ldots, x^{s-1}, y, \ldots, y^{t-1}, \gamma_i^s, \delta_j^t \text{ for } i = 1,\ldots,p \text{ and } j = 2,\ldots,q\right\}$$

is a K-basis of $\mathrm{HH}^0(\Lambda(p,q;s,t))$.

Proof. We note that $\gamma_i\beta_j = 0 = \beta_j\gamma_i$ and $\delta_j\alpha_i = 0 = \alpha_i\delta_j$ for all $i = 1,\ldots,p$, $j = 1,\ldots,q$ so $x, y \in Z(\Lambda(p,q;s,t))$. Moreover $\Lambda(p,q;s,t)$ is weakly symmetric so all socle elements are central, namely, γ_i^s and δ_j^t are central for $i = 1,\ldots,p$, $j = 2,\ldots,q$. The result now follows. □

We remark that Proposition 3.1 may be simplified if $p = q = 1$, as then $\Lambda(1,1;s,t)$ is the commutative algebra $K[\alpha,\beta]/(\alpha\beta, \alpha^s - \beta^t)$ with $s \geqslant 2, t \geqslant 2$. Thus $\mathrm{HH}^0(\Lambda(1,1;s,t)) = \Lambda(1,1;s,t)$ which has K-basis $\{1, \alpha, \ldots, \alpha^s, \beta, \ldots, \beta^{t-1}\}$ and dimension $s+t$.

In order to compute the first Hochschild cohomology group, we now fix a minimal set of generators of the ideal $J_{(s,t)}$, and denote this set by g^2.

Proposition 3.2. *Consider the algebra $\Lambda(p,q;s,t)$ with $1 \leqslant p \leqslant q$.*

If $p \geqslant 2$ then the following elements form a minimal set of generators for the ideal $J_{(s,t)}$:

$$\begin{array}{lcl} g_1^2 & = & \gamma^s - \delta^t \\ g_i^2 & = & \alpha_i\cdots\alpha_p\gamma^{s-1}\alpha_1\cdots\alpha_i \text{ for all } 2 \leqslant i \leqslant p-1 \\ g_p^2 & = & \alpha_p\beta_1 \\ g_{p+1}^2 & = & \beta_q\alpha_1 \\ g_{p+j}^2 & = & \beta_j\cdots\beta_q\delta^{t-1}\beta_1\cdots\beta_j \text{ for all } 2 \leqslant j \leqslant q-1. \end{array}$$

If $p = 1$ then the following elements form a minimal set of generators for the ideal $J_{(s,t)}$:

$$\begin{array}{lcl} g_0^2 & = & \alpha^s - \delta^t \\ g_1^2 & = & \alpha\beta_1 \\ g_2^2 & = & \beta_q\alpha \\ g_{j+1}^2 & = & \beta_j\cdots\beta_q\delta^{t-1}\beta_1\cdots\beta_j \text{ for all } 2 \leqslant j \leqslant q-1. \end{array}$$

We now compute the first Hochschild cohomology group $\mathrm{HH}^1(\Lambda(p,q;s,t))$. We use the explicit description of the start of a minimal projective bimodule resolution (P^*, d^*) for $\Lambda(p,q;s,t)$ as given in [7]. All tensors are over K so we write $\otimes$ for $\otimes_K$. For ease of notation, write

Λ for $\Lambda(p,q;s,t)$. Let $P^2 = \bigoplus_k \Lambda\mathfrak{o}(g_k^2) \otimes \mathfrak{t}(g_k^2)\Lambda$, $P^1 = \bigoplus_{a \text{ arrow}} \Lambda\mathfrak{o}(a) \otimes \mathfrak{t}(a)\Lambda$ and $P^0 = \bigoplus_{v=1}^{p+q-1} \Lambda e_v \otimes e_v \Lambda$. Then the minimal projective bimodule resolution of Λ begins

$$\cdots \longrightarrow P^2 \xrightarrow{d^2} P^1 \xrightarrow{d^1} P^0 \xrightarrow{d^0} \Lambda \longrightarrow 0$$

with the following maps. The map d^0 is the usual multiplication map. The map $d^1 \colon P^1 \to P^0$ is given by

$$d^1 \colon \mathfrak{o}(a) \otimes \mathfrak{t}(a) \mapsto \mathfrak{o}(a) \otimes a - a \otimes \mathfrak{t}(a)$$

where the first term $\mathfrak{o}(a) \otimes a$ lies in the summand $\Lambda\mathfrak{o}(a) \otimes \mathfrak{o}(a)\Lambda$ and the second term $a \otimes \mathfrak{t}(a)$ lies in the summand $\Lambda\mathfrak{t}(a) \otimes \mathfrak{t}(a)\Lambda$. Now, each element of g^2 is a linear combination of paths in $K\mathcal{Q}_{(p,q)}$ so, for $x \in g^2$, we may write

$$x = \sum_{j=1}^{r} c_j a_{1j} \dots a_{kj} \dots a_{s_j j}$$

where $c_j \in K$ and the a_{kj} are arrows in $\mathcal{Q}_{(p,q)}$. With this notation for $x \in g^2$, the map $d^2 \colon P^2 \to P^1$ is given by

$$d^2 \colon \mathfrak{o}(x) \otimes \mathfrak{t}(x) \mapsto \sum_{j=1}^{r} c_j \sum_{k=1}^{s_j} a_{1j} \dots a_{k-1j} \otimes a_{k+1j} \dots a_{s_j j}$$

where the term $a_{1j} \dots a_{k-1j} \otimes a_{k+1j} \dots a_{s_j j}$ lies in the summand $\Lambda\mathfrak{o}(a_{kj}) \otimes \mathfrak{t}(a_{kj})\Lambda$ of P^1.

We now apply $\mathrm{Hom}_{\Lambda^e}(-,-)$ to this resolution, where $\Lambda^e = \Lambda \otimes \Lambda^{op}$ is the enveloping algebra of Λ. Let $\partial^1 : \mathrm{Hom}_{\Lambda^e}(P^1,\Lambda) \to \mathrm{Hom}_{\Lambda^e}(P^2,\Lambda)$ be the map induced by d^2 and let $\partial^0 : \mathrm{Hom}_{\Lambda^e}(P^0,\Lambda) \to \mathrm{Hom}_{\Lambda^e}(P^1,\Lambda)$ be the map induced by d^1. Then $\mathrm{HH}^1(\Lambda) = \mathrm{Ker}\,\partial^1 / \mathrm{Im}\,\partial^0$.

Proposition 3.3. *If $q \geqslant 2$ then*

$$\dim \mathrm{HH}^1(\Lambda(p,q;s,t)) = \begin{cases} s+t & \text{if } \mathrm{char}\, K \mid \gcd(s,t) \\ s+t-1 & \text{otherwise.} \end{cases}$$

If $q = 1$ then $p = 1$ and

$$\dim \mathrm{HH}^1(\Lambda(1,1;s,t)) = \begin{cases} s+t+1 & \text{if } \mathrm{char}\, K \mid \gcd(s,t) \\ s+t & \text{otherwise.} \end{cases}$$

Proof. There are three cases to consider.
Case I: $p \geqslant 2$

We start by calculating $\mathrm{Im}\,\partial^0$. Let $\varphi \in \mathrm{Hom}(P^0, \Lambda(p,q;s,t))$ so that $\partial^0(\varphi) = \varphi d^1$. Suppose that φ is given by

$$\begin{array}{rrl} \varphi : & e_1 \otimes e_1 & \mapsto\ c_{1,0}e_1 + c_{1,1}\gamma_1 + \cdots + c_{1,s}\gamma_1^s + d_{1,1}\delta_1 + \cdots + d_{1,t-1}\delta_1^{t-1} \\ & e_i \otimes e_i & \mapsto\ c_{i,0}e_i + c_{i,1}\gamma_i + \cdots + c_{i,s}\gamma_i^s \ \text{ for } i = 2,\dots,p \\ & e_{p-1+i} \otimes e_{p-1+i} & \mapsto\ d_{i,0}e_{p-1+i} + d_{i,1}\delta_i + \cdots + d_{i,t}\delta_i^t \ \text{ for } i = 2,\dots,q \end{array}$$

where $c_{i,j}, d_{i,j} \in K$.

We have

$$\begin{array}{rcl} \varphi d^1(\mathfrak{o}(\alpha_1) \otimes \mathfrak{t}(\alpha_1)) & = & \varphi(e_1 \otimes \alpha_1 - \alpha_1 \otimes e_2) \ = \ \varphi(e_1 \otimes e_1)\alpha_1 - \alpha_1\varphi(e_2 \otimes e_2) \\ & = & (c_{1,0} - c_{2,0})\alpha_1 + (c_{1,1} - c_{2,1})\gamma_1\alpha_1 + \cdots + (c_{1,s-1} - c_{2,s-1})\gamma_1^{s-1}\alpha_1. \end{array}$$

In a similar way we get

$$\begin{aligned}
\varphi d^1(\mathfrak{o}(\alpha_2)\otimes\mathfrak{t}(\alpha_2)) &= \varphi(e_2\otimes\alpha_2-\alpha_2\otimes e_3)\\
&= (c_{2,0}-c_{3,0})\alpha_2+(c_{2,1}-c_{3,1})\gamma_2\alpha_2+\cdots+(c_{2,s-1}-c_{3,s-1})\gamma_2^{s-1}\alpha_2\\
&\vdots\\
\varphi d^1(\mathfrak{o}(\alpha_p)\otimes\mathfrak{t}(\alpha_p)) &= \varphi(e_p\otimes\alpha_p-\alpha_p\otimes e_1)\\
&= (c_{p,0}-c_{1,0})\alpha_p+(c_{p,1}-c_{1,1})\gamma_p\alpha_p+\cdots+(c_{p,s-1}-c_{1,s-1})\gamma_p^{s-1}\alpha_p\\
\varphi d^1(\mathfrak{o}(\beta_1)\otimes\mathfrak{t}(\beta_1)) &= \varphi(e_1\otimes\beta_1-\beta_1\otimes e_{p+1})\\
&= (c_{1,0}-d_{2,0})\beta_1+(d_{1,1}-d_{2,1})\delta_1\beta_1+\cdots+(d_{1,t-1}-d_{2,t-1})\delta_1^{t-1}\beta_1\\
&\vdots\\
\varphi d^1(\mathfrak{o}(\beta_q)\otimes\mathfrak{t}(\beta_q)) &= \varphi(e_{p+q-1}\otimes\beta_q-\beta_q\otimes e_1)\\
&= (d_{q,0}-c_{1,0})\beta_q+(d_{q,1}-d_{1,1})\delta_q\beta_q+\cdots+(d_{q,t-1}-d_{1,t-1})\delta_q^{t-1}\beta_q.
\end{aligned}$$

Thus $\dim\operatorname{Im}\partial^0=(p-1)s+(q-1)t$.

Now let $\psi\in\operatorname{Ker}\partial^1$ so that $\psi d^2=0$, and suppose that $\psi\in\operatorname{Hom}(P^1,\Lambda(p,q;s,t))$ is given by

$$\begin{array}{rll}
\psi: & \mathfrak{o}(\alpha_i)\otimes\mathfrak{t}(\alpha_i) & \mapsto\ c_{i,0}\alpha_i+c_{i,1}\gamma_i\alpha_i+\cdots+c_{i,s-1}\gamma_i^{s-1}\alpha_i\\
& \mathfrak{o}(\beta_j)\otimes\mathfrak{t}(\beta_j) & \mapsto\ d_{j,0}\beta_j+d_{j,1}\delta_j\beta_j+\cdots+d_{j,t-1}\delta_j^{t-1}\beta_j
\end{array}$$

for $i=1,\dots,p$, $j=1,\dots,q$ and where $c_{k,l},d_{k,l}\in K$.

From Proposition 3.2 with $p\geqslant 2$, and recalling that $\gamma=\gamma_1$ and $\delta=\delta_1$, it is easy to see that $\psi d^2(\mathfrak{o}(g_k^2)\otimes\mathfrak{t}(g_k^2))$ is immediately zero for $k=2,\dots,p+q-1$ and so we do not get any restrictions on the constants in the cases where g_k^2 is a monomial. It remains to consider $g_1^2=\gamma^s-\delta^t$. The condition $\psi d^2(\mathfrak{o}(g_1^2)\otimes\mathfrak{t}(g_1^2))=0$ gives that

$$(s(c_{1,0}+c_{2,0}+\cdots+c_{p,0})-t(d_{1,0}+d_{2,0}+\cdots+d_{q,0}))\gamma^s=0$$

so that $s(c_{1,0}+c_{2,0}+\cdots+c_{p,0})-t(d_{1,0}+d_{2,0}+\cdots+d_{q,0})=0$. Hence

$$\dim\operatorname{Ker}\partial^1=\begin{cases} ps+qt & \text{if } \operatorname{char}K\mid\gcd(s,t)\\ ps+qt-1 & \text{otherwise.}\end{cases}$$

Thus, for $p\geqslant 2$, we have

$$\dim\operatorname{HH}^1(\Lambda(p,q;s,t))=\begin{cases} s+t & \text{if } \operatorname{char}K\mid\gcd(s,t)\\ s+t-1 & \text{otherwise.}\end{cases}$$

Case II: $p=1$ and $q\geqslant 2$

To calculate $\operatorname{Im}\partial^0$, let $\varphi\in\operatorname{Hom}(P^0,\Lambda(1,q;s,t))$ so that $\partial^0(\varphi)=\varphi d^1$. Suppose that $\varphi\in\operatorname{Hom}(P^0,\Lambda(1,q;s,t))$ is given by

$$\begin{array}{rll}
\varphi: & e_1\otimes e_1 & \mapsto\ c_{1,0}e_1+c_{1,1}\alpha+\cdots+c_{1,s}\alpha^s+d_{1,1}\delta_1+\cdots+d_{1,t-1}\delta_1^{t-1}\\
& e_i\otimes e_i & \mapsto\ d_{i,0}e_i+d_{i,1}\delta_i+\cdots+d_{i,t}\delta_i^t
\end{array}$$

for $i=2,\dots,q$ and where $c_{1,j},d_{1,j},d_{i,j}\in K$.

We have

$$\varphi d^1(\mathfrak{o}(\alpha)\otimes\mathfrak{t}(\alpha))=\varphi(e_1\otimes\alpha-\alpha\otimes e_1)=\varphi(e_1\otimes e_1)\alpha-\alpha\varphi(e_1\otimes e_1)=0$$

since $\alpha \in Z(\Lambda(1,q;s,t))$. Now,

$$\begin{aligned}\varphi d^1(\mathfrak{o}(\beta_1)\otimes\mathfrak{t}(\beta_1)) &= \varphi(e_1\otimes\beta_1-\beta_1\otimes e_2)\\ &= (c_{1,0}-d_{2,0})\beta_1+(d_{1,1}-d_{2,1})\delta_1\beta_1+\cdots+(d_{1,t-1}-d_{2,t-1})\delta_1^{t-1}\beta_1\end{aligned}$$

$$\vdots$$

$$\begin{aligned}\varphi d^1(\mathfrak{o}(\beta_q)\otimes\mathfrak{t}(\beta_q)) &= \varphi(e_q\otimes\beta_q-\beta_q\otimes e_1)\\ &= (d_{q,0}-c_{1,0})\beta_q+(d_{q,1}-d_{1,1})\delta_q\beta_q+\cdots+(d_{q,t-1}-d_{1,t-1})\delta_q^{t-1}\beta_q.\end{aligned}$$

Thus $\dim \operatorname{Im}\partial^0=(q-1)t$.

Now let $\psi\in\operatorname{Ker}\partial^1$ so that $\psi d^2=0$, and suppose that $\psi\in\operatorname{Hom}(P^1,\Lambda(1,q;s,t))$ is given by

$$\begin{aligned}\psi:\quad \mathfrak{o}(\alpha)\otimes\mathfrak{t}(\alpha) &\mapsto c_{1,0}e_1+c_{1,1}\alpha+\cdots+c_{1,s}\alpha^s+\tilde{c}_{1,1}\delta_1+\cdots+\tilde{c}_{1,t-1}\delta_1^{t-1}\\ \mathfrak{o}(\beta_i)\otimes\mathfrak{t}(\beta_i) &\mapsto d_{i,0}\beta_i+d_{i,1}\delta_i\beta_i+\cdots+d_{i,t-1}\delta_i^{t-1}\beta_i\end{aligned}$$

for $i=1,\dots,q$ and where $c_{1,j},\tilde{c}_{1,j},d_{i,j}\in K$.

From Proposition 3.2, the minimal generating set for $J_{(s,t)}$ is

$$\{g_0^2=\alpha^s-\delta^t,\ g_1^2=\alpha\beta_1,\ g_2^2=\beta_q\alpha,\ g_{j+1}^2=\beta_j\cdots\beta_q\delta^{t-1}\beta_1\cdots\beta_j\mid 2\leqslant j\leqslant q-1\}$$

where we recall that $\delta=\delta_1$. Starting with g_1^2, the equation $\psi d^2(\mathfrak{o}(g_1^2)\otimes\mathfrak{t}(g_1^2))=0$ gives that

$$\begin{aligned}0 &= \psi(\mathfrak{o}(\alpha)\otimes\mathfrak{t}(\alpha))\beta_1+\alpha\psi(\mathfrak{o}(\beta_1)\otimes\mathfrak{t}(\beta_1))\\ &= (c_{1,0}e_1+\tilde{c}_{1,1}\delta_1+\cdots+\tilde{c}_{1,t-1}\delta_1^{t-1})\beta_1.\end{aligned}$$

Hence $c_{1,0}=\tilde{c}_{1,1}=\cdots=\tilde{c}_{1,t-1}=0$. So we may immediately simplify our expression for ψ as

$$\begin{aligned}\psi:\quad \mathfrak{o}(\alpha)\otimes\mathfrak{t}(\alpha) &\mapsto c_{1,1}\alpha+\cdots+c_{1,s}\alpha^s\\ \mathfrak{o}(\beta_i)\otimes\mathfrak{t}(\beta_i) &\mapsto d_{i,0}\beta_i+d_{i,1}\delta_i\beta_i+\cdots+d_{i,t-1}\delta_i^{t-1}\beta_i\end{aligned}$$

for $i=1,\dots,q$. It then follows that $\psi d^2(\mathfrak{o}(g_j^2)\otimes\mathfrak{t}(g_j^2))$ is zero for $j=2,\dots,q$ and so we do not get any restrictions on the constants here. Finally,

$$0=\psi d^2(\mathfrak{o}(g_0^2)\otimes\mathfrak{t}(g_0^2))=(sc_{1,1}-t(d_{1,0}+d_{2,0}+\cdots+d_{q,0}))\alpha^s$$

so that $sc_{1,1}-t(d_{1,0}+d_{2,0}+\cdots+d_{q,0})=0$. Hence

$$\dim\operatorname{Ker}\partial^1=\begin{cases}s+qt & \text{if } \operatorname{char}K\mid\gcd(s,t)\\ s+qt-1 & \text{otherwise.}\end{cases}$$

Thus, for $q\geqslant 2$, we have

$$\dim\operatorname{HH}^1(\Lambda(1,q;s,t))=\begin{cases}s+t & \text{if } \operatorname{char}K\mid\gcd(s,t)\\ s+t-1 & \text{otherwise.}\end{cases}$$

Case III: $p=1=q$

To calculate $\operatorname{Im}\partial^0$, let $\varphi\in\operatorname{Hom}(P^0,\Lambda(1,1;s,t))$ so that $\partial^0(\varphi)=\varphi d^1$. We have

$$\varphi d^1(\mathfrak{o}(\alpha)\otimes\mathfrak{t}(\alpha))=\varphi(e_1\otimes\alpha-\alpha\otimes e_1)=\varphi(e_1\otimes e_1)\alpha-\alpha\varphi(e_1\otimes e_1)=0$$

since $\Lambda(1,1;s,t)$ is commutative. Similarly $\varphi d^1(\mathfrak{o}(\beta)\otimes\mathfrak{t}(\beta))=0$. Thus $\operatorname{Im}\partial^0=(0)$.

Hence $\operatorname{HH}^1(\Lambda(1,1;s,t))=\operatorname{Ker}\partial^1$. Let $\psi\in\operatorname{Ker}\partial^1$ so that $\psi d^2=0$, and suppose that $\psi\in\operatorname{Hom}(P^1,\Lambda(1,1;s,t))$ is given by

$$\begin{aligned}\psi:\quad \mathfrak{o}(\alpha)\otimes\mathfrak{t}(\alpha) &\mapsto c_{1,0}e_1+c_{1,1}\alpha+\cdots+c_{1,s}\alpha^s+d_{1,1}\beta+\cdots+d_{1,t-1}\beta^{t-1}\\ \mathfrak{o}(\beta)\otimes\mathfrak{t}(\beta) &\mapsto c_{2,0}e_1+c_{2,1}\alpha+\cdots+c_{2,s}\alpha^s+d_{2,1}\beta+\cdots+d_{2,t-1}\beta^{t-1}\end{aligned}$$

where $c_{i,j},d_{i,j}\in K$.

From Proposition 3.2, the minimal generating set for $J_{(s,t)}$ is $\{g_0^2=\alpha^s-\beta^t,\ g_1^2=\alpha\beta,\ g_2^2=\beta\alpha\}$. Starting with g_1^2, the equation $\psi d^2(\mathfrak{o}(g_1^2)\otimes\mathfrak{t}(g_1^2))=0$ gives that

$$c_{1,0}\beta+d_{1,1}\beta^2+\cdots+d_{1,t-2}\beta^{t-1}+c_{2,0}\alpha+c_{2,1}\alpha^2+\cdots+c_{2,s-2}\alpha^{s-1}+(d_{1,t-1}+c_{2,s-1})\alpha^s=0.$$

Hence $c_{1,0} = d_{1,1} = \cdots = d_{1,t-2} = c_{2,0} = c_{2,1} = \cdots = c_{2,s-2} = 0$ and $d_{1,t-1} + c_{2,s-1} = 0$. So we may simplify our expression for ψ as

$$\begin{array}{rcl} \psi : \mathfrak{o}(\alpha) \otimes \mathfrak{t}(\alpha) & \mapsto & c_{1,1}\alpha + \cdots + c_{1,s}\alpha^s + d_{1,t-1}\beta^{t-1} \\ \psi : \mathfrak{o}(\beta) \otimes \mathfrak{t}(\beta) & \mapsto & -d_{1,t-1}\alpha^{s-1} + c_{2,s}\alpha^s + d_{2,1}\beta + \cdots + d_{2,t-1}\beta^{t-1}. \end{array}$$

We then have that

$$0 = \psi d^2(\mathfrak{o}(g_0^2) \otimes \mathfrak{t}(g_0^2)) = sc_{1,1}\alpha^s - td_{2,1}\beta^t = (sc_{1,1} - td_{2,1})\alpha^s$$

and hence $sc_{1,1} - td_{2,1} = 0$. The final equation $\psi d^2(\mathfrak{o}(g_0^2) \otimes \mathfrak{t}(g_0^2)) = 0$ gives no new information. Hence $\psi \in \operatorname{Ker} \partial^1$ is given by

$$\begin{array}{rcl} \psi : \mathfrak{o}(\alpha) \otimes \mathfrak{t}(\alpha) & \mapsto & c_{1,1}\alpha + \cdots + c_{1,s}\alpha^s + d_{1,t-1}\beta^{t-1} \\ \psi : \mathfrak{o}(\beta) \otimes \mathfrak{t}(\beta) & \mapsto & -d_{1,t-1}\alpha^{s-1} + c_{2,s}\alpha^s + d_{2,1}\beta + \cdots + d_{2,t-1}\beta^{t-1} \end{array}$$

with the additional linear dependency that $sc_{1,1} - td_{2,1} = 0$. Therefore

$$\dim \mathrm{HH}^1(\Lambda(1,1;s,t)) = \dim \operatorname{Ker} \partial^1 = \begin{cases} s+t+1 & \text{if } \operatorname{char} K \mid \gcd(s,t) \\ s+t & \text{otherwise.} \end{cases}$$

This completes the proof. □

3.2. $\mathrm{HH}^0(A)$ **and** $\mathrm{HH}^1(A)$ **for the algebras** $A = \Gamma(p,q;r)$**.** We now turn to the algebras $\Gamma(p,q;r) = K\mathcal{Q}_{(p,q)}/I_r$. Set $\eta_i = \alpha_i \cdots \alpha_p \delta \alpha_1 \cdots \alpha_{i-1}$ for $1 \leqslant i \leqslant p$ (so that $\eta_1 = \gamma\delta$) and $\theta_j = \beta_j \cdots \beta_q \gamma \beta_1 \cdots \beta_{j-1}$ for $1 \leqslant j \leqslant q$ (so that $\theta_1 = \delta\gamma$).

Proposition 3.4. *Let $p \leqslant q$ be positive integers and consider the algebra $\Gamma(p,q;r)$. Set $z = \sum_{i=1}^{p} \eta_i + \sum_{j=1}^{q} \theta_j$. Then*

$$\dim \mathrm{HH}^0(\Gamma(p,q;r)) = \begin{cases} p+q+r-1 & \text{if } p > 1 \\ q+r+1 & \text{if } p = 1 \text{ and } q > 1 \\ r+3 & \text{if } p = 1 = q, \end{cases}$$

and a basis for $\mathrm{HH}^0(\Gamma(p,q;r))$ is given by

$$\begin{aligned} &\left\{1; (\gamma\delta)^r; z^k, 1 \leqslant k \leqslant r-1; \eta_i^r; \theta_j^r \text{ for } 2 \leqslant i \leqslant p \text{ and } 2 \leqslant j \leqslant q\right\} \text{ if } p > 1 \\ &\left\{1; (\gamma\delta)^r; z^k, 1 \leqslant k \leqslant r-1; (\delta\gamma)^{r-1}\delta; \theta_j^r \text{ for } 2 \leqslant j \leqslant q\right\} \text{ if } p = 1 \text{ and } q > 1 \\ &\left\{1; (\gamma\delta)^r; z^k, 1 \leqslant k \leqslant r-1; (\gamma\delta)^{r-1}\gamma; (\delta\gamma)^{r-1}\delta\right\} \text{ if } p = 1 = q. \end{aligned}$$

Proof. It is clear that $(\gamma\delta)^r$, η_i^r and θ_j^r are in the centre of $\Gamma(p,q;r)$ since they are socle elements, and it is easy to check that $(\gamma\delta)^{r-1}\gamma$ and $(\delta\gamma)^{r-1}\delta$ are in the centre in the appropriate cases.

Conversely, a central element ζ must be in $\bigoplus_{v=1}^{p+q-1} e_v\Gamma(p,q;r)e_v$ and therefore is a linear combination of $(\gamma\delta)^k$ for $0 \leqslant k \leqslant r$, $(\delta\gamma)^k$ for $1 \leqslant k \leqslant r-1$, $(\gamma\delta)^k\gamma$ and $(\delta\gamma)^k\delta$ for $0 \leqslant k \leqslant r-1$, and η_i^k and θ_j^k for $2 \leqslant i \leqslant p$, $2 \leqslant j \leqslant q$ and $1 \leqslant k \leqslant r$. Writing the equations $\alpha_i\zeta = \zeta\alpha_i$ and $\beta_j\zeta = \zeta\beta_j$ gives the result (noting that $\eta_i^k = \alpha_i \cdots \alpha_p(\delta\gamma)^{k-1}\delta\alpha_1 \cdots \alpha_{i-1}$ and $\theta_j^k = \beta_j \cdots \beta_q(\gamma\delta)^{k-1}\gamma\beta_1 \cdots \beta_{j-1}$). □

We now use the same method as for $\Lambda(p,q;s,t)$ to compute $\mathrm{HH}^1(\Gamma(p,q;r))$, starting with a minimal set g^2 of generators of I_r.

Proposition 3.5. *Consider the algebra $\Gamma(p,q;r)$ with $1 \leqslant p \leqslant q$.*

If $p \geqslant 2$ then the following elements form a minimal set of generators for the ideal I_r:

$$\begin{array}{rcl} g_1^2 & = & (\gamma\delta)^r - (\delta\gamma)^r \\ g_i^2 & = & \eta_i^r\alpha_i \text{ for all } 2 \leqslant i \leqslant p-1 \\ g_p^2 & = & \alpha_p\alpha_1 \\ g_{p+1}^2 & = & \beta_q\beta_1 \\ g_{p+j}^2 & = & \theta_j^r\beta_j \text{ for all } 2 \leqslant j \leqslant q-1. \end{array}$$

If $p = 1$ then the following elements form a minimal set of generators for the ideal I_r:

$$\begin{array}{lcl} g_0^2 & = & (\gamma\delta)^r - (\delta\gamma)^r \\ g_1^2 & = & \alpha^2 \\ g_2^2 & = & \beta_q\beta_1 \\ g_{j+1}^2 & = & \theta_j^r\beta_j \text{ for all } 2 \leqslant j \leqslant q-1. \end{array}$$

The proof of the next result giving the first Hochschild cohomology group is a similar calculation to that of Proposition 3.3 and so is left to the reader.

Proposition 3.6. *If $p \geqslant 2$ then*

$$\dim \mathrm{HH}^1(\Gamma(p,q;r)) = r+1.$$

If $p = 1$ and $q \geqslant 2$ then

$$\dim \mathrm{HH}^1(\Gamma(1,q;r)) = \begin{cases} r+4 & \text{if } \mathrm{char}\, K = 2 \\ r+2 & \text{if } \mathrm{char}\, K \neq 2. \end{cases}$$

If $q = 1$ then $p = 1$ and

$$\dim \mathrm{HH}^1(\Gamma(1,1;r)) = \begin{cases} 2r+6 & \text{if } \mathrm{char}\, K = 2 \\ 2r+2 & \text{if } \mathrm{char}\, K \neq 2. \end{cases}$$

3.3. **Higher Hochschild cohomology groups for $\Gamma(p,q;r)$.** In order to distinguish the algebras of the form $\Gamma(p,q;r)$ up to derived equivalence we need the dimensions of the Hochschild cohomology groups up to $\mathrm{HH}^{2p-2}(\Gamma(p,q;r))$. If $p = 1$, this is just $\mathrm{HH}^0(\Gamma(1,q;r))$ which we already know, so we shall assume that $p > 1$ in the remainder of this section. We begin by giving the start of a projective bimodule resolution of $\Gamma(p,q;r)$ to enable us to find these groups. For ease of notation, set $\Gamma = \Gamma(p,q;r)$.

The projective bimodules P^n in a minimal projective bimodule resolution of Γ are known from [8], specifically that the multiplicity of $\Gamma e_i \otimes e_j\Gamma$ as a direct summand of P^n is equal to the dimension of $\mathrm{Ext}^n_\Gamma(S_i, S_j)$, where S_k is the simple module at the vertex k. We thus define projective Γ-Γ-bimodules (equivalently Γ^e-modules) $P^0, P^1, \ldots, P^{2p}$ which will be the projectives in our minimal projective bimodule resolution for Γ.

Definition 3.7. Let $\Gamma = \Gamma(p,q;r)$ with $p > 1$. We define projective Γ-Γ-bimodules $P^0, P^1, \ldots, P^{2p}$ as follows.

$$P^0 = \bigoplus_{i=1}^{p+q-1} \Gamma e_i \otimes e_i\Gamma,$$

$$P^1 = \bigoplus_{i=1}^{p-1} \Gamma e_i \otimes e_{i+1}\Gamma \oplus \Gamma e_p \otimes e_1\Gamma \oplus \bigoplus_{j=2}^{q-1} \Gamma e_{p+j-1} \otimes e_{p+j}\Gamma \oplus \Gamma e_{p+q-1} \otimes e_1\Gamma \oplus \Gamma e_1 \otimes e_{p+1}\Gamma,$$

$$\begin{aligned} P^{2n} = {} & \bigoplus_{i=2}^{p-n} \Gamma e_i \otimes e_{i+n}\Gamma \oplus \bigoplus_{i=p-n+1}^{p-1} \Gamma e_i \otimes e_{i+n+1-p}\Gamma \oplus \Gamma e_p \otimes e_{n+1}\Gamma \oplus \bigoplus_{j=2}^{q-n} \Gamma e_{p+j-1} \otimes e_{p+j+n-1}\Gamma \\ & \oplus \bigoplus_{j=q-n+1}^{q-1} \Gamma e_{p+j-1} \otimes e_{p+j+n-q}\Gamma \oplus \Gamma e_{p+q-1} \otimes e_{p+n}\Gamma \oplus \Gamma e_1 \otimes e_1\Gamma, \text{ for } 1 \leqslant n < p, \end{aligned}$$

$$\begin{aligned} P^{2n-1} = {} & \bigoplus_{i=2}^{p-n} \Gamma e_i \otimes e_{i+n}\Gamma \oplus \bigoplus_{i=p-n+1}^{p-1} \Gamma e_i \otimes e_{i+n-p}\Gamma \oplus \Gamma e_p \otimes e_n\Gamma \oplus \Gamma e_1 \otimes e_{n+1}\Gamma \\ & \oplus \bigoplus_{j=2}^{q-n} \Gamma e_{p+j-1} \otimes e_{p+j+n-1}\Gamma \oplus \bigoplus_{j=q-n+2}^{q-1} \Gamma e_{p+j-1} \otimes e_{p+j+n-q-1}\Gamma \oplus \Gamma e_{p+q-1} \otimes e_{p+n-1}\Gamma \\ & \oplus \Gamma e_1 \otimes e_{p+n}\Gamma \oplus \Gamma e_{p+q-n} \otimes e_1\Gamma, \text{ for } 2 \leqslant n < p \end{aligned}$$

$$P^{2p-1} = \bigoplus_{i=1}^{p} \Gamma e_i \otimes e_i\Gamma \oplus \bigoplus_{j=2}^{q-p} \Gamma e_{p+j-1} \otimes e_{2p+j-1}\Gamma \oplus \bigoplus_{j=q-p+2}^{q} \Gamma e_{p+j-1} \otimes e_{2p+j-q-1}\Gamma$$
$$\oplus \Gamma e_q \otimes e_1\Gamma \oplus \Gamma e_1 \otimes e_{2p}\Gamma,$$
$$P^{2p} = \bigoplus_{i=1}^{p-1} \Gamma e_i \otimes e_{i+1}\Gamma \oplus \bigoplus_{j=2}^{q-p} \Gamma e_{p+j-1} \otimes e_{2p+j-1}\Gamma \oplus \bigoplus_{j=q-p+1}^{q} \Gamma e_{p+j-1} \otimes e_{2p+j-q}\Gamma$$
$$\oplus \Gamma e_p \otimes e_1\Gamma \oplus \Gamma e_1 \otimes e_1\Gamma.$$

The first maps $d^i : P^i \to P^{i-1}$, for $i = 1, 2, 3$ of a minimal projective bimodule resolution are given in [7]. We extend the resolution in [7] for our algebra Γ, in a similar way to [14], to make the following definition of maps $d^i : P^i \to P^{i-1}$ for $i = 1, \ldots, 2p$.

Definition 3.8. Let $\Gamma = \Gamma(p, q; r)$ with $p > 1$. We define Γ-Γ-bimodule homomorphisms $d^i : P^i \to P^{i-1}$, for $i = 1, \ldots, 2p$, as follows.

➢ $\boldsymbol{d^{2n-1} : P^{2n-1} \to P^{2n-2}}$, for $1 \leqslant n < p$,

- $e_i \otimes e_{i+n} \mapsto e_i \otimes \alpha_{i+n-1} - \alpha_i \otimes e_{i+n}$ for $2 \leqslant i \leqslant p - n$
- $e_i \otimes e_{i+n-p} \mapsto$
$$(-1)^{i+p+1} \Big[e_i \otimes \eta^r_{i+n-p} + (-1)^{(p-i)(n-1)} \alpha_i \cdots \alpha_p \otimes \alpha_1 \cdots \alpha_{n-p+i-1} - \eta^r_i \otimes e_{i+n-p}$$
$$- (-1)^n (-1)^{(p-i)(n-1)} \sum_{m=1}^{p-i} (-1)^{m(n-1)} \alpha_i \cdots \alpha_{p-m} \otimes \alpha_{n-m+1} \cdots \alpha_p \delta (\gamma\delta)^{r-1} \alpha_1 \cdots \alpha_{n-p+i-1}$$
$$+ (-1)^n (-1)^{(p-i)(n-1)} \sum_{m=p-i+2}^{n-1} (-1)^{m(n-1)} \alpha_i \cdots \alpha_p (\delta\gamma)^{r-1} \delta \alpha_1 \cdots \alpha_{p-m} \otimes \alpha_{n-m+1} \cdots \alpha_{n-p+i-1} \Big]$$
for $p - n + 2 \leqslant i \leqslant p$
- $e_{p-n+1} \otimes e_1 \mapsto$
$$e_{p-n+1} \otimes \alpha_p + (-1)^n \sum_{m=1}^{n-1} (-1)^{m(n-1)} \alpha_{p-n+1} \cdots \alpha_{p-m} \otimes \alpha_{n-m+1} \cdots \alpha_p \delta (\gamma\delta)^{r-1} - \alpha_{p-n+1} \cdots \alpha_p \otimes e_1$$
- $e_1 \otimes e_{n+1} \mapsto$
$$(-1)^{n-1} \left[e_1 \otimes \alpha_1 \cdots \alpha_n + (-1)^n \sum_{m=1}^{n-1} (-1)^{m(n-1)} \delta (\gamma\delta)^{r-1} \alpha_1 \cdots \alpha_{p-m} \otimes \alpha_{n-m+1} \cdots \alpha_n + (-1)^n \alpha_1 \otimes e_{n+1} \right]$$
- $e_{p+j-1} \otimes e_{p+j+n-1} \mapsto -e_{p+j-1} \otimes \beta_{j+n-1} + \beta_j \otimes e_{p+j+n-1}$ for $2 \leqslant j \leqslant q - n$
- $e_{p+j-1} \otimes e_{p+j+n-q-1} \mapsto$
$$(-1)^{j+q} \Big[e_{p+j-1} \otimes \theta^r_{j+n-q} + (-1)^{(q-j)(n-1)} \beta_j \cdots \beta_q \otimes \beta_1 \cdots \beta_{n+j-q-1} - \theta^r_j \otimes e_{p+j+n-q-1}$$
$$- (-1)^n (-1)^{(q-j)(n-1)} \sum_{m=1}^{q-j} (-1)^{m(n-1)} \beta_j \cdots \beta_{q-m} \otimes \beta_{n-m+1} \cdots \beta_q (\gamma\delta)^{r-1} \gamma \beta_1 \cdots \beta_{n-q+j-1}$$
$$+ (-1)^n (-1)^{(q-j)(n-1)} \sum_{m=q-j+2}^{n-1} (-1)^{m(n-1)} \beta_j \cdots \beta_q \gamma (\delta\gamma)^{r-1} \beta_1 \cdots \beta_{q-m} \otimes \beta_{n-m+1} \cdots \beta_{n-q+j-1} \Big]$$
for $q - n + 2 \leqslant j \leqslant q$
- $e_{p+q-n} \otimes e_1 \mapsto$
$$- \left[e_{p+q-n} \otimes \beta_q + (-1)^n \sum_{m=1}^{n-1} (-1)^{m(n-1)} \beta_{q-n+1} \cdots \beta_{q-m} \otimes \beta_{n-m+1} \cdots \beta_q (\gamma\delta)^{r-1} \gamma - \beta_{q-n+1} \cdots \beta_q \otimes e_1 \right]$$
- $e_1 \otimes e_{p+n} \mapsto$
$$(-1)^n \left[e_1 \otimes \beta_1 \cdots \beta_n + (-1)^n \sum_{m=1}^{n-1} (-1)^{m(n-1)} (\gamma\delta)^{r-1} \gamma \beta_1 \cdots \beta_{q-m} \otimes \beta_{n-m+1} \cdots \beta_n + (-1)^n \beta_1 \otimes e_{p+n} \right]$$

➢ $d^{2n} : P^{2n} \to P^{2n-1}$, for $1 \leqslant n < p$,

- $e_i \otimes e_{i+n} \mapsto$

$$\sum_{k=0}^{r} \eta_i^k \otimes \eta_{i+n}^{r-k} + \sum_{k=0}^{r-1} \Bigg[\sum_{m=0}^{p-i-n} \eta_i^k \alpha_i \cdots \alpha_{i+m} \otimes \alpha_{i+m+n+1} \cdots \alpha_p \delta \alpha_1 \cdots \alpha_{i+n-1} \eta_{i+n}^{r-k-1}$$
$$+ \sum_{m=0}^{i-2} \eta_i^k \alpha_i \cdots \alpha_p \delta \alpha_1 \cdots \alpha_m \otimes \alpha_{m+n+1} \cdots \alpha_{i+n-1} \eta_{i+n}^{r-k-1}$$
$$+ (-1)^n \sum_{j=0}^{q-n} \eta_i^k \alpha_i \cdots \alpha_p \beta_1 \cdots \beta_j \otimes \beta_{j+n+1} \cdots \beta_q \alpha_1 \cdots \alpha_{i+n-1} \eta_{i+n}^{r-k-1} \Bigg]$$

  for $2 \leqslant i \leqslant p-n$

- $e_i \otimes e_{i+n+1-p} \mapsto e_i \otimes \alpha_{i+n-p} - (-1)^n \alpha_i \otimes e_{i+n+1-p}$ for $p+1-n \leqslant i \leqslant p$

- $e_1 \otimes e_1 \mapsto$

$$\sum_{k=0}^{r-1} \Bigg[\sum_{i=0}^{p-n} \Big(\delta(\gamma\delta)^k \alpha_1 \cdots \alpha_i \otimes \alpha_{i+n+1} \cdots \alpha_p (\delta\gamma)^{r-k-1} + (-1)^n (\gamma\delta)^k \alpha_1 \cdots \alpha_i \otimes \alpha_{i+n+1} \cdots \alpha_p (\delta\gamma)^{r-k-1} \delta \Big)$$
$$+ \sum_{j=0}^{q-n} \Big(\gamma(\delta\gamma)^k \beta_1 \cdots \beta_j \otimes \beta_{j+n+1} \cdots \beta_q (\gamma\delta)^{r-k-1} + (-1)^n (\delta\gamma)^k \beta_1 \cdots \beta_j \otimes \beta_{j+n+1} \cdots \beta_q (\gamma\delta)^{r-k-1} \gamma \Big) \Bigg]$$

- $e_{p+j-1} \otimes e_{p+j+n-q} \mapsto e_{p+j-1} \otimes \beta_{j+n-q} - (-1)^n \beta_j \otimes e_{p+j+n-q}$ for $q+1-n \leqslant j \leqslant q$

- $e_{p+j-1} \otimes e_{p+j+n-1} \mapsto \sum_{k=0}^{r} \theta_j^k \otimes \theta_{j+n}^{r-k} + \sum_{k=0}^{r-1} \Bigg[\sum_{m=0}^{q-j-n} \theta_j^k \beta_j \cdots \beta_{j+m} \otimes \beta_{j+m+n+1} \cdots \beta_q \gamma \beta_1 \cdots \beta_{j+n-1} \theta_{j+n}^{r-k-1}$
$$+ \sum_{m=0}^{j-2} \theta_j^k \beta_j \cdots \beta_q \gamma \beta_1 \cdots \beta_m \otimes \beta_{m+n+1} \cdots \beta_{j+n-1} \theta_{j+n}^{r-k-1}$$
$$+ (-1)^n \sum_{i=0}^{p-n} \theta_j^k \beta_j \cdots \beta_q \alpha_1 \cdots \alpha_i \otimes \alpha_{i+n+1} \cdots \alpha_p \beta_1 \cdots \beta_{j+n-1} \theta_{j+n}^{r-k-1} \Bigg]$$

  for $2 \leqslant j \leqslant q-n$

➢ $d^{2p-1} : P^{2p-1} \to P^{2p-2}$

- $e_i \otimes e_i \mapsto$

$$(-1)^i \Big[e_i \otimes \eta_i^r + (-1)^{(p-i)(p-1)} \alpha_i \cdots \alpha_p \otimes \alpha_1 \cdots \alpha_{i-1} - \eta_i^r \otimes e_i$$
$$- (-1)^p (-1)^{(p-i)(p-1)} \sum_{m=1}^{p-i} (-1)^{m(p-1)} \alpha_i \cdots \alpha_{p-m} \otimes \alpha_{p-m+1} \cdots \alpha_p \delta (\gamma\delta)^{r-1} \alpha_1 \cdots \alpha_{i-1}$$
$$+ (-1)^p (-1)^{(p-i)(p-1)} \sum_{m=p-i+2}^{p-1} (-1)^{m(p-1)} \alpha_i \cdots \alpha_p (\delta\gamma)^{r-1} \delta \alpha_1 \cdots \alpha_{p-m} \otimes \alpha_{p-m+1} \cdots \alpha_{i-1} \Big]$$

  for $2 \leqslant i \leqslant p$

- $e_1 \otimes e_1 \mapsto$

$$e_1 \otimes \gamma - \sum_{m=1}^{p-1} (-1)^{m(p-1)} \alpha_1 \cdots \alpha_{p-m} \otimes \alpha_{p-m+1} \cdots \alpha_p \delta (\gamma\delta)^{r-1}$$
$$+ (-1)^p \sum_{m=1}^{p-1} (-1)^{m(p-1)} \delta (\gamma\delta)^{r-1} \alpha_1 \cdots \alpha_{p-m} \otimes \alpha_{p-m+1} \cdots \alpha_p + (-1)^p \gamma \otimes e_1$$

- $e_{p+j-1} \otimes e_{2p+j-1} \mapsto -e_{p+j-1} \otimes \beta_{p+j-1} + \beta_j \otimes e_{2p+j-1}$ for $2 \leqslant j \leqslant q-p$

- $e_{p+j-1}\otimes e_{2p+j-q-1}\mapsto$

$$(-1)^{j+q}\Big[e_{p+j-1}\otimes\theta^r_{p+j-q}+(-1)^{(q-j)(p-1)}\beta_j\cdots\beta_q\otimes\beta_1\cdots\beta_{p+j-q-1}-\theta^r_j\otimes e_{2p+j-q-1}$$
$$-(-1)^p(-1)^{(q-j)(p-1)}\sum_{m=1}^{q-j}(-1)^{m(p-1)}\beta_j\cdots\beta_{q-m}\otimes\beta_{p-m+1}\cdots\beta_q(\gamma\delta)^{r-1}\gamma\beta_1\cdots\beta_{p-q+j-1}$$
$$+(-1)^p(-1)^{(q-j)(p-1)}\sum_{m=q-j+2}^{p-1}(-1)^{m(p-1)}\beta_j\cdots\beta_q\gamma(\delta\gamma)^{r-1}\beta_1\cdots\beta_{q-m}\otimes\beta_{p-m+1}\cdots\beta_{p-q+j-1}\Big]$$

for $q-p+2\leqslant j\leqslant q$

- $e_q\otimes e_1\mapsto$

$$-\left[e_q\otimes\beta_q+(-1)^p\sum_{m=1}^{p-1}(-1)^{m(p-1)}\beta_{q-p+1}\cdots\beta_{q-m}\otimes\beta_{p-m+1}\cdots\beta_q(\gamma\delta)^{r-1}\gamma-\beta_{q-p+1}\cdots\beta_q\otimes e_1\right]$$

- $e_1\otimes e_{2p}\mapsto$

$$(-1)^p\left[e_1\otimes\beta_1\cdots\beta_p+(-1)^p\sum_{m=1}^{p-1}(-1)^{m(p-1)}(\gamma\delta)^{r-1}\gamma\beta_1\cdots\beta_{q-m}\otimes\beta_{p-m+1}\cdots\beta_p+(-1)^p\beta_1\otimes e_{2p}\right]$$

➢ $d^{2p}: P^{2p}\to P^{2p-1}$

- $e_i\otimes e_{i+1}\mapsto e_i\otimes\alpha_i-(-1)^p\alpha_i\otimes e_{i+1}\qquad$ for $1\leqslant i\leqslant p-1$
- $e_p\otimes e_1\mapsto e_p\otimes\alpha_p-(-1)^p\alpha_p\otimes e_1$
- $e_1\otimes e_1\mapsto$

$$\sum_{k=0}^{r-1}\Big[(-1)^p\delta(\gamma\delta)^k\otimes(\delta\gamma)^{r-k-1}+(\gamma\delta)^k\otimes(\delta\gamma)^{r-k-1}\delta$$
$$-\sum_{j=0}^{q-p}\Big(\gamma(\delta\gamma)^k\beta_1\cdots\beta_j\otimes\beta_{j+p+1}\cdots\beta_q(\gamma\delta)^{r-k-1}+(-1)^p(\delta\gamma)^k\beta_1\cdots\beta_j\otimes\beta_{j+p+1}\cdots\beta_q(\gamma\delta)^{r-k-1}\gamma\Big)\Big]$$

- $e_{p+j-1}\otimes e_{p+j+p-q}\mapsto e_{p+j-1}\otimes\beta_{j+p-q}-(-1)^p\beta_j\otimes e_{p+j+p-q}\qquad$ for $q+1-p\leqslant j\leqslant q$
- $e_{p+j-1}\otimes e_{2p+j-1}\mapsto$

$$\sum_{k=0}^{r}\theta^k_j\otimes\theta^{r-k}_{j+p}+\sum_{k=0}^{r-1}\Bigg[\sum_{m=0}^{q-j-p}\theta^k_j\beta_j\cdots\beta_{j+m}\otimes\beta_{j+m+p+1}\cdots\beta_q\gamma\beta_1\cdots\beta_{j+p-1}\theta^{r-k-1}_{j+p}$$
$$+\sum_{m=0}^{j-2}\theta^k_j\beta_j\cdots\beta_q\gamma\beta_1\cdots\beta_m\otimes\beta_{m+p+1}\cdots\beta_{j+p-1}\theta^{r-k-1}_{j+p}-\theta^k_j\beta_j\cdots\beta_q\otimes\beta_1\cdots\beta_{j+p-1}\theta^{r-k-1}_{j+p}\Bigg]$$

for $2\leqslant j\leqslant q-p$.

It remains to show that the projective bimodules and homomorphisms that we have defined do indeed give the start of a minimal projective bimodule resolution of Γ.

Theorem 3.9. *With the above notation,*

$$\cdots\longrightarrow P^{2p}\xrightarrow{d^{2p}}P^{2p-1}\xrightarrow{d^{2p-1}}\qquad\cdots\longrightarrow P^1\xrightarrow{d^1}P^0\longrightarrow\Gamma\longrightarrow 0$$

is the beginning of a minimal projective resolution of Γ *as a* Γ-Γ*-bimodule (when* $p>1$*).*

Proof. It may be verified directly from the definitions that $d^2=0$ and thus we have a complex. The strategy for proving exactness is identical to that of [14, Theorem 1.6] (and see [7, Proposition 2.8]), whereby we apply $(\Gamma/\operatorname{rad}(\Gamma)\otimes-)$ to the complex and show that the resulting sequence is a minimal projective resolution of $\Gamma/\operatorname{rad}(\Gamma)$ as a right Γ-module. Minimality is then immediate since we know that the projectives are those of a minimal projective resolution of Γ as a Γ-Γ-bimodule from [8]. □

We are now in a position to give the dimensions of the Hochschild cohomology groups up to $\mathrm{HH}^{2p-2}(\Gamma)$. We only give those in even degree since we shall not need the others. The details of the proof are left to the reader.

Theorem 3.10. *For $2 \leqslant n < p \leqslant q$ we have*

$$\dim \mathrm{HH}^{2n-2}(\Gamma) = \begin{cases} r & \text{if } n \text{ is odd and } \operatorname{char} K \nmid 2r \\ r+1 & \text{if } n \text{ is odd and } \operatorname{char} K \mid 2r \\ r & \text{if } n \text{ is even and } \operatorname{char} K \neq 2 \\ r+1 & \text{if } n \text{ is even and } \operatorname{char} K = 2, \end{cases}$$

and for $2 \leqslant p < q$ we have

$$\dim \mathrm{HH}^{2p-2}(\Gamma) = \begin{cases} r-1 & \text{if } p \text{ is odd and } \operatorname{char} K \nmid 2r \\ r & \text{if } p \text{ is odd, } \operatorname{char} K \neq 2 \text{ and } \operatorname{char} K \mid 2r \\ r+1 & \text{if } p \text{ is even and } \operatorname{char} K \neq 2 \\ r+2 & \text{if } \operatorname{char} K = 2. \end{cases}$$

4. Derived equivalence classes

It was shown in [5] that two algebras of the form $A(p,q) = \Gamma(p,q;1)$ or $\Lambda(n) = \Lambda(1,n;2,2)$ are derived equivalent if and only if they are isomorphic. The main result in this section is to extend this to all algebras of the form $\Gamma(p,q;r)$ and $\Lambda(p,q;s,t)$, and hence to all basic indecomposable finite-dimensional K-algebras A which are symmetric special biserial algebras with at most one non-uniserial indecomposable projective module.

We start with some properties of these algebras, all of which are invariants under derived equivalence.

Proposition 4.1. *Suppose $1 \leqslant p \leqslant q$. The algebras $\Gamma(p,q;r)$ and $\Lambda(p,q;s,t)$ have the following properties.*

(1) *The number of simples of* $\begin{cases} \Gamma(p,q;r) \text{ is } p+q-1, \\ \Lambda(p,q;s,t) \text{ is } p+q-1. \end{cases}$

(2) *The Cartan invariants of* $\begin{cases} \Gamma(p,q;r) \text{ are } \begin{cases} \underbrace{1,1,\ldots,1}_{p+q-2},4r & \text{if } r(p+q-2) \text{ is even,} \\ \underbrace{1,1,\ldots,1}_{p+q-3},2,2r & \text{if } r(p+q-2) \text{ is odd,} \end{cases} \\ \Lambda(p,q;s,t) \text{ are } \underbrace{1,1,\ldots,1}_{p+q-2},s+t+(p+q-2)st. \end{cases}$

(3) *The Cartan determinant of* $\begin{cases} \Gamma(p,q;r) \text{ is } 4r, \\ \Lambda(p,q;s,t) \text{ is } s+t+(p+q-2)st. \end{cases}$

Proof. **(1)** This is immediate from the number of vertices of the quiver $\mathcal{Q}_{(p,q)}$.

(2) Let $\mathcal{I}_m$ be the $m \times m$ identity matrix, let $\mathcal{J}_n$ be the $n \times n$ matrix with all entries equal to 1, and set $u = p+q-2$. We start with the algebra $\Gamma(p,q;r)$. The Cartan matrix of $\Gamma(p,q;r)$ is the $(p+q-1) \times (p+q-1)$ matrix

$$C_\Gamma = \begin{bmatrix} 4r & 2r & \cdots & 2r \\ 2r & & & \\ \vdots & & \mathcal{I}_u + r\mathcal{J}_u & \\ 2r & & & \end{bmatrix}.$$

The Smith normal form for C_Γ is

$$\begin{cases} \begin{bmatrix} \mathcal{I}_u & 0 \\ 0 & 4r \end{bmatrix} & \text{if } ru \text{ is even} \\ \\ \begin{bmatrix} \mathcal{I}_{u-1} & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2r \end{bmatrix} & \text{if } ru \text{ is odd,} \end{cases}$$

and thus the Cartan invariants of $\Gamma(p,q;r)$ are $\begin{cases} \underbrace{1,1,\ldots,1}_{p+q-2},4r & \text{if } ru \text{ is even} \\ \underbrace{1,1,\ldots,1}_{p+q-3},2,2r & \text{if } ru \text{ is odd.} \end{cases}$

Now consider the algebra $\Lambda(p,q;s,t)$. The Cartan matrix of $\Lambda(p,q;s,t)$ is the $(p+q-1)\times(p+q-1)$ matrix

$$C_\Lambda = \begin{bmatrix} s+t & t & \ldots & t & s & \ldots & s \\ t & & & & & & \\ \vdots & & \mathcal{I}_{p-1}+t\mathcal{J}_{p-1} & & & 0 & \\ t & & & & & & \\ s & & & & & & \\ \vdots & & 0 & & & \mathcal{I}_{q-1}+s\mathcal{J}_{q-1} & \\ s & & & & & & \end{bmatrix}.$$

The Smith normal form for C_Λ is

$$\begin{bmatrix} \mathcal{I}_u & 0 \\ 0 & s+t+ust \end{bmatrix},$$

so the Cartan invariants of $\Lambda(p,q;s,t)$ are $\underbrace{1,1,\ldots,1}_{p+q-2},s+t+(p+q-2)st$.

(3) This is immediate from (2). □

We now consider isomorphism classes of algebras of the same form. It is clear that $\Lambda(q,q;s,t) = \Lambda(q,q;t,s)$, and the next result shows that, with this one exception, two algebras both of the form $\Gamma(p,q;r)$ or of the form $\Lambda(p,q;s,t)$ are pairwise non-isomorphic.

Theorem 4.2. **(1)** *The algebras of the form $\Gamma(p,q;r)$ (with $1 \leqslant p \leqslant q$) are pairwise non-isomorphic.*

(2) *The algebras of the form $\Lambda(p,q;s,t)$ (with $1 \leqslant p \leqslant q$) are pairwise non-isomorphic with the exception that $\Lambda(q,q;s,t) = \Lambda(q,q;t,s)$.*

Proof. **(1)** First, suppose that the algebras $\Gamma(p,q;r)$ and $\Gamma(p',q';r')$ are isomorphic with $1 \leqslant p \leqslant q$, $1 \leqslant p' \leqslant q'$. Since both algebras are basic, the quivers are uniquely determined and hence $\mathcal{Q}_{(p,q)} = \mathcal{Q}_{(p',q')}$. Thus $p = p'$ and $q = q'$. From the Cartan determinant in Proposition 4.1**(3)**, we have that $r = r'$.

(2) Suppose that $\Lambda(p,q;s,t) \cong \Lambda(p',q';s',t')$ with $1 \leqslant p \leqslant q$, $1 \leqslant p' \leqslant q'$. Since both algebras are basic, we again have that $p = p'$ and $q = q'$. Then, using the zero-th Hochschild cohomology group from Proposition 3.1, we have $s+t = s'+t'$. Equality of the Cartan invariants from Proposition 4.1**(2)** gives that $st = s't'$. Hence s', t' are the two roots of the equation $x^2-(s+t)x+st = 0$. Thus we have either $s = s'$ and $t = t'$ (which gives us the algebra $\Lambda(p,q;s,s)$), or $s = t'$ and $t = s'$. In the latter case we have the algebras $\Lambda(p,q;s,t)$ and $\Lambda(p,q;t,s)$, and it remains to show $p = q$ when $s \neq t$. The K-dimension of $\Lambda(p,q;s,t)$ is $tp^2+sq^2+p+q-2$ and that of $\Lambda(p,q;t,s)$ is $sp^2+tq^2+p+q-2$. Thus $tp^2+sq^2 = sp^2+tq^2$ so that $(t-s)(p+q)(p-q) = 0$. Since $p+q > 0$ and $s \neq t$ we have $p = q$, which is precisely the case $\Lambda(q,q;s,t) = \Lambda(q,q;t,s)$. □

Our final theorem classifies up to derived equivalence, all basic indecomposable finite-dimensional K-algebras A which are symmetric special biserial algebras with at most one non-uniserial indecomposable projective module.

Theorem 4.3. **(1)** *An algebra of the form $\Lambda(p,q;s,t)$ (with $1 \leqslant p \leqslant q$) is derived equivalent to exactly one algebra in the following list:*

(a) *$\Lambda(1,p+q-1;s,t)$ with $2 \leqslant s \leqslant t$,*

(b) *N_M^{p+q-1} with $p+q > 2$ and $\min(s,t) = 1$, $\max(s,t) = M$.*

(2) *An algebra of the form* $\Gamma(p,q;r)$ *(with* $1 \leqslant p \leqslant q$*) is derived equivalent to an algebra of the form* $\Lambda(p,q;s,t)$ *if and only if they are isomorphic. This is only the case for* $\Gamma(1,1;1) \cong \Lambda(1,1;2,2)$ *and* $\operatorname{char} K \neq 2$.

(3) *The algebras* $\Gamma(p,q;r)$ *and* $\Gamma(p',q';r')$ *(with* $1 \leqslant p \leqslant q$ *and* $1 \leqslant p' \leqslant q'$*) are derived equivalent if and only if* $(p,q,r) = (p',q',r')$.

Proof. **(1)** The algebra $\Lambda(p,q;s,t)$ is the generalised Brauer tree algebra associated to the Brauer tree in Figure 1:

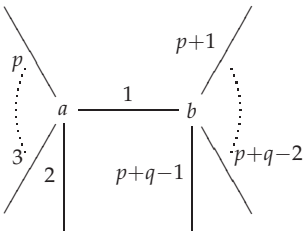


Figure 1

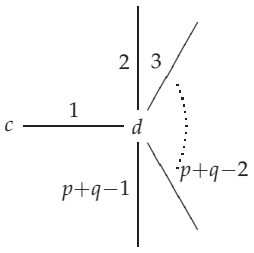


Figure 2

in which the vertices a and b have multiplicities respectively s and t (we refer to [4, Section 4.18] or [11] for the definition of a Brauer tree algebra and a generalised Brauer tree algebra).

It was proved in [11, Theorem 9.7] that generalised Brauer tree algebras up to derived equivalence depend only on the number of edges in the graph and the set of multiplicities. Therefore $\Lambda(p,q;s,t)$ is derived equivalent to the generalised Brauer tree algebra associated to the Brauer tree in Figure 2, in which the vertices c and d have multiplicities respectively $m = \min(s,t)$ and $M = \max(s,t)$ and $\{a,b\} = \{c,d\}$. This algebra is equal to

- either $K\Delta_{p+q-1}/L_M = N_M^{p+q-1}$ if $m = 1$, that is, if $(s,t) = (M,1)$ or $(s,t) = (1,M)$, with $M \geqslant 1$,
- or $\Lambda(1,p+q-1;m,M)$ if $m > 1$, that is, if $s \geqslant 2$ and $t \geqslant 2$.

Moreover, none of these algebras are derived equivalent, again by [11, Theorem 9.7].

(2) First, we show that the algebras $\Gamma(1,1;1)$ and $\Lambda(1,1;2,2)$ are isomorphic when $\operatorname{char} K \neq 2$. Since K is algebraically closed, let ε be a square root of -1 in K. Then the map

$$\varphi : \Gamma(1,1;1) \to \Lambda(1,1;2,2) \text{ given by } \begin{cases} \alpha \mapsto \alpha + \varepsilon\beta \\ \beta \mapsto \alpha - \varepsilon\beta \end{cases}$$

is an isomorphism of algebras.

Suppose that there is a derived equivalence between the algebras $\Gamma(p,q;r)$ and $\Lambda(p',q';s,t)$. Then the algebras have the same number of simple modules so, from Proposition 4.1, we have $p+q = p'+q'$. From **(1)**, the algebra $\Lambda(p',q';s,t)$ is derived equivalent to exactly one algebra in the list above. Moreover, $s+t = m+M$ where $m = \min(s,t)$ and $M = \max(s,t)$.

Case I: $m = \min(s,t) = 1$.

Let $\Gamma = \Gamma(p,q;r)$ and $\Lambda = \Lambda(p',q';s,t)$. From Propositions 3.1 and 3.3, $\dim \mathrm{HH}^0(\Lambda) = p+q+M-1$ and $\dim \mathrm{HH}^1(\Lambda) = M$. We first assume that $p = 1$, so that $q > 1$. From Proposition 3.4, $\dim \mathrm{HH}^0(\Gamma) = q+r+1$ so that $M = r+1$. However, Proposition 3.6 gives

$$\dim \mathrm{HH}^1(\Gamma) = \begin{cases} r+4 & \text{if } \operatorname{char} K = 2 \\ r+2 & \text{if } \operatorname{char} K \neq 2 \end{cases}$$

which is a contradiction. So $\Gamma(1,q;r)$ is not derived equivalent to $\Lambda(p',q';s,t)$.

On the other hand, if $p > 1$ then $\dim \mathrm{HH}^0(\Gamma) = p+q+r-1$ so that $M = r$. But $\dim \mathrm{HH}^1(\Gamma) = r+1$ so that $M = r+1$, a contradiction. Again $\Gamma(p,q;r)$ is not derived equivalent to $\Lambda(p',q';s,t)$.

Case II: $m = \min(s,t) > 1$.

We use Propositions 3.1, 3.3, 3.4 and 3.6 without further comment. We begin with the case $p = 1$. If $q = 1$ then $\dim \mathrm{HH}^0(\Lambda) = s + t$ and $\dim \mathrm{HH}^0(\Gamma) = r + 3$. From Proposition 4.1**(3)**, the Cartan determinant of Γ is $4r$ and of Λ is $s + t$. Hence $r + 3 = s + t = 4r$ so that $r = 1$ and $s = 2 = t$. If $\mathrm{char}\, K \neq 2$ then we have an isomorphism $\Gamma(1, 1; 1) \cong \Lambda(1, 1; 2, 2)$ from above. If $\mathrm{char}\, K = 2$, then $\dim \mathrm{HH}^1(\Lambda) = 5$ and $\dim \mathrm{HH}^1(\Gamma) = 8$, which is a contradiction and $\Gamma(1, 1; 1)$ is not derived equivalent to $\Lambda(1, 1; 2, 2)$. In the case where $q > 1$ (keeping $p = 1$), a similar consideration of the zero-th and first Hochschild cohomology groups and the Cartan determinant shows that $\Gamma(1, q; r)$ is not derived equivalent to $\Lambda(p', q'; s, t)$.

Now suppose that $p > 1$. We have $\dim \mathrm{HH}^0(\Lambda) = p + q + s + t - 2$ and $\dim \mathrm{HH}^0(\Gamma) = p + q + r - 1$ so that $r = s + t - 1$. So $\dim \mathrm{HH}^1(\Gamma) = r + 1 = s + t$, and we must have $\mathrm{char}\, K = \ell | \gcd(s, t)$. From Proposition 4.1**(3)**, the Cartan determinant of Γ is $4r$ and the Cartan determinant of Λ is $s + t + (p + q - 2)st$ so that $3(s + t) = (p + q - 2)st - 4$. Thus $\ell | 4$ so that $\ell = 2$, and s, t are both even. Thus we are in the situation where $\Gamma(p, q; r)$ is derived equivalent to $\Lambda(p', q'; s, t)$, $\mathrm{char}\, K = 2$, both s, t are even, $r = s + t - 1$, and $p + q = p' + q'$. We shall use Külshammer invariants and the same arguments as in [10, Subsection 4.5.2] for this case. Recall from Proposition 2.1 and its proof that the algebras $\Lambda(p', q'; s, t)$ and $\Gamma(p, q; r)$ are symmetric; moreover, once we have fixed a K-basis of paths for the socle of a symmetric algebra A and completed it to a K-basis $\mathcal{B}_A$ of paths for A, the linear map $f : A \to K$ that is defined on $\mathcal{B}_A$ by sending socle elements to 1 and the rest to 0 defines a symmetric non-degenerate associative bilinear form on A. Orthogonals will be taken with respect to this bilinear form. Let $\kappa(A)$ be the commutator subspace of A and, for any non-negative integer n, define $T_n(A) = \left\{x \in A; x^{2^n} \in \kappa(A)\right\}$. It was proved in [15] that the generalised Reynolds ideals (or Külshammer invariants) $T_n(A)^\perp$ are derived invariant. Note that $\mathrm{soc}(A) \subseteq T_n(A)^\perp \subsetneq Z(A)$ for every n. It is well-known that the centre $Z(A)$ is a derived invariant. Given a vector space V, let $\mathcal{B}_V$ denote a basis of V.

- We start with the algebra $\Gamma = \Gamma(p, q; r)$. A basis $\mathcal{B}_\Gamma$ of paths of Γ is given by the union over all i, j with $1 \leqslant i, j \leqslant p + q - 1$ of all paths from e_i to e_j of length at most $(p + q)r$ except $(\delta\gamma)^r$. Recall from Proposition 2.1 the basis $\mathcal{B}_{\mathrm{soc}(\Gamma)} = \left\{(\gamma\delta)^r; \eta_i^r, 2 \leqslant i \leqslant p; \theta_i^r, 2 \leqslant j \leqslant q\right\} \subset \mathcal{B}_\Gamma$ of $\mathrm{soc}(\Gamma)$ and from Proposition 3.4 the basis $\mathcal{B}_{Z(\Gamma)} = \left\{1; (\gamma\delta)^r; z^k, 1 \leqslant k \leqslant r - 1; \eta_i^r; \theta_j^r \text{ for } 2 \leqslant i \leqslant p \text{ and } 2 \leqslant j \leqslant q\right\}$ of $Z(\Gamma)$ where $z = \sum_{i=1}^p \eta_i + \sum_{j=1}^q \theta_j$. Then $\dim \kappa(\Gamma) = r\left((p + q)^2 - 1\right) - 1$ and a basis of $\kappa(\Gamma)$ is given by

$$\mathcal{B}_{\kappa(\Gamma)} = \{\gamma(\delta\gamma)^k; \delta(\gamma\delta)^k; (\gamma\delta)^{k+1} - (\delta\gamma)^{k+1}; \eta_i^{k+1} - \eta_1^{k+1}, 1 \leqslant i \leqslant p; \theta_j^{k+1} - \theta_1^{k+1}, 2 \leqslant j \leqslant q; \\ \text{for } 0 \leqslant k \leqslant r - 1\} \cup \{b \in \mathcal{B}_\Gamma; \mathfrak{o}(b) \neq \mathfrak{t}(b)\}.$$

  Now, as in [10, Subsection 4.5.2], work in $\Gamma/\kappa(\Gamma)$ to find a basis $\mathcal{B}_{T_1(\Gamma)} = \mathcal{B}_{\kappa(\Gamma)} \cup \left\{(\gamma\delta)^k; \frac{r+1}{2} \leqslant k \leqslant r\right\}$ of $T_1(\Gamma)$ (recall that r is odd), then work in $Z(\Gamma)/\mathrm{soc}(\Gamma)$ to find a basis $\mathcal{B}_{T_1(\Gamma)^\perp} = \mathcal{B}_{\mathrm{soc}(\Gamma)} \cup \left\{z^k; \frac{r+1}{2} \leqslant k \leqslant r - 1\right\}$ for $T_1(\Gamma)^\perp$ so that $\Gamma' := Z(\Gamma)/T_1(\Gamma)^\perp$ has basis $\mathcal{B}_{\Gamma'} = \left\{1; z^k; 1 \leqslant k \leqslant \frac{r-1}{2}\right\}$. We now consider the Jacobson radical $\mathfrak{r}_{\Gamma'}$ of the algebra Γ', which is spanned by $\left\{z^k; 1 \leqslant k \leqslant \frac{r-1}{2}\right\}$, and its square $\mathfrak{r}_{\Gamma'}^2$, which is spanned by $\left\{z^k; 2 \leqslant k \leqslant \frac{r-1}{2}\right\}$ so that $\dim_K \mathfrak{r}_{\Gamma'}/\mathfrak{r}_{\Gamma'}^2 = 1$.
- We now turn to the algebra $\Lambda = \Lambda(p', q'; s, t)$. Since they are derived equivalent, we may assume that $\Lambda = \Lambda(1, n; m, M)$ with $n = p + q - 1$ (to simplify notation). Note that m and M are even. Set $\alpha = \alpha_1$. We follow the same method as for Γ, using

Propositions 2.1 and 3.1.

$$\mathcal{B}_\Lambda = \bigcup_{1\leqslant i,j\leqslant p+q-1} \{\text{all paths from } e_i \text{ to } e_j \text{ of length at most } nM \text{ that do not contain } \alpha\} \cup \left\{\alpha^k, 1 \leqslant k \leqslant m-1\right\}$$

$$\mathcal{B}_{\mathrm{soc}(\Lambda)} = \{\delta_i^m, 1 \leqslant i \leqslant n\}$$

$$\mathcal{B}_{Z(\Lambda)} = \left\{1; \alpha^k, 1 \leqslant k \leqslant m; y^\ell, 1 \leqslant \ell \leqslant t-1; \delta_j^M, 2 \leqslant j \leqslant q\right\} \text{ where } y = \sum_{j=1}^{q} \delta_j$$

$$\mathcal{B}_{\kappa(\Lambda)} = \left\{\delta^\ell - \delta_j^\ell; 2 \leqslant j \leqslant q, 1 \leqslant \ell \leqslant M\right\} \cup \{b \in \mathcal{B}_\Lambda; \mathfrak{o}(b) \neq \mathfrak{t}(b)\}$$

$$\mathcal{B}_{T_1(\Lambda)} = \mathcal{B}_{\kappa(\Lambda)} \cup \left\{\alpha^k, \frac{m}{2} < k \leqslant m; \delta^\ell, \frac{M}{2} < \ell \leqslant M; \alpha^{m/2} + \delta^{M/2}\right\}$$

$$\mathcal{B}_{T_1(\Lambda)^\perp} = \mathcal{B}_{\mathrm{soc}(\Lambda)} \cup \left\{\alpha^k, \frac{m}{2} < k \leqslant m-1; y^\ell, \frac{M}{2} < \ell \leqslant M-1; \alpha^{m/2} + y^{M/2}\right\}.$$

So the Jacobson radical $\mathfrak{r}_{\Lambda'}$ of $\Lambda' := Z(\Lambda)/T_1(\Lambda)^\perp$ has basis $\left\{\alpha^k, 1 \leqslant k \leqslant \frac{m}{2}; y^\ell, 1 \leqslant \ell < \frac{M}{2}\right\}$ and $\mathfrak{r}_{\Lambda'}^2$ has basis $\left\{\alpha^k, 2 \leqslant k \leqslant \frac{m}{2}; y^\ell, 2 \leqslant \ell < \frac{M}{2}\right\}$ so that

$$\dim_K \mathfrak{r}_{\Lambda'}/\mathfrak{r}_{\Lambda'}^2 = \begin{cases} 2 \text{ if } M \geqslant 4 \\ 1 \text{ if } M = 2 \text{ (and therefore } m = 2). \end{cases}$$

Since we assumed that Λ and Γ are derived equivalent, the algebras Λ' and Γ' are isomorphic and hence we have $\dim_K \mathfrak{r}_{\Gamma'}/\mathfrak{r}_{\Gamma'}^2 = \dim_K \mathfrak{r}_{\Lambda'}/\mathfrak{r}_{\Lambda'}^2$, which implies that $m = 2 = M$, that is, $s = 2 = t$. Therefore $r = s + t - 1 = 3$. We now use the Cartan determinants: $0 = \det C_\Gamma - \det C_\Lambda = 4r - (s + t + (p + q - 2)st) = 12 - (4 + (p + q - 2)4) = 4(4 - (p + q))$ so that $p + q = 4$. Since $1 < p \leqslant q$, we must have $p = 2 = q$. Therefore $\Gamma = \Gamma(2, 2; 3)$ and Λ is derived equivalent to $\Lambda(2, 2; 2, 2)$. However, it was shown in [9, Section 3] that $\Gamma(2, 2; 3) = \mathcal{D}(3\mathcal{A})_1^3$ and $\Lambda(2, 2; 2, 2) = \mathcal{D}(3\mathcal{A})_2^{2,2}$ are not derived equivalent. Therefore Λ and Γ are not derived equivalent.

(3) If the algebras $\Gamma(p, q; r)$ and $\Gamma(p', q'; r')$ (with $p \leqslant q$ and $p' \leqslant q'$) are derived equivalent, then from the Cartan determinant and number of simples (Proposition 4.1) we know that $r = r'$ and $p + q = p' + q'$. Assume for contradiction that $(p, q) \neq (p', q')$. We may suppose that $p < p'$. It follows that $p < q$ (for otherwise $p = q$ and hence $q' = (p' + q') - p' = (p + q) - p' = 2p - p' < 2p' - p' = p'$, a contradiction). Using Theorem 3.10 and Proposition 3.4, we then have

$$\mathrm{HH}^{2p-2}(\Gamma(p, q; r)) \neq \mathrm{HH}^{2p-2}(\Gamma(p', q'; r'))$$

which contradicts the fact that the algebras are derived equivalent. Thus $(p, q) = (p', q')$. □

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